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FUZZY SOFT POSITIVE IMPLICATIVE HYPER *BCK*-IDEALS OF SEVERAL TYPES

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Abstract. Fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$ are introduced, and their relations are investigated. Relations between fuzzy soft strong hyper *BCK*-ideal and fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$ are discussed. We prove that the level set of fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$ are positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$, respectively. Conditions for a fuzzy soft set to be a fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$, respectively, are founded, and conditions for a fuzzy soft set to be a fuzzy soft weak hyper *BCK*-ideal are considered.

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Keywords: hyper *BCK*-algebra, fuzzy soft (weak, strong) hyper *BCK*-ideal, fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$

1. INTRODUCTION

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced in 1934 by the French mathematician F. Marty [13] when Marty defined hypergroups, began to analyze their properties, and applied them to groups and relational algebraic functions (see [13]). Since then, many papers and several books have been written on this topic. Nowadays, hyperstructures have a lot of applications in several branches of mathematics and computer sciences etc. (see [1,4,11,12]). In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. In [9], Jun et al. applied the hyperstructures to *BCK*-algebras, and introduced the concept of a hyper *BCK*-algebra which is a generalization of a *BCK*-algebra. Sine then, Jun et al. studied more notions and results in [5], and [8]. Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social science etc. These problems cannot be dealt with by classical methods, because classical methods have inherent difficulties. To overcome these difficulties, Molodtsov [14] proposed a new approach, which was called soft set theory, for modeling uncertainty. Jun applied the notion of soft sets to the theory of BCK/BCI-algebras, and Jun et al. [5] studied ideal theory of BCK/BCI-algebras based on soft set theory. Maji et al. [15] extended the study of soft sets to fuzzy soft sets. They introduced the concept of fuzzy soft sets as a generalization of the standard soft sets, and presented an application of fuzzy soft sets in a decision making problem. Jun et al. applied fuzzy soft set to BCK/BCI-algebras. Khademan et al. [10] applied the notion of fuzzy soft sets by Maji et al. to the theory of hyper BCK-algebras. They introduced the notion of fuzzy soft positive implicative hyper BCK-ideal, and investigated several properties. They discussed the relation between fuzzy soft positive implicative hyper BCK-ideal and fuzzy soft hyper BCK-ideal, and provided characterizations of fuzzy soft positive implicative hyper BCK-ideal. Using the notion of positive implicative hyper BCK-ideal, they established a fuzzy soft weak (strong) hyper BCK-ideal.

In this paper, we introduce the notion of fuzzy soft positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$, $(\ll,\ll,\subseteq)$ and $(\subseteq,\ll,\subseteq)$, and investigate their relations and properties. We discuss relations between fuzzy soft strong hyper BCK-ideal and fuzzy soft positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$ and $(\ll,\ll,\subseteq)$. We prove that the level set of fuzzy soft positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$, $(\ll,\ll,\subseteq)$ and $(\subseteq,\ll,\subseteq)$ are positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$, $(\ll,\ll,\subseteq)$ and $(\subseteq,\ll,\subseteq)$, respectively. We find conditions for a fuzzy soft set to be a fuzzy soft positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$, $(\ll,\ll,\subseteq)$ and $(\subseteq,\ll,\subseteq)$, respectively. We also consider conditions for a fuzzy soft set to be a fuzzy soft weak hyper BCK-ideal.

2. Preliminaries

Let H be a nonempty set endowed with a hyper operation "$\circ$", that is, "$\circ$" is a function from $H\times H$ to $\mathcal{P}^*(H)=\mathcal{P}(H)\setminus\{\varnothing\}$. For two subsets A and B of H, denote by $A\circ B$ the set $\cup\{a\circ b \mid a\in A, b\in B\}$. We shall use $x\circ y$ instead of $x\circ\{y\}$, $\{x\}\circ y$, or $\{x\}\circ\{y\}$.

By a hyper BCK-algebra (see [9]) we mean a nonempty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(H1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,
(H2) $(x\circ y)\circ z=(x\circ z)\circ y$,
(H3) $x\circ H\ll\{x\}$,
(H4) $x\ll y$ and $y\ll x$ imply $x=y$,

for all $x,y,z\in H$, where $x\ll y$ is defined by $0\in x\circ y$ and for every $A,B\subseteq H$, $A\ll B$ is defined by $\forall a\in A, \exists b\in B$ such that $a\ll b$.

In a hyper BCK-algebra H, the condition (H3) is equivalent to the condition:

$$x\circ y\ll\{x\}. \tag{2.1}$$

In any hyper *BCK*-algebra H, the following hold (see [9]):

$$x \circ 0 \ll \{x\},\ 0 \circ x \ll \{0\},\ 0 \circ 0 \ll \{0\}, \tag{2.2}$$
$$(A \circ B) \circ C = (A \circ C) \circ B,\ A \circ B \ll A,\ 0 \circ A \ll \{0\}, \tag{2.3}$$
$$0 \circ 0 = \{0\}, \tag{2.4}$$
$$0 \ll x,\ x \ll x,\ A \ll A, \tag{2.5}$$
$$A \subseteq B \text{ implies } A \ll B, \tag{2.6}$$
$$0 \circ x = \{0\},\ 0 \circ A = \{0\}, \tag{2.7}$$
$$A \ll \{0\} \text{ implies } A = \{0\}, \tag{2.8}$$
$$x \in x \circ 0, \tag{2.9}$$
$$x \circ 0 = \{x\},\ A \circ 0 = A, \tag{2.10}$$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H.
A subset I of a hyper *BCK*-algebra H is called a *hyper BCK-ideal* of H (see [9]) if it satisfies

$$0 \in I \tag{2.11}$$
$$(\forall x, y \in H)\ (x \circ y \ll I,\ y \in I \Rightarrow x \in I) \tag{2.12}$$

A subset I of a hyper *BCK*-algebra H, is called a *strong hyper BCK-ideal* of H (see [8]) if it satisfies (2.11) and

$$(\forall x, y \in H)\ ((x \circ y) \cap I \neq \varnothing,\ y \in I \Rightarrow x \in I). \tag{2.13}$$

Recall that every strong hyper *BCK*-ideal is a hyper *BCK*-ideal, but the converse may not be true (see [8]). A subset I of a hyper *BCK*-algebra H is called a *weak hyper BCK-ideal* of H (see [9]) if it satisfies (2.11) and

$$(\forall x, y \in H)\ (x \circ y \subseteq I,\ y \in I \Rightarrow x \in I) \tag{2.14}$$

Every hyper *BCK*-ideal is a weak hyper *BCK*-ideal, but the converse may not be true.
A subset I of a hyper *BCK*-algebra H is said to be

- *reflexive* if $(x \circ x) \subseteq I$ for all $x \in H$,
- *closed* if the following assertion is valid.

$$(\forall x \in H)(\forall y \in I)(x \ll y\ \Rightarrow\ x \in I).$$

Given a subset I of H and $x, y, z \in H$, we consider the following conditions:

$$(x \circ y) \circ z \subseteq I,\ y \circ z \subseteq I \Rightarrow x \circ z \subseteq I \tag{2.15}$$
$$(x \circ y) \circ z \subseteq I,\ y \circ z \ll I \Rightarrow x \circ z \subseteq I \tag{2.16}$$
$$(x \circ y) \circ z \ll I,\ y \circ z \subseteq I \Rightarrow x \circ z \subseteq I \tag{2.17}$$
$$(x \circ y) \circ z \ll I,\ y \circ z \ll I \Rightarrow x \circ z \subseteq I \tag{2.18}$$

Definition 1 ([3, 6])**.** Let I be a nonempty subset of a hyper BCK-algebra H and $0 \in I$. If it satisfies (2.15) (resp. (2.16), (2.17) and (2.18)), then we say that I is a positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ (resp. $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$) for all $x, y, z \in H$.

Molodtsov ([14]) defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denote the power set of U and $A \subseteq E$.

Definition 2 ([14])**.** A pair (λ, A) is called a *soft set* over U, where λ is a mapping given by

$$\lambda : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\lambda(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (λ, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [14].

Definition 3 ([15])**.** Let U be an initial universe set and E be a set of parameters. Let $\mathcal{F}(U)$ denote the set of all fuzzy sets in U. Then a pair $(\tilde{\lambda}, A)$ is called a *fuzzy soft set* over U where $A \subseteq E$ and $\tilde{\lambda}$ is a mapping given by $\tilde{\lambda} : A \to \mathcal{F}(U)$.

In general, for every parameter u in A, $\tilde{\lambda}[u]$ is a fuzzy set in U and it is called *fuzzy value set* of parameter u.

Given a fuzzy set μ in a hyper BCK-algebra H and a subset T of H, by $\mu_*(T)$ and $\mu^*(T)$ we mean

$$\mu_*(T) = \inf_{a \in T} \mu(a) \text{ and } \mu^*(T) = \sup_{a \in T} \mu(a). \tag{2.19}$$

Definition 4 ([2])**.** A fuzzy soft set $(\tilde{\lambda}, A)$ over a hyper BCK-algebra H is called

- a *fuzzy soft hyper BCK-ideal* based on a paramenter $u \in A$ over H (briefly, u-fuzzy soft hyper BCK-ideal of H) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0, 1]$ of u satisfies the following conditions:

$$(\forall x, y \in H) \left(x \ll y \;\Rightarrow\; \tilde{\lambda}[u](x) \geq \tilde{\lambda}[u](y) \right), \tag{2.20}$$

$$(\forall x, y \in H) \left(\tilde{\lambda}[u](x) \geq \min\{\tilde{\lambda}[u]_*(x \circ y), \tilde{\lambda}[u](y)\} \right). \tag{2.21}$$

- a *fuzzy soft weak hyper BCK-ideal* based on a paramenter $u \in A$ over H (briefly, *u-fuzzy soft weak hyper BCK-ideal* of H) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0, 1]$ of u satisfies condition (2.21) and

$$(\forall x \in H) \left(\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x) \right). \tag{2.22}$$

- a *fuzzy soft strong hyper BCK-ideal* over H based on a paramenter u in A (briefly, u-fuzzy soft strong hyper BCK-ideal of H) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies the following conditions:

$$(\forall x, y \in H) \left(\tilde{\lambda}[u](x) \geq \min\{\tilde{\lambda}[u]^*(x \circ y), \tilde{\lambda}[u](y)\} \right), \tag{2.23}$$

$$(\forall x \in H) \left(\tilde{\lambda}[u]_*(x \circ x) \geq \tilde{\lambda}[u](x) \right). \tag{2.24}$$

If $(\tilde{\lambda}, A)$ is a fuzzy soft (weak, strong) hyper BCK-ideal based on a paramenter u over H for all $u \in A$, we say that $(\tilde{\lambda}, A)$ is a *fuzzy soft (weak, strong) hyper BCK-ideal* of H.

3. Fuzzy soft positive implicative hyper *BCK*-ideals

In what follows, let H be a hyper BCK-algebra unless otherwise specified.

Definition 5. Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H. Then $(\tilde{\lambda}, A)$ is called

- a *fuzzy soft positive implicative hyper BCK-ideal* of type $(\subseteq, \subseteq, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies the following conditions:

$$(\forall x, y \in H)\ (x \ll y \Rightarrow \tilde{\lambda}[u](x) \geq \tilde{\lambda}[u](y)), \tag{3.1}$$

$$(\forall x, y, z \in H)(\tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\}). \tag{3.2}$$

- a *fuzzy soft positive implicative hyper BCK-ideal* of type $(\subseteq, \ll, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies (3.1) and

$$(\forall x, y, z \in H)\ (\tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\}). \tag{3.3}$$

- a *fuzzy soft positive implicative hyper BCK-ideal* of type $(\ll, \subseteq, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies (3.1) and

$$(\forall x, y, z \in H)\ (\tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\}). \tag{3.4}$$

- a *fuzzy soft positive implicative hyper BCK-ideal* of type $(\ll, \ll, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies (3.1) and

$$(\forall x, y, z \in H)\ (\tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\}). \tag{3.5}$$

Theorem 1. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H.*

(1) *If* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\ll, \subseteq, \subseteq)$ *or type* $(\subseteq, \ll, \subseteq)$*, then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\subseteq, \subseteq, \subseteq)$*.*

(2) *If* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\ll, \ll, \subseteq)$*, then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\ll, \subseteq, \subseteq)$ *and* $(\subseteq, \ll, \subseteq)$*.*

Proof. (1) Assume that $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ or type $(\subseteq, \ll, \subseteq)$. Then

$$\begin{aligned}\tilde{\lambda}[u]_*(x \circ z) &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\} \\ &\geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\}\end{aligned}$$

or

$$\begin{aligned}\tilde{\lambda}[u]_*(x \circ z) &\geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\} \\ &\geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\},\end{aligned}$$

respectively. Thus $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$.

(2) Suppose that $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Then

$$\begin{aligned}\tilde{\lambda}[u]_*(x \circ z) &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\} \\ &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\}\end{aligned}$$

and

$$\begin{aligned}\tilde{\lambda}[u]_*(x \circ z) &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\} \\ &\geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\}.\end{aligned}$$

Therefore $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$. □

Corollary 1. *If* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\ll, \ll, \subseteq)$*, then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\subseteq, \subseteq, \subseteq)$*.*

The following example shows that any fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Example 1. Consider a hyper BCK-algebra $H = \{0, a, b, c\}$ with the hyper operation "$\circ$" in Table 1.
Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 2. Then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$.

TABLE 1. Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b,c\}$	$\{0,b,c\}$

TABLE 2. Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b	c
x	0.9	0.8	0.5	0.3
y	0.9	0.7	0.6	0.4

Since

$$\tilde{\lambda}[x]_*(c\circ 0) = 0.3 < 0.5 = \min\left\{\tilde{\lambda}[x]^*((c\circ b)\circ 0), \tilde{\lambda}[x]_*(b\circ 0)\right\},$$

it is not an x-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll,\subseteq,\subseteq)$, and thus it is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll,\subseteq,\subseteq)$.

Question.

> Is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq,\subseteq,\subseteq)$ a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq,\ll,\subseteq)$?

The following example shows that any fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq,\ll,\subseteq)$ is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll,\subseteq,\subseteq)$ or $(\ll,\ll,\subseteq)$.

Example 2. Consider a hyper BCK-algebra $H = \{0,a,b\}$ with the hyper operation "$\circ$" in Table 3.

TABLE 3. Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

Given a set $A = \{x,y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 4.

TABLE 4. Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.3
y	0.8	0.7	0.1

Then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. Since

$$\tilde{\lambda}[x]_*(b \circ b) = 0.3 < 0.9 = \min\left\{\tilde{\lambda}[x]^*((b \circ a) \circ b), \tilde{\lambda}[x]^*(a \circ b)\right\},$$

it is not an x-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and so not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Also, since

$$\tilde{\lambda}[y]_*(b \circ b) = 0.1 < 0.8 = \min\left\{\tilde{\lambda}[y]^*((b \circ 0) \circ b), \tilde{\lambda}[y]^*(0 \circ b)\right\},$$

it is not a y-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$ and so not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$.

Question.

> Is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ or $(\ll, \ll, \subseteq)$?

Lemma 1 ([10]). *Every fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ is a fuzzy soft hyper BCK-ideal.*

The converse of Lemma 1 is not true (see [10, Example 3.6]). Using Theorems 1 and Lemma 1, we have the following corollary.

Corollary 2. *Every fuzzy soft positive implicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of types $(\ll, \subseteq, \subseteq)$, $(\subseteq, \ll, \subseteq)$ or $(\ll, \ll, \subseteq)$ is a fuzzy soft hyper BCK-ideal.*

We can check that the fuzzy soft set $(\tilde{\lambda}, A)$ in Example 1 is a fuzzy soft hyper BCK-ideal of H, but it is not a fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$. This shows that any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$. Also, we know that the fuzzy soft set $(\tilde{\lambda}, A)$ in Example 2 is a fuzzy soft hyper BCK-ideal of H, but it is a fuzzy soft hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Thus any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Let $(\tilde{\lambda}, A)$ be a fuzzy soft hyper BCK-ideal of H. If $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of type $(\subseteq, \ll, \subseteq)$, then it is a fuzzy soft positive implicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of type $(\subseteq, \subseteq, \subseteq)$ by Theorem 1(1). Hence every fuzzy soft hyper BCK-ideal of H is a fuzzy soft positive implicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of type $(\subseteq, \subseteq, \subseteq)$. But this is contradictory to [10, Example 3.6]. Therefore we know that any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$.

We consider relation between a fuzzy soft positive implicative hyper BCK-ideal of any type and a fuzzy soft strong hyper BCK-ideal.

Theorem 2. *Every fuzzy soft positive implicative hyper BCK-ideal of type* $(\ll, \subseteq, \subseteq)$ *is a fuzzy soft strong hyper BCK-ideal of H.*

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and let u be any parameter in A. Since $x \circ x \ll x$ for all $x \in H$, it follows from (3.1) that

$$\tilde{\lambda}[u]_*(x \circ x) \geq \tilde{\lambda}[u]_*(x) = \tilde{\lambda}[u](x).$$

Taking $z = 0$ in (3.4) and using (2.10) imply that

$$\begin{aligned}\tilde{\lambda}[u](x) &= \tilde{\lambda}[u]_*(x \circ 0)\\ &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ 0), \tilde{\lambda}[u]_*(y \circ 0)\}\\ &= \min\{\tilde{\lambda}[u]^*(x \circ y), \tilde{\lambda}[u](y)\}.\end{aligned}$$

Therefore $(\tilde{\lambda}, A)$ is a fuzzy soft strong hyper BCK-ideal of H. □

Corollary 3. *Every fuzzy soft positive implicative hyper BCK-ideal of type* $(\ll, \ll, \subseteq)$ *is a fuzzy soft strong hyper BCK-ideal of H.*

The following example shows that the converse of Theorem 2 and Corollary 3 is not true in general.

Example 3. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given in Table 5. Given a set $A = \{x, y\}$ of parameters, we define a fuzzy

TABLE 5. Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

soft set $(\tilde{\lambda}, A)$ by Table 6.
Then $(\tilde{\lambda}, A)$ is a fuzzy soft strong hyper BCK-ideal of H. Since

$$\tilde{\lambda}[x]_*(b \circ b) = 0.5 < 0.9 = \min\left\{\tilde{\lambda}[x]^*((b \circ 0) \circ b), \tilde{\lambda}[x]_*(0 \circ b)\right\},$$

TABLE 6. Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.1	0.5
y	0.7	0.2	0.6

we know that $(\tilde{\lambda}, A)$ is not an x-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and so it is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Also

$$\tilde{\lambda}[y]_*(b \circ b) = 0.6 < 0.7 = \min\left\{\tilde{\lambda}[y]^*((b \circ b) \circ b), \tilde{\lambda}[y]^*(b \circ b)\right\},$$

and so $(\tilde{\lambda}, A)$ it is not a y-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Thus it is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Therefore any fuzzy soft strong hyper BCK-ideal of H may not be a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ or $(\ll, \ll, \subseteq)$.

Consider the hyper BCK-algebra $H = \{0, a, b, c\}$ in Example 1 and a set $A = \{x, y\}$ of parameters. We define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 2 in Example 1. Then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$. But $(\tilde{\lambda}, A)$ is not a fuzzy soft strong hyper BCK-ideal of H since

$$\tilde{\lambda}[y](c) = 0.4 < 0.6 = \min\left\{\tilde{\lambda}[y]^*(c \circ b), \tilde{\lambda}[y](b)\right\}.$$

Hence we know that any fuzzy soft positive implicative hyper BCK-ideal of types $(\subseteq, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$ is not a fuzzy soft strong hyper BCK-ideal of H.

Given a fuzzy soft set $(\tilde{\lambda}, A)$ over H and $t \in [0,1]$, we consider the following set

$$U(\tilde{\lambda}[u]; t) := \left\{x \in H \mid \tilde{\lambda}[u](x) \geq t\right\} \tag{3.6}$$

where u is a parameter in A, which is called *level set* of $(\tilde{\lambda}, A)$.

Lemma 2. *If a fuzzy soft set $(\tilde{\lambda}, A)$ over H satisfies the condition (3.1), then $0 \in U(\tilde{\lambda}[u]; t)$ for all $t \in [0,1]$ and any parameter u in A with $U(\tilde{\lambda}[u]; t) \neq \varnothing$.*

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H which satisfies the condition (3.1). For any $t \in [0,1]$ and any parameter u in A, assume that $U(\tilde{\lambda}[u]; t) \neq \varnothing$. Since $0 \ll x$ for all $x \in H$, it follows from (3.1) that $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in H$. Hence $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in U(\tilde{\lambda}[u]; t)$, and so $\tilde{\lambda}[u](0) \geq t$. Thus $0 \in U(\tilde{\lambda}[u]; t)$. □

Lemma 3 ([2]). *A fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft hyper BCK-ideal of H if and only if the set $U(\tilde{\lambda}[u]; t)$ in (3.6) is a hyper BCK-ideal of H for all $t \in [0,1]$ and any parameter u in A with $U(\tilde{\lambda}[u]; t) \neq \varnothing$.*

Theorem 3. *If a fuzzy soft set* $(\tilde{\lambda}, A)$ *over* H *is a fuzzy soft positive implicative hyper BCK-ideal of type* $(\subseteq, \ll, \subseteq)$*, then the set* $U(\tilde{\lambda}[u];t)$ *in* (3.6) *is a positive implicative hyper BCK-ideal of type* $(\subseteq, \ll, \subseteq)$ *for all* $t \in [0,1]$ *and any parameter* u *in* A *with* $U(\tilde{\lambda}[u];t) \neq \varnothing$*.*

Proof. Assume that a fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. Then $0 \in U(\tilde{\lambda}[u];t)$ by Lemma 2. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \subseteq U(\tilde{\lambda}[u];t)$ and $y \circ z \ll U(\tilde{\lambda}[u];t)$. Then

$$\tilde{\lambda}[u](a) \geq t \text{ for all } a \in (x \circ y) \circ z \tag{3.7}$$

and

$$(\forall b \in y \circ z)(\exists c \in U(\tilde{\lambda}[u];t))(b \ll c). \tag{3.8}$$

The condition (3.7) implies $\tilde{\lambda}[u]_*((x \circ y) \circ z) \geq t$, and the condition (3.8) implies from (3.1) that $\tilde{\lambda}[u](b) \geq \tilde{\lambda}[u](c) \geq t$ for all $b \in y \circ z$. Let $d \in x \circ z$. Using (3.3), we have

$$\tilde{\lambda}[u](d) \geq \tilde{\lambda}[u]_*(x \circ z) \geq \min\left\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\right\} \geq t.$$

Thus $d \in U(\tilde{\lambda}[u];t)$, and so $x \circ z \subseteq U(\tilde{\lambda}[u];t)$. Therefore $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. □

The following example shows that the converse of Theorem 3 is not true in general.

Example 4. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" in Table 7.

TABLE 7. Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 8.

TABLE 8. Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.8
y	0.8	0.3	0.6

Then

$$U(\tilde{\lambda}[x];t)=\begin{cases}\varnothing & \text{if } t\in(0.9,1],\\ \{0\} & \text{if } t\in(0.8,0.9],\\ \{0,b\} & \text{if } t\in(0.5,0.8],\\ H & \text{if } t\in[0,0.5]\end{cases}$$

and

$$U(\tilde{\lambda}[y];t)=\begin{cases}\varnothing & \text{if } t\in(0.8,1],\\ \{0\} & \text{if } t\in(0.6,0.8],\\ \{0,b\} & \text{if } t\in(0.3,0.6],\\ H & \text{if } t\in[0,0.3],\end{cases}$$

which are positive implicative hyper BCK-ideals of type $(\subseteq,\ll,\subseteq)$. Note that $a\ll b$ and $\tilde{\lambda}[u](a)<\tilde{\lambda}[u](b)$ for all $u\in A$. Thus $(\tilde{\lambda},A)$ is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq,\ll,\subseteq)$.

Lemma 4 ([3]). *Every positive implicative hyper BCK-ideal of type $(\subseteq,\subseteq,\subseteq)$ is a weak hyper BCK-ideal of H.*

Lemma 5 ([8]). *Let I be a reflexive hyper BCK-ideal of H. Then*

$$(\forall x,y\in H)((x\circ y)\cap I\neq\varnothing\ \Rightarrow\ x\circ y\ll I). \tag{3.9}$$

Lemma 6. *If any subset I of H is closed and satisfies the condition (2.14), then the condition (2.12) is valid.*

Proof. Assume that $x\circ y\ll I$ and $y\in I$ for all $x,y\in H$. Let $a\in x\circ y$. Then there exists $b\in I$ such that $a\ll b$. Since I is closed, we have $a\in I$ and thus $x\circ y\subseteq I$. It follows from (2.14) that $x\in I$. □

Theorem 4. *Let A be a fuzzy soft set over H satisfying the condition (3.1) and*

$$(\forall T\in\mathcal{P}(H))(\exists x_0\in T)\left(\tilde{\lambda}[u](x_0)=\tilde{\lambda}[u]^*(T)\right). \tag{3.10}$$

If the set $U(\tilde{\lambda}[u];t)$ in (3.6) is a reflexive positive implicative hyper BCK-ideal of type $(\subseteq,\ll,\subseteq)$ for all $t\in[0,1]$ and any parameter u in A with $U(\tilde{\lambda}[u];t)\neq\varnothing$, then $(\tilde{\lambda},A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq,\ll,\subseteq)$.

Proof. For any $x,y,z\in H$ let

$$t:=\min\{\tilde{\lambda}[u]_*((x\circ y)\circ z),\tilde{\lambda}[u]^*(y\circ z)\}.$$

Then $\tilde{\lambda}[u]_*((x\circ y)\circ z)\geq t$ and so $\tilde{\lambda}[u](a)\geq t$ for all $a\in(x\circ y)\circ z$. Since $\tilde{\lambda}[u]^*(y\circ z)\geq t$, it follows from (3.10) that $\tilde{\lambda}[u](b_0)=\tilde{\lambda}[u]^*(y\circ z)\geq t$ for some $b_0\in y\circ z$. Hence $b_0\in U(\tilde{\lambda}[u];t)$, and thus $U(\tilde{\lambda}[u];t)\cap(y\circ z)\neq\varnothing$. Since $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\subseteq,\ll,\subseteq)$ and hence of type $(\subseteq,\subseteq,\subseteq)$, $U(\tilde{\lambda}[u];t)$ is a weak hyper BCK-ideal of H by Lemma 4. Let $x,\in H$ be such that $x\ll y$. If $y\in U(\tilde{\lambda}[u];t)$, then $\tilde{\lambda}[u](x)\geq\tilde{\lambda}[u](y)\geq t$ by (3.1) and so $x\in U(\tilde{\lambda}[u];t)$, that is, $U(\tilde{\lambda}[u];t)$ is closed. Hence $U(\tilde{\lambda}[u];t)$ is a hyper BCK-ideal of H by Lemma 6.

Since $U(\tilde{\lambda}[u];t)$ is reflexive, it follows from Lemma 5 that $y \circ z \ll U(\tilde{\lambda}[u];t)$. Hence $x \circ z \subseteq U(\tilde{\lambda}[u];t)$ since $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper *BCK*-ideal of type $(\subseteq, \ll, \subseteq)$. Hence

$$\tilde{\lambda}[u](a) \geq t = \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\}$$

for all $a \in x \circ z$, and thus

$$\tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\}$$

for all $x, y, z \in H$. Therefore $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper *BCK*-ideal of type $(\subseteq, \ll, \subseteq)$. □

Corollary 4. *Let A be a fuzzy soft set over H satisfying the condition* (3.1) *and* (3.10). *For any* $t \in [0,1]$ *and any parameter u in A, assume that* $U(\tilde{\lambda}[u];t)$ *is nonempty and reflexive. Then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper BCK-ideal of type* $(\subseteq, \ll, \subseteq)$ *if and only if* $U(\tilde{\lambda}[u];t)$ *is a positive implicative hyper BCK-ideal of type* $(\subseteq, \ll, \subseteq)$.

Theorem 5. *If a fuzzy soft set* $(\tilde{\lambda}, A)$ *over H is a fuzzy soft positive implicative hyper BCK-ideal of type* $(\ll, \subseteq, \subseteq)$*, then the set* $U(\tilde{\lambda}[u];t)$ *in* (3.6) *is a positive implicative hyper BCK-ideal of type* $(\ll, \subseteq, \subseteq)$ *for all* $t \in [0,1]$ *and any parameter u in A with* $U(\tilde{\lambda}[u];t) \neq \varnothing$.

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft positive implicative hyper *BCK*-ideal of type $(\ll, \subseteq, \subseteq)$. Then $0 \in U(\tilde{\lambda}[u];t)$ by Lemma 2. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \ll U(\tilde{\lambda}[u];t)$ and $y \circ z \subseteq U(\tilde{\lambda}[u];t)$. Then

$$(\forall a \in (x \circ y) \circ z)(\exists b \in U(\tilde{\lambda}[u];t))(a \ll b), \tag{3.11}$$

which implies from (3.1) that $\tilde{\lambda}[u](a) \geq \tilde{\lambda}[u](b)$ for all $a \in (x \circ y) \circ z$. Since $y \circ z \subseteq U(\tilde{\lambda}[u];t)$, we have

$$\tilde{\lambda}[u](a) \geq t \text{ for all } a \in y \circ z. \tag{3.12}$$

Let $c \in x \circ z$. Then

$$\tilde{\lambda}[u](c) \geq \tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\} \geq t$$

for all $x, y, z \in H$ by (3.4), and thus $c \in U(\tilde{\lambda}[u];t)$. Hence $x \circ z \subseteq U(\tilde{\lambda}[u];t)$. Therefore $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper *BCK*-ideal of type $(\ll, \subseteq, \subseteq)$. □

The converse of Theorem 5 is not true as seen in the following example.

Example 5. Consider the hyper BCK-algebra $H = \{0, a, b\}$ and the fuzzy soft set $(\tilde{\lambda}, A)$ in Example 2. Then

$$U(\tilde{\lambda}[x];t) = \begin{cases} \varnothing & \text{if } t \in (0.9, 1], \\ \{0\} & \text{if } t \in (0.5, 0.9], \\ \{0, a\} & \text{if } t \in (0.3, 0.5], \\ H & \text{if } t \in [0, 0.3] \end{cases}$$

and

$$U(\tilde{\lambda}[y];t) = \begin{cases} \varnothing & \text{if } t \in (0.8, 1], \\ \{0\} & \text{if } t \in (0.7, 0.8], \\ \{0, a\} & \text{if } t \in (0.1, 0.7], \\ H & \text{if } t \in [0, 0.1], \end{cases}$$

which are positive implicative hyper BCK-ideals of type $(\ll, \subseteq, \subseteq)$. But we know $(\tilde{\lambda}, A)$ is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Lemma 7 ([8])**.** *Every reflexive hyper BCK-ideal I of H satisfies the following implication:*

$$(\forall x, y \in H)\ ((x \circ y) \cap I \neq \varnothing \Rightarrow x \circ y \subseteq I)$$

Lemma 8 ([7])**.** *Every positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ is a hyper BCK-ideal.*

We provide conditions for a fuzzy soft set to be a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Theorem 6. *Let A be a fuzzy soft set over H satisfying the condition* (3.10)*. If the set $U(\tilde{\lambda}[u];t)$ in* (3.6) *is a reflexive positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u];t) \neq \varnothing$, then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.*

Proof. Assume that $U(\tilde{\lambda}[u];t) \neq \varnothing$ for all $t \in [0, 1]$ and any parameter u in A. Suppose that $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Then $U(\tilde{\lambda}[u];t)$ is a hyper BCK-ideal of H by Lemma (8). It follows from Lemma (3) that $(\tilde{\lambda}, A)$ is a fuzzy soft hyper BCK-ideal of H. Thus the condition (3.1) is valid. Now let $t = \min\left\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\right\}$ for $x, y, z \in H$. Since $(\tilde{\lambda}, A)$ satisfies the condition (3.10), there exists $x_0 \in (x \circ y) \circ z$ such that $\tilde{\lambda}[u](x_0) = \tilde{\lambda}[u]^*((x \circ y) \circ z) \geq t$ and so $x_0 \in U(\tilde{\lambda}[u];t)$. Hence $((x \circ y) \circ z) \cap U(\tilde{\lambda}[u];t) \neq \varnothing$ and so $(x \circ y) \circ z \ll U(\tilde{\lambda}[u];t)$ by Lemma 7 and (2.6). Moreover $\tilde{\lambda}[u](c) \geq \tilde{\lambda}[u]_*(y \circ z) \geq t$ for all $c \in y \circ z$, and hence $c \in U(\tilde{\lambda}[u];t)$ which shows that $y \circ z \subseteq U(\tilde{\lambda}[u];t)$. Since $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$, it follows that $x \circ z \subseteq U(\tilde{\lambda}[u];t)$. Thus $\tilde{\lambda}[u](a) \geq t$ for all $a \in x \circ z$, and so

$$\tilde{\lambda}[u]_*(x \circ z) \geq t = \min\left\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\right\}.$$

Consequently, $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. □

Corollary 5. *Let A be a fuzzy soft set over H satisfying the condition (3.10). For any $t \in [0,1]$ and any parameter u in A, assume that $U(\tilde{\lambda}[u];t)$ is nonempty and reflexive. Then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ if and only if $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.*

Using a positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ (resp., $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$), we establish a fuzzy soft weak hyper BCK-ideal.

Theorem 7. *Let I be a positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ (resp., $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$) and let $z \in H$. For a fuzzy soft set $(\tilde{\lambda}, A)$ over H and any parameter u in A, if we define the fuzzy value set $\tilde{\lambda}[u]$ by*

$$\tilde{\lambda}[u] : H \to [0,1],\ x \mapsto \begin{cases} t & \text{if } x \in I_z, \\ s & \text{otherwise,} \end{cases} \tag{3.13}$$

where $t > s$ in $[0,1]$ and $I_z := \{y \in H \mid y \circ z \subseteq I\}$, then $(\tilde{\lambda}, A)$ is a u-fuzzy soft weak hyper BCK-ideal of H.

Proof. It is clear that $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in H$. Let $x, y \in H$. If $y \notin I_z$, then $\tilde{\lambda}[u](y) = s$ and so

$$\tilde{\lambda}[u](x) \geq s = \min\left\{\tilde{\lambda}[u](y), \tilde{\lambda}[u]_*(x \circ y)\right\}. \tag{3.14}$$

If $x \circ y \nsubseteq I_z$, then there exists $a \in x \circ y \setminus I_z$, and thus $\tilde{\lambda}[u](a) = s$. Hence

$$\min\left\{\tilde{\lambda}[u](y), \tilde{\lambda}[u]_*(x \circ y)\right\} = s \leq \tilde{\lambda}[u](x). \tag{3.15}$$

Assume that $x \circ y \subseteq I_z$ and $y \in I_z$. Then

$$(x \circ y) \circ z \subseteq I \text{ and } y \circ z \subseteq I. \tag{3.16}$$

If I is of type $(\subseteq, \subseteq, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Thus

$$\tilde{\lambda}[u](x) = t \geq \min\left\{\tilde{\lambda}[u](y), \tilde{\lambda}[u]_*(x \circ y)\right\}. \tag{3.17}$$

The condition (3.16) implies that $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ by (2.6). Hence, if I is of type $(\ll, \ll, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Therefore we have (3.17). From the condition (3.16), we have $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$. If I is of type $(\subseteq, \ll, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Therefore we have (3.17). From the condition (3.16), we have $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$. If I is of type $(\ll, \subseteq, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Therefore we have (3.17). Therefore $(\tilde{\lambda}, A)$ is a u-fuzzy soft weak hyper BCK-ideal of H. □

Theorem 8. *Let $(\tilde{\lambda},A)$ be a fuzzy soft set over H in which the nonempty level set $U(\tilde{\lambda}[u];t)$ of $(\tilde{\lambda},A)$ is reflexive for all $t \in [0,1]$. If $(\tilde{\lambda},A)$ is a fuzzy soft positive implicative hyper BCK-ideal of H of type $(\ll,\subseteq,\subseteq)$, then the set*

$$\tilde{\lambda}[u]_z := \{x \in H \mid x \circ z \subseteq U(\tilde{\lambda}[u];t)\} \tag{3.18}$$

is a (weak) hyper BCK-ideal of H for all $z \in H$.

Proof. Assume that $(\tilde{\lambda},A)$ is a fuzzy soft positive implicative hyper BCK-ideal of H of type $(\ll,\subseteq,\subseteq)$. Obviously $0 \in \tilde{\lambda}[u]_z$. Then $(\tilde{\lambda},A)$ is a fuzzy soft hyper BCK-ideal of H, and so $U(\tilde{\lambda}[u];t)$ is a hyper BCK-ideal of H. Let $x,y \in H$ be such that $x \circ y \subseteq \tilde{\lambda}[u]_z$ and $y \in \tilde{\lambda}[u]_z$. Then $(x \circ y) \circ z \subseteq U(\tilde{\lambda}[u];t)$ and $y \circ z \subseteq U(\tilde{\lambda}[u];t)$ for all $t \in [0,1]$. Using (2.6), we know that $(x \circ y) \circ z \ll U(\tilde{\lambda}[u];t)$. Since $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of H of type $(\ll,\subseteq,\subseteq)$, it follows from (2.17) that $x \circ z \subseteq U(\tilde{\lambda}[u];t)$, that is, $x \in \tilde{\lambda}[u]_z$. This shows that $\tilde{\lambda}[u]_z$ is a weak hyper BCK-ideal of H. Let $x,y \in H$ be such that $x \circ y \ll \tilde{\lambda}[u]_z$ and $y \in \tilde{\lambda}[u]_z$, and let $a \in x \circ y$. Then there exists $b \in \tilde{\lambda}[u]_z$ such that $a \ll b$, that is, $0 \in a \circ b$. Thus $(a \circ b) \cap U(\tilde{\lambda}[u];t) \neq \varnothing$. Since $U(\tilde{\lambda}[u];t)$ is a reflexive hyper BCK-ideal of H, it follows from (H1) and Lemma 7 that $(a \circ z) \circ (b \circ z) \ll a \circ b \subseteq U(\tilde{\lambda}[u];t)$ and so that $a \circ z \subseteq U(\tilde{\lambda}[u];t)$ since $b \circ z \subseteq U(\tilde{\lambda}[u];t)$. Hence $a \in \tilde{\lambda}[u]_z$, and so $x \circ y \subseteq \tilde{\lambda}[u]_z$. Since $\tilde{\lambda}[u]_z$ is a weak hyper BCK-ideal of H, we get $x \in \tilde{\lambda}[u]_z$. Consequently $\tilde{\lambda}[u]_z$ is a hyper BCK-ideal of H. □

Corollary 6. *Let $(\tilde{\lambda},A)$ be a fuzzy soft set over H in which the nonempty level set $U(\tilde{\lambda}[u];t)$ of $(\tilde{\lambda},A)$ is reflexive for all $t \in [0,1]$. If $(\tilde{\lambda},A)$ is a fuzzy soft positive implicative hyper BCK-ideal of H of type $(\ll,\ll,\subseteq)$, then the set*

$$\tilde{\lambda}[u]_z := \{x \in H \mid x \circ z \subseteq U(\tilde{\lambda}[u];t)\} \tag{3.19}$$

is a (weak) hyper BCK-ideal of H for all $z \in H$.

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