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NILPOTENT COVERS AND NON-NILPOTENT SUBSETS OF FINITE GROUPS OF LIE TYPE

AZIZOLLAH AZAD, JOHN R. BRITNELL AND NICK GILL

ABSTRACT. Let G be a finite group, and c an element of $\mathbb{Z}^+ \cup \{\infty\}$. A subgroup H of G is said to be *c-nilpotent* if it is nilpotent, and has nilpotency class at most c . A subset X of G is said to be *non-c-nilpotent* if it contains no two elements x and y such that the subgroup $\langle x, y \rangle$ is *c-nilpotent*. In this paper we study the quantity $\omega_c(G)$, defined to be the size of the largest non-*c-nilpotent* subset of G .

In the case that L is a finite group of Lie type, we identify covers of L by *c-nilpotent* subgroups, and we use these covers to construct large non-*c-nilpotent* sets in L . We prove that for groups L of fixed rank r , there exist constants D_r and E_r such that $D_r N \leq \omega_\infty(L) \leq E_r N$, where N is the number of maximal tori in L .

In the case of groups L with twisted rank 1, we provide exact formulae for $\omega_c(L)$ for all $c \in \mathbb{Z}^+ \cup \{\infty\}$. If we write q for the level of the Frobenius endomorphism associated with L and assume that $q > 5$, then these formulae may be expressed as polynomials in q with coefficients in $\mathbb{Z}[\frac{1}{2}]$.

1. INTRODUCTION AND RESULTS

Let G be a group and c an element of $\mathbb{Z}^+ \cup \{\infty\}$. For $c \in \mathbb{Z}^+$, we define G to be *c-nilpotent* if G is nilpotent of class at most c . We define G to be *∞ -nilpotent* if G is nilpotent.

1.1. Non-*c-nilpotent* subsets. A subset X of G is said to be *non-c-nilpotent* if, for any two distinct elements x and y in X , the subgroup $\langle x, y \rangle$ of G which they generate is not *c-nilpotent*. The subset X is said to be *non-nilpotent* if it is non- ∞ -nilpotent.

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Define $\omega_c(G)$ to be the maximum size of a non- c -nilpotent subset of G . It is a trivial observation that for any group G , we have

$$\omega_1(G) \geq \omega_2(G) \geq \cdots \geq \omega_\infty(G) \geq 1.$$

Furthermore, if the class of a nilpotent subgroup of G is bounded above by an integer d , then clearly

$$\omega_d(G) = \omega_{d+1}(G) = \cdots = \omega_\infty(G).$$

Certainly such a bound exists whenever G is finite. Furthermore if G is a nilpotent group of class d , then $\omega_c(G) = 1$ for $c \geq d$, and for $c = \infty$.

Our main interest in this paper will be the quantity $\omega_\infty(G)$, where G is a finite group of Lie type. However we shall also consider $\omega_c(G)$ for various finite values of c . In particular the case when $c = 1$ is of interest, since $\omega_1(G)$ is the maximum size of a non-commuting subset (i.e. a subset X of G such that $[x, y] \neq 1$ for all $x, y \in X$).

1.2. Nilpotent covers. There is a close connection between the non- c -nilpotent subsets of maximum size in a group G , and c -nilpotent covers of G . Let \mathcal{N} be a family of c -nilpotent subgroups of G . We shall be interested in two possible properties of \mathcal{N} :

1. **Covering:** If for every $g \in G$ there exists $X \in \mathcal{N}$ such that $g \in X$, then we say that \mathcal{N} is a c -nilpotent cover of G , or that \mathcal{N} covers G . If $c = \infty$, we say that \mathcal{N} is a nilpotent cover of G .
2. **2-minimality:** If for every subgroup $X_i \in \mathcal{N}$ there is an element g_i (called a *distinguished element*), such that $i \neq j$ implies that $\langle g_i, g_j \rangle$ is not c -nilpotent, then we say that \mathcal{N} is 2-minimal.

We remark that every group admits a c -nilpotent cover for all $c \in \mathbb{Z}^+ \cup \{\infty\}$, since the family consisting of all cyclic subgroups is one such. But not every group admits a 2-minimal c -nilpotent cover: examples are the symmetric groups S_n for $n \geq 15$ in the case $c = 1$ [Bro88, Bro91].

Suppose that \mathcal{N} is a 2-minimal c -nilpotent cover, and that $X_1, X_2 \in \mathcal{N}$ contain distinguished elements g_1 and g_2 respectively. We note that if $X_2 \neq X_1$, then $g_1 \notin X_2$ (otherwise the subgroup $\langle g_1, g_2 \rangle$ would be a subgroup of X_2 , and hence c -nilpotent; this contradicts the condition for 2-minimality). It follows that the removal from \mathcal{N} of any one of its members

results in a family which is not a cover of G (since the distinguished element of the removed member is not in any other member). Hence a 2-minimal c -nilpotent cover of G is, in particular, a *minimal* c -nilpotent cover. The converse is false, however: it is not necessarily true that a minimal c -nilpotent cover of G is 2-minimal. This is clear from the fact that there exist finite groups with no 2-minimal c -nilpotent covers.

The significance of these properties to the calculation of $\omega_c(G)$, and in particular the value of finding a 2-minimal cover of G , is shown by the following proposition.

Proposition 1.1. *Let \mathcal{N} be a family of c -nilpotent subgroups of G .*

- (1) *If \mathcal{N} covers G then $|\mathcal{N}| \geq \omega_c(G)$.*
- (2) *If \mathcal{N} is 2-minimal then $|\mathcal{N}| \leq \omega_c(G)$.*
- (3) *If \mathcal{N} is a 2-minimal cover of G , then $|\mathcal{N}| = \omega_c(G)$.*

Proof. For the first part, suppose X is a set of size greater than $|\mathcal{N}|$. If \mathcal{N} covers G , then there exist two elements of X lying in one member of \mathcal{N} , and so X cannot be non- c -nilpotent. The second part is obvious, since the distinguished elements of a 2-minimal family form a non- c -nilpotent set. The third part of the proposition follows immediately from the other two. \square

Lemmas 3.1 and 3.2 below provide useful generalizations of the inequalities in the first two parts of Proposition 1.1.

1.3. Results and structure. The paper is structured as follows. In §2 we give some background results on linear algebraic groups. In §3 we set out the group-theoretic notation we will use throughout the paper, and we prove a number of basic lemmas pertaining to non- c -nilpotent groups.

In §4 we consider the case where $L = G^F$, where G is a simple linear algebraic group of rank r , and F is a Frobenius endomorphism of G . Theorem 4.3 offers a general method for producing lower bounds for $\omega_\infty(L)$, by counting the Sylow subgroups of L for certain prime divisors of $|L|$. In the course of proving this theorem, we provide in Lemma 4.1 a generalization and strengthening of the main result of [Aza11].

In §4 we also prove the following theorem, which is the main result of this paper.

Theorem 1.2. *For every $r > 0$, there exist constants $D_r, E_r > 0$, such that for any simple linear algebraic group G of rank r , and for any Frobenius endomorphism F of G , we have*

$$D_r N(G^F) \leq \omega_\infty(G^F) \leq E_r N(G^F),$$

where $N(G^F)$ is the number of F -stable maximal tori in G .

The number $N(G^F)$ is given by a result of Steinberg (Proposition 2.5 below) as $q^{|\Phi|}$, where q is the level of the Frobenius endomorphism F , and Φ is a root system for G . In fact it would be possible, in the statement of Theorem 1.2, to take $N(G^F)$ instead to be the number of maximal tori in G^F itself; this point is discussed in the proof of Proposition 4.6 below.

It seems reasonable to conjecture that an upper bound of the same form exists not just for $\omega_\infty(G^F)$ but for $\omega_1(G^F)$, and more speculatively, that the dependence on rank can be removed. We may formulate the following conjecture.

Conjecture 1.3. *There exist absolute constants $D, E > 0$ such that, for any simple linear algebraic group G and any Frobenius endomorphism F of G , we have*

$$DN(G^F) \leq \omega_\infty(G^F) \leq \cdots \leq \omega_1(G) \leq EN(G^F),$$

where $N(G^F)$ is the number of F -stable maximal tori in G .

Theorem 1.2 confirms the existence of a rank-dependent constant D . In particular the existence of a constant D is confirmed for groups of bounded rank, which includes all the exceptional groups. The existence of a non-rank-dependent constant E for the groups $\mathrm{PGL}_n(q)$ follows from the main result of [AIPS11]; and we note that the full conjecture is also confirmed in the case of groups of rank 1 by the results of §5 of this paper.

In §5 we assume that G^F is a finite group of Lie type where G is simple and the (twisted) Lie rank of G^F is 1. We prove that, for all $c \in \mathbb{Z}^+ \cup \{\infty\}$, G^F admits a 2-minimal c -nilpotent cover. Furthermore we construct explicit examples of such covers and we calculate their size, thereby producing exact formulae for $\omega_c(G^F)$ in each case.

The nature of these formulae is somewhat remarkable, and we discuss some of their characteristics in §6. We suggest a number of questions and

conjectures arising from these observations, and from other results in the paper.

1.4. Background results. The value of $\omega_\infty(G)$ has been studied for various groups; it has usually been denoted $\omega(\mathcal{N}_G)$. Endimioni has proved that if a finite group G satisfies $\omega_\infty(G) \leq 3$ then G is nilpotent, while if $\omega_\infty(G) \leq 20$ then G is solvable; furthermore these bounds cannot be improved [End94]. Tomkinson has shown that if G is a finitely generated solvable group such that $\omega_\infty(G) = n$, then $|G/Z^*(G)| \leq n^{n^4}$, where $Z^*(G)$ is the hypercentre of G [Tom92]. Also, for a finite insoluble group G , it has been proved by the first author and Hassanabadi that G satisfies the condition $\omega_\infty(G) = 21$ if and only if $G/Z^*(G) \cong A_5$ [AMH04, Theorem 1.2].

In addition to the results in [AMH04], the computation of $\omega_\infty(G)$ for particular classes of groups G has recently started to garner attention. In particular, the first author has given lower bounds for $\omega_\infty(G)$ [Aza11] when $G = \text{GL}_n(q)$. In a forthcoming paper by the second and third authors [BG], a nilpotent cover of $\text{GL}_n(q)$ is constructed which is 2-minimal when $q > n$; this construction will establish the exact value of $\omega_\infty(\text{GL}_n(q))$ for almost all values of n and q , and will give an upper bound in the remaining cases.

The particular statistic $\omega_1(G)$ has attracted considerable recent attention, and been calculated for various groups G ; much of this work has concentrated on the case of almost simple groups G [AAMHZ10, AIPS11, AP09, Bro88, Bro91].

The study of non-commuting sets in a group G , including the study of $\omega_1(G)$, goes back many years. In 1976, B. H. Neumann famously answered the following question of Erdős from a few years earlier: if all non-commuting sets in a group G are finite, does there exist an upper bound $n = n(G)$ for the size of a non-commuting set in G (i.e. is $\omega_1(G)$ finite)? Neumann answered this question affirmatively by showing that if all non-commuting sets in a group G are finite, then $|G : Z(G)|$ is finite [Neu76]. Pyber subsequently gave a strong upper bound for $|G : Z(G)|$, subject to the same condition on G [Pyb87].

Related to this area of study is the problem of calculating, for a finite 2-generator group G , the size $\mu(G)$ of the largest subset $X \subseteq G$ such that any pair of elements of X generate G . Such sets are closely related to covers

of G by proper subgroups (in just the same way that we have seen that non- c -nilpotent sets are related to c -nilpotent covers). The statistic $\mu(G)$ has been studied for the symmetric and alternating groups in [Bla06], and for the groups $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$ in [BEG⁺08].

1.5. Connections to perfect graphs. We note a connection with a generalization of the commuting graph of a finite group G . The commuting graph of G is the graph $\Gamma_1(G)$ whose vertices are the elements of G , with an edge joining vertices x and y if and only if x and y commute in G . (An alternative definition excludes central elements of G ; the distinction is unimportant here.) There is an obvious correspondence between the maximal abelian subgroups of G , and the maximal cliques in the commuting graph $\Gamma_1(G)$. From this fact it follows that the minimal size of a covering of G by abelian subgroups is equal to the *clique cover number* of $\Gamma_1(G)$, i.e. the minimal number of cliques required to cover its vertices.

Suppose that G has a 2-minimal abelian cover. Then the set of distinguished elements form an independent set in $\Gamma_1(G)$, and it follows that the clique cover number and the independence number (being the maximal size of an independent set of vertices) are the same for $\Gamma_1(G)$. (It is obvious that the clique cover number is at least as big as the independence number.)

A graph is *perfect* if the clique cover number and the independence number coincide for every induced subgraph. It appears that it is not known which finite groups have perfect commuting graphs.¹

There is an obvious generalization of $\Gamma_1(G)$ as follows: for $c \in \mathbb{Z}^+ \cup \{\infty\}$ define $\Gamma_c(G)$ to be the graph whose vertices are the elements of G , with an edge joining vertices x and y if and only if the subgroup $\langle x, y \rangle$ is c -nilpotent. We can generalize the earlier observation connecting abelian covers to properties of $\Gamma_1(G)$ in the following way.

Proposition 1.4. *Suppose that a finite group G has a 2-minimal c -nilpotent cover \mathcal{N} . Then the clique cover number and the independence number of $\Gamma_c(G)$ coincide.*

¹Peter Cameron has recently discussed this problem on his blog, at <http://cameroncounts.wordpress.com/2011/02/01/perfectness-of-commuting-graphs/>.

Proof. It is clear that the clique cover number can be no larger than $|\mathcal{N}|$. The proposition therefore follows from the observation that the set of distinguished elements in members of \mathcal{N} forms an independent set of size $|\mathcal{N}|$. \square

Note that Proposition 1.4 implies that a necessary condition for $\Gamma_c(G)$ to be perfect is that G admits a 2-minimal c -nilpotent cover.

We remark that, when $c > 1$, it is not necessarily true that the maximal cliques in the resulting graph $\Gamma_c(G)$ correspond to maximal c -nilpotent subgroups of G ; for instance, there exist two 3-nilpotent groups of order 64 for which the graph $\Gamma_2(G)$ has maximal cliques of size 40. Thus the proof of Proposition 1.4 yields some extra information: we can conclude that if G has a 2-minimal c -nilpotent cover \mathcal{N} , then $\Gamma_c(G)$ admits a minimal covering by (not necessarily maximal) cliques such that each clique corresponds to a maximal c -nilpotent subgroup of G .

In a different direction, one might define the graph $\Gamma_c(G)$ to be *predictable* if all maximal cliques in $\Gamma_c(G)$ correspond to maximal c -nilpotent subgroups of G . One can then ask for which groups G , and which values of c , is the graph $\Gamma_c(G)$ predictable. In particular is it true that if G is a simple group, then $\Gamma_c(G)$ will be predictable for all $c \in \mathbb{Z}^+ \cup \{\infty\}$?

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2. BACKGROUND ON LINEAR ALGEBRAIC GROUPS

The material in this section is drawn primarily from [Car85]. Throughout the section, G is a connected reductive linear algebraic group over an algebraically closed field K of characteristic $p > 0$.

Since G is a linear algebraic group we can write $G \leq \mathrm{GL}_n(K)$ for some integer n . An element $g \in G$ is then said to be *semisimple* if g is diagonalizable in $\mathrm{GL}_n(K)$, and it is said to be *unipotent* if its only eigenvalue is 1. It is a fact that the condition for g to be semisimple (respectively unipotent) is independent of the embedding of G into $\mathrm{GL}_n(K)$, and so we can say

that g is semisimple (respectively unipotent) element without reference to any particular embedding.

A *unipotent subgroup* of G is a closed subgroup, all of whose elements are unipotent. A *Borel subgroup* of G is a maximal connected closed solvable subgroup of G . Borel subgroups always exist, and all Borel subgroups of G are conjugate [Car85, p. 16]. We can write $B = UT$ where U is the unipotent radical of B (i.e. the maximal connected normal unipotent subgroup of B) and T is a maximal torus of B (and hence of G). Note that the union of all Borel subgroups of G is the whole of G , and in particular, any semisimple element of G lies in a maximal torus of G .

The *rank* of G , written $\text{rank}(G)$, is defined to be the dimension of a maximal torus in G . We write W for the *Weyl group* of G and recall that if T is a maximal torus of G then $N_G(T)/T \cong W$.

2.1. Regular elements and centralizers. Any unattributed statements in this section can be found in [Car85, §1.14]. We note first that $\dim C_G(g) \geq \text{rank}(G)$ for all $g \in G$, where $C_G(g)$ denotes the centralizer of g in G . An element g of G is said to be *regular* if $\dim C_G(g) = \text{rank}(G)$. If g is regular in G , then $C_G(g)^0$, the connected component of $C_G(g)$, is commutative.

The key facts for our purposes concerning regular semisimple elements are given by the following result from [Ste65].

Proposition 2.1. *Let G be a semisimple group and let $s \in G$ be semisimple. Then the following conditions are equivalent.*

- (1) s is regular.
- (2) $C_G(s)^0$ is a maximal torus in G .
- (3) s is contained in a unique maximal torus of G .

Every connected reductive group G contains regular unipotent elements and any two are conjugate in G . Let $u \in G$ be regular and unipotent; then every semisimple element in $C_G(u)$ lies in $Z(G)$, the centre of G [Car85, Prop. 5.1.5]. The following result will be useful [Car85, Prop. 5.1.3].

Proposition 2.2. *Let G be a connected reductive group and let $u \in G$ be unipotent. Then the following conditions on u are equivalent.*

- (1) u is regular.
- (2) u lies in a unique Borel subgroup of G .

The final proposition of this section follows from [GLS98, Theorem 1.12.5].

Proposition 2.3. *Let G be a simple algebraic group, and let W be the Weyl group of G . If t is a prime divisor of $|Z(G)|$, then t divides $|W|$.*

2.2. Groups of Lie type. A *Frobenius endomorphism* of a reductive linear algebraic group G is an algebraic endomorphism F of G with the property that there is an algebraic group embedding of G into $\mathrm{GL}_n(K)$, for an algebraically closed field K and $n \in \mathbb{N}$, and a power of F , say F^m , that is equal to the map on G induced by a Galois automorphism of the field K . Let q_0 be the size of the fixed field of F^m . Then the *level* of F is defined to be $q_0^{1/m}$ (cf. [GLS98, Definition 2.1.9]).

In what follows, we shall usually adopt the convention that q is the level of the Frobenius map F . When dealing with the rank 1 Ree groups in Section 5, however, it will be convenient to take q to be the square of the level of F (which is q_0 in the notation above).

A Frobenius endomorphism is always an automorphism of G , considered as an abstract group, though the inverse map need not be algebraic. We write G^F for the fixed points of G under F . A *finite group of Lie type* is a group of the form G^F , where G is a reductive linear algebraic group and F is a Frobenius endomorphism of G .

The *simple groups of Lie type* are defined in [GLS98, Definition 2.2.8]; they are subgroups of G^F , where G is a simple linear algebraic group and F is a Frobenius endomorphism. A consequence of this terminology is that a simple group of Lie type is not necessarily a finite group of Lie type. For the purposes of studying non-nilpotent sets, however, the next result renders the distinction irrelevant (see the discussion following Lemma 3.6).

Lemma 2.4. *Let L be a simple group of Lie type. Then $L \cong G^F/Z(G^F)$ for some simple linear algebraic group G and Frobenius endomorphism F .*

Proof. This follows from [GLS98, Theorem 2.2.6(f)] and [GLS98, Definition 2.5.10]. \square

If F is a Frobenius endomorphism of the group G , and if H is a closed subgroup of G , then we write H^F for the set of points of H fixed by F .

When G is simple, the group G^F is said to be *untwisted* if F is induced by a Galois automorphism of the field, and *twisted* otherwise. When G^F is

untwisted we define its *rank* to be equal to that of G . When G^F is twisted there are two standard notions of rank (cf. [GLS98, Remark 2.3.3]), but we shall need only one: the *twisted rank* of G . If G^F is a generic group of Lie type we shall refer to its (*twisted*) *rank* to mean just the rank if G^F is untwisted, and the twisted rank otherwise.

2.3. Borel subgroups and tori in G^F . We have already mentioned Borel subgroups and tori in the group G . We extend this notion to the group G^F as in [Car85, §1.18]. Note first that a subgroup H of G which satisfies $F(H) = H$ is said to be *F-stable*.

A *Borel subgroup* of G^F is a subgroup of the form B^F , where B is an F -stable Borel subgroup of G . Any two Borel subgroups of G^F are conjugate in G^F ; what is more, if we write $B = UT$ as above, then [CR87, (69.10)] implies that $N_{G^F}(U^F) = B^F$.

A *maximal torus* of G^F is a subgroup of the form T^F , where T is an F -stable maximal torus of G . An F -stable maximal torus of G is said to be *maximally split* if it lies in an F -stable Borel subgroup of G , and a maximal torus of G^F is said to be *maximally split* if it has the form T^F for some maximally split torus T in G . Any two maximally split tori of G^F are conjugate in G^F .

It follows from the Lang-Steinberg theorem that the group G contains an F -stable Borel subgroup and that any F -stable Borel subgroup contains an F -stable maximal torus of G . Thus G^F contains both a Borel subgroup and a maximally split maximal torus.

The following result of Steinberg gives a precise count for the number of F -stable maximal tori in G (cf. [Car85, Theorem 3.4.1]; we remark that the result as stated there applies also to the Ree and Suzuki groups). Recall that q is defined to be the level of the Frobenius endomorphism F .

Proposition 2.5. *Let Φ be the root system for G . The number of F -stable maximal tori in G is $q^{|\Phi|}$.*

We have already introduced the notation $N(G^F)$ for the number of F -stable maximal tori in G (see Theorem 1.2). Note that if q is sufficient large, then the number of maximal tori in G^F is equal to the number of F -stable maximal tori in G [Car85, Proposition 3.6.6].

2.4. Sylow t -subgroups in G^F . We are interested in the Sylow structure of the group G^F , for which we refer to [GLS98] and [SS70]. Note first the elementary facts that an element $g \in G^F$ is unipotent if and only if it has order p^ℓ for some $\ell \geq 0$, while g is semisimple if and only if p does not divide the order of g .

In order to study the structure of a Sylow p -subgroup, we consider an F -stable Borel subgroup B of G . If we write $B = UT$ as above, then since B, U and T are all F -stable, we have $B^F = U^F \rtimes T^F$. Now [GLS98, Theorem 2.3.4] implies that U^F is a Sylow p -subgroup of G^F .

We shall require the following fact, which follows from [SS70, II.5.19].

Proposition 2.6. *Let t be a prime dividing $|G^F|$, with $t \neq p$, and suppose that X is a Sylow t -subgroup of G^F . Then X lies inside $N_G(T)$, where T is an F -stable maximal torus of G . If t is coprime to $|W|$, then X lies inside T itself (and so, in particular, X lies inside T^F , a maximal torus of G^F).*

Let x and n be positive integers. A *primitive prime divisor* (also sometimes called a *Zsigmondy prime*) of $x^n - 1$ is a prime divisor s which does not divide $x^k - 1$ for any $k < n$. A well known theorem of Zsigmondy [Zsi92] (in part also due to Bang [Ban86]) states that if $x > 1$ and $n > 2$, then there exists a primitive prime divisor of $x^n - 1$ except in the case $(x, n) = (2, 6)$. A primitive prime divisor exists also when $n = 2$, unless $x = 2^b - 1$ for some b .

We shall need the following simple inequality.

Lemma 2.7. *Let p be prime, and let $q = p^a$ for some $a \in \mathbb{N}$. Let $n > 1$ be such that a primitive prime divisor of $q^n - 1$ exists. Then there is a primitive prime divisor r of $q^n - 1$ such that $r > an$.*

Proof. It is clear from Zsigmondy's Theorem that if a primitive prime divisor of $q^n - 1$ exists, and if $q^n \neq 64$, then a primitive prime divisor to the base p also exists; that is, a divisor r of $p^{an} - 1$ which does not divide $p^b - 1$ for any $b < an$. Now Euler's Theorem states that $p^{\phi(r)} \equiv 1 \pmod{r}$, where ϕ is the totient function. It follows that $\phi(r) \geq an$, and hence that $r > an$. It is easy to check that the result holds also for the exceptional case in Zsigmondy's Theorem, arising when $q^n = 64$. \square

2.5. Classical groups. Suppose that G is simple of rank n , and let F be a Frobenius endomorphism of level q . If G is from one of the families

L	Φ	$ \Phi $	$ W $
$\mathrm{PSL}_{n+1}(q)$	A_n	$n(n+1)$	$(n+1)!$
$\mathrm{PSU}_{n+1}(q)$	2A_n	$n(n+1)$	$(n+1)!$
$\mathrm{P}\Omega_{2n+1}(q)$	B_n	$2n^2$	$2^n n!$
$\mathrm{PSp}_{2n}(q)$	C_n	$2n^2$	$2^n n!$
$\mathrm{P}\Omega_{2n}^+(q)$	D_n	$2n(n-1)$	$2^{n-1} n!$
$\mathrm{P}\Omega_{2n}^-(q)$	2D_n	$2n(n-1)$	$2^{n-1} n!$

TABLE 1. Sizes of root systems Φ and Weyl groups W for the families of groups L of classical type.

A_n, B_n, C_n or D_n , then G^F can be identified as a ‘classical group’, as follows. G^F is isomorphic to the quotient of a group of isometries over a field

We shall need some standard facts about the sizes of root systems and Weyl groups for various classical groups. These are collected together in Table 1.

3. LEMMAS ON NON-NILPOTENT SETS

In this section H is a group and c is an element of $\mathbb{Z}^+ \cup \{\infty\}$. We begin with some results which are useful for giving upper and lower bounds for $\omega_c(H)$. The first result, which is based on [AP09, Lemma 2.4], requires us first to extend the definition of ω_c to subsets of groups: for $X \subseteq H$, we write $\omega_c(X)$ for size of the largest non- c -nilpotent subset of X .

Lemma 3.1. *Let Y_1, \dots, Y_k be subsets of H such that $Y_1 \cup \dots \cup Y_k = H$. Then*

$$\omega_c(H) \leq \omega_c(Y_1) + \dots + \omega_c(Y_k).$$

Proof. Suppose that Ω is a non- c -nilpotent set in H . Then clearly $|\Omega \cap Y_i| \leq \omega_c(Y_i)$ for $i = 1, \dots, k$, and the result follows. \square

Lemma 3.2. *Let $X_1, \dots, X_k, Y_1, \dots, Y_k$ be subsets of H . Suppose that the following hold:*

- (1) for $i = 1, \dots, k$, X_i is a non- c -nilpotent subset of Y_i of maximum size;
- (2) if $1 \leq i < j \leq k$ and $x_i \in X_i, x_j \in X_j$, then $\langle x_i, x_j \rangle$ is not c -nilpotent.

Then $X_1 \cup \cdots \cup X_k$ is a non- c -nilpotent subset of H , and

$$\omega_c(H) \geq \omega_c(Y_1) + \cdots + \omega_c(Y_k).$$

Proof. The fact that $X_1 \cup \cdots \cup X_k$ is a non- c -nilpotent subset of H follows immediately from our suppositions, and the lower bound stated for $\omega_c(H)$ is an obvious consequence. \square

As mentioned in the introduction, Lemmas 3.1 and 3.2 put into a more general form the connection between nilpotent covers and non-nilpotent sets already established in Proposition 1.1. These lemmas have a number of useful special cases: we refer in particular to [Aza11, Lemma 3.2] and [Aza11, Lemma 3.3]. The following result is a slightly refined version of the latter. For a prime p , we write $\nu_p(H)$ for the number of Sylow p -subgroups of a finite group H .

Lemma 3.3. *Let H be finite and let p be a prime number dividing $|H|$. Let $P = P_1, P_2, \dots, P_{\nu_p(H)}$ be the Sylow p -subgroups of H . Suppose that*

$$(3.1) \quad P \setminus \bigcup_{i=2}^{\nu_p(H)} P_i \neq \emptyset.$$

Then the set $\{P_1, \dots, P_{\nu_p(H)}\}$ is 2-minimal, and there exists a non-nilpotent set $\Omega \subseteq H$ such that all elements of Ω are p -elements, and such that $|\Omega| = \nu_p(H)$.

Proof. Note first that the condition (3.1) is equivalent to the assertion that H contains elements which lie in a unique Sylow p -subgroup of H . Since the Sylow p -subgroups of H are conjugate, we see that every Sylow p -subgroup P_i of H contains an element w_i which lies in no other Sylow p -subgroup. Let Ω be the set $\{w_i \mid i = 1, \dots, \nu_p(H)\}$. Then clearly $|\Omega| = \nu_p(H)$.

Let w_i and w_j be distinct elements of Ω , and let $W = \langle w_i, w_j \rangle$. Since no p -subgroup of H contains both w_i and w_j , it follows that the Sylow p -subgroups of W are not normal in W , and so W is not nilpotent. We conclude that Ω is a non-nilpotent set, and hence that the family $\{P_1, \dots, P_{\nu_p(H)}\}$ is 2-minimal. \square

We will use Lemma 3.3 in association with the following result.

Lemma 3.4. *Let H be finite and suppose that p_1, \dots, p_n are distinct prime divisors of $|H|$. For $i = 1, \dots, n$, let Ω_i be a non- c -nilpotent subset of H such that the orders of the elements of Ω_i are powers of p_i . Suppose that for $w_i \in \Omega_i$ and $w_j \in \Omega_j$, we have $[w_i, w_j] \neq 1$ whenever $i \neq j$. Then*

$$\omega_c(H) \geq |\Omega_1| + |\Omega_2| + \dots + |\Omega_n|.$$

Proof. The result follows easily from Lemma 3.2, once we have made the following observation: if w_i and w_j are group elements of prime power order for distinct primes, then the subgroup $\langle w_i, w_j \rangle$ is nilpotent if and only if $[w_i, w_j] = 1$. \square

We end with two related results. The first result is a restated (and slightly generalized) version of [Aza11, Lemma 3.1], which we state here without further proof. The second is a useful corollary to the first.

Lemma 3.5. *Let N be a normal subgroup of H and let Ω be a non- c -nilpotent subset of H/N . Any set of representatives of Ω in H is a non- c -nilpotent subset of H .*

Lemma 3.6. *Let Y be a central subgroup of H . A set $\Omega \subseteq H$ is non- c -nilpotent if and only if the set*

$$\Omega/Y = \{hY \mid h \in \Omega\}$$

is non- c -nilpotent in H/Y and $|\Omega| = |\Omega/Y|$.

Proof. Suppose that Ω is non- c -nilpotent. Suppose that $g, h \in \Omega$ are such that $gY = hY$. Then $h = gy$ for some $y \in Y$, and so $\langle g, h \rangle = \langle g, y \rangle$ is abelian; it follows that $g = h$, and we conclude that $|\Omega| = |\Omega/Y|$. If $\langle gY, hY \rangle$ is c -nilpotent then $\langle g, h, Y \rangle$ is c -nilpotent (since Y is central), and once again we see that $g = h$. So Ω/Y is non- c -nilpotent, as required.

Now suppose that Ω/Y is non- c -nilpotent and $|\Omega| = |\Omega/Y|$. Then Ω is a set of representatives of Ω/Y in H and the result follows from Lemma 3.5. \square

Lemma 3.6 has an important consequence: it tells us that in order to study non- c -nilpotent sets in a finite simple group of Lie type J , it is sufficient to study them in L where L is any group such that $L/Z(L) \cong J$. Now Lemma 2.4 implies that there is a simple linear algebraic group G

and a Frobenius endomorphism F such that $G^F/Z(G^F) \cong K$; therefore it is sufficient to study non- c -nilpotent sets in G^F . For instance, to understand non-nilpotency in $\mathrm{PSL}_2(q)$, we can study non- c -nilpotent sets in $\mathrm{SL}_2(q) = G^F$, where $G = \mathrm{SL}_2(K)$. Similarly, studying non- c -nilpotent sets in $\mathrm{GL}_n(q)$ will tell us about non- c -nilpotent sets in $\mathrm{PGL}_n(q) = G^F$ where $G = \mathrm{PGL}_n(K)$.

In the reverse direction, Lemma 3.6 implies that a knowledge of non- c -nilpotent sets in finite simple groups provides information on the non- c -nilpotent sets in all quasi-simple groups.

4. SOME LOWER BOUNDS

Throughout this section G is a simple linear algebraic group over an algebraically closed field K of characteristic p , and F is a Frobenius endomorphism for G .

4.1. Non-nilpotent sets given by Sylow subgroups. Our first result generalizes the main result of [Aza11].

Lemma 4.1. *Let C be a conjugacy class of G^F such that elements of C are regular unipotent. There is a set $\Omega \subseteq C$ such that $|\Omega| = \nu_p(G^F)$ and Ω is non-nilpotent.*

Proof. We can construct such an Ω by including one element of C from every Sylow p -subgroup of G^F . Since regular unipotent elements lie in a unique Borel subgroup of G , they lie in a unique Sylow p -subgroup of G^F . Now the result follows from Lemma 3.3. \square

Lemma 4.2. *Suppose that C is a conjugacy class of regular semisimple elements in G^F . Suppose also that the elements of C have order t^a , where t is a prime that does not divide $|W|$, and a is a positive integer. Then C has a non-nilpotent subset Ω of size $\nu_t(G^F)$.*

Proof. Since t does not divide $|W|$, any Sylow t -subgroup of G^F lies inside a maximal torus of G^F . Since a regular semisimple element lies in a unique maximal torus of G (and hence a unique maximal torus of G^F), it follows that each regular semisimple element of order t^a lies in a unique Sylow t -subgroup of G^F . Now the result follows from Lemma 3.3. \square

Theorem 4.3. *Let T_1, \dots, T_k be maximal tori in G^F , and suppose that they are pairwise non-conjugate. Suppose that for $i = 1, \dots, k$, the torus T_i contains a regular element g_i of order t_i , where t_i is a prime that does not divide $|W|$. Then*

$$\omega_\infty(G^F) \geq \nu_p(G^F) + \nu_{t_1}(G^F) + \dots + \nu_{t_k}(G^F).$$

Proof. Let Ω_0 be the non-nilpotent set of regular unipotent elements given in Lemma 4.1. Let Ω_i be the non-nilpotent set of regular semisimple elements in conjugates of T_i given in Lemma 4.2. Let $\Omega = \Omega_0 \cup \Omega_1 \cup \dots \cup \Omega_k$.

Let $i \in \{1, \dots, k\}$. Since t_i does not divide $|W|$ we see that T_i contains a Sylow t_i -subgroup of G^F . Since g_i is regular of order t_i we know that it lies inside a unique maximal torus of G^F , and hence $t_i \neq t_j$ for $i \neq j$. Furthermore, since g_i is regular, we see that $[g_i, h] \neq 1$ for all $h \notin T_i$. In particular $[g_i, h] \neq 1$ for any $h \in \Omega \setminus \Omega_i$.

Similarly, since an element $g \in \Omega_0$ is regular unipotent, if h is a semisimple element of G and $[g, h] = 1$ then $h \in Z(G)$. Since $t_i \nmid |W|$, we see from Proposition 2.3 that t_i does not divide $|Z(G)|$, and hence that $[g, h] \neq 1$ for all $g \in \Omega_0$ and $h \in \Omega \setminus \Omega_0$. Now the result follows by Lemma 3.4. \square

4.2. Rank dependent bounds for $\omega_\infty(G)$. In this section we complete the proof of Theorem 1.2. It is convenient to begin by describing a particular class of torus in G^F which we will focus on when we come to the proof. Our method of doing this differs depending on whether G is classical or exceptional.

Revised up to here. The next paragraphs are a mess.

Suppose first that G^F is a classical group of Lie type, and let \mathbb{F}_{q^α} be the field over which G^F is defined (so $\alpha = 2$ if G is unitary, and otherwise $\alpha = 1$.) We know that G^F is a central quotient of some group L that acts as a group of similarities for a non-degenerate or zero form on a vector space V over \mathbb{F}_{q^α} . Write d for the dimension of V .

Recall that a Singer cycle in $\mathrm{GL}(V)$ is a cyclic subgroup of order $q^{\alpha d} - 1$, and that the action of such a group on V is necessarily irreducible.

If G^F is of type A_n , C_n or 2D_n , or of type ${}^2A_{n-1}$ when n is even, then there is a Singer cycle X (that is, a cyclic subgroup of order $q^{\alpha d} - 1$) in $\mathrm{GL}_d(q^\alpha)$ such that the intersection $X \cap L$ acts irreducibly on V [Hup70];

and a maximal torus of G^F . We assume in what follows that we have chosen X so that T_0 has these properties.

On the other hand, suppose that G^F is of type B_n or D_n , or of type ${}^2A_{n-1}$ for odd n . Let U be a non-degenerate subspace of dimension $d-1$, and let $X \leq \mathrm{GL}_d(q^\alpha)$ be the product of a Singer cycle on U and a copy of $q^{\alpha \times}$ acting on U^\perp . Again, if X is chosen appropriately, then $T_0 = X \cap G^F$ is a maximal torus of G^F which acts irreducibly on U , and we assume that we have chosen X so that T has these properties.

Suppose next that G^F is an exceptional group of Lie type. We take a different approach here, along lines indicated by [BPS09].

Definition 4.4. A non-trivial torus T of the group G^F is said to be *sharp* if $C_{G^F}(t) = T$ for every non-identity $t \in T$.

The next result is [BPS09, Theorem 3.1].

Proposition 4.5. *All of the exceptional simple groups of Lie type except for $E_7(q)$ contain sharp maximal tori.*

We shall require the following fact.

Proposition 4.6. *Let $r > 1$. There is a constant C_r such that for any algebraic group G of rank r , for any Frobenius endomorphism F of G , and for any G^F -conjugacy class \mathcal{T} of maximal tori of G^F , we have $N(G^F) \leq C_r |\mathcal{T}|$, where $N(G^F)$ is the number of F -stable maximal tori in G .*

Proof. For notational convenience in what follows, we shall write $f(r)$ as a token for an unspecified function of r , with the understanding that separate instances of the token may refer to distinct functions.

Recall that, by Proposition 2.5, $N(G^F) = q^{|\Phi|}$ where Φ is a set of roots associated with G . If q is bounded above by a function of r , then $N(G^F) < f(r)$ and the proposition holds trivially. We may assume, then, that $q > |\alpha| + 1$ for all roots α associated with G . It now follows, by [Car85, Proposition 3.6.6], that all maximal tori in G^F are non-degenerate, and so $N(G^F)$ is equal to the number of maximal tori in G^F .

Checking the orders of the finite groups of Lie type we note that $|G^F| \leq 2^r q^{|\Phi|+r}$. Since all tori in G^F are non-degenerate, [Car85, Corollary 3.6.5] implies that $|N_{G^F}(T^F) : T^F| < f(r)$. Thus to prove the proposition it is

sufficient to show that there is a function $f(r)$ such that, if T is a maximal torus in G^F , then $|T| \geq f(r)q^r$.

Let $T \in \mathcal{T}$, and let $Y = \text{Hom}(G_m, T)$, the cocharacter group of algebraic homomorphisms $G_m(K) \rightarrow T$. Then F acts on Y in a natural way: for $\alpha \in Y$, $\alpha^F = F \circ \alpha$. Since F is a Frobenius map, there is a map $F_0 : Y \rightarrow Y$ such that $F = [q]F_0$, where $[q]$ is the map induced by the field automorphism $x \mapsto x^q$.

Now [Car85, Proposition 3.3.5] implies that $|T^F| = \det_{Y_0 \otimes \mathbb{R}}([q] - F_0^{-1}w)$ for some w in the Weyl group of T . Since $F_0^{-1}w$ has finite order, its eigenvalues m_1, \dots, m_r on $Y_0 \otimes \mathbb{C}$ are roots of unity. Thus $[q] - F_0^{-1}w$ has eigenvalues $q - m_1, \dots, q - m_r$, and hence

$$|T^F| = (q - m_1)(q - m_2) \cdots (q - m_r).$$

In particular $|T^F|$ is a polynomial of degree q that is a product of cyclotomic polynomials in q . For $\ell \in \mathbb{Z}^+$, the ℓ -th cyclotomic polynomial has degree $\phi(\ell)$ where ϕ is the totient function. It is well-known that $\phi(\ell)$ is bounded below by a strictly increasing function of ℓ (see, for instance, [MV07]), thus there are at most $f(r)$ ways to choose the eigenvalues m_1, \dots, m_r . We conclude that $|T^F| \geq f(r)q^r$ as required. \square

Proof of Theorem 1.2: lower bound. We begin by establishing the lower bound for $\omega_\infty(G^F)$. By Proposition 4.6 it suffices to show that

$$\omega_\infty(G^F) \geq |\mathcal{T}|,$$

where \mathcal{T} is any class of maximal tori.

Suppose that G^F is an exceptional group, other than $E_7(q)$. Then G^F has a class \mathcal{T} of sharp tori. Let T_1 and T_2 be distinct tori in \mathcal{T} . Then clearly $T_1 \cap T_2 = \{1\}$. But now for any non-identity $x_1 \in T_1$ and $x_2 \in T_2$ we see that $\langle x_1, x_2 \rangle$ is centreless, and hence not nilpotent. It follows immediately that $\omega_\infty(G^F) \geq |\mathcal{T}|$.

Now suppose that G^F is of classical type. Let T be the distinguished torus identified in Section 2.4; that is: T is a maximal irreducible cyclic subgroup if G^F admits one; otherwise T has an irreducible cyclic action on a subspace of codimension 1.

Letting d be the dimension of the natural module for G^F , we see that T has a element x of order t , where t is a primitive prime divisor either of

$q^d - 1$ or of $q^{d-1} - 1$. Since $\langle x \rangle$ acts irreducibly on a space of dimension d or $d - 1$, we see that x is regular. By Lemma 2.7 we may suppose that $t > (d - 1) \log_p q$, and so for all but finitely many values of q we see that t is coprime with the order of the Weyl group of G . Now by Proposition 2.6, it is clear that every Sylow t -subgroup of G^F lies inside a conjugate of T .

Let \mathcal{T} be the conjugacy class of T . The order of the intersection of two conjugates of T is not divisible by t , and it follows that distinct Sylow t -subgroups of G^F have trivial intersection. Let \mathcal{C} be a set of elements of G^F containing one element of order t from each torus in \mathcal{T} . Then clearly any two elements of \mathcal{C} generate a non-nilpotent group. Therefore $\omega_\infty(G^F) \geq |\mathcal{T}|$ as required.

It remains only to deal with the groups $E_7(q)$. We use two facts, both established in the proof of Theorem 3.4 in [BPS09]: First, that $E_7(q)$ contains a class \mathcal{T} of maximal tori of order $\frac{1}{\delta}(q^7 - 1)$ where $\delta = \text{hcf}(2, q - 1)$; second, that for $T \in \mathcal{T}$, for t a primitive prime divisor of $q^7 - 1$, and for $g \in T$ an element of order t , the centralizer of g in $E_7(q)$ is T . Now t does not divide $|E_7(q) : T|$, and it follows that the Sylow t -subgroups of $E_7(q)$ intersect trivially, and that each is contained in a unique member of \mathcal{T} . It is clear from these facts that $\omega_\infty(E_7(q)) \geq |\mathcal{T}|$, as required. \square

It is perhaps worth mentioning that for groups G of a fixed Lie type, it may well be possible to establish a rank-independent lower bound for $\omega_\infty(G)$ by a more careful count of maximal tori. We have not attempted to carry out this project, however.

Proof of Theorem 1.2: upper bound. We now turn to the task of proving an upper bound for $\omega_\infty(G^F)$. We revive the convention, used in the proof of Proposition 4.6, of writing $f(r)$ as a token for an unspecified function of r .

Observe first, that all semisimple elements of G^F are contained in a maximal torus of G^F ; these are abelian, and by definition there are at most $N(G^F)$ of them. On the other hand, a result of Steinberg [Car85, Theorem 6.6.1] states that there are $N(G^F)$ unipotent elements in G^F . So taking the set of maximal tori of G^F together with the set of cyclic groups generated by a unipotent element, we have at most $2N(G^F)$ abelian subgroups of G^F , covering all unipotent and semisimple elements of G^F .

We therefore need only to construct a family of nilpotent subgroups of G^F , of size at most $f(r)N(G^F)$, whose members together contain all the elements of G^F that are neither unipotent nor semisimple. Let g be such an element. Write $g = su$ for the Jordan decomposition of g ; so $s \in G^F$ is semisimple, $u \in G^F$ is unipotent, and $[s, u] = 1$. Note that u lies in the centralizer of s ; indeed, by [MT11, Proposition 14.7], we have that u belongs to $C_G(s)^0$, the connected component of $C_G(s)$.

Now [MT11, Corollary 14.7] implies that there are in G at most $f(r)$ G -conjugacy classes of centralizers of semisimple elements. Furthermore [Car78, §1] implies that if $s \in G^F$, then $C_G(s)$ is an F -stable reductive subgroup of maximal rank, and moreover, that the number of G^F -conjugates of such centralizers is at most $f(r)$.

It will therefore be sufficient to prove that if C is an F -stable subgroup of maximal rank, equal to $C_G(s)^0$ for some $s \in G^F$, then the unipotent elements of the G^F -conjugates of C^F can be covered by a set of p -subgroups of size $f(r)N(G^F)$. But the number of subgroups needed is equal to

$$|\{G^F\text{-conjugates of } C^F\}| \cdot |\{\text{Sylow } p\text{-subgroups of } C^F\}| \leq \frac{|G^F|}{|C^F|} \cdot \frac{|C^F|}{|B^F|} = \frac{|G^F|}{|B^F|},$$

where B is a Borel subgroup of C . In particular B contains a maximal torus T of G . To finish the proof we cite the facts, seen already in the proof of Proposition 4.6, that $|T^F| \geq f(r)q^r$, and that $|G^F| \leq f(r)q^{|\Phi|+r}$. The result now follows. \square

5. EXACT RESULTS FOR RANK 1 GROUPS

In this section we calculate the value $\omega_c(G)$ exactly for every $c \in \mathbb{Z}^+ \cup \{\infty\}$, and for every finite group of Lie type G^F , where G is simple, and the (twisted) Lie rank of G^F is 1. Some of these statistics have been calculated previously [Aza11, AAMHZ10]. In every case we establish the exact value of $\omega_c(G)$ by constructing a 2-minimal c -nilpotent covering of G .

The simple groups of (twisted) rank 1 belong to the families $A_1(q)$, ${}^2A_2(q)$, ${}^2B_2(2^{2m+1})$ and ${}^2G_2(3^{2m+1})$. For the family $A_1(q)$, when q is odd, there are two isogeny types to deal with, namely the groups $\text{SL}_2(q)$ and $\text{PGL}_2(q)$. Similarly, the family ${}^2A_2(q)$ comprises the groups $\text{SU}_3(q)$ and $\text{PGU}_3(q)$ which are distinct when $q \equiv -1 \pmod{3}$.

Before proceeding, we require a definition. Let $\mathcal{P} = \{X_1, \dots, X_k\}$ be a set of subgroups of a group G . We say that \mathcal{P} is a *partition* of G if $G = X_1 \cup \dots \cup X_k$, and if $X_i \cap X_j = \{1\}$ for all $1 \leq i < j \leq k$.

We will need the following preliminary proposition, which is a summary of results from [Sch94, §3.5].

Proposition 5.1. *Let G be an almost simple group admitting a partition \mathcal{P} . Then G is one of $\mathrm{PSL}_2(q)$ with $q > 3$; $\mathrm{PGL}_2(q)$ with $q > 3$; or ${}^2B_2(2^{2m+1})$ with $m > 0$. Furthermore each group has a partition of form $\mathcal{P}_T \cup \mathcal{P}_p$, where \mathcal{P}_T is the set of maximal tori in G , and \mathcal{P}_p is the set of Sylow p -subgroups of G .*

Note that, by Proposition 2.5, we have $|\mathcal{P}_T| = q^{|\Phi|} = (|G|_p)^2$.

Several of the arguments in this part of the paper fail for certain small values of q , and in some of these cases we have relied on direct computation. In every such case, we have established the value of $\omega_c(G)$ by finding a 2-minimal c -nilpotent cover for G . These calculations were performed using the Magma computer algebra package [BCP97].

5.1. **$\mathrm{PGL}_2(q)$.** The main result in this section is the following.

Theorem 5.2. *Let $G = \mathrm{PGL}_2(q)$, and let $c \in \mathbb{Z}^+ \cup \{\infty\}$. Then G has a 2-minimal c -nilpotent cover, and*

$$\omega_c(G) = \begin{cases} 4 & \text{if } q = 2, \\ 7 & \text{if } c > 1 \text{ and } q = 3, \\ 10 & \text{if } c = 1 \text{ and } q = 3, \\ q^2 + q + 1 & \text{if } q \geq 4. \end{cases}$$

Before we prove Theorem 5.2 we state a lemma, which will be useful also in the next section when we study the groups $\mathrm{SL}_2(q)$.

Lemma 5.3. *Let X be a nilpotent subgroup of $\mathrm{PGL}_2(q)$. Then one of the following holds.*

- (1) X is cyclic;
- (2) X is a dihedral 2-group.

Proof. It is well known that $\mathrm{PGL}_2(2) \cong S_3$ and that $\mathrm{PGL}_2(3) \cong S_4$; in these cases it is easy to verify the result directly. For $q > 3$ we refer to [Kin05],

which gives a list of the maximal subgroups of $\mathrm{PGL}_2(q)$ and $\mathrm{PSL}_2(q)$; the result can readily be derived from this information. \square

Proof of Theorem 5.2. When $q = 2$ or $q = 3$, the result can be calculated directly. We therefore suppose that $q > 3$. Let $\mathcal{P} = \mathcal{P}_T \cup \mathcal{P}_p$ be the partition given in Proposition 5.1, and observe that

$$|\mathcal{P}| = |\mathcal{P}_T| + |\mathcal{P}_p| = q^2 + q + 1.$$

The subgroups in \mathcal{P} are abelian, and so by Lemma 3.1, we have $\omega_1(G) \leq |\mathcal{P}|$. In each member of \mathcal{P} , we choose any element of order greater than 2 to be a distinguished element; suitable elements exist since $q > 3$, and by construction they are regular.

Let g and h be two distinct distinguished elements. We need to check that $\langle g, h \rangle$ is not nilpotent. It will suffice to check that g and h do not commute, by Lemma 5.3, since both g and h have order greater than 2. But the centralizer of g is precisely that member of \mathcal{P} which contains g , and this subgroup does not contain h . So $\langle g, h \rangle$ is not nilpotent, and we see from Lemma 3.2 that $\omega_\infty(G) \geq |\mathcal{P}|$. Now the result follows, since $\omega_1(G) \geq \omega_\infty(G)$. \square

5.2. $\mathrm{SL}_2(q)$. Consider the situation when $G = \mathrm{SL}_2(q)$. If q is even, then $\mathrm{SL}_2(q) \cong \mathrm{PGL}_2(q)$ and the result of the previous section applies. Thus in this section we assume that q is odd. Our results generalize the main result of [Aza11].

Theorem 5.4. *Let q be odd, and let $G = \mathrm{SL}_2(q)$. For every $c \in \mathbb{Z}^+ \cup \{\infty\}$, the group G has a 2-minimal c -nilpotent cover, and*

$$\omega_c(G) = \begin{cases} 5 & \text{if } q = 3, \\ 21 & \text{if } q = 5, \\ q^2 + q + 1 & \text{if } q > 5. \end{cases}$$

Proof. If $q = 3$ or 5 we calculate the value of $\omega_c(G)$ directly. Assume that $q > 5$ and let $H = G/Z(G) \cong \mathrm{PSL}_2(q)$. Let $\mathcal{P} = \mathcal{P}_T \cup \mathcal{P}_p$ be the partition of H given in Proposition 5.1; we observe that

$$|\mathcal{P}| = |\mathcal{P}_T| + |\mathcal{P}_p| = q^2 + q + 1.$$

For each member, \hat{N} , of \mathcal{P} , define N to be the unique subgroup of G that contains $Z(G)$ and satisfies $N/Z(G) = \hat{N}$. Let \mathcal{N} be the set of all such N

and observe that $|\mathcal{N}| = |\mathcal{P}| = q^2 + q + 1$, that \mathcal{N} covers G , that all members of \mathcal{N} are abelian, and that the intersection of any two members of \mathcal{N} equals $Z(G)$.

Now in every member N of \mathcal{N} we choose an element g such that $gZ(G) \in H$ has order greater than 2. From here the proof is identical to that of Theorem 5.2. \square

5.3. **SU₃(q).** Our main result in this section is the following.

Theorem 5.5. *Let $G = \text{SU}_3(q)$. For every $c \in \mathbb{Z}^+ \cup \{\infty\}$, the group G has a 2-minimal c -nilpotent cover, and*

$$\omega_c(G) = \begin{cases} 31 & \text{if } c \geq 2 \text{ and } q = 2, \\ 10 & \text{if } c = 1 \text{ and } q = 2, \\ 757 & \text{if } c \geq 2 \text{ and } q = 3, \\ q^6 + q^5 + q^3 + q^2 + 1 & \text{if } c \geq 2 \text{ and } q > 3, \\ q^6 + q^5 + q^4 + q^3 + q^2 + q + 1 & \text{if } c = 1 \text{ and } q > 2. \end{cases}$$

Proof. When $q \leq 3$ the number $w_c(G)$ can be computed directly. We assume therefore that $q > 3$ and we construct a 2-minimal c -nilpotent cover of G . Let U be a Sylow p -subgroup of G where $q = p^a$ for some positive integer a . If $g \in Z(U)$, then $C_G(g) = U \rtimes C_{q+1}$. Let h be an element of $C_G(g)$ of order $q+1$; one can check that $L := C_G(h) \cong \text{GU}_2(q) \cong \text{SL}_2(q) \rtimes C_{q+1}$. Now g lies in a unique Sylow p -subgroup of L and so we conclude that $M := C_L(h)$ is the only maximal nilpotent subgroup containing gh (in fact M is abelian). Since $Z(U)$ is a characteristic subgroup of M , we conclude that $N_G(M) \leq N_G(Z(U)) = U \rtimes C_{q^2-1}$. It is easy to check that $N_G(M) = Z(G) \rtimes C_{q^2-1}$ and so

$$|M^G| = q^2(q^3 + 1).$$

If $g \in U \setminus Z(U)$ then $C_G(g) = U_0 \times Z(G)$, where U_0 is an abelian subgroup of U of order q^2 . We conclude that any nilpotent subgroup containing g must equal $Q \times Z(G)$, where Q is a p -group. Indeed it is clear that g lies in a unique Sylow p -subgroup of G , namely U . Observe that U has nilpotency class 2 and, since $N_G(U) = U \rtimes C_{q^2-1}$, we see that

$$|U^G| = q^3 + 1.$$

Similarly it is clear that $C_G(g') = U_0$ for all elements g' of $U_0 \setminus Z(U)$. It follows, in particular, that U_0 is the only maximal abelian subgroup of G containing these elements. Since $N_G(U_0) = U \rtimes C_{q-1}$, we conclude that

$$|U_0^G| = q^4 + q^3 + q + 1.$$

Suppose next that g is a mixed element of G (i.e. one that is neither unipotent nor semisimple). Then $g = su$, where s is semisimple and u is unipotent, and where $[s, u] = 1$. By the above remarks, we obtain that $u \in Z(U)$ and that $s \in C_G(u)$. It follows that g lies in some conjugate of M .

We are left with the task of finding a set of c -nilpotent subgroups of G containing all semisimple elements of G . Every semisimple element of G lies in a maximal torus of G ; each of these tori is isomorphic to one of C_{q^2-1} , $C_{q+1} \times C_{q+1}$ or C_{q^2-q+1} . Since $q > 2$, Zsigmondy's Theorem implies that there exists a primitive prime divisor t of $q^6 - 1$. Immediately from its definition, we see that t divides $q^2 - q + 1$, and is coprime to $q^2 - 1$. This implies, in particular, that a Sylow t -subgroup of G is a cyclic subgroup of a maximal torus isomorphic to C_{q^2-q+1} . We call this torus T_2 . Moreover, if g is a generator of the Sylow t -subgroup, then g is regular, i.e. $C_G(g) = T_2$; so any nilpotent subgroup of G containing g must be a subgroup of T_2 .

Next, let T_0 be a split maximal torus, i.e. one isomorphic to C_{q^2-1} . Suppose, first, that $q - 1$ is divisible by an odd prime t . Then t does not divide $|G|/|T_0|$ and so T_0 contains a Sylow t -subgroup of G . Moreover, any element g of order t must be regular, i.e. $C_G(g) = T_0$. This implies, in particular, that any nilpotent subgroup of G containing g is a subgroup of T_0 . Suppose, on the other hand, that $q - 1$ is not divisible by an odd prime; so $q - 1 = 2^a$ for some positive integer a . Then $N_G(T_0)$ is a nilpotent group containing a Sylow 2-subgroup S of G . Now S is dihedral; and if $g \in S$ is an element of order greater than 2, then g lies in T and g is regular, i.e. $C_G(g) = T$. Note that, since $q > 3$, such an element g exists. This implies, in particular, that any nilpotent subgroup of G containing g is a subgroup of $ST_0 = N_G(T_0)$, and so $\langle g, g' \rangle$ is not nilpotent for any conjugate g' of g which is not contained in T_0 .

Now let T_1 be a maximal torus congruent to $C_{q+1} \times C_{q+1}$. Suppose that $q + 1$ is divisible by a prime $t > 3$. Then, just as in the previous case, it follows that T_1 contains a Sylow t -subgroup of G , and a regular element g

of order t , such that any nilpotent subgroup of G containing g is a subgroup of T_1 .

Suppose that $q + 1$ is divisible by 9. Then T_1 contains a regular element g of order 9 and $C_G(g) = T_1$. On the other hand observe that if S is a Sylow 3-subgroup of G , then S is a subgroup of the normalizer of some conjugate of T_1 , and all elements of order 9 lie in that conjugate. In this case we conclude that any nilpotent subgroup of G that contains g lies in $N_G(T_1)$. In particular if g' is any conjugate of g that does not lie in T_1 then $\langle g, g' \rangle$ is not nilpotent.

Suppose next that $q + 1$ is divisible by 12. Then T_1 contains a regular element g of order 12 such that both g^3 and g^4 are regular, i.e. $C_G(g) = C_G(g^3) = C_G(g^4) = T_1$. This implies, in particular that if N is a nilpotent subgroup of G containing g , then all odd order subgroups of N lie in T_1 , and all subgroups of order coprime to 3 lie in T_1 ; in particular N itself is a subgroup of T_1 .

Assume next that $q + 1 = 6$. One can verify computationally that if g is a regular element of order 6 in T_1 , and if g' is any G -conjugate of g , then $\langle g, g' \rangle$ is nilpotent if and only if $g' \in T_1$. Thus any nilpotent subgroup of G that contains g is a subgroup of T_1 .

Assume, finally, that $q + 1 = 2^a$ for some a , and let g be a regular element in T_1 ; then $C_G(g) = T_1$ is a 2-group, and so any nilpotent subgroup of G that contains g must lie in some Sylow 2-subgroup S of G . By comparing orders, we see that S contains a conjugate of T_1 as a subgroup of index 2; without loss of generality, we assume that S contains T_1 . Let $h \in S \setminus T_1$ and let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of h . An easy calculation implies that, up to relabelling, $\lambda_2 = -\lambda_1$. Let ζ be a generator of the cyclic subgroup of \mathbb{F}_{q^2} of order $q + 1$. We may take g to be an element with eigenvalues $\zeta, \zeta^2, \zeta^{q-2}$. Observe that $\det g = 1$, that the eigenvalues are distinct (so g is regular), and that none of these eigenvalues is equal to -1 times any of the others. This implies that if S' is any Sylow 2-subgroup of G containing g , then g lies in the unique maximal torus of G contained as an index 2 subgroup in S' . Since g therefore lies in a unique maximal torus of G , namely T_1 , we conclude that any nilpotent subgroup of G that contains g must be a subgroup of one of the three Sylow 2-subgroups of G that normalize T_1 .

Our calculations now yield a 2-minimal c -nilpotent cover, \mathcal{N} , of G . If $c \geq 2$, set

$$\mathcal{N} = U^G \cup M^G \cup T_0^G \cup T_1^G \cup T_2^G.$$

If $c = 1$, set

$$\mathcal{N} = U_0^G \cup M^G \cup T_0^G \cup T_1^G \cup T_2^G.$$

We have already observed that $U_0^G \cup M^G$ (and hence $U^G \cup M^G$) contains every non-semisimple element of G , while it is well-known that the set of all maximal tori, $T_0^G \cup T_1^G \cup T_2^G$, between them contain all semisimple elements of G . Thus \mathcal{N} is, in each case, a c -nilpotent cover of G . On the other hand our calculations imply that each member N of \mathcal{N} contains an element g such that any c -nilpotent subgroup containing g must lie in N (or, in a couple of exceptional cases, in a small overgroup of N). Furthermore no member of \mathcal{N} is a subgroup of any other member of \mathcal{N} (or, in the exceptional cases, a subgroup of the relevant overgroups of any other member of \mathcal{N}). Hence \mathcal{N} is 2-minimal, as required.

In order to calculate the size of \mathcal{N} recall that $|T_0^G \cup T_1^G \cup T_2^G| = q^6$ by Proposition 2.5. The result follows. \square

5.4. $\text{PGU}_3(\mathbf{q})$. In this section we assume that $q \equiv -1 \pmod{3}$, since otherwise $\text{PGU}_3(q) \cong \text{PSU}_3(q)$ and the results of the previous section, combined with Lemma 3.6, yield the value of $\omega_c(G)$.

Theorem 5.6. *Let $G = \text{PGU}_3(q)$ with $q \equiv -1 \pmod{3}$. For every $c \in \mathbb{Z}^+ \cup \{\infty\}$, the group G has a 2-minimal c -nilpotent cover, and*

$$\omega_c(G) = \begin{cases} 49 & \text{if } c \geq 2 \text{ and } q = 2, \\ 71 & \text{if } c = 1 \text{ and } q = 2, \\ q^6 + q^5 + q^3 + q^2 + 1 & \text{if } c \geq 2 \text{ and } q > 2, \\ q^6 + q^5 + q^4 + q^3 + q^2 + q + 1 & \text{if } c = 1 \text{ and } q > 2. \end{cases}$$

Proof. For $q = 2$ we compute the value of $\omega_c(G)$ directly. For $q > 2$ our argument is virtually identical to that of the previous subsection, and so we give only a brief summary here. We define a family \mathcal{N} of subgroups of G as follows. If $c \geq 2$, set

$$\mathcal{N} = (\mathcal{U})^G \cup M_0^G \cup \mathcal{T}.$$

If $c = 1$, set

$$\mathcal{N} = (\mathcal{U}_0)^G \cup M_0^G \cup \mathcal{T}.$$

Here \mathcal{T} is the set of maximal tori of G ; these come in three conjugacy classes, just as in $\mathrm{SU}_3(q)$. The group \hat{U} is a Sylow p -subgroup of G while \hat{U}_0 is the subgroup of $\mathrm{PSU}(3, q)$ equal to the projection of the group U_0 given in $\mathrm{SU}_3(q)$.

Finally one constructs M_0 by taking $g \in Z(\hat{U})$. Then $C_G(g) = U \rtimes C_{q+1}$. Let C be a cyclic subgroup of $C_G(g)$ of order $q + 1$. Then define $M_0 := Z(\hat{U}) \times C$. To show now that \mathcal{N} is a 2-minimal c -nilpotent cover of G , one follows the line of argument of the previous section. \square

5.5. ${}^2\mathbf{B}_2(2^{2m+1})$. In this section we deal with the Suzuki groups. Our main result generalizes [AAMHZ10, Theorem 1.2] which gives a value for $\omega_1(G)$.

It will be convenient to redefine the variable q , which hitherto has been defined as the level of the Frobenius endomorphism F . For the group ${}^2\mathbf{B}_2(2^{2m+1})$ this would give the fractional power $q = 2^{m+\frac{1}{2}}$. For a clearer exposition in this section, we shall take q to be the square of that value, i.e. $q = 2^{2m+1}$.

Theorem 5.7. *Let $q = 2^{2m+1}$ for $m \geq 0$, and let G be the Suzuki group ${}^2\mathbf{B}_2(q)$. Then, for all $c \in \mathbb{Z}^+ \cup \{\infty\}$, the group G has a 2-minimal c -nilpotent cover. Furthermore, if $m \geq 1$, then*

$$\omega_c(G) = \begin{cases} q^4 + q^2 + 1 & \text{if } c \geq 2, \\ q^4 + q^3 - q^2 + q - 1 & \text{if } c = 1. \end{cases}$$

If $m = 0$, then $\omega_c(G) = 6$ for all $c \in \mathbb{Z}^+ \cup \{\infty\}$.

Proof. If $m = 0$, then ${}^2\mathbf{B}_2(q) \cong C_5 \rtimes C_4$ and the result can be computed directly. Assume $m > 0$ and let $\mathcal{P} = \mathcal{P}_T \cup \mathcal{P}_p$ be the partition given in Prop. 5.1. The value of $|\mathcal{P}_T|$ is given by Proposition 2.5 as q^4 (taking into account the modified definition of q currently in force). The value of $|\mathcal{P}_p|$ is given by [Suz62, Theorem 9] as $q^2 + 1$. Thus

$$|\mathcal{P}| = |\mathcal{P}_T| + |\mathcal{P}_p| = q^4 + q^2 + 1.$$

The members of \mathcal{P}_T are abelian, and the members of \mathcal{P}_p are nilpotent of class 2. So we have $\omega_2(G) \leq |\mathcal{P}|$, by Lemma 3.1.

Every member of \mathcal{P} is a maximal nilpotent subgroup of G . For each $g \in H \setminus \{1\}$, the only maximal nilpotent subgroup containing g is H itself

[Suz82, Theorem 9]. It follows from Lemma 3.2 that $\omega_\infty(G) \geq |\mathcal{P}|$, and this is enough to establish the first part of the theorem.

We are left with the task of calculating $\omega_1(G)$. Let P be a Sylow p -subgroup of G , and let $g \in \mathcal{P} \setminus Z(P)$. Then the centralizer of g in G is $\langle g, Z(P) \rangle$. If we choose g_1, \dots, g_{q-1} to be representatives from the $q-1$ non-trivial cosets of $Z(P)$ in P , then we see that $[g_i, g_j] \neq 1$ whenever $1 \leq i < j \leq q-1$. Define $H_i = \langle g_i, Z(P) \rangle$ for $i = 1, \dots, q-1$; then the subgroups H_i form a 2-minimal abelian cover of P , with the elements g_i as distinguished elements. We may repeat this process for every member of \mathcal{P}_p . The resulting abelian subgroups, together with those of \mathcal{P}_T , form a 2-minimal abelian cover of G . The size of this cover is given by

$$|\mathcal{P}_T| + (q-1)|\mathcal{P}_p| = q^4 + (q-1)(q^2 + 1),$$

and the result follows. \square

5.6. ${}^2\mathbf{G}_2(3^{2m+1})$. The final family of groups to deal with is that of the Ree groups of type ${}^2G_2(3^{2m+1})$. As with the Suzuki groups, it is convenient here to redefine q to be the square of its value in earlier sections. Thus, in what follows, we set $q = 3^{2m+1}$.

We will make extensive use of the structural information about the group ${}^2G_2(q)$ given by the main theorem of [War66]; and we rely on the classification of the maximal subgroups of ${}^2G_2(q)$ provided by [Kle88, Theorem C]. Facts stated without proof in this section are taken from the statements of these two theorems.

We also make use of the following parametrization of a Sylow 3-subgroup P of ${}^2G_2(q)$, which can be found in [CCN⁺85]. We write elements of P as triples $(x, y, z) \in \mathbb{F}_q^3$ with the multiplication given by $(x_1, y_1, z_1) \cdot (x_2, y_2, z_2)$ equal to

$$(5.1) \quad (x_1 + x_2, y_1 + y_2 + x_1x_2^s - x_1^s x_2, z_1 + z_2 + y_1x_2 + x_1^s x_2^2 + x_1x_2^{s+1} - x_1^2 x_2^s),$$

where $s = 3^{m+1}$. Under this parametrization it is not hard to see that

$$\begin{aligned} Z(P) &= \{(x, y, z) \mid x = y = 0\}, \\ Z_2(P) &= \{(x, y, z) \mid x = 0\}, \end{aligned}$$

where $Z_2(P)$ is the second term of the upper central series for P . Both of these groups are elementary abelian, with orders q and q^2 respectively.

Theorem 5.8. *Let $q = 3^{2m+1}$ for $m \geq 0$, and let G be the Ree group ${}^2G_2(q)$. Then, for all $c \in \mathbb{Z}^+ \cup \{\infty\}$, the group G has a 2-minimal c -nilpotent cover.*

Furthermore

$$\omega_c(G) = \begin{cases} 316 & \text{if } c \geq 2 \text{ and } q = 3, \\ 372 & \text{if } c = 1 \text{ and } q = 3, \\ q^6 + q^5 + q^3 + q^2 + 1 & \text{if } c \geq 3 \text{ and } q > 3, \\ q^6 + q^5 + \frac{1}{2}q^4 - \frac{1}{2}q^3 + q^2 + \frac{1}{2}q - \frac{1}{2} & \text{if } c = 2 \text{ and } q > 3, \\ q^6 + \frac{3}{2}q^5 - \frac{1}{2}q^4 + \frac{3}{2}q^2 - \frac{1}{2}q & \text{if } c = 1 \text{ and } q > 3. \end{cases}$$

Proof. In the case that $m = 0$, and hence $q = 3$, we have the isomorphism ${}^2G_2(q) \cong \text{PSL}_2(8).3$, and the result can be computed directly. Assume that $m > 0$ and observe that the set of prime divisors of $|G|$ can be partitioned naturally into six sets:

- (1) the two singleton sets $\{2\}$ and $\{3\}$,
- (2) the two sets \mathcal{R}^\pm of prime divisors of $q \pm \sqrt{3q} + 1$.
- (3) the set \mathcal{S} of prime divisors of $\frac{q-1}{2}$, and
- (4) the set \mathcal{T} of prime divisors $\frac{q+1}{4}$,

We say that an element g of G belongs to one of the sets \mathcal{R}^\pm , \mathcal{S} or \mathcal{T} if all of the prime divisors of the order of g belong to that set. If the element g belongs to any of these sets, then there is a unique maximal nilpotent subgroup $M(g)$ containing g , which is equal to the centralizer in G of g .

- (1) If g belongs to \mathcal{R}^\pm then $M(g)$ is cyclic of order $q \pm \sqrt{3q} + 1$. We write C_\pm for the subgroup $M(g)$ (defined up to conjugacy).
- (2) If g belongs to \mathcal{S} then $M(g)$ is cyclic of order $q - 1$. We write D for the subgroup $M(g)$ in this case.
- (3) If g belongs to \mathcal{T} then $M(g)$ is isomorphic to $C_{\frac{q+1}{2}} \times C_2$. We write E for the subgroup $M(g)$.

Define \mathcal{N}_T to be the set of all conjugates in G of C_\pm , D or E . So \mathcal{N}_T is a set of abelian subgroups of G , and every element of G whose order is divisible by a prime greater than 3 lies in a member of \mathcal{N}_T . Furthermore if g and h are elements of maximal order in distinct members of \mathcal{N}_T , then $\langle g, h \rangle$ is clearly not nilpotent.

It remains to deal with elements of G whose order is not divisible by a prime greater than 3; these have order 2, 3, 6 or 9. Let P be a Sylow 3-subgroup of G and let g_P be an element in P of the maximal order 9.

Since $P \cap Q = \{1\}$ for distinct Sylow 3-subgroups of G , and since P contains the centralizer of g , we see that P is the only maximal nilpotent subgroup containing g .

Note next that there is a unique conjugacy class of involutions in G ; it follows easily from the centralizer structure that we have outlined, by straightforward Sylow arguments, that $\mathcal{N}_T \cup P^G$ contains all elements of G of order other than 6.

Now if $g \in G$ has order 6, then the only maximal nilpotent subgroup containing g is the centralizer in G of g , which we shall call F . The subgroup F is isomorphic to $C_2 \times A(q)$, where $A(q)$ is an elementary abelian group of order q . We see that $\mathcal{N} = \mathcal{N}_T \cup P^G \cup F^G$ covers every element of G .

For every element N of \mathcal{N} , we have seen that N has an element g_N such that N is the only maximal nilpotent subgroup containing g_N . It follows that \mathcal{N} is a 2-minimal nilpotent cover of G . We note, furthermore, that every member of \mathcal{N} is abelian, apart from the Sylow 3-subgroups of G . So \mathcal{N} is a 2-minimal 3-nilpotent cover of G . Now Proposition 1.1 implies that for $c \geq 3$, we have

$$\omega_c(G) = |\mathcal{N}| = |\mathcal{N}_T| + |P^G| + |F^G|.$$

The set \mathcal{N}_T is in 1–1 correspondence with the maximal tori of G and thus, by Proposition 2.5, it has size q^6 . Thus

$$\begin{aligned} \omega_C(G) &= q^6 + |P^G| + |F^G| \\ &= q^6 + \frac{|G|}{q^3(q-1)} + \frac{|G|}{q(q-1)} \\ &= q^6 + q^5 + q^3 + q^2 + 1. \end{aligned}$$

We now consider the case $c = 2$. We construct a 2-minimal 2-nilpotent cover \mathcal{N}_2 as follows. From the set \mathcal{N} constructed above, we retain the sets \mathcal{N}_T and F^G , since the subgroups in these sets are abelian. Let $g \in P \setminus Z_2(P)$ be an element of order 9, and define

$$P_g = \langle g, Z_2(P) \rangle.$$

Using the parametrization of P described above, we write $g = (x_0, y_0, z_0)$. We see that

$$P_g = \{(x, y, z) \mid x \in \{0, x_0, 2x_0\}, y, z \in \mathbb{F}_q\}.$$

Now clearly $Z(P_g) = Z(P)$, and $P_g/Z(P_g)$ is abelian. We conclude that P_g is a non-abelian group of order $3q^2$ and nilpotency class 2.

For $i = 1, \dots, \frac{q-1}{2}$, let $g_i = (x_i, y_i, z_i)$ be elements of P , chosen so that $x_1, \dots, x_{\frac{q-1}{2}} \in \mathbb{F}_q$ are pairwise linearly independent as vectors over \mathbb{F}_3 ; i.e. the set of subspaces

$$\left\{ \langle x_i \rangle \mid i = 1 \dots, \frac{q-1}{2} \right\}$$

contains every 1-subspace of \mathbb{F}_q . It is easy to see that the corresponding subgroups P_{g_i} cover the Sylow 3-group P ; there are $\frac{q-1}{2}$ of these subgroups.

We claim that the groups P_{g_i} are a 2-minimal 2-nilpotent cover of P . To see this we must show that for distinct i and j , the group $P_{i,j} := \langle g_i, g_j \rangle$ has nilpotency class 3. Using (5.1), one can check that $Z(P_{i,j}) = P_{i,j} \cap Z(P)$; and since $P_{i,j}/Z(P_{i,j})$ is clearly non-abelian, we conclude that $P_{i,j}$ has nilpotency class 3, as required. The elements g_i will be our distinguished elements.

The construction just described may be repeated inside any Sylow 3-subgroup of G , to obtain in each a set of $\frac{q-1}{2}$ nilpotent subgroups of class 2. Let $\mathcal{N}_{2,p}$ be the collection of all these subgroups. We recall that $P \cap P^g = \{1\}$ whenever P and P^g are distinct, and so any pair of distinguished elements from distinct members of $\mathcal{N}_{2,p}$ generate either a non-nilpotent group (if they belong to distinct Sylow 3-subgroups), or a group of nilpotency class 3 (otherwise).

We now define $\mathcal{N}_2 = \mathcal{N}_T \cup F^G \cup \mathcal{N}_{2,p}$, and observe that the members of $\mathcal{N}_{2,p}$ cover G . We have seen that each member N of $\mathcal{N}_T \cup F^G$ contains an element g_N which lie in no other maximal nilpotent subgroup of G . On the other hand, for each member N of $\mathcal{N}_{2,p}$ there is an element $g_N \in N$, such that the only maximal nilpotent subgroup containing g_N is a Sylow 3-subgroup of G ; furthermore, if $N_1, N_2 \in \mathcal{N}_{2,p}$ are distinct, then the elements g_{N_1} and g_{N_2} generate either a non-nilpotent group or a group of nilpotency class 3. It follows that \mathcal{N}_2 is a 2-minimal 2-nilpotent cover of G .

We now calculate that

$$\begin{aligned} |\mathcal{N}_2| &= |\mathcal{N}_T| + |F^G| + |\mathcal{N}_{2,p}| \\ &= |\mathcal{N}_T| + |F^G| + \frac{1}{2}(q-1)|P^G| \\ &= q^6 + q^5 + q^2 + \frac{1}{2}(q-1)(q^3+1) \end{aligned}$$

and the claimed result for $c = 2$ follows.

It remains to deal with the case $c = 1$. We construct a 2-minimal 1-nilpotent (abelian) cover, and as a consequence obtain a maximal non-commuting set in G . As we did for $c = 2$, we retain the sets \mathcal{N}_T and F^G of abelian subgroups.

Let $h = (x_0, y_0, z_0) \in P \setminus Z_2(P)$ be an element of order 9. Define $Q_h = \langle h, Z(P) \rangle$, and observe that

$$Q_h = \{(x, y, z) \mid (x, y) \in \{(0, 0), (x_0, y_0), (2x_0, 2y_0)\}, z \in \mathbb{F}_q\}.$$

Clearly Q_h is abelian of order $3q$.

For $i = 1, \dots, \frac{q(q-1)}{2}$, let $h_i = (x_i, y_i, z_i)$ be elements of $P \setminus Z_2(P)$, chosen so that $(x_1, y_1), \dots, (x_{\frac{q-1}{2}}, y_{\frac{q-1}{2}}) \in \mathbb{F}_q \times \mathbb{F}_q$ are pairwise linearly independent as vectors over \mathbb{F}_3 . So the set

$$\left\{ \langle (x_i, y_i) \rangle \mid i = 1 \dots, \frac{q(q-1)}{2} \right\}$$

contains all of those 1-spaces in $\mathbb{F}_q \times \mathbb{F}_q$ with non-zero first coordinate. It is easy to see that the set of corresponding subgroups Q_{h_i} covers all elements of $P \setminus Z_2(P)$, as well as all elements of $Z(P)$. There are $\frac{q(q-1)}{2}$ subgroups in this set. Using (5.1), it is clear that for distinct i and j , we have $[h_i, h_j] \neq 1$, and so $\langle h_i, h_j \rangle$ is non-abelian.

As in the case $c = 2$, we can repeat the construction inside every Sylow 3-subgroup of G , obtaining in each case a collection of $\frac{q-1}{2}$ abelian subgroups. Let $\mathcal{N}_{1,p}$ be the set of all the subgroups so obtained, and let

$$\mathcal{N}_1 = \mathcal{N}_T \cup F^G \cup \mathcal{N}_{1,p}.$$

We have seen that $\mathcal{N}_T \cup F^G$ contains all elements of G whose order is not a power of 3. Now $\mathcal{N}_{1,p}$ contains all 3-elements, apart from those conjugate to an element of $Z_2(P) \setminus Z(P)$. But such an element is contained in a member of F^G , and so we conclude that \mathcal{N}_1 covers G .

We must now establish the 2-minimality of \mathcal{N}_1 . We have seen how to find distinguished elements of the subgroups in $\mathcal{N}_T \cup F^G$. Every member N of $\mathcal{N}_{1,p}$ contains an element g_N , such that the only maximal nilpotent subgroup containing g_N is a Sylow 3-subgroup of G ; furthermore, any two such elements generate a non-abelian group. It follows that \mathcal{N}_1 is a 2-minimal abelian cover of G .

Finally, we observe that

$$\begin{aligned} |\mathcal{N}_1| &= |\mathcal{N}_T| + |F^G| + |\mathcal{N}_{1,p}| \\ &= |\mathcal{N}_T| + |F^G| + \frac{1}{2}q(q-1)|P^G| \\ &= q^6 + q^5 + q^2 + \frac{1}{2}q(q-1)(q^3+1) \end{aligned}$$

and the result follows. \square

6. QUESTIONS AND CONJECTURES

The main results of this paper suggest a number of interesting questions which we discuss below. We suppose throughout that G is a finite group, although some of the questions we discuss also apply to infinite groups.

6.1. Questions about exact formulae. Let G be a simple algebraic group, and G^F a finite group of Lie type with (twisted) Lie rank equal to 1. This is the situation treated in §5 above, and our results from that section, collectively, have a number of suggestive properties.

- Provided that $q > 5$, the isogeny class of G does not affect the value of $\omega_c(G^F)$.
- Provided that $q > 5$, the value of $\omega_c(G)$ is a polynomial in q . Furthermore, unless $c \leq 2$ and $G^F = {}^2G_2(3^{2m+1})$, all of the coefficients in this polynomial are equal to 1 or -1 .
- Provided that $q > 3$, the values of $\omega_\infty(\mathrm{SU}_3(q))$ and $\omega_\infty({}^2G_2(q))$ coincide.

We are naturally interested in how far these phenomena generalize. For instance, let us continue to suppose that G^F is a finite group of Lie type, where G is simple, but let us drop the assumption that G has rank 1. Then we have the following questions.

Question 6.1. *Is $\omega_c(G^F)$ independent of the isogeny class of G for large enough q ?*

Question 6.2. *Is $\omega_c(G^F)$ a polynomial in q for large enough q ? If so, are the coefficients of $\omega_\infty(G^F)$ always equal to 1 or -1 ?*

Question 6.3. *Can we classify (and explain) any coincidences in the value of $\omega_c(G^F)$ for different families of groups G^F ?*

6.2. Intersection with conjugacy classes. In this paper we have focused on the problem of calculating the size of a maximal non-nilpotent set. We are also interested in the possible structure of such a set and an obvious first approach is to study the interaction of non-nilpotent sets with conjugacy classes. With this in mind, then, we define a non-trivial conjugacy class which is a non-nilpotent set to be a *non-nilpotent class*. We pose the following question:

Question 6.4. *Suppose that G contains a non-nilpotent class. Can we describe the structure of G ?*

A natural first question concerning the structure of such a group G would be to ask if it can be simple: we conjecture below that this is impossible. Some evidence for this conjecture can be found in the following result. Recall that $O_p(G)$ is the largest normal p -group in G .

Lemma 6.5. *Suppose that G contains a non-nilpotent class of elements of prime order. Then G is not simple.*

Proof. Let C be a non-nilpotent class of elements of prime order p in G and let $x, y \in C$. The group $\langle x, y \rangle$ will be nilpotent precisely if it is a p -group. Thus the class C will be non-nilpotent if and only if no two elements of C lie in the same Sylow p -subgroup of G . But in this case the elements of C are *isolated* in the Sylow p -subgroup in which they lie. Now the Z_p^* theorem (due to Glauberman [Gla66] when $p = 2$, and to Guralnick and Robinson [GR93] for odd p) implies that $C \subset O_p(G)$, and hence that the group is not simple. \square

So much for negative information about groups containing non-nilpotent conjugacy classes. On the other hand there are certainly many groups containing such a class. (In what follows, for x an element of a group H , we write x^H for the conjugacy class of H containing x .)

Lemma 6.6. *Suppose that G is a Frobenius group with complement H . Let $x \in H$ and suppose that $|x^H| = 1$. Then x^G is a non-nilpotent set in G .*

Note that, in particular, a Frobenius group with abelian complement contains a conjugacy class which is a non-nilpotent set. A relevant example

is $\mathrm{PSL}_2(3) \cong A_4$ in which both conjugacy classes of elements of order 3 are non-nilpotent classes.

Proof. Write K for the Frobenius kernel of G and note that $(|K|, |H|) = 1$. Let y be a conjugate of x . If $\langle x, y \rangle \cap K = \{1\}$, then x and y lie in the same complement which is a contradiction. Hence $\langle x, y \rangle \cap K \neq \{1\}$. But $C_G(x) \cap K = \{1\}$ and $(o(x), K) = 1$, thus $\langle x, y \rangle$ is not nilpotent. \square

Rather than restricting our attention to non-nilpotent classes, one can ask the more general question of how conjugacy classes of a group interact with non-nilpotent sets. In this respect, a classical theorem of Baer and Suzuki is relevant (cf. [Hup67, p. 298], [Suz62]).

Theorem 6.7. *Let C be a conjugacy class in G such that for all $x, y \in G$, $\langle x, y \rangle$ is nilpotent. Then $C \subset F(G)$, the Fitting group of G .*

In particular this result implies that, in a simple group, every conjugacy class contains a non-nilpotent set of size at least 2. Other results of this ilk—focussed on the property of solvability rather than nilpotency—can be found in [DGHP12]. It is unclear, however how much the value 2 can be increased.

Question 6.8. *Let G be a simple group. Can one state a minimum bound for the quantity*

$$(6.1) \quad \min \{ \omega_\infty(C) \mid C \text{ is a non-trivial conjugacy class in } G \}?$$

One might also ask about upper bounds for the quantity (6.1) for a simple group G . In this regard we posit the following conjecture, the truth of which would imply, in particular, that a simple group cannot contain a non-nilpotent class.

Conjecture 6.9. *Let G be a simple group. Then*

$$(6.2) \quad \max \left\{ \frac{\omega_\infty(C)}{|C|} \mid C \text{ is a non-trivial conjugacy class in } G \right\} \leq \frac{1}{2}.$$

It is clear that the bound $\frac{1}{2}$ cannot be improved: in the group A_5 , the class of elements of order 3, and the two classes of elements of order 5, all contain non-nilpotent sets of half their size. The same is true in $\mathrm{PSL}_2(7)$ of the conjugacy class of elements of order 3.

We remark, finally, that one might ask similar questions for *non-commuting classes*. (We say that a conjugacy class C in a group G is non-commuting if, for all distinct g, h in C , the group $\langle g, h \rangle$ is non-abelian.) In particular it seems likely that the previous conjecture remains true in this more general context, i.e. with ω_∞ replaced in the statement by ω_1 .

6.3. Rank and nilpotency class. Let N be a nilpotent subgroup of G , a finite group of Lie type. The connection between the nilpotency class of N and the rank of G appears to be slightly subtle.

We note, first of all, that the nilpotency class of N cannot, in general, be bounded above by a function of the rank of G . One illustrative example is the case that N is a dihedral 2-subgroup of $G = \mathrm{PSL}_2(p)$, of the largest possible order 2^k . The value of k is unbounded as p varies across the primes, and since N has nilpotency class $k-1$, the non-existence of a bound in terms of rank of G is demonstrated.

On the other hand let ℓ be the minimum number such that $\omega_i(G) = \omega_\ell(G)$ for all $i \geq \ell$. Since G is finite such a number ℓ exists; but in light of the remarks of the previous paragraph, there is no *a priori* reason why ℓ should be bounded above by any function of the rank. It is nevertheless tempting to conjecture that such a bound exists.

Conjecture 6.10. *For all $r \in \mathbb{Z}^+$, there exists $\ell \in \mathbb{Z}^+$ such that if G is a finite group of Lie type of rank r , then $\omega_i(G) = \omega_\ell(G)$ for all $i \geq \ell$.*

Our results for rank 1 groups provide some evidence for the plausibility of this conjecture. Note, in particular, that the presence of nilpotent subgroups of unbounded rank does not prevent the conjecture being true for the group $\mathrm{PSL}_2(q)$.

In another direction one might ask whether the example of the dihedral 2-subgroups of $\mathrm{PSL}_2(q)$ is in some respects atypical. In particular, one could pose the following question.

Question 6.11. *Is it the case that for all $r \in \mathbb{Z}^+$, there exists $\ell \in \mathbb{Z}^+$, such that if G is a finite group of Lie type of rank r , and if N is a nilpotent subgroup of G of odd order, then the nilpotency class of N is less than ℓ ?*

An obvious way of addressing this question would be to undertake a detailed study of the maximal nilpotent subgroups of finite groups of Lie type.

Interesting results in this direction already exist: for instance Vdovin has classified, for every finite simple group, the nilpotent subgroups of maximal size [Vdo00]; we also mention the work of Wehrfritz, who has studied nilpotency in many settings (see, amongst many others, [Weh73]).

6.4. Non-solvable subsets. A natural variant of the statistics we have studied in this paper replaces nilpotence with solvability. In this context we study the derived length of a subgroup rather than its nilpotency class. Let G be a group and c an element of $\mathbb{Z}^+ \cup \{\infty\}$. If $c \in \mathbb{Z}^+$ then we define G to be c -solvable if G is solvable with derived series of length c . We define G to be ∞ -solvable if G is solvable.

A subset X of G is said to be *non- c -solvable* if, for any two distinct elements x and y in X , $\langle x, y \rangle$ is a subgroup of G which is not c -solvable. Define $\beta_c(G)$ to be the maximum size of a non- c -solvable set in G . One can study the behaviour of $\beta_c(G)$ in much the same way as we have done here for $\omega_c(G)$, as well as defining the notion of a *2-minimal c -solvable cover* in the obvious way.

In the case that $c = 1$, the two notions of c -solvability and c -nilpotency are identical. At the other end of the spectrum, however, for large c , one might expect significant difference in their behaviour. We might ask the same sort of question in this connection as Question 6.11 above:

Question 6.12. *Is it the case that, for all $r \in \mathbb{Z}^+$, there exists $\ell \in \mathbb{Z}^+$ such that if G is a finite group of Lie type of rank r , and if N is a solvable subgroup of G , then the derived series of N has length less than ℓ ?*

REFERENCES

- [AAMHZ10] A. Abdollahi, A. Azad, A. Mohammadi Hassanabadi, and M. Zarrin, *On the clique numbers of non-commuting graphs of certain groups*, Algebra Colloq. **17** (2010), no. 4, 611–620.
- [AIPS11] Azizollah Azad, Mohammad A. Iranmanesh, Cheryl E. Praeger, and Pablo Spiga, *Abelian coverings of finite general linear groups and an application to their non-commuting graphs*, J. Algebraic Combin. **34** (2011), no. 4, 683–710.
- [AMH04] A. Abdollahi and A. Mohammadi Hassanabadi, *Finite groups with a certain number of elements pairwise generating a non-nilpotent subgroup*, Bull. Iranian Math. Soc. **30** (2004), no. 2, 1–20, 99–100.

- [AP09] Azizollah Azad and Cheryl E. Praeger, *Maximal subsets of pairwise non-commuting elements of three-dimensional general linear groups*, Bull. Aust. Math. Soc. **80** (2009), no. 1, 91–104.
- [Aza11] Azizollah Azad, *On nonnilpotent subsets in general linear groups*, Bull. Aust. Math. Soc. **83** (2011), no. 3, 369–375.
- [Ban86] A.S. Bang, *Talteoretiske undersøgelser*, Tidsskrift Math. **5** (1886), no. 4, 130–137.
- [BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, 1997.
- [BEG⁺08] J. R. Britnell, A. Evseev, R. M. Guralnick, P. E. Holmes, and A. Maróti, *Sets of elements that pairwise generate a linear group*, J. Combin. Theory Ser. A **115** (2008), no. 3, 442–465.
- [BG] John R. Britnell and Nick Gill, *Small nilpotent covers and large non-nilpotent subsets of $GL_n(q)$* , 2012. In preparation.
- [Bla06] Simon R. Blackburn, *Sets of permutations that generate the symmetric group pairwise*, Combin. Theory Ser. A **113** (2006), no. 7, 1572–1581.
- [BPS09] László Babai, Péter P. Pálffy, and Jan Saxl, *On the number of p -regular elements in finite simple groups*, LMS J. Comput. Math. **12** (2009), 82–119.
- [Bro88] Ron Brown, *Minimal covers of S_n by abelian subgroups and maximal subsets of pairwise noncommuting elements*, J. Combin. Theory Ser. A **49** (1988), no. 2, 294–307.
- [Bro91] ———, *Minimal covers of S_n by abelian subgroups and maximal subsets of pairwise noncommuting elements. II*, J. Combin. Theory Ser. A **56** (1991), no. 2, 285–289.
- [Car78] R. W. Carter, *Centralizers of semisimple elements in finite groups of Lie type*, Proc. London Math. Soc. (3) **37** (1978), no. 3, 491–507.
- [Car85] Roger W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1985, Conjugacy classes and complex characters, A Wiley-Interscience Publication.
- [CCN⁺85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, 1985.
- [CR87] Charles W. Curtis and Irving Reiner, *Methods of representation theory. Vol. II*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1987, With applications to finite groups and orders, A Wiley-Interscience Publication.
- [DGHP12] Silvio Dolfi, Robert M. Guralnick, Marcel Herzog, and Cheryl E. Praeger, *A new solvability criterion for finite groups*, J. Lond. Math. Soc. (2) **85** (2012), no. 2, 269–281.
- [End94] Gérard Endimioni, *Groupes finis satisfaisant la condition (\mathcal{N}, n)* , C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), no. 12, 1245–1247.

- [Gla66] George Glauberman, *Central elements in core-free groups*, J. Algebra **4** (1966), 403–420.
- [GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple K -groups.
- [GR93] Robert M. Guralnick and Geoffrey R. Robinson, *On extensions of the Baer-Suzuki theorem*, Israel J. Math. **82** (1993), no. 1-3, 281–297.
- [Hup67] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.
- [Hup70] ———, *Singer-zyklen in klassischen gruppen*, Math. Z. **117** (1970), 141–150.
- [Kin05] Oliver H. King, *The subgroup structure of finite classical groups in terms of geometric configurations*, Surveys in combinatorics 2005, London Math. Soc. Lecture Note Ser., vol. 327, Cambridge Univ. Press, Cambridge, 2005, pp. 29–56.
- [Kle88] Peter B. Kleidman, *The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$ and their automorphism groups*, J. Algebra **117** (1988), no. 1, 30–71.
- [MT11] Gunter Malle and Donna Testerman, *Linear algebraic groups and finite groups of Lie type*, Cambridge Studies in Advanced Mathematics, vol. 133, Cambridge University Press, Cambridge, 2011.
- [MV07] Hugh L. Montgomery and Robert C. Vaughan, *Multiplicative number theory. I. Classical theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007.
- [Neu76] B. H. Neumann, *A problem of Paul Erdős on groups*, J. Austral. Math. Soc. Ser. A **21** (1976), no. 4, 467–472.
- [Pyb87] L. Pyber, *The number of pairwise noncommuting elements and the index of the centre in a finite group*, J. London Math. Soc. (2) **35** (1987), no. 2, 287–295.
- [Sch94] Roland Schmidt, *Subgroup lattices of groups*, de Gruyter Expositions in Mathematics, vol. 14, Walter de Gruyter & Co., Berlin, 1994.
- [SS70] T. A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N. J. , 1968/69), Lecture Notes in Mathematics, Vol. 131, Springer, Berlin, 1970, pp. 167–266.
- [Ste65] Robert Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 49–80.
- [Suz62] Michio Suzuki, *On a class of doubly transitive groups*, Ann. of Math. **75** (1962), no. 1, 105–145.
- [Suz82] M. Suzuki, *Group theory I*, Springer-Verlag, 1982.

- [Tom92] M. J. Tomkinson, *Hypercentre-by-finite groups*, Publ. Math. Debrecen **40** (1992), no. 3-4, 313–321.
- [Vdo00] E. P. Vdovin, *Large nilpotent subgroups of finite simple groups*, Algebra Log. **39** (2000), no. 5, 526–546, 630.
- [War66] Harold N. Ward, *On Ree's series of simple groups*, Trans. Amer. Math. Soc. **121** (1966), 62–89.
- [Weh73] B. A. F. Wehrfritz, *Infinite linear groups. An account of the group-theoretic properties of infinite groups of matrices*, Springer-Verlag, New York, 1973, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 76.
- [Zsi92] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. Phys. **3** (1892), no. 1, 265–284.

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