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Kerr-AdS Black Hole Behaviors from Dark Energy

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Abstract

We investigate the critical behaviors of four-dimensional Kerr-AdS black holes from quintessential Dark Energy (DE) contributions. Using a moduli space, coordinated by the DE state parameter ω and the quintessence field intensity α, we deal with three different (ω)-models. By **elaborating analytical formulas of** relevant thermodynamical quantities denoted by $X^{(\omega)}$, we find significant similarities and distinctions. Precisely, for the ($-\frac{1}{3}$)-model, we show that DE contributions stabilise such black holes. For the (-1)-model, however, we get a reversed DE effect. In the ($-\frac{2}{3}$)-model, Kerr-AdS black holes reveal a resistance regarding the usual DE effects. **Exploiting the explicit formulas of such thermodynamical quantities, we give certain physical interpretations for thermal behaviors**. Although such relevant distinctions, we show that the (ω)-models involve similar universal ratios associated with certain critical thermodynamical quantities. **Then, we analyse the photon orbits in the presence of DE**.

Keywords: Kerr-AdS black holes, Thermodynamics, Quintessential dark energy.

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Contents

1 Introduction

Recently, black hole thermodynamic systems have been extensively investigated using different approaches including analytic and numerical ones [1, 2, 3, 4, 5, 6, 7]. These efforts have been supported by the fact that such systems have various similarities with known thermodynamic models. Such similarities appear naturally in the study of black holes in AdS geometries in different dimensions. The corresponding thermodynamical properties have been discussed first in [8]. After, this has been generalised to other solutions including charged and rotating AdS black holes. In particular, it has been revealed that the charged AdS black holes involve Van der Waals like phase transitions. Concretely, various investigations have been elaborated by interpreting the cosmological constant as the pressure and its conjugate as the volume [9, 10, 11]. In this way, it has been obtained a nice relation between the behavior of the RN-AdS black hole systems and the Van der Waals fluids [12, 13, 14]. Precisely, the associated P-V criticality can be linked to the liquid-gas statistical systems. This type of criticality, depending on the AdS black hole space-time dimension, provides non trivial behavioral results [15, 16, 17]. Furthermore, several phase transitions of various black holes have been explored. Concretely, the second order phase transition of four-dimensional Kerr-AdS black holes has been investigated in [18, 19]. Critical phenomena of extended Kerr-AdS black holes in the phase space have been also studied in [20]. Specifically, many numerical approaches have been developed to compute the involved thermodynamical quantities including the Gibbs free energy. The latter has been considered as a relevant quantity to discuss the corresponding Kerr-AdS black hole phase transitions.

More recently, dark matter (DM) and dark energy (DE) have been implemented in the study of black hole thermodynamics by extending the associated space parameter describing non trivial contributions [21, 22, 23, 24, 25]. It is realised that DE, being still an open subject question, has been approached using the ratio quantity $\omega = \frac{p_{dark}}{\rho_{dark}}$, interpreted as the equation of state. Various DE contributions have been investigated in terms of such a ratio belonging to the range $[-1, 0[$. Effectively, the quintessence, with intensity α, corresponding to the range $\left[-1, -\frac{1}{3}\right]$ considered as a scalar field having negative pressure has been introduced to investigate the DE effects on the black hole physics [26, 27, 28]. Among others, it has been suggested that DE can be considered as a cooling mechanism affecting the black hole thermodynamical behaviors [29, 30].

The aim of this work is to contribute to these activities by studying critical behaviors of the four-dimensional Kerr-AdS black holes from quintessential DE. The present study is made in terms of the DE moduli space parameterized by α and ω. These two parameters, together with the angular velocity Ω, control the four-dimensional Kerr-AdS black hole transition behaviors. More precisely, we investigate three different models based on particular values of ω by **giving analytical formulas of** the corresponding thermodynamical quantities referred to as X^{ω}. Among others, we obtain significant similarities and distinctions for such (ω)-models. For the ($-\frac{1}{3}$)-models, DE contributions stabilise the four-dimensional Kerr-AdS black hole. However, for both ($-\frac{2}{3}$) and (-1)-models, we observe significant distinctions. For instance, the (-1)-model exhibits a reversed DE effect regarding the associated thermodynamical quantities. Dealing with ($-\frac{2}{3}$)-model, we find that the Kerr-AdS black holes resist the DE effects. Analysing the (ω)-model results, **we discuss the associated thermal behaviour and** certain universalities of critical phase transition points. **Moreover, we investigate the DE effect on the photon orbits for the three (ω)-models and make contacts with some known results**. In this work, we use dimensionless units $\hbar = c = G = k_\beta = 1$.

The organisation of the paper is as follows. In section 2, we present a concise review on the ordinary properties of four-dimensional Kerr-AdS black holes which will be exploited to discuss the DE effect in section 3. Some results and discussions on **thermal behaviors** and certain phase transition points, in terms of universality ratios, are given in section 4. **In section 5, we inspect the DE effect on the photon orbits for the three (ω)-models and make contact with some known results**. The last section is devoted to conclusion and open questions.

2 Ordinary Kerr-AdS black hole phase transitions

Before studying the behavior of four-dimensional Kerr-AdS black holes in the presence of quintessential DE, we first give a concise review on its ordinary thermodynamic phase structure properties. Following [31, 32, 33], the Kerr-AdS black hole is considered as a rotating solution which can be derived from the Gibbons-Hawking action. According to [34], the Kerr-AdS metric of the four dimensional rotating black hole reads as

$$ds^2 = \frac{\Sigma^2}{\Delta_r}dr^2 + \frac{\Sigma^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta sin^2\theta}{\Sigma^2}(a\frac{dt}{\Xi} - (r^2 - a^2)\frac{d\phi}{\Xi})^2 - \frac{\Delta_r}{\Sigma^2}(\frac{dt}{\Xi} - a\sin^2\theta\frac{d\phi}{\Xi})^2, \quad (2.1)$$

where one has the following form

$$\begin{aligned} \Delta_r =& \ r^2 - 2Mr + a^2 + \frac{r^2}{\ell^2}(r^2 + a^2), & \Delta_\theta = 1 - \frac{a^2}{\ell^2}\cos^2\theta, \\ \Xi =& \ 1 - \frac{a^2}{\ell^2}, & \Sigma^2 = r^2 + a^2\cos^2\theta. \end{aligned} \quad (2.2)$$

In thermodynamical activities, ℓ representing the AdS curvature can be related to the pressure P given by

$$P = \frac{3}{8\pi\ell^2} = -\frac{\Lambda}{8\pi}. \tag{2.3}$$

The metric parameters m and a, appearing in the previous equations, are linked to the black hole mass M and the angular momentum J, respectively

$$M = \frac{m}{\Xi^2}, \qquad J = \frac{am}{\Xi^2}. \tag{2.4}$$

The mass of the black hole M can be determined by imposing the constraint $\Delta_r(r_+) = 0$. Indeed, we find

$$M = \frac{(r_+^2 + a^2)(r_+^2 + \ell^2)}{2r_+\ell^2}. \tag{2.5}$$

Following [35], the Bekenstein-Hawking entropy reads as

$$S = \pi\frac{(r_+^2 + a^2)}{\Xi}. \tag{2.6}$$

Using $\Delta_r(r_+) = 0$, (2.4) and (2.6), one can get an extended Smarr formula for the Kerr-AdS black hole in four-dimensions. Concretely, the calculation gives the following result

$$M(S,J) = \left(\frac{\pi}{4S}\left\{\frac{4SJ^2}{\pi\ell^2} + 4J^2 + \left[\frac{S^2}{\pi^2\ell^2} + \frac{S}{\pi}\right]^2\right\}\right)^{\frac{1}{2}}. \tag{2.7}$$

To get elegant relations, it is convenient to use a suitable scaling redefinition $T\ell \to T$, $\frac{M}{\ell} \to M$, $\frac{J}{\ell^2} \to J$, $\frac{S}{\ell^2} \to S$, $\Omega\ell \to \Omega$. In this way, ℓ no longer appears in the thermodynamical quantities [18]. To study the Kerr-AdS balck hole thermodynamical properties, one should compute the associated quantities using known laws [36]. It is noted that the first law of thermodynamics of such a black hole is expressed as

$$dM = TdS + \Omega dJ, \tag{2.8}$$

where T is the Hawking temperature. Here, Ω denotes the difference between the angular velocities at the event horizon (Ω_h) and at infinity (Ω_∞) [37]. Following [18], the temperature, for instance, reads as

$$T(S,J) = \frac{1}{8\pi M}\left(1 - \frac{4\pi^2J^2}{S^2} + \frac{4S}{\pi} + \frac{3S^2}{\pi^2}\right). \tag{2.9}$$

Using the above scale redefinitions, the other needed quantities can be computed. According to [18], they are listed as follows

$$M^2(S,J) = \frac{S}{4\pi} + \frac{\pi J^2}{S} + J^2 + \frac{S}{2\pi}\left(\frac{S}{\pi} + \frac{S^2}{2\pi^2}\right), \quad (2.10)$$

$$\Omega = \frac{\pi J}{MS}\left(1 + \frac{S}{\pi}\right), \quad (2.11)$$

$$\frac{J}{S} = \frac{M\Omega}{\pi + S}, \quad (2.12)$$

$$a = \frac{J}{M} = \frac{S\Omega}{\pi}\left(1 + \frac{S}{\pi}\right)^{-1}. \quad (2.13)$$

Substituting (2.12) in (2.10), one can write M^2 in terms of S and Ω

$$M^2(S,\Omega) = \frac{S}{4\pi}\frac{1 + \frac{2S}{\pi}(1 + \frac{S}{2\pi})}{1 - \frac{\Omega^2 S}{\pi(1+\frac{S}{\pi})}}. \quad (2.14)$$

Putting (2.14) in (2.9), one gets the semi-classical temperature

$$T(S,\Omega) = \sqrt{\frac{S(\pi+S)^3}{(\pi + S - S\Omega^2)}}\left[\frac{\pi^2 - 2\pi S(\Omega^2 - 2) - 3S^2(\Omega^2 - 1)}{4\pi^{\frac{3}{2}} S(\pi + S)^2}\right]. \quad (2.15)$$

The reality condition of T being required by

$$\pi + S - S\Omega^2 > 0 \quad (2.16)$$

constraints $\Omega^2 < 1 + \frac{\pi}{S}$. This yields a restriction on Ω for the entropy fixed values. Other thermodynamical properties of the black hole can be approached by exploiting the semi-classical specific heat at constant angular velocity [18]. Applying the relation $C_\Omega = T\left(\frac{\partial S}{\partial T}\right)_\Omega = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_\Omega}$, one obtains

$$C_\Omega(S,\Omega) = \frac{2S(\pi+S)(\pi+S-S\Omega^2)\left(\pi^2 - 2\pi S(\Omega^2-2) - 3S^2(\Omega^2-1)\right)}{(\pi+S)^3(3S-\pi) - 6S^2(\pi+S)^2\Omega^2 + S^3(4\pi+3S)\Omega^4}. \quad (2.17)$$

To examine the phase transition phenomena, the above temperature expression (2.15) and the specific heat capacity (2.17) will be used. The corresponding behaviors are illustrated in figure 1.

It follows from figure 1 that the semi-classical Hawking temperature T is continuous as a function of the semi-classical entropy S. Moreover, it is observed that, for a fixed Ω value, there is no first order phase transition occurring in the Kerr-AdS black hole physics. However, it involves a minimum at the point $S_{min} = 1.09761$ corresponding to

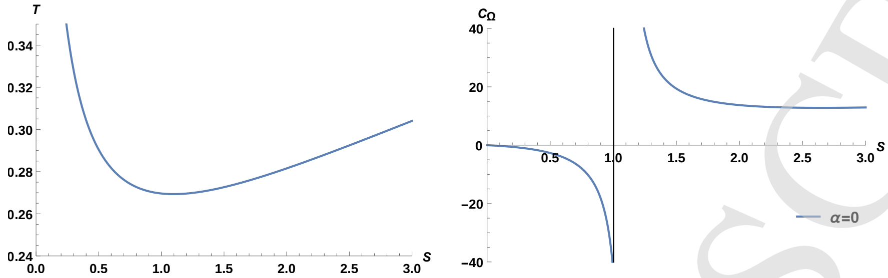

Figure 1: The semi-classical Hawking temperature (T) and the specific heat (C_Ω) in terms of the entropy (S) for $\Omega = 0.3$.

the minimal temperature $T_{min} = 0.26933$ under which no black hole can survive. Furthermore, the discontinuity of the specific heat C_Ω appears at the critical value of the entropy $S_c = S_{min} = 1.09761$. At such a critical value, the specific heat C_Ω changes from negative infinity to positive infinity. Since the entropy is proportional to the square of the black hole mass, the critical point S_c separates two branches of the Kerr-AdS black holes. The first one associated with a small black hole mass is considered thermodynamically as an unstable one possessing a negative specific heat ($C_\Omega < 0$). However, the second branch concerning the large black hole mass is interpreted as a stable phase involving a positive specific heat ($C_\Omega > 0$).
Having discussed the case of the Kerr-AdS black holes in the absence of DE, we are situated to investigate the effect of DE by considering different quintessential contributions.

3 Kerr-AdS black hole phase transitions from quintessence contributions

In this section, we study DE effects on the previous four dimensional Kerr-AdS black hole physics. This study will be done in terms of new parameters related to DE contributions. In particular, a close inspection shows that the parameter space of the black hole $\mathcal{M}$ can be factorized in two sectors

$$\mathcal{M} = \mathcal{M}_{OBH} \times \mathcal{M}_{ec}. \tag{3.1}$$

The first sector is associated with the parameters of the ordinary balck holes (OBH)

$$\mathcal{M}_{OBH} \equiv \{J, M, Q\}, \tag{3.2}$$

where Q is the charge. For the neutral black hole ($Q = 0$), this can be reduced to the Kerr black hole parameterized only by J and M

$$\mathcal{M}_{OBH} \equiv \{J, M\}, \tag{3.3}$$

that we are interested in. The second sector $\mathcal{M}_{ec}$ concerns extra contributions associated with outside horizon contributions including DE, DM and other non trivial ones.
In the present work, we consider only DE contributions via a quintessence scalar field. In this way, the second sector $\mathcal{M}_{ec}$ is controlled by two parameters α and ω corresponding to the quintessence intensity and the DE state parameter, respectively. For generic values of α, the dealt with models should depend only on certain values of ω. Here, however, we pay attention to some particular cases. General values of ω, needing non trivial reflections, could be considered as an investigation projet. We hope to come back to it in the future.

In the rest of the paper, we would like to examine three different (ω)-models of the Kerr-AdS black hole behaviors. To determine the corresponding metric, we first consider the usual results associated with spherically symmetric Schwarzschild black hole solution in quintessential dark energy. Then, we use Newman-Janis algorithm to get Kerr black hole metric surrounded by quintessential DE. For more interpretations, discussions of this method and its physical implications can be found in [38, 39, 40, 41]. The Schwarzschild black hole can be described by the following metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2, \tag{3.4}$$

where the metric function $f(r)$ is given by

$$f(r) = 1 - \frac{2M}{r} - \frac{\alpha}{r^{3\omega+1}}. \tag{3.5}$$

More details on such calculations can be found in [42]. The expression of the associated metric is written as

$$ds^2 = \frac{\Sigma^2}{\Delta_r}dr^2 + \frac{\Sigma^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta sin^2\theta}{\Sigma^2}(a\frac{dt}{\Xi} - (r^2 - a^2)\frac{d\phi}{\Xi})^2 - \frac{\Delta_r}{\Sigma^2}(\frac{dt}{\Xi} - a\sin^2\theta\frac{d\phi}{\Xi})^2, \tag{3.6}$$

In this case, one has

$$\Delta_r = r^2 - 2Mr + a^2 + \frac{r^2}{\ell^2}(r^2 + a^2) - \alpha r^{1-3\omega}. \tag{3.7}$$

In what follows, the corresponding thermodynamical quantities that we will obtain involve a subscript indicating the values of ω. We refer to them as $X^{(\omega)}(S, \Omega, \alpha)$ where α is a positive normalisation factor associated with DE intensity.

3.1 Model with DE state parameter $\omega = -\frac{1}{3}$

To study the behavior of the four dimensional Kerr-AdS black holes in the presence of DE for the value $\omega = -\frac{1}{3}$, we should first elaborate the generalized Smarr formula by using the expressions (2.4), (2.6) and the condition $\Delta_r(r_+) = 0$. For $(-\frac{1}{3})$-model, the computations give

$$M^{(-1/3)}(S, J, \alpha) = \left(\frac{\pi}{4S}\left\{\frac{4SJ^2}{\pi\ell^2} + 4J^2 + \left[\frac{S^2}{\pi^2\ell^2} + \frac{S}{\pi} - \frac{\alpha S}{\pi}\right]^2\right\}\right)^{\frac{1}{2}}. \tag{3.8}$$

Exploiting the previous scaling redefinitions, the thermal temperature of the Kerr-AdS black holes, in the presence of DE, reads as

$$T^{(-1/3)}(S, J, \alpha) = \frac{\partial M}{\partial S} = \frac{1}{8\pi M}\left((\alpha - 1)^2 - \frac{4\alpha S}{\pi} - \frac{4\pi^2 J^2}{S^2} + \frac{4S}{\pi} + \frac{3S^2}{\pi^2}\right). \tag{3.9}$$

Similarly, the other involved thermodynamics quantities can be computed. In particular, the mass square is given by

$$\left(M^{(-1/3)}(S, J, \alpha)\right)^2 = \frac{S}{4\pi} + \frac{\pi J^2}{S} + J^2 + \frac{S}{2\pi}\left(\frac{S}{\pi} + \frac{S^2}{2\pi^2} + \frac{\alpha^2}{2} - \alpha\left(\frac{S}{\pi} + 1\right)\right). \tag{3.10}$$

Putting the angular momentum, given in (2.12), into (3.10), we can obtain the expression of $\left(M^{(-1/3)}(S, J, \alpha)\right)^2$ in terms of S and Ω. Concretely, we find

$$\left(M^{(-1/3)}(S, \Omega, \alpha)\right)^2 = \frac{S}{4\pi}(\pi + S)\frac{\left((\alpha - 1)^2 + \frac{2S}{\pi}(1 + \frac{S}{2\pi} - \alpha)\right)}{(\pi + S - \Omega^2 S)}. \tag{3.11}$$

Turning on the effect of DE, we derive the temperature $T^{(-1/3)}$ and the specific heat $C_\Omega^{(-1/3)}$ in terms of S, α and Ω of the Kerr-AdS black holes. By replacing (3.11) into (3.9), we get

$$T^{(-1/3)}(S, \Omega, \alpha) = \left[\frac{\left(\pi^2 + 3S^2 - 4S\pi(\alpha - 1) - (2\alpha - \alpha^2)\pi^2 - \frac{S\pi(\pi+S-\pi\alpha)^2\Omega^2}{(\pi+S)(\pi+S-S\Omega^2)}\right)}{4\pi^{3/2}\sqrt{\frac{S(\pi+S)(\pi+S-\pi\alpha)^2}{(\pi+S-S\Omega^2)}}}\right]. \tag{3.12}$$

It is worth noting that we can recover the ordinary results by turning off DE contributions. Taking $\alpha = 0$, we get (2.15) which matches perfectly with the results reported in [18]. However, to examine the thermodynamical behaviors of such Kerr-AdS black holes, we calculate the semi-classical specific heat at constant angular velocity Ω. In the presence of DE, the specific heat takes the simplified form

$$C_\Omega^{(-1/3)}(S, \Omega, \alpha) = \frac{2S(\pi + S)(\pi + S - S\Omega^2)\left((\pi + S)\left(\pi^2 - 2\pi S(\Omega^2 - 2) - 3S^2(\Omega^2 - 1)\right) - \alpha\beta_1\right)}{(\pi + S)\left((\pi + S)^3(3S - \pi) - 6S^2(\pi + S)^2\Omega^2 + S^3(4\pi + 3S)\Omega^4\right) + \alpha\beta_2}, \tag{3.13}$$

where the β_i terms are given by

$$\beta_1 = \left(\pi^3 + \pi S^2(1-\Omega^2) - 2\pi^2 S(-1+\Omega^2)\right), \quad (3.14)$$

$$\beta_2 = \left(\pi^5 + 4\pi^4 S - 6\pi^3 S^2 + (\Omega^2-1)^2(\pi S^4 + 4\pi^2 S^3)\right). \quad (3.15)$$

For $\alpha = 0$, this reproduces the ordinary specific heat given in (2.17), as reported in [18]. Using (3.12) and (3.13), we plot the variation of the entropy S in terms of the temperature $T^{(-1/3)}$ and the specific heat ($C_\Omega^{(-1/3)}$) for a fixed value of Ω (0.3) in figure 2.

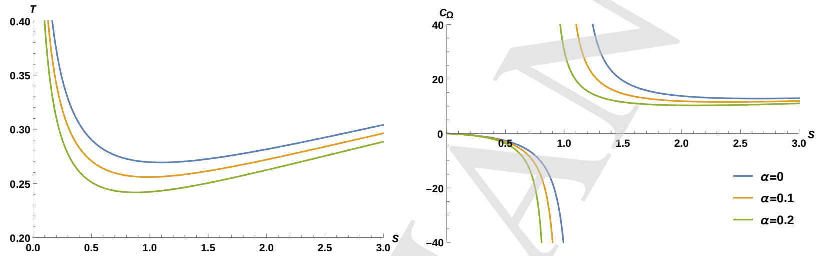

Figure 2: The semi-classical Hawking temperature ($T^{(-1/3)}$) and the Specific heat ($C_\Omega^{(-1/3)}$) in terms of the entropy (S) for fixed $\Omega = 0.3$ and $\omega = -1/3$.

It has been realized that the minimal temperature $T_{\min}^{(-1/3)}$ is quite complicated. However, in order to obtain an explicit expression for such a minimal temperature one can use an optimisation method. Setting $(\frac{\partial T}{\partial S})_{\alpha,\Omega} = 0$, we find a polynomial of a fifth order with respect to S, being a non trivial task. However, we need several approximations and simplifications. The first approximation that one should take is to remove the fifth order. Indeed, when the effect of DE vanishes, we get an identical polynomial to the one found in the absence of DE. After solving the polynomial in the presence of DE, an examination shows that the solutions could be simplified. Since we are dealing with objets that have a small value of intensity α, the higher orders greater than two will be omitted. In this way, the approximated entropy which corresponds to the minimal temperature reads as

$$S_{min}^{(-1/3)}(\Omega,\alpha) = \frac{\pi}{\sqrt{6}(3+\alpha)}\left[\frac{\sqrt{6}(2+\alpha-\Omega^2-\alpha\Omega^2)}{(-1+\Omega^2)} + \gamma_2 + \sqrt{\gamma_3 - \frac{6\sqrt{6}\,(2-3\Omega^2-3\Omega^4+2\Omega^6+\alpha(9-6\Omega^2-9\Omega^4+6\Omega^6))}{(-1+\Omega^2)^3\,\gamma_2}}\right], \tag{3.16}$$

where the γ_i terms are given by

$$\gamma_1 = \Big(\,(-1+\Omega^2)^2\,(-\Omega^4+\alpha^2(3-4\Omega^2+\Omega^4)-\alpha(-4+2\Omega^2+\Omega^4)+ \sqrt{\Omega^8+2\alpha\Omega^4(-4+2\Omega^2+\Omega^4)-\alpha^2(-16+16\Omega^2+10\Omega^4-12\Omega^6+\Omega^8)}\,)\Big)^{1/3}, \tag{3.17}$$

$$\gamma_2 = \sqrt{\frac{20\alpha^2(-1+\Omega^2)^2+3\gamma_1\,(2+3\gamma_1-2\Omega^2+2\Omega^4)+3\alpha\,(\gamma_1^2+4\gamma_1\Omega^2(-1+\Omega^2)+8(-1+\Omega^2)^2)}{\gamma_1(-1+\Omega^2)^2}}, \tag{3.18}$$

$$\gamma_3 = \frac{(-20\alpha^2(1+\Omega^4)+3\gamma_1(4-3\gamma_1-4\Omega^2+4\Omega^4)-3\alpha(\gamma_1^2-8\gamma_1\Omega^2(-1+\Omega^2)+8(-1+\Omega^2)^2)\,)}{\gamma_1(-1+\Omega^2)^2}. \tag{3.19}$$

Substituting (3.16) into (3.12), we find the corresponding minimum temperature

$$T_{min}^{(-1/3)}(\Omega,\alpha) = \left[\frac{3^{5/4}\left(1+\frac{\delta^2}{2}-2\sqrt{\frac{2}{3}}\,\delta\,(-1+\alpha)-2\alpha+\alpha^2+\frac{\delta\,(6+\sqrt{6}\,\delta-6\,\alpha)^2\Omega^2}{\sqrt{6}\,(6+\sqrt{6}\,\delta)\left(-6+\sqrt{6}\,\delta\,(-1+\Omega^2)\right)}\right)}{2^{3/4}\,\pi\,\sqrt{-\frac{\delta(6+\sqrt{6}\,\delta)\,(6+\sqrt{6}\,\delta-6\,\alpha)^2}{(-6+\sqrt{6}\,\delta\,(-1+\Omega^2))}}}\right], \tag{3.20}$$

where we have

$$\delta = \frac{1}{(3+\alpha)}\left[\frac{\sqrt{6}(2+\alpha-\Omega^2-\alpha\Omega^2)}{(-1+\Omega^2)} + \gamma_2 + \sqrt{\gamma_3 - \frac{6\sqrt{6}\,(2-3\Omega^2-3\Omega^4+2\Omega^6+\alpha(9-6\Omega^2-9\Omega^4+6\Omega^6))}{(-1+\Omega^2)^3\,\gamma_2}}\right]. \tag{3.21}$$

At this stage, it is possible to provide an interpretation concerning the two branches shown in figure 2. In $S-T$ plane, we observe from the curve that both of these phases exist above the minimal temperature $T_{min}^{(-1/3)}$. Since the entropy is proportional to the black hole mass, the first branch which corresponds to a lower mass is considered a thermodynamically unstable. The second one which corresponds to a higher mass is considered as a thermodynamically stable. We notice from the decrease of $S_{min}^{(-1/3)}$ when the intensity α increases that the stable phase becomes relevant.

The curve in the $S - C_\Omega$ plane, appearing in the figure 2, shows two phases separated by a critical entropy $S_c^{(-1/3)} = S_{min}^{(-1/3)}$. Indeed, the first phase associated with a smaller mass black hole posses a negative heat capacity ($C_\Omega < 0$). It is considered as an unstable one. However, the second phase corresponding to the larger mass black hole involving a positive heat capacity ($C_\Omega > 0$) is considered as a stable one. Besides, the equality between the critical entropy $S_c^{(-1/3)}$ and $S_{min}^{(-1/3)}$ yields directly the decrease of the critical entropy $S_c^{(-1/3)}$ when the intensity α increases. This makes the stable phase larger. According to [43] and using the formula $G = M - T\cdot S - J\cdot\Omega$, we obtain the Gibbs free energy which reads as

$$G^{(-1/3)}(S,\Omega,\alpha) = \frac{\sqrt{S}\,\left(\pi^3\,(1-\alpha) + \pi^2\,(S - 2S\alpha) - S^3(1-\Omega^2) - \pi\,S^2(1+\alpha)\,(1-\Omega^2)\,\right)}{4\pi^{\frac{3}{2}}(\pi+S)^{3/2}(\pi+S-S\Omega^2)^{1/2}}. \tag{3.22}$$

Now, we investigate the effects of DE on the Kerr-AdS black hole phase transitions by plotting the Gibbs free energy as a function of the entropy in figure 3.

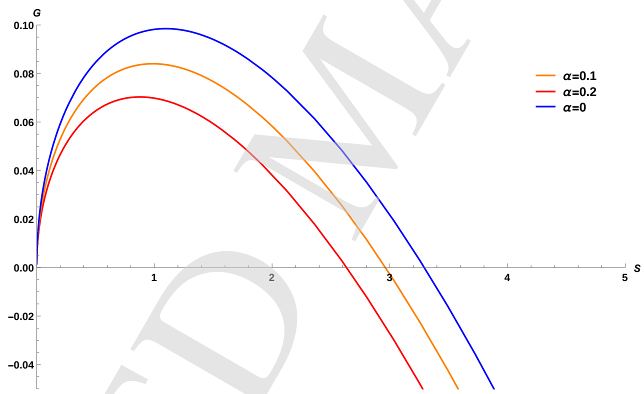

Figure 3: The Gibbs free energy ($G^{(-1/3)}$) plotted for Kerr-AdS black hole with respect to the entropy ($S^{(-1/3)}$) for fixed $\Omega = 0.3$, $\omega = -1/3$ and different values of α.

From figure 3, we notice that the Gibbs free energy changes its sign. The positive values of $G^{(-1/3)}(S,\Omega,\alpha)$ correspond to unstable black holes while the negative values are associated with stable black holes. In this way, one can remark that DE decreases the unstable phase. Thus, DE provides a more stable black hole. Moreover, the point where the Gibbs free energy changes its sign corresponds to the Hawking-Page phase transition. At this point, the entropy is given by

$$S_{HP}^{(-1/3)}(\Omega,\alpha) = \frac{\pi\,\left(4 - 5\Omega^2 + \Omega^4 - A(1-\Omega^2) + A^2 - \alpha(1-\Omega^2)(4+2\Omega^2+A)\,\right)}{3\,A\,(1-\Omega^2)}, \tag{3.23}$$

where the quantity A takes the following form

$$A(\Omega,\alpha) = \Big(8 - 15\Omega^2 + 6\Omega^4 + \Omega^6 + 3\alpha(-4+\Omega^2)(-1+\Omega^2)^2 \\ -3\sqrt{3}\sqrt{\Omega^2\,(1-\Omega^2)^3(\Omega^2 + 2\alpha(4-\Omega^2)\,)}\Big)^{\frac{1}{3}}. \tag{3.24}$$

This value of the entropy corresponds to the temperature

$$T_{HP}^{(-1/3)}(\Omega,\alpha) = \frac{1}{4\pi^2\sqrt{q(3\pi+\pi q-\pi q\Omega^2)}\,(\pi+\frac{\pi q}{3})^{3/2}} \times \Big[3\pi^3\,(1-\alpha) + \frac{1}{3}\pi^3\,q^3\,(1-\Omega^2) \\ +\frac{1}{3}\pi^3\,q^2\,(7-\alpha-5\Omega^2+\alpha\Omega^2) + \pi^3\,q\,(5-2\Omega^2+2\alpha\,(-1+\Omega^2)\,)\Big], \tag{3.25}$$

where the quantity q is

$$q(\Omega,\alpha) = \frac{(4 - A + A^2 - 5\Omega^2 + A\,\Omega^2 + \Omega^4 - \alpha(1-\Omega^2)(4 + A + 2\Omega^2)\,)}{A\,(1-\Omega^2)}. \tag{3.26}$$

In figure 4, we plot the Gibbs free energy as a function of the temperature in order to compare the different phases for several values of DE intensity α.

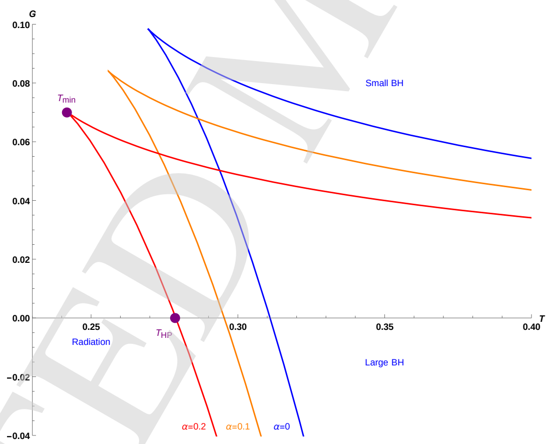

Figure 4: the Gibbs free energy ($G^{(-1/3)}$) plotted for Kerr-AdS black hole with respect to temperature ($T^{(-1/3)}$) for $\Omega = 0.3$, $\omega = -1/3$ and different values of α.

From figure 4, we notice that the radiation phase, which is the phase where the black hole can decay into a pure thermal AdS space, and the phase, where no black hole can exist, decrease as the parameter α increases. Furthermore, we remark that this figure data confirms the idea that DE stabilises the Kerr-AdS black hole. To confirm such behaviors, one may think about the heat capacity $C_\Omega^{(-1/3)}$ as a function of the temperature. This is illustrated in figure 5.

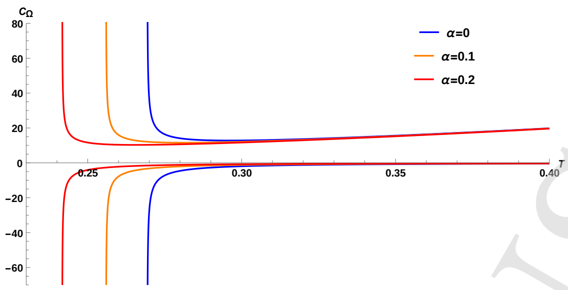

Figure 5: The specific heat $C_\Omega^{(-1/3)}$ illustrated for Kerr-AdS black holes with respect to the temperature ($T^{(-1/3)}$) for $\Omega = 0.3$, $\omega = -1/3$ and different values of α.

It follows from figure 5 that the heat capacity changes its sign at the minimal temperature $T_{min}^{(-1/3)}$. We also find that the phase where no black hole can exist decreases when the intensity α increases.
In $(-\frac{1}{3})$-model, we conclude that the decrease of $S_{min}^{(-1/3)}$ confirms that the stable phase becomes relevant when the intensity α increases, which makes the black hole more stable. This analysis matches perfectly with the trivial mechanism of DE.

3.2 Model with DE state parameter $\omega = -\frac{2}{3}$

To further investigate the DE effect, we investigate four dimensional Kerr-AdS behaviors for the value $\omega = -\frac{2}{3}$. As the previous model, we first give the generalised Smarr formula for such a $(-\frac{2}{3})$-model by using the expressions (2.4) and (2.6). Solving the condition $\Delta_r(r_+) = 0$, we get

$$M^{(-2/3)}(S,J,\alpha) = \left(\frac{\pi}{4S}\left\{\frac{4SJ^2}{\pi\ell^2} + 4J^2 + \left[\frac{S^2}{\pi^2\ell^2} + \frac{S}{\pi} - \alpha\left(\frac{S}{\pi}\right)^{\frac{3}{2}}\right]^2\right\}\right)^{\frac{1}{2}}. \tag{3.27}$$

Taking into account the above scaling redefinitions, the thermal temperature of the Kerr-AdS black holes become

$$T^{(-2/3)}(S,J,\alpha) = \frac{1}{8\pi M}\left(1 - \frac{4\pi^2 J^2}{S^2} + \frac{4S}{\pi} + \frac{3S^2}{\pi^2} - \alpha\left(\frac{5S^{3/2}}{\pi^{3/2}} + \frac{3S^{1/2}}{\pi^{1/2}}\right) + \frac{2\alpha^2 S}{\pi}\right). \tag{3.28}$$

Similary, the square mass is given by

$$\left(M^{(-2/3)}(S,J,\alpha)\right)^2 = \frac{S}{4\pi} + \frac{\pi J^2}{S} + J^2 + \frac{S}{2\pi}\left(\frac{S^2}{2\pi^2} + \frac{S}{\pi} + \frac{\alpha^2 S}{2\pi} - \alpha\frac{S^{1/2}}{\pi^{1/2}}(\frac{S}{\pi} + 1)\right). \tag{3.29}$$

Using (2.12) and substituting J in (3.29), we write $\left(M^{(-2/3)}\right)^2$ in terms of S and Ω as follows

$$\left(M^{(-2/3)}(S,\Omega,\alpha)\right)^2 = \frac{S}{4\pi}\left(\pi+S\right)\frac{\left(1+\frac{2S}{\pi}(1+\frac{S}{2\pi}+\frac{\alpha^2}{2}-\alpha(\frac{S^{1/2}}{\pi^{1/2}}+\frac{S^{-1/2}}{\pi^{-1/2}}))\right)}{(\pi+S-\Omega^2 S)}. \quad (3.30)$$

To examine the effect of DE, we put (3.30) into (3.28) which can give the semi-classical temperature in terms of S and Ω. For $(-\frac{2}{3})$-model, the calculations produce

$$T^{(-2/3)}(S,\Omega,\alpha) = \left[\frac{\pi^2+3S^2+4S\,\pi-\frac{\sqrt{S}\,(3\pi+5S)\alpha}{\pi^{-1/2}}+2S\,\alpha^2\,\pi-\frac{S\,\pi\,(\pi+S-\sqrt{\pi}\,\sqrt{S}\,\alpha)^2\,\Omega^2}{(\pi+S)(\pi+S-S\,\Omega^2)}}{4\pi^{\frac{3}{2}}\sqrt{\frac{S(\pi+S)(\pi+S-\sqrt{\pi}\,\sqrt{S}\,\alpha)^2}{(\pi+S-S\Omega^2)}}}\right]. \quad (3.31)$$

Taking $\alpha = 0$, we recover the ordinary temperature given in (2.15), as obtained in [18]. For $\alpha \neq 0$, the thermodynamical behaviors of the corresponding black hole can be approached by exploiting the semi-classical specific heat. At constant angular velocity, the computations lead to

$$C_{\Omega}^{(-2/3)}(S,\Omega,\alpha) = \frac{2S(\pi+S)(\pi+S-S\Omega^2)\left((\pi+S)\left(\pi^2-2\pi S(\Omega^2-2)-3S^2(\Omega^2-1)\right)-\alpha\beta_3\right)}{(\pi+S)\left((\pi+S)^3(3S-\pi)-6S^2(\pi+S)^2\Omega^2+S^3(4\pi+3S)\Omega^4\right)+\alpha\beta_4}, \quad (3.32)$$

where the β_i terms are given by

$$\beta_3 = \left(2\pi^{5/2}\sqrt{S}-S^{3/2}\pi^{3/2}(-4+3\Omega^2)-2\sqrt{\pi}S^{5/2}(-1+\Omega^2)\right), \quad (3.33)$$

$$\beta_4 = \left(4\pi^{7/2}S^{3/2}\Omega^2-\pi^{5/2}S^{5/2}\Omega^2(-4+3\Omega^2)\right). \quad (3.34)$$

Turning of the DE effects ($\alpha = 0$), this result reduces to the ordinary specific heat appearing in [18]. To specify the DE effect ($\alpha \neq 0$), we discuss the temperature (3.31) and the specific heat (3.32) with respect to the entropy S for a fixed value of $\Omega = 0.3$. This is illustrated in figure 6.

As mentioned before, the expression of the minimal temperature is complicated. However, the corresponding computations may need certain relevant approximations. In fact, there are many ways to reach the desired results. Setting $\left(\frac{\partial T}{\partial S}\right)_{\alpha,\Omega} = 0$, we find a polynomial with rational orders in S. It is known that this situation is not an easy task. To handle this rational order, we exploit the following limit $(1-x^n) \approx 1-nx$ for small values of x. Besides, we use similar technics explored in the previous $(-\frac{1}{3})$-model. After calculations, the approximated entropy, which corresponds to the minimal entropy, can

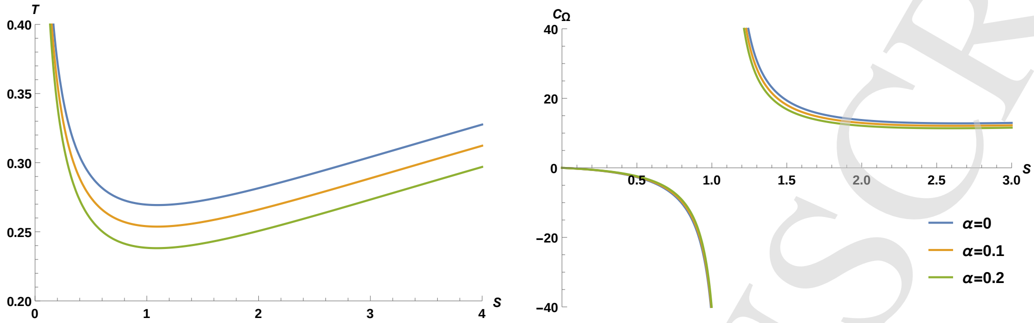

Figure 6: The semi-classical Hawking temperature ($T^{(-2/3)}$) and the specific heat ($C_\Omega^{(-2/3)}$) as function of the entropy (S) for $\Omega = 0.3$ and $\omega = -2/3$.

be written as

$$S_{min}^{(-2/3)}(\Omega, \alpha) = \frac{1}{6}\Bigg[3\gamma_6 - \frac{2\pi(-2+\Omega^2)}{(-1+\Omega^2)} +3\sqrt{\left(\gamma_7 + \frac{(-108\pi\,\alpha\,\Omega^2(-1+\Omega^2) + 45\alpha\,\Omega^2(4-7\Omega^2+3\Omega^4) - 8\pi^3(2-3\Omega^2-3\Omega^4+2\Omega^6))}{27\gamma_6\,(-1+\Omega^2)^3}\right)}\Bigg], \tag{3.35}$$

where the γ_i parameters are given by

$$\gamma_4 = \Big(54\,(-1+\Omega^2)^2\,\gamma_5 + \big(55296\,\pi^3\,\alpha^3\,\Omega^6(20+12\pi-15\Omega^2)^3\,(2-3\Omega^2+\Omega^4)^3 + 2916\,\Omega^4\,(-1+\Omega^2)^4\,\gamma_5^2\big)^{1/2}\Big)^{1/3}, \tag{3.36}$$

$$\gamma_5 = \big(-64\,\pi^6\,\Omega^2 + 432\,\pi^2\,\alpha^2\,\Omega^2 + 75\,\alpha^2\,\Omega^2(4-3\Omega^2)^2 + 192\,\pi^4\,\alpha(-2+\Omega^2) -360\pi\,\alpha^2\,\Omega^2(-4+3\,\Omega^2) - 80\,\pi^3\,\alpha(8-10\Omega^2+3\Omega^4)\big)\,, \tag{3.37}$$

$$\gamma_6 = \sqrt{\frac{(2^{-4/3}\,\gamma_4^2 - 12\times 2^{1/3}\,\pi\,\alpha\,\Omega^2(20+12\,\pi-15\Omega^2)\,(2-3\Omega^2+\Omega^4) + 4\gamma_4\,\pi^2\,(1-\Omega^2+\Omega^4))}{9\,\gamma_4\,(-1+\Omega^2)^2}}, \tag{3.38}$$

$$\gamma_7 = \frac{(-2^{-4/3}\,\gamma_4^2 + 12\times 2^{1/3}\,\pi\,\alpha\,\Omega^2(20+12\,\pi-15\Omega^2)\,(2-3\Omega^2+\Omega^4) + 8\gamma_4\,\pi^2\,(1-\Omega^2+\Omega^4))}{9\,\gamma_4\,(-1+\Omega^2)^2}. \tag{3.39}$$

Substituting (3.35) into (3.31), we find the corresponding minimum temperature given by

$$T_{min}^{(-2/3)}(\Omega,\alpha)=\left[\frac{\sqrt{\frac{3}{2}}\,\left(12+8\xi+\xi^2-2\sqrt{6}\,(3+\frac{5\xi}{6})\sqrt{\xi}\,\alpha+4\,\xi\,\alpha^2+\frac{2\,\xi(6+\xi-\sqrt{6}\,\sqrt{\xi}\,\alpha)^2\,\Omega^2}{(6+\xi)(-6+\xi\,(-1+\Omega^2))}\right)}{4\pi\sqrt{-\frac{\xi\,(6+\xi)\,(6+\xi-\sqrt{6}\,\sqrt{\xi}\,\alpha)^2}{(-6+\xi(-1+\Omega^2))}}}\right], \tag{3.40}$$

where we have

$$\xi(\Omega,\alpha)=\frac{1}{\pi}\left[3\gamma_6-\frac{2\pi(-2+\Omega^2)}{(-1+\Omega^2)}\right.$$
$$\left.+3\sqrt{\left(\gamma_7+\frac{(-108\pi\,\alpha\,\Omega^2(-1+\Omega^2)+45\alpha\,\Omega^2(4-7\Omega^2+3\Omega^4)-8\pi^3(2-3\Omega^2-3\Omega^4+2\Omega^6))}{27\gamma_6\,(-1+\Omega^2)^3}\right)}\right]. \tag{3.41}$$

Now, we are in position to discuss the corresponding Kerr-AdS black hole phase transitions. From figure 6, we observe that the curve in the plane $(S-T)$ involves two branches separated by a minimal temperature (3.40). Indeed, the first branch associated with a small black hole mass is thermodynamically unstable. However, the second one corresponding to a large black hole mass is stable.

In $(S-C_\Omega)$ curve, two phases separated by a critical entropy $S_c^{(-\frac{2}{3})}=S_{min}^{(-\frac{2}{3})}$ appear. Furthermore, the first phase corresponding to a small black hole mass having a negative heat capacity $(C_\Omega<0)$ is unstable. The second one, which corresponds to a large black hole mass possessing a positive heat capacity $(C_\Omega>0)$, is considered as a stable one. It is observed that the heat capacity is not affected by DE contributions.

To understand more such non trivial behaviors, one may introduce the Gibbs free energy. For the present model, this energy can be written as

$$G^{(-2/3)}(S,\Omega,\alpha)=\frac{\sqrt{S}(\pi^3+\pi^2S-\pi^{\frac{3}{2}}S^{\frac{3}{2}}\alpha\Omega^2-\pi S^2(1-\Omega^2)-S^3(1-\Omega^2))}{4\pi^{\frac{3}{2}}(\pi+S)^{\frac{3}{2}}\,(\pi+S-S\Omega^2)^{1/2}}. \tag{3.42}$$

In figure 7, we plot the Gibss free energy as a function of the entropy in order to get more information about the effect of DE on the black hole phase transitions. From this figure, we remark that the Gibbs free energy changes its sign. The positive values of $G^{(-2/3)}(S,\alpha,\Omega)$ correspond to unstable black holes, while the negative values are associated with stable black holes. We also realise that for the $(-\frac{2}{3})$-model, there is no impact of intensity α on the Gibbs free energy. This could be understood from the fact that the α multiplication coefficient is very small with respect to the ordinary case. For $\Omega=0.3$, its numerical value is arround 10^{-4} which could be ignored. Moreover, the Hawking-Page transition corresponds to the point where the Gibbs free energy changes

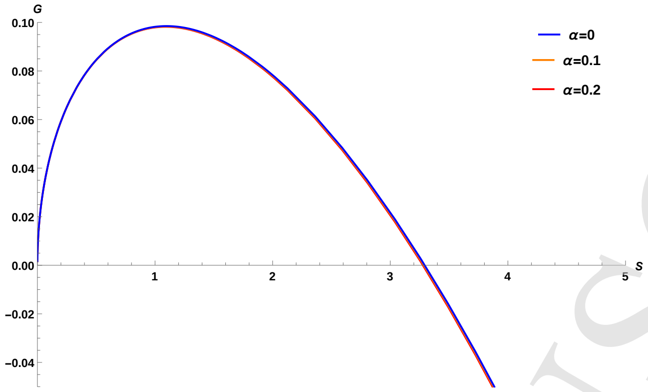

Figure 7: The Gibbs free energy $G^{(-2/3)}$ plotted for Kerr-AdS black hole with respect to the entropy S for $\Omega = 0.3$, $\omega = -2/3$.

its sign. The associated entropy reads as

$$S_{HP}^{(-2/3)}(\Omega,\alpha) = \frac{\sqrt{\pi}\left(2^{\frac{1}{3}}z^2 - \sqrt{\pi}(1-\Omega^2)(2z + 9\times 2^{\frac{2}{3}}\,\alpha\,\Omega^2) + 2\times 2^{\frac{2}{3}}\pi(4-\Omega^2)(1-\Omega^2)\right)}{6z(1-\Omega^2)}, \tag{3.43}$$

where we have

$$z(\alpha,\Omega) = \Big(27\pi(1+\pi)\,\alpha\,\Omega^2(1-2\Omega^2) + 4\pi^{\frac{3}{2}}(1-\Omega^2)^2(8+\Omega^2) \\ -6\sqrt{6}\,\Omega\,\pi^{\frac{3}{4}}\sqrt{\alpha(8-31\Omega^2+\pi(24-87\Omega^2)) + 2\pi^{\frac{2}{3}}\,\Omega^2(1-\Omega^2)^3}\Big)^{\frac{1}{3}}. \tag{3.44}$$

This entropy value leads to the following temperature

$$T_{HP}^{(-2/3)}(\Omega,\alpha) = \frac{1}{4\sqrt{6}\pi^{5/2}\sqrt{d(d+6\sqrt{\pi}-d\Omega^2)}(d+6\sqrt{\pi})^{3/2}}\times \\ \Big(216\pi^{3/2} - 72\sqrt{6d}\,\pi^{5/4}\,\alpha + 3d^3\,(1-\Omega^2) - 2\sqrt{6}\,d^{5/2}\,\pi^{1/4}\,\alpha\,(1-\Omega^2) \\ +36d\,\pi\,(5-2\Omega^2) - 6\sqrt{6}\,d^{3/2}\,\pi^{3/4}\,\alpha(4-3\Omega^2) + 6d^2\sqrt{\pi}(7-5\Omega^2)\Big), \tag{3.45}$$

where d is given by

$$d(\alpha,\Omega) = \frac{\left(2^{1/3}z^2 - \sqrt{\pi}(1-\Omega^2)(2z + 9\times 2^{2/3}\,\alpha\,\Omega^2) + 2\times 2^{2/3}\pi(4-\Omega^2)(1-\Omega^2)\right)}{z\,(1-\Omega^2)}. \tag{3.46}$$

To compare the different phases for several values of DE intensity α, we plot the Gibbs free energy as a function of temperature in figure 8.

From figure 8, we observe that the radiation phase, which is the phase where the black hole decay into a pure thermal AdS space, and the phase where no black hole can

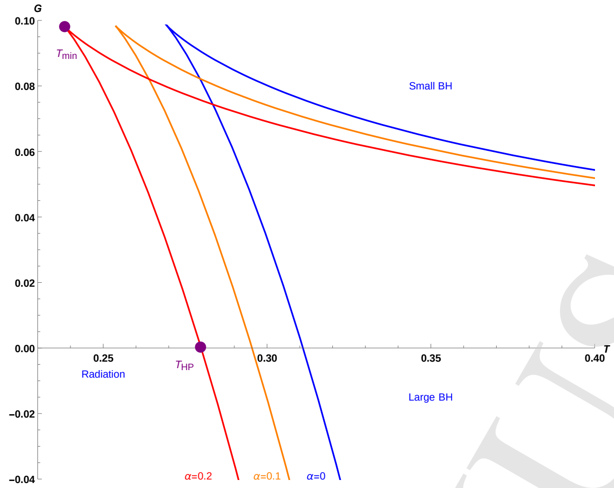

Figure 8: The Gibbs free energy $(G^{(-2/3)})$ plotted for Kerr-AdS black hole with respect to the temperature $(T^{(-2/3)})$ for $\Omega = 0.3$, $\omega = -2/3$ and different values of α.

exist decrease as the parameter α increases. Moreover, it is observed that the stable and the unstable phase are not affected by the DE contributions.
To examine DE effects on the heat capacity (3.32), we plot such a quantity as a function of the temperature for different DE intensity values in figure 9.

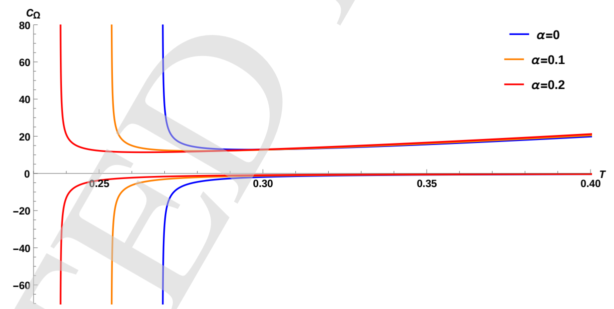

Figure 9: The specific heat $C_\Omega^{(-2/3)}$ plotted for Kerr-AdS black hole with respect to the temperature $(T^{(-2/3)})$ for $\Omega = 0.3$, $\omega = -2/3$ and different values of α.

It follows from figure 9 that the heat capacity changes its sign at the minimal temperature $T_{min}^{(-2/3)}$. We also find that the phase where no black hole can exist decreases when the intensity α increases. For the $(-\frac{2}{3})$-model, we find that DE does not have any affect on the relevant thermodynamical quantities. However, the phase where no black hole can exist and the radiation phase become small.

3.3 Model with DE state parameter $\omega = -1$

Here, we study the (-1)-model, being the last one. A fast examination reveals that this model can be associated with a possible cosmological scaling. Forgetting about other terms, the metric relevant one is given by

$$\Delta_\alpha^{AdS} = \Delta_\alpha + \Delta^{AdS} = -r^4\left(\frac{\Lambda}{3}+\alpha\right) = -r^4\Lambda_{eff}. \quad (3.47)$$

Analysing the sign of such a cosmological effective constant Λ_{eff}, we expect a reversed DE effect. We believe that the sign will be crucial in the forth coming discussions even by absorbing the cosmological constant in the previous scaling given in (2.8). To discuss the properties of this model, we should find the generalised Smarr formula for such a four dimensional Kerr-AdS black hole model. Using the expressions (2.4), (2.6) and solving the constraint $\Delta_r(r_+) = 0$, we get

$$M^{(-1)}(S,J,\alpha) = \left(\frac{\pi}{4S}\left\{\frac{4SJ^2}{\pi\ell^2} + 4J^2 + \left[\frac{S^2}{\pi^2\ell^2} + \frac{S}{\pi} - \alpha\frac{S^2}{\pi^2}\right]^2\right\}\right)^{\frac{1}{2}}. \quad (3.48)$$

In terms of DE intensity, the thermal temperature of the four-dimensional Kerr-AdS black holes reads as

$$T^{(-1)}(S,J,\alpha) = \frac{1}{8\pi M}\left(1 - \frac{4\pi^2J^2}{S^2} + \frac{4S}{\pi} + \frac{3S^2}{\pi^2} - \alpha\left(\frac{4S}{\pi} + \frac{6S^2}{\pi^2}\right) + \frac{3\alpha^2S^2}{\pi^2}\right). \quad (3.49)$$

Using the previous scaling redefinitions, the mass square takes the following form

$$\left(M^{(-1)}(S,J,\alpha)\right)^2 = \frac{S}{4\pi} + \frac{\pi J^2}{S} + J^2 + \frac{S}{2\pi}\left(\frac{S^2}{2\pi^2} + \frac{S}{\pi} + \frac{\alpha^2S^2}{2\pi^2} - \alpha\frac{S}{\pi}\left(\frac{S}{\pi}+1\right)\right). \quad (3.50)$$

Substituting (2.12) in (3.50), we can express $\left(M^{(-1)}\right)^2$ in terms of S and Ω. This yields

$$\left(M^{(-1)}(S,\Omega,\alpha)\right)^2 = \frac{S}{4\pi}(\pi+S)\frac{\left(1 + \frac{2S}{\pi}(1 + \frac{S}{2\pi} + \frac{\alpha^2 S}{2\pi} - \alpha(\frac{S}{\pi}+1))\right)}{\pi + S - \Omega^2 S}. \quad (3.51)$$

Similarly, the semi-classical temperature is given by

$$T^{(-1)}(S,\Omega,\alpha) = \left[\frac{\pi^2 + 3S^2 + 4S\pi + 3S^2\alpha^2 - \alpha S(4\pi+6S) - \frac{S\pi\Omega^2(\pi+S-S\alpha)^2}{(\pi+S)\,(\pi+S-S\Omega^2)}}{4\pi^{\frac{3}{2}}\sqrt{\frac{S(\pi+S)(\pi+S-S\,\alpha)^2}{(\pi+S-S\Omega^2)}}}\right]. \quad (3.52)$$

It is noted that we recover the ordinary temperature obtained in (2.15) by sending α to zero. To get further insight into the thermodynamical behaviour of the black hole, we can compute the semi-classical specific heat at constant angular velocity. Concretely, we find

$$C_\Omega^{(-1)}(S,\Omega,\alpha) = \frac{2S(\pi+S)^2(\pi+S-S\Omega^2)\left((\pi+S)\left(\pi^2-2\pi S(\Omega^2-2)-3S^2(\Omega^2-1)\right)+\alpha S\beta_5\right)}{(\pi+S)^2((\pi+S)^3(3S-\pi)-6S^2\Omega^2(\pi+S)^2+S^3\Omega^4(4\pi+3S))+S\alpha\beta_6}, \tag{3.53}$$

where the β_i parameters are

$$\beta_5 = \left(\frac{\pi S\Omega^2(2\pi - S(-2+\alpha))}{(\pi+S)} + (\pi+S-S\Omega^2)(-4\pi+3S(-2+\alpha))\right), \tag{3.54}$$

$$\begin{aligned}\beta_6 = {} & (-2\pi^5+3S^5(-2+\alpha)(-1+\Omega^2)^2-4\pi^3S^2(-1+\Omega^2)(-9+3\alpha^2+2\Omega^2)\\ & +\pi^4S(-14+3\alpha+12\Omega^2)+2\pi^2S^3(-1+\Omega^2)(22-10\Omega^2+\alpha(-9+4\Omega^2))\\ & +2\pi S^4(-1+\Omega^2)(13-9\Omega^2+\alpha(-6+4\Omega^2))).\end{aligned} \tag{3.55}$$

To study the black hole phase transitions, we use the standard thermodynamical tools. Indeed, we numerically examine the previous thermodynamical quantities, given in (3.52) and (3.53), with respect to the entropy S. These quantities can be illustrated in figure 10.

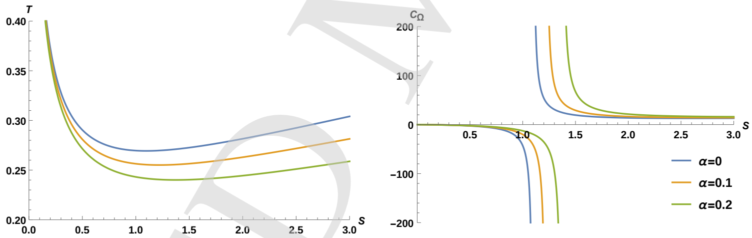

Figure 10: The semi-classical Hawking temperature ($T^{(-1)}$) and the specific heat ($C_\Omega^{(-1)}$) in terms of entropy (S) for $\Omega = 0.3$ and $\omega = -1$.

To interpret this numerical result, we should compute the minimal temperature. Indeed, we use the same steps exploited in the previous (ω)-models. In this way, the approximated entropy, which corresponds to the minimal temperature $T_{min}^{(-1)}$, can be written as

$$S_{min}^{(-1)}(\Omega,\alpha) = \frac{1}{6}\left[3\,\gamma_9 - \frac{2\,\pi\,(-2+\Omega^2)}{(-1+\Omega^2)} + 3\sqrt{\gamma_{10} - \frac{8\,\pi^3(2-3\,\Omega^2-3\,\Omega^4+2\Omega^6)}{27\gamma_9\,(-1+\Omega^2)^3}}\,\right], \tag{3.56}$$

where the γ_i parameters are given by

$$\gamma_8 = \Big((-1+\Omega^2)^2(\sqrt{27\,\alpha^3\,(-1+\Omega^2)^2 + (\Omega^4+3\,\alpha(1-\Omega^2+\Omega^4))^2} - (\Omega^4+3\,\alpha(1-\Omega^2+\Omega^4))\,)\Big)^{1/3}, \tag{3.57}$$

$$\gamma_9 = \sqrt{\frac{2\,\pi^2\,\left(3\,\gamma_8^2 - 9\,\alpha\,(-1+\Omega^2)^2 + 2\,\gamma_8\,(1-\Omega^2+\Omega^4)\,\right)}{9\,\gamma_8\,(-1+\Omega^2)^2}}, \tag{3.58}$$

$$\gamma_{10} = \frac{2\,\pi^2\,\left(-3\,\gamma_8^2 + 9\,\alpha\,(-1+\Omega^2)^2 + 4\,\gamma_8\,(1-\Omega^2+\Omega^4)\,\right)}{9\,\gamma_8\,(-1+\Omega^2)^2}. \tag{3.59}$$

Putting (3.56) in (3.52), we obtain the expression of the minimal temperature

$$T_{min}^{(-1)}(\Omega,\alpha) = \left[\frac{\sqrt{\frac{3}{2}}\,\left(12+8\,\chi+\chi^2-2\chi(4+\chi)\,\alpha+\chi^2\,\alpha^2+\frac{2\,\chi\,(-6+\chi\,(-1+\alpha)\,)^2\,\Omega^2}{(6+\chi)\,(-6+\chi(-1+\Omega^2)\,)}\right)}{4\,\pi\,\sqrt{-\frac{\chi\,(6+\chi)\,(-6+\chi(-1+\alpha)\,)^2}{(-6+\chi(-1+\Omega^2)\,)}}}\right], \tag{3.60}$$

where we have

$$\chi(\Omega,\alpha) = \frac{1}{\pi}\left[3\,\gamma_9 - \frac{2\,\pi\,(-2+\Omega^2)}{(-1+\Omega^2)} + 3\sqrt{\gamma_{10} - \frac{8\,\pi^3(2-3\,\Omega^2-3\,\Omega^4+2\Omega^6)}{27\gamma_9\,(-1+\Omega^2)^3}}\right]. \tag{3.61}$$

It is possible now to interpret the result given in the figure 10. The curve of the $(T-S)$ plane reveals two branches separated by the minimal temperature given in (3.60). The first branch associated with the small black hole mass is an unstable one. While, the second phase which corresponds to the large black hole mass is considered as a stable one. However, we remark that the entropy $S_{min}^{(-1)}$ increases when the intensity α increases. In this way, DE makes the unstable phase relevant.

From the curve of the $C_\Omega - S$ plane, the enhancement of the critical entropy $S_c^{(-1)} = S_{min}^{(-1)}$, when the intensity α increases makes the unstable phase larger. Indeed, the stable phase with a positive specific heat is getting smaller.

To investigate the associated phase transitions, we can exploit the Gibbs free energy in terms of the entropy. Concretely, the calculation provides

$$G^{(-1)}(S,\Omega,\alpha) = \frac{\sqrt{S}(\pi^3+\pi^2 S(1+\alpha)-\pi S^2(1-2\alpha)(1-\Omega^2)-S^3(1-\alpha)(1-\Omega^2))}{4\pi^{\frac{3}{2}}(\pi+S)^{\frac{3}{2}}\,(\pi+S-S\Omega^2)^{1/2}}, \tag{3.62}$$

which is illustrated in figure 11 for certain values of α.

It follows from the figure 11 that the Gibbs free energy changes its sign. The positive values of $G^{(-1)}(S,\Omega,\alpha)$ correspond to unstable black holes while the negative ones

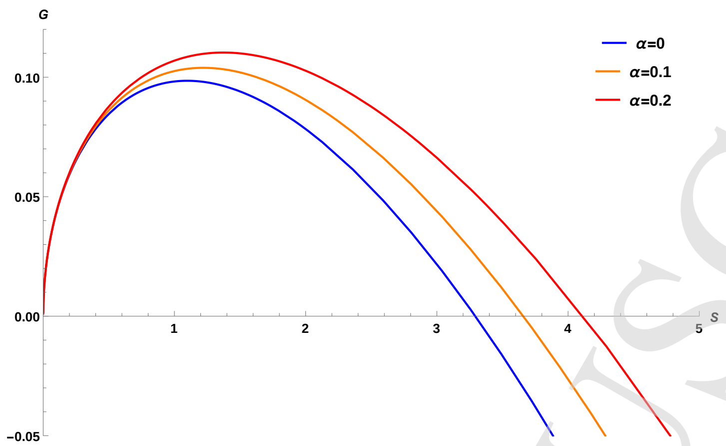

Figure 11: The Gibbs free energy ($G^{(-1)}$) plotted for Kerr-AdS black hole with respect to the entropy (S) for $\Omega = 0.3$, $\omega = -1$ and different values of α.

are associated with stable phases. We observe that DE increases the unstable phase. Moreover, the point where the Gibbs free energy changes its sign corresponds to the Hawking-Page phase transition. At this point, the entropy takes the form

$$S_{HP}^{(-1)} = \frac{\left(2^{2/3}X^2 - 2\pi X(1-2\alpha)(1-\Omega^2) - 2\times 2^{1/3}\pi^2(1-\Omega^2)(-4+3\alpha^2+\Omega^2+4\alpha(1-\Omega^2))\right)}{6X\,(1-\alpha)(1-\Omega^2)}, \tag{3.63}$$

where the quantity X is

$$X(\Omega,\alpha) = \pi\left(2(1-\Omega^2)^2(8+\Omega^2-6\alpha\,(2+\Omega^2))\right)^{1/3}. \tag{3.64}$$

This entropy generates the following temperature

$$T_{HP}^{(-1)}(\alpha,\Omega) = \frac{1}{4\pi^{3/2}\sqrt{2Y\left(2\pi+\frac{Y}{3(1-Y)}\right)}\,(Y+6\pi(1-\alpha)(1-\Omega^2)\,)^{3/2}}$$
$$\times\left[Y^3+72\pi^3(1-\alpha)^2(1-\Omega^2)^2+12a\,\pi^2(1-\alpha)(1-\Omega^2)(5-3\alpha-2\Omega^2)+2a^2\,\pi\,(7-5\Omega^2-\alpha(6-4\Omega^2))\right], \tag{3.65}$$

where the term Y is given by

$$Y(\alpha,\Omega) = \frac{\left(2^{2/3}X^2 - 2\pi X(1-2\alpha)(1-\Omega^2) - 2\times 2^{1/3}\,\pi^2\,(1-\Omega^2)(-4+3\alpha^2+\Omega^2-4\alpha\,(-1+\Omega^2))\right)}{X}. \tag{3.66}$$

To compare the different phases for several values of DE intensity α, we plot the Gibbs free energy as a function of the temperature in figure 12.

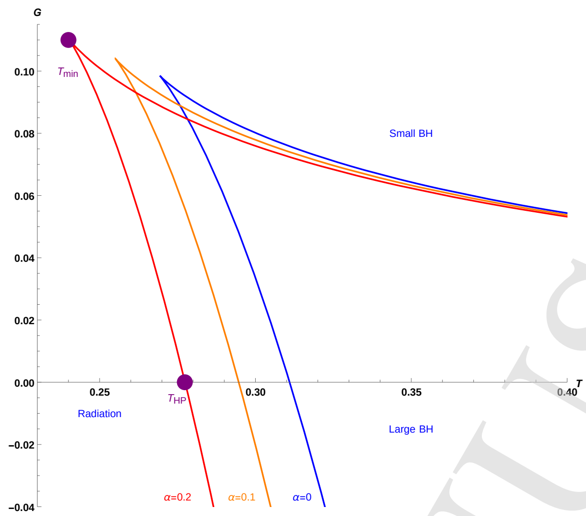

Figure 12: The Gibbs free energy ($G^{(-1)}$) plotted for Kerr-AdS black hole with respect to the temperature ($T^{(-1)}$) for $\Omega = 0.3$, $\omega = -1$ and different values of α.

It follows from figure 12 that the radiation phase, which is the phase where the black hole decay into a pure thermal AdS space, and the phase where no black hole can exist decrease as the parameter α increases. This can confirm that DE destabilise the (-1)-model black hole.
In order to more understand the DE effect on the heat capacity $C_\Omega^{(-1)}$, we illustrate in figure 13 its temperature dependence. An examination shows that the heat capacity changes its sign at the minimal temperature $T_{min}^{(-1)}$. We also find that the phase where no black hole can exist decreases when the intensity α increases.

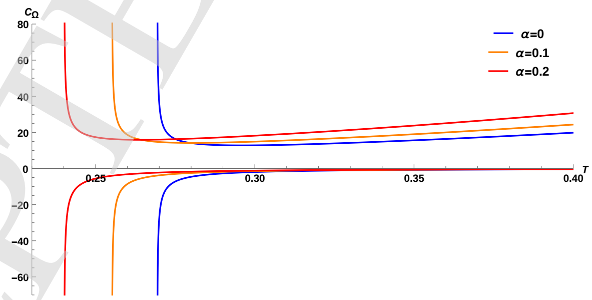

Figure 13: The specific heat $C_\Omega^{(-1)}$ plotted for Kerr-AdS black hole with respect to temperature ($T^{(-1)}$) for $\Omega = 0.3$, $\omega = -1$ and different values of α.

For this last model ((-1)-model), the obtained result of Kerr-AdS black holes seem to be non trivial. As expected, the Gibbs free energy and the heat capacity undergo a reversed DE behaviors, which could be associated with the sign of the DE factor. This feature could be in fact due to the cosmological constant contribution. This remark may deserve deeper thinking in future.

4 Results and discussions

In this section, we analyse and discuss the obtained results by giving certain proper physical meanings.

4.1 Thermal behaviors

We first start by the temperature behaviours in the studied models. Indeed, we plot the temperature as a function of the entropy for such three models in figure 14.

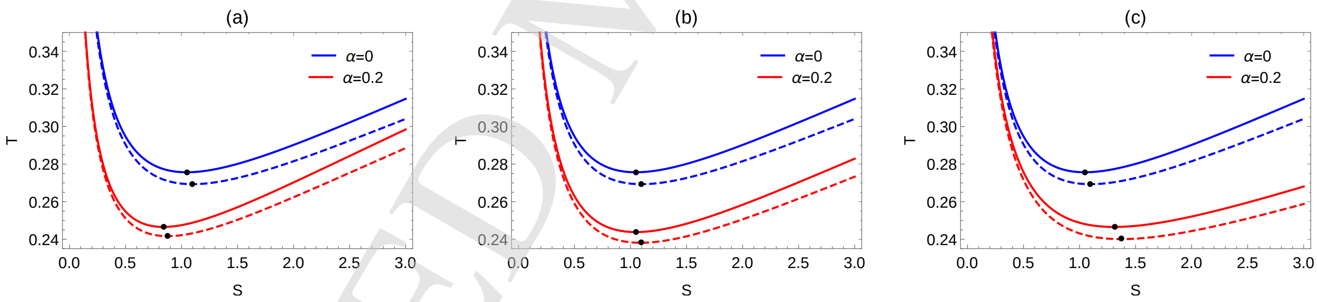

Figure 14: The temperature T plotted with respect to the entropy S for $\Omega = 0$ (straight line) and $\Omega = 0.3$ (dashed line).(a): $(T - S)$ for $\omega = -\frac{1}{3}$. (b): $(T - S)$ for $\omega = -\frac{2}{3}$. (c): $(T - S)$ for $\omega = -1$. The dots in the figure denotes the minimal entropy $S^{(\omega)}_{min}$.

It follows from this figure that we observe that the angular velocity Ω decreases the temperature and increases the unstable phase for all the cases ($\alpha = 0, \alpha \neq 0$). Besides, we remark a special behavior regarding the minimal entropy. In figure (a), we notice that the minimal entropy $S^{(-1/3)}_{min}$ decreases as

a function of DE intensity α. In figure (b), however, $S_{min}^{(-2/3)}$ has the same values appearing in the absence of DE. Furthermore, the figure (c) shows an opposite behavior to the one illustrated in figure (a). A close examination reveals that the Gibbs free energy G exhibits a similar behavior, being presented in figure 15. For the $(-\frac{1}{3})$-model illustrated in figure(d), the Gibbs free energy G

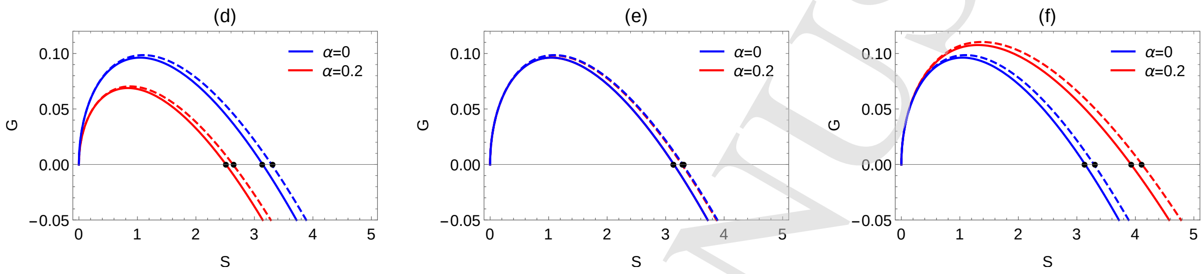

Figure 15: The Gibbs free energy G plotted with respect to the entropy S for $\Omega = 0$ (straight line) and $\Omega = 0.3$ (dashed line).(a): $(G - S)$ for $\omega = -\frac{1}{3}$. (b): $(G - S)$ for $\omega = -\frac{2}{3}$. (c): $(G - S)$ for $\omega = -1$. The dots in the figure denotes the Hawking-Page entropy $S_{HP}^{(\omega)}$.

decreases when the DE intensity α increases. However, the (-1)-model of figure (f) involves an opposite behavior. Besides, we remark that DE does not have any effect on the Gibbs free energy for $(-\frac{2}{3})$-model shown in figure (e). This matches perfectly with result associated with the Schwarzschild AdS black hole $(\Omega = 0)$, being characterised by the straight line in figure 15. This can be supported by the expression of G. Indeed, we give the Gibbs free energy for Schwarzschild AdS black hole

$$G^{(-1/3)}(S, \alpha) = G_0 - \frac{\alpha}{4}\sqrt{\frac{S}{\pi}},$$
$$G^{(-2/3)}(S, \alpha) = G_0,$$
$$G^{(-1)}(S, \alpha) = G_0 + \frac{\alpha}{4}\left(\frac{S}{\pi}\right)^{3/2},$$

where $G_0 = \frac{\sqrt{S}(\pi - S)}{4\pi^{3/2}}$ indicates the ordinary Gibbs free energy. Physically, this behavior of G and $S_{min}^{(\omega)}$ values can be understood from the fact that these

physical quantities ares considered as extensive parameters. It is worth noting that the DE intensity α decreases always the Hawking-Page temperature (T_{HP}), while ω increases the T_{HP} when $-1 \leq \omega \leq -\frac{2}{3}$. However, T_{HP} is decreased when $-\frac{2}{3} \leq \omega \leq -\frac{1}{3}$.

4.2 Universality of certain critical quantities

Here, we discuss the obtained results for the three (ω)-models showing relevant similarities and distinctions. However, we focus on some critical thermodynamical quantities of four-dimensional Kerr-AdS black holes by probing possible universal relations. To show that, we list first the corresponding involved quantities for certain values of DE parameters for a fixed value of Ω i.e 0.3. For simplicity reasons, we vary α between 0.05 and 0.2 by assuming that other values bring similar behaviors. For (-1)-model, we collect the associated numerical results in table 1.

	S_{min}	T_{min}	S_{HP}	T_{HP}
$\alpha = 0.05$	1.1556	0.2623	3.4656	0.3029
$\alpha = 0.1$	1.2200	0.2551	3.6587	0.2946
$\alpha = 0.15$	1.2919	0.2477	3.8273	0.2853
$\alpha = 0.2$	1.3726	0.2400	4.1142	0.2774

Table 1: Critical values for the (-1)-model.

The relevant thermodynamical quantities of ($-\frac{2}{3}$)-model are listed in table 2.

	S_{min}	T_{min}	S_{HP}	T_{HP}
$\alpha = 0.05$	1.0959	0.2615	3.2917	0.3032
$\alpha = 0.1$	1.0942	0.2537	3.2855	0.2954
$\alpha = 0.15$	1.0925	0.2459	3.2840	0.2877
$\alpha = 0.2$	1.0908	0.2381	3.2777	0.2798

Table 2: Critical values for the ($-\frac{2}{3}$)-model.

The results of ($-\frac{1}{3}$)-model are given in table 3.

In all three (ω)-models, we first remark that there exist nice universal ratio quantities for four-dimensional Kerr-AdS black holes regarding the entropy. In particular,

	S_{min}	T_{min}	S_{HP}	T_{HP}
$\alpha = 0.05$	1.0423	0.2627	3.1300	0.3033
$\alpha = 0.1$	0.9870	0.2559	2.9635	0.2953
$\alpha = 0.15$	0.9316	0.2488	2.7975	0.2872
$\alpha = 0.2$	0.8762	0.2416	2.6330	0.2788

Table 3: Critical values for the $(-\frac{1}{3})$-model.

we consider the ratio $\frac{S^{(\alpha)}_{min}}{S^{(\alpha)}_{HP}}$. For the (-1)-model ($(-\frac{1}{3})$-model), we observe that when α increases, the quantity $S^{(\alpha)}_{min}$ and $S^{(\alpha)}_{HP}$ increase (decreases). However, for $(-\frac{2}{3})$-model, the $S^{(\alpha)}_{min}$ and $S^{(\alpha)}_{HP}$ keep approximatively the same values which are arround 1.09 and 3.28, respectively. Even the relevant distinctions of these (ω)-models, the ratio $\frac{S^{(\alpha)}_{min}}{S^{(\alpha)}_{HP}}$ remains almost unchanged. Indeed, it is given by

$$\frac{S^{(\alpha)}_{min}}{S^{(\alpha)}_{HP}} \approx 0.33. \tag{4.1}$$

Similarly for the temperature, we compute numerically the ratio $\frac{T^{(\alpha)}_{min}}{T^{(\alpha)}_{HP}}$ for all the studied (ω)-models. Inspecting such calculations, we find the second universal relation given by

$$\frac{T^{(\alpha)}_{min}}{T^{(\alpha)}_{HP}} \approx 0.86. \tag{4.2}$$

It has been remarked that two critical ratios are independent of DE parameters. These two different values could be explained by physical properties of the involved thermodynamical quantities. We believe that this nice feature needs deeper investigations. This will be investigated in future works.

5 Behaviours of shadow Kerr AdS black holes surrounded by a quintessential field

In this section, we investigate the Kerr-AdS black hole shadow for the three (ω)-models. To start, we consider the Lagrangian given by

$$\mathcal{L} = \frac{1}{2}\, g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \tag{5.1}$$

where the over dot is the derivative with respect to the affine parameter Γ, while $g_{\mu\nu}$ is the metric tensor [44]. In this way, the canonical conjugated

momentum for such a Kerr-AdS black hole surrounded by quintessential dark energy can be computed from the following relation

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = g_{\alpha\,\mu}\dot{x}^\mu. \tag{5.2}$$

Indeed, this equation generates

$$p_t = g_{t\,t}\dot{t} + g_{t\,\phi}\dot{\phi} = -E, \tag{5.3}$$

$$p_\phi = g_{\phi\,t}\dot{t} + g_{\phi\,\phi}\dot{\phi} = L, \tag{5.4}$$

where E and L represent the energy and the angular momentum, respectively. The associated metric components are given by

$$g_{t\,t} = \frac{a^2\Delta_\theta \sin^2\theta - \Delta_r}{\Xi^2\,\Sigma^2}, \tag{5.5}$$

$$g_{\phi\,t} = g_{t\,\phi} = \frac{a\sin^2\theta\left[\Delta_r - a^2\Delta_\theta\sin^2\theta\right]}{\Xi^2\,\Sigma^2}, \tag{5.6}$$

$$g_{\phi\,\phi} = \frac{\sin^2\theta\left[\left(a^2+r^2\right)\Delta_\theta - a^2\Delta_r\sin^2\theta\right]}{\Xi^2\,\Sigma^2}. \tag{5.7}$$

Multiplying (5.3) by $(-g_{\phi\,\phi})$, (5.4) by $(g_{t\,\phi})$ and grouping the two equations together, we obtain

$$\dot{t} = \frac{E g_{\phi\,\phi} + L g_{t\,\phi}}{\left(g_{t\,\phi}\right)^2 - g_{t\,t}g_{\phi\,\phi}}. \tag{5.8}$$

Replacing each metric component by its expression, we get

$$\Sigma^2\dot{t} = \Sigma^2\frac{d\,t}{d\Gamma} = \frac{\Xi^2 E\left[\Delta_\theta\left(r^2+a^2\right)\left(r^2+a^2-a\lambda\right) + a\Delta_r\left(\lambda - a\sin^2\theta\right)\right]}{\Delta_r\Delta_\theta}, \tag{5.9}$$

with $\lambda = L/E$. However, the second components of the light rays velocity read as

$$\dot{\phi} = -\frac{E g_{t\,\phi} + L g_{t\,t}}{\left(g_{t\,\phi}\right)^2 - g_{t\,t}g_{\phi\,\phi}}, \tag{5.10}$$

$$\Sigma^2\frac{d\,\phi}{d\Gamma} = \Xi^2 E\left[\frac{a\left(r^2+a^2\right) - a\lambda}{\Delta_r} + \frac{\lambda - a\sin^2\theta}{\sin^2\theta\Delta_\theta}\right]. \tag{5.11}$$

In order to get the remaining two components of the velocity light rays, one should exploit the Hamilton-Jacobi equation governing the geodesic motion in space-time. This equation can be formulated as follows

$$\frac{\partial S}{\partial \Gamma} = \mathcal{H} = -\frac{1}{2}g^{\mu\,\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}. \tag{5.12}$$

Concretely, the action is given by

$$S = \frac{1}{2}m_0^2\,\Gamma - E\,t + L\,\phi + S_r(r) + S_\theta(\theta), \tag{5.13}$$

where m_0 is the particle mass [45]. The contravariant metric tensor is

$$g^{\mu\nu} = \begin{pmatrix} \frac{\Xi^2\left[\Delta_r a^2\sin^2\theta - \Delta_\theta\left(a^2+r^2\right)^2\right]}{\Delta_r\Delta_\theta\Sigma^2} & 0 & 0 & -\frac{\Xi^2\left[a\Delta_\theta\left(r^2+a^2\right)-a\Delta_r\right]}{\Delta_r\Delta_\theta\Sigma^2} \\ 0 & \frac{\Delta_r}{\Sigma^2} & 0 & 0 \\ 0 & 0 & \frac{\Delta_\theta}{\Sigma^2} & 0 \\ -\frac{\Xi^2\left[a\Delta_\theta\left(r^2+a^2\right)-a\Delta_r\right]}{\Delta_r\Delta_\theta\Sigma^2} & 0 & 0 & \frac{\Xi^2\left[-a^2\Delta_\theta\sin^2\theta+\Delta_r\right]}{\Delta_r\Delta_\theta\Sigma^2\sin^2\theta} \end{pmatrix}. \tag{5.14}$$

The computation shows that the explicit form of the Hamilton-Jacobi equation takes the following form

$$\begin{aligned} m_0^2 = 2\frac{\partial S}{\partial \Gamma} = & \frac{\Xi^2}{\Sigma^2\Delta_r}\left[\left(r^2+a^2\right)\frac{\partial S}{\partial t} + a\frac{\partial S}{\partial \phi}\right]^2 \\ & - \frac{\Xi^2}{\Sigma^2\Delta_\theta\sin^2\theta}\left[a\sin^2\theta\,\frac{\partial S}{\partial t} + \frac{\partial S}{\partial \phi}\right]^2 \\ & - \frac{\Delta_r}{\Sigma^2}\left(\frac{\partial S}{\partial r}\right)^2 - \frac{\Delta_\theta}{\Sigma^2}\left(\frac{\partial S}{\partial \theta}\right)^2. \end{aligned} \tag{5.15}$$

Using the notations $\frac{\partial S}{\partial t} = -E$, $\frac{\partial S}{\partial \phi} = L$ and considering photon particles with $m_0 = 0$, we obtain

$$\begin{aligned} 0 = & \left\{\Delta_r\left(\frac{\partial S}{\partial r}\right)^2 - \frac{\Xi^2}{\Delta_r}\left[\left(r^2+a^2\right)E - a\,L\right]^2 + \frac{\Xi^2}{\Delta_\theta}\left(L-aE\right)^2\right\} \\ & + \left\{\frac{\Delta_\theta}{\Sigma^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\Xi^2}{\Delta_\theta}\cos^2\theta\left[-E^2a^2 + \frac{L^2}{sin^2\theta}\right]\right\}. \end{aligned} \tag{5.16}$$

Factorizing these two equations, where the separating constant is $C = \mathcal{K} - \frac{\Xi^2}{\Delta_\theta}\left(L-aE\right)^2$, we get

$$\begin{cases} R = \Xi^2\left[\left(r^2+a^2\right)E - La\right]^2 - \mathcal{K}\Delta_r = E^2\left\{\Xi^2\left[\left(r^2+a^2\right) - \lambda a\right]^2 - k\Delta_r\right\}, \\ \Theta = \mathcal{K}\Delta_\theta - \Xi^2\left(\frac{L - aE\sin^2\theta}{\sin\theta}\right)^2 = E^2\left\{k\Delta_\theta - \Xi^2\left(\frac{\lambda - a\sin^2\theta}{\sin\theta}\right)^2\right\}, \end{cases} \tag{5.17}$$

where $\mathcal{K}$ is the carter constant and $k = \mathcal{K}/E^2$. The equations of motion expression for the photon around a rotating AdS black hole surrounded by

DE can be obtained. Precisely, they are expressed as follows

$$\Sigma^2 \frac{d\,t}{d\Gamma} = \frac{\Xi^2 E\left[\Delta_\theta\left(r^2+a^2\right)\left(r^2+a^2-a\lambda\right)+a\Delta_r\left(\lambda-a\sin^2\theta\right)\right]}{\Delta_r\Delta_\theta}, \tag{5.18}$$

$$\Sigma^2 \frac{d\,r}{d\Gamma} = \sqrt{R}, \tag{5.19}$$

$$\Sigma^2 \frac{d\,\theta}{d\Gamma} = \sqrt{\Theta}, \tag{5.20}$$

$$\Sigma^2 \frac{d\,\phi}{d\Gamma} = \Xi^2 E\left[\frac{a\left(r^2+a^2\right)-a\lambda}{\Delta_r}+\frac{\lambda-a\sin^2\theta}{\sin^2\theta\Delta_\theta}\right]. \tag{5.21}$$

To analyse the photon orbits, we should introduce two dimensionless impact parameters denoted as $\eta \equiv k$ and $\xi \equiv \lambda$. In this way, the unstable circular orbit can determine the boundary of black hole geometric shape using the equation given by

$$R = 0 = \frac{\partial R}{\partial r}. \tag{5.22}$$

After calculations, we find

$$\eta = \left.\frac{r^2\Xi^2\left(16a^2\Delta_\theta\Delta_r-16\Delta_r^2+8r\Delta_r\Delta_r'-r^2\left(\Delta_r'\right)^2\right)}{a^2\Delta_\theta\left(\Delta_r'\right)^2}\right|_{r=r_0}, \tag{5.23}$$

$$\xi = \left.\frac{-4r\Delta_r+\left(r^2+a^2\right)\Delta_r'}{a\Delta_r'}\right|_{r=r_0}, \tag{5.24}$$

where r_0 is the circular orbit radius of the photon and $\Delta_r' = \frac{\partial \Delta_r}{\partial r}$. Linking the characteristic length of the AdS space with the cosmological constant by the relation $\Lambda = -3/\ell^2$, we can elaborate the shadow of the Kerr-AdS black hole. Taking the limit $\Lambda = c = 0$,

$$\eta = \frac{r_0^3\left(4a^2M-r_0\left(3M-r_0\right)^2\right)}{a^2\left(r_0-M\right)^2}, \tag{5.25}$$

$$\xi = \frac{r_0^2\left(3M-r_0\right)-a^2\left(M+r_0\right)}{a\left(r_0-M\right)}, \tag{5.26}$$

we recover the results associated with Kerr black hole reported in [46]. The Schwarzschild black hole can be also obtained by taking the limit $a = 0$. For $M = 1$, the photon sphere can be described by the equation

$$\eta + \xi^2 = 27 \tag{5.27}$$

with $r_0 = 3$ [47]. In figure 16, the orbits are illustrated in terms of the associated celestial coordinates (α, β), reported in [46], for the three (ω)-models.

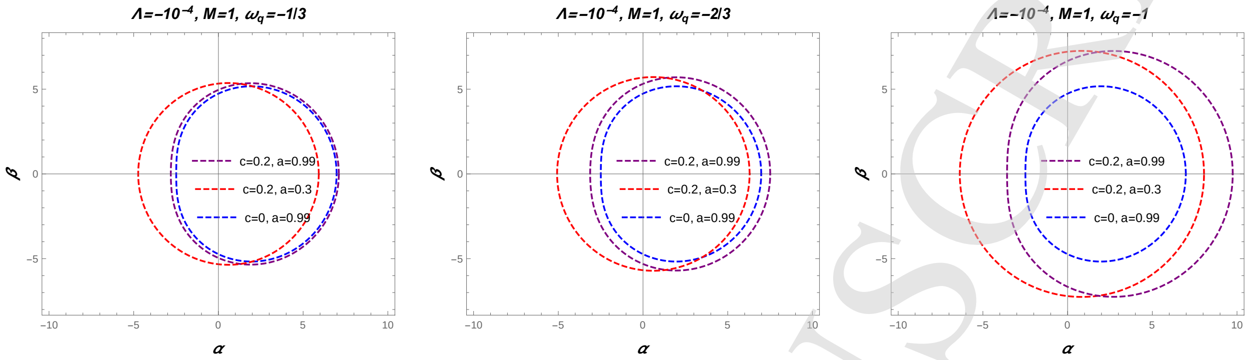

Figure 16: The shadow of Kerr-AdS black hole for (ω)-models in celestial coordinates (α, β). The ordinary case without DE is plotted with the blue color.

It follows from this figure that the decrease of ω implies the increase of the shadow. However, the later increases with high values of the DE intensity c. Besides, we notice that the D-shape of the shadow, which is a characteristic of the Kerr black hole, becomes relevant for the (-1)-model and gets even more deformed for high values of the DE intensity. The same behaviour is observed for the spherical shadow of Schwarzschild-AdS black hole [47]. Moreover, it has been remarked that when the value of ω decreases the size of the shadow gets bigger. We believe that such non trivial behaviors need deeper investigations. We hope to come back to these behaviors in connection with features in future works.

6 Conclusions and open questions

In this paper, we have studied the critical behaviors of four dimensional Kerr-AdS black holes from quintessential DE contributions. Exploiting a moduli space coordinated by the DE state parameter ω and the quintessence field intensity α, we have investigated three (ω)-models. In particular, we have computed relevant thermodynamical quantities needed to approach the corresponding phase transitions. Among others, we have shown that such models involve crucial similarities and distinctions regarding such quantities. For instance, we have found, in the $(-\frac{1}{3})$-model, that DE stabilise the Kerr-AdS black holes. However, the remaining two models have shown surprising behaviors. Indeed, the $(-\frac{2}{3})$-model has revealed a resistance to DE effects while (-1)-model has brought reversed DE effects. **Moreover, certain thermal behaviors have been discussed**. After an examination, we have remarked that the parameter ω can be put as

$$\omega = -1 + \varpi, \tag{6.1}$$

where ϖ is a new parameter restricted by $0 \leq \varpi \leq \frac{2}{3}$. This new parameter could control such distinctions. Starting from an anti-DE effect associated with $\varpi = 0$, the behaviors completely change by increasing the values of ϖ. Even such visible distinctions in the (ϖ)-models, we have observed similarities associated with universal properties in terms of four dimensional Kerr-AdS black hole critical quantities. **Besides, we have showed that the decrease (increase) of $\omega\,(\alpha)$ enhances the size of the black hole shadow**.
This work comes up with many open questions. The natural one concerns the understanding of the founded results, in terms of ϖ. Besides, higher dimensional models could be considered as a future work by investigating the effect of space-time dimensions on Kerr-AdS black hole behaviors in the presence of non trivial contributions including DE.

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