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Stable periodic orbits for the Mackey–Glass equation

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Abstract

We study the classical Mackey–Glass delay differential equation

$$x'(t) = -ax(t) + bf_n(x(t-1))$$

where a, b, n are positive reals, and $f_n(\xi) = \xi/[1+\xi^n]$ for $\xi \geq 0$. As a limiting ($n \to \infty$) case we also consider the discontinuous equation

$$x'(t) = -ax(t) + bf(x(t-1))$$

where $f(\xi) = \xi$ for $\xi \in [0, 1)$, $f(1) = 1/2$, and $f(\xi) = 0$ for $\xi > 1$. First, for certain parameter values $b > a > 0$, an orbitally asymptotically stable periodic orbit is constructed for the discontinuous equation. Then it is shown that for large values of n, and with the same parameters a, b, the Mackey–Glass equation also has an orbitally asymptotically stable periodic orbit near to the periodic orbit of the discontinuous equation.

Although the obtained periodic orbits are stable, their projections $\mathbb{R} \ni t \mapsto (x(t), (x(t-1))) \in \mathbb{R}^2$ can be complicated.

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Keywords: Mackey–Glass equation; Limiting nonlinearity; Periodic orbit; Return map; Computer–assisted proof

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1. Introduction

The Mackey–Glass equation

$$y'(t) = -ay(t) + b\frac{y(t-\tau)}{1+y^n(t-\tau)}$$

with positive parameters a, b, τ, n was introduced in 1977 by Michael Mackey and Leon Glass [28] as a model of feedback control of blood cells. This simple-looking differential equation with a single delay attracted the attention of many mathematicians since its hump-shaped nonlinearity causes entirely different dynamics compared to the case where the nonlinearity is monotone. See the work [25] of Lasota for a similar model. There exist several rigorous mathematical results, numerical and experimental studies on the Mackey-Glass equation showing convergence, oscillations of solutions, and complicated behavior, see e.g. [1,6,14,16,26,27,35]. Despite the intense research, the dynamics is not fully understood yet.

By rescaling the time we may assume $\tau = 1$. Therefore, we consider

$$y'(t) = -ay(t) + bf_n(y(t-1)) \tag{E_n}$$

where $a > 0$, $b > 0$, $n \geq 4$, and $f_n(\xi) = \xi/[1+\xi^n]$ for $\xi \geq 0$. The natural phase space to study (E_n) is $C^+ = C([-1,0],(0,\infty))$, see [8,10]. For each $\psi \in C^+$ there is a unique solution $y^{n,\psi} : [-1,\infty) \to (0,\infty)$ with $y^{n,\psi}(s) = \psi(s)$, $-1 \leq s \leq 0$. The solutions define the continuous semiflow $F^n : [0,\infty) \times C^+ \ni (t,\psi) \mapsto y_t^{n,\psi} \in C^+$, where $y_t^{n,\psi}(s) = y^{n,\psi}(t+s)$, $-1 \leq s \leq 0$.

If $a \geq b > 0$ then it is elementary to show that $y^\psi(t) \to 0$ as $t \to \infty$ for all $\psi \in C^+$. In the rest of the paper we assume $b > a > 0$. Then there is a global attractor $\mathcal{A} \subset C^+$, that is, $\mathcal{A} \subset C^+$ is compact and nonempty, $F^n(t,\mathcal{A}) = \mathcal{A}$ for all $t \geq 0$, and $\mathcal{A}$ attracts all bounded subsets of C^+.

The unique positive zero ζ_n^0 of $[0,\infty) \ni \xi \mapsto -a\xi + bf_n(\xi) \in \mathbb{R}$ defines the unique equilibrium point $\hat{\zeta}_n^0 \in C^+$ of F^n by $\hat{\zeta}_n^0(s) = \zeta_n^0$, $-1 \leq s \leq 0$. It is easy to see that, for fixed $b > a > 0$, there exists an $N(a,b) \geq 4$ so that $\hat{\zeta}_n^0$ is unstable for $n \geq N(a,b)$, and, as n increases, $\hat{\zeta}_n^0$ is a source of periodic orbits via local Hopf bifurcations, see e.g. [44]. On the other hand, the papers [15] by Karakostas et al. and [9] by Gopalsamy et al. give conditions for the global attractivity of the unique positive equilibrium of (E_n) for $b > a > 0$, and n is below a certain constant given in terms of a, b. Our result is valid for some $b > a > 0$ and n is large.

The maximum of f_n is at $\xi_n^0 = 1/\sqrt[n]{n-1}$. Assuming $n > b/(b-a)$, the inequality $\xi_n^0 < \zeta_n^0$ follows. Liz, Röst and Wu [26,35] gave conditions to guarantee $\mathcal{A} \subset \{\psi \in C^+ : \psi(s) > \xi_n^0\}$, that is $\mathcal{A}$ is in the region where the feedback is monotone decreasing. This means that the structure of $\mathcal{A}$ can be studied by using the Poincaré–Bendixson type theorem of Mallet-Paret and Sell [31], see also [17,41].

If $\xi_n^0 < \zeta_n^0$ (a consequence of $n > b/(b-a)$) then there is a unique $\zeta_n^1 \in (0,\xi_n^0)$ with $bf_n(\zeta_n^1) = a\zeta_n^0$, and $(\xi - \zeta_n^0)(-a\zeta_n^0 + bf_n(\xi)) < 0$ for $\xi \in (\zeta_n^1,\infty) \setminus \{\zeta_n^0\}$. Consequently, if $t > 0$, $y(t) = \zeta_n^0$ and $y(t-1) \in (\zeta_n^1, \zeta_n^0)$ then $y'(t) > 0$, and if $t > 0$, $y(t) = \zeta_n^0$ and $y(t-1) > \zeta_n^0$ then $y'(t) < 0$. This means a negative feedback condition in the region (ζ_n^1,∞) with respect to ζ_n^0. Then the inclusion $\mathcal{A} \subset \{\psi \in C^+ : \psi(s) > \zeta_n^1\}$ (which can be guaranteed by following [26,35]) allows to apply the results obtained for equations with negative feedback resulting in a Morse decomposition of $\mathcal{A}$, see [30,32]. In particular, periodic orbits and some connections between them can be obtained in this way. In addition, under the negative feedback condition complicated dynamics is possible, see [11,24]. The work of Lani-Wayda [23] shows chaos for an equation with a hump-shaped nonlinearity, similar to f_n, however, the result is not applicable for (E_n). A major problem in delay differential equations is to prove complicated dynamics for the Mackey–Glass equation (E_n).

We emphasize that, in general, for equation (E_n) the global attractor $\mathcal{A}$ is not in a region of C^+ where the negative feedback condition holds with respect to the positive equilibrium.

The aim of this paper is to construct periodic orbits for equation (E_n) for some parameter values a, b, n. The obtained periodic orbits are hyperbolic, orbitally stable, exponentially attractive with asymptotic phase. In the proof we use the limiting Mackey–Glass equation

$$x'(t) = -ax(t) + bf(x(t-1)) \tag{E_∞}$$

where $f(\xi) = \lim_{n\to\infty} f_n(\xi)$, that is, $f(\xi) = \xi$ for $\xi \in [0, 1)$, $f(1) = 1/2$, and $f(\xi) = 0$ for $\xi > 1$. Theorem 3.1 shows that if the parameters $b > a > 0$ are given so that the hypothesis

(H) equation (E_∞) has an ω-periodic solution $p : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $p(0) = 1$, $p(t) > 1$ for all $t \in [-1, 0)$,
(ii) $(p(t), p(t-1)) \neq (1, a/b)$ for all $t \in [0, \omega]$

holds then there exists an $n_* \geq 4$ such that, for all $n \geq n_*$, equation (E_n) has a periodic solution $p^n : \mathbb{R} \to \mathbb{R}$ with period $\omega^n > 0$ so that the periodic orbits $\mathcal{O}^n = \{p^n_t : t \in [0, \omega^n]\}$ are hyperbolic, orbitally stable, exponentially attractive with asymptotic phase, and $\omega^n \to \omega$, $\operatorname{dist}\{\mathcal{O}^n, \mathcal{O}\} \to 0$ as $n \to \infty$, where $\mathcal{O} = \{p_t : t \in [0, \omega]\}$.

In order to get periodic orbits for equation (E_n) by the application of Theorem 3.1, hypothesis (H) needs to be verified. We have two types of results in this direction: analytical and computer-assisted proofs. First, Proposition 4.1 shows that (H) is satisfied if b is large comparing to a, namely $b > \max\{ae^a, e^a - e^{-a}\}$. Another result, whose proof can be found in [18], is Proposition 4.2 when b is sufficiently close to a. More precisely, for every $a > 0$ there exists an $\varepsilon_0 = \varepsilon_0(a) > 0$ such that for the parameters a, b with $b \in (a, a + \varepsilon_0)$ condition (H) holds. If $a > 0$ is fixed and $b_k \to a+$ as $k \to \infty$ then the minimal period of the obtained periodic solution tends to ∞. The most interesting examples for a, b, such that (H) is valid, are obtained by applying rigorous computer-assisted techniques, see Proposition 4.5.

The results of this paper can be summarized as follows.

Theorem 1.1. *If the pair of parameters a, b satisfies either the condition of Proposition 4.1, or Proposition 4.2, or a and b are given in Proposition 4.5, then there exists an $n_* = n_*(a, b) \geq 4$ such that for all $n \geq n_*$ equation (E_n) has a hyperbolic, orbitally stable, exponentially attractive periodic orbit with asymptotic phase.*

The shapes of the obtained periodic solutions (depending on a, b, n) vary from simple looking slowly oscillating solutions (with respect to ζ^0_n) to solutions with complex structures, see the figures at the end of the paper. For some of the periodic solutions x guaranteed by Theorem 1.1 the projections $\mathbb{R} \ni t \mapsto (x(t), x(t-1)) \in \mathbb{R}^2$ produce complicated looking structures, and the figures are similar to those obtained in numerical studies and were believed to be a sign of chaotic dynamics generated by (E_n)

As the proofs are technical, we give a brief overview of the key steps. A solution of (E_∞) is defined as a continuous function $x : [-1, t_*) \to \mathbb{R}$ with $0 < t_* \le \infty$ such that the map $[0, t_*) \ni t \mapsto f(x(t-1)) \in \mathbb{R}$ is locally integrable and

$$x(t) = e^{-a(t-\tau)}x(\tau) + \int_\tau^t e^{-a(t-s)} f(x(s-1))\, ds$$

holds for all $0 \le \tau < t < t_*$. It is easy to see that for any initial function $\varphi \in C^+$ there is a unique solution $x = x^\varphi : [-1, \infty) \to (0, \infty)$ satisfying $x_0 = \varphi$. However, comparing solutions with $\varphi > 1$ and $\varphi \equiv 1$, one sees that there is no continuous dependence on initial data in C^+. By choosing

$$C_r^+ = \{\psi \in C^+ : \psi^{-1}(c) \text{ is finite for all } c \in (0, 1]\}$$

as a phase space, the solutions of (E_∞) define the continuous semiflow $[0, \infty) \ni (t, \varphi) \mapsto x_t^\varphi \in C_r^+$.

If $\varphi \in C_r^+$ and the solution $x = x^\varphi$ of (E_∞) satisfies the additional property

$$(x(t), x(t-1)) \neq \left(1, \frac{a}{b}\right) \tag{1.1}$$

for all t in a fixed interval $[0, M]$, then Proposition 2.6 shows that, for large n, the solution $y^{n,\psi}$, $\psi \in C^+$, of (E_n) remains close to x^φ on the interval $[0, M]$ provided they are close on the interval $[0, 1]$. Condition (1.1) guarantees that, if $x(t) = x^\varphi(t)$ is close to 1 and the derivative $x'(t)$ exists, then $x'(t) = -ax(t) + bf(x(t-1))$ is not close to zero, and this makes it possible to show that the measure of the set $\{t \in [-1, M-1] : |x^\varphi(t) - 1| < \delta\}$ is bounded by $K\delta$ for a fixed $K > 0$, and for all small $\delta > 0$. Therefore, for most of the times $t \in [0, M]$, one has $x^\varphi(t-1) \notin (1-\delta, 1+\delta)$, that is, $x^\varphi(t-1)$ is not close to the discontinuity point of f allowing the application of perturbation type arguments. Another key fact is that, for each $\delta \in (0, 1)$, $f_n(\xi) \to f(\xi)$ as $n \to \infty$ uniformly in $\xi \in [0, 1-\delta] \cup [1+\delta, \infty)$. This convergence ensures that, for large n, $\max_{t\in[0,1]} |x^\varphi(t) - y^{n,\psi}(t)|$ is small provided $\varphi(s) \ge 1+\delta$, $\psi(s) \ge 1+\delta$, $s \in [-1, 0]$. See Propositions 2.5 and 2.6 for the precise results.

Hypothesis (H) implies the existence of a small $\gamma > 0$ so that for a translate q of p one has $q_0 \in S_\gamma = \{\psi \in C^+ : \psi(s) \ge \psi(0) = 1+\gamma,\ s \in [-1, 0]\}$. Theorem 3.1 states that, for large n, the solution curves $[0, \infty) \ni t \mapsto y_t^{n,\psi} \in C^+$ of (E_n) starting in S_γ return to S_γ, that is, $y_t^{n,\psi} \in S_\gamma$ for some $t > 0$. A crucial property of the Mackey–Glass nonlinearity is that, for $n \ge 4$, the polynomial bound $\sup_{\xi>0} |f_n'(\xi)| \le n/4$ is valid, while for a fixed $\gamma > 0$, $\sup_{\xi\ge 1+\gamma} |f_n'(\xi)| \le n/[1+(1+\gamma)^n]$ tending to zero exponentially as $n \to \infty$. This guarantees that, for large n, the return map $S_\gamma \ni \psi \mapsto y_t^{n,\psi} \in S_\gamma$ is a contraction, and its fixed point determines a stable periodic orbit of (E_n).

The paper is organized as follows. Section 2 shows the basic properties of the solutions of equation (E_∞) in the phase space C_r^+. In addition, it gives the technical conditions ensuring that the solutions of (E_∞) with certain regularity properties are approximated on compact intervals by the solutions (E_n) for large n. Section 3 proves that, for large n, under hypothesis (H) equation (E_n) has a stable periodic orbit close to the orbit guaranteed by (H). Section 4 contains analytical and rigorous computer-assisted tools to find parameter values $b > a > 0$ with hypothesis (H).

The paper is concluded by figures demonstrating the variety of the obtained periodic orbits of the Mackey–Glass equation (E_n).

We remark that a well known and widely applied technique in the study of a delay differential equation of the form $y'(t) = -ay(t) + bg_n(y(t-1))$, with a parameter n, is to consider the limiting equation $x'(t) = -ax(t) + bg(x(t-1))$ with the assumption that the limiting nonlinearity $g(\xi) = \lim_{n\to\infty} g_n(\xi)$ is a step function with finite steps. The idea turned out to be very successful to prove a variety of dynamical properties, see the papers [4,11,19–21,34,36,40,42,43,45]. For example, the search for periodic orbits can be reduced to a finite dimensional problem in this way. For an equation with a piece-wise linear limiting nonlinearity g, like f in equation (E_∞), the search for periodic orbits is still an infinite dimensional problem. For a delay differential equation, different from (E_n), Mackey et al. [29] introduced the limiting nonlinearity f as in this paper, and constructed stable periodic orbits for the original equation. The technique of Mackey et al. [29] applies to equation (E_n) provided b is large comparing to a, and yields a periodic solution of (E_n) with a relatively simple shape. In the present paper the situation of [29] is covered by Proposition 4.1 (together with Theorem 3.1).

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2. Preliminary results

Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{N}_0$ denote the set of real numbers, complex numbers, positive integers, nonnegative integers, respectively. Let C be the Banach space $C([-1,0],\mathbb{R})$ equipped with the norm $\|\varphi\| = \max_{s\in[-1,0]} |\varphi(s)|$. For a continuous function $u : I \to \mathbb{R}$ defined on an interval I, and for $t, t-1 \in I$, $u_t \in C$ is given by $u_t(s) = u(t+s)$, $s \in [-1,0]$. Introduce the subsets

$$C^+ = \{\psi \in C : \psi(s) > 0 \text{ for all } s \in [-1,0]\},$$

$$C_r^+ = \left\{\psi \in C^+ : \psi^{-1}(c) \text{ is finite for all } c \in (0,1]\right\}$$

of C where $\psi^{-1}(c) = \{s \in [-1,0] : \psi(s) = c\}$. C^+ and C_r^+ are metric spaces with the metric $d(\varphi,\psi) = \|\varphi - \psi\|$. For a finite set S let $\#S$ denote the number of elements of the set S.

Let a, b, n be real positive parameters with $b > a > 0$ and $n \geq 4$. Define the function

$$f_n : [0,\infty) \ni \xi \mapsto \frac{\xi}{1+\xi^n} \in \mathbb{R},$$

and its limit when $n \to \infty$

$$f(\xi) = \lim_{n\to\infty} f_n(\xi) = \lim_{n\to\infty} \frac{\xi}{1+\xi^n} = \begin{cases} \xi & \text{if } 0 \le \xi < 1 \\ \frac{1}{2} & \text{if } \xi = 1 \\ 0 & \text{if } \xi > 1 \end{cases}$$

Set

$$\xi_n^0 = \frac{1}{\sqrt[n]{n-1}}, \qquad \xi_n^1 = \sqrt[n]{\frac{n+1}{n-1}}.$$

Then $0 < \xi_n^0 < 1 < \xi_n^1$, and $\xi_n^0,\ \xi_n^1$ are the only zeros of $f_n',\ f_n''$ in $(0, \infty)$, respectively.

Proposition 2.1. *Let* $n \ge 4$.

(i) For each $\varepsilon \in (0, 1)$, $f_n(\xi) \to f(\xi)$, *as* $n \to \infty$, *uniformly in* $\xi \in [0, \infty) \setminus (1-\varepsilon, 1+\varepsilon)$.
(ii) If $\xi > 0$ *then* $|f_n'(\xi)| \le n/4$.
(iii) If $\xi > 1$ *then* $|f_n'(\xi)| \le n/(1+\xi^n)$.

Proof. (i) If $0 \le \xi \le 1-\varepsilon$ then

$$|f_n(\xi) - f(\xi)| = \frac{\xi^{n+1}}{1+\xi^n} \le (1-\varepsilon)^{n+1}.$$

If $\xi \ge 1+\varepsilon$ then

$$|f_n(\xi) - f(\xi)| = f_n(\xi) = \frac{1}{1/\xi + \xi^{n-1}} \le (1+\varepsilon)^{1-n}.$$

(ii) From $f_n'(\xi) = [1-(n-1)\xi^n]/(1+\xi^n)^2$ it is easy to see that $f_n'(\xi) \in [0, 1)$ for $\xi \in (0, \xi_n^0]$, and $f_n'(\xi) < 0$ for $\xi > \xi_n^0$. From

$$f_n''(\xi) = \frac{n\xi^{n-1}[(n-1)\xi^n - (n+1)]}{(1+\xi^n)^3}$$

it follows that f_n' has a minimum at ξ_n^1. For the minimum value $f_n'(\xi_n^1) = -(n-1)^2/(4n) \ge -n/4$ holds. Therefore, $-n/4 \le f_n'(\xi) < 1$ for all $\xi > 0$. This proves (ii).

(iii) If $\xi > 1$ then

$$|f_n'(\xi)| = \frac{(n-1)\xi^n - 1}{(1+\xi^n)^2} < \frac{(n-1)\xi^n}{(1+\xi^n)^2} < \frac{n-1}{1+\xi^n} < \frac{n}{1+\xi^n}$$

and (iii) holds. □

Consider the equations

$$y'(t) = -ay(t) + bf_n(y(t-1)) \tag{E_n}$$

and

$$x'(t) = -ax(t) + bf(x(t-1)). \tag{E_∞}$$

A solution of equation (E_n) on $[-1,\infty)$ with initial function $\psi \in C^+$ is a continuous function $y : [-1,\infty) \to \mathbb{R}$ so that $y_0 = \psi$, the restriction $y|_{(0,\infty)}$ is differentiable, and equation (E_n) holds for all $t > 0$. The solutions are easily obtained from the variation-of-constants formula for ordinary differential equations on successive intervals of length one,

$$y(t) = e^{-a(t-k)} y(k) + b \int_k^t e^{-a(t-s)} f_n(y(s-1))\, ds, \tag{2.1}$$

where $k \in \mathbb{N}_0$, $k \le t \le k+1$. Hence one can find that each $\psi \in C^+$ uniquely determines a solution $y = y^{n,\psi} : [-1,\infty) \to \mathbb{R}$ with $y_0^{n,\psi} = \psi$, and $y^{n,\psi}(t) > 0$ for all $t \ge 0$. In addition, one sees that $y^{n,\psi}$ satisfies the integral equation

$$y^{n,\psi}(t) = e^{-a(t-\tau} y^{n,\psi}(\tau) + b \int_\tau^t e^{-a(t-s)} f(y^{n,\psi}(s-1))\, ds \qquad (0 \le \tau < t < \infty). \tag{2.2}$$

The solutions define the continuous semiflow

$$F^n : [0,\infty) \times C^+ \ni (t,\psi) \mapsto y_t^{n,\psi} \in C^+.$$

For equation (E_∞) with the discontinuous f, we use formula (2.1) with f instead of f_n to define solutions for (E_∞). A solution of equation (E_∞) with initial function $\varphi \in C^+$ is a continuous function $x = x^\varphi : [-1, t_\varphi) \to \mathbb{R}$ with some $0 < t_\varphi \le \infty$ such that $x_0 = \varphi$, the map $[0, t_\varphi) \ni s \mapsto f(x(s-1)) \in \mathbb{R}$ is locally integrable, and

$$x(t) = e^{-a(t-k)} x(k) + b \int_k^t e^{-a(t-s)} f(x(s-1))\, ds, \tag{2.3}$$

holds for all $k \in \mathbb{N}_0$ and $t \in [0, t_\varphi)$ with $k \le t \le k+1$.

It is not difficult to show that, for any $\varphi \in C^+$, there is a unique solution x^φ of equation (E_∞) on $[-1,\infty)$. However, comparing solutions with initial functions $\varphi > 1$, $\varphi \equiv 1$, one sees that there is no continuous dependence on initial data in C^+. Therefore we restrict our attention to the subset C_r^+ of C^+. The choice of C_r^+ as a phase space guarantees not only continuous dependence on initial data, but also allows to compare certain solutions of equations (E_∞) and (E_n) for large n.

Proposition 2.2 below shows that, for all $\varphi \in C_r^+$, equation (E_∞) has a unique solution x^φ on $[-1,\infty)$ with $x_t^\varphi \in C_r^+$ for all $t \ge 0$. Once we have the existence of $x^\varphi : [-1,\infty) \to \mathbb{R}$, it is elementary to obtain the integral equation

$$x^\varphi(t) = e^{-a(t-\tau)} x^\varphi(\tau) + b \int_\tau^t e^{-a(t-s)} f(x^\varphi(s-1))\, ds \qquad (0 \le \tau < t < \infty). \tag{2.4}$$

Proposition 2.2. *For each $\varphi \in C_r^+$ there is a unique maximal solution $x^\varphi : [-1, \infty) \to \mathbb{R}$ of equation (E_∞). The maximal solution x^φ satisfies:*

(i) $x_t^\varphi \in C_r^+$ *for all* $t \geq 0$,
(ii) if $t > 0$ *and* $x^\varphi(t-1) \neq 1$, *then* x^φ *is differentiable at* t, *and equation* (E_∞) *holds at* t.

The map

$$F : [0, \infty) \times C_r^+ \ni (t, \varphi) \mapsto x_t^\varphi \in C_r^+$$

is a continuous semiflow.

Proof. *Step 1.* Let $\varphi \in C_r^+$ be given. Then there exists a sequence $(s_l)_{l=0}^L$ so that $-1 \leq s_0 < s_1 < \ldots < s_L \leq 0$ and $\varphi^{-1}(1) = \{s_0, \ldots, s_L\}$. Set $J = \{t_0, \ldots, t_L\}$ with $t_l = s_l + 1$, $l \in \{0, \ldots, L\}$. The function $[0, 1] \ni s \mapsto f(\varphi(s-1)) \in [0, 1]$ is bounded, and continuous at all $\xi \in [0, 1] \setminus J$. Consequently, it is integrable on $[0, 1]$. It follows that the definition

$$x_0 = \varphi, \quad x(t) = e^{-at}x(0) + b\int_0^t e^{-a(t-s)} f(x(s-1))\, ds, \quad t \in [0, 1] \tag{2.5}$$

of $x : [-1, 1] \to \mathbb{R}$ gives a continuous function. Moreover, x is differentiable at each point of $(0, 1] \setminus J$, and equation (E_∞) holds for all $t \in (0, 1] \setminus J$

Step 2. Assume $x_t \in C^+$ is not satisfied for all $t \in [0, 1]$. Then, by $x_0 = \varphi \in C^+$, there is a minimal $t_* \in (0, 1]$ with $x(t_*) = 0$. As J is a finite set, there is an $\varepsilon > 0$ so that $[t_* - \varepsilon, t_*) \cap J = \emptyset$, and ($E_\infty$) holds for all $t \in (t_* - \varepsilon, t_*)$. Clearly,

$$x'(t) = -ax(t) + bf(x(t-1)) \geq -ax(t) \quad \text{for all } t \in (t_* - \varepsilon, t_*),$$

and it follows that

$$x(t) \geq x(t_* - \varepsilon)e^{-a(t-t_*+\varepsilon)} \geq x(t_* - \varepsilon)e^{-a\varepsilon} \quad \text{for all } t \in (t_* - \varepsilon, t_*).$$

Hence, by the continuity of x, we obtain the contradiction $x(t_*) > 0$. Therefore $x_t \in C^+$ for all $t \in [0, 1]$.

Step 3. If $x_t \in C_r^+$ is not true for all $t \in [0, 1]$, then there exists a $c \in (0, 1]$ so that $\{t \in [0, 1] : x(t) = c\}$ is an infinite set. As J is finite, we may choose an open interval $I \subset (0, 1)$ so that $I \cap J = \emptyset$ and $I_c = \{t \in I : x(t) = c\}$ is infinite. Note that x is differentiable on I, (E_∞) holds on I.

By $\varphi \in C_r^+$, one may assume that $x(t-1) \notin \{ac/b, 1\}$ for all $t \in I$. Observe $ac/b < 1$. This fact and the continuity of $x_0 = \varphi$ allow us to distinguish three cases.

Case 1: $x(t-1) > 1$ for all $t \in I$. Then $x'(t) = -ax(t) < 0$ for all $t \in I$, and hence x is strictly decreasing on I, a contradiction to the fact that I_c is infinite.

Case 2: $x(t-1) < ac/b$ for all $t \in I$. Since I_c is infinite, there is a $t^* \in I$ with $x(t^*) = c$ and $x'(t^*) \geq 0$. On the other hand, we have

$$x'(t^*) = -ax(t^*) + bf(x(t^*-1)) = -ac + bx(t^*-1)) < -ac + b\frac{ac}{b} = 0,$$

a contradiction.

Case 3: $ac/b < x(t-1) < 1$ for all $t \in I$. Similarly to Case 2, as I_c is infinite, there is $t^{**} \in I$ with $x(t^{**}) = c$ and $x'(t^*) \leq 0$. From (E_∞) it follows that

$$x'(t^{**}) = -ax(t^{**}) + bf(x(t^{**}-1)) = -ac + bx(t^{**}-1) > -ac + b\frac{ac}{b} = 0,$$

a contradiction.

Therefore $x_t \in C_r^+$ for all $t \in [0, 1]$.

Step 4. So far we proved that, for any $\varphi \in C_r^+$, (2.5) defines a continuous extension x of φ to $[-1, 1]$, and $x_t \in C_r^+$ for all $t \in [0, 1]$. This procedure can be repeated to find a unique continuous function $x : [-1, \infty) \to \mathbb{R}$ such that $x_0 = \varphi$, $x_t \in C_r^+$ for all $t \geq 0$, and equation (2.3) holds for all $k \in \mathbb{N}_0$ and $t \in [k, k+1]$. Therefore, a unique solution x^φ exists on $[-1, \infty)$, and statement (i) is satisfied as well. Moreover, according to the remark preceding the proposition, the integral equation (2.4) holds for x^φ.

If $x^\varphi(\bar{t}-1) \neq 1$ then $x^\varphi(s-1) \neq 1$ for all $s \in [\tau, t]$ provided $\tau < \bar{t} < t$ and τ, t are close to $\bar{t}$. As the only discontinuity of f is at $\xi = 1$, by applying the fundamental theorem of calculus, statement (ii) follows from the integral equation (2.4).

Step 5. Let $t_1 \geq 0$, $t_2 > 0$, $\varphi \in C_r^+$. We claim that the semigroup property $F(t_1 + t_2, \varphi) = F(t_2, F(t_1, \varphi))$ holds. It is sufficient to show the semigroup property for $t_2 \in (0, 1]$, since the case $t_2 > 1$ can be obtained from repeated application of the case $t_2 \in (0, 1]$. So, let $t_2 \in (0, 1]$.

Let $\psi = F(t_1, \varphi) = x^\varphi_{t_1}$. We have to show $x^\psi_{t_2} = F(t_2, \psi) = F(t_1 + t_2, \varphi) = x^\varphi_{t_1+t_2}$, that is, $x^\psi(t_2 + \theta) = x^\varphi(t_1 + t_2 + \theta)$, $\theta \in [-1, 0]$.

If $t_2 + \theta \leq 0$, then $x^\psi(t_2 + \theta) = \psi(t_2 + \theta) = x^\varphi(t_1 + t_2 + \theta)$. If $t_2 + \theta > 0$ then, by (2.4) with $\tau = 0$, $t = t_2 + \theta$ and $\psi = x^\varphi_{t_1}$,

$$x^\psi(t_2+\theta) = e^{-a(t_2+\theta)}\psi(0) + b\int_0^{t_2+\theta} e^{-a(t_2+\theta-s)} f(\psi(s-1))\,ds. \tag{2.6}$$

By (2.4) with $\tau = 0$, $t = t_1$, one has

$$\psi(0) = x^\varphi(t_1) = e^{-at_1}x(0) + b\int_0^{t_1} e^{-a(t_1-s)} f(x^\varphi(s-1))\,ds. \tag{2.7}$$

Substituting (2.7) into (2.6) and using $\psi(s-1) = x^\varphi(t_1 + s - 1)$, we obtain

$$\begin{aligned} x^\psi(t_2+\theta) &= e^{-a(t_1+t_2+\theta)}x(0) + b\int_0^{t_1+t_2+\theta} e^{-a(t_1+t_2+\theta-s)} f(x^\varphi(s-1))\,ds \\ &= x^\varphi(t_1+t_2+\theta). \end{aligned}$$

Therefore, $x^\psi_{t_2} = x^\varphi_{t_1+t_2}$.

Step 6. For any fixed $\varphi \in C_r^+$, the continuity of $[0, \infty) \ni t \mapsto F(t, \varphi) \in C_r^+$ follows from the uniform continuity of x^φ on compact subintervals of $[-1, \infty)$.

Step 7. Now we show that, for each fixed $t \ge 0$, the map $C_r^+ \ni \varphi \mapsto F(t,\varphi) \in C_r^+$ is continuous. Suppose $t \in [0,1]$. Let $\varepsilon >$ be given, and let $\varphi^m \to \varphi$ in C_r^+. The finite set $\varphi^{-1}(1)$ can be covered by open intervals with total length less than $\varepsilon/(3b)$. Let U be the union of these open intervals. Clearly, $f(\varphi^m(s)) \to f(\varphi(s))$ uniformly in $s \in [-1,0] \setminus U$ as $m \to \infty$. Thus, there is $m_0 \in \mathbb{N}$ so that, for all $m \ge m_0$, $\sup_{s\in[-1,0]\setminus U} |f(\varphi^m(s)) - f(\varphi(s))| < \varepsilon/(3b)$ and $\|\varphi^m - \varphi\| < \varepsilon/3$. Then, from (2.3) we obtain that, for every $t \in [0,1]$ and $m \ge m_0$,

$$\begin{aligned}\|F(t,\varphi^m)-F(t,\varphi)\| &\le \sup_{t\in[-1,1]} |x^{\varphi^m}(t) - x^{\varphi}(t)| \\ &\le \|\varphi^m - \varphi\| + b\left(\int\limits_U + \int\limits_{[-1,0]\setminus U}\right)|f(\varphi^m(s)) - f(\varphi(s))|\,ds \\ &< \frac{\varepsilon}{3} + b\frac{\varepsilon}{3b} + b\frac{\varepsilon}{3b} = \varepsilon.\end{aligned}$$

Therefore, $\varphi^m \to \varphi$ in C_r^+ implies $F(t,\varphi^m) \to F(t,\varphi)$ as $m \to \infty$ uniformly in $t \in [0,1]$. This fact combined with the semigroup property gives that, in case $\varphi^m \to \varphi$ in C_r^+, $F(t,\varphi^m) \to F(t,\varphi)$ as $m \to \infty$ uniformly on t in compact subintervals of $[0,\infty)$.

Step 8. The continuity of F in t from Step 6 and the continuity of F in φ uniformly on t in compact subintervals of $[0,\infty)$ from Step 7 together yield the continuity of F jointly in $(t,\varphi) \in [0,\infty) \times C_r^+$. This completes the proof. □

Next we prove a boundedness property.

Proposition 2.3. *Let $\varphi \in C_r^+$, $\psi \in C^+$ and $n \ge 4$.*

(i) If φ and ψ are in $C([-1,0],(0,b/a))$ then the segments x_t^{φ} and $y_t^{n,\psi}$ are in $C([-1,0],(0,b/a))$ for all $t \ge 0$. The same holds for $C([-1,0],(0,b/a])$.
(ii) There exist $t_*(\varphi) \ge 0$, $t_{**}(\psi) \ge 0$ such that $x_t^{\varphi} \in C([-1,0],(0,b/a))$ for all $t \ge t_*(\varphi)$, and $y_t^{n,\psi} \in C([-1,0],(0,b/a))$ for all $t \ge t_{**}(\varphi)$.
(iii) If $y^{n,\psi}(t_0) \le b/a$ for some $t_0 \ge 0$ then $|(y^{n,\psi})'(t)| < 2b$ holds for all $t > t_0$. Similarly, if $x^{\varphi}(t_1) \le b/a$ for some $t_1 \ge 0$ then $|(x^{\varphi})'(t)| < 2b$ holds for all $t > t_1$ for which $(x^{\varphi})'(t)$ exists.

Proof. Let $\varphi \in C_r^+$ and $\psi \in C^+$ be given. Let z denote either x^{φ} or $y^{n,\psi}$. Then, by (2.2) and (2.4), for $0 \le \tau < t < \infty$, we have

$$z(t) = e^{-a(t-\tau)}z(\tau) + b\int\limits_{\tau}^{t} e^{-a(t-s)}g(z(s-1))\,ds$$

where $g = f$ if $z = x^{\varphi}$, and $g = f_n$ if $z = y^{n,\psi}$.

If $z(\tau) \le b/a$ for some $\tau \ge 0$ then, by $g \le 1$, for all $t \ge \tau$

$$z(t) \le \frac{b}{a} e^{-a(t-\tau)} + b \int_{\tau}^{t} e^{-a(t-s)}\, ds = \frac{b}{a}.$$

If $z(\tau) < b/a$ then $z(t) < b/a$ follows for all $t > \tau$. Hence, statement (i) is immediate.

In order to show (ii), by the first part of the proof it suffices to find a $t_0 \ge 0$ with $z(t_0) < b/a$. Assuming $z(t) \ge b/a$ for all $t \ge 0$, from the integral equation for z, for $t > 1$ one gets

$$\begin{aligned} z(t) &= e^{-a(t-1)} z(1) + b \int_{1}^{t} e^{-a(t-s)} g(z(s-1))\, ds \\ &\le e^{-a(t-1)} z(1) + \frac{b}{a}\left[1 - e^{-a(t-1)}\right] \sup_{\xi \ge b/a} g(\xi). \end{aligned}$$

Hence, by using $b > a > 0$ and $\sup_{\xi \ge b/a} g(\xi) < 1$, it follows that $\limsup_{t\to\infty} z(t) < b/a$, a contradiction. Therefore, (ii) holds.

Statement (iii) is obvious from equations (E_n), (E_∞) and Proposition 2.2. $\square$

For $\gamma > 0$ define

$$\Sigma_\gamma = \{\varphi \in C : \varphi(s) \ge 1 + \gamma \text{ fo all } s \in [-1, 0]\}.$$

Clearly, $\Sigma_\gamma \subset C^+$. The difference of two solutions of (E_n) with initial functions from Σ_γ can be estimated as follows.

Proposition 2.4. *Let $\gamma > 0$, $\psi \in \Sigma_\gamma$, $\chi \in \Sigma_\gamma$, and $n \ge 4$. Let $y = y^{n,\psi}$ and $z = z^{n,\chi}$ denote the solutions of (E_n) on $[-1, \infty)$ with initial functions ψ and χ, respectively. Then, for each integer $M \ge 0$, we have*

$$|y(t) - z(t)| \le \left(|\psi(0) - \chi(0)| + \frac{bn}{1 + (1+\gamma)^n} \|\psi - \chi\|\right)\left(1 + b\frac{n}{4}\right)^M$$

for all $t \in [0, M+1]$.

Proof. For $t \in [0, 1]$ from the integral equation (2.1) for y and z with $k = 0$, by using Proposition 2.1,

$$\begin{aligned} |y(t) - z(t)| &\le e^{-at} |\psi(0) - \chi(0)| + b \sup_{\xi \ge 1+\gamma} |f'(\xi)| \int_0^t e^{-a(t-s)} |y(s-1) - z(s-1)|\, ds \\ &\le \left(|\psi(0) - \chi(0)| + \frac{bn}{1 + (1+\gamma)^n} \|\psi - \chi\|\right) \end{aligned}$$

follows. This means that the statement holds for $M = 0$.

Let $\Delta = |\psi(0) - \chi(0)| + \big[bn/(1+(1+\gamma)^n)\big] \|\psi - \chi\|$. Suppose that $j \geq 1$ is an integer and for all $t \in [0, j]$ the inequality

$$|y(t) - z(t)| \leq \Delta \left(1 + \frac{bn}{4}\right)^{j-1}$$

is valid. Then

$$\|y_j - z_j\| \leq \Delta \left(1 + \frac{bn}{4}\right)^{j-1}$$

holds as well. Using Proposition 2.1 and the last inequality, from the integral equations (2.1) for y and z with $t \in [j, j+1]$ and $k = j$, we obtain

$$\begin{aligned} |y(t) - z(t)| \leq & e^{-a(t-j)}|y(j) - z(j)| + b \sup_{\xi>0} |f'(\xi)| \int_j^t e^{-a(t-s)} |y(s-1) - z(s-1)|\, ds \\ \leq & \left(1 + \frac{bn}{4}\right) \|y_j - z_j\| \\ \leq & \Delta \left(1 + \frac{bn}{4}\right)^j, \end{aligned}$$

and the proof is complete. $\square$

For $\varphi \in C_r^+$, $\tau \geq 0$ and $\delta \in (0, 1)$ define

$$\Delta(\varphi, \tau, \delta) = \{t \in (\tau, \tau + 1) : |x^\varphi(t) - 1| < \delta\}.$$

By the continuity of x^φ, the set $\Delta(\varphi, \tau, \delta)$ is the union of disjoint open intervals. Let $|\Delta(\varphi, \tau, \delta)|$ denote the sum of the lengths of these open intervals.

For $\varphi \in C_r^+$, $\tau \geq 0$ and $\delta_0 \in (0, \min\{a/b, 1 - a/b\})$ let

$$\Gamma(\varphi, \tau, \delta_0) = \{t \in [\tau - 1, \tau] : x^\varphi(t) \in \{1, a/b + \delta_0, a/b - \delta_0\}\}.$$

By Proposition 2.2, the set $\Gamma(\varphi, \tau, \delta_0)$ is finite, that is $\#\Gamma(\varphi, \tau, \delta_0) < \infty$.

For $\delta_0 \in (0, \min\{a/b, 1 - a/b\})$ set

$$N_{\delta_0} = [1 - \delta_0, 1 + \delta_0] \times \left[\frac{a}{b} - \delta_0, \frac{a}{b} + \delta_0\right].$$

The next result guarantees that the solution x^φ of equation (E_∞) spends relatively little time in a neighborhood of the discontinuity $\xi = 1$ of f, that is, $|\Delta(\varphi, \tau, \delta)|$ is small.

Proposition 2.5. *Let $\varphi \in C_r^+$, $M \geq 1$, $\delta_0 \in (0, \min\{a/b, 1 - a/b\})$ and $\gamma_0 \in \mathbb{N}$ be given such that*

$$\big(x^\varphi(t), x^\varphi(t-1)\big) \notin N_{\delta_0} \quad \textit{for all } t \in [0, M] \tag{2.8}$$

and

$$\#\Gamma(\varphi,\tau,\delta_0)\le\gamma_0 \quad \textit{for all } \tau\in[0,M-1] \tag{2.9}$$

hold. Define $K_0=2(1+\gamma_0)/[(b-a)\delta_0]$.

Then, for any $\delta\in(0,\delta_0]$,

$$|\Delta(\varphi,\tau,\delta)|\le K_0\delta \quad \textit{for all } \tau\in[0,M-1]. \tag{2.10}$$

Proof. Set $x=x^{\varphi}$, and fix $\tau\in[0,M-1]$ and $\delta\in(0,\delta_0]$.

Define the subsets

$$\begin{aligned}
U&=\{t\in(\tau,\tau+1):\, x(t-1)\neq 1 \text{ and } x(t-1)\notin[a/b-\delta_0,a/b+\delta_0]\},\\
V&=\{t\in(\tau,\tau+1):x(t-1)\in[a/b-\delta_0,a/b+\delta_0]\},\\
W&=\{t\in(\tau,\tau+1):x(t-1)=1\}
\end{aligned}$$

of $(\tau,\tau+1)$. Clearly, U, V, W are disjoint, and $(\tau,\tau+1)=U\cup V\cup W$.

Setting

$$\Delta=\Delta(\varphi,\tau,\delta)=\{t\in(\tau,\tau+1):|x(t)-1|<\delta\},$$

we have

$$\Delta=\Delta\cap(\tau,\tau+1)=(\Delta\cap U)\cup(\Delta\cap V)\cup(\Delta\cap W).$$

Observe $\Delta\cap V=\emptyset$ by (2.8), and W is finite by Proposition 2.2, and $\Delta\cap U$ is an open subset of $(\tau,\tau+1)$. It follows that $|\Delta|=|\Delta\cap U|$.

By (2.9) the set $\Gamma=\Gamma(\varphi,\tau,\delta_0)$ is a finite subset of $[\tau,\tau+1]$. Therefore the open set U can be written as

$$U=\bigcup_{j=1}^{N}(\alpha_j,\beta_j)$$

where

$$\tau\le\alpha_1<\beta_1\le\alpha_2<\beta_2\le\ldots\le\alpha_N<\beta_N\le\tau+1.$$

The set $\Gamma=\Gamma(\varphi,\tau,\delta_0)$ contains the points $\beta_1,\alpha_2,\beta_2,\ldots,\alpha_{N-1},\beta_{N-1},\alpha_N$, and possibly more points. Since $\beta_j=\alpha_{j+1}$ can happen for $j\in\{1,\ldots,N-1\}$, the set Γ contains at least $N-1$ points. Thus,

$$N\le 1+\#\Gamma\le 1+\gamma_0.$$

Clearly

$$\Delta \cap U = \bigcup_{j=1}^{N} \left(\Delta \cap (\alpha_j, \beta_j)\right).$$

For a given (α_j, β_j) there are three cases according to whether $x(t-1) > 1$, or $a/b + \delta_0 < x(t-1) < 1$, or $0 < x(t-1) < a/b - \delta_0$ for all $t \in (\alpha_j, \beta_j)$. In all cases, $x'(t)$ exists and equation (E_∞) holds for all $t \in (\alpha_j, \beta_j)$ since $x(t-1) \neq 1$.

Case 1: $x(t-1) > 1$ for all $t \in (\alpha_j, \beta_j)$. If $t \in \Delta \cap (\alpha_j, \beta_j)$ then $x(t) > 1 - \delta$, and, by $\delta_0 < a/b$,

$$x'(t) = -ax(t) \leq -a(1-\delta) \leq -a(1-\delta_0) < -(b-a)\delta_0.$$

Case 2: $a/b + \delta_0 < x(t-1) < 1$ for all $t \in (\alpha_j, \beta_j)$. If $t \in \Delta \cap (\alpha_j, \beta_j)$ then $x(t) < 1 + \delta$, and, by $\delta_0 < a/b$,

$$x'(t) = -ax(t) + bx(t-1) > -a(1+\delta) + b(a/b + \delta_0) > (b-a)\delta_0.$$

Case 3: $0 < x(t-1) < a/b - \delta_0$ for all $t \in (\alpha_j, \beta_j)$. If $t \in \Delta \cap (\alpha_j, \beta_j)$ then $x(t) > 1 - \delta$, and, by $\delta_0 < a/b$,

$$x'(t) = -ax(t) + bx(t-1) < -a(1-\delta) + b(a/b - \delta_0) < -(b-a)\delta_0.$$

CLAIM. Either $\Delta \cap (\alpha_j, \beta_j) = \emptyset$, or $\Delta \cap (\alpha_j, \beta_j) = \left(\widehat{\alpha_j}, \widehat{\beta_j}\right)$ for some $\widehat{\alpha_j}, \widehat{\beta_j}$ with $\alpha_j \leq \widehat{\alpha_j} < \widehat{\beta_j} \leq \beta_j$.

Proof of the Claim. Suppose $\Delta \cap (\alpha_j, \beta_j) \neq \emptyset$. Since $\Delta \cap (\alpha_j, \beta_j)$ is open, it suffices to show that for all t_1, t_2, t_3 in (α_j, β_j) with $t_1 < t_2 < t_3$ and $t_1 \in \Delta$, $t_3 \in \Delta$ we have $t_2 \in \Delta$.

Assume $t_2 \notin \Delta$. Then either $x(t_2) \geq 1+\delta$, or $x(t_2) \leq 1-\delta$. If $x(t_2) \geq 1+\delta$ then, by $t_1, t_3 \in \Delta$, $t_2 \in (t_1, t_3)$ and continuity, there exist $t_2^* \in (t_1, t_2]$ and $t_2^{**} \in [t_2, t_3)$ so that

$$x(t_2^*) = x(t_2^{**}) = 1 + \delta, \ x(t) < 1 + \delta \quad \text{for all } t \in (t_1, t_2^*) \cup (t_2^{**}, t_3).$$

On the other hand, in Cases 1, 2, 3 we obtain

$$x'(t_2^*) < 0, \ x'(t_2^{**}) > 0, \ x'(t_2^*) < 0,$$

respectively, which is a contradiction. The possibility $x(t_2) \leq 1 - \delta$ similarly leads to a contradiction. Therefore, $t_2 \in \Delta$, and the Claim holds.

According to the above Claim, assume $\Delta \cap (\alpha_j, \beta_j) = \left(\widehat{\alpha_j}, \widehat{\beta_j}\right)$ with $\alpha_j \leq \widehat{\alpha_j} < \widehat{\beta_j} \leq \beta_j$. Then one of the Cases 1–3 holds for all $t \in (\widehat{\alpha_j}, \widehat{\beta_j})$, and

$$|x'(t)| \geq (b-a)\delta_0 \quad \text{for all } t \in (\widehat{\alpha_j}, \widehat{\beta_j}).$$

In particular, $x'(t)$ does not change sign in $(\widehat{\alpha_j}, \widehat{\beta_j})$. Observe that $(\widehat{\alpha_j}, \widehat{\beta_j}) \subset \Delta$ implies $|x(\widehat{\beta_j}) - x(\widehat{\alpha_j})| \leq 2\delta$. Combining the above facts we obtain

$$2\delta \geq |x(\widehat{\beta_j}) - x(\widehat{\alpha_j})| = |\int_{\widehat{\alpha_j}}^{\widehat{\beta_j}} x'(t)\,dt| = \int_{\widehat{\alpha_j}}^{\widehat{\beta_j}} |x'(t)|\,dt \geq (b-a)\delta_0(\widehat{\beta_j} - \widehat{\alpha_j}).$$

Hence

$$|\Delta \cap (\alpha_j, \beta_j)| = \widehat{\beta_j} - \widehat{\alpha_j} \leq \frac{2\delta}{(b-a)\delta_0}.$$

Since $j \in \{1, \ldots, N\}$ was arbitrary,

$$|\Delta| = |\Delta \cap U| = \sum_{j=1}^{N} |\Delta \cap (\alpha_j, \beta_j)| \leq N \frac{2\delta}{(b-a)\delta_0} \leq \frac{2(1+\gamma_0)}{(b-a)\delta_0}\delta = K_0\delta,$$

and the proof is complete. □

By Proposition 2.1, for each $\delta \in (0, 1)$ there exists $n_1 = n_1(\delta) \geq 4$ so that

$$|f_n(\xi) - f(\xi)| < \delta \text{ provided } n \geq n_1(\delta),\ \xi \geq 0,\ \text{ and } |\xi - 1| \geq \delta.$$

Now we are able to guarantee that, for large n, the solutions of equation (E_n) remain close to a solution of equation (E_∞) on a compact interval, provided they are close on the interval $[0, 1]$.

Proposition 2.6. *Let $\varphi \in C_r^+$, an integer $M > 1$, $\delta_0 \in (0, \min\{a/b, 1 - a/b\})$ and $\gamma_0 \in \mathbb{N}$ be given so that conditions (2.8) and (2.9) are satisfied. Let*

$$K_0 = \frac{2}{(b-a)\delta_0}(1+\gamma_0),\ B = 1 + 2b(K_0+1) \text{ and } \delta_1 = \frac{\delta_0}{2}B^{-M}.$$

Then for all $\delta \in (0, \delta_1]$, for all $n \geq n_1(\delta)$, and for all $\psi \in C^+$, for the solutions $x = x^\varphi$ of (E_∞) and $y = y^{n,\psi}$ of (E_n),

$$\|y_1 - x_1\| < \delta \text{ implies } |y(t) - x(t)| < \delta B^M \quad (t \in [0, M+1]).$$

Proof. Let $\delta \in (0, \delta_1]$, $n \geq n_1(\delta)$, and $\psi \in C^+$ be fixed. Set $x = x^\varphi$ and $y = y^{n,\psi}$.

Suppose that

$$\|y_1 - x_1\| < \delta.$$

It is sufficient to show that

$$\|y_{j+1} - x_{j+1}\| < \delta B^j \quad \text{for all } j \in \{0, \ldots, M\}.$$

We prove by induction. Assume that $j \in \{1, \ldots, M\}$ is given, and

$$\|y_j - x_j\| < \delta B^{j-1}.$$

Observe that this is true for $j = 1$ by our assumption. It suffices to verify

$$\|y_{j+1} - x_{j+1}\| < \delta B^j.$$

For $t \geq j$, the integral equations

$$x(t) = e^{-a(t-j)}x(j) + b\int_j^t e^{-a(t-s)} f(x(s-1))\,ds,$$

$$y(t) = e^{-a(t-j)}y(j) + b\int_j^t e^{-a(t-s)} f_n(y(s-1))\,ds$$

hold. Hence

$$\begin{aligned}\|y_{j+1} - x_{j+1}\| \leq & \max_{t\in[j,j+1]} e^{-a(t-j)}|y(j) - x(j)| \\ & + b \max_{t\in[j,j+1]} \int_j^t e^{-a(t-s)}|f_n(y(s-1)) - f(x(s-1))|\,ds, \\ & \leq |y(j) - x(j)| + b\int_{j-1}^{j} |f_n(y(s)) - f(x(s))\,ds.\end{aligned}$$

Define the set

$$J_\delta = \Delta(\varphi, j-1, 2\delta B^{j-1}) = \{t \in (j-1, j) : |x(t) - 1| < 2\delta B^{j-1}\}.$$

From $\delta \leq \delta_1$ it follows that

$$2\delta B^{j-1} \leq 2\delta_1 B^{j-1} = \delta_0 B^{-M} B^{j-1} \leq \delta_0.$$

Then estimation (2.10) of Proposition 2.5 with $\tau = j - 1$ and $2\delta B^{j-1}$ instead of δ applies to get

$$|J_\delta| = |\Delta(\varphi, j-1, 2\delta B^{j-1})| \leq 2\delta B^{j-1} K_0.$$

For $s \in J_\delta$ one has $f_n(y(s)) \in (0, 1)$ and $f(x(s)) \in [0, 1]$, and hence

$$|f_n(y(s)) - f(x(s))| \leq 1.$$

Suppose $s \in (j-1, j) \setminus J_\delta$. Then $|x(s) - 1| \geq 2\delta B^{j-1}$, and

$$\text{either } x(s) \leq 1 - 2\delta B^{j-1} \text{ or } x(s) \geq 1 + 2\delta B^{j-1}.$$

Using $|y(s) - x(s)| \le \|y_j - x_j\| < \delta B^{j-1}$ for all $s \in [j-1, j]$, it follows that $|y(s) - 1| \ge \delta B^{j-1} \ge \delta$. Then, by $n > n_1(\delta)$,

$$|f_n(y(s)) - f(y(s))| < \delta.$$

Now we distinguish two cases.

Case 1: $x(s) \le 1 - 2\delta B^{j-1}$. Then $y(s) \le 1 - \delta B^{j-1}$ holds as well, and

$$\begin{aligned}|f_n(y(s)) - f(x(s))| &\le |f_n(y(s)) - f(y(s))| + |f(y(s)) - f(x(s))| \\ &\le \delta + |y(s) - x(s)| \\ &\le \delta + \delta B^{j-1} \le 2\delta B^{j-1}.\end{aligned}$$

Case 2: $x(s) \ge 1 + 2\delta B^{j-1}$. In this case we have $y(s) \ge 1 + \delta B^{j-1}$, and hence

$$\begin{aligned}|f_n(y(s)) - f(x(s))| &\le |f_n(y(s)) - f(y(s))| + |f(y(s)) - f(x(s))| \\ &= |f_n(y(s)) - f(y(s))| + 0 \\ &\le \delta \le 2\delta B^{j-1}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|y_{j+1} - x_{j+1}\| &\le \delta B^{j-1} + b \int_{j-1}^{j} |f_n(y(s)) - f(x(s))|\, ds \\ &\le \delta B^{j-1} + b \int_{J_\delta} 1\, ds + b \int_{(j-1,j)\setminus J_\delta} 2\delta B^{j-1}\, ds \\ &\le \delta B^{j-1} + b|J_\delta| + b2\delta B^{j-1} \\ &\le \delta B^{j-1} + b2\delta B^{j-1} K_0 + b2\delta B^{j-1} \\ &= \delta B^{j-1}(1 + +2bK_0 + 2b) = \delta B^j.\end{aligned}$$

This completes the proof. □

3. Periodic orbits assuming hypothesis (H)

In this section we prove the existence of periodic orbits under the hypothesis:

(H) The parameters $b > a > 0$ are given such that equation (E_∞) has an ω-periodic solution $p : \mathbb{R} \to \mathbb{R}$ with the properties:

(H1) $p(0) = 1$, $p(t) > 1$ for all $t \in [-1, 0)$;
(H2) $(p(t), p(t-1)) \ne (1, a/b)$ for all $t \in [0, \omega]$.

Remarks.

(i) From (E_∞) it follows that $p(t) = e^{-at}$ for $t \in [0,1]$. Then one finds $\omega > 2$. Note that $p_0 \in C_r^+$. Then, by Proposition 2.2 and periodicity, $p_t \in C_r^+$ for all $t \in \mathbb{R}$.

(ii) Let $\varphi \in C^+$ be arbitrary with $\varphi(0) = 1$ and $\varphi(s) > 1$ for $s \in [-1,0)$. Then $\varphi \in C_r^+$. Proposition 2.2 gives the unique solution $x^\varphi : [-1,\infty) \to \mathbb{R}$ of (E_∞) with $x_t^\varphi \in C_r^+$ for all $t \geq 0$. Suppose that there exists an $\omega > 2$ with $x^\varphi(\omega) = 1$, and $x^\varphi(\omega + s) > 1$ for all $s \in [-1,0)$. Define $\psi = x_\omega^\varphi$. By $f(\xi) = 0$ for $\xi > 1$, it follows that $x^\varphi(t) = x^\psi(t)$ for all $t \geq 0$, and $x^\psi(t+\omega) = x^\psi(t)$ for all $t \geq -1$. Thus x^ψ can be extended to an ω-periodic solution of (E_∞) satisfying condition (H1). Condition (H2) requires to show that $p(t) = 1$ implies $p(t-1) \neq a/b$ for all $t \in [0,\omega]$. These observations give a relatively straightforward way to verify condition (H), see the next section.

Theorem 1.1 is a consequence of the following result together with sufficient conditions for (H).

Theorem 3.1. *Suppose $b > a > 0$ are given such that condition (H) holds. Then there exists an $n_* \geq 4$ such that, for all $n \geq n_*$, equation (E_n) has a periodic solution $p^n : \mathbb{R} \to \mathbb{R}$ with period $\omega^n > 0$ so that the periodic orbits*

$$\mathcal{O}^n = \{p_t^n : t \in [0,\omega^n]\}$$

are hyperbolic, orbitally stable, exponentially attractive with asymptotic phase, and $\omega^n \to \omega$, $\operatorname{dist}\{\mathcal{O}^n, \mathcal{O}\} \to 0$ *as $n \to \infty$.*

Proof. The proof is divided into several steps.

Step 1. Preparation.

Let M be the largest integer with $M \leq \omega + 1/2$. Then $M \in \{2,3,\ldots\}$ and

$$M - \frac{1}{2} \leq \omega < M + \frac{1}{2}. \tag{3.1}$$

The continuity and periodicity of p, and condition (H2) guarantee the existence of

$$\delta_0 \in \left(0, \min\left\{\frac{a}{b}, 1 - \frac{a}{b}\right\}\right) \tag{3.2}$$

such that

$$(p(t), p(t-1)) \notin N_{\delta_0} \quad \text{for all } t \in \mathbb{R}, \tag{3.3}$$

where

$$N_{\delta_0} = [1-\delta_0, 1+\delta_0] \times \left[\frac{a}{b} - \delta_0, \frac{a}{b} + \delta_0\right].$$

Hypothesis (H) and the continuity of p yield an $\alpha \in (0, 1/2)$ such that

$$\min_{t \in [-1-\alpha, -\alpha]} p(t) > 1.$$

By using (H), equation (E_∞) and the choice of α, one gets $p(t) = e^{-at}$ for all $t \in [-\alpha, 1]$. Define

$$\varepsilon = \min\left\{\frac{a}{2}, \frac{\delta_0}{2}, \frac{1}{3}\left(\frac{b}{a} - 1\right), \frac{1}{3}\left(e^{a\alpha} - 1\right), \frac{1}{3}\left(\min_{t\in[-1-\alpha,-\alpha]} p(t) - 1\right)\right\}.$$

Set

$$\sigma_0 = \frac{1}{a}\log\frac{1+3\varepsilon}{1+2\varepsilon}, \quad \sigma_1 = \frac{1}{a}\log\frac{1+2\varepsilon}{1+\varepsilon}, \quad \sigma_2 = \frac{1}{a}\log(1+2\varepsilon).$$

Clearly, $\sigma_0 + \sigma_2 = (1/a)\log(1+3\varepsilon) \le \alpha < 1/2$, and

$$p(-\sigma_0 - \sigma_2) = 1 + 3\varepsilon, \quad p(-\sigma_2) = 1 + 2\varepsilon, \quad p(\sigma_1 - \sigma_2) = 1 + \varepsilon.$$

By the choices of M, α and ε, we have

$$M - 1 \le \omega - \frac{1}{2} < \omega - (\sigma_0 + \sigma_2) < \omega - \sigma_0 < \omega + \sigma_1 < \omega + \sigma_2 < \omega + \frac{1}{2} < M + 1. \tag{3.4}$$

It is convenient to define the shifted version of p by

$$q : \mathbb{R} \ni t \mapsto p(t - \sigma_2) \in \mathbb{R}.$$

Then q is an ω-periodic solution of (E_∞) satisfying

$$q(t) = (1+2\varepsilon)e^{-at} \quad \text{for all } t \in [-\sigma_0, \sigma_2], \tag{3.5}$$

$$q(-\sigma_0) = 1 + 3\varepsilon, \quad q(0) = 1 + 2\varepsilon, \quad q(\sigma_1) = 1 + \varepsilon, \quad q(\sigma_2) = 1, \tag{3.6}$$

$$q(t) \ge 1 + 3\varepsilon \quad \text{for all } t \in [-1 - \sigma_0, -\sigma_0]. \tag{3.7}$$

Set $s_0 = 4(e-1)/b$, and define

$$h : [s_0, \infty) \ni s \mapsto \left(b\left(1 + \frac{4b}{a}\right)s\left(1 + \frac{bs}{4}\right)^M\right)^{1/s} - 1 \in \mathbb{R}.$$

Clearly, $\lim_{s\to\infty} h(s) = 0$. For $s > s_0$ we have

$$\begin{aligned}
\frac{d}{ds}\log[h(s)+1] &= \frac{d}{ds}\left[\frac{1}{s}\log\left(b\left(1+\frac{4b}{a}\right)s\right) + \frac{M}{s}\log\left(1+\frac{bs}{4}\right)\right] \\
&= -\frac{1}{s^2}\left[\log\left(b\left(1+\frac{4b}{a}\right)s\right) - 1\right] - \frac{M}{s^2}\left[\log\left(1+\frac{bs}{4}\right) - \frac{bs}{4+bs}\right] \\
&\le -\frac{1}{s^2}\left[\log\left(b\left(1+\frac{4b}{a}\right)s\right) - 1\right] - \frac{M}{s^2}\left[\log\left(1+\frac{bs}{4}\right) - 1\right] < 0.
\end{aligned}$$

Consequently, h strictly decreases on $[s_0, \infty)$ from $h(s_0) > 0$ to 0. Then there exists an $n_0 = n_0(\varepsilon) \ge s_0$ so that

$$h(n) < \varepsilon \quad \text{for all } n \geq n_0. \tag{3.8}$$

Observe that, by the monotonicity of h and the definition of n_0,

$$\frac{[1+h(n)]^n}{1+(1+\varepsilon)^n} \leq \left(\frac{1+h(n)}{1+\varepsilon}\right)^n \leq \left(\frac{1+h(n_0)}{1+\varepsilon}\right)^{n_0} < 1 \quad \text{for all } n \geq n_0. \tag{3.9}$$

From $p_0 \in C_r^+$, by periodicity and Proposition 2.2, $p_t \in C_r^+$ follows for all $t \in \mathbb{R}$. Then, by using the periodicity of p again, it is obvious that

$$\begin{aligned}\gamma_0 &:= \sup_{\tau\in\mathbb{R}} \#\Gamma(p_0, \tau, \delta_0) \\ &= \max_{\tau\in\mathbb{R}} \#\{t \in [\tau, \tau+1] : p(t-1) \in \{1, a/b+\delta_0, a/b-\delta_0\}\} < \infty.\end{aligned}$$

Let

$$K_0 = \frac{2}{(b-a)\delta_0}(1+\gamma_0), \quad B = 1 + 2b(K_0+1), \delta_1 = \frac{\delta_0}{2} B^{-M}.$$

Fixing $\delta = \varepsilon B^{-M}$, we have

$$\delta \leq \frac{\delta_0}{2} B^{-M} = \delta_1.$$

Recall from Section 2 that $n_1 = n_1(\delta) = n_1(\varepsilon B^{-M})$ is given so that

$$|f_n(\xi) - f(\xi)| < \delta \quad \text{provided } \xi \geq 0,\ |\xi - 1| \geq \delta,\ n \geq n_1. \tag{3.10}$$

By the definition of q, δ_0, γ_0, M, the conditions (2.8) and (2.9) of Proposition 2.5 hold with $\varphi = q_0$. Proposition 2.6 can be applied with $\varphi = q_0$ and an arbitrary $\psi \in C^+$ to obtain that, for the solution q of (E_∞) and for the solution $y = y^{n,\psi}$ of (E_n),

$$\|q_1 - y_1\| < \delta = \varepsilon B^{-M} \text{ implies } |q(t) - y(t)| < \delta B^M = \varepsilon \text{ for all } t \in [0, M+1] \tag{3.11}$$

is satisfied provided $n \geq n_1(\delta) = n_1(\varepsilon B^{-M})$.

It is straightforward to see that there is an $n_2 = n_2(\varepsilon)$ so that

$$b f_n(1+\varepsilon) < \varepsilon B^{-M} \quad \text{for all } n \geq n_2. \tag{3.12}$$

Note that $B^{-M} < 1$, and hence (3.12) implies $b f_n(1+\varepsilon) < \varepsilon$ for all $n \geq n_2$.

Define

$$n_* = n_*(\varepsilon) = \max\left\{n_0(\varepsilon), n_1(\varepsilon B^{-M}), n_2(\varepsilon)\right\}.$$

Step 2. A return map.

Introduce the subsets

$$S = \{\psi \in C : \psi(0) = 1 + 2\varepsilon,\ 1 + \varepsilon \le \psi(s) \le b/a \text{ for all } s \in [-1, 0]\},$$
$$S^0 = \{\psi \in S : 1 + \varepsilon < \psi(s) < b/a \text{ for all } s \in [-1, 0]\}$$

of C. Let $n \ge n_*$ be fixed. For given $\psi \in S$ let $y = y^{n,\psi}$ be the solution of equation (E_n) with $y_0 = \psi$. Our aim is to show that the solution curve $[0, \infty) \ni t \mapsto y_t \in C$ returns to S, that is, $y_\tau \in S$ for some $\tau > 0$.

First we estimate $q(t) - y(t)$. For $t \in [0, 1]$,

$$q(t) = e^{-at} q(0) + b \int_0^t e^{-a(t-s)} f(q(s-1))\, ds = e^{-at}(1 + 2\varepsilon)$$

$$y(t) = e^{-at}(1 + 2\varepsilon) + b \int_0^t e^{-a(t-s)} f_n(y(s-1))\, ds.$$

Hence, by $y_0 = \psi \in S$ and (3.12), for all $t \in [0, 1]$,

$$|q(t) - y(t)| \le b \int_0^t e^{-a(t-s)} f_n(y(s-1))\, ds \le b f_n(1 + \varepsilon) < \varepsilon B^{-M}, \tag{3.13}$$

that is, $\|q_1 - y_1\| < \varepsilon B^{-M} = \delta$. Applying (3.11)

$$|q(t) - y(t)| < \varepsilon \quad \text{for all } t \in [0, M + 1] \tag{3.14}$$

follows. In addition, Proposition 2.3 yields

$$y(t) \le \frac{b}{a} \quad \text{for all } t \in [-1, \infty). \tag{3.15}$$

The ω-periodicity of q, $\omega > 2$, the properties (3.5), (3.6), (3.7) of q and (3.14) imply

$$y(\omega - \sigma_0) > q(\omega - \sigma_0) - \varepsilon = q(-\sigma_0) - \varepsilon = 1 + 2\varepsilon, \tag{3.16}$$

$$y(\omega + \sigma_1) < q(\omega + \sigma_1) + \varepsilon = q(\sigma_1) + \varepsilon = 1 + 2\varepsilon, \tag{3.17}$$

$$y(t) > q(t) - \varepsilon \ge 1 + \varepsilon - \varepsilon = 1 \quad \text{for all } t \in [\omega - \sigma_0, \omega + \sigma_1], \tag{3.18}$$

$$y(t) > q(t) - \varepsilon \ge 1 + 2\varepsilon \quad \text{for all } t \in [\omega - 1 - \sigma_0, \omega - \sigma_0]. \tag{3.19}$$

We claim that y strictly decreases on the interval $[\omega - \sigma_0, \omega + \sigma_1]$. Indeed, by (3.18), (3.19), (3.12) and the choice of ε,

$$
\begin{aligned}
y'(t) &= -ay(t) + bf_n(y(t-1)) \leq -a + bf_n(1+2\varepsilon) \\
&< -a + bf_n(1+\varepsilon) < -a + \varepsilon B^{-M} < -a + \varepsilon \\
&\leq -\frac{a}{2} \qquad \text{for all } t \in [\omega - \sigma_0, \omega + \sigma_1],
\end{aligned}
\tag{3.20}
$$

and by (3.16), (3.17), there is a unique $\tau = \tau(n, \psi) \in (\omega - \sigma_0, \omega + \sigma_1)$ so that $y(\tau) = 1 + 2\varepsilon$. Taking into account (3.19) and (3.15) as well,

$$
y_\tau \in S
$$

follows. The return map R is defined by

$$
R : S \ni \psi \mapsto F^n(\tau, \psi) \in S.
$$

Note that $p(t) = e^{-at}$, $t \in [0, 1]$, implies $q(t) \leq 1$ for all $t \in [\sigma_2, 1 + \sigma_2]$. This fact combined with (3.14) gives $y(t) < 1 + \varepsilon < b/a$ for all $t \in [\sigma_2, 1 + \sigma_2]$. Hence Proposition 2.3 and (3.19), (3.20), $\tau = \tau(n, \psi) \in (\omega - \sigma_0, \omega + \sigma_1)$, $y(\tau) = 1 + 2\varepsilon$ combined guarantee $y_\tau \in S^0$. Therefore

$$
R(S) \subset S^0.
$$

Step 3. R is a contraction.

Let $n \geq n_*$, $\psi, \chi \in S$, $y = y^{n,\psi}$, $z = y^{n,\chi}$, $\tau_y = \tau(n, \psi)$, $\tau_z = \tau(n, \chi)$. Observing $\psi(0) = \chi(0)$, Proposition 2.4 with $\gamma = \varepsilon$ gives

$$
|y(t) - z(t)| \leq \frac{bn}{1 + (1+\varepsilon)^n} \left(1 + \frac{bn}{4}\right)^M \|\psi - \chi\| \quad (t \in [0, M+1]). \tag{3.21}
$$

In order to estimate $\tau_y - \tau_z$, note that $y(\tau_y) = 1 + 2\varepsilon = z(\tau_z)$ and $\tau_y, \tau_z \in (\omega - \sigma_0, \omega + \sigma_1)$. Then, by (3.20) and (3.21),

$$
\begin{aligned}
\frac{a}{2}|\tau_y - \tau_z| &\leq \left| \int_{\tau_y}^{\tau_z} y'(t)\, dt \right| = |y(\tau_z) - y(\tau_y)| \\
&= |y(\tau_z) - z(\tau_z)| \\
&\leq \frac{bn}{1 + (1+\varepsilon)^n} \left(1 + \frac{bn}{4}\right)^M \|\psi - \chi\|.
\end{aligned}
$$

Hence

$$
|\tau_y - \tau_z| \leq \frac{2}{a} \frac{bn}{1 + (1+\varepsilon)^n} \left(1 + \frac{bn}{4}\right)^M \|\psi - \chi\|. \tag{3.22}
$$

By Proposition 2.3, $|y'(t)| < 2b$ for all $t > 0$, and thus

$$
\|y_{\tau_y} - y_{\tau_z}\| \leq 2b|\tau_y - \tau_z|. \tag{3.23}
$$

Now, (3.22), (3.23) and (3.21) combined yield

$$
\begin{aligned}
\|R(\psi) - R(\chi)\| &= \|y_{\tau_y} - z_{\tau_z}\| \\
&\le \|y_{\tau_y} - y_{\tau_z}\| + \|y_{\tau_z} - z_{\tau_z}\| \\
&\le \left(1 + \frac{4b}{a}\right) \frac{bn}{1 + (1+\varepsilon)^n} \left(1 + \frac{bn}{4}\right)^M \|\psi - \chi\|.
\end{aligned}
\tag{3.24}
$$

Recall the function h and inequality (3.9) from Step 1, and set

$$
\kappa = \left(\frac{1 + h(n_0)}{1 + \varepsilon}\right)^{n_0}.
$$

As $\psi, \chi \in S$ were arbitrary, estimation (3.24) and inequality (3.9) imply

$$
\|R(\psi) - R(\chi)\| \le \kappa \|\psi - \chi\| \text{ for all } \psi, \chi \in S. \tag{3.25}
$$

By $\kappa < 1$ the map $R : S \to S$ is a contraction. The closed subset S of C is a complete metric space with the metric induced by the norm of C. Then R has a unique fixed point in S, denoted by $\eta(n)$. By the remark at the end of Step 2, $\eta(n) = S(\eta(n)) \in S^0$. The fixed point $\eta(n)$ determines a periodic solution $p^n : \mathbb{R} \to \mathbb{R}$ of equation (E_n) with period $\omega^n \in (\omega - \sigma_0, \omega + \sigma_1)$. Clearly, $p^n(t) = y^{n,\eta(n)}(t)$ for $t \ge -1$, and $\omega^n = \tau(n, \eta(n))$. Let

$$
\mathcal{O}^n = \left\{p_t^n : 0 \le t \le \omega^n\right\}
$$

be the corresponding periodic orbit.

Step 4. Hyperbolicity and attraction of $\mathcal{O}^n$.

The results of Chaper XIV in [8] or the Appendix of [22] will be applied.

Let $n \ge n_*$ be fixed, and let $p^n : \mathbb{R} \to \mathbb{R}$ be the periodic solution of (E_n) obtained in Step 3. Recall that $p_0^n \in S^0$, and for the period ω^n of p^n the relation $\omega^n \in (\omega - \sigma_0, \omega + \sigma_1) \subset (3/2, \infty)$ holds.

The restriction of the continuous semiflow $F^n : [0, \infty) \times C^+ \to C^+$ to the set $(1, \infty) \times C^+$ is continuously differentiable. Define the closed subspace $H = \{\psi \in C : \psi(0) = 0\}$ of C, and the continuous linear functional $l^* : C \ni \psi \mapsto \psi(0) \in \mathbb{R}$. Then $(l^*)^{-1}(0) = H$. We look for solutions $(t, \psi) \in (1, \infty) \times C$ of the equation $G(t, \psi) = 0$, where $G : (1, \infty) \times C \ni (t, \psi) \mapsto l^*\left(F^n(t, \psi) - p_0^n\right) \in \mathbb{R}$. Note that $G(\omega^n, p_0^n) = 0$. Moreover, by inequality (3.20) with $y = p^n$ and $\omega^n \in (\omega - \sigma_0, \omega + \sigma_1)$, one sees $D_1 G(\omega^n, p_0^n)1 = l^*\left(D_1 F^n(\omega^n, p_0^n)1\right) = \left(p_{\omega^n}^n\right)'(0) = (p^n)'(\omega^n) < -a/2$. Using the Implicit Function Theorem, there exist an open neighborhood U of p_0^n in C, a $\nu \in (0, \min\{\omega^n - (\omega - \sigma_0), \omega + \sigma_1 - \omega^n\})$, and a C^1-map $\zeta : U \to \mathbb{R}$ with $\zeta(p_0^n) = \omega^n$ and $\zeta(U) \subset (\omega^n - \nu, \omega^n + \nu)$ so that, for every $(t, \psi) \in (\omega^n - \nu, \omega^n + \nu) \times U$, the equality $G(t, \psi) = 0$ holds if and only if $t = \zeta(\psi)$.

By Step 2, for any $\psi \in S$ there is a unique $\tau(n, \psi) \in (\omega - \sigma_0, \omega + \sigma_1)$ with $y^{n,\psi}(\tau(n, \psi)) = F^n(\tau(n, \psi), \psi)(0) = 1 + 2\varepsilon$, or equivalently, $G(\tau(n, \psi), \psi) = 0$. From $(\omega^n - \nu, \omega^n + \nu) \subset (\omega - \sigma_0, \omega + \sigma_1)$ it follows that, for all $\psi \in U \cap S$, we have $\zeta(\psi) = \tau(n, \psi)$. Notice that $p_0^n \in S^0$, and S^0 is open in the hyperplane $p_0^n + H$. Then $U \cap S^0$ is an open neighborhood of p_0^n in $p_0^n + H$. Therefore the restriction of the return map R to $U \cap S^0$ coincides with the restriction of the C^1-map

$$U \ni \psi \mapsto F^n(\zeta(\psi), \psi) \in C$$

to $U \cap S^0$. Consequently, R is C^1-smooth on $U \cap S^0$.

There is an open ball V in $p_0^n + H$ with center at p_0^n so that $V \subset U \cap S^0$. From $\kappa < 1$ and (3.25) it follows for all $k \in \mathbb{N}$ and $\psi, \chi \in V$ that

$$\|R^k(\psi) - R^k(\chi)\| \le \kappa^k \|\psi - \chi\|.$$

Set $L = DR(p_0^n)$. Clearly $\|L^k\| = \|DR^k(p_0^n)\| \le \kappa^k$ for all $k \in \mathbb{N}$.

As H is a real Banach space, the spectrum of $L = DR(p_0^n)$ is defined as the spectrum of its complexification $L_{\mathbb{C}} : H_{\mathbb{C}} \to H_{\mathbb{C}}$. Recall (see Section III.7 in [8]) that the norm $\|\cdot\|$ on $H_{\mathbb{C}}$ satisfies the admissibility condition $\max\{\|u\|, \|v\|\} \le \|u + iv\| \le \|u\| + \|v\|$ for all u, v in H. A consequence of this condition is that for any linear bounded operator $T : H \to H$ and its complexification we have $\|T_{\mathbb{C}}\| \le 2\|T\|$. In particular

$$\|(L_{\mathbb{C}})^k\| = \|(L^k)_{\mathbb{C}}\| \le 2\|L^k\| \le 2\kappa^k$$

for all $k \in \mathbb{N}$.

By the spectral radius formula

$$\rho(L_{\mathbb{C}}) = \lim_{k\to\infty} \|(L_{\mathbb{C}})^k\|^{1/k} \le \lim_{k\to\infty} (2\kappa^k)^{1/k} = \kappa < 1,$$

and this shows that the spectrum of the derivative $DR(p_0^n) : H \to H$ is in the open unit disk of $\mathbb{C}$.

The results of Chapter XIV in [8] apply to conclude that $\mathcal{O}^n$ is hyperbolic, orbitally stable, and exponentially attractive with asymptotic phase.

Step 5. The limit of $\mathcal{O}^n$ as $n \to \infty$.

Let $p^n : \mathbb{R} \to \mathbb{R}$ be the periodic solution of (E_n) obtained in Step 3 for all $n \ge n_*$.

Inequality (3.13) with $y(t) = p^n(t)$ gives that

$$|q(t) - p^n(t)| \le b f_n(1+\varepsilon) \ \text{ for all } t \in [0, 1].$$

Then Proposition 2.6 with $y(t) = p^n(t)$ yields

$$|q(t) - p^n(t)| < b f_n(1+\varepsilon) B^M \ \text{ for all } t \in [0, M+1].$$

If $n \to \infty$ then $f_n(1+\varepsilon) \to 0$. Consequently,

$$|q(t) - p^n(t)| \to 0 \text{ as } n \to \infty \text{ uniformly in } t \in [0, M+1].$$

Hence, by $\omega^n \in (\omega - \sigma_0, \omega + \sigma_1)$, it is easy to get that $\omega^n \to \omega$ and $\mathrm{dist}(\mathcal{O}^n, \mathcal{O}) \to 0$ as $n \to \infty$.

This completes the proof. □

4. Hypothesis (H)

In this section we show that there exist parameters $b > a > 0$ so that condition (H) holds. In Subsection 4.1 analytic tools are used to give parameters $b > a > 0$ satisfying (H). A computer-assisted technique is applied in Subsection 4.2.

4.1. Analytic tools to verify (H)

We have two types of results. In the first case b is large comparing to a.

Proposition 4.1. *If $a > 0$, $b > 0$ satisfy $b > \max\{ae^a, e^a - e^{-a}\}$ then (H) holds.*

Proof. By Remark (ii) in Section 3, in order to have an ω-periodic solution of (E_∞) with (H1) it suffices to consider $\varphi \in C^+$ with $\varphi(0) = 1$, $\varphi(s) > 1$ for $s \in [-1, 0)$, and to find an $\omega > 2$ with $x^\varphi(\omega) = 1$ and $x^\varphi(\omega + s) > 1$, $s \in [-1, 0)$.

Assume $b > \max\{ae^a, e^a - e^{-a}\}$. Let $\varphi \in C^+$ be given with $\varphi(0) = 1$, $\varphi(s) > 1$ for $s \in [-1, 0)$, and set $x = x^\varphi$. Then $x(t) = e^{-at}$ for $t \in [0, 1]$.

By the integral equation (2.4) with $\tau = 1$, $x(\tau) = e^{-a}$, $f(x(s-1)) = x(s-1) = e^{-a(s-1)}$ one finds

$$\begin{aligned} x(t) &= e^{-a(t-1)}e^{-a} + b\int_1^t e^{-a(t-s)}e^{-a(s-1)}\,ds \\ &= e^{-at}\left[1 + be^a(t-1)\right] \quad (1 \le t \le 2). \end{aligned} \tag{4.1}$$

Then, $x(2) = e^{-2a}(1 + be^a) > 1$ because of $b > e^a - e^{-a}$. Equation (4.1) gives

$$x'(t) = e^{-at}\left[be^a - a - abe^a(t-1)\right] \quad (1 < t < 2).$$

Hence x' can have at most one zero in $(1, 2)$. Then, by $e^{-a} = x(1) < 1 < x(2)$, there is a unique $t_1 \in (1, 2)$ with $x(t_1) = 1$. Clearly, $x(t) < 1$ for $t \in [1, t_1)$, and $x(t) > 1$ for $t \in (t_1, 2]$.

By using the integral equation (2.4) with $\tau = 2$, equation (4.1), and $f(x(s-1)) = x(s-1)$ for $s \in [2, t_1 + 1)$, it follows that

$$\begin{aligned} x(t) &= e^{-a(t-2)}x(2) + b\int_2^t e^{-a(t-s)}e^{-a(s-1)}\left[1 + be^a(s-2)\right]ds \\ &= e^{-at}\left[1 + be^a + be^a(t-2) + \frac{1}{2}b^2e^{2a}(t-2)^2\right] \quad (2 \le t \le t_1 + 1), \end{aligned} \tag{4.2}$$

and from equation (4.2) one finds

$$x'(t) = e^{-at}\Big[-a - (a-1)be^a + be^a(be^a - a)(t-2) - \frac{1}{2}ab^2e^{2a}(t-2)^2\Big] \quad (2 < t < t_1 + 1). \tag{4.3}$$

The definition of $t_1 \in (1, 2)$ gives

$$1 + be^a(t_1 - 1) = e^{at_1}. \tag{4.4}$$

By (4.2) with $t = t_1 + 1$, and by using (4.4),

$$
\begin{aligned}
x(t_1+1) &= e^{-a(t_1+1)}\left[1+be^a+be^a(t_1-1)+\frac{1}{2}b^2e^{2a}(t_1-1)^2\right] \\
&= e^{-a(t_1+1)}\left[\frac{1}{2}+be^a+\frac{1}{2}\left(1+be^a(t_1-1)\right)^2\right] \\
&= e^{-a(t_1+1)}\left[be^a+\frac{1}{2}-\frac{1}{2}e^{2a}\right]+\frac{1}{2}\left[e^{-a(t_1-1)}+e^{a(t_1-1)}\right] \\
&> 1
\end{aligned}
\tag{4.5}
$$

because $be^a+(1/2)-(1/2)e^{2a}>0$ by $b>e^a-e^{-a}$, and $e^{-a(t_1-1)}+e^{a(t_1-1)}\geq 2$.

We claim $x(t)>1$ for all $t\in[2,t_1+1]$. Since $x(2)>1$ and $x(t_1+1)>1$, in case the claim does not hold there exists $t_2\in(2,t_1+1)$ with $x'(t_2)=0$ and $x(t_2)\leq 1$. From (4.3) one gets

$$1+be^a+be^a(t_2-2)+\frac{1}{2}b^2e^{2a}(t_2-2)^2=\frac{1}{a}\left[be^a+b^2e^{2a}(t_2-2)\right],$$

and then (4.2) with the above equality gives

$$x(t_2)=\frac{b}{a}e^{-a(t_2-2)}\left[e^{-a}+b(t_2-2)\right].$$

Define

$$A:[0,1]\ni s\mapsto \frac{b}{a}e^{-as}\left[e^{-a}+bs\right]\in\mathbb{R}.$$

From $b>ae^a$ and $b>e^a-e^{-a}>1-e^{-a}$ one obtains

$$A(0)=\frac{b}{ae^a}>1,\ \ A(1)=\frac{b}{ae^a}\left(e^{-a}+b\right)>1.$$

If $A(s)>1$ for all $s\in[0,1]$ fails then $A'(s_*)=0$ and $A(s_*)\leq 1$ for some $s_*\in(0,1)$. The equality $A'(s_*)=0$ implies $aA(s_*)=b^2e^{-as_*}/a$. Hence, by using $b>ae^a$, $A(s_*)=(b/a)^2e^{-as_*}>(b/a)^2e^{-a}>e^{2a}e^{-a}=e^a>1$, a contradiction. Consequently, $A(s)>1$ for all $s\in[0,1]$. As $t_2\in(2,t_1+1)\subset(2,3)$, we conclude with $s=t_2-2$ that $x(t_2)=A(t_2-2)>1$, a contradiction. Thus, $x(t)>1$ for all $t\in[2,t_1+1]$.

Now we have $x(t)>1$ for all $t\in(t_1,t_1+1]$. Setting $\omega=t_1+1+(1/a)\log x(t_1+1)$, it is straightforward to see that $x'(t)=-ax(t)$ and $x(t)=x(t_1+1)e^{-a(t-t_1-1)}>1$ for all $t\in(t_1+1,\omega)$ and $x(\omega)=1$. Then the ω-periodic extension p of the restriction $x|_{[0,\omega]}$ is an ω-periodic solution of equation (E_∞) satisfying (H1).

It remains to verify (H2). The equality $p(t)=1$ in $[0,\omega]$ holds at $t=0$, $t=t_1$ and $t=\omega$. (H1) and the periodicity imply $p(-1)=p(\omega-1)>1>a/b$. By $b>ae^a$ and $t_1\in(1,2)$, $p(t_1-1)=x(t_1-1)=e^{-a(t_1-1)}>e^{-a}>a/b$. Thus (H2) holds as well. □

The second case, where we have an analytic proof for (H), is when b is sufficiently close to a.

Proposition 4.2. *For every $a>0$ there exists an $\varepsilon_0=\varepsilon_0(a)>0$ such that for the parameters a,b with $b\in(a,a+\varepsilon_0)$ condition (H) holds.*

In particular, for the periodic solution $p = p(a, b)$ of equation (E_∞) the minimal period $\omega = \omega(a, b)$ satisfies $\omega > 5$, and there exists a $\sigma = \sigma(a, b) \in (4, \omega - 1)$ so that

$$0 < p(t) < 1 \text{ for all } t \in (0, \sigma); \;\; p(t) > 1 \text{ for all } t \in (\sigma, \omega).$$

Moreover, if $a > 0$ is fixed and $(b_k)_{k=1}^{\infty}$ is a sequence in $(a, a + \varepsilon_0(a))$ with $\lim_{k\to\infty} b_k = b$ then $\sigma(a, b_k) \to \infty$, $\omega(a, b_k) \to \infty$ as $k \to \infty$.

The proof of Proposition 4.2 is given in [18].

Remarks.

(i) The periodic solutions p of equation (E_∞) obtained in Propositions 4.1 and 4.2 have relatively simple shapes. In fact, they are slowly oscillating around 1 in the sense that $|t_1 - t_2| > 1$ for any two times $t_1 \neq t_2$ with $p(t_1) = p(t_2) = 1$. Three consecutive times in $p^{-1}(1)$ determine the minimal period $\omega(a, b)$. Moreover, from the proofs of Propositions 4.1 and 4.2 it is easily seen that $p(\mathbb{R}) \subset [e^{-a}, \infty)$ holds for the range of the periodic solution p.

(ii) The periodic solution p guaranteed by Proposition 4.1 has the additional property that its range $p(\mathbb{R}) \subset [e^{-a}, \infty)$ is in the interval $(a/b, \infty)$ since $b > ae^a$. For solutions $x(t)$ of equation (E_∞) with range in $(a/b, \infty)$ a negative feedback property holds with respect to $\xi = 1$, that is, in case $x(t) = 1$ we have

$$x'(t) = -ax(t) + bf(x(t-1)) = -a + bx(t-1) > 0 \text{ provided } \frac{a}{b} < x(t-1) < 1,$$
$$x'(t) = -ax(t) + bf(x(t-1)) = -a < 0 \text{ provided } x(t-1) > 1.$$

In the range of the periodic solution p given by Proposition 4.2 the negative feedback property with respect to $\xi = 1$ does not hold for equation (E_∞).

(iii) Proposition 4.2 gives a periodic solution p with large minimal period $\omega(a, b)$ if b is close to a. For the minimal period $\omega(a, b)$ of the periodic solution p in Proposition 4.1, by the proof we have $\omega = t_1 + 1 + (1/a)\log x(t_1 + 1)$, $t_1 \in (1, 2)$, and then (4.5) easily implies the estimation

$$\frac{1}{a}\log\left(1 + be^a\right) < \omega(a, b) < \frac{2}{a}\log\left(1 + be^a\right).$$

4.2. A computer-assisted proof of (H)

First, recall that both (E_n) and (E_∞) are delay differential equations (DDE) with constant delay. A major difference is that, albeit the latter has a seemingly more appealing delayed term, the right–hand side of (E_∞) is non–smooth.

Our goal now is to utilize rigorous computations in order to verify that (H) holds for some pairs of a, b. Such techniques have been readily applied to DDEs both using global representations [3,12,5,13] and by propagating multiple local Taylor expansions of the solution over a single delay interval [37,38]. In the later approach, the state within the phase space is represented by a finite number of polynomials. This structure is well suited for solving the delayed system, however, e.g. computing Poincaré maps will readily increase the inaccuracy of the computations.

This is caused by the necessity of using intermediate values from the past that were not "stored" but may be obtained solely from one of those local expansions at an intermediate point.

The approach we choose is similar to [37,38], integrating forward in time. We have used the rigorous package CAPD as the foundation for our software [7]. The aforementioned non–smooth nature of (E_∞) gives rise to difficulties when directly applying methods of [37,38] as controlling the crossings of 1 is essentially a series of Poincaré maps that need to be computed. In order to overcome these difficulties, we leverage the specific form of (E_∞). The key observation is that the solution segments may be explicitly constructed when we restrict our attention to a certain subset of the phase space C_r^+.

We need some new notations. First, for a $\psi \in C([\gamma,\delta],\mathbb{R})$ and for $[\alpha,\beta] \subseteq [\gamma,\delta]$, let us introduce

$$\psi_{[\alpha,\beta]} : [0,\beta-\alpha] \to \mathbb{R}^+, \qquad \psi_{[\alpha,\beta]}(s) = \psi(\alpha+s).$$

Then, define the sets $C^{>1}([0,\delta])$, $C^+_{\mathrm{pol}}([0,\delta])$, $C^{\leq 1}_{\mathrm{pol}}([0,\delta])$, and $C^+_{\mathrm{seg}}([0,\delta])$ as

$$\begin{aligned}
C^{>1}([0,\delta]) &= \{\psi \in C([0,\delta],[1,\infty)) : \ \psi(s) > 1 \text{ for all } s \in (0,\delta)\}, \\
C^+_{\mathrm{pol}}([0,\delta]) &= \Big\{\psi \in C([0,\delta],\mathbb{R}^+) : \ \psi(s) = e^{-as}\sum_{k=0}^{n} c_k \frac{(bs)^k}{k!} \\
&\qquad \text{for some reals } c_0, c_1, \ldots, c_n \text{ with } s \in [0,\delta]\Big\}, \\
C^{\leq 1}_{\mathrm{pol}}([0,\delta]) &= \Big\{\psi \in C^+_{\mathrm{pol}}([0,\delta]) : \ \psi(s) \leq 1 \text{ for all } s \in [0,\delta]\Big\}, \text{ and} \\
C^+_{\mathrm{seg}}([0,\delta]) &= C^{>1}([0,\delta]) \ \cup \ C^{\leq 1}_{\mathrm{pol}}([0,\delta]), \text{ respectively.}
\end{aligned}$$

Finally, the space $C^+_{r,\mathrm{comp}}$ we use for rigorous computations is chosen as

$$\begin{aligned}
C^+_{r,\mathrm{comp}} = \Big\{\psi \in C_r^+ : \ \exists \{d_i\}_{i=0}^m \subset [-1,0] \text{ s.t. } -1 = d_0 < \ldots < d_i < \ldots < d_m = 0 \\
\text{and } \psi_{[d_i,d_{i+1}]} \in C^+_{\mathrm{seg}}([0,d_{i+1}-d_i]) \text{ for } i = 0,\ldots,m-1\Big\}.
\end{aligned} \tag{4.6}$$

That is, we consider those functions in C_r^+ that are comprised of finitely many subsegments being either above 1 or expressible as a polynomial multiplied with e^{-at} in local coordinates (*i.e.* time is shifted to start from 0).

Lemma 4.3. *Let $\psi \in C^+_{r,\mathrm{comp}}$ and $d \in (-1,0]$ such that $\psi_{[-1,d]} \in C^+_{\mathrm{seg}}([0,d+1])$. Then, $x^{\psi}_{[0,d+1]} \in C^+_{\mathrm{pol}}([0,d+1])$.*

In particular, if $\psi_{[-1,d]} \in C^{>1}([0,d+1])$, then $x^{\psi}_{[0,d+1]}(s) = e^{-as}\psi(0)$. On the other hand, if $\psi_{[-1,d]} \in C^{\leq 1}_{\mathrm{pol}}([0,d+1])$ with

$$\psi_{[-1,d]}(s) = e^{-as}\sum_{k=0}^{n} c_k \frac{(bs)^k}{k!},$$

then

$$x^{\psi}_{[0,d+1]}(s) = e^{-as}\left[\psi(0) + \sum_{k=1}^{n+1} c_{k-1}\frac{(bs)^k}{k!}\right]. \tag{4.7}$$

Proof. First, assume that $\psi_{[-1,d]} \in C^{>1}([0, d+1])$. Then, ($E_\infty$) implies

$$\frac{d}{dt}x^{\psi}(t) = -ax^{\psi}(t)$$

for $t \in [0, d+1]$ with $x^{\psi}(0) = \psi(0)$. Clearly, $x^{\psi}_{[0,d+1]}(t) = e^{-at}\psi(0)$ follows.

Second, consider the case $\psi_{[-1,d]} \in C^{\le 1}_{\text{pol}}([0, d+1])$ with

$$\psi_{[-1,d]}(s) = e^{-as}\sum_{k=0}^{n} c_k\frac{(bs)^k}{k!}.$$

Thus, (E_∞) is transformed into

$$\frac{d}{dt}x^{\psi}(t) = -ax^{\psi}(t) + be^{-at}\sum_{k=0}^{n} c_k\frac{(bt)^k}{k!}.$$

By the method of variation of constants, we readily obtain that

$$x^{\psi}_{[0,d+1]}(t) = e^{-at}\left[\psi(0) + \int_0^t b\sum_{k=0}^{n} c_k\frac{(bs)^k}{k!}ds\right] = e^{-at}\left[\psi(0) + \sum_{k=1}^{n+1} c_{k-1}\frac{(bt)^k}{k!}\right].$$

The claim $x^{\psi}_{[0,d+1]} \in C^{+}_{\text{pol}}([0, d+1])$ directly follows from the formulae derived above. □

Clearly, for any $\varphi \in C^{+}_{\text{pol}}([0, \delta])$ the set $\{s \in [0, \delta] : \varphi(s) = 1\}$ is finite. Therefore, a consequence of Lemma 4.3 is that $C^{+}_{r,\text{comp}}$ is invariant, that is, $F(t, C^{+}_{r,\text{comp}}) \subseteq C^{+}_{r,\text{comp}}$ for all $t \ge 0$.

Lemma 4.3 provides the basis for the rigorous computational procedure as it gives exact formulae for the subsegments of the solution. Note that the coefficients c_k propagate unchanged (just re-indexed) to the next subsegment. This is notably beneficial for interval methods.

Given an initial condition $\psi \in C^{+}_{r,\text{comp}}$, in order to compute the solution we still need to subdivide the continuing segment from Lemma 4.3 into subsegments where the solution is above or below 1. This procedure consists of two parts. First, all crossings of 1 have to be found and localized with sufficient precision. Then, for all subsegments where the solution is below 1, the coefficients in the polynomial part have to be obtained.

For localizing crossings we have used the rigorous Newton method capable of proving the existence of zeros of functions [33,39]. For the latter task, the following lemma provides a simple computational scheme.

Lemma 4.4. *Let* $\varphi \in C^+_{\mathrm{pol}}([0,\delta])$ *and* $\tau \in [0,\delta]$. *If*

$$\varphi(t) = e^{-at}\sum_{k=0}^{n} c_k \frac{(bt)^k}{k!},$$

then

$$\varphi_{[\tau,\delta]}(t) = e^{-at}\sum_{k=0}^{n}\left(e^{-a\tau}\cdot\frac{d^k}{dt^k}\frac{1}{b^k}e^{at}\varphi(t)\bigg|_{t=\tau}\right)\frac{(bt)^k}{k!}.$$

Proof. First, recall that for a polynomial

$$p(t) = \sum_{k=0}^{n}\eta_k t^k,$$

it is well known that

$$\eta_k = \frac{1}{k!}\frac{d^k}{dt^k}p(t)\bigg|_{t=0}.$$

Now, by definition,

$$\varphi_{[\tau,\delta]}(t) = \varphi(\tau+t) \qquad \text{for } t \in [0,\delta-\tau]$$

holds. Hence, as $e^{at}\varphi(t)$ is a polynomial of degree n, so is

$$e^{at}\varphi_{[\tau,\delta]}(t) = e^{-a\tau}e^{a(\tau+t)}\varphi_{[\tau,\delta]}(t) = e^{-a\tau}e^{a(\tau+t)}\varphi(\tau+t).$$

Therefore,

$$\begin{aligned}
e^{at}\varphi_{[\tau,\delta]}(t) = e^{-a\tau}e^{a(\tau+t)}\varphi(\tau+t) &= e^{-a\tau}\sum_{k=0}^{n}\left(\frac{1}{k!}\frac{d^k}{dt^k}e^{a(\tau+t)}\varphi(\tau+t)\bigg|_{t=0}\cdot t^k\right)\\
&= \sum_{k=0}^{n}\left(e^{-a\tau}\cdot\frac{d^k}{dt^k}\frac{1}{b^k}e^{a(\tau+t)}\varphi(\tau+t)\bigg|_{t=0}\cdot\frac{(bt)^k}{k!}\right)\\
&= \sum_{k=0}^{n}\left(e^{-a\tau}\cdot\frac{d^k}{dt^k}\frac{1}{b^k}e^{at}\varphi(t)\bigg|_{t=\tau}\cdot\frac{(bt)^k}{k!}\right). \quad \square
\end{aligned}$$

Using the results presented in this section, we may formulate the rigorous integration procedure for (E_∞) within $C^+_{r,\mathrm{comp}}$. In the following, let $\mathbb{IR}$ denote the set of real intervals. We note that all computations are performed using interval arithmetic, the differentiation in Lemma 4.4 is carried out using automatic differentiation, hence, all computational results are rigorous.

Algorithm 1: Rigorous integration of (E_∞).

Data: $a, b \in \mathbb{IR}$, $\psi \in C^+_{r,\text{comp}}$
Step 1: Obtain d_1 from ψ;
Step 2: Compute $x^{\psi}_{[0,d_1+1]}(t)$ by Lemma 4.3;
Step 3: Find all $\tau \in [0, d_1 + 1]$ s.t. $x^{\psi}_{[0,d_1+1]}(t)$ crosses 1 transversely at $t = \tau$;
Step 3b: If non-transverse / uncertain crossings exist, then Abort;
Data: Transverse crossings $\tau_1, \tau_2, \dots, \tau_M$
Step 4: Compute the proper representations over
$[0, \tau_1], \dots, [\tau_k, \tau_{k+1}], \dots, [\tau_M, d_1 + 1]$ using Lemma 4.4;
Result: $x^{\psi}_{[d_1,d_1+1]} \in C^+_{r,\text{comp}}$

Algorithm 1 describes one-step forward integration from an initial $\psi \in C^+_{r,\text{comp}}$. The length of this step equals to the first element of the time subdivision in the representation of $\psi \in C^+_{r,\text{comp}}$. Clearly, we may repeat Algorithm 1 and obtain a rigorous enclosure of $x^{\psi}(t)$.

Using an initial condition $\psi \in C^+_{r,\text{comp}}$ satisfying $\psi(t) > 1$ for $t \in [-1, 0)$ and $\psi(0) = 1$, we may repeatedly execute our algorithm. If, at any point, we can guarantee that there exists $\tau > 0$ such that $x^{\psi}_{[\tau,\tau+1]}(t) > 1$, then, we have proved the existence of a periodic orbit and, at the same time, obtained a rigorous enclosure of its trajectory.

In the following, we present the results of our computations, namely, pairs of a and b for which the proof was successful. The code for our software may be found at [2].

Proposition 4.5. *System (E_∞) attains a periodic solution satisfying condition (H) for parameter values*

- $a = 2$, $b = 15$ *with* $\omega \in [2.493021, 2.495419]$.
- $a = 7$, $b = 7.5$ *with* $\omega \in [36.564308, 36.565093]$.
- $a = 4.22$, $b = 7$ *with* $\omega \in [16.689452, 16.693050]$.
- $a = 4$, $b = 10$ *with* $\omega \in [6.847648, 6.851446]$.
- $a = 5.95$, $b = 10$ *with* $\omega \in [10.101741, 10.104210]$.
- $a = 7.04$, $b = 10$ *with* $\omega \in [19.837935, 19.840255]$.

Figs. 1–6 present the corresponding trajectories and phase portraits. Figs. 1 and 2 illustrate the typical simple looking periodic solutions guaranteed by Propositions 4.1 and 4.2, respectively. Figs. 3–6 show more complicated shapes.

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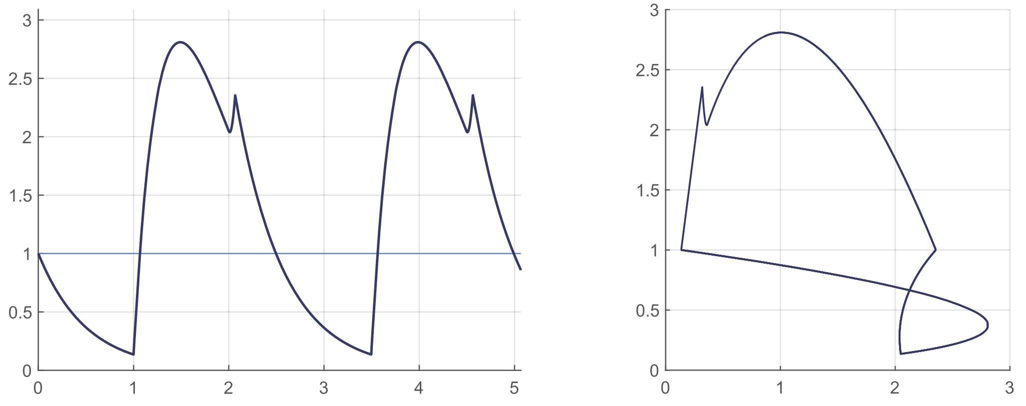

Fig. 1. Periodic solution for $a = 2$, $b = 15$. Left: trajectory. Right: $(x(t), x(t-1))$ phase portrait.

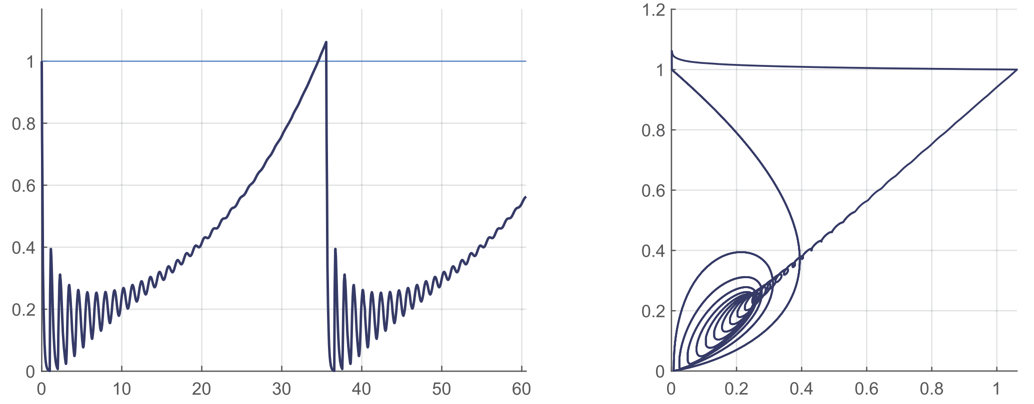

Fig. 2. Periodic solution for $a = 7$, $b = 7.5$. Left: trajectory. Right: $(x(t), x(t-1))$ phase portrait.

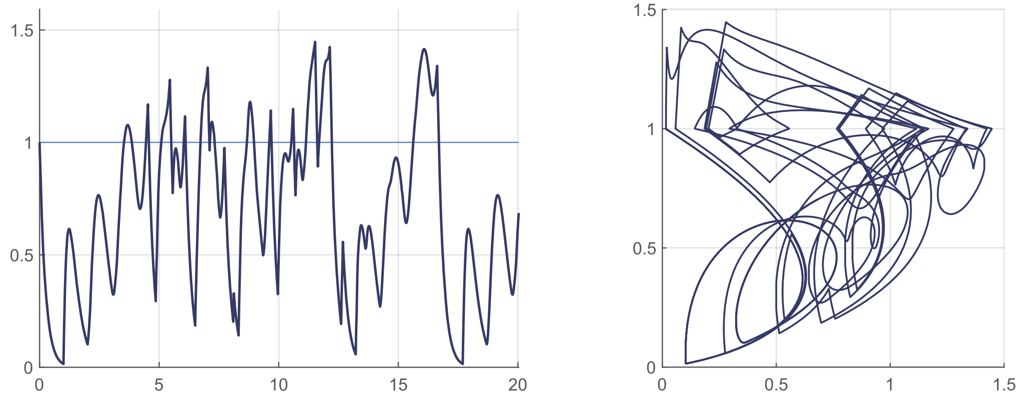

Fig. 3. Periodic solution for $a = 4.22$, $b = 7$. Left: trajectory. Right: $(x(t), x(t-1))$ phase portrait.

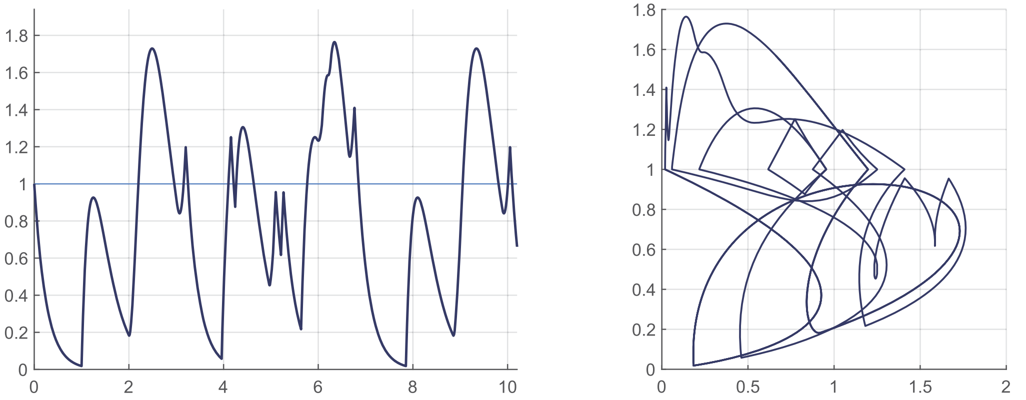

Fig. 4. Periodic solution for $a = 4$, $b = 10$. Left: trajectory. Right: $(x(t), x(t-1))$ phase portrait.

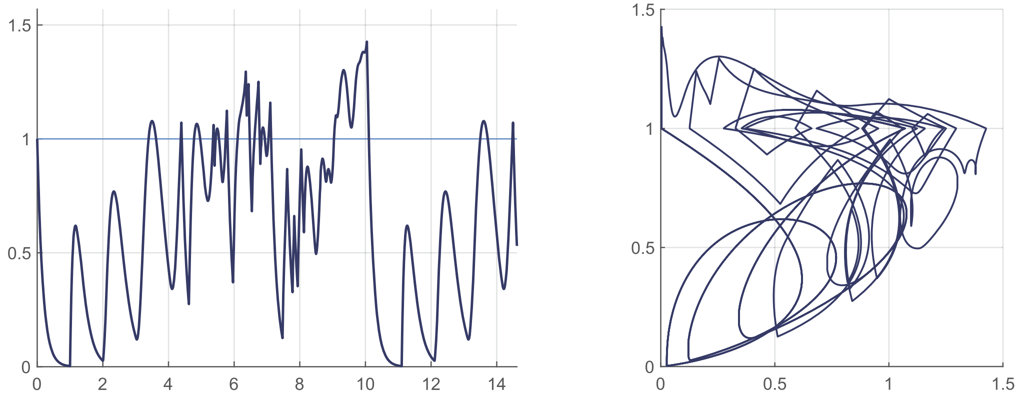

Fig. 5. Periodic solution for $a = 5.95$, $b = 10$. Left: trajectory. Right: $(x(t), x(t-1))$ phase portrait.

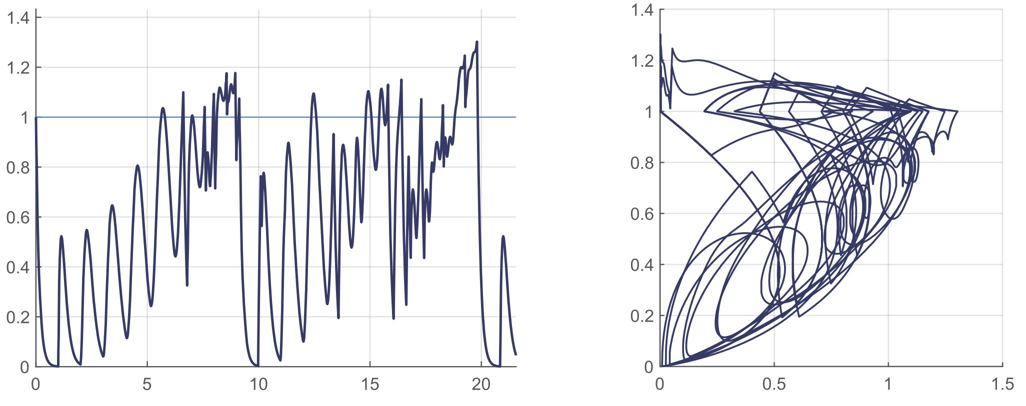

Fig. 6. Periodic solution for $a = 7.04$, $b = 10$. Left: trajectory. Right: $(x(t), x(t-1))$ phase portrait.

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