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RADII OF k-STARLIKENESS OF ORDER α OF STRUVE AND LOMMEL FUNCTIONS

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Abstract. In the present work our main objective is to determine the radii of $k-$ starlikeness of order α of the some normalized Struve and Lommel functions of the first kind. Furthermore it has been shown that the obtained radii satisfy some functional equations. The main key tool of our proofs are the Mittag-Leffler expansions of the Struve and Lommel functions of the first kind and minimum principle for harmonic functions. Also we take advantage of some basic inequalities in the complex analysis.

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1. INTRODUCTION

It is well-known that there are numerous connections between geometric function theory and special functions. Due to these close relationships many authors studied on some geometric properties of special functions like Bessel, Struve, Lommel, Wright and Mittag-Leffler functions. Especially, the authors in the papers [3–5, 7, 14–16, 19] have investigated univalence, starlikeness, convexity and close-to convexity of the above mentioned functions. Actually, the beginning of these studies is based on the papers [6, 12, 21] written by Brown, Kreyszig and Todd and Wilf, respectively. Also the authors who studied the geometric properties of special functions have used some properties of zeros of the mentioned special functions. For comprehensive information about the zeros of these functions, we refer to the studies [17, 18, 20]. Motivated by the earlier investigations on this field our main goal is to determine the radii of k-starlikeness of the normalized Struve and Lommel functions of the first kind. Morever, we show that our obtained radii are the smallest positive roots of some functional equations. Also, for some special values of k and α we obtain some earlier results given by [1–3].

Now we would like to remind some basic concepts in geometric function theory.

Let $\mathbb{D}_r$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$ with radius $r > 0$ and $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{A}$ denote the class of analytic functions $f : \mathbb{D}_r \to \mathbb{C}$,

$$f(z) = z + \sum_{n\geq 2} a_n z^n,$$

which satisfies the normalization conditions $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$ we mean the class of functions belonging to $\mathcal{A}$ which are univalent in $\mathbb{D}_r$. The class of k-starlike functions of order α is denoted by $\mathcal{ST}(k,\alpha)$, where $k \geq 0$ and $0 \leq \alpha < 1$. This class of functions was introduced by Kanas and Wiśniowska [10, 11] which generalizes the class of uniformly convex functions introduced by Goodman in [8]. On the other hand, Kanas and Srivastava defined a linear operator and determined some conditions on the parameters for which this linear operator maps the classes of starlike and univalent functions onto the classes $k-$uniformly convex functions and $k-$starlike functions in [9]. Very recently, Srivastava gave comprehensive information about the usages of $q-$analysis in geometric function theory of complex analysis in his *survey-cum-expository* article [13]. Srivastava's work in particular inspired us to prepare this paper.

Analytic characterization of the class k-starlike functions of order α is

$$\mathcal{ST}(k,\alpha) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha, k \geq 0, 0 \leq \alpha < 1, z \in \mathbb{D} \right\}.$$

Also, the real number

$$r(f) = \sup\left\{ r > 0 : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha \text{ for all } z \in \mathbb{D} \right\}$$

is called the radius of $k-$starlikeness of order α of the function f.

The Struve and Lommel functions are defined as the infinite series

$$\mathbf{H}_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu+1}, \ -\nu - \frac{3}{2} \notin \mathbb{N},$$

and

$$s_{\mu,\nu}(z) = \frac{(z)^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} \sum_{n\geq 0} \frac{(-1)^n}{\left(\frac{\mu-\nu+3}{2}\right)_n\left(\frac{\mu+\nu+3}{2}\right)_n} \left(\frac{z}{2}\right)^{2n}, \ \frac{1}{2}(-\mu \pm \nu - 3) \notin \mathbb{N},$$

where $z, \mu, \nu \in \mathbb{C}$. Also, we know that the Struve and Lommel functions are the solutions of the inhomogeneous Bessel differential equations

$$zw''(z) + zw'(z) + (z^2 - \nu^2)w(z) = \frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}$$

and

$$zw''(z) + zw'(z) + (z^2 - \nu^2)w(z) = z^{\mu+1},$$

respectively. One can find comprehensive information about these functions in [20].

Since the functions $\mathbf{H}_\nu$ and $s_{\mu,\nu}$ do not belong to the class $\mathcal{A}$, first we consider the following six normalized forms:

$$u_\nu(z) = \left(\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)\right)^{\frac{1}{\nu+1}}, \quad \nu \neq -1, \tag{1.1}$$

$$v_\nu(z) = \sqrt{\pi}2^\nu z^{-\nu}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z), \tag{1.2}$$

$$w_\nu(z) = \sqrt{\pi}2^\nu z^{\frac{1-\nu}{2}}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(\sqrt{z}), \tag{1.3}$$

$$f_\mu(z) = \left(\mu(\mu+1)s_{\mu-\frac{1}{2},\frac{1}{2}}(z)\right)^{\frac{1}{\mu+\frac{1}{2}}}, \quad \mu \in \left(-\frac{1}{2},1\right), \quad \mu \neq 0, \tag{1.4}$$

$$g_\mu(z) = \mu(\mu+1)z^{-\mu+\frac{1}{2}}s_{\mu-\frac{1}{2},\frac{1}{2}}(z) \tag{1.5}$$

and

$$h_\mu(z) = \mu(\mu+1)z^{\frac{3-2\mu}{4}}s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{z}). \tag{1.6}$$

As a consequence, all functions considered above belong to the analytic functions class $\mathcal{A}$.

2. Main Results

Our first main result is related to the normalized Struve functions as follows.

Theorem 1. *Let $|\nu| \leq \frac{1}{2}$, $0 \leq \alpha < 1$ and $k \geq 0$. Then, the following assertions are true:*

- **i.** *The radius r_u is the radius of $k-$starlikeness of order α of the normalized Struve function $z \mapsto u_\nu$ and it is the smallest positive root of the equation*

$$r(1+k)\mathbf{H}'_\nu(r) - (k+\alpha)(\nu+1)\mathbf{H}_\nu(r) = 0 \tag{2.1}$$

  in $(0, h_{\nu,1})$, where $h_{\nu,1}$ is the first positive zero of Struve function $\mathbf{H}_\nu$.
- **ii.** *The radius r_v is the radius of $k-$starlikeness of order α of the normalized Struve function $z \mapsto v_\nu$ and it is the smallest positive root of the equation*

$$r(1+k)\mathbf{H}'_\nu(r) - [\nu(1+k)+(k+\alpha)]\mathbf{H}_\nu(r) = 0 \tag{2.2}$$

  in $(0, h_{\nu,1})$.
- **iii.** *The radius r_w is the radius of $k-$starlikeness of order α of the normalized Struve function $z \mapsto w_\nu$ and it is the smallest positive root of the equation*

$$(1+k)\sqrt{r}\mathbf{H}'_\nu(\sqrt{r}) + (1-\nu-k-\nu k-2\alpha)\mathbf{H}_\nu(\sqrt{r}) = 0 \tag{2.3}$$

  in $(0, h^2_{\nu,1})$.

Proof. We know that the zeros of the functions $\mathbf{H}_\nu(z)$ and $\mathbf{H}'_\nu(z)$ are real and simple when $|\nu| \leq \frac{1}{2}$, (see [4, 17]). Also the zeros of the function $\mathbf{H}_\nu(z)$ and its derivative interlace when $|\nu| \leq \frac{1}{2}$, according to [4]. In addition, it is known from [4] that the Struve function $\mathbf{H}_\nu(z)$ has the following infinite product representation:

$$\sqrt{\pi}2^\nu z^{-\nu-1}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{h_{\nu,n}^2}\right), \tag{2.4}$$

where $h_{\nu,n}$ denotes $n-$th positive zero of the Struve function $\mathbf{H}_\nu$. Using this product representation one can easily see that

$$\frac{zu'_\nu(z)}{u_\nu(z)} = 1-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2}, \tag{2.5}$$

$$\frac{zv'_\nu(z)}{v_\nu(z)} = 1-2\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2} \tag{2.6}$$

and

$$\frac{zw'_\nu(z)}{w_\nu(z)} = 1-\sum_{n\geq 1}\frac{z}{h_{\nu,n}^2-z}. \tag{2.7}$$

On the other hand, it is known from [19] that the inequality

$$\Re\left(\frac{z}{\theta-z}\right) \leq \frac{|z|}{\theta-|z|} \tag{2.8}$$

holds true for $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$ such that $|z| < \theta$. Now, by using inequality (2.8) in (2.5), (2.6) and (2.7), respectively, we get

$$\begin{aligned}\Re\left(\frac{zu'_\nu(z)}{u_\nu(z)}\right) &= \Re\left(1-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2}\right)\\ &\geq 1-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{|z|^2}{h_{\nu,n}^2-|z|^2}\\ &= \frac{|z|\,u'_\nu(|z|)}{u_\nu(|z|)},\end{aligned} \tag{2.9}$$

$$\Re\left(\frac{zv'_\nu(z)}{v_\nu(z)}\right) = \Re\left(1-2\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2}\right) \geq 1-2\sum_{n\geq 1}\frac{|z|^2}{h_{\nu,n}^2-|z|^2} = \frac{|z|\,v'_\nu(|z|)}{v_\nu(|z|)} \tag{2.10}$$

and

$$\Re\left(\frac{zw'_\nu(z)}{w_\nu(z)}\right) = \Re\left(1-\sum_{n\geq 1}\frac{z}{h_{\nu,n}^2-z}\right) \geq 1-\sum_{n\geq 1}\frac{|z|}{h_{\nu,n}^2-|z|} = \frac{|z|\,w'_\nu(|z|)}{w_\nu(|z|)}. \tag{2.11}$$

Also, from the reverse triangle inequality

$$|z_1-z_2|| \geq ||z_1|-|z_2||$$

we have

$$\left|\frac{zu_\nu'(z)}{u_\nu(z)}-1\right|=\left|-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2}\right|\leq\frac{2}{\nu+1}\sum_{n\geq 1}\frac{|z|^2}{h_{\nu,n}^2-|z|^2}=1-\frac{|z|\,u_\nu'(|z|)}{u_\nu(|z|)}, \tag{2.12}$$

$$\left|\frac{zv_\nu'(z)}{v_\nu(z)}-1\right|=\left|-2\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2}\right|\leq 2\sum_{n\geq 1}\frac{|z|^2}{h_{\nu,n}^2-|z|^2}=1-\frac{|z|\,v_\nu'(|z|)}{v_\nu(|z|)} \tag{2.13}$$

and

$$\left|\frac{zw_\nu'(z)}{w_\nu(z)}-1\right|=\left|-\sum_{n\geq 1}\frac{z}{h_{\nu,n}^2-z}\right|\leq\sum_{n\geq 1}\frac{|z|}{h_{\nu,n}^2-|z|}=1-\frac{|z|\,w_\nu'(|z|)}{w_\nu(|z|)}. \tag{2.14}$$

As a result of the above inequalities, one can easily obtain that

$$\Re\left(\frac{zu_\nu'(z)}{u_\nu(z)}\right)-k\left|\frac{zu_\nu'(z)}{u_\nu(z)}-1\right|-\alpha\geq(1+k)\frac{|z|\,u_\nu'(|z|)}{u_\nu(|z|)}-(k+\alpha), \tag{2.15}$$

$$\Re\left(\frac{zv_\nu'(z)}{v_\nu(z)}\right)-k\left|\frac{zv_\nu'(z)}{v_\nu(z)}-1\right|-\alpha\geq(1+k)\frac{|z|\,v_\nu'(|z|)}{v_\nu(|z|)}-(k+\alpha), \tag{2.16}$$

and

$$\Re\left(\frac{zw_\nu'(z)}{w_\nu(z)}\right)-k\left|\frac{zw_\nu'(z)}{w_\nu(z)}-1\right|-\alpha\geq(1+k)\frac{|z|\,w_\nu'(|z|)}{w_\nu(|z|)}-(k+\alpha). \tag{2.17}$$

It is important to emphasize here that the equalities in the last three inequalities hold true for $z=|z|=r$. If we consider the minimum principle for harmonic functions in the inequalities (2.15), (2.16) and (2.17), then we can say that these inequalities are valid if and only if $|z|<r_u$, $|z|<r_v$ and $|z|<r_w$, where r_u, r_v and r_w are the smallest positive roots of the following equations

$$(1+k)\frac{ru_\nu'(r)}{u_\nu(r)}-(k+\alpha)=0,$$

$$(1+k)\frac{rv_\nu'(r)}{v_\nu(r)}-(k+\alpha)=0$$

and

$$(1+k)\frac{rw_\nu'(r)}{w_\nu(r)}-(k+\alpha)=0,$$

respectively. Taking into account the definitions of the functions u_ν, v_ν and w_ν, it can be easily seen that the last three equations are equivalent to (2.1), (2.2) and (2.3), respectively. Now, we would like to show that equation (2.1) has an unique root on the interval $(0,h_{\nu,1})$. To show this, let us consider the function $\Psi_\nu:(0,h_{\nu,1})\mapsto\mathbb{R}$,

$$\Psi_\nu(r)=(1+k)\frac{ru_\nu'(r)}{u_\nu(r)}-(k+\alpha)=(1+k)\left(1-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{r^2}{h_{\nu,n}^2-r^2}\right)-(k+\alpha).$$

The function $r \mapsto \Psi_\nu(r)$ is strictly decreasing since

$$\Psi_\nu'(r) = -\frac{4r(1+k)}{\nu+1}\sum_{n\geq 1}\frac{h_{\nu,n}^2}{\left(h_{\nu,n}^2 - r^2\right)^2} < 0.$$

Morever, we have

$$\lim_{r\searrow 0}(1+k)\left(1 - \frac{2}{\nu+1}\sum_{n\geq 1}\frac{r^2}{h_{\nu,n}^2 - r^2}\right) - (k+\alpha) = 1-\alpha > 0$$

and

$$\lim_{r\nearrow h_{\nu,1}}(1+k)\left(1 - \frac{2}{\nu+1}\sum_{n\geq 1}\frac{r^2}{h_{\nu,n}^2 - r^2}\right) - (k+\alpha) = -\infty.$$

As a result of these limit relations, we can say that equation (2.1) has an unique root in $(0, h_{\nu,1})$. Similarly, it can be shown that equations (2.2) and (2.3) have a root in $(0, h_{\nu,1})$ and $(0, h_{\nu,1}^2)$, respectively. □

The following main result is regarding the normalized Lommel functions of the first kind.

Theorem 2. *The following assertions are true:*

i. *Let $\mu \in (-\frac{1}{2}, 1)$ and $\mu \neq 0$. Then, the radius r_f is the radius of $k-$starlikeness of order α of the normalized Lommel function $z \mapsto f_\mu$ and it is the smallest positive root of the equation*

$$r(1+k)s'_{\mu-\frac{1}{2},\frac{1}{2}}(r) - (k+\alpha)(\mu+\frac{1}{2})s_{\mu-\frac{1}{2},\frac{1}{2}}(r) = 0 \tag{2.18}$$

in $(0, l_{\mu,1})$, where $l_{\mu,1}$ is the first positive zero of Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$.

ii. *Let $\mu \in (-1, 1)$ and $\mu \neq 0$. Then, the radius r_g is the radius of $k-$starlikeness of order α of the normalized Lommel function $z \mapsto g_\mu$ and it is the smallest positive root of the equation*

$$r(1+k)s'_{\mu-\frac{1}{2},\frac{1}{2}}(r) + \left((1+k)(\frac{1}{2}-\mu) - (k+\alpha)\right)s_{\mu-\frac{1}{2},\frac{1}{2}}(r) = 0 \tag{2.19}$$

in $(0, l_{\mu,1})$.

iii. *Let $\mu \in (-1, 1)$ and $\mu \neq 0$. Then, the radius r_h is the radius of $k-$starlikeness of order α of the normalized Lommel function $z \mapsto h_\mu$ and it is the smallest positive root of the equation*

$$2\sqrt{r}(1+k)s'_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r}) + ((1+k)(3-2\mu) - 4(k+\alpha))\,s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r}) = 0 \tag{2.20}$$

in $(0, l_{\mu,1}^2)$.

Proof. It is known from [4, 18] that the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and its derivative $s'_{\mu-\frac{1}{2},\frac{1}{2}}$ have only real and simple zeros when $\mu \in (-1,1)$ and $\mu \neq 0$. Morever, the zeros of the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and its derivative $s'_{\mu-\frac{1}{2},\frac{1}{2}}$ interlace under the same conditions, according to [4]. Also, the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$ can be written as the product (see [4])

$$s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \frac{z^{\mu+\frac{1}{2}}}{\mu(\mu+1)} \prod_{n\geq 1}\left(1-\frac{z^2}{l^2_{\mu,n}}\right), \tag{2.21}$$

where $l_{\mu,n}$ denotes $n-$th positive zero of the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$. Using equality (2.21), it can be easily seen that

$$\frac{zf'_\mu(z)}{f_\mu(z)} = 1-\frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2}, \tag{2.22}$$

$$\frac{zg'_\mu(z)}{g_\mu(z)} = 1-2\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2} \tag{2.23}$$

and

$$\frac{zh'_\mu(z)}{h_\mu(z)} = 1-\sum_{n\geq 1}\frac{z}{l^2_{\mu,n}-z}. \tag{2.24}$$

Now, if we consider inequality (2.8) in the equalities (2.22), (2.23) and (2.24), respectively, then we have that

$$\begin{aligned}\Re\left(\frac{zf'_\mu(z)}{f_\mu(z)}\right) &= \Re\left(1-\frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2}\right)\\ &\geq 1-\frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{|z|^2}{l^2_{\mu,n}-|z|^2}\\ &= \frac{|z|\,f'_\mu(|z|)}{f_\mu(|z|)},\end{aligned} \tag{2.25}$$

$$\Re\left(\frac{zg'_\mu(z)}{g_\mu(z)}\right) = \Re\left(1-2\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2}\right) \geq 1-2\sum_{n\geq 1}\frac{|z|^2}{l^2_{\mu,n}-|z|^2} = \frac{|z|\,g'_\mu(|z|)}{g_\nu(|z|)} \tag{2.26}$$

and

$$\Re\left(\frac{zh'_\mu(z)}{h_\mu(z)}\right) = \Re\left(1-\sum_{n\geq 1}\frac{z}{l^2_{\nu,n}-z}\right) \geq 1-\sum_{n\geq 1}\frac{|z|}{l^2_{\mu,n}-|z|} = \frac{|z|\,h'_\mu(|z|)}{h_\mu(|z|)}. \tag{2.27}$$

By using the reverse triangle inequality again we can write that

$$\left|\frac{zf'_\mu(z)}{f_\mu(z)}-1\right| = \left|-\frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2}\right| \leq \frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{|z|^2}{l^2_{\mu,n}-|z|^2} = 1-\frac{|z|\,f'_\mu(|z|)}{f_\mu(|z|)}, \tag{2.28}$$

$$\left|\frac{zg'_\mu(z)}{g_\mu(z)}-1\right| = \left|-2\sum_{n\geq1}\frac{z^2}{l^2_{\mu,n}-z^2}\right| \leq 2\sum_{n\geq1}\frac{|z|^2}{l^2_{\mu,n}-|z|^2} = 1-\frac{|z|\,g'_\mu(|z|)}{g_\mu(|z|)} \tag{2.29}$$

and

$$\left|\frac{zh'_\mu(z)}{h_\mu(z)}-1\right| = \left|-\sum_{n\geq1}\frac{z}{l^2_{\mu,n}-z}\right| \leq \sum_{n\geq1}\frac{|z|}{l^2_{\mu,n}-|z|} = 1-\frac{|z|\,h'_\mu(|z|)}{h_\mu(|z|)}. \tag{2.30}$$

As consequences of the above inequalities, it can be easily obtained that

$$\Re\left(\frac{zf'_\mu(z)}{f_\mu(z)}\right)-k\left|\frac{zf'_\mu(z)}{f_\mu(z)}-1\right|-\alpha \geq (1+k)\frac{|z|\,f'_\mu(|z|)}{f_\mu(|z|)}-(k+\alpha), \tag{2.31}$$

$$\Re\left(\frac{zg'_\mu(z)}{g_\mu(z)}\right)-k\left|\frac{zg'_\mu(z)}{g_\mu(z)}-1\right|-\alpha \geq (1+k)\frac{|z|\,g'_\mu(|z|)}{g_\mu(|z|)}-(k+\alpha) \tag{2.32}$$

and

$$\Re\left(\frac{zh'_\mu(z)}{h_\mu(z)}\right)-k\left|\frac{zh'_\mu(z)}{h_\mu(z)}-1\right|-\alpha \geq (1+k)\frac{|z|\,h'_\mu(|z|)}{h_\mu(|z|)}-(k+\alpha). \tag{2.33}$$

It is worth mentioning that the equalities in the inequalities (2.31), (2.32) and (2.33) hold true for $z=|z|=r$. Also, if we consider the minimum principle for harmonic functions in these inequalities, then we can say that these inequalities are valid if and only if $|z|<r_f$, $|z|<r_g$ and $|z|<r_h$, where r_f, r_g and r_h are the smallest positive roots of the following equations

$$(1+k)\frac{rf'_\mu(r)}{f_\mu(r)}-(k+\alpha)=0,$$

$$(1+k)\frac{rg'_\mu(r)}{g_\mu(r)}-(k+\alpha)=0$$

and

$$(1+k)\frac{rh'_\mu(r)}{h_\mu(r)}-(k+\alpha)=0,$$

respectively. Taking into account the definitions of the functions f_μ, g_μ and h_μ, it can be easily seen that the last three equations are equivalent to (2.18), (2.19) and (2.20), respectively. In addition, we can easily show that equations (2.18) and (2.19) have one root in the interval $(0,l_{\mu,1})$, while equation (2.20) has a root in $(0,l^2_{\mu,1})$. Because the proof of these assertions are similar to the proof of the previous theorem, details are omitted. □

Remark 1. For $k=0$ and $k=\alpha=0$, Theorem 1 and Theorem 2 reduce to some earlier results given by [1–3], respectively.

Now, we would like present some applications regarding our main results. For this, we consider the following relationships between Struve and elementary trigonometric functions:

$$\mathbf{H}_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin z \text{ and } \mathbf{H}_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}(1-\cos z).$$

Using these relationships for $\nu=-\frac{1}{2}$ and $\nu=\frac{1}{2}$, we have

$$u_{\frac{1}{2}}(z)=\left(\frac{2(1-\cos z)}{\sqrt{z}}\right)^{\frac{2}{3}},\ v_{\frac{1}{2}}(z)=\frac{2(1-\cos z)}{z},\ w_{\frac{1}{2}}(z)=2(1-\cos\sqrt{z})$$

and

$$u_{-\frac{1}{2}}(z)=\frac{\sin^2 z}{z},\ v_{-\frac{1}{2}}(z)=\sin z,\ w_{-\frac{1}{2}}(z)=\sqrt{z}\sin\sqrt{z}.$$

Corollary 1. *The following statements are true.*

i. *The radius of $k-$starlikeness of order α of the function $u_{\frac{1}{2}}(z)=\left(\frac{2(1-\cos z)}{\sqrt{z}}\right)^{\frac{2}{3}}$ is the smallest positive root of the equation*

$$2(1+k)r\sin r+(1+4k+3\alpha)(\cos r-1)=0$$

in $\left(0,h_{\frac{1}{2},1}\right)$.

ii. *The radius of $k-$starlikeness of order α of the function $v_{\frac{1}{2}}(z)=\frac{2(1-\cos z)}{z}$ is the smallest positive root of the equation*

$$(1+k)r\sin r+(1+2k+\alpha)(\cos r-1)=0$$

in $\left(0,h_{\frac{1}{2},1}\right)$.

iii. *The radius of $k-$starlikeness of order α of the function $w_{\frac{1}{2}}(z)=2(1-\cos\sqrt{z})$ is the smallest positive root of the equation*

$$(1+k)\sqrt{r}\sin\sqrt{r}+2(k+\alpha)(\cos\sqrt{r}-1)=0$$

in $\left(0,h^2_{\frac{1}{2},1}\right)$.

iv. *The radius of $k-$starlikeness of order α of the function $u_{-\frac{1}{2}}(z)=\frac{\sin^2 z}{z}$ is the smallest positive root of the equation*

$$2(1+k)r\cos r-(1+2k+\alpha)\sin r=0$$

in $\left(0,h_{-\frac{1}{2},1}\right)$.

v. *The radius of $k-$starlikeness of order α of the function $v_{-\frac{1}{2}}(z)=\sin z$ is the smallest positive root of the equation*

$$(1+k)r\cos r-(k+\alpha)\sin r=0$$

in $\left(0,h_{-\frac{1}{2},1}\right)$.

vi. *The radius of $k-$starlikeness of order α of the function $w_{-\frac{1}{2}}(z) = \sqrt{z}\sin\sqrt{z}$ is the smallest positive root of the equation*

$$(1+k)\sqrt{r}\cos\sqrt{r} - (k+2\alpha-1)\sin\sqrt{r} = 0$$

in $\left(0, h^2_{-\frac{1}{2},1}\right)$.

Now, by taking $k = \alpha = 0$ in Corollary 1 we get the following result.

Corollary 2. *The following assertions are true.*

- **i.** *The radius of starlikeness of the function $u_{\frac{1}{2}}(z) = \left(\frac{2(1-\cos z)}{\sqrt{z}}\right)^{\frac{2}{3}}$ is $r \cong 2.7865$ and it is the smallest positive root of the equation $2r\sin r + \cos r - 1 = 0$.*
- **ii.** *The radius of starlikeness of the function $v_{\frac{1}{2}}(z) = \frac{2(1-\cos z)}{z}$ is $r \cong 2.33112$ and it is the smallest positive root of the equation $2r\sin r + 2\cos r - 1 = 0$.*
- **iii.** *The radius of starlikeness of the function $w_{\frac{1}{2}}(z) = 2(1-\cos\sqrt{z})$ is $r \cong 9.8696$ and it is the smallest positive root of the equation $\sqrt{r}\sin\sqrt{r} = 0$.*
- **iv.** *The radius of starlikeness of the function $u_{-\frac{1}{2}}(z) = \frac{\sin^2 z}{z}$ is $r \cong 1.16556$ and it is the smallest positive root of the equation $2r\cos r - \sin r = 0$.*
- **v.** *The radius of starlikeness of the function $v_{-\frac{1}{2}}(z) = \sin z$ is $r \cong 1.5708$ and it is the smallest positive root of the equation $r\cos r = 0$.*
- **vi.** *The radius of starlikeness of the function $w_{-\frac{1}{2}}(z) = \sqrt{z}\sin\sqrt{z}$ is $r \cong 4.11586$ and it is the smallest positive root of the equation $\sqrt{r}\cos\sqrt{r} + \sin\sqrt{r} = 0$.*

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