Miskolc Mathematical Notes
Vol. 22 (2021), No. 1, pp. 211–244
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2021.3050

S^p-ALMOST PERIODIC AND S^p-ALMOST AUTOMORPHIC SOLUTIONS OF AN INTEGRAL EQUATION WITH INFINITE DELAY

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Received 26 September, 2019

Abstract. We state sufficient conditions for the existence and uniqueness of Stepanov-like pseudo almost periodic and Stepanov-like pseudo almost automorphic solutions for a class of nonlinear Volterra integral with infinite delay of the form

$$x(t) = f(t,x(t),x(t-r(t))) - \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s.$$

Our approach is based on Bochner's transform, some analytic techniques, and a Banach fixed point theorem. Then we apply these results to a nonlinear differential equation when the delay is time-dependent and the force function is continuous

$$x'(t) = ax(t) + \alpha x'(t-r(t)) - q(t,x(t),x(t-r(t))) + h(t).$$

2010 *Mathematics Subject Classification:* 34A12; 34C27; 43A60

Keywords: integral equation, almost periodic, almost automorphic, pseudo almost periodic, pseudo almost automorphic, infinite delay

1. INTRODUCTION

The existence and stability of almost periodic solutions of some models are among the most attractive topics in the qualitative theory of differential and integral equations due to their applications in physical science, mathematical biology, population growth... Hence, in the literature, several studies have been conducted on Bohr's almost periodicity and Bochner's almost automorphic to establish sufficient conditions for the existence and uniqueness of various types of differential and integral equations. For instance, one can see [1–3, 6, 20] and the references therein. In particular, it can be noted that several qualitative studies of various differential and integral equations have been carried out in recently published articles [7, 17–19, 25]. The notion of pseudo almost periodicity functions which is the central issue in this paper is a new concept introduced a few years ago by Zhang [27] as a generalization of the classical notion of Bohr's almost periodicity. Also, the notion of almost automorphic (Stepanov) was then defined firstly by N'G uérékata and Pankov [24] as an extension of the classical and well-known almost automorphic concept. It should be mentioned that the study of the existence of almost periodic solutions of the integral equation with a discrete delay was initiated in [16], where Fink and Gatica established the existence of a positive almost periodic solution to the following equation

$$x(t)=\int_{t-\tau}^{t} f(s,x(s))\mathrm{d}s, \tag{1.1}$$

which arises in models for the spread of epidemics. Since then, many works related to the sufficient conditions on the delay and the function f in order to establish the existence of almost periodic solutions to equation (1.1).

In 1997, Ait Dads and Ezzinbi [11] studied the existence of positive almost periodic solutions for the following neutral integral equation

$$x(t)=\gamma x(t-\tau)+(1-\gamma)\int_{t-\tau}^{t} f(s,x(s))\mathrm{d}s. \tag{1.2}$$

Later, Ait Dads et al. [10] established the existence of positive pseudo almost periodic solutions in the case of infinite delay for the equation

$$x(t)=\int_{-\infty}^{t} a(t,t-s)f(s,x(s))\mathrm{d}s, \quad t\in\mathbb{R}. \tag{1.3}$$

Afterwards, Ding et al. [15] developed the above results to the following integral equation with neutral delay

$$x(t)=\alpha x(t-\beta)+\int_{-\infty}^{t} a(t,t-s)f(s,x(s))\mathrm{d}s+h(t,x(t)), \quad t\in\mathbb{R}. \tag{1.4}$$

Equations similar to (1.4) arise in the study of [28] where the authors established the existence and uniqueness of almost periodic and pseudo almost periodic solutions of the integral equation given by

$$x(t)=\alpha(t)x(t-\sigma(t))+\int_{-\infty}^{t} \beta(t,t-s)f(s,x(s),x'(s))\mathrm{d}s, \quad t\in\mathbb{R}, \tag{1.5}$$

where $\sigma(t)$ is almost periodic (respectively pseudo almost periodic). Recently, the authors in [12] consider two variants of Eq. (1.5), a variant where the delay $\sigma(t)$ is compact almost automorphic in time and another variant where the delay is state-dependent. Also, the existence and uniqueness of periodic solutions of a more general model were established via three fixed point theorems by Islam [21].

Hence, one of the still topical subjects in the study of integral equations and/or differential equations is that if the force functions and/or the coefficients possess a specific property, are we going to find the same characteristics in the solution? Roughly speaking, if the considered functions are Stepanov-like pseudo almost periodic, will the expected solutions of the differential or integral equation be of the same type? The aim of this work is to study the existence and uniqueness of Stepanov-like pseudo almost periodic and Stepanov-like pseudo almost automorphic solutions for the following integral equation

$$x(t) = f(t, x(t), x(t - r(t))) - \int_t^{+\infty} c(t,s) g(s, x(s), x(s - r(s))) \mathrm{d}s, \tag{1.6}$$

where $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $c : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, $r(\cdot)$ is a time-dependent delay. To the best of our knowledge, there are no papers published on the S^p-pseudo almost periodic solutions and/or S^p-pseudo almost automorphic solutions of this class of Volterra equation.

Our main contributions in this paper are:

(1) The existence and uniqueness of Stepanov-like pseudo almost periodic solution for system (1.6) are proved.
(2) A new proof for the composition theorem in the space $PAPS^p$ is given based mainly on the Banach's transform.
(3) The existence and uniqueness of Stepanov-like pseudo almost automorphic solution for system (1.6) are proved.
(4) The existence and uniqueness of Stepanov-like pseudo almost periodic solutions and Stepanov-like pseudo almost automorphic of a class of logistic differential equation are established.

The organization of this work is as follows. In Section 2, we present some definitions and lemmas that will be used later. In Section 3, we state our main results. More precisely, we give sufficient conditions for the existence and the uniqueness of S^p-pseudo almost automorphic and S^p-pseudo almost periodic solutions of the integral equation (1.6). Our approach is based mainly on Bochner's transform, using analytic techniques and Banach's fixed point theorem. Finally, in Section 4, we study the validity of our theoretical result, therefore we give an illustrating application. It should be mentioned that the main results of this paper include Theorems 1, 2, 3 and 4.

2. Preliminaries: spaces of functions

Throughout this article $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ and $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ denote Banach spaces and $C(\mathbb{E}, \mathbb{F})$ the Banach space of continuous functions from $\mathbb{E}$ to $\mathbb{F}$. We denote by $BC(\mathbb{R}, \mathbb{E})$ the Banach space of bounded and continuous defined functions on $\mathbb{R}$ with the sup norm defined by

$$\|f\| = \sup_{t \in \mathbb{R}} \|f(t)\|. \tag{2.1}$$

Definition 1 ([4])**.** A set D of real numbers is said to be relatively dense if there exists a number $\ell > 0$ such that any interval of length ℓ contains at least one number of D.

Definition 2 ([4])**.** A function $f \in C(\mathbb{R}, \mathbb{E})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ the set $T(f, \varepsilon) = \{\tau : f(t + \tau) - f(t)\}$ is relatively dense, i.e. for any $\varepsilon > 0$

there exists $l = l(\varepsilon) > 0$ such that every interval of length l contains a number τ with the property that

$$\|f(t+\tau) - f(t)\| < \varepsilon, \quad t \in \mathbb{R}.$$

The collection of all such functions will be denoted by $AP(\mathbb{R},\mathbb{E})$.

Definition 3 ([26]). A function $f \in C(\mathbb{R}\times\mathbb{E},\mathbb{F})$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $y \in K$ where $K \subset \mathbb{E}$ is any compact subset if for each $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that every interval of length l contains a number τ with the property that

$$\|f(t+\tau,y) - f(t,y)\| < \varepsilon, \quad t \in \mathbb{R},\ y \in K.$$

The collection of such functions will be denoted by $AP(\mathbb{R}\times\mathbb{E},\mathbb{F})$.

Lemma 1 ([5]). *Let $f \in AP(\mathbb{R}\times\mathbb{E},\mathbb{F})$ and $\phi \in AP(\mathbb{R},\mathbb{E})$ then the function $[t \longmapsto F(t,\phi(t))] \in AP(\mathbb{R},\mathbb{F})$.*

Definition 4 ([27]). A continuous function $f : \mathbb{R} \longrightarrow \mathbb{E}$ is called pseudo almost periodic if it can be written as $f = h + \phi$ where $h \in AP(\mathbb{R},\mathbb{E})$ and $\phi \in PAP_0(\mathbb{R},\mathbb{E})$ where the space $PAP_0(\mathbb{R},\mathbb{E})$ is defined by

$$PAP_0(\mathbb{R},\mathbb{E}) = \left\{ f \in BC(\mathbb{R},\mathbb{E}), \mathcal{M}(\|f\|) = \lim_{T\longrightarrow\infty} \frac{1}{2T}\int_{-T}^{T} \|f(t)\|\mathrm{d}t = 0 \right\}.$$

The functions h and ϕ in above definition are respectively called the almost periodic components and the ergodic perturbation of the pseudo-almost periodic function f. The collection of all pseudo almost periodic functions which map from $\mathbb{R}$ to $\mathbb{E}$ will be denoted by $PAP(\mathbb{R},\mathbb{E})$.

Definition 5 ([13]). The Bochner transform $f^b(t,s)$ with $t \in \mathbb{R}, s \in [0,1]$ of a function $f : \mathbb{R} \mapsto \mathbb{E}$ is defined by $f^b(t,s) := f(t+s)$.

Definition 6 ([13]). The Bochner transform $F^b : \mathbb{R}\times[0,1]\times\mathbb{E} \mapsto \mathbb{E}$ of a function $F : \mathbb{R}\times\mathbb{E} \mapsto \mathbb{E}$ is defined by $F^b(t,s,u) := F(t+s,u)$ for each $t \in \mathbb{R},\ s \in [0,1]$, and $u \in \mathbb{E}$.

Definition 7 ([13]). Let $p \in [1,\infty[$. The space $BS^p(\mathbb{R},\mathbb{E})$ of all Stepanov-like bounded functions, with exponent p, consists of all measurable functions $f : \mathbb{R} \mapsto \mathbb{E}$ such that $f^b \in L^\infty(\mathbb{R}, L^p((0,1),\mathbb{E}))$. This is a Banach space with the norm

$$\|f\|_{BS^p(\mathbb{R},\mathbb{E})} := \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \mathrm{d}\tau\right)^{1/p}.$$

Definition 8 ([14]). A function $f \in BS^p(\mathbb{R},\mathbb{E})$ is called Stepanov-like almost periodic if $f^b \in AP(\mathbb{R}, L^p((0,1),\mathbb{E}))$. The collection of these functions will be denoted by $APS^p(\mathbb{R},\mathbb{E})$.

Definition 9 ([14])**.** A function $f:\mathbb{R}\times\mathbb{E}\to\mathbb{F},(t,u)\mapsto f(t,u)$ with $f(\cdot,u)\in BS^p(\mathbb{R},)$, for each $u\in\mathbb{E}$, is called Stepanov almost periodic function in $t\in\mathbb{R}$ uniformly for $u\in\mathbb{E}$ if for each $\varepsilon>0$ and each compact set $K\subset\mathbb{E}$ there exists a relatively dense set $P=P(\varepsilon,f,K)\subset\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}}\Big(\int_0^1\|f(t+s+\tau,u)-f(t+s,u)\|\mathrm{d}s\Big)^{1/p}<\varepsilon, \tag{2.2}$$

for each $\tau\in P$, $u\in K$. We denote by $APS^p(\mathbb{R}\times\mathbb{E},\mathbb{F})$ the set of such functions.

Definition 10 ([13])**.** Let $p\geq 1$. A function $f\in BS^p(\mathbb{R},\mathbb{E})$ is called S^p-pseudo almost periodic (or Stepanov-like pseudo almost periodic) if it can be expressed as

$$f=h+\phi, \tag{2.3}$$

where $h^b\in AP(L^p((0,1),\mathbb{E}))$ and $\phi^b\in PAP_0(L^p((0,1),\mathbb{E}))$. In other words, a function $f\in L^p(\mathbb{R},\mathbb{E})$ is said to be S^p-pseudo almost periodic if its Bochner transform $f^b:\mathbb{R}\longrightarrow L^p((0,1),\mathbb{E})$ is pseudo almost periodic in the sense that there exist two functions $h,\phi:\mathbb{R}\to\mathbb{E}$ such that $f=h+\phi$, where $h^b\in AP(L^p((0,1),\mathbb{E}))$ and $\phi^b\in PAP_0(L^p((0,1),\mathbb{E}))$ that is,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\|\varphi(\sigma)\|^p\mathrm{d}\sigma\Big)^{\frac{1}{p}}\mathrm{d}t=0. \tag{2.4}$$

The collection of such functions will be denoted by $PAPS^p(\mathbb{R},\mathbb{E})$.

Definition 11 ([23])**.** A continuous function $f:\mathbb{R}\longrightarrow\mathbb{E}$ is almost automorphic if for every sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t)=\lim_{n\to+\infty}f(t+s_n) \tag{2.5}$$

is well defined for each $t\in\mathbb{R}$, and

$$\lim_{n\to+\infty}g(t-s_n)=f(t) \tag{2.6}$$

for each $t\in\mathbb{R}$. The collection of all almost automorphic functions which map from $\mathbb{R}$ to $\mathbb{E}$ is denoted by $AA(\mathbb{R},\mathbb{E})$.

Definition 12 ([5])**.** A function $f:\mathbb{R}\times\mathbb{E}\longrightarrow\mathbb{E}$ $(t,x)\longmapsto f(t,x)$ is said to be almost automorphic in $t\in\mathbb{R}$ for each $u\in\mathbb{E}$ when it satisfies the two following conditions:

(1) For all $x\in\mathbb{E}$, the function $f(\cdot,x)\in AA(\mathbb{R},\mathbb{E})$.
(2) For all subset compact K of $\mathbb{E}$, for all $\varepsilon>0$ there exists $\delta=\delta(k,\varepsilon)>0$ such that, for all $x,z\in k$, if $d(x,z)\leq\delta$ then we have $d(f(x,t),f(z,t))\leq\varepsilon$ for all $t\in\mathbb{R}$.

The collection of such functions will be denoted by $AA(\mathbb{E}\times\mathbb{R},\mathbb{F})$.

Lemma 2 ([5]). *Let $f \in AA(\mathbb{R}\times\mathbb{E},\mathbb{F})$ and $u \in AA(\mathbb{R},\mathbb{E})$, then we have*

$$[t \longmapsto f(t,u(t))] \in AA(\mathbb{R},\mathbb{F}).$$

Lemma 3 ([9]). *If the functions $x(\cdot) \in PAPS^p(\mathbb{R},\mathbb{R})$ and $r(\cdot) \in APS^p(\mathbb{R},\mathbb{R})$ then we have $x(\cdot - r(\cdot)) \in PAPS^p(\mathbb{R},\mathbb{R})$.*

Definition 13 ([14]). A function $f \in BS^p(\mathbb{R},\mathbb{E})$ is called S^p-almost automorphic if $f^b \in AA(L^p((0,1),\mathbb{E}))$. The collection of such functions will be denoted by $AAS^p(\mathbb{R},\mathbb{E})$.

Definition 14 ([14]). A function $f \in BS^p(\mathbb{R}\times\mathbb{E},\mathbb{F})$, $(t,u)\mapsto F(t,u)$ where $F(\cdot,u) \in L^p(\mathbb{R},\mathbb{E})$ for each $u \in \mathbb{E}$, is called S^p-pseudo almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{E}$ if $t \mapsto F(t,u)$ is S^p-pseudo automorphic for each $u \in K$ where $K \subset \mathbb{E}$ is a bounded subset. The collection of such functions will be denoted by $PAAS^p$ $(\mathbb{R}\times\mathbb{E},\mathbb{F})$.

3. Main results

3.1. *Stepanov-like pseudo almost periodic solutions*

In this section, we consider the following integral equation

$$x(t) = f(t,x(t),x(t-r(t))) - \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s, \tag{3.1}$$

where $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $c,r:\mathbb{R}\to\mathbb{R}$ are continuous functions. We give sufficient conditions which guarantee the existence of S^p-pseudo almost periodic solutions for equation (3.1).

(H1) $f:\mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic, i.e. $f^b = h^b + \phi^b$, where $h^b \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $\phi^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|\phi(\sigma,u)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t = 0, \tag{3.2}$$

uniformly in $u \in \mathbb{R}^2$.

(H2) f is Lipschitz i.e. $\exists L_f^1, L_f^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le L_f^1|x_1-y_1| + L_f^2|x_2-y_2|. \tag{3.3}$$

(H3) $g:\mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic, i.e. $g^b = g_1^b + g_2^b$, where $g_1^b \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $g_2^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|g_2(\sigma,u)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t = 0, \tag{3.4}$$

uniformly for all $u \in \mathbb{R}^2$.

(H4) g is Lipschitz i.e. $\exists L_g^1, L_g^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \le L_g^1|x_1 - y_1| + L_g^2|x_2 - y_2|. \tag{3.5}$$

(H5) There exists a constant $\lambda > 0$ such that $c(t,s) \le e^{\lambda(t-s)}$, for all $s \ge t$.

(H6) The function $t \mapsto r(t) \in APS^p(\mathbb{R},\mathbb{R}) \cap C^1(\mathbb{R},\mathbb{R})$ with

$$0 \le r(t) \le \bar{r},\ r(t) \le r^* < 1, \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

Lemma 4. *Assume that (H1)-(H3) hold, if $x(\cdot) \in PAPS^p(\mathbb{R},\mathbb{R})$, then the function $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\beta(\cdot) = f(\cdot,x(\cdot),x(\cdot - r(\cdot)))$ belongs to $PAPS^p(\mathbb{R},\mathbb{R})$.*

Proof. Let $f = h + \phi$ where $h^b \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and the function $\phi^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$. Similarly, let $x^b(\cdot) = x_1^b(\cdot) + x_2^b(\cdot)$ where the function $x_1^b \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_2^b \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ that is

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} |x_2(\sigma)|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.7}$$

for all $t \in \mathbb{R}$. By Lemma 3 we get $x(\cdot - r(\cdot)) \in PAPS^p(\mathbb{R},\mathbb{R})$, then

$$x^b(\cdot - r(\cdot)) = x_1^b(\cdot - r^b(\cdot)) + x_2^b(\cdot - r^b(\cdot)), \tag{3.8}$$

where $x_1^b(\cdot - r^b(\cdot)) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_2^b(\cdot - r^b(\cdot)) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ that is

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} |x_2(\sigma - r(\sigma))|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0. \tag{3.9}$$

Since $f^b : \mathbb{R} \longrightarrow L^p((0,1),\mathbb{R})$ we decompose f^b as follows

$$\begin{aligned}
&f^b(\cdot,x^b(\cdot),x^b(\cdot - r(t)))\\
&= h^b(\cdot,x_1^b(\cdot),x_1^b(\cdot - r(t))) + f^b(\cdot,x^b(\cdot),x^b(\cdot - r(t))) - h^b(\cdot,x_1^b(\cdot),x_1^b(\cdot - r(t)))\\
&= h^b(\cdot,x_1^b(\cdot),x_1^b(\cdot - r(t))) + f^b(\cdot,x^b(\cdot),x^b(\cdot - r(t))) - f^b(\cdot,x_1^b(\cdot),x_1^b(\cdot - r(t)))\\
&\quad + \phi^b(\cdot,x_1^b(\cdot),x_1^b(\cdot - r(t))).
\end{aligned}$$

Let us prove that $h^b(\cdot,x_1^b(\cdot),x_1^b(\cdot - r(\cdot))) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$. First, the function $x_1^b(\cdot) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_1^b(\cdot - r^b(\cdot)) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Then the function $u_1^b(\cdot) = (x_1^b(\cdot), x_1^b(\cdot - r(t))) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}^2))$. Indeed, $\forall\varepsilon > 0$, $\exists\ell > 0$, $\forall a \in \mathbb{R}$, $\exists\tau \in [a, a+\ell]$ such that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\left(\int_0^1 \left\|u_1^b(t+\tau) - u_1^b(t)\right\|_\infty^p \mathrm{d}s\right)^{\frac{1}{p}}\\
&= \sup_{t\in\mathbb{R}}\left(\int_0^1 \left(\max\left(|x_1^b(t+\tau) - x_1^b(t)|, |x_1^b(t+\tau - r^b(t)) - x_1^b(t - r^b(t))|\right)\right)^p \mathrm{d}s\right)^{\frac{1}{p}}\\
&\le \varepsilon.
\end{aligned}$$

Since the function $h^b \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $u_1^b(\cdot) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}^2))$ then, we can apply the composition theorem of almost periodic functions 1, thus $h^b(\cdot,u_1^b(\cdot)) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Now, set

$$G^b(\cdot) = f^b(\cdot,x^b(\cdot),x^b(\cdot - r^b(\cdot))) - f^b(\cdot,x_1^b(\cdot),x_1^b(\cdot - r^b(\cdot))). \tag{3.10}$$

$G^b(\cdot) \in PAP_0(\mathbb{R},(L^p((0,1),\mathbb{R}))$. Indeed, let $T > 0$, we have

$$\begin{aligned}
&\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|G^b(\sigma)\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t\\
&\le \lim_{T\to+\infty}\frac{L_f^1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x^b(\sigma)-x_1^b(\sigma)\right|^p d\sigma\right)^{\frac{1}{p}}\mathrm{d}t\\
&\quad+\lim_{T\to+\infty}\frac{L_f^2}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x^b(\sigma-r^b(\sigma))-x_1^b(\sigma-r^b(\sigma))\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t\\
&\le \lim_{T\to+\infty}\frac{L_f^1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x_2^b(\sigma)\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t\\
&\quad+\lim_{T\to+\infty}\frac{L_f^2}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x_2^b(\sigma-r^b(\sigma))\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t.
\end{aligned}$$

Using (3.7) and (3.9) we get $\displaystyle\frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|G_2^b(\sigma)\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t = 0.$

Moreover, using the composition theorem of ergodic functions (cf. [22]) we have $\phi^b(\cdot,u_1^b(\cdot)) \in PAP_0(\mathbb{R},L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|\phi^b(\sigma,u_1^b(\sigma))\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t = 0. \tag{3.11}$$

□

Lemma 5. *Assume that (H3)-(H5) hold. If* $x(\cdot) \in PAPS^p(\mathbb{R},\mathbb{R})$*, then the function* $\Theta : t \longmapsto \displaystyle\int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s \in PAPS^p(\mathbb{R},\mathbb{R})$ *for all* $s \in \mathbb{R}$.

Proof. Using Lemma 4 and the hypothesis (H5), we obtain that the integral is convergent and consequently $t \longmapsto \displaystyle\int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ is well defined. Otherwise, since $[s \longmapsto g(s,x(s),x(s-r(s))] \in PAPS^p(\mathbb{R},\mathbb{R})$, one can write

$$g = g_1 + g_2 \tag{3.12}$$

with $g_1 \in APS^p(\mathbb{R},\mathbb{R})$ i.e. for each $\varepsilon' > 0$, there exists $\ell > 0$ such that every interval of length ℓ contains a τ such that $\|g_1(t+\tau) - g_1(t)\|_{S^p} < \varepsilon'$ and the function

$g_2 \in PAP_0(\mathbb{R}, L^p((t,t+1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|g_2(\sigma)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t = 0. \tag{3.13}$$

Then

$$\Theta(t) = \int_t^{+\infty} c(t,s)g_1(s)\mathrm{d}s + \int_t^{+\infty} c(t,s)g_2(s)\mathrm{d}s = \Theta_1(t)+\Theta_2(t).$$

Now, we shall study the S^p-almost periodicity of $\Theta_1(\cdot)$. We have

$$\begin{aligned}
|\Theta_1(t+\tau)-\Theta_1(t)| &\le \left|\int_{t+\tau}^{+\infty} \mathrm{e}^{\lambda(t+\tau-s)}g_1(s)\mathrm{d}s - \int_t^{+\infty}\mathrm{e}^{\lambda(t-s)}g_1(s)\mathrm{d}s\right| \\
&= \left|\int_t^{+\infty} \mathrm{e}^{\lambda(t-\xi)}g_1(\xi+\tau)\mathrm{d}\xi - \int_t^{+\infty}\mathrm{e}^{\lambda(t-s)}g_2(s)\mathrm{d}s\right| \\
&\le \int_t^{+\infty}\mathrm{e}^{\lambda(t-s)}|g_1(s+\tau)-g_1(s)|\mathrm{d}s.
\end{aligned}$$

According to Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ one has for all $\tau\in\mathbb{R}$,

$$\begin{aligned}
|\Theta_1(t+\tau)-\Theta_1(t)| &\le \int_0^{+\infty}\mathrm{e}^{-\lambda s}|g_1(s+t+\tau)-g_1(s+t)|\mathrm{d}s \\
&\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_0^{+\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}|g_1(s+t+\tau)-g_1(s+t)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}$$

Using Fubini's theorem, we get for all $\tau\in\mathbb{R}$

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}|\Theta_1(t+\tau)-\Theta_1(t)|^p\mathrm{d}t\right)^{\frac{1}{p}} \\
&\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}\int_0^{+\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}|g_1(s+t+\tau)-g_1(s+t)|^p\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}} \\
&\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_0^{+\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{x\in\mathbb{R}}\int_x^{x+1}|g_1(s+t+\tau)-g_1(s+t)|^p\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}$$

As g_1 is S^p almost periodic, for $\varepsilon' = C\varepsilon > 0$ we have

$$\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}|g_1(t++s+\tau)-g_1(t+s)|^p\mathrm{d}t\right)^{\frac{1}{p}} < \varepsilon' = C\varepsilon, \tag{3.14}$$

where $C = \left(\frac{\lambda p}{2}\right)^{\frac{1}{p}}\left(\frac{\lambda q}{2}\right)^{\frac{1}{q}}$. Then $\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}|\Theta_1(t+\tau)-\Theta_1(t)|^p\mathrm{d}t\right)^{\frac{1}{p}} \le \varepsilon$. This proves the S^p-almost periodicity of Θ_1. Now let's show the ergodicity of $\Theta_2(\cdot)$. Since

$\Theta_2(\cdot) \in BC(\mathbb{R},\mathbb{R})$, it remains to show that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_{t}^{t+1} |\Theta_2(\sigma)|^p \, \mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t = 0. \tag{3.15}$$

Let $q \in [1,\infty[$ such that $\frac{1}{p}+\frac{1}{q}=1$ then, by Hölder's inequality and Fubini's theorem we obtain

$$\begin{aligned}
&\int_{-T}^{T} \left(\int_{t}^{t+1} |\Theta_2(\sigma)|^p \, \mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t \\
&\leq (2T)^{\frac{1}{q}} \left[\int_{-T}^{T} \left(\int_{t}^{t+1} |\Theta_2(\sigma)|^p \, \mathrm{d}\sigma \right) \mathrm{d}t \right]^{\frac{1}{p}} \\
&\leq |\Theta_2|_\infty^{\frac{1}{q}} (2T) \left[\frac{1}{2T} \int_{-T}^{T} \left(\int_{t}^{t+1} \left| \int_{\sigma}^{+\infty} \mathrm{e}^{\lambda(\sigma-s)} g_2(s) \mathrm{d}s \right| \mathrm{d}\sigma \right) \mathrm{d}t \right]^{\frac{1}{p}},
\end{aligned}$$

which gives

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_{t}^{t+1} |\Theta_2(\sigma)|^p \, \mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t \\
&\qquad \leq \lim_{T\to+\infty} |\Theta_2|_\infty^{\frac{1}{q}} \left[\frac{1}{2T} \int_{-T}^{T} \left(\int_{t}^{t+1} \left| \int_{\sigma}^{+\infty} \mathrm{e}^{\lambda(\sigma-s)} g_2(s) \mathrm{d}s \right| \mathrm{d}\sigma \right) \mathrm{d}t \right]^{\frac{1}{p}}.
\end{aligned}$$

On the other hand, we get

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_{t}^{t+1} \left| \int_{\sigma}^{+\infty} \mathrm{e}^{\lambda(\sigma-s)} g_2(s) \mathrm{d}s \right| \mathrm{d}\sigma \right) \mathrm{d}t \leq I+J, \tag{3.16}$$

with

$$I = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_{t}^{t+1} \left| \int_{\sigma}^{T} \mathrm{e}^{\lambda(\sigma-s)} g_2(s) \mathrm{d}s \right| \mathrm{d}\sigma \right) \mathrm{d}t \tag{3.17}$$

and

$$J = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_{t}^{t+1} \left| \int_{T}^{+\infty} \mathrm{e}^{\lambda(\sigma-s)} g_2(s) \mathrm{d}s \right| \mathrm{d}\sigma \right) \mathrm{d}t. \tag{3.18}$$

Further

$$I \leq \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_{t}^{t+1} \mathrm{e}^{\lambda\sigma} \int_{\sigma}^{T} \mathrm{e}^{-\lambda s} |g_2(s)| \, \mathrm{d}s \mathrm{d}\sigma \right) \mathrm{d}t.$$

Now, we have $[s \longmapsto \mathrm{e}^{-\lambda s}]$ and $[s \longmapsto |g_2(s)|]$ are two continuous functions on $[\sigma, T]$, furthermore $[s \longmapsto |g_2(s)|]$ keep a constant sign, so $\exists\, \xi \in]\sigma, T[$ such that,

$$\begin{aligned}
I &= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_{t}^{t+1} \left(\mathrm{e}^{\lambda\sigma} |g_2(\xi)| \int_{\sigma}^{T} \mathrm{e}^{-\lambda s} \mathrm{d}s \right) \mathrm{d}\sigma \right) \mathrm{d}t \\
&= \lim_{T\to+\infty} \frac{1}{2T\lambda} \int_{-T}^{T} \left(\int_{t}^{t+1} \left(|g_2(\xi)| \left[1-\mathrm{e}^{-\lambda(T-\sigma)} \right] \right) \mathrm{d}\sigma \right) \mathrm{d}t.
\end{aligned}$$

Now as $1-\mathrm{e}^{-\lambda(T-\sigma)} \leq 1$, one has $I \leq \lim_{T\to+\infty} \frac{1}{2T\lambda}\int_{-T}^{T}\left(\int_{t}^{t+1}|g_2(\xi)|\,\mathrm{d}\sigma\right)\mathrm{d}t$. Since $\xi \in]\sigma, T[$, then $\xi = (1-\alpha)\sigma+\alpha T$, where $\alpha \in]0,1[$. Hence,

$$I = \lim_{T\to+\infty} \frac{1}{2T\lambda}\int_{-T}^{T}\left(\int_{t}^{t+1}(|g_2((1-\alpha)\sigma+\alpha T)|)\,\mathrm{d}\sigma\right)\mathrm{d}t. \tag{3.19}$$

On the other hand, since $t \leq \sigma \leq t+1$, we get

$$(1-\alpha)t+\alpha T \leq (1-\alpha)\sigma+\alpha T \leq (1-\alpha)t+(1-\alpha)+\alpha T.$$

Besides, $1-\alpha < 1$, thus

$$(1-\alpha)t+\alpha T \leq (1-\alpha)\sigma+\alpha T \leq (1-\alpha)t+\alpha T+1.$$

Set $z = (1-\alpha)\sigma+\alpha T$ and $u = (1-\alpha)\sigma+\alpha T$. We obtain,

$$I \leq \lim_{T\to+\infty} \frac{1}{2T\lambda}\int_{-T}^{T}\left(\int_{z}^{z+1}|g_2(u)|\,du\right)\mathrm{d}t. \tag{3.20}$$

According to the hypothesis $g_2 \in PAP_0(\mathbb{R}, L^p((t,t+1),\mathbb{R}))$ we conclude that $I = 0$. Meanwhile, by applying Fubini's theorem we obtain

$$\begin{aligned} J &= \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\left|\int_{T}^{+\infty}\mathrm{e}^{\lambda(\alpha-s)}g_2(s)\mathrm{d}s\right|\mathrm{d}\alpha\right)\mathrm{d}t \\ &= \lim_{T\to+\infty} \frac{1}{2T\lambda}\int_{-T}^{T}\left(\int_{T}^{+\infty}|g_2(s)|\,\mathrm{e}^{-s\lambda}\left[\mathrm{e}^{\lambda(t+1)}-\mathrm{e}^{\lambda t}\right]\mathrm{d}s\right)\mathrm{d}t. \end{aligned}$$

Thus, $J \leq J_1 + J_2$ with

$$J_1 = \lim_{T\to+\infty} \frac{\mathrm{e}^{\lambda}}{2T\lambda}\int_{-T}^{T}\int_{T}^{+\infty}|g_2(s)|\,\mathrm{e}^{\lambda(t-s)}\mathrm{d}s\mathrm{d}t, \tag{3.21}$$

and

$$J_2 = \lim_{T\to+\infty} \frac{1}{2T\lambda}\int_{-T}^{T}\int_{T}^{+\infty}|g_2(s)|\,\mathrm{e}^{\lambda(t-s)}\mathrm{d}s\mathrm{d}t. \tag{3.22}$$

Let $\xi = s-t$, then

$$J_1 \leq \lim_{T\to+\infty} \frac{\mathrm{e}^{\lambda}|g_2|_\infty}{2T\lambda}\int_{-T}^{T}\int_{T-t}^{+\infty}\mathrm{e}^{-\lambda\xi}\mathrm{d}\xi\mathrm{d}t = \lim_{T\to+\infty} \frac{\mathrm{e}^{\lambda}|g_2|_\infty}{2T\lambda^3}\left[1-\mathrm{e}^{-2\lambda T}\right] = 0.$$

Similarly, it is easy to see that $J_2 = 0$ which implies that $J = 0$. Then we have

$$\lim_{T\to+\infty}|\Theta_2|_\infty^{\frac{1}{q}}\left[\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\left|\int_{\sigma}^{+\infty}\mathrm{e}^{\lambda(\sigma-s)}g_2(s)\mathrm{d}s\right|\mathrm{d}\sigma\right)\mathrm{d}t\right]^{\frac{1}{p}} = 0. \tag{3.23}$$

Consequently,

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}|\Theta_2(\sigma)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t = 0. \tag{3.24}$$

Therefore the function $\Theta : t \longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ belongs to $PAPS^p(\mathbb{R},\mathbb{R})$. □

Now, we are able to establish the existence and uniqueness of the Stepanov-like pseudo almost periodic solutions of (3.1).

Theorem 1. *We assume (H1)-(H5) hold. If $m < 1$ then, (3.1) has a unique S^p-pseudo almost periodic solution with*

$$m = \max\left(L_f^1, L_f^2(1-r^*)^{-\frac{1}{p}}, L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}, L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\right).$$

Proof. Define the operator on $PAPS^p(\mathbb{R},\mathbb{R})$ by

$$\Gamma(x)(t) = f(t,x(t),x(t-r(t))) - \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s, \quad t \in \mathbb{R}. \tag{3.25}$$

Using Lemma 3 and (H1) we get that the function $t \mapsto f(t,x(t),x(t-r(t)))$ is continuous. Furthermore, by Lemma 4 and the hypothesis (H5) we get that the integral defined by $t \longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ exists. Thus, Γx is well defined. Moreover, from Lemmas 4 and 5 we deduce that

$$\Gamma : PAPS^p(\mathbb{R},\mathbb{R}) \longrightarrow PAPS^p(\mathbb{R},\mathbb{R}).$$

Let $x, y \in PAPS^p(\mathbb{R},\mathbb{R})$, according to Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, we get

$$\begin{aligned}
|\Gamma x(t) - \Gamma y(t)| &\leq |f(t,x(t),x(t-r(t))) - f(t,y(t),y(t-r(t)))| \\
&\quad + \left|\int_t^{+\infty} c(t,s)\Big(g(s,y(s),y(s-r(s)))\mathrm{d}s - g(s,x(s),x(s-r(s)))\Big)\mathrm{d}s\right| \\
&= |f(t,x(t),x(t-r(t))) - f(t,y(t),y(t-r(t)))| \\
&\quad + \left(\frac{2}{q\lambda}\right)^{\frac{1}{q}}\left(\int_0^{+\infty} \mathrm{e}^{\frac{-\lambda ps}{2}}|g(s+t,y(s+t),y(s+t-r(t))) \right.\\
&\quad \left. -\ g(s+t,x(s+t),x(s+t-r(t)))|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}$$

Then using Fubini's theorem and Minkowski's inequality, we get

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\Gamma x(t)-\Gamma y(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\mathrm{d}t\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_0^\infty \mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{\xi\in\mathbb{R}}\int_\xi^{\xi+1}\left(L_g^1|y(t+s)-x(t+s)|\right.\right.\\
&\left.\left.+\ L_g^2|y(t+s-r(s))-x(t+s-r(s))|\right)^p \mathrm{d}t\ \mathrm{d}s\right)^{\frac{1}{p}}\\
\le\ &\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\times\left(\int_0^\infty \mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left(L_g^1|y(t')-x(t')|\right.\right.\\
&\left.\left.+\ L_g^2|y(t'-r(t))-x(t'-r(t))|\right)^p\mathrm{d}t'\mathrm{d}s\right)^{\frac{1}{p}}\\
\le\ &\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\left(\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left(L_g^1|y(t')-x(t')|\right.\right.\\
&\left.\left.+\ L_g^2|y(t'-r(t'))-x(t'-r(t'))|\right)^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
\le\ &L_f^1\|x-y\|_{S^p}+L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\|x-y\|_{S^p}\\
&+L_f^2\left(1-r'(t)\right)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
&+L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\left(1-r'(t)\right)^{-\frac{1}{p}}\times\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
\le\ &L_f^1\|x-y\|_{S^p}+L_f^2(1-r^*)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar r}^{\xi+1-\bar r}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
&+L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\|x-y\|_{S^p}\\
&+L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar r}^{\xi+1-\bar r}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
\le\ &m\|x-y\|_{S^p},
\end{aligned}
$$

where

$$m=\max\left(L_f^1, L_f^2(1-r^*)^{-\frac{1}{p}}, L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}, L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\right).$$

Since $m<1$, the operator $\Gamma:(PAPS^p(\mathbb{R},\mathbb{R}),\|\cdot\|_{S^p})\longrightarrow(PAPS^p(\mathbb{R},\mathbb{R}),\|\cdot\|_{S^p})$ is a contraction. Therefore, by applying the Banach fixed point theorem, there is a unique $x_*\in PAPS^p(\mathbb{R},\mathbb{R})$ such that $\Gamma(x_*)=x_*$, which corresponds to the unique S^p-almost periodic pseudo solution of equation (3.1). $\square$

3.2. *Stepanov like (pseudo) almost automorphic solutions*

In this section, we establish the existence of pseudo almost automorphic solutions of equation (3.1). For this study, we make the following assumptions:

(H1) $f:\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is S^p-pseudo almost automorphic, i.e. $f^b=h^b+\phi^b$, where the function $h^b\in AA(\mathbb{R}\times\mathbb{R}^2,L^p((0,1),\mathbb{R}))$ and the function $\phi^b\in PAP_0$ $((\mathbb{R}\times\mathbb{R}^2,L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|\phi(\sigma,u)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0, \tag{3.26}$$

uniformly for all $u\in\mathbb{R}^2$.

(H2) f is a Lipschitz function, i.e. $\exists L_f^1, L_f^2>0$ such that $\forall x_1,x_2,y_1,y_2\in\mathbb{R}$,

$$|f(t,x_1,x_2)-f(t,y_1,y_2)|\le L_f^1|x_1-y_1|+L_f^2|x_2-y_2|. \tag{3.27}$$

(H3) $g:\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is S^p-pseudo almost automorphic, i.e. $g^b=g_1^b+g_2^b$, where $g_1^b\in AA(\mathbb{R}\times\mathbb{R}^2,L^p((0,1),\mathbb{R}))$ and $g_2^b\in PAP_0(\mathbb{R}\times\mathbb{R}^2,L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|g_2(\sigma,u)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0, \tag{3.28}$$

uniformly for all $u\in\mathbb{R}^2$.

(H4) g is Lipschitz , i.e. $\exists L_g^1, L_g^2>0$ such that $\forall x_1,x_2,y_1,y_2\in\mathbb{R}$,

$$|g(t,x_1,x_2)-g(t,y_1,y_2)|\le L_g^1|x_1-y_1|+L_g^2|x_2-y_2|. \tag{3.29}$$

(H5) There exists a constant $\lambda>0$ such that $c(t,s)\le \mathrm{e}^{\lambda(t-s)}$, for all $s\ge t$.

(H6) The function $t\mapsto r(t)\in C^1(\mathbb{R},\mathbb{R})$ with

$$0\le r(t)\le\bar{r},\quad r(t)\le r^*<1. \tag{3.30}$$

Lemma 6. *Assume that (H6) holds. If $x(\cdot)\in PAAS^p(\mathbb{R},\mathbb{R})$ then $x(\cdot-r(\cdot))\in PAAS^p(\mathbb{R})$.*

Proof. Since $x(\cdot) \in PAAS^p(\mathbb{R},\mathbb{R})$, then $x(\cdot)$ can be written as $x = x_1 + x_2$, where $x_1^b(\cdot) \in AA(\mathbb{R}, L^p([0,1],\mathbb{R}))$ and $x_2^b(\cdot) \in PAP_0(\mathbb{R}, L^p([0,1],\mathbb{R}))$, such that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |x_2(\sigma)|^p \, d\sigma \right)^{\frac{1}{p}} dt = 0. \tag{3.31}$$

Let

$$x(\cdot - r(\cdot)) = x_1(\cdot - r(\cdot)) + x(\cdot - r(\cdot)) - x_1(\cdot - r(\cdot)) = \Psi_1(\cdot) + \Psi_2(\cdot), \tag{3.32}$$

where $\Psi_1(\cdot) = x_1(\cdot - r(\cdot))$ and $\Psi_2(\cdot) = x(\cdot - r(\cdot)) - x_1(\cdot - r(\cdot))$. Note that the function $\Psi_2^b(\cdot) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ (cf. [9]). Hence, it only remains to show that $x_1^b(\cdot - r^b(\cdot)) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Since $x_1^b(\cdot) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$ then for any sequence $(s_n')_{n\in\mathbb{R}}$ there exists a subsequence $(s_n)_{n\in\mathbb{R}}$ and a function $g \in L^p_{loc}(\mathbb{R},\mathbb{R})$ such that

$$\left(\int_t^{t+1} |x_1(s+s_n) - g(s)|^p \, ds \right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0 \tag{3.33}$$

and

$$\left(\int_t^{t+1} |g(s-s_n) - x_1(s)|^p \, ds \right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.34}$$

Thus we could find

$$\begin{aligned} &\left(\int_t^{t+1} |x_1(s+s_n - r(s+s_n)) - g(s-r(s))|^p \, ds \right)^{\frac{1}{p}} \\ &\leq \left(\int_t^{t+1} |x_1(s+s_n - r(s+s_n)) - g(s-r(s+s_n)|^p \, ds \right)^{\frac{1}{p}} \\ &\quad + \left(\int_t^{t+1} |g(s-r(s+s_n) - g(s-r(s))|^p \, ds \right)^{\frac{1}{p}}. \end{aligned}$$

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence such that $g_n \to g$ as $n \to \infty$ in $BS^p(\mathbb{R},\mathbb{R})$ which is dominated by some integrable function w, then

$$\left(\int_t^{t+1} |g(s-r(s+s_n) - g(s-r(s))|^p \, ds \right)^{\frac{1}{p}} \leq I + J + K,$$

where

$$I = \left(\int_t^{t+1} |g(s-r(s+s_n) - g_n(s-r(s+s_n)|^p \, ds \right)^{\frac{1}{p}}, \tag{3.35}$$

$$J = \left(\int_t^{t+1} |g_n(s-r(s+s_n) - g_n(s-r(s))|^p \, ds \right)^{\frac{1}{p}} \tag{3.36}$$

and

$$K = \left(\int_t^{t+1} |g_n(s-r(s)) - g(s-r(s))|^p \, ds \right)^{\frac{1}{p}}. \tag{3.37}$$

Let us show that $I=0$. For that, letting $s'=s-r(s+s_n)$ one obtains

$$\begin{aligned}I &= \big(1-r'(s+s_n)\big)^{-\frac{1}{p}}\left(\int_{t-r(t+s_n)}^{t+1-r(t+1-s_n)}\big|g(s')-g_n(s')\big|^p\,\mathrm{d}s'\right)^{\frac{1}{p}}\\ &\le (1-r^*))^{-\frac{1}{p}}\left(\int_{t-\bar{r}}^{t+1-\bar{r}}\big|g(s')-g_n(s')\big|^p\,\mathrm{d}s'\right)^{\frac{1}{p}}\\ &\le (1-r^*))^{-\frac{1}{p}}\left(\int_{t}^{t+1}\big|g(s')-g_n(s')\big|^p\,\mathrm{d}s'\right)^{\frac{1}{p}}\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

In addition, by applying the dominated convergence theorem we get $J=0$. Moreover, let $s'=s-r(s)$, then

$$\begin{aligned}K &= \big(1-r'(s)\big)^{-\frac{1}{p}}\left(\int_{t-r(t)}^{t+1-r(t+1)}\big|g(s')-g_n(s')\big|^p\,\mathrm{d}s'\right)^{\frac{1}{p}}\\ &\le (1-r^*))^{-\frac{1}{p}}\left(\int_{t-\bar{r}}^{t+1-\bar{r}}\big|g(s')-g_n(s')\big|^p\,\mathrm{d}s'\right)^{\frac{1}{p}}\\ &\le (1-r^*))^{-\frac{1}{p}}\left(\int_{t}^{t+1}\big|g(s')-g_n(s')\big|^p\,\mathrm{d}s'\right)^{\frac{1}{p}}\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

What is left to show that

$$\left(\int_t^{t+1}|x_1(s+s_n-r(s+s_n))-g(s-r(s+s_n)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\underset{n\to\infty}{\longrightarrow}0. \tag{3.38}$$

For this purpose, we set $s-r(s+s_n)=s'$, then

$$\begin{aligned}&\left(\int_t^{t+1}|x_1(s+s_n-r(s+s_n))-g(s-r(s+s_n)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &= \big(1-r'(s+s_n)\big)^{-\frac{1}{p}}\left(\int_{t-r(t+s_n)}^{t+1-r(t+1-s_n)}\big|x_1(s'+s_n)-g(s')\big|^p\,\mathrm{d}s'\right)^{\frac{1}{p}}\\ &\le (1-r^*))^{-\frac{1}{p}}\left(\int_{t-\bar{r}}^{t+1-\bar{r}}\big|x_1(s'+s_n)-g(s')\big|^p\,\mathrm{d}s'\right)^{\frac{1}{p}}\\ &\le (1-r^*))^{-\frac{1}{p}}\left(\int_{t}^{t+1}\big|x_1(s'+s_n)-g(s')\big|^p\,\mathrm{d}s'\right)^{\frac{1}{p}}\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

Similarly, we can get

$$\left(\int_t^{t+1}|g(s-s_n+r(s-s_n))-x_1(s-r(s))|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\underset{n\to\infty}{\longrightarrow}0. \tag{3.39}$$

Consequently,

$$x_1^b(\cdot-r^b(\cdot))\in AA(\mathbb{R},L^p([0,1],\mathbb{R})).$$

Therefore the function

$$x(\cdot - r(\cdot)) \in PAAS^p(\mathbb{R}).$$

□

Lemma 7. *We assume that (H1)-(H2) hold. If $x(\cdot) \in PAAS^p(\mathbb{R},\mathbb{R})$, then the function $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\beta(\cdot) = f(\cdot, x(\cdot), x(\cdot - r(\cdot)))$ belongs to $PAAS^p(\mathbb{R},\mathbb{R})$.*

Proof. By (H1), we have $f^b = h^b + \phi^b$ where $h^b \in AA(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $\phi^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} |\phi(\sigma,u)|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.40}$$

uniformly for all $u \in \mathbb{R}^2$. Similarly, $x^b = x_1^b + x_2^b$ where $x_1^b(\cdot) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_2^b(\cdot) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} |x_2(\sigma)|^p \,d\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.41}$$

for all $t \in \mathbb{R}$. Then, by Lemma 6, we get $x(\cdot - r(\cdot)) \in PAAS^p(\mathbb{R},\mathbb{R})$, i.e.

$$x^b(\cdot - r(\cdot)) = x_1^b(\cdot - r^b(\cdot)) + x_2^b(\cdot - r^b(\cdot)),$$

where $x_1^b(\cdot - r^b(\cdot)) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_2^b(\cdot - r^b(\cdot)) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} |x_2(\sigma - r(\sigma))|^p \,d\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.42}$$

for all $t \in \mathbb{R}$. In addition, $f^b : \mathbb{R} \longrightarrow L^p((0,1),\mathbb{R})$. Now decompose f^b as follows

$$\begin{aligned}
&f^b(\cdot, x^b(\cdot), x^b(\cdot - r^b(\cdot)))\\
&= h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))) + f^b(\cdot, x^b(\cdot), x^b(\cdot - r^b(\cdot))) - h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot)))\\
&= h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))) + f^b(\cdot, x^b(\cdot), x^b(\cdot - r^b(\cdot))) - f^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot)))\\
&\quad + \phi^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))).
\end{aligned}$$

We start by demonstrating that $h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Since $x_1^b(\cdot) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$, i.e. there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $y_1^b(\cdot) \in L^p_{loc}(\mathbb{R},\mathbb{E})$ such that

$$\left(\int_t^{t+1} |x_1(s+s_n) - y_1(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0, \tag{3.43}$$

and

$$\left(\int_t^{t+1} |y_1(s-s_n) - x_1(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.44}$$

Similarly, $x_1^b(\cdot - r^b(\cdot)) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$, i.e. there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $y_1^b(\cdot - r^b(\cdot)) \in L^p_{loc}(\mathbb{R},\mathbb{E})$ such that

$$\left(\int_t^{t+1} |x_1(s+s_n - r(s+s_n)) - y_1(s-r(s))|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0, \tag{3.45}$$

and

$$\left(\int_t^{t+1} |y_1(s-s_n - r(s-s_n)) - x_1(s-r(s))|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.46}$$

Then $u_1^b(\cdot) = (x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))) \in AA(R, L^p((0,1),\mathbb{R}^2))$. Indeed, let $(s'_n)_{n'\in\mathbb{N}}$ a sequence has a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $w_1^b(\cdot) = (y_1^b(\cdot), y_1^b(\cdot - r^b(\cdot))) \in L^p_{loc}(\mathbb{R},\mathbb{R}^2)$ such that

$$\begin{aligned}
&\left(\int_t^{t+1} \|u_1(s+s_n) - w_1(s)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\
\Leftrightarrow &\left(\int_t^{t+1} \|(x_1(s+s_n), x_1(s+s_n - r(s+s_n))) - (y_1(s), y_1(s-r(s)))\|_\infty^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\
\Leftrightarrow &\left(\int_t^{t+1} \max\left(|x_1(s+s_n) - y_1(s)|^p, |x_1(s+s_n - r(s+s_n) - y_1(s-r(s))|^p\right) \mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}$$

We deduce from (3.43) and (3.45) that

$$\lim_{n\to\infty}\left(\int_t^{t+1} \max\left(|x_1(s+s_n) - y_1(s)|^p, |x_1(s+s_n - r(s+s_n) - y_1(s-r(s))|^p\right) \mathrm{d}s\right)^{\frac{1}{p}} = 0.$$

Moreover, since $h^b \in AA(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and by applying Lemma 2, it is easy to see that the function $h^b(.,u_1^b(.)) \in AA(\mathbb{R}, L^p([0,1],\mathbb{R}))$. Now set

$$G^b(.) = f^b(\cdot, x^b(\cdot), x^b(\cdot - rs + s_n(\cdot))) - f^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))). \tag{3.47}$$

$G^b(\cdot) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Indeed, $T > 0$ using the fact that f is Lipschitz and Minkowski's inequality we get

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|G^b(\sigma)\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\leq \lim_{T\to+\infty} \frac{L_f^1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x^b(\sigma) - x_1^b(\sigma)\right|^p d\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\quad + \lim_{T\to+\infty} \frac{L_f^2}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x^b(\sigma - r^b(\sigma)) - x_1^b(\sigma - r^b(\sigma))\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\leq \lim_{T\to+\infty} \frac{L_f^1}{2T}\int_{-T}^{T}\left(\int_0^1 |x_2^b(\sigma)|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t
\end{aligned}$$

$$+\lim_{T\to+\infty}\frac{L_f^2}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x_2^b(\sigma-r^b(\sigma))\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t.$$

By (3.41) and (3.42) we obtain

$$\frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|G_2^b(\sigma)\right|^p d\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{3.48}$$

It remains to show that $\phi^b(\cdot,x_1^b(\cdot),x_1^b(\cdot-r^b(\cdot)))\in PAP_0(L^p((0,1),\mathbb{R}))$. We have already shown that the function $u_1^b(\cdot)=(x_1^b(\cdot),x_1^b(\cdot-r^b(\cdot)))\in AP(R,L^p((0,1),\mathbb{R}^2))$. Since the function $\phi^b\in PAP_0(\mathbb{R}\times\mathbb{R}^2,L^p((0,1),\mathbb{R}))$, hence by applying the composition theorem of ergodic functions (cf. [22]), we get $\phi^b(\cdot,u_1^b(\cdot))\in PAP_0(L^p([0,1],\mathbb{R})$ such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|\phi^b(\sigma,u_1^b(\sigma))\right|^p d\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{3.49}$$

□

Lemma 8. *Suppose that assumptions (H3)-(H5) hold. If $x(\cdot)\in PAAS^p(\mathbb{R},\mathbb{R})$ then the function Θ defined by $\Theta: t\longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ belongs to $PAAS^p(\mathbb{R},\mathbb{R})$.*

Proof. Using Lemma 7 and hypothesis (H5) one can easily check that the integral $t\longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ is well defined. Since the function $[\,s\longmapsto g(s,x(s),x(s-r(s))]\in PAAS^p(\mathbb{R},\mathbb{R})$, we can write $g=g_1+g_2$ where $g_1^b\in AA(\mathbb{R},L^p((0,1),\mathbb{R}))$ and $g_2^b\in PAP_0(\mathbb{R},L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|g_2(\sigma)|^p d\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{3.50}$$

Then

$$\Theta(t)=\int_t^{+\infty}c(t,s)g_1(s)\mathrm{d}s+\int_t^{+\infty}c(t,s)g_2(s)\mathrm{d}s=\Theta_1(t)+\Theta_2(t). \tag{3.51}$$

We have already shown that $[t\longmapsto \theta_2(t)]\in PAP_0(\mathbb{R},L^p([0,1],\mathbb{R}))$ (see Lemma 5). So to prove the S^p-pseudo almost periodicity of the function $\theta(\cdot)$, it suffices to show that $[t\longmapsto \theta_1(t)]\in SAA(\mathbb{R},\mathbb{R})$. Let $(s_n')_{n\in\mathbb{N}}$ a sequence of real numbers and g_1 is S^p-almost automorphic function then there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g_1^*\in L^p_{loc}(\mathbb{R},\mathbb{E})$ such that

$$\left(\int_0^1 |g_1(s_n+s+t)-g_1^*(s+t)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\underset{n\to\infty}{\longrightarrow}0, \tag{3.52}$$

and

$$\left(\int_0^1 |g_1^*(t+s-s_n)-g_1(t+s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.53}$$

Set

$$\theta_1^*(t)=\int_t^{+\infty} c(t,s)g_1^*(s)\mathrm{d}s. \tag{3.54}$$

Then we have

$$\begin{aligned}
|\theta_1(u+s_n)-\theta_1^*(u)| &= \left|\int_{u+s_n}^{+\infty} c(u+s_n,s)g_1(s)\mathrm{d}s-\int_u^{+\infty} c(u,s)g_1^*(s)\mathrm{d}s\right|\\
&= \left|\int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}g_1(s_n+s)ds-\int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}g_1^*(s)\mathrm{d}s\right|\\
&= \int_0^{+\infty} \mathrm{e}^{-s\lambda}|g_1(s_n+s+u)-g_1^*(s+u)|\mathrm{d}s.
\end{aligned}$$

Using Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$$|\theta_1(u+s_n)-\theta_1^*(u)| \le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_0^{\infty} \mathrm{e}^{\frac{-s\lambda p}{2}}|g_1(s_n+s+u)-g_1^*(s+u)|^p\mathrm{d}s\right)^{\frac{1}{p}}.$$

Then

$$\begin{aligned}
&\left(\int_t^{t+1} |\theta_1(u+s_n)-\theta_1^*(u)|^p\,\mathrm{d}u\right)^{\frac{1}{p}}\\
&\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_t^{t+1}\int_0^{\infty} \mathrm{e}^{\frac{-\lambda ps}{2}}|g_1(s_n+s+u)-g_1^*(s+u)|^p\,\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{p}}\\
&\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\left(\int_{t'}^{t'+1} |g_1(s_n+u')-g_1^*(u')|^p\mathrm{d}u'\right)^{\frac{1}{p}}.
\end{aligned}$$

Since

$$\lim_{n\to+\infty}\left(\int_{t'}^{t'+1} |g_1(s_n+u)-g_1^*(u)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}=0, \tag{3.55}$$

we obtain

$$\lim_{n\to+\infty}\left(\int_t^{t+1} |\theta_1(u+s_n)-\theta_1^*(u)|^p\,\mathrm{d}u\right)^{\frac{1}{p}}=0. \tag{3.56}$$

On the other hand, we have

$$\begin{aligned}
|\theta_1^*(u-s_n)-\theta_1(u)| &\le \left|\int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}g_1^*(s-s_n)\mathrm{d}s-\int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}g_1(s)\mathrm{d}s\right|\\
&\le \int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}|g_1^*(s-s_n)-g_1(s)|\mathrm{d}s
\end{aligned}$$

$$= \int_0^{+\infty} \mathrm{e}^{-\lambda s} |g_1^*(s+u-s_n) - g_1(s+u)| \mathrm{d}s.$$

Using Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ we get

$$|\theta_1^*(u-s_n) - \theta_1(u)| \leq \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}} \left(\int_0^{+\infty} \mathrm{e}^{\frac{-s\lambda p}{2}} |g_1(s-s_n+u) - g_1^*(s+u)|^p \mathrm{d}s\right)^{\frac{1}{p}}.$$

Now using Fubini theorem we get

$$\begin{aligned}
&\left(\int_t^{t+1} |\theta_1^*(u-s_n) - \theta_1(u)|^p \mathrm{d}u\right)^{\frac{1}{p}} \\
&\leq \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}} \left(\int_t^{t+1} \int_0^{\infty} \mathrm{e}^{\frac{-\lambda p s}{2}} |g_1^*(s-s_n+u) - g_1(s+u)|^p \mathrm{d}s \mathrm{d}u\right)^{\frac{1}{p}} \\
&= \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}} \left(\frac{2}{\lambda p}\right)^{\frac{1}{p}} \left(\int_{t'}^{t'+1} |g_1^*(s_n+u') - g_1(u')|^p \mathrm{d}u'\right)^{\frac{1}{p}}.
\end{aligned}$$

Using the fact that

$$\lim_{n\to+\infty} \left(\int_{t'}^{t'+1} \left|g_1^*(u'-s_n) - g_1(u')\right|^p \mathrm{d}u'\right)^{\frac{1}{p}} = 0, \tag{3.57}$$

we get

$$\lim_{n\to+\infty} \left(\int_{t'}^{t'+1} |\theta_1^*(u-s_n) - \theta_1(u)|^p \mathrm{d}u\right)^{\frac{1}{p}} = 0. \tag{3.58}$$

Therefore, $\Theta : t \longmapsto \int_t^{+\infty} c(t,s)\, g(s,x(s),x(s-r(s)))\, \mathrm{d}s \in PAAS^p(\mathbb{R},\mathbb{R})$. □

Theorem 2. *Assume that (H1)-(H5) hold. If $m<1$, then (3.1) has a unique S^p-pseudo almost automorphic solution with*

$$m = \max\left(L_f^1, L_f^2(1-r^*)^{-\frac{1}{p}}, L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}, L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\right).$$

Proof. Let us consider the operator Γ defined on $PAAS^p(\mathbb{R},\mathbb{R})$ by

$$\Gamma(x)(t) = f(t,x(t),x(t-r(t))) - \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s, \quad t \in \mathbb{R}.$$

By Lemma 6 and (H1) we obtain that the function $t \longmapsto f(t,x(t),x(t-r(t)))$ is continuous. Furthermore, using Lemma 7 and (H5), we get that the integral defined by $t \longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ exists. Thus, Γx is well defined. Moreover, from Lemmas 7 and 8 we deduce that

$$\Gamma : PAAS^p(\mathbb{R},\mathbb{R}) \to PAAS^p(\mathbb{R},\mathbb{R}).$$

Let $x, y \in PAAS^p(\mathbb{R},\mathbb{R})$, by making a change of variables and according to Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ we get

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\Gamma x(t)-\Gamma y(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_0^{\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{\xi\in\mathbb{R}}\int_{\xi}^{\xi+1}\left(L_g^1|y(t+s)-x(t+s)|\right.\right.\\
&\quad\left.\left.+\ L_g^2|y(t+s-r(s))-x(t+s-r(s))|\right)^p\mathrm{d}t\ \mathrm{d}s\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\times\left(\int_0^{\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left(L_g^1|y(t')-x(t')|\right.\right.\\
&\quad\left.\left.+\ L_g^2|y(t'-r(t))-x(t'-r(t))|\right)^p\mathrm{d}t'\mathrm{d}s\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\left(\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left(L_g^1|y(t')-x(t')|\right.\right.\\
&\quad\left.\left.+\ L_g^2|y(t'-r(t'))-x(t'-r(t'))|\right)^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq L_f^1\|x-y\|_{S^p}+L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\|x-y\|_{S^p}\\
&\quad+L_f^2\left(1-r'(t)\right)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
&\quad+L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\left(1-r'(t)\right)^{-\frac{1}{p}}\times\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\le L_f^1\|x-y\|_{S^p} + L_f^2(1-r^*)^{-\frac{1}{p}} \sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar{r}}^{\xi+1-\bar{r}} |x(\rho)-y(\rho)|^p\, d\rho\right)^{\frac{1}{p}}\\
&\quad + L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\|x-y\|_{S^p}\\
&\quad + L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}} \sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar{r}}^{\xi+1-\bar{r}} |x(\rho)-y(\rho)|^p\, d\rho\right)^{\frac{1}{p}}\\
&\le m\|x-y\|_{S^p},
\end{aligned}
$$

where

$$
m = \max\left(L_f^1, L_f^2(1-r^*)^{-\frac{1}{p}}, L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}, L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\right).
$$

Since $m<1$, the operator $\Gamma : (PAAS^p(\mathbb{R},\mathbb{R}), \|\cdot\|_{S^p}) \longrightarrow (PAAS^p(\mathbb{R},\mathbb{R}), \|\cdot\|_{S^p})$ is a contraction. Therefore, by applying the Banach fixed point theorem there is a unique $x_* \in PAAS^p(\mathbb{R},\mathbb{R})$ such that $\Gamma(x_*) = x_*$, which corresponds to the unique S^p-almost periodic pseudo solution of the equation (3.1). □

4. Application

The purpose of this section is to show the existence and uniqueness of the S^p-pseudo almost periodic and S^p-pseudo almost automorphic solutions of the following logistic differential equation

$$
x'(t) = ax(t) + \alpha x'(t-r(t)) - q(t, x(t), x(t-r(t))) + h(t). \tag{4.1}
$$

But rather than dealing with equation (4.1) we will study the existence and uniqueness of S^p-pseudo almost periodic and S^p-pseudo almost automorphic solutions of the following integral equation

$$
x(t) = \alpha x(t-r(t)) - \int_t^{+\infty}[q(s,x(s),x(s-r(s))) - a\alpha x(s-r(s))]\,\mathrm{e}^{a(t-s)}\mathrm{d}s + p(t), \tag{4.2}
$$

where $h:\mathbb{R}\longrightarrow\mathbb{R}$, $q:\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are continuous functions, $p:\mathbb{R}\longrightarrow\mathbb{R}$ a differentiable function, $a>0$, $0\le|\alpha|<1$ are respectively constants and $r(\cdot)$ is a time-dependent delay. Indeed, let x a solution of (4.2) then

$$
\begin{aligned}
x'(t) &= \alpha x'(t-r(t)) - q(t,x(t),x(t-r(t))) + a\alpha x(t-r(t))\\
&\quad - a\int_t^{\infty}[q(s,x(s),x(s-r(s))) - a\alpha x(s-r(s))]\mathrm{e}^{a(t-s)}\mathrm{d}s + p'(t)\\
&= a\left[\alpha x(t-r(t)) - \int_t^{\infty}[q(s,x(s),x(s-r(s))) - a\alpha x(s-r(s))]\,\mathrm{e}^{a(t-s)}\mathrm{d}s\right]\\
&\quad + \alpha x'(t-r(t)) - q(t,x(t),x(t-h)) + p'(t)
\end{aligned}
$$

$$= ax(t) + \alpha x'(t - r(t)) - q(t, x(t), x(t - r(t))) + h(t),$$

where $h(t) = p'(t)$. Then, the solutions of equation (4.1) are exactly those of the integral equation (4.2).

4.1. *Stepanov-like pseudo almost periodic solutions*

We will study the S^p-pseudo almost periodic solutions of (4.2). For this study, we formulate the following assumptions

(H1) $q : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic function, i.e.
$$q^b = q_1^b + q_2^b, \tag{4.3}$$
with $q_1^b \in AP(\mathbb{R} \times \mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $q_2^b \in PAP_0(\mathbb{R} \times \mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that
$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |q_2(\sigma,u)|^p \,\mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{4.4}$$
uniformly for all $u \in \mathbb{R}^2$.

(H2) q is Lipschitz, i.e. $\exists L_q^1, L_q^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$
$$|q(t,x_1,x_2) - q(t,y_1,y_2)| \le L_q^1 |x_1 - y_1| + L_q^2 |x_2 - y_2|. \tag{4.5}$$

(H3) $p : \mathbb{R} \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic function, i.e.
$$p^b = p_1^b + p_2^b \tag{4.6}$$
where $p_1^b \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $p_2^b \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ such that
$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |p_2(\sigma)|^p \,\mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t = 0. \tag{4.7}$$

(H4) The function $t \mapsto r(t) \in APS^p(\mathbb{R},\mathbb{R}) \cap C^1(\mathbb{R},\mathbb{R})$, with
$$0 \le r(t) \le \bar{r}, \quad r(t) \le r^* < 1. \tag{4.8}$$

Theorem 3. *Assume that (H1)-(H3) hold. If $m_1 < 1$ then (4.2) has a unique S^p-pseudo almost perodic solution, where*
$$m_1 = \max\left(\alpha(1-r^*), L_q^1 \left(\frac{2}{aq}\right)^{\frac{1}{q}} \left(\frac{2}{ap}\right)^{\frac{1}{p}}, (1-r^*)\left(L_q^2 - a\alpha\right) \left(\frac{2}{aq}\right)^{\frac{1}{q}} \left(\frac{2}{ap}\right)^{\frac{1}{p}} \right).$$

Proof. Let the operator Λ defined on $PAPS^p(\mathbb{R},\mathbb{R})$ by
$$\Lambda(x)(t) = \alpha x(t - r(t)) - \int_t^{+\infty} [q(s, x(s), x(s - r(s))) - a\alpha x(s - r(s))] \mathrm{e}^{a(t-s)} \mathrm{d}s + p(t). \tag{4.9}$$

$\Lambda x \in PAPS^p(\mathbb{R},\mathbb{R})$. In fact, pose
$$f(\cdot, x(\cdot), x(\cdot - r(\cdot))) = \alpha x(\cdot - r(\cdot)) + p(\cdot), \tag{4.10}$$

and

$$g(\cdot,x(\cdot),x(\cdot-r(\cdot))) = [q(\cdot,x(\cdot),x(\cdot-r(\cdot))) - a\alpha x(\cdot-r(\cdot))]. \tag{4.11}$$

Since, $[t\mapsto x(t)] \in PAAS^p(\mathbb{R},\mathbb{R})$, Lemma 3 implies that

$$[t\mapsto x(t-r(t))] \in PAPS^p(\mathbb{R},\mathbb{R}),$$

for all $t\in\mathbb{R}$. Thus,

$$[t\longmapsto \alpha x(t-r(t))] \in PAAS^p(\mathbb{R},\mathbb{R}),$$

for all $t\in\mathbb{R}$. In accordance with Lemma 4, the function

$$[t\mapsto f(t,x(t),x(t-r(t)))] \in PAPS^p(\mathbb{R},\mathbb{R}).$$

Moreover, under (H1)-(H2) and using Lemma 4, we obtain that the function

$$[s\to q(s,x(s),x(s-r(s)))] \in PAPS^p(\mathbb{R},\mathbb{R}).$$

As previously we show that the function $[s\longmapsto -a\alpha x(s-r(s))] \in PAPS^p(\mathbb{R},\mathbb{R})$, for all $t\in\mathbb{R}$. Then, $[s\longmapsto g(s,x(s),x(s-r(s)))] \in PAPS^p(\mathbb{R},\mathbb{R})$, as being the sum of two S^p-pseudo almost periodic functions. It follows from Lemma 5 that

$$\left[t\longmapsto \int_t^{+\infty} [q(s,x(s),x(s-r(s))) - a\alpha x(s-r(s))]\,\mathrm{e}^{a(t-s)}\mathrm{d}s\right] \in PAPS^p(\mathbb{R},\mathbb{R}).$$

Therefore we deduce that $\Lambda x \in PAPS^p(\mathbb{R},\mathbb{R})$.

Let $x,\, y \in PAPS^p(\mathbb{R},\mathbb{R})$, then

$$\begin{aligned}
&|\Lambda x(t) - \Lambda y(t)| \\
&\le |\alpha x(t-r(t)) - \alpha y(t-r(t))| \\
&\quad + \int_t^{+\infty} \mathrm{e}^{a(t-s)}\,|q(s,y(s),y(s-r(s))) - q(s,x(s),x(s-r(s))) \\
&\quad + a\alpha x(s-r(s)) - a\alpha y(s-r(s))|\,\mathrm{d}s \\
&\le |\alpha x(t-r(t)) - \alpha y(t-r(t))| + \int_0^{+\infty} \mathrm{e}^{-as}\,|q(s+t,y(s+t),y(s+t-r(s+t))) \\
&\quad - q(s+t,x(s+t),x(s+t-r(s+t))) \\
&\quad + a\alpha x(s+t-r(s+t)) - a\alpha y(s+t-r(s+t))|\,\mathrm{d}s \\
&\le |\alpha x(t-r(t)) - \alpha y(t-r(t))| + \left(\frac{2}{qa}\right)^{\frac{1}{q}} \\
&\quad \times \int_0^{\infty} \mathrm{e}^{\frac{-aps}{2}}\,|q(s+t,y(s+t),y(s+t-r(s+t))) + a\alpha x(s+t-r(s+t)) \\
&\quad - q(s+t,x(s+t),x(s+t-r(s+t))) - a\alpha y(s+t-r(s+t))|^p\,\mathrm{d}s.
\end{aligned}$$

So, using Fubini's theorem and Minkowski's inequality we have

$$\sup_{\xi\in\mathbb{R}} \left(\int_\xi^{\xi+1} |\Lambda x(t) - \Lambda y(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

$$\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}$$

$$\times\left(\int_{0}^{+\infty}\mathrm{e}^{\frac{-aps}{2}}\sup_{\xi\in\mathbb{R}}\int_{\xi}^{\xi+1}|a\alpha x(s+t-r(s+t))+q(s+t,y(s+t),y(s+t-r(s+t)))\right.$$

$$\left.-q(s+t,x(s+t),x(s+t-r(s+t)))-a\alpha y(s+t-r(s+t))|^{p}\,\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}$$

$$\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}$$

$$\left(\int_{0}^{+\infty}\mathrm{e}^{\frac{-aps}{2}}\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left|a\alpha x(t'-r(t'))\right.\right.$$

$$\left.\left.+q(t',y(t'),y(t'-r(t')))-q(t',x(t'),x(t'-r(t')))-a\alpha y(t'-r(t'))\right|^{p}\mathrm{d}t'\mathrm{d}s\right)^{\frac{1}{p}}$$

$$\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}$$

$$\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|a\alpha x(t'-r(t'))+q(t',y(t'),y(t'-r(t')))\right.\right.$$

$$\left.\left.-\;q(t',x(t'),x(t'-r(t')))-a\alpha y(t'-r(t'))\right|^{p}\mathrm{d}t'\right)^{\frac{1}{p}}$$

$$\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}$$

$$\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left(L_q^1\left|y(t')-x(t')\right|+(L_q^2-a\alpha)\left|y(t'-r(t'))-x(t'-r(t'))\right|\right)^{p}\mathrm{d}t'\right)^{\frac{1}{p}}$$

$$\leq \alpha\sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|x(t-r(t))-y(t-r(t))|^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1$$

$$\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|y(t')-x(t')\right|^{p}\mathrm{d}t'\right)^{\frac{1}{p}}$$

$$+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|y(t'-r(t'))-x(t'-r(t'))\right|^{p}\mathrm{d}t'\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&\leq \alpha\left(1-r'(t)\right)\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}\left|x(t')-y(t')\right|^p \mathrm{d}t'\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\|x-y\|_{S^p}\\
&\quad+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)\left(1-r(t')\right)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'-r(\xi')}^{\xi'+1-r(\xi'+1)}|y(t)-x(t)|^p\,\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq \alpha(1-r^*)\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar{r}}^{\xi+1-\bar{r}}\left|x(t')-y(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\|x-y\|_{S^p}\\
&\quad+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)(1-r^*)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'-\bar{r}}^{\xi'+1-\bar{r}}|y(t)-x(t)|^p\,\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq m_1\|x-y\|_{S^p},
\end{aligned}
$$

with

$$
m_1=\max\left(\alpha(1-r^*),L_q^1\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}},(1-r^*)\left(L_q^2-a\alpha\right)\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}\right).
$$

As $m_1<1$, the operator $\Lambda:(PAPS^p(\mathbb{R},\mathbb{R}),\|\cdot\|_{S^p})\longrightarrow(PAPS^p(\mathbb{R},\mathbb{R}),\|\cdot\|_{S^p})$ is a contraction. □

4.2. *Example 1*

Let us consider the following logistic differential equation

$$
x'(t)=3x(t)+\frac{x'(t-r(t))}{2}-q(t,x(t),x(t-r(t)))+h(t), \tag{4.12}
$$

where $q:\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by:

$$
\begin{aligned}
q(t,\sin(t),\sin(t-\cos(t)))&=\left(\sin(t)+\sin(\sqrt{2}t)\right)[\sin(t)+\sin(t-\cos(t))]\\
&\quad+\frac{[\sin(t)+\sin(t-\cos(t))]}{1+t^2}\\
&=q_1(t,\sin(t),\sin(t-\cot(t)))+q_2(t,\sin(t),\sin(t-\cos(t))).
\end{aligned}
$$

$h:\mathbb{R}\longrightarrow\mathbb{R}$ defined by $h(t)=h_1(t)+h_2(t)$, with

$$
h_1(t)=\begin{cases}-\sin(t) & \text{for} t\neq k\pi\\ 0 & \text{for } t=k\pi\end{cases} \tag{4.13}
$$

and

$$h_2(t) = \arctan(t). \tag{4.14}$$

Solving (4.12) returns to work out the following integral equation

$$x(t) = \frac{x(t-r(t))}{2} - \int_t^{+\infty} \left[q(s,x(s),x(s-r(s))) - \frac{3x(s-r(s))}{2} \right] \mathrm{e}^{3(t-s)} \mathrm{d}s + p(t), \tag{4.15}$$

where the function $p : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $p(t) = p_1(t) + p_2(t)$, with

$$p_1(t) = \begin{cases} \cos(t) & \text{for } t \neq k\pi \\ k & \text{for } t = k\pi \end{cases} \tag{4.16}$$

and

$$p_2(t) = \frac{1}{1+t^2}. \tag{4.17}$$

Firstly, the function $q_1^b(\cdot) \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$. In addition, the function $q_2^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$. Then, the function $q : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic, thus hypothesis (H1) holds. Secondly, the function q is Lipschitz. Then hypothesis (H2) holds. Meanwhile, it is well-known that $[t \longmapsto p_1(t)] \in SAP(\mathbb{R},\mathbb{R})$ (cf. [8]). Moreover, $p_2^b(\cdot) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Then $p(\cdot) \in PAPS^p(\mathbb{R},\mathbb{R})$, which implies that (H3) holds. Now, by virtue of Theorem 3 equation (4.15) admits a unique S^p-pseudo almost periodic solution when $m_1 < 1$, with

$$m_1 = \max\left(\frac{1}{2}, 3\left(\frac{2}{3q}\right)^{\frac{1}{q}} \left(\frac{2}{3p}\right)^{\frac{1}{p}}, \frac{3}{2}\left(\frac{2}{3q}\right)^{\frac{1}{q}} \left(\frac{2}{3p}\right)^{\frac{1}{p}} \right).$$

4.3. *Stepanov-like pseudo almost automorphic solutions*

We will study the S^p-pseudo almost automorphic solutions of (4.2). For this study, we formulate the following assumptions

(H1) $q : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost automorphic, i.e.

$$q^b = q_1^b + q_2^b, \tag{4.18}$$

where $q_1^b \in AA(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $q_2^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |q_2(\sigma,u)|^p \,\mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{4.19}$$

uniformly in $u \in \mathbb{R}^2$.

(H2) q is Lipschitz, i.e. $\exists L_q^1, L_q^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$|q(t,x_1,x_2) - q(t,y_1,y_2)| \leq L_q^1|x_1-y_1| + L_q^2|x_2-y_2|. \tag{4.20}$$

(H3) $p:\mathbb{R}\longrightarrow\mathbb{R}$ is S^p-pseudo almost automorphic, i.e. $p^b=p_1^b+p_2^b$ where $p_1^b\in AP(\mathbb{R},L^p((0,1),\mathbb{R}))$ and $p_2^b\in PAP_0(\mathbb{R},L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|p_2(\sigma)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{4.21}$$

(H6) The function $t\mapsto r(t)\in C^1(\mathbb{R},\mathbb{R})$ with

$$0\le r(t)\le\bar{r},\quad r(t)\le r^*<1. \tag{4.22}$$

Theorem 4. *Assume that (H1)-(H3) hold. If $m_2<1$ then (4.2) has a unique S^p-pseudo almost automorphic solution where*

$$m_2=\max\left(\alpha(1-r^*),L_q^1\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}},(1-r^*)\left(L_q^2-a\alpha\right)\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}\right).$$

Proof. Let the operator Λ_2 defined on $PAAS^p(\mathbb{R},\mathbb{R})$ by

$$\Lambda_2(x)(t)=\alpha x(t-r(t))-\int_t^{+\infty}[q(s,x(s),x(s-r(s)))-a\alpha x(s-r(s))]\,\mathrm{e}^{a(t-s)}\mathrm{d}s+p(t).$$

Now, showing that $\Lambda_2(x)\in PAAS^p(\mathbb{R},\mathbb{R})$, set the functions

$$f(\cdot,x(\cdot),x(\cdot-r(t)))=\alpha x(\cdot-r(t))+p(\cdot), \tag{4.23}$$

and

$$g(\cdot,x(\cdot),x(\cdot-r(t)))=[q(\cdot,x(\cdot),x(\cdot-r(t)))-a\alpha x(\cdot-r(t)).] \tag{4.24}$$

Since, $x(\cdot)\in PAAS^p(\mathbb{R},\mathbb{R})$, Lemma 6 implies that

$$[t\longmapsto x(t-r(t))]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

Then

$$[t\mapsto\alpha x(t-r(t))]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

By Lemma 7 we obtain that

$$[t\longmapsto f(t,x(t),x(t-r(t)))]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

Assumptions (H1)-(H2) and Lemma 7, yield that

$$[s\to q(s,x(s),x(s-r(s)))]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

Moreover, as previously we show, $[s\longmapsto -a\alpha x(s-r(s))]\in PAAS^p(\mathbb{R},\mathbb{R})$. Hence, the function $[s\longmapsto g(s,x(s),x(s-r(s)))]\in PAAS^p(\mathbb{R},\mathbb{R})$, as the sum of two S^p-pseudo almost periodic functions. Then, it follows from Lemma 8 that

$$\left[t\longmapsto\int_t^{+\infty}[q(s,x(s),x(s-r(s)))-a\alpha x(s-r(s))]\,\mathrm{e}^{a(t-s)}\mathrm{d}s\right]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

We deduce that $\Lambda_2 x\in PAAS^p(\mathbb{R},\mathbb{R})$. It remains to show that Λ_2 admits a unique fixed point. Let $x,\,y\in PAAS^p(\mathbb{R},\mathbb{R})$

$$|\Lambda_2 x(t)-\Lambda_2 y(t)|\le|\alpha x(t-r(t))-\alpha y(t-r(t))|$$

$$
\begin{aligned}
&+\int_t^{+\infty} \mathrm{e}^{a(t-s)}\,|q(s,y(s),y(s-r(s)))-q(s,x(s),x(s-r(s)))\\
&+a\alpha x(s-r(s))-a\alpha y(s-r(s))|\,\mathrm{d}s\\
&\leq |\alpha x(t-r(t))-\alpha y(t-r(t))|+\int_0^{+\infty}\mathrm{e}^{-as}\,|q(s+t,y(s+t),y(s+t-r(s+t)))\\
&-q(s+t,x(s+t),x(s+t-r(s+t)))\\
&+a\alpha x(s+t-r(s+t))-a\alpha y(s+t-r(s+t))|\,\mathrm{d}s\\
&\leq |\alpha x(t-r(t))-\alpha y(t-r(t))|+\left(\frac{2}{qa}\right)^{\frac{1}{q}}\\
&\times\int_0^{\infty}\mathrm{e}^{\frac{-aps}{2}}\,|q(s+t,y(s+t),y(s+t-r(s+t)))+a\alpha x(s+t-r(s+t))\\
&-q(s+t,x(s+t),x(s+t-r(s+t)))-a\alpha y(s+t-r(s+t))|^p\,\mathrm{d}s.
\end{aligned}
$$

So, using Fubini's theorem and Minkowski's inequality we have

$$
\begin{aligned}
&\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\Lambda_2 x(t)-\Lambda_2 y(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\\
&\times\left(\int_0^{+\infty}\mathrm{e}^{\frac{-aps}{2}}\sup_{\xi\in\mathbb{R}}\int_\xi^{\xi+1}|a\alpha x(s+t-r(s+t))+q(s+t,y(s+t),y(s+t-r(s+t)))\right.\\
&\left.-q(s+t,x(s+t),x(s+t-r(s+t)))-a\alpha y(s+t-r(s+t))|^p\,\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\\
&\times\left(\int_0^{+\infty}\mathrm{e}^{\frac{-aps}{2}}\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left|a\alpha x(t'-r(t'))\right.\right.\\
&\left.\left.+q(t',y(t'),y(t'-r(t')))-q(t',x(t'),x(t'-r(t')))-a\alpha y(t'-r(t'))\right|^p\mathrm{d}t'\mathrm{d}s\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|a\alpha x(t'-r(t'))+q(t',y(t'),y(t'-r(t')))\right.\right.\\
&\left.\left.-\ q(t',x(t'),x(t'-r(t')))-a\alpha y(t'-r(t'))\right|^p \mathrm{d}t'\right)^{\frac{1}{p}}\\
&\le \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}\\
&\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left(L_q^1\left|y(t')-x(t')\right|+(L_q^2-a\alpha)\left|y(t'-r(t'))-x(t'-r(t'))\right|\right)^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\le \alpha\sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|x(t-r(t))-y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\\
&\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|y(t')-x(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|y(t'-r(t'))-x(t'-r(t'))\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\le \alpha\left(1-r'(t)\right)\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}\left|x(t')-y(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\|x-y\|_{S^p}\\
&+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)\left(1-r(t')\right)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'-r(\xi')}^{\xi'+1-r(\xi'+1)}\left|y(t')-x(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\le \alpha(1-r^*)\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar r}^{\xi+1-\bar r}\left|x(t')-y(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\|x-y\|_{S^p}\\
&+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)(1-r^*)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'-\bar r}^{\xi'+1-\bar r}\left|y(t')-x(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\le m_2\|x-y\|_{S^p},
\end{aligned}
$$

with

$$
m_2=\max\left(\alpha(1-r^*),L_q^1\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}},(1-r^*)\left(L_q^2-a\alpha\right)\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}\right).
$$

As $m_2 < 1$, the operator $\Lambda_2 : (PAAS^p(\mathbb{R},\mathbb{R}), \|\cdot\|_{S^p}) \longrightarrow (PAAS^p(\mathbb{R},\mathbb{R}), \|\cdot\|_{S^p})$ is a contraction and the result holds by Banach's fixed point theorem. □

4.4. *Example 2*

In order to illustrate Theorem 4, we consider the following logistic differential equation

$$x'(t) = 2x(t) + \frac{x'(t-r(t))}{4} - q(t,x(t),x(t-r(t))) + h(t), \tag{4.25}$$

where $q : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by:

$$\begin{aligned} q(t,\sin(t),\sin(t-4)) &= \frac{[\sin(t)+\sin(t-\cos(t))]}{2+\cos(t)+\cos(\sqrt{2}t)} + \frac{[\sin(t)+\sin(t-\cos(t))]}{1+t^2} \\ &= q_1(t,\sin(t),\sin(t-\cos(t))) + q_2(t,\sin(t),\sin(t-\cos(t))), \end{aligned}$$

and $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(t) = h_1(t) + h_2(t)$ where

$$h_1(t) = \frac{\sin(t)+\sqrt{2}\sin(\sqrt{2}t)}{(2+\cos(t)+\cos(\sqrt{2}t))^2}\cos\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right)$$

and

$$h_2(t) = \arctan(t). \tag{4.26}$$

Rather than dealing with (4.25) we will study the following integral equation

$$x(t) = \frac{x(t-r(t))}{2} - \int_t^{+\infty}\left[q(s,x(s),x(s-r(s))) - \frac{3x(s-r(s))}{2}\right]e^{3(t-s)}ds + p(t), \tag{4.27}$$

the function q is S^p-pseudo almost automorphic and lipschtizian. In addition, the function $p : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $p(t) = p_1(t) + p_2(t)$ where

$$p_1(t) = \sin\left(\frac{1}{2+\cos(t)+\sin(\sqrt{2}t)}\right) \tag{4.28}$$

and

$$p_2(t) = \frac{1}{1+t^2}, \tag{4.29}$$

belongs to $PAAS^p(\mathbb{R},\mathbb{R})$. Hence, one can deduce that all the assumptions (H1), (H2) and (H3) of Theorem 4 are satisfied and thus equation (4.25) has a unique S^p-pseudo almost automorphic solution.

Acknowledgement

The authors would like to thank the anonymous reviewers and the editors for their constructive comments, which greatly improved the quality of the original version.

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