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ON THE PRIME GRAPH OF A FINITE GROUP

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Abstract. Let G be a finite group. We define the prime graph $\Gamma(G)$ of G as follows: The vertices of $\Gamma(G)$ are the primes dividing the order of G and two distinct vertices p, q are joined by an edge, denoted by $p \sim q$, if there is an element in G of order pq. We denote by $\pi(G)$, the set of all prime divisors of $|G|$. The degree $deg(p)$ of a vertex p of $\Gamma(G)$ is the number of edges incident with p. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ where $p_1 < p_2 < ... < p_k$, then we define $D(G) = (deg(p_1), deg(p_2), ..., deg(p_k))$, which is called the degree pattern of G. Given a finite group M, if the number of non-isomorphic groups G such that $|G| = |M|$ and $D(G) = D(M)$ is equal to r, then M is called r-fold OD-characterizable. Also a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper we give some results on characterization of finite groups by prime graphs and OD-characterizability of finite groups. In particular we apply our results to show that the simple groups $G_2(7)$, $B_3(5)$, $\mathbb{A}_{11}$, and $\mathbb{A}_{19}$ are OD-characterizable.

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1. INTRODUCTION

Throughout this paper, groups under consideration are finite. For any group G, we denote by $\pi(G)$ the set of prime divisors of $|G|$. We denote the set of elements of G by $\pi_e(G)$. We associate to $\pi_e(G)$ a graph called prime graph of G, denoted by $\Gamma(G)$. The vertex set of this graph is $\pi(G)$ and two distinct vertices p, q are joined by an edge, denote by $p \sim q$, if $pq \in \pi_e(G)$. The connected components of $\Gamma(G)$ is denoted by π_1, π_2, ..., $\pi_{t(G)}$, where $t(G)$ is the number of connected components of $\Gamma(G)$. If the order of G is even, the notation is chosen so that $2 \in \pi_1$. Clearly the order of G can be expressed as the product of m_1, m_2, ..., $m_{t(G)}$, where $\pi(m_i) = \pi_i$, $1 \leq i \leq t(G)$.

The degree $deg(p)$ of a vertex p of $\Gamma(G)$ is the number of edges incident with p. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$, then we define

$$D(G) = (deg(p_1), deg(p_2), ..., deg(p_k)),$$

which is called the degree pattern of G. Given a finite group M, if the number of non-isomorphic groups G such that $|G| = |M|$ and $D(G) = D(M)$ is equal to r, then M is called r-fold OD-characterizable. Also a 1-fold OD-characterizable group is simply called OD-characterizable.

We call a directed graph strongly connected if there is a directed path from each vertex in the graph to every other vertex. Given an integer a and a positive integer n with $(a,n)=1$, the multiplicative order of a modulo n is the smallest positive integer k such that $a^k \equiv 1(mod n)$. We denote the order of a modulo n by $Ord_n(a)$. It is easy to see that if $a^l \equiv 1(mod n)$, then $Ord_n(a)\big| l$. Let G be a finite group with $|G| = {p_1}^{\alpha_1}{p_2}^{\alpha_2}...{p_k}^{\alpha_k}$, where $p_1 < p_2 < ... < p_k$ are prime numbers. We define a directed graph $\gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct vertices p_i, p_j are joined by an edge, denote by $p_i \sim p_j$, whenever $p_i \nsim p_j$ in $\Gamma(G)$ and $Ord_{{p_j}^{\alpha_j}}(p_i) > \alpha_i$.

The problem of OD-characterizability of simple groups was raised in [2] for the first time. Then many researchers paid attention to characterize finite simple groups by orders and degree patterns of their prime graphs, to mention a few references we will quote [8] and [7].

In this paper we consider the prime graph of a finite group G and prove results which will be used to prove the OD-characterizability of the simple groups $G_2(7)$, $B_3(5)$, $\mathbb{A}_{11}$, and $\mathbb{A}_{19}$. Of course there are many other simple groups whose OD-characterizability can be proved using the results of this paper.

If m and l are natural numbers and p is a prime number, the notation $p^m \parallel n$ means that $p^m \big| n$ and $p^{m+1} \nmid n$. For a prime number r and a positive integer n, n_r denotes the r-part of n, i.e. type n_r is a power of r and $n = mn_r$, where $(m,r)=1$.

2. PRELIMINARIES

Lemma 1. *Let $a>1$ and n be natural numbers and r be a prime number. If $2 \neq r^n \parallel a-1$, then $r^{n+1} \parallel (a^r - 1)$.*

Proof. See [3], 3.2. □

Lemma 2. *Let p_i and p_j be two distinct prime numbers, $p_j \neq 2$, $Ord_{p_j}(p_i) = m$ and $p_j \parallel {p_i}^m - 1$, then $Ord_{{p_j}^d}(p_i) = m{p_j}^{d-1}$, where d is a positive integer.*

Proof. By Lemma 1 and induction on t we see that

$$ {p_j}^t \parallel {p_i}^{m{p_j}^{t-1}} - 1, \tag{2.1}$$

where t is an arbitrary natural number. Now we prove the lemma by induction on d. If $d=1$, then clearly the lemma holds.

Suppose that $Ord_{{p_j}^k}(p_i) = m{p_j}^{k-1}$. Set $s = Ord_{{p_j}^{k+1}}(p_i)$. Thus ${p_j}^{k+1}\big|{p_i}^s - 1$ and so ${p_j}^k\big|{p_i}^s - 1$. Hence $m{p_j}^{k-1}\big|s$, because $Ord_{{p_j}^k}(p_i) = m{p_j}^{k-1}$. On the other hand by (2.1) we have ${p_j}^{k+1}\big|{p_i}^{m{p_j}^k} - 1$ and since $Ord_{{p_j}^{k+1}}(p_i) = s$, $s\big|m{p_j}^k$. It follows that $m{p_j}^{k-1}\big|\ s\big|m{p_j}^k$. This means that $s = m{p_j}^{k-1}$ or $s = m{p_j}^k$. If $s = m{p_j}^{k-1}$, then we have ${p_j}^{k+1}\big|{p_i}^{m{p_j}^{k-1}} - 1$. But by (2.1) ${p_j}^k \parallel {p_i}^{m{p_j}^{k-1}} - 1$. This contradiction shows that $Ord_{{p_j}^{k+1}}(p_i) = s = m{p_j}^k$. Therefore $Ord_{{p_j}^{k+1}}(p_i) = m{p_j}^k$ and the lemma is proved. □

Lemma 3. *Let G be a finite group with $t(G) \geq 2$. If $N \trianglelefteq G$ is a π_i-group, then* $(\prod_{j=1, j\neq i}^{t(G)} m_j) \big| |N| - 1$.

Proof. See Lemma 8 of [1]. □

Lemma 4. *Let G be a finite group with $|G| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \ldots {p_n}^{\alpha_n}$, $p_1 < p_2 < \ldots < p_n$ where p_i is a prime number, $1 \leq i \leq n$. Also assume that M is an arbitrary normal subgroup of G. Then the following holds:*

1) *If p_i, $p_j \in \pi(G)$ and $p_i \sim p_j$ in $\gamma(G)$, then $p_i \big| |M|$ implies that $p_j \big| |M|$, where p_i, p_j are distinct prime numbers.*
2) *Let p_i, $p_j \in \pi(M)$, $p_i \nsim p_j$ in $\Gamma(G)$ and ${p_j}^{\alpha_j} \nmid \prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})$. If $[\prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j} \big| {p_j}^{\alpha_j}$, then $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}} \big| |M|$.*
3) *If p_i, $p_j \in \pi(M)$, $p_i \nsim p_j$ in $\Gamma(G)$ and $Ord_{{p_j}^d}(p_i) > \alpha_i$ for some integer $1 \leq d \leq \alpha_j$, then ${p_j}^{\alpha_j+1-d} \big| |M|$.*

Proof. 1) Since $p_i \sim p_j$ in $\gamma(G)$, we conclude that $p_i \nsim p_j$ in $\Gamma(G)$ and $Ord_{{p_j}^{\alpha_j}}(p_i) > \alpha_i$. We suppose that $p_i \big| |M|$. By Frattini argument $N_G(M_{p_i})M = G$, where M_{p_i} is a Sylow p_i-subgroup of M. If $p_j \nmid |M|$, then since ${p_j}^{\alpha_j} \big| |G|$, we have ${p_j}^{\alpha_j} \big| |N_G(M_{p_i})|$ and so $N_G(M_{p_i})$ has a subgroup, say L of order ${p_j}^{\alpha_j}$. $M_{p_i} \trianglelefteq N_G(M_{p_i})$ implies that $LM_{p_i} \leq N_G(M_{p_i})$. On the other hand there is an positive integer $\beta \leq \alpha_i$ such that $|LM_{p_i}| = {p_j}^{\alpha_j} {p_i}^{\beta}$ and since $p_i \nsim p_j$ in $\Gamma(G)$, the prime graph of LM_{p_i} is not connected. Also $M_{p_i} \trianglelefteq LM_{p_i}$. Thus ${p_j}^{\alpha_j} \big| {p_i}^{\beta} - 1$ by Lemma 3. Hence $Ord_{{p_j}^{\alpha_j}}(p_i) \big| \beta$. In particular we have $Ord_{{p_j}^{\alpha_j}}(p_i) \leq \alpha_i$ and this is a contradiction and so $p_j \big| |M|$.

2) We have $N_G(M_{p_i})M = G$. Thus $\frac{{p_j}^{\alpha_j}}{|N_G(M_{p_i})|_{p_j}} \big| |M|$. Moreover if N is a minimal normal subgroup of $N_G(M_{p_i})$ such that $N \leq M_{p_i}$, then N is isomorphic to a direct product of cyclic groups $\mathbb{Z}_{p_i}$. Assume that N is isomorphic to a direct product of r cyclic group $\mathbb{Z}_{p_i}$. ($N \cong \mathbb{Z}_{p_i} \times \ldots \times \mathbb{Z}_{p_i}$). Since $\frac{N_G(M_{p_i})}{C_{N_G(M_{p_i})}(N)} \hookrightarrow Aut(N)$, we have $\frac{|N_G(M_{p_i})|}{|C_{N_G(M_{p_i})}(N)|} \big| |Aut(N)| = |Aut({\mathbb{Z}_{p_i}}^r)| = |Gl_r(p_i)| = \prod_{k=1}^{r} ({p_i}^r - {p_i}^{k-1})$. This implies that $|N_G(M_{p_i})| \big| |C_{N_G(M_{p_i})}(N)| \prod_{k=1}^{r} ({p_i}^r - {p_i}^{k-1})$. But since $p_i \nsim p_j$ in $\Gamma(G)$, $p_j \nmid C_{N_G(M_{p_i})}(N)|$. (Note that N is a p_i-group).
Thus $|N_G(M_{p_i})|_{p_j} \big| [\prod_{k=1}^{r} ({p_i}^r - {p_i}^{k-1})]_{p_j}$. Also since $r \leq \alpha_i$,

$$[\prod_{k=1}^{r} ({p_i}^r - {p_i}^{k-1})]_{p_j} \Big| [\prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}.$$

Therefore $|N_G(M_{p_i})|_{p_j} \big| [\prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}$.

Now from $N_G(M_{p_i})M = G$, we conclude that $|G| = |N_G(M_{p_i})M| = \frac{|N_G(M_{p_i})||M|}{|N_G(M_{p_i})\cap M|}$ and so $|G|\big||N_G(M_{p_i})||M|$. Thus $p_j^{\alpha_j} = |G|_{p_j}\big||N_G(M_{p_i})|_{p_j}|M|_{p_j}$ and since

$$|N_G(M_{p_i})|_{p_j}\Big|[\prod_{k=1}^{\alpha_i}(p_i^{\alpha_i} - p_i^{k-1})]_{p_j},\ p_j^{\alpha_j}\Big|[\prod_{k=1}^{\alpha_i}(p_i^{\alpha_i} - p_i^{k-1})]_{p_j}|M|_{p_j}.$$

By assumption $[\prod_{k=1}^{\alpha_i}(p_i^{\alpha_i} - p_i^{k-1})]_{p_j}\big|p_j^{\alpha_j}$ and so $\frac{p_j^{\alpha_j}}{[\prod_{k=1}^{\alpha_i}(p_i^{\alpha_i}-p_i^{k-1})]_{p_j}}\big||M|_{p_j}\big||M|$.

3) We will prove that $p_j^d \nmid |N_G(M_{p_i})|$.

If $p_j^d\big||N_G(M_{p_i})|$, then $N_G(M_{p_i})$ has a subgroup, say J of order p_j^d.

Since $M_{p_i} \trianglelefteq JM_{p_i}$ and the prime graph of JM_{p_i} is not connected ($p_i \nsim p_j$ in $\Gamma(G)$) by Lemma 3, we have $p_j^d\big|p_i^e - 1$ for a positive integer $e \leq \alpha_i$. It means that $p_i^e \equiv 1(mod p_j^d)$. It follows that $Ord_{p_j^d}(p_i) \leq \alpha_i$, which is a contradiction. Thus $p_j^d \nmid |N_G(M_{p_i})|$ and so $|N_G(M_{p_i})|_{p_j}\big|p_j^{d-1}$. But since $N_G(M_{p_i})M = G$, we conclude that $|G|\big||N_G(M_{p_i})||M|$, which implies that $p_j^{\alpha_j} = |G|_{p_j}\big||N_G(M_{p_i})|_{p_j}|M|_{p_j}|p_j^{d-1}|M|_{p_j}$ and so $p_j^{\alpha_j+1-d} = p_j^{\alpha_j-(d-1)}\big||M|$. The proof is completed. □

3. Characterization of finite groups by prime graph and order of the group

Theorem 1. *Let G be a finite group with $|G| = p_1^{\alpha_1}p_2^{\alpha_2}...p_n^{\alpha_n}$, $p_1 < p_2 < \cdots < p_n$ where p_i is a prime number, $1 \leq i \leq n$. If the directed graph $\gamma(G)$ is strongly connected, then the following assertions hold.*

1) *There is a simple group S such that $S \trianglelefteq G \leq Aut(S)$ and $\pi(S) = \pi(G)$. Also if $p_i \nsim p_j$ in $\Gamma(G)$, then $p_i \nsim p_j$ in $\Gamma(S)$ too and if $p_i \sim p_j$ in $\Gamma(G)$, then $p_i \sim p_j$ in $\Gamma(Aut(S))$ too.*
2) *Let $p_i, p_j \in \pi(G)$, $p_i \nsim p_j$ in $\Gamma(G)$ and $p_j^{\alpha_j} \nmid \prod_{k=1}^{\alpha_i}(p_i^{\alpha_i} - p_i^{k-1})$. If $[\prod_{k=1}^{\alpha_i}(p_i^{\alpha_i} - p_i^{k-1})]_{p_j}\big|p_j^{\alpha_j}$, then $\frac{p_j^{\alpha_j}}{[\prod_{k=1}^{\alpha_i}(p_i^{\alpha_i}-p_i^{k-1})]_{p_j}}\big||S|$.*
3) *If p_i, $p_j \in \pi(G)$, $p_i \nsim p_j$ in $\Gamma(G)$ and for some integer $1 \leq d \leq \alpha_j$, $Ord_{p_j^d}(p_i) > \alpha_i$, then $p_j^{\alpha_j+1-d}\big||S|$.*

Proof. Assume that L is a minimal normal subgroup of G. Thus $L \neq 1$ and so there is a prime number $p_i \in \pi(G)$ such that $p_i\big||L|$. Since $\gamma(G)$ is strongly connected, for all $p_j \in \pi(G)$ there exists a directed path from p_i to p_j. So by Lemma 4 and induction on the length of path we can easily see that $p_j\big||L|$ for all $p_j \in \pi(G)$. Therefore $\pi(L) = \pi(G)$ and since $\gamma(G)$ is strongly connected, clearly $\Gamma^c(G)$ is connected, where $\Gamma^c(G)$ denotes the complement of the graph $\Gamma(G)$. Now if L is a direct product of more than one isomorphic simple groups, then since $\pi(L) = \pi(G)$, $\Gamma(G)$ is a complete graph and so $\Gamma(G)^c$ is not connected, a contradiction. Hence L is a simple group. On the other hand if for some $q \in \pi(G)$, $q\big||C_G(L)|$, then $q \sim t$ in $\Gamma(G)$ for all $t \in \pi(G) - \{q\}$ and so $\Gamma^c(G)$ is not connected, which is contradiction. Thus $C_G(L) = 1$ and since

$\frac{G}{C_G(L)} \hookrightarrow Aut(L)$, we conclude that $G \hookrightarrow Aut(L)$. So the proof of Part 1 is completed. We conclude Part 2 and 3 of the Theorem from Lemma 4. □

Theorem 2. *Let G be a finite group, $|G| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_n}^{\alpha_n}$, where $p_1 < p_2 < \cdots < p_n$ and p_i is a prime number, $1 \leq i \leq n$. If γ_1 is a strongly connected directed subgraph of the graph $\gamma(G)$ and V_1 is the vertex set of γ_1, then the following assertions hold.*

1) *There is a simple group S such that $S \trianglelefteq \frac{G}{O_{\pi(G)-V_1}(G)} \leq Aut(S)$, $V_1 \subseteq \pi(S) \subseteq \pi(G)$ and if p_i, $p_j \in V_1$ and $p_i \nsim p_j$ in $\Gamma(G)$, then $p_i \nsim p_j$ in $\Gamma(S)$ and if $p_i \sim p_j$ in $\Gamma(G)$, then $p_i \sim p_j$ in $\Gamma(Aut(S))$ ($O_{\pi(G)-V_1}(G)$ is the largest normal subgroup N with $\pi(N) = \pi(G) - V_1$).*
2) *Let p_i, $p_j \in V_1$, $p_i \nsim p_j$ in $\Gamma(G)$ and ${p_j}^{\alpha_j} \nmid \prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})$. If $[\prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j} \big| {p_j}^{\alpha_j}$, then $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}} \big| |S|$.*
3) *If p_i, $p_j \in V_1$ and $p_i \nsim p_j$ in $\Gamma(G)$ and for some integer $1 \leq d \leq \alpha_j$, $Ord_{{p_j}^d}(p_i) > \alpha_i$, then ${p_j}^{\alpha_j + 1 - d} \big| |S|$.*

Proof. Set $L = O_{\pi(G)-V_1}(G)$ and $\bar{G} = \frac{G}{L}$. Suppose that S is a minimal normal subgroup of $\bar{G}$. Thus for a normal subgroup of G, say M_1, we have $S = \frac{M_1}{L}$, where $L \leq M_1$. It is obvious that there is a prime number $q \in V_1$, such that $q \big| |M_1|$. But there exists a path between q and t for all $t \in V_1 - \{q\}$. Therefore by Lemma 4 and induction on length we see that $V_1 \subseteq \pi(M_1)$. It follows that $V_1 \subseteq \pi(S) \subseteq \pi(G)$. Since γ_1 is a strongly connected subgraph of $\gamma(G)$, for all $p_i \in V_1$, there exists $p_j \in V_1$ such that $p_i \nsim p_j$ in $\Gamma(G)$ and so S is not a direct product of more than one isomorphic simple groups. Hence S is a simple group. Now we prove that $C_{\bar{G}}(S) = 1$. Assume that $C_{\bar{G}}(S) \neq 1$. Thus there is a subgroup of G, say K such that $C_{\bar{G}}(S) = \frac{K}{L}$, $L \neq K$. It follows that there is a prime number $r \in V_1$ such that $r \big| |K|$. It means that $r \big| |C_{\bar{G}}(S)|$. Moreover since $V_1 \subseteq \pi(S)$, we conclude that $r \sim t$ in $\Gamma(\bar{G})$ for all $t \in V_1 - \{r\}$. It is easy to see that $r \sim t$ in $\Gamma(G)$ for all $t \in V_1 - \{r\}$ and so $r \nsim t$ in $\gamma(G)$, in particular in γ_1 for all $t \in V_1 - \{r\}$, but this is a contradiction with γ_1 being a strongly connected graph and thus $C_{\bar{G}}(S) = 1$. Hence $S \trianglelefteq \bar{G} = \frac{\bar{G}}{1} = \frac{\bar{G}}{C_{\bar{G}}(S)} \leq Aut(S)$.

Now we assume that p_i, $p_j \in V_1$ and $p_i \nsim p_j$ in $\Gamma(G)$ and ${p_j}^{\alpha_j} \nmid \prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})$. Also suppose that $[\prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j} \big| {p_j}^{\alpha_j}$. By using Part 2 of Lemma 4 for $M_1 \trianglelefteq G$, we conclude that $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}} \big| |M_1|$ and since $p_j \in V_1$, $p_j \nmid |L|$ and so $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i} ({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}} \big| |\frac{M_1}{L}| = |S|$, thus the proof of Part 2 is completed.

Similar arguments prove Part 3.

If p_i, $p_j \in V_1$, $p_i \sim p_j$ in $\Gamma(S)$, then clearly $p_i \sim p_j$ in $\Gamma(\bar{G})$ and so in $\Gamma(G)$. Thus if $p_i \nsim p_j$ in $\Gamma(G)$, then $p_i \nsim p_j$ in $\Gamma(S)$. Also if p_i, $p_j \in V_1$ and $p_i \sim p_j$ in $\Gamma(G)$, then there is an element $g \in G$, such that $g^{p_i p_j} = 1$ and $o(g) = p_i p_j$. Thus $g^{p_i p_j} \in L$. Since $o(g) = p_i p_j \nmid |L|$, $g \notin L$. If $g^{p_i} \in L$, then since $\pi(L) \subseteq \pi(G) - V_1$ and p_i, $p_j \in V_1$, we conclude that there is a positive integer m such that $(p_i p_j, m) = 1$

and $(g^{p_i})^m = g^{p_i m} = 1$. This implies that $p_i p_j \big| p_i m$, because $o(g) = p_i p_j$, thus $p_j \big| m$, a contradiction. Therefore $g^{p_i} \notin L$. Similarly $g^{p_j} \notin L$ and so $o(gL) = p_i p_j$. Thus $p_i \sim p_j$ in $\Gamma(\bar{G})$. But since $\bar{G} \leq Aut(S)$, $p_i \sim p_j$ in $\Gamma(Aut(S))$ and the proof is completed. □

4. OD-CHARACTERIZABILITY OF FINITE GROUPS

Let G be a finite group and $|G| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} ... {p_n}^{\alpha_n}$, where $p_1 < p_2 < \cdots < p_n$ and p_i is a prime number, $1 \leq i \leq n$. For $i = 1, 2, ..., n$, set $R(p_i) = |\{p_j \in \pi(G) | p_i \neq p_j$, $Ord_{{p_j}^{\alpha_j}}(p_i) > \alpha_i$ and $Ord_{{p_i}^{\alpha_i}}(p_j) > \alpha_j\}|$. We have the following three propositions.

Proposition 1. *Let G be a finite group with $|G| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} ... {p_n}^{\alpha_n}$, $p_1 < p_2 < \cdots < p_n$, p_i is a prime number, $1 \leq i \leq n$. Assume that there is $p_m \in \pi(G)$ such that $deg(p_m) = 0$ in $\Gamma(G)$ and $R(p_m) = n - 1$. Then the following assertions hold.*

1) *There is a simple group S such that $S \trianglelefteq G \leq Aut(S)$, $\pi(S) = \pi(G)$. Also we have $deg_{\Gamma(S)}(p_i) \leq deg_{\Gamma(G)}(p_i) \leq deg_{\Gamma(Aut(S))}(p_i)$, $1 \leq i \leq n$.*
2) *If $p_l \in \pi(G)$, ${p_l}^{\alpha_l} \nmid \prod_{k=1}^{\alpha_m} ({p_m}^{\alpha_m} - {p_m}^{k-1})$ and $[\prod_{k=1}^{\alpha_m} ({p_m}^{\alpha_m} - {p_m}^{k-1})]_{p_l} \big| {p_l}^{\alpha_l}$, then $\frac{{p_l}^{\alpha_l}}{[\prod_{k=1}^{\alpha_m} ({p_m}^{\alpha_m} - {p_m}^{k-1})]_{p_l}} \big| |S|$.*
3) *If $p_l \in \pi(G)$, ${p_m}^{\alpha_m} \nmid \prod_{k=1}^{\alpha_l} ({p_l}^{\alpha_l} - {p_l}^{k-1})$ and $[\prod_{k=1}^{\alpha_l} ({p_l}^{\alpha_l} - {p_l}^{k-1})]_{p_m} \big| {p_m}^{\alpha_m}$, then $\frac{{p_m}^{\alpha_m}}{[\prod_{k=1}^{\alpha_l} ({p_l}^{\alpha_l} - {p_l}^{k-1})]_{p_m}} \big| |S|$.*
4) *If $p_l \in \pi(G)$ and $Ord_{{p_l}^d}(p_m) > \alpha_m$ for some integer $1 \leq d \leq \alpha_l$, then ${p_l}^{\alpha_l + 1 - d} \big| |S|$.*
5) *If $p_l \in \pi(G)$ and $Ord_{{p_m}^d}(p_l) > \alpha_l$. for some integer $1 \leq d \leq \alpha_m$, then ${p_m}^{\alpha_m + 1 - d} \big| |S|$.*

Proof. By Theorem 1 it is sufficient to prove that $\gamma(G)$ is strongly connected. Since $deg(p_m) = 0$ in $\Gamma(G)$, $p_i \nsim p_m$ in $\Gamma(G)$ for all $i \neq m$, $1 \leq i \leq n$ and since $R(p_m) = n - 1$, $Ord_{{p_m}^{\alpha_m}}(p_i) > \alpha_i$ and $Ord_{{p_i}^{\alpha_i}}(p_m) > \alpha_m$ for all $i \neq m$, $1 \leq i \leq n$. Hence there is a directed edge from p_i to p_m and from p_m to p_i for all $i \neq m$, $1 \leq i \leq n$.

Now assume that p_a, p_b are two arbitrary vertices in $\gamma(G)$. Then by above discussion there is a directed edge from p_a to p_m and from p_m to p_a in $\gamma(G)$. Also there is a directed edge from p_m to p_b and from p_b to p_m. Thus there is a directed path from p_a to p_b. Therefore $\gamma(G)$ is strongly connected.

Since for all $q \in \pi(G) - \{p_m\}$, $q \nsim p_m$ in $\Gamma(G)$, 2, 3, 4 and 5 are concluded from Theorem 1 Part 2 and 3. □

Proposition 2. *Let G be a finite group with $|G| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} ... {p_n}^{\alpha_n}$, $p_1 < p_2 < \cdots < p_n$, p_i is a prime number, $1 \leq i \leq n$. Assume that there exists $p_m \in \pi(G)$ such that $deg(p_m) = 1$ in $\Gamma(G)$ and $R(p_m) = n - 1$. Then the following assertions hold.*

1) *There exists a simple group S and a prime number $p_r \in \pi(G) - \{p_m\}$ such that $S \trianglelefteq \frac{G}{O_{p_r}(G)} \leq Aut(S)$ and $\pi(G) - \{p_r\} \subseteq \pi(S) \subseteq \pi(G)$. ($O_{p_r}(G)$ is the largest normal subgroup N of G with $\pi(N) = \{p_r\}$).*

2) a) *If* $p_s \in \pi(G)$ *and* $deg(p_s) = n-1$ *in* $\Gamma(G)$, *then there is a simple group* S *such that* $S \trianglelefteq \frac{G}{O_{p_s}(G)} \le Aut(S)$ *and* $\pi(G) - \{p_s\} \subseteq \pi(S) \subseteq \pi(G)$.

b) *If* $p_t \in \pi(G) - \{p_s, p_m\}$, ${p_t}^{\alpha_t} \nmid \prod_{k=1}^{\alpha_m}({p_m}^{\alpha_m} - {p_m}^{k-1})$ *and* $[\prod_{k=1}^{\alpha_m}({p_m}^{\alpha_m} - {p_m}^{k-1})]_{p_t} | {p_t}^{\alpha_t}$, *then* $\frac{{p_t}^{\alpha_t}}{[\prod_{k=1}^{\alpha_m}({p_m}^{\alpha_m} - {p_m}^{k-1})]_{p_t}} \Big| |S|$.

c) *If* $p_t \in \pi(G) - \{p_s, p_m\}$, ${p_m}^{\alpha_m} \nmid \prod_{k=1}^{\alpha_t}({p_t}^{\alpha_t} - {p_t}^{k-1})$ *and* $[\prod_{k=1}^{\alpha_t}({p_t}^{\alpha_t} - {p_t}^{k-1})]_{p_m} | {p_m}^{\alpha_m}$, *then* $\frac{{p_m}^{\alpha_m}}{[\prod_{k=1}^{\alpha_t}({p_t}^{\alpha_t} - {p_t}^{k-1})]_{pm}} \Big| |S|$.

d) *If* $p_t \in \pi(G) - \{p_s, p_m\}$ *and* $Ord_{{p_t}^d}(p_m) > \alpha_m$ *for some integer* $1 \le d \le \alpha_t$, *then* ${p_t}^{\alpha_t+1-d} \Big| |S|$.

e) *If* $p_t \in \pi(G) - \{p_s, p_m\}$ *and* $Ord_{{p_m}^d}(p_t) > \alpha_t$ *for some integer* $1 \le d \le \alpha_m$, *then* ${p_m}^{\alpha_m+1-d} \Big| |S|$.

Proof. 1) By Theorem 2 it is sufficient to prove that $\gamma(G)$ has a strongly connected subgraph with $n-1$ vertices. Since $deg(p_m) = 1$ in $\Gamma(G)$, there exists $p_r \in \pi(G)$ such that $p_r \sim p_m$ in $\Gamma(G)$. If p_i is an arbitrary vertex of the directed graph $\gamma(G)$ such that $p_i \neq p_r, p_m$, then since $R(p_m) = n-1$, we conclude that $Ord_{{p_i}^{\alpha_i}}(p_m) > \alpha_m$ and $Ord_{{p_m}^{\alpha_m}}(p_i) > \alpha_i$. On the other hand $p_i \nsim p_m$ in $\Gamma(G)$ and so there is an edge from p_i to p_m and from p_m to p_i in $\gamma(G)$.

Now if p_a, p_b are two arbitrary vertices of $\gamma(G)$ such that $p_a, p_b \neq p_r, p_m$, then there is an edge from p_a to p_m and from p_m to p_a, also from p_b to p_m and from p_m to p_b in $\gamma(G)$. Thus there is a path from p_a to p_b. Hence there is a strongly connected subgraph of $\gamma(G)$ such that its vertex set is equal to $\pi(G) - \{p_r\}$. Therefore by Theorem 2, there is a simple group S such that $S \trianglelefteq \frac{G}{O_{p_r}(G)} \le Aut(S)$ and $\pi(G) - \{p_r\} \subseteq \pi(S) \subseteq \pi(G)$.

2) Assume that there exists $p_s \in \pi(G)$ such that $deg(p_s) = n-1$ in $\Gamma(G)$. So p_s is joint to all vertices in $\Gamma(G)$. In particular $p_s \sim p_m$ in $\Gamma(G)$.

By similar argument as in Part 1 we can see that $\gamma(G)$ has a strongly connected subgraph such that its vertex set is equal to $\pi(G) - \{p_s\}$. Thus by Theorem 2, there is a simple group S such that $S \trianglelefteq \frac{G}{O_{p_s}(G)} \le Aut(S)$ and $\pi(G) - \{p_s\} \subseteq \pi(S) \subseteq \pi(G)$. Also b, c, d, e are concluded from Theorem 2 Part 2 and 3. ☐

We define $(m)^*$ for all $m \in \mathbb{Z}$ by $(m)^* = \begin{cases} m & \text{for} \quad m > 0 \\ 0 & \text{for} \quad m \le 0. \end{cases}$

Proposition 3. *Let* G *be a finite group with* $|G| = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_n}^{\alpha_n}$, $p_1 < p_2 < \dots < p_n$, p_i *is a prime number,* $1 \le i \le n$. *We set* $M = max\{(R(p_i) - deg(p_i))^* | 1 \le i \le n\}$ *and* $m = min\{(R(p_i) - deg(p_i))^* | 1 \le i \le n\}$. *If* $M + m \ge n-1$, *then there is a simple group* S *such that* $S \trianglelefteq G \le Aut(S)$ *and* $\pi(S) = \pi(G)$. *Also* $deg_{\Gamma(S)}(q) \le deg_{\Gamma(G)}(q) \le deg_{\Gamma(Aut(S))}(q)$ *for all* $q \in \pi(G)$.

Proof. By Theorem 1 it is sufficient to prove that $\gamma(G)$ is strongly connected. So assume that p_d is an arbitrary vertex of $\gamma(G)$. We define A_d and B_d as follows:

$$A_d = \{p_i \in \pi(G) | p_i \neq p_d, \ p_i \nsim p_d \text{ in } \Gamma(G)\},$$

$$B_d = \{p_j \in \pi(G) | p_j \neq p_d,\ Ord_{p_j^{\alpha_j}}(p_d) > \alpha_d \text{ and } Ord_{p_d^{\alpha_d}}(p_j) > \alpha_j\}.$$

Thus $|A_d| = n-1-deg(p_d)$, $|B_d| = R(p_d)$, where $deg(p_d)$ is the degree of p_d in $\Gamma(G)$.

Moreover $A_d \cap B_d$ is equal to set of all vertices in $\gamma(G)$ that are joined to p_d and also p_d is joined to them by an edge. Since $p_d \notin A_d \cup B_d$, $A_d \cup B_d \subseteq \pi(G) - \{p_d\}$ and so $|A_d \cup B_d| \leq n-1$. Therefore we have $|A_d \cup B_d| = n-1-deg(p_d)+R(p_d)-|A_d \cap B_d| \leq n-1$. Hence $|A_d \cap B_d| \geq R(p_d)-deg(p_d)$. Since $|A_d \cap B_d| \geq 0$, we have $|A_d \cap B_d| \geq (R(p_d)-deg(p_d))^*$.

But $(R(p_d)-deg(p_d))^* \geq m$, which implies that $|A_d \cap B_d| \geq m$. Thus there exist m vertices in $\gamma(G)$ that are joined to p_d and also p_d is joined to them by an edge, where p_d is an arbitrary vertex of $\gamma(G)$. Denote the set of all these m vertices by E_d.

Now we assume that $p_c \in \pi(G)$ and $M = (R(p_c)-deg(p_c))^*$. Then by a similar argument we see that there exist M vertices in $\gamma(G)$ that are joined to p_c and also p_c is joined to them by an edge. Denote the set of all these M vertices by F_c.

We will show that if $p_u \in \pi(G)$ is different from p_c, then there is a directed path from p_u to p_c and from p_c to p_u. Since $p_u \neq p_c$, $p_u \nsim p_c$ and $p_c \nsim p_u$ in $\gamma(G)$. We know that $p_u \sim q$ and $q \sim p_u$ in $\gamma(G)$ for all $q \in E_u$. If $E_u \cap F_c = \varnothing$, then since $\{p_u\} \cup E_u \cup \{p_c\} \cup F_c \subseteq \pi(G)$, $p_u \neq p_c$, $p_u \nsim p_c$ and $p_c \nsim p_u$ in $\gamma(G)$, we have $|\{p_u\} \cup E_u \cup \{p_c\} \cup F_c| = 1+m+1+M \leq n$, which is a contradiction with assumption, ($M+m \geq n-1$). Thus $E_u \cap F_c \neq \varnothing$. Suppose that $p_v \in E_u \cap F_c$. It follows that $p_u \sim p_v$, $p_v \sim p_u$, $p_c \sim p_v$ and $p_v \sim p_c$. Hence $p_u \to p_v \to p_c$ is a directed path from p_u to p_c and $p_c \to p_v \to p_u$ is a directed path from p_c to p_u. So we proved that for all $p_u \in \pi(G)$ there exists a directed path from p_u to p_c and there is a directed path from p_c to p_u.

Now we assume that p_a, p_b are two arbitrary vertices of $\gamma(G)$. Thus by the above discussion there is a path from p_c to p_a and from p_a to p_c, also there is a path from p_c to p_b and from p_b to p_c. Therefore there is a path from p_a to p_b and so $\gamma(G)$ is strongly connected and the proof is completed. □

5. Applications

We give some examples of characterization of finite groups by prime graph and OD-characterization of them.

We note that the following examples are proved in [4] and a few more papers. But our proofs are based on Theorems 1 and 2 and Propositions 1, 2 and 3. The prime graphs of all groups considered are obtained by [6].

Example 1. We consider the simple group $C_2(7)$. We know that $|C_2(7)| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ and $2 \sim 3$, $2 \sim 7$, $3 \sim 7$, $5 \nsim 2$, $5 \nsim 7$ and $5 \nsim 3$ in $\Gamma(C_2(7))$. Since $Ord_{5^2}(2) > 8$, $Ord_{5^2}(3) > 2$, $Ord_{2^8}(5) > 2$, $Ord_{3^2}(5) > 2$, we deduce that $E_{\gamma(C_2(7))} \supseteq \{(2,5),(5,2),(3,5),(5,3)\}$, where $E_{\gamma(C_2(7))}$ is the edge set of $\gamma(G)$. Hence there exists a strongly connected subgraph of $\gamma(C_2(7))$ that its vertex set is $\{2,3,5\}$.

Now if G is a finite group with $|G| = |C_2(7)|$ and $\Gamma(G) = \Gamma(C_2(7))$, then $\gamma(G) = \gamma(C_2(7))$ and so there exists a strongly connected subgraph of $\gamma(G)$ that its vertex set is $\{2,3,5\}$. Thus by Theorem 2 there is a simple group S such that $S \trianglelefteq \frac{G}{O_7(G)} \leq Aut(S)$ and $\{2,3,5\} \subseteq \pi(S) \subseteq \pi(G) = \{2,3,5,7\}$. Since $3 \nsim 5$ in $\Gamma(C_2(7)) = \Gamma(G)$ and $Ord_5(3) > 2$ by Part 3 of Theorem 2, we conclude that $5^{2+1-1} = 5^2 \big| |S|$. Similarly since $2 \nsim 5$ in $\Gamma(G)$ and $Ord_{2^4}(5) > 2$, $2^{8+1-4} = 2^5 \big| |S|$. Now by Table 4 of [5] we see that $S \cong B_2(7)$ or $S \cong C_2(7)$ and since $S \leq \frac{G}{O_7(G)}$ and $\frac{|G|}{|O_7(G)|} \big| |G| = |C_2(7)|$, we conclude that $O_7(G) = 1$ and $G = S$ and so $G \cong B_2(7)$ or $G \cong C_2(7)$.

Hence if $\Gamma(G) = \Gamma(C_2(7))$ and $|G| = |C_2(7)|$, then $G \cong C_2(7)$ or $G \cong B_2(7)$.

Example 2. We consider the simple group $B_3(5)$. We know that $|B_3(5)| = 2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \cdot 31$ and $2 \sim 3$, $2 \sim 5$, $2 \sim 13$ $2 \sim 31$, $3 \sim 5$, $3 \sim 7$, $3 \sim 13$ and $5 \sim 13$ in $\Gamma(B_3(5))$ and $7 \nsim i$, $31 \nsim j$ for $i \in \{2,5,13,31\}$ and $j \in \{3,5,7,13\}$ in $\Gamma(G)$. We have $Ord_{31}(3) > 4$, $Ord_{3^4}(31) > 1$, $Ord_7(31) > 1$, $Ord_{31}(7) > 1$, $Ord_{31}(13) > 1$ and $Ord_{13}(31) > 1$. Thus $E_{\gamma(B_3(5))} \supseteq \{(31,3),(3,31),(31,7),(7,31),(31,13),(13,31)\}$, where $E_{\gamma(B_3(5))}$ is the edge set of $\gamma(B_3(5))$. Therefore there exists a strongly connected subgraph of $\gamma(B_3(5))$ that its vertex set is $\{3,7,13,31\}$. Now if G is a finite group with $|G| = |B_3(5)|$ and $\Gamma(G) = \Gamma(B_3(5))$, then $\gamma(G) = \gamma(B_3(5))$ and so there exists a strongly connected subgraph of $\gamma(G)$ that its vertex set is $\{3,7,13,31\}$ Thus by Theorem 2, there is a simple group S such that $S \trianglelefteq \frac{G}{O_{\{2,5\}}(G)} \leq Aut(S)$ and $\{3,7,13,31\} \subseteq \pi(S) \subseteq \{2,3,5,7,13,31\}$. But since $3 \nsim 31$ in $\Gamma(G) = \Gamma(B_3(5))$ and $3^4 \nmid 31-1$ by Theorem 2 Part 2 we have $\frac{3^4}{[31-1]_3} = 3^3 \big| |S|$ and so $3^3 \cdot 7 \cdot 13 \cdot 31 \big| |S|$. Now by Table 4 of [5], we conclude that $S \cong B_3(5)$ or $S \cong C_3(5)$. Thus $O_{\{2,5\}}(G) = 1$ and since $|G| = |B_3(5)|$, we conclude that $G \cong B_3(5)$ or $G \cong C_3(5)$.

Hence if $\Gamma(G) = \Gamma(B_3(5))$ and $|G| = |B_3(5)|$, then $G \cong B_3(5)$ or $G \cong C_3(5)$.

Example 3. We consider the simple group $\mathbb{A}_{11}$. We know that $|\mathbb{A}_{11}| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$. We can easily see that $deg(11) = 0$ in $\Gamma(\mathbb{A}_{11})$. Assume that G is a finite group with $D(G) = D(\mathbb{A}_{11})$ and $|G| = |\mathbb{A}_{11}|$.

Since $Ord_{11}(2) > 7$, $Ord_{11}(3) > 4$, $Ord_{11}(5) > 2$, $Ord_{11}(7) > 1$, $Ord_{2^7}(11) > 1$, $Ord_{3^4}(11) > 1$, $Ord_{5^2}(11) > 1$ and $Ord_7(11) > 1$, we conclude that $R(11) = 4$ and since $|\pi(G)| = 5$ by Proposition 1 there is a simple group S such that $S \trianglelefteq G \leq Aut(S)$ and $\pi(S) = \pi(G) = \{2,3,5,7,11\}$. Since $2^7 \nmid 11-1 = 10$ and $[10]_2 \big| 2^7$ by Part 2 of Proposition 1 we have $\frac{2^7}{[10]_2} = 2^6 \big| |S|$. Similarly $3^4 \big| |S|$. Thus $|S| = 2^a \cdot 3^4 \cdot 5^b \cdot 7 \cdot 11$, where $6 \leq a \leq 7$, $1 \leq b \leq 2$ and so by Table 4 of [5] S is isomorphic to $\mathbb{A}_{11}$ and since $S \leq G$, $|G| = |\mathbb{A}_{11}|$, we conclude that $G \cong \mathbb{A}_{11}$.

Hence $\mathbb{A}_{11}$ is OD-characterizable.

Example 4. We consider the simple group $\mathbb{A}_{19}$. We know that $|\mathbb{A}_{19}| = 2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. Obviously $deg(19) = 0$ in $\Gamma(\mathbb{A}_{19})$. Now assume that G is a finite group with $D(G) = D(\mathbb{A}_{19})$ and $|G| = |\mathbb{A}_{19}|$.

We have $Ord_{19}(2) > 15$, $Ord_{19}(3) > 8$, $Ord_{19}(5) > 3$, $Ord_{19}(7) > 2$, $Ord_{19}(11) > 1$, $Ord_{19}(13) > 1$, $Ord_{19}(17) > 1$, $Ord_{2^{15}}(19) > 1$, $Ord_{3^8}(19) > 1$, $Ord_{5^3}(19) > 1$, $Ord_{7^2}(19) > 1$, $Ord_{11}(19) > 1$, $Ord_{13}(19) > 1$ and $Ord_{17}(19) > 1$, thus $R(19) = 7$ and since $|\pi(G)| = 8$ by Proposition 1 there is a simple group S such that $S \trianglelefteq G \leq Aut(S)$ and $\pi(S) = \pi(G) = \{2,3,5,7,11,13,17,19\}$. Since $2^{15} \nmid 19-1 = 18$, $[18]_2 \big| 2^{15}$ by Part 2 of Proposition 1 we have $\frac{2^{15}}{[18]_2} = 2^{14} \big| |S|$. Similarly $\frac{3^8}{[18]_3} = 3^6 \big| |S|$, $5^3 \big| |S|$ and $7^2 \big| |S|$. Thus $|S| = 2^a \cdot 3^b \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, where $14 \leq a \leq 15$, $6 \leq b \leq 8$ and so by Table 4 of [5] $S \cong \mathbb{A}_{19}$ and since $S \leq G$, $|G| = |\mathbb{A}_{19}|$, we conclude that $G \cong \mathbb{A}_{19}$.

Hence $\mathbb{A}_{19}$ is OD-characterizable.

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