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Abelian covers of alternating groups

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ABELIAN COVERS OF ALTERNATING GROUPS

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ABSTRACT. Let $G = A_n$, a finite alternating group. We study the commuting graph of G and establish, for all possible values of n barring $13, 14, 17$ and 19, whether or not the independence number is equal to the clique-covering number.

1. INTRODUCTION

Throughout this paper G is a finite group.

1.1. **Definitions and main result.** Let Ω be a subset of the group G. A *non-commuting subset* of Ω is a subset N of Ω such that
$$x, y \in N \implies xy \neq yx.$$
We define $\delta(\Omega)$ to be the maximum possible cardinality of a non-commuting subset of Ω. A non-commuting subset that has cardinality $\delta(\Omega)$ will be called a *maximal non-commuting subset* of Ω.

An *abelian cover* of Ω is a set $\mathcal{C}$ of abelian subgroups of G whose union contains Ω. We define $\Delta(\Omega)$ to be the minimum possible cardinality of an abelian cover of Ω. An abelian cover that has cardinality $\Delta(\Omega)$ will be called a *minimal abelian cover* of Ω.

A simple application of the pigeon-hole principle yields the following fact:

Lemma 1.1. *For any finite group G and a subset $\Omega \subseteq G$, $\delta(\Omega) \leqslant \Delta(\Omega)$.*

In this paper we study the case where G is a finite alternating group; we are interested in ascertaining when $\delta(G) = \Delta(G)$. Our main result gives almost complete information.

Theorem 1.2. *Let A_n be the alternating group on n letters, with n a positive integer.*
  (1) *If $n \leqslant 11$ or $n = 15$, then $\delta(A_n) = \Delta(A_n)$;*
  (2) *If $n = 12, 16$ or 18 or $n \geqslant 20$, then $\delta(A_n) \neq \Delta(A_n)$.*

1.2. **The commuting graph.** The definitions just given can be recast in terms of a particular graph, as follows. Let G be a group and $\Omega \subseteq G$. Then the *commuting graph* of Ω, denoted $\Gamma(\Omega)$, is the graph whose vertex set is Ω and with vertices connected if and only if they commute. Now a cover of Ω by abelian subgroups of G corresponds to a cover of $\Gamma(\Omega)$ by cliques, and so $\Delta(\Omega)$ is the *clique-covering number* of $\Gamma(\Omega)$. Similarly, a non-commuting subset of Ω is an independent set in $\Gamma(\Omega)$, and $\delta(\Omega)$ is the *independence number* of the graph.

1.3. **Notation.** Our notation is entirely standard. The *support* of a permutation $g \in S_n$ is the set S of those elements j in $\{1, 2, 3, \ldots, n\}$ satisfying $g(j) \neq j$. Two permutations $g, h \in S_n$ are *disjoint* if they have disjoint supports. Let $n_1, n_2, \ldots, n_r$ be positive integers such that $n_1 + n_2 + \cdots + n_r \leqslant n$. We say that a permutation $g \in S_n$ has cycle type $n_1 - n_2 - \cdots - n_r$ if g can be written as a product of disjoint cycles $g = g_1 g_2 \cdots g_r$, where the cycle g_i is an n_i-cycle.

For $g \in G$, we write $Cl_G(g)$ for the conjugacy class of g in G (we will omit the subindex G, when the group G is clear from the context.) Also we will write $C_G(g)$ for the centralizer of g in G.

1.4. **Context.** The following result of Brown is a starting point for our research [Bro88, Bro91].

Theorem 1.3. *Let S_n be the symmetric group on n letters, with n a positive integer.*
  (1) *If $n \leqslant 7, n = 9$ or $n = 11$, then $\delta(S_n) = \Delta(S_n)$;*
  (2) *If $n = 8$, $n = 13$ or $n \geqslant 15$, then $\delta(S_n) \neq \Delta(S_n)$.*

One can view our main result, Theorem 1.2, as an analogue of Theorem 1.3 for the alternating groups. The study of groups via their commuting graph goes back many years. Perhaps the most famous result in this line of study concerns groups that are not finite: in 1976, B. H. Neumann answered a question of Erdős by proving that if all non-commuting sets in a group G are finite, then $|G : Z(G)|$ is finite [Neu76].

The question of whether or not $\delta(G) = \Delta(G)$ has been studied by various authors for various groups G (see, for instance, [AAMHZ10, AFO13, AP09, ABG]). To our knowledge, Theorem 1.2 is the first result that asserts that $\delta(G) \neq \Delta(G)$ for some finite *simple* group G.

2. Background

In this section we record some basic results and definitions that will be needed in the sequel. The lemmas that we will need are little more than observations. The proof of the first is left to the reader.

Lemma 2.1. *There exists an abelian cover of G of size $\Delta(G)$ and consisting entirely of maximal abelian subgroups.*

Lemma 2.2. *Let $g \in G$, and suppose that $C_G(g)$ is abelian. Then*
  (1) *$C_G(g)$ is a maximal abelian subgroup of G and it is the unique maximal abelian subgroup of G that contains g;*
  (2) *There exists an abelian cover of G of size $\Delta(G)$ containing $C_G(g)$.*

Proof. Let A be a maximal abelian subgroup of G that contains $C_G(g)$. In particular, A contains g and so, since A is abelian, $A \leqslant C_G(g)$. Thus $A = C_G(g)$ and (1) is proved. Item (2) is a consequence of Lemma 2.1 and (1). $\square$

Now we need a number of definitions; the ensuing lemmas will highlight their significance. Let $\mathcal{C}_a(G)$ be the set of all the abelian centralizers in G, $Y_a(G)$ its union and $Y_b(G)$ the complement of $Y_a(G)$. Hence

$$\mathcal{C}_a(G) = \{X \leqslant G \mid X \text{ is abelian, and } X = C_G(g) \text{ for some } g \in G\};$$
$$Y_a(G) = \bigcup_{X \in \mathcal{C}_a(G)} X; \qquad Y_b(G) = G \setminus Y_a(G).$$

We remark that both $Y_a(G)$ and $Y_b(G)$ are unions of conjugacy classes of G.

Next, for every element $X \in \mathcal{C}_a(G)$, choose an element $g_X \in G$ such that $C_G(g_X) = X$. Now define

$$N_a(G) = \{g_X \mid X \in \mathcal{C}_a(G)\}.$$

We caution that the set $N_a(G)$ is **not** uniquely defined, since there may be more than one choice of g_X for any given $X \in \mathcal{C}_a(G)$. In what follows we will refer to 'a set $N_a(G)$', by which we will mean a set constructed in the given way. For all of the above definitions – $\mathcal{C}_a(G), Y_a(G), Y_b(G), N_a(G)$ – when the group G is obvious from the context, we may drop the (G) from the name. Thus, for example, we will write $\mathcal{C}_a$ for $\mathcal{C}_a(G)$.

The next two lemmas connect the above definitions; the first is immediate.

Lemma 2.3. *The set $\mathcal{C}_a(G)$ is an abelian cover of $Y_a(G)$, and a set $N_a(G)$ is a non-commuting subset of $Y_a(G)$. In particular, $\delta(Y_a(G)) = \Delta(Y_a(G))$.*

Lemma 2.4.   (1) *There is an abelian cover of G of size $\Delta(G)$ for which $\mathcal{C}_a(G)$ is a subset.*
  (2) *Let X be a set $N_a(G)$. There is a non-commuting subset of G of size $\delta(G)$ for which X is a subset.*

Proof. For (1) take a cover of maximal abelian subgroups (which we can do by Lemma 2.1). Now take a set $N_a(G)$ and observe that, by Lemma 2.2 (1), this set must be covered by $\mathcal{C}_a(G)$. For (2), suppose that N is a non-commuting subset of G and let $V = N \cap Y_a$. Clearly $|V| \leqslant \delta(Y_a(G)) = |X|$, since $Y_a(G)$ is covered by $\delta(Y_a(G))$ abelian subgroups (Lemma 2.3). Thus if we remove V from N and put X in its

place, then N will not have diminished in size. What is more, by construction, N is still non-commuting (since any element of X does not commute with any element outside Y_a). $\qquad\square$

Lemma 2.5. *A group G has an abelian cover of size $\delta(G)$ if and only if $\Delta(Y_b(G)) = \delta(Y_b(G))$.*

Proof. Suppose, first, that G has an abelian cover $\mathcal{A}$ of size $\delta(G)$. By Lemma 2.4 we can assume that $\mathcal{C}_a$ is a subset of $\mathcal{A}$. Let N be a non-commuting subset of G of size $\delta(G)$; then there is a subset, N_a, of N of size $|\mathcal{C}_a|$ whose elements lie inside elements of $\mathcal{C}_a$.

Now let $N_b = N \setminus N_a$, and $\mathcal{C}_b = \mathcal{A} \setminus \mathcal{C}_a$. Then N_b is a non-commuting subset of Y_b, $\mathcal{C}_b$ is an abelian cover of Y_b, and $|N_b| = |\mathcal{C}_b|$, as required.

For the converse, suppose that $\Delta(Y_b) = \delta(Y_b)$, let N_b be a maximal non-commuting subset of Y_b, and let $\mathcal{C}_b$ be a minimal abelian cover of Y_b. Now let X be a set $N_a(G)$ and observe that $X \cup N_b$ is a maximal non-commuting subset of G, and that $\mathcal{C}_a \cup \mathcal{C}_b$ is a minimal abelian cover of G of size $\delta(G)$. $\qquad\square$

We close by making an elementary remark. Suppose that G is a group for which $\delta(G) = \Delta(G)$, that N is a non-commuting subset of G of size $\delta(G)$, and that $\mathcal{C}$ is an abelian cover of G of size $\Delta(G)$. Then every element of $\mathcal{C}$ contains a unique element of N (indeed the same is true for G replaced in our suppositions by any subset Ω of G).

3. Results on alternating groups

We will make heavy use of the following elementary lemma, a proof of which can be found in [Bro91].

Lemma 3.1. *Let σ be a product of nontrivial disjoint cycles $\sigma_1, ..., \sigma_k$ no two of which have the same length. Then every element of S_n which commutes with σ is a product $\tau\rho$, where τ is a product of powers of the cycles σ_i, $1 \leqslant i \leqslant k$ and ρ and σ have disjoint support.*

The treatment that follows is split into three cases – when n is even, when n is odd, and when n is small. The strategy in the first two cases is identical, and strongly resembles the method of Brown [Bro91]. However some of the details differ and so, for clarity, the cases are written separately.

3.1. n **is even.** We assume here that n is even, and let

$$\sigma = (4,5)(6,7,8)(9,10,\ldots,n) \in A_n.$$

We are interested in the set $Cl_{A_n}(\sigma)$, the conjugacy class of σ in A_n and we note first that an element of $Cl_{A_n}(\sigma)$ does not have an abelian centralizer in A_n.

Lemma 3.2. *Suppose that $n = 12$ or n is even and $n \geqslant 16$, $g \in Cl_{A_n}(\sigma)$, K is a maximal abelian subgroup of A_n containing g, and h is an element of K. Then one of the following occurs:*

    (1) *h lies in an abelian centralizer.*
    (2) *h has cycle type $2-3-\underbrace{d-d-\cdots-d}_{k}$ where $1 \leqslant d, k \leqslant n-8$ and $dk = n-8$.*

Proof. It is sufficient to prove the result for the case $g = \sigma = (4,5)(6,7,8)(9,...,n)$. Now we write $a = (4,5)$, $b = (6,7,8)$, $c = (9,10,...,n)$ and observe that $g = abc$.

The restrictions on n mean that we can apply Lemma 3.1 to the element g, and conclude that every permutation h that commutes with g will be of the form $a^i b^j c^k \rho$ where $0 \leqslant i \leqslant 1$, $0 \leqslant j \leqslant 2$, $0 \leqslant k \leqslant n-8$, and the support of ρ is a subset of $\{1,2,3\}$.

Observe that h will satisfy condition (2) of the lemma if and only if one of the following holds:

    • $i = 1$, $j = 1$, and $\rho = (1)$;
    • $i = 1$, $j = 2$, and $\rho = (1)$;
    • $i = 1$, $j = 0$, and ρ is a 3-cycle;
    • $i = 0$, $j = 1$, and ρ is a 2-cycle.

In the first column below we list a representative h of every S_n-conjugacy class that has a non-trivial intersection with K, and does not satisfy condition (2) of the lemma. In the second column we list a permutation f that, provided $n \neq 12$, has abelian centralizer, and which commutes with h.

Permutation h	Permutation f
$(4,5)(9,...,n)^k$	$(1,6,2,7,3,8)(9,...,n)$
$(6,7,8)(9,...,n)^k$	$(1,2,3,4)(6,7,8)(9,...,n).$
$(1,2)(4,5)(9,...,n)^k$	$(1,4,2,5)(6,7,8)(9,...,n)$
$(1,2,3)(6,7,8)(9,...,n)^k$	$(1,6,2,7,3,8)(9,...,n)$
$(1,2)(4,5)(6,7,8)(9,...,n)^k$	$(1,4,2,5)(6,7,8)(9,..,n)$
$(1,2,3)(4,5)(6,7,8)(9,...,n)^k$	$(1,6,2,7,3,8)(9,...,n)$

We caution that not all of the permutations h listed above lie in K since, for certain values of k, the listed permutation will not be even. However the given list certainly includes all of the required permutations. Furthermore it is easy to check, using Lemma 3.1, that $C_G(f)$ is indeed abelian.

We must deal with the case when $n = 12$. Again we list a representative h of every S_{12}-conjugacy class that has a non-trivial intersection with K, and does not satisfy condition (2) of the lemma. Again, in the second column we list a permutation f that has abelian centralizer, and which commutes with h.

Permutation h	Permutation f
$(4,5)(9,10,11,12)$	$(1,6,2,7,3,8)(9,10,11,12)$
$(6,7,8)(9,11)(10,12)$	$(1,2,3,4,5)(6,7,8)(9,11)(10,12)$
$(1,2)(4,5)(9,11)(10,12)$	$(3,6,7,8)(1,9,4,10,2,11,5,12)$
$(1,2,3)(6,7,8)(9,11)(10,12)$	$(1,6,2,7,3,8)(9,10,11,12)$
$(1,2)(4,5)(6,7,8)(9,11)(10,12)$	$(6,7,8)(1,4,9,2,5,11)(10,12)$
$(1,2,3)(4,5)(6,7,8)(9,10,11,12)$	$(1,6,2,7,3,8)(9,10,11,12)$

A quick computation checks that the centralizer for $(1,2,3,4,5)(6,7,8)(9,11)(10,12)$ is abelian, and by Lemma 3.1 the rest are obviously abelian as well. □

We need an easy lemma concerning permutations that satisfy condition (2) of the previous lemma.

Lemma 3.3. *Suppose that $n = 12$, or n is even and $n \geqslant 16$, and that $h \in A_n$ has cycle type $2{-}3{-}\underbrace{d{-}d{-}\cdots{-}d}_{k}$ where $1 \leqslant d, k \leqslant n - 8$ and $dk = n - 8$. Then $C_{A_n}(h)$ contains a permutation f with cycle type $2{-}3{-}(n{-}8)$ and, moreover, $C_{A_n}(f) \leqslant C_{A_n}(h)$.*

Proof. Write $h = abc_1$ where a is a 2-cycle, b is a 3-cycle, c_1 is a product of k d-cycles, and a, b and c have disjoint support. Then c_1 is a power of an $(n{-}8)$-cycle, c, and $f = abc \in C_{A_n}(h)$ has cycle type $2{-}3{-}(n{-}8)$.

Now the restrictions on n mean that we can apply Lemma 3.1 to the element f to conclude that any element f_1 that commutes with f will be of the form $a^i b^j c^k \rho$ where $0 \leqslant i \leqslant 1$, $0 \leqslant j \leqslant 2$, $0 \leqslant k \leqslant n - 8$, and ρ and f have disjoint supports. Now one can check directly that such an element commutes with h, and we conclude that $C_{A_n}(f) \leqslant C_{A_n}(h)$. □

Proposition 3.4. *Suppose that $n = 12$ or n is even and $n \geqslant 16$. Then $\delta(A_n) \neq \Delta(A_n)$.*

Proof. Assume, for a contradiction, that $\delta(A_n) = \Delta(A_n)$. Let E be a maximal set of non-commuting elements in A_n. By Lemma 2.4, we can (and do) assume that E contains a set $N_a(A_n)$. Let $\mathcal{A}$ be a minimal abelian cover of A_n. By Lemma 2.4, we can (and do) assume that $\mathcal{A}$ contains $\mathcal{C}_a$. As before, the set $Cl_{A_n}(\sigma)$ is the conjugacy class of $\sigma = (1,2)(3,4,5)(9,10,\ldots,n)$ in A_n, and every element in $Cl_{A_n}(\sigma)$ has cycle type $2 - 3 - (n - 8)$.

Now let γ be a fixed $(n - 8)$-cycle. For $g \in Cl_{A_n}(\sigma)$, write $\mathrm{Long}(g)$ for the $(n - 8)$-cycle that forms part of the cycle decomposition of g. Now define

$$Cl_\gamma := \{g \in Cl_{A_n}(\sigma) \mid \mathrm{Long}(g) = \gamma\};$$
$$\mathcal{A}_\gamma := \{A \in \mathcal{A} \mid h \in A \text{ for some } h \in Cl_\gamma\};$$
$$E_\gamma := E \cap Cl_\gamma.$$

Since $\delta(A_n) = \Delta(A_n)$, every element of $\mathcal{A}$ contains an element of E. Let $A \in \mathcal{A}_\gamma$ and let h be the unique element in $A \cap E$. Then, since $\mathcal{A}$ contains $\mathcal{C}_a$, Lemma 3.2 implies that h has cycle type $2{-}3{-}\underbrace{d - d - \cdots - d}_{k}$

where $1 \leqslant d, k \leqslant n-8$ and $dk = n-8$. Now Lemma 3.3 implies that we can replace h in E by an element $f \in A$ with cycle type $2-3-(n-8)$, and E will still be non-commuting. Hence we can (and do) assume that, for every $A \in \mathcal{A}_\gamma$ the unique element in $A \cap E$ is of type $2-3-(n-8)$.

Since $\bigcup_{A \in \mathcal{A}_\gamma} A \supseteq Cl_\gamma$ we conclude that $|E_\gamma| = |\mathcal{A}_\gamma|$ and E_γ is precisely the set of elements in E that lie in some element of $\mathcal{A}_\gamma$.

Now one can check, firstly, that $|Cl_\gamma| = 1120$ and, secondly, that if $\Omega \subset Cl_\gamma$ is a set of commuting permutations, then $|\Omega| \leqslant 4$. So that every element $A \in \mathcal{A}_\gamma$ contains at most 4 permutations of the given type and, in consequence, $|\mathcal{A}_\gamma| \geqslant 1120/4 = 280$. Thus $|E_\gamma| \geqslant 280$.

Now consider $\mathcal{B}$, the set of groups generated by 4 disjoint cycles a, b, c, γ where a is a $2-$cycle, both b and c are $3-$cycles and γ is as above. We can easily see that $\mathcal{B}$ is an abelian cover of $\mathcal{C}_\gamma$. What is more $|\mathcal{B}| = 280$. Therefore $|E_\gamma| \leqslant 280$ and so $|E_\gamma| = 280 = |\mathcal{A}_\gamma|$.

Now let $A \in \mathcal{A}_\gamma$. Since $|\mathcal{A}_\gamma| = 280$ and A contains at most 4 elements of Cl_γ which has size 1120, we see that A contains exactly 4 elements of Cl_γ. Let g be an element of Cl_γ and write $g = ab\gamma$ where a is a 2-cycle, b is a 3-cycle, and a, b and γ are disjoint. One can easily check that g lies in two maximal abelian subgroups:

- $A_1 = \langle a, b, \gamma, \rho_1 \rangle$ where ρ_1 is a 2-cycle that is disjoint from a, b and γ ;
- $A_2 = \langle a, b, \gamma, \rho_2 \rangle$ where ρ_2 is a 3-cycle that is disjoint from a, b and γ;

Now A_1 contains 2 elements of Cl_γ, while A_2 contains 4 elements of Cl_γ. We conclude that $A_2 \in \mathcal{A}_\gamma$. It now follows that $\mathcal{A}_\gamma$ is equal to the set $\mathcal{B}$, defined in the previous paragraph.

Now observe that the following groups all lie in $\mathcal{A}_\gamma$:

$$A_1 := \langle (1,2,3), (4,5,6), (7,8), \gamma \rangle; \quad A_2 := \langle (1,2,3), (5,7,8), (4,6), \gamma \rangle;$$
$$A_3 := \langle (5,7,8), (2,4,6), (1,3), \gamma \rangle; \quad A_4 := \langle (2,4,6), (1,3,5), (7,8), \gamma \rangle;$$
$$A_5 := \langle (1,3,5), (6,7,8), (2,4), \gamma \rangle; \quad A_6 := \langle (1,2,3), (4,7,8), (5,6), \gamma \rangle;$$
$$A_7 := \langle (4,7,8), (1,5,6), (2,3), \gamma \rangle; \quad A_8 := \langle (2,3,4), (1,5,6), (7,8), \gamma \rangle;$$
$$A_9 := \langle (2,3,4), (6,7,8), (1,5), \gamma \rangle.$$

Let a_i be the unique element in $A_i \cap E$ for $i = 1, \ldots, 9$. Without loss of generality, we may assume that $a_1 = (1,2,3)(7,8)\gamma$. Now for a_2 not to commute with a_1 it must be of the form $(5,7,8)^i(4,6)\gamma^j$, where $i = 1, 2$ and j and $n-8$ are coprime. Notice that the choice of i and j does not affect the set of permutations in Cl_γ that commute with it. Thus, without lost of generality we may assume that $a_2 = (5,7,8)(4,6)\gamma$.

Following the same logic, we may take $a_3 = (2,4,6)(1,3)\gamma$, $a_4 = (1,3,5)(7,8)\gamma$ and $a_5 = (6,7,8)(2,4)\gamma$. Now starting from the fact that $a_1 = (1,2,3)(7,8)\gamma$, and following the same logic as above, we can also deduce that $a_6 = (4,7,8)(5,6)\gamma$, $a_7 = (1,5,6)(2,3)\gamma$ and $a_8 = (2,3,4)(7,8)\gamma$.

Now we find that we cannot choose a_9, since any element we choose will commute either with $a_8 = (2,3,4)(7,8)\gamma$ or $a_5 = (6,7,8)(2,4)\gamma$. We have the contradiction that we sought. $\square$

3.2. n is odd. The case where n is odd will be proven in essentially the same way as the even case but considering instead permutations of cyclic type $2-3-8-(n-16)$ since the permutations of cyclic type $2-3-(n-8)$ aren't in A_n when n is odd. Let $\tau = (4,5)(6,7,8)(9,10,\ldots,16)(17,18,\ldots,n) \in A_n$. We consider the set $Cl_{A_n}(\tau)$ and observe, as before, that an element of $Cl_{A_n}(\tau)$ does not have an abelian centralizer in A_n. We will have the same situation as in the even case:

Lemma 3.5. *Suppose that n is odd and $n \geqslant 21$, $g \in Cl_{A_n}(\tau)$, K is a maximal abelian group containing g, and h and element of K. Then one of the following occurs:*

(1) *h lies in an abelian centralizer.*
(2) *h has cycle type $2-3-\underbrace{d-d-\cdots-d}_{k}-\underbrace{e-e-\cdots-e}_{j}$ where $1 \leqslant d, k \leqslant 8$, $1 \leqslant e, j \leqslant n-16$, $dk = 8$, and $ej = n-16$.*

Proof. Again it is sufficient to choose

$$g = \tau = (4,5)(6,7,8)(9,10,11,12,13,14,15,16)(17,18,\ldots,n)$$

and check that every permutation that commutes with g satisfies the lemma. By Lemma 3.1 every permutation that commutes with g will be of the form $a^i b^j c^k d^l \rho$ with $a = (4,5)$, $b = (6,7,8)$, $c = (9,10,...,16)$, $d = (17,18,...,n)$ and the support of ρ is $\{1,2,3\}$.

As in the even case we list a representative h of every S_n-conjugacy class that has a non-trivial intersection with K, and does not satisfy condition (2) of the lemma. In the second column we list a permutation f that, provided $n \neq 19$, has abelian centralizer, and which commutes with h.

Permutation h	Permutation f
$(4,5)(9,10,...,16)^k(17,...,n)^l$	$(1,6,2,7,3,8)(9,10,...,16)(17,...,n)$
$(6,7,8)(9,10,...,16)^k(17,...,n)^l$	$(1,4,2,5)(6,7,8)(9,10,...,16)(17,...,n)$
$(1,2)(4,5)(9,10,...,16)^k(17,...,n)^l$	$(1,4,2,5)(6,7,8)(9,10,...,16)(17,...,n)$
$(1,2,3)(6,7,8)(9,10,...,16)^k(17,...,n)^l$	$(1,6,2,7,3,8)(9,10,...,16)(17,...,n)$
$(1,2)(4,5)(6,7,8)(9,10,...,16)^k(7,...,n)^l$	$(1,4,2,5)(6,7,8)(9,10,...,16)(17,..,n)$
$(1,2,3)(4,5)(6,7,8)(9,10,...,16)^k(17,...,n)^l$	$(1,6,2,7,3,8)(9,10,...,16)(17,...,n)$

$\square$

Now to deal with the permutations that satisfy condition (2) we will use the next lemma. The statement and proof are analogous to Lemma 3.3, and so the proof is omitted.

Lemma 3.6. *Suppose that n is odd and $n \geqslant 21$, and that $h \in A_n$ has cycle type* $2{-}3{-}\underbrace{d{-}d{-}\cdots{-}d}_{k}{-}\underbrace{e{-}e{-}\cdots{-}e}_{j}$
where $1 \leqslant d, k \leqslant 8$, $1 \leqslant e, j \leqslant n-16$, $dk = 8$, and $ej = n - 16$. Then $C_{A_n}(h)$ contains a permutation f with cycle type $2{-}3{-}8{-}(n{-}16)$ and, moreover, $C_{A_n}(f) \leqslant C_{A_n}(h)$.

Proposition 3.7. *Suppose that n is odd and $n \geqslant 21$. Then $\delta(A_n) \neq \Delta(A_n)$.*

Proof. We do essentially the same thing as in the even case. Assume $\delta(A_n) = \Delta(A_n)$. Let E a non commuting subset of size $\delta(A_n)$ such that $N_a(A_n) \subset E$. Let $Cl_{\gamma,\theta}$ be the set of permutations of cycle type $2{-}3{-}8{-}(n{-}16)$ where the 8-cycle is a given cycle γ and the $(n{-}16)$-cycle is a given cycle θ, with the support of γ and θ disjoint from $\{1,2,3,4,5,6,7,8\}$. Define

$$\mathcal{A}_{\gamma,\theta} := \{A \in \mathcal{A} \mid h \in A \text{ for some } h \in Cl_{\gamma,\theta}\};$$
$$E_{\gamma,\theta} := E \cap Cl_{\gamma,\theta}.$$

Again we see that to cover $Cl_{\gamma,\theta}$ we need at least 280 abelian groups, so that $|E_{\gamma,\theta}| = |\mathcal{A}_{\gamma,\theta}| \geqslant 280$. By Lemma 3.5 these permutations will satisfy condition (2) of said lemma. By Lemma 3.6 we can take these elements to be of the type $2{-}3{-}8{-}(n{-}16)$.

Now consider $\mathcal{B}$ to be the set of groups generated by five disjoint cycles a, b, c, γ, θ where a is a 2-cycle, b and c are 3-cycles, and γ, θ as above. This is a cover of $Cl_{\gamma,\theta}$ by 280 abelian groups and so each one must contain exactly one element of $E_{\gamma,\theta}$.

Now we assume that $(1,2,3)(7,8)\gamma\theta$ is in $E_{\gamma,\theta}$ and consider in order the representatives for

$$\langle (1,2,3),(4,5,6),(7,8),\gamma,\theta \rangle; \quad \langle (1,2,3),(5,7,8),(4,6),\gamma,\theta \rangle;$$
$$\langle (5,7,8),(2,4,6),(1,3),\gamma,\theta \rangle; \quad \langle (2,4,6),(1,3,5),(7,8),\gamma,\theta \rangle;$$
$$\langle (1,3,5),(6,7,8),(2,4),\gamma,\theta \rangle; \quad \langle (1,2,3),(4,7,8),(5,6),\gamma,\theta \rangle;$$
$$\langle (4,7,8),(1,5,6),(2,3),\gamma,\theta \rangle; \quad \langle (2,3,4),(1,5,6),(7,8),\gamma,\theta \rangle.$$

As before, we may assume that the following elements are in $E_{\gamma,\theta}$:

$$(5,7,8)(4,6)\gamma\theta; \quad (2,4,6)(1,3)\gamma\theta;$$
$$(1,3,5)(7,8)\gamma\theta; \quad (6,7,8)(2,4)\gamma\theta;$$
$$(4,7,8)(5,6)\gamma\theta; \quad (1,5,6)(2,3)\gamma\theta;$$
$$(2,3,4)(7,8)\gamma\theta.$$

Now we must choose a representative for $\langle (2,3,4),(6,7,8),(1,5),\gamma,\theta \rangle$, but we can not do this, since any element we choose will commute either with $(2,3,4)(7,8)\gamma\theta$ or $(6,7,8)(2,4)\gamma\theta$ both of which are already in $E_{\gamma,\theta}$, this contradiction implies that $\delta(A_n) \neq \Delta(A_n)$. $\square$

3.3. **Small n.** To complete our understanding of the situation in alternating groups, we need to consider the cases $n \leqslant 11$ and $n = 13, 14, 15, 19$. We will not give full information here – to prove Theorem 1.2 it is sufficient to prove that $\delta(A_n) = \Delta(A_n)$ for $n \leqslant 11$ and $n = 15$.

Lemma 3.8. *Let n be a positive integer. Then every element of A_n lies in an abelian centralizer if and only if $n \leqslant 7$ or $n = 10$.*

Moreover if $n = 8$ or 9, then the only elements of A_n that do not lie in an abelian centralizer have cycle type $4{-}4$. Similarly, the only elements of A_{11} that do not lie in an abelian centralizer have cycle type $4{-}4{-}3$.

Proof. The statements for $n \leqslant 11$ can be checked by direct computation. Suppose now that $n \geqslant 12$. One can check that if n is even (resp. odd), then elements of type $4{-}4{-}(n{-}9)$ (resp. $4{-}4{-}(n{-}8)$) do not lie in an abelian centralizer in A_n. $\square$

In the notation of §2, the lemma asserts that, for $n \leqslant 7$ and $n = 10$, we have $Y_a(A_n) = A_n$. Now Lemma 2.3 immediately yields the following corollary.

Corollary 3.9. *If $n \leqslant 7$ or $n = 10$, then $\delta(A_n) = \Delta(A_n)$.*

The following lemmas will help us deal with some of the remaining cases.

Lemma 3.10. *Let n be a positive integer with $n \geqslant 8$, and let k and ℓ be positive integers such that $n - (4k + \ell) < \min(\ell, 2k)$ and $\ell \neq 2k$, $\ell \neq 4k$. Let $\sigma, \tau \in S_n$ of cycle type $2k{-}2k{-}\ell$ such that $\sigma\tau = \tau\sigma$. Then,*

1. *The ℓ-cycle of τ is a power of the ℓ-cycle of σ.*
2. *The supports of σ and τ are equal.*

Proof. For the first part, let $r \in \{1, 2, \ldots, n\}$, we consider the orbit of r under $\langle \sigma \rangle$, this is $Orb_\sigma(r) = \{\sigma^i(r) : i \in \mathbb{Z}\}$. Suppose that $|Orb_\sigma(r)| = \ell$. We have that $\sigma^i \tau^j(r) = \tau^j \sigma^i(r)$ for all $i, j \in \mathbb{Z}$, and so

$$Orb_\sigma(\tau^j(r)) = \tau^j\left(Orb_\sigma(r)\right),$$

which implies that

$$Orb_\sigma(\tau^j(r)) = Orb_\sigma(r),$$

because there is only one orbit by σ of size ℓ. We conclude that $\tau^j(r) \in Orb_\sigma(r)$ for all $j \in \mathbb{Z}$, and then $Orb_\tau(r) \subseteq Orb_\sigma(r)$.

Since τ has cycle type $2k{-}2k{-}\ell$ there are three possibilities: $|Orb_\tau(r)| = \ell$, $|Orb_\tau(r)| = 1$ and $|Orb_\tau(r)| = 2k$. In the first case clearly $Orb_\tau(r) = Orb_\sigma(r)$, so that both ℓ-cycles have the same support. Lemma 3.1 then tells us that one ℓ-cycle must be the power of the other.

Suppose one of the other possibilities holds: then if $|Orb_\tau(r) = 1|$ we have that $\tau(r) = r$, then $\tau\sigma^j(r) = \sigma^j\tau(r) = \sigma^j(r)$, so $\sigma^j(r)$ is fixed by τ for all $j = 1, 2, 3, \ldots, \ell$, i.e. τ has ℓ fixed points. But this is impossible, because $n - (4k + \ell) < \ell$.

This means that we must have $|Orb_\tau(r)| = 2k$ for all r's in the support of σ's ℓ-cycle. Since τ has cycle type $2k{-}2k{-}\ell$ the support of σ's ℓ-cycle must be the same as the support for one of the $2k$-cycles of σ, or the union of the supports of the two $2k$-cycles of σ. So we have only two possibilities: $\ell = 2k$ or $\ell = 4k$, neither one possible by hypothesis. This ends the proof of the first part.

For the second part, let $r \in \{1, 2, 3, \ldots, n\}$, and suppose (for a contradiction) that $\tau(r) = r$ and $\sigma(r) \neq r$. This implies that $Orb_\sigma(r)$ must have $2k$ or ℓ elements. But now observe that, for all i,

$$\tau\sigma^i(r) = \sigma^i\tau(r) = \sigma^i(r)$$

and we conclude that every element in $Orb_\sigma(r)$ is fixed by τ. But this is impossible, because τ only fixes $n - (4k + \ell)$ elements. We conclude that every element fixed by τ is fixed by σ. The same reasoning yields that every element fixed by σ is fixed by τ and the result holds. $\square$

Lemma 3.11. *Let n a positive integer with $n \geqslant 8$, and suppose that $n = 4k$ for some integer k. Let Λ be the conjugacy class of elements of cycle type $2k{-}2k$ in A_n. Then the commutator graph $\Gamma(\Lambda)$ is disconnected. Furthermore if Ω is (the vertex set of) a connected component of $\Gamma(\Lambda)$, then $\Omega \subset H < S_n$, with $H \cong S_k \wr K_4$, where K_4 is the normal Klein 4-subgroup of S_4.*

Proof. Let $g = (1, 2, \ldots, 2k)(2k + 1, 2k + 2, \ldots, 4k)$ and consider the following partition of the set $\{1, 2, \ldots, n\}$ into four subsets of size k:

$$\{1, 3, 5, \ldots, 2k - 1\}, \quad \{2k + 1, 2k + 3, 2k + 5, \ldots, 4k - 2\},$$
$$\{2, 4, 6, \ldots, 2k\}, \quad \{2k + 2, 2k + 4, 2k + 6, \ldots, 4k\}.$$

It is easy to see that this is the unique partition of type $k{-}k{-}k{-}k$ on which g acts as a double-transposition. What is more direct computation confirms that the same is true of any element of Λ that commutes with g. Let Ω be the connected component of $\Gamma(\Lambda)$ that contains g. We conclude that Ω lies inside the subgroup generated by all elements of Λ that act as double-transpositions on Λ. All such elements lie inside H and we are done. $\square$

Corollary 3.12. *Let n be a positive integer, $n \geqslant 8$, and let k, ℓ be positive integers such that $n - (4k + \ell) < \min(2k, \ell)$. Let Λ be the conjugacy class of elements of cycle type $2k{-}2k{-}\ell$ in A_n. Then the commutator graph $\Gamma(\Lambda)$ is disconnected. Furthermore if Ω is (the vertex set of) a connected component of $\Gamma(\Lambda)$, then $\Omega \subset H < S_n$ with $H \cong (S_k \wr K_4) \times \mathbb{Z}/\ell\mathbb{Z}$.*

Proof. Let $\sigma, \tau \in \Omega$, the vertex set of a connected component of $\Gamma(\Lambda)$. By Lemma 3.10 the ℓ-cycle of τ is a power of the ℓ-cycle of σ, and their $2k{-}2k$ parts have the same support. By Lemma 3.11 $\Omega \subset H$ with $H \cong (S_k \wr K_4) \times \mathbb{Z}/\ell\mathbb{Z}$. $\square$

Corollary 3.13. *If $n \leqslant 11$, then $\delta(A_n) = \Delta(A_n)$.*

Proof. By Corollary 3.9, we may (and we do) assume that $n = 8$, 9 or 11. By Lemma 2.5, we must prove that $\delta(Y_b(A_n)) = \Delta(Y_b(A_n))$ in each case. By Lemma 3.8, $Y_b(A_n)$ is equal to the conjugacy class of cycle type $4{-}4$ for $n = 8$ or 9, and $4{-}4{-}3$ for $n = 11$.

Now, by Corollary 3.12, the commutator graph of $Y_b(A_n)$ is disconnected and each connected component lies inside a subgroup of S_n isomorphic to $S_2 \wr K_4$ for $n = 8$ or $n = 9$, and $(S_2 \wr K_4) \times \mathbb{Z}/3\mathbb{Z}$ for $n = 11$. Let Ω be such a connected component; clearly if we can show that $\delta(\Omega) = \Delta(\Omega)$, then we are done.

We consider first the cases $n = 8$ and $n = 9$, without loss of generality, we take Ω to be a connected component inside "the" wreath product $H = S_2 \wr S_4$. Direct computation reveals that H contains 12 elements of cycle type $4 - 4$; they are the following elements and their inverses:

$$(1, 3, 2, 4)(5, 7, 6, 8), \quad (1, 5, 2, 6)(3, 7, 4, 8), \quad (1, 7, 2, 8)(3, 5, 4, 6),$$
$$(1, 3, 2, 4)(5, 8, 6, 7), \quad (1, 5, 2, 6)(3, 8, 4, 7), \quad (1, 7, 2, 8)(3, 6, 4, 5).$$

Now observe that the three elements in the first row form a non-commuting set, while the elements in each column generate an abelian group. Taking Ω to be this set of 12 elements, we conclude that $\delta(\Omega) = \Delta(\Omega) = 3$, and we are done.

For the case $n = 11$, we observe that H has 24 elements of cycle type $4 - 4 - 3$ namely

$$(1,3,2,4)(5,7,6,8)(9,10,11), \quad (1,5,2,6)(3,7,4,8)(9,10,11), \quad (1,7,2,8)(3,5,4,6)(9,10,11),$$
$$(1,3,2,4)(5,8,6,7)(9,10,11), \quad (1,5,2,6)(3,8,4,7)(9,10,11), \quad (1,7,2,8)(3,6,4,5)(9,10,11),$$
$$(1,3,2,4)(5,7,6,8)(9,11,10), \quad (1,5,2,6)(3,7,4,8)(9,11,10), \quad (1,7,2,8)(3,5,4,6)(9,11,10),$$
$$(1,3,2,4)(5,8,6,7)(9,11,10), \quad (1,5,2,6)(3,8,4,7)(9,11,10), \quad (1,7,2,8)(3,6,4,5)(9,11,10).$$

and their inverses. Again, observe that the three elements in the first row form a non-commuting set, while the elements in each column generate an abelian group. Taking Ω to be this set of 24 elements, we conclude that $\delta(\Omega) = \Delta(\Omega) = 3$, and we are done. $\square$

Lemma 3.14. $\delta(A_{15}) = \Delta(A_{15})$.

Proof. Direct computation shows that $Y_b(A_{15})$ consists of the elements of cycle type $7{-}4{-}4$, $6{-}6{-}3$, $6{-}4{-}2{-}2$, and $6{-}3{-}3{-}2$. Notice that among these classes no two elements in different classes commute. Hence we may set Λ to be each class in turn, and show that in eacy case $\delta(\Lambda) = \Delta(\Lambda)$.

Class $7{-}4{-}4$: In this case Corollary 3.12 implies that $\Gamma(\Lambda)$ is disconnected. Taking Ω to be the maximal connected component that contains $(1, 3, 2, 4)(5, 7, 6, 8)(9, 10, 11, 12, 13, 14, 15)$ and setting $\sigma = (9, 10, 11, 12, 13, 14, 15)$ one can check that this component has 72 elements, as follows:

$$(1,3,2,4)(5,7,6,8)\sigma^i, \quad (1,5,2,6)(3,7,4,8)\sigma^i, \quad (1,7,2,8)(3,5,4,6)\sigma^i,$$
$$(1,3,2,4)(5,8,6,7)\sigma^i, \quad (1,5,2,6)(3,8,4,7)\sigma^i, \quad (1,7,2,8)(3,6,4,5)\sigma^i.$$

(Here i can take any value between 1 and 6, and an element is either on the list or its inverse is on the list.)

Now the elements of each column generate an abelian group and these 3 groups cover Ω. On the other hand, fixing i and taking the elements of any row we get a non-commuting set of size 3, as required.

Class $6-6-3$: Reasoning along the lines of Corollary 3.12, we can deduce that $\Gamma(\Lambda)$ is disconnected in this situation, with a maximal connected component lying inside a subgroup of S_n isomorphic to $(S_3 \wr K_4) \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Further a maximal connected component contains 432 elements, with one example as follows:

$$
\begin{array}{lll}
(1,10,2,11,3,12)(4,8,5,9,6,7)\sigma^i, & (1,8,3,7,2,9)(4,11,6,10,5,12)\sigma^i, & (1,4,2,5,3,6)(7,10,8,11,9,12)\sigma^i, \\
(1,10,3,12,2,11)(4,8,6,7,5,9)\sigma^i, & (1,7,3,9,2,8)(4,11,6,10,5,12)\sigma^i, & (1,4,3,6,2,5)(7,11,9,10,8,12)\sigma^i, \\
(1,4,3,6,2,5)(7,10,9,12,8,11)\sigma^i, & (1,7,2,8,3,9)(4,10,5,11,6,12)\sigma^i, & (1,11,3,10,2,12)(4,7,6,9,5,8)\sigma^i, \\
(1,5,3,4,2,6)(7,12,9,11,8,10)\sigma^i, & (1,8,3,7,2,9)(4,12,6,11,5,10)\sigma^i, & (1,10,2,11,3,12)(4,9,5,7,6,8)\sigma^i, \\
(1,4,2,5,3,6)(7,12,8,10,9,11)\sigma^i, & (1,7,3,9,2,8)(4,12,6,11,5,10)\sigma^i, & (1,10,3,12,2,11)(4,9,6,8,5,7)\sigma^i, \\
(1,11,3,10,2,12)(4,9,6,8,5,7)\sigma^i, & (1,7,2,8,3,9)(4,12,5,10,6,11)\sigma^i, & (1,5,3,4,2,6)(7,10,9,12,8,11)\sigma^i, \\
(1,11,3,10,2,12)(4,8,6,7,5,9)\sigma^i, & (1,7,2,8,3,9)(4,11,5,12,6,10)\sigma^i, & (1,4,2,5,3,6)(7,11,8,12,9,10)\sigma^i, \\
(1,10,3,12,2,11)(4,7,6,9,5,8)\sigma^i, & (1,7,3,9,2,8)(4,10,6,12,5,11)\sigma^i, & (1,5,3,4,2,6)(7,11,9,10,8,12)\sigma^i, \\
(1,10,2,11,3,12)(4,7,5,8,6,9)\sigma^i, & (1,8,3,7,2,9)(4,10,6,12,5,11)\sigma^i, & (1,4,3,6,2,5)(7,12,9,11,8,10)\sigma^i, \\
(1,5,3,4,2,6)(7,11,8,12,9,10)\sigma^i, & (1,7,3,8,2,9)(4,12,6,10,5,11)\sigma^i, & (1,11,2,10,3,12)(4,9,5,8,6,7)\sigma^i, \\
(1,5,2,6,3,4)(7,11,9,10,8,12)\sigma^i, & (1,7,2,9,3,8)(4,12,5,11,6,10)\sigma^i, & (1,10,2,12,3,11)(4,9,5,8,6,7)\sigma^i, \\
(1,4,2,5,3,6)(7,10,9,12,8,11)\sigma^i, & (1,7,2,9,3,8)(4,10,5,12,6,11)\sigma^i, & (1,10,3,11,2,12)(4,9,6,7,5,8)\sigma^i, \\
(1,5,3,4,2,6)(7,12,8,10,9,11)\sigma^i, & (1,8,2,7,3,9)(4,11,5,10,6,12)\sigma^i, & (1,11,2,10,3,12)(4,8,5,7,6,9)\sigma^i, \\
(1,5,2,6,3,4)(7,12,9,11,8,10)\sigma^i, & (1,7,3,8,2,9)(4,10,6,11,5,12)\sigma^i, & (1,10,2,12,3,11)(4,8,5,7,6,9)\sigma^i, \\
(1,4,2,5,3,6)(7,12,9,11,8,10)\sigma^i, & (1,8,2,7,3,9)(4,10,5,12,6,11)\sigma^i, & (1,10,3,11,2,12)(4,7,6,8,5,9)\sigma^i, \\
(1,7,3,8,2,9)(4,11,6,12,5,10)\sigma^i, & (1,4,2,5,3,6)(7,11,9,10,8,12)\sigma^i, & (1,10,3,11,2,12)(4,8,6,9,5,7)\sigma^i, \\
(1,8,2,7,3,9)(4,12,5,11,6,10)\sigma^i, & (1,5,2,6,3,4)(7,10,9,12,8,11)\sigma^i, & (1,10,2,12,3,11)(4,7,5,9,6,8)\sigma^i, \\
(1,7,2,9,3,8)(4,11,5,10,6,12)\sigma^i, & (1,5,3,4,2,6)(7,10,8,11,9,12)\sigma^i, & (1,11,2,10,3,12)(4,7,5,9,6,8)\sigma^i, \\
(1,7,2,8,3,9)(4,10,6,12,5,11)\sigma^i, & (1,10,3,11,2,12)(4,8,5,7,6,9)\sigma^i, & (1,4,2,6,3,5)(7,10,8,12,9,11)\sigma^i, \\
(1,7,3,9,2,8)(4,10,5,11,6,12)\sigma^i, & (1,11,3,12,2,10)(4,8,5,7,6,9)\sigma^i, & (1,4,3,5,2,6)(7,10,9,11,8,12)\sigma^i, \\
(1,8,3,7,2,9)(4,11,5,12,6,10)\sigma^i, & (1,11,2,10,3,12)(4,7,6,8,5,9)\sigma^i, & (1,4,3,5,2,6)(7,12,9,10,8,11)\sigma^i, \\
(1,7,2,8,3,9)(4,12,6,11,5,10)\sigma^i, & (1,10,3,11,2,12)(4,9,5,8,6,7)\sigma^i, & (1,6,3,4,2,5)(7,11,9,12,8,10)\sigma^i, \\
(1,7,3,9,2,8)(4,12,5,10,6,11)\sigma^i, & (1,11,3,12,2,10)(4,9,5,8,6,7)\sigma^i, & (1,4,2,6,3,5)(7,12,8,11,9,10)\sigma^i, \\
(1,8,3,7,2,9)(4,12,5,10,6,11)\sigma^i, & (1,11,2,10,3,12)(4,9,6,7,5,8)\sigma^i, & (1,6,3,4,2,5)(7,12,9,10,8,11)\sigma^i, \\
(1,8,3,7,2,9)(4,10,5,11,6,12)\sigma^i, & (1,11,2,10,3,12)(4,8,6,9,5,7)\sigma^i, & (1,4,2,6,3,5)(7,11,8,10,9,12)\sigma^i, \\
(1,7,3,9,2,8)(4,11,5,12,6,10)\sigma^i, & (1,11,3,12,2,10)(4,7,5,9,6,8)\sigma^i, & (1,6,3,4,2,5)(7,10,9,11,8,12)\sigma^i, \\
(1,7,2,8,3,9)(4,11,6,10,5,12)\sigma^i, & (1,10,3,11,2,12)(4,7,5,9,6,8)\sigma^i, & (1,4,3,5,2,6)(7,11,9,12,8,10)\sigma^i, \\
(1,7,2,9,3,8)(4,10,6,11,5,12)\sigma^i, & (1,10,3,12,2,11)(4,9,5,7,6,8)\sigma^i, & (1,4,2,6,3,5)(7,10,9,11,8,12)\sigma^i, \\
(1,12,3,11,2,10)(4,9,5,7,6,8)\sigma^i, & (1,7,3,8,2,9)(4,10,5,12,6,11)\sigma^i, & (1,4,3,5,2,6)(7,10,8,12,9,11)\sigma^i, \\
(1,7,3,8,2,9)(4,11,5,10,6,12)\sigma^i, & (1,12,2,10,3,11)(4,9,6,8,5,7)\sigma^i, & (1,6,3,4,2,5)(7,12,8,11,9,10)\sigma^i, \\
(1,10,3,12,2,11)(4,7,5,8,6,9)\sigma^i, & (1,9,3,7,2,8)(4,12,5,11,6,10)\sigma^i, & (1,4,2,6,3,5)(7,11,9,12,8,10)\sigma^i, \\
(1,7,2,9,3,8)(4,11,6,12,5,10)\sigma^i, & (1,12,3,11,2,10)(4,7,5,8,6,9)\sigma^i, & (1,4,3,5,2,6)(7,12,8,11,9,10)\sigma^i, \\
(1,9,3,7,2,8)(4,11,5,10,6,12)\sigma^i, & (1,12,2,10,3,11)(4,8,6,7,5,9)\sigma^i, & (1,6,3,4,2,5)(7,11,8,10,9,12)\sigma^i, \\
(1,12,2,10,3,11)(4,7,6,9,5,8)\sigma^i, & (1,7,2,9,3,8)(4,12,6,10,5,11)\sigma^i, & (1,6,3,4,2,5)(7,10,8,12,9,11)\sigma^i, \\
(1,9,3,7,2,8)(4,10,5,12,6,11)\sigma^i, & (1,12,3,11,2,10)(4,8,5,9,6,7)\sigma^i, & (1,4,3,5,2,6)(7,12,8,11,9,10)\sigma^i, \\
(1,7,3,8,2,9)(4,12,5,11,6,10)\sigma^i, & (1,10,3,12,2,11)(4,8,5,9,6,7)\sigma^i, & (1,4,2,6,3,5)(7,12,9,10,8,11)\sigma^i, \\
\end{array}
$$

(Here i can take any value between 1 and 6, and an element is either on the list or its inverse is on the list.)

In the list above the elements of every row generate an abelian group and so these 36 groups cover the connected component. On the other hand, fixing i and taking the elements of the first column we get a non-commuting set of size 36, as required.

Class $6-4-2-2$: a maximal connected component in $\Gamma(\Lambda)$ contains 72 pemutations. One example is as follows:

$$
\begin{array}{lll}
(1,2,3,4)(5,8)(6,7)\sigma^i, & (1,2,3,4)(5,6)(7,8)\sigma^i, & (1,2,3,4)(5,7)(6,8)\sigma^i, \\
(1,2)(3,4)(5,6,7,8)\sigma^i, & (1,3)(2,4)(5,6,7,8)\sigma^i, & (1,4)(2,3)(5,6,7,8)\sigma^i, \\
(1,2,4,3)(5,6)(7,8)\sigma^i, & (1,2,4,3)(5,7)(6,8)\sigma^i, & (1,2,4,3)(5,8)(6,7)\sigma^i, \\
(1,3)(2,4)(5,6,8,7)\sigma^i, & (1,4)(2,3)(5,6,8,7)\sigma^i, & (1,2)(3,4)(5,6,8,7)\sigma^i, \\
(1,3,2,4)(5,7)(6,8)\sigma^i, & (1,3,2,4)(5,8)(6,7)\sigma^i, & (1,3,2,4)(5,6)(7,8)\sigma^i, \\
(1,4)(2,3)(5,7,6,8)\sigma^i, & (1,2)(3,4)(5,7,6,8)\sigma^i, & (1,3)(2,4)(5,7,6,8)\sigma^i, \\
\end{array}
$$

(Here $\sigma = (9,10,11,12,13,14,15)$, i can be either 1 or 5, and an element is either on the list or its inverse is on the list.)

In the list above the elements of every row generate an abelian group and so these 6 groups cover the connected component. On the other hand, fixing i and taking the elements of the first column we get a non-commuting set of size 6, as required.

Class $6-3-3-2$: a maximal connected component in $\Gamma(\Lambda)$ contains 96 pemutations. One example is as follows:

$(1,2,5,6,3,4)(7,8,9)(10,12,11),$ $(1,2,5,6,3,4)(7,8,9)(10,12,11),$ $(1,5,3)(2,6,4)(7,12,8,11,9,10),$ $(1,2,5,6,3,4)(7,9,8)(10,11,12),$
$(1,3,5)(2,4,6)(7,11,9,12,8,10),$ $(1,4,5,2,3,6)(7,8,9)(10,12,11),$ $(1,5,3)(2,6,4)(7,11,9,12,8,10),$ $(1,4,5,2,3,6)(7,9,8)(10,11,12),$
$(1,3,5)(2,4,6)(7,11,8,10,9,12),$ $(1,2,3,4,5,6)(7,8,9)(10,12,11),$ $(1,5,3)(2,6,4)(7,11,8,10,9,12),$ $(1,2,3,4,5,6)(7,9,8)(10,11,12),$
$(1,4,3,2,5,6)(7,8,9)(10,12,11),$ $(1,3,5)(2,6,4)(7,11,8,10,9,12),$ $(1,4,3,2,5,6)(7,9,8)(10,11,12),$ $(1,5,3)(2,4,6)(7,11,9,12,8,10),$
$(1,2,3,6,5,4)(7,8,9)(10,12,11),$ $(1,3,5)(2,6,4)(7,11,9,12,8,10),$ $(1,2,3,6,5,4)(7,9,8)(10,11,12),$ $(1,5,3)(2,4,6)(7,11,9,12,8,10),$
$(1,2,5,4,3,6)(7,8,9)(10,12,11),$ $(1,3,5)(2,6,4)(7,12,8,11,9,10),$ $(1,2,5,4,3,6)(7,9,8)(10,11,12),$ $(1,5,3)(2,4,6)(7,12,8,11,9,10),$
$(1,3,5)(2,6,4)(7,10,9,12,8,11),$ $(1,4,3,2,5,6)(7,8,9)(10,11,12),$ $(1,5,3)(2,4,6)(7,10,9,12,8,11),$ $(1,4,3,2,5,6)(7,9,8)(10,12,11),$
$(1,3,5)(2,6,4)(7,12,9,11,8,10),$ $(1,2,5,4,3,6)(7,8,9)(10,11,12),$ $(1,5,3)(2,4,6)(7,12,9,11,8,10),$ $(1,2,5,4,3,6)(7,9,8)(10,12,11),$
$(1,3,5)(2,6,4)(7,12,8,10,9,11),$ $(1,2,3,6,5,4)(7,8,9)(10,11,12),$ $(1,5,3)(2,4,6)(7,12,8,10,9,11),$ $(1,2,3,6,5,4)(7,9,8)(10,12,11),$
$(1,2,5,6,3,4)(7,8,9)(10,11,12),$ $(1,3,5)(2,4,6)(7,10,9,12,8,11),$ $(1,2,5,6,3,4)(7,9,8)(10,12,11),$ $(1,5,3)(2,6,4)(7,10,9,12,8,11),$
$(1,4,5,2,3,6)(7,8,9)(10,11,12),$ $(1,3,5)(2,4,6)(7,12,8,10,9,11),$ $(1,4,5,2,3,6)(7,9,8)(10,12,11),$ $(1,5,3)(2,6,4)(7,12,8,10,9,11),$
$(1,2,3,4,5,6)(7,8,9)(10,11,12),$ $(1,3,5)(2,4,6)(7,12,9,11,8,10),$ $(1,2,3,4,5,6)(7,9,8)(10,12,11),$ $(1,5,3)(2,6,4)(7,12,9,11,8,10).$

(Here, to save space, we have omitted the cycle $(13,14)$ from every permutation. As always every element is either on the list or its inverse is on the list.)

In the list above the elements of every row generate an abelian group and so these 12 groups cover the connected component. On the other hand, taking the elements of the first column we get a non-commuting set of size 12, as required. $\qquad\square$

Theorem 1.2 now follows from Propositions 3.4 and 3.7, Corollary 3.13 and Lemma 3.14.

4. Further work

Note that both Theorems 1.2 and 1.3 have numbers for which information is not given. In particular, the following groups are not covered: $S_{10}, S_{12}, S_{14}, A_{13}, A_{14}, A_{17}$ and A_{19}.

Results from this paper can be used to study these seven groups: for instance, one can check that $Y_b(S_{10})$ consists of the single conjugacy class $4-2-2$. Thus Lemma 2.5 and a consideration of the abelian groups that contain elements from this class imply that $\delta(S_{10}) = \Delta(S_{10})$ if and only if one can find 9450 noncommuting permutations in S_{10} of cycle type $4-2-2$.

In [Bro88, Bro91], Brown studies the asymptotics of the sequence $\Delta(S_n)/\delta(S_n)$ and, in particular, shows that this sequence has no limit and takes on infinitely many distinct values arbitrarily close to 1. It seems reasonable to think that the same is true of the sequence $\Delta(A_n)/\delta(A_n)$, but this has not yet been established.

Graphs analogous to the commuting graph have been studied in various contexts. In particular, in [ABG], a graph $\Gamma_c(G)$ is defined for any finite group G and any $c \in \mathbb{Z}^+ \cup \{\infty\}$ as follows: vertices are the elements of G, with two vertices $a, b \in G$ joined by an edge if and only if $\langle a, b \rangle$ is nilpotent of class at most c. Observe that the commuting graph is simply the graph $\Gamma_1(G)$.

A natural extension to the work in the current paper would be to establish whether or not $\delta(\Gamma_c(G)) = \Delta(\Gamma_c(G))$ for G alternating or symmetric, and $c \neq 1$.

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