# Études in Homotopical Thinking

## $\mathbb{F}_1$-Geometry, Concurrent Computing, and Motivic Measures

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# ABSTRACT

This thesis weaves together three papers, each of which provides a use of homotopical intuition in a different field of mathematics. The first applies it to the study of various models of $\mathbb{F}_1$-geometry, focusing mainly on the Bost-Connes algebra. The second endeavors to compare two homotopical models for concurrent computing before introducing a new one as well. Finally, the last paper provides a construction for obtaining derived motivic measures from an abstract six functors formalism and, in particular, applies this idea to obtain a lift of the Gillet-Soulé motivic measure.

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*Chapter 1*

# INTRODUCTION

This thesis consists of three different chapters which, on the surface, seem somewhat disconnected. The underlying theme connecting all three, however, is an attempt to understand the different ways that abstract homotopy theory and homotopical thinking may be applied to different areas of mathematics. All three chapters are relatively self-contained, and each has a short introduction to its mathematical prerequisites, as well as its own citations. In this section, we merely describe the most important results of each chapter.

## 1.1 $\mathbb{F}_1$-Geometry

In the initial chapter, our focus is on importing homotopy into the study of $\mathbb{F}_1$-geometry. In it, we build on the work of **ManMar2** extending much of it to an arbitrary base scheme $(S, \sigma)$ equipped with an action of $\hat{\mathbb{Z}}$ which factors through some finite quotient.

**Definition 1.1.1.** *Let $(S, \alpha_S)$ be a scheme with a good effectively finite action of $\hat{\mathbb{Z}}$. Let $Z_n = Spec(\mathbb{Q}^n)$ and let $\Phi_n(\alpha_S)$ be the action of $\hat{\mathbb{Z}}$ on $S \times Z_n$ as in* (2.2.12) *and* (2.2.13)*. Given a class $[f : (X, \alpha_X) \to (S, \alpha_S)]$ in $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$, with $\alpha_X$ the compatible $\hat{\mathbb{Z}}$-action on $X$, let*

$$\sigma_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f : (X, \alpha_X \circ \sigma_n) \to (S, \alpha_S \circ \sigma_n)] \tag{1.1.1}$$

$$\tilde{\rho}_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f \times id : (X \times Z_n, \Phi_n(\alpha_X)) \to (S \times Z_n, \Phi_n(\alpha_S))]. \tag{1.1.2}$$

**Proposition 1.1.2.** *For all $n \in \mathbb{N}$ the $\sigma_n$ defined in* (2.2.14) *are ring homomorphisms*

$$\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S \circ \sigma_n)}) \tag{1.1.3}$$

*and the $\tilde{\rho}_n$ defined in* (2.2.15) *are group homomorphisms*

$$\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \Phi_n(\alpha_S))}), \tag{1.1.4}$$

*with compositions satisfying*

$$\tilde{\rho}_n \circ \sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \alpha_S \times \alpha_n)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$$

$$\sigma_n \circ \tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)^{\oplus n}}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}),$$

*with $\sigma_n \circ \tilde{\rho}_n = n\, id$ and $\tilde{\rho}_n \circ \sigma_n$ is the product by $(Z_n, \alpha_n)$.*

From here, we go on to describe several other relevant constructions, including K-theory spectra associated with torified varieties, which are themselves equippable with $\hat{\mathbb{Z}}$-actions as above.

**Proposition 1.1.3.** *For $a = s, o, w$, the category $C^a_{\mathcal{T}}$ has objects that are pairs $(X, T)$ of a torifiable variety and a torification, with morphisms the locally closed embeddings that are, respectively, strong, ordinary, or weak morphisms of torified varieties. The Grothendieck topology is generated by the covering families*

$$\{(Y, T_Y) \hookrightarrow (X, T_X), (X \smallsetminus Y, T_{X \smallsetminus Y}) \hookrightarrow (X, T_X)\} \tag{1.1.5}$$

*where both embeddings are strong, ordinary, or weak morphisms, respectively. The category $C^a_{\mathcal{T}}$ is an assembler with spectrum $K(C^a_{\mathcal{T}})$ satisfying $\pi_0 K(C^a_{\mathcal{T}}) = K_0(\mathcal{T})^a$. Similarly, for $G = \mathbb{Q}/\mathbb{Z}$ or $G = \hat{\mathbb{Z}}$ let $C^{G,a}_{\mathcal{T}}$ be the category with objects $(X, T, \alpha)$ given by a torifiable variety $X$ with a torification $T$ and a $G$-action $\alpha$ of the kind discussed in §2.4 and morphisms the locally closed embeddings that are $G$-equivariant strong, ordinary, or weak morphisms. The Grothendieck topology is generated by covering families* (2.4.1) *with $G$-equivariant embeddings. The category $C^{G,a}_{\mathcal{T}}$ is also an assembler, whose associated spectrum $K(C^{G,a}_{\mathcal{T}})$ satisfies $\pi_0 K(C^{G,a}_{\mathcal{T}}) = K_0^G(\mathcal{T})^a$.*

All of the above ideas are in service of geometrizing and describing spectral lifts of Bost-Connes systems.

We then describe zeta functions for torified varieties analogous to the Hasse-Weil zeta function for $\mathbb{F}_q$-varieties, except instead of counting $q^k$-points for varying $k$, we are counting $\mathbb{F}_{1^k}$-points. This ends up resulting in various polylogarithmic formulae.

**Lemma 1.1.4.** *Let $X$ be a variety over $\mathbb{Z}$ with torified Grothendieck class*

$$[X] = \sum_{i=0}^{N} a_i \mathbb{T}^i \tag{1.1.6}$$

*with coefficients $a_i \in \mathbb{Z}_+$ and $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1$. Then the number of points over $\mathbb{F}_{1^m}$ of $X$ is given by*

$$\#X(\mathbb{F}_{1^m}) = \sum_{i=0}^{N} a_i\, m^i. \tag{1.1.7}$$

*In particular, $\#X(\mathbb{F}_1) = a_0 = \chi(X)$ the Euler characteristic.*

**Lemma 1.1.5.** *Let $X$ be a variety over $\mathbb{Z}$ with a torified Grothendieck class $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$ with $a_k \in \mathbb{Z}_+$. Then the $\mathbb{F}_1$-zeta function is given by*

$$\log Z_{\mathbb{F}_1}(X,t) = \sum_{k=0}^{N} a_k \, Li_{1-k}(t), \tag{1.1.8}$$

*where $Li_s(t)$ is the polylogarithm function with $Li_1(t) = -\log(1-t)$ and for $k \geq 1$*

$$Li_{1-k}(t) = (t\frac{d}{dt})^{k-1} \frac{t}{1-t}.$$

Next, we describe another approach to a derived lift of the classical Hasse-Weil zeta function distinct from that found in **CaWoZa** . This approach is interesting because it relies on a map directly out of the assembler category of $k$-varieties for finite $k$, as opposed to making use of the of SW-categories (which yield equivalent K-theory spectra, but are optimized for easily mapping into Waldhausen categories).

**Lemma 1.1.6.** *A factorizable motivic measure $\mu : K_0(\mathcal{V}) \to R$, as in Definition 2.6.7, determines a functor $\Phi_\mu : C_{\mathcal{V}} \to \mathcal{E}_R^{\pm}$ where $C_{\mathcal{V}}$ is the assembler category encoding the scissor-congruence relations of the Grothendieck ring $K_0(\mathcal{V})$ and $\mathcal{E}_R^{\pm}$ is the $\mathbb{Z}/2\mathbb{Z}$-graded endomorphism category.*

**Theorem 1.1.7.** *The functor $\Phi_\mu : C_{\mathcal{V}} \to \mathcal{E}_R^{\pm}$ defined above induces a map of $\Gamma$-spaces and of the associated spectra $\Phi_\mu : K(\mathcal{V}) \to F_{\mathcal{E}_R^{\pm}}(\mathbb{S})$. The induced maps on the homotopy groups has the property that the composition*

$$K_0(\mathcal{V}) \xrightarrow{\Phi_\mu} K_0(\mathcal{E}_R^{\pm}) \xrightarrow{\delta} K_0(\mathcal{E}_R) \to K_0(\mathcal{E}_R)/K_0(R) = W_0(R) \tag{1.1.9}$$

*with $\delta$ as in Lemma 2.6.4, is given by the zeta function $\zeta_\mu : K_0(\mathcal{V}) \to W_0(R)$. Furthermore, this yields a corresponding map on K-theory spectra.*

Finally, we come to perhaps the heart of this first chapter. In particular, it is another endeavor to categorify and geometrize the Bost-Connes system, this time invoking the notion of a Nori diagram category. Here we consider $\hat{\mathbb{Z}}$-equivariant Nori motives, and construct a fiber functor to a categorification of the Bost-Connes algebra due to **MaTa** This provides a categorical lift of the motivic measure constructed in **ManMar2** .

**Theorem 1.1.8.** *The $\sigma_n$ and $\tilde{\rho}_n$ of* (2.7.2) *and* (2.7.3) *determine a Bost-Connes system on the category $C(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$ of Nori motives associated to the diagram*

*$D(\mathcal{V}^{\hat{\mathbb{Z}}})$. The representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \to Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ constructed above has the property that the induced functor*

$$C(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T) \to Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$$

*intertwines the endofunctors $\sigma_n$ and $\tilde{\rho}_n$ of the Bost-Connes system on $C(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$ and the Frobenius $F_n$ and Verschiebung $V_n$ of the Bost-Connes structure on $Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$.*

## 1.2 Concurrent Computing

The second chapter comprises an application of model categories to the theory of concurrent computing. Over the course of the past thirty or so years, there have been many different attempts to construct a homotopical framework suited to the needs of this theory. In the first portion of this chapter, we prove that two different such models cannot give rise to the same homotopy theory under reasonable assumptions.

**Theorem 1.2.1.** *Suppose that there is a model structure on the category of precubical sets along with a realization functor $L : \boldsymbol{PrSh}(\square) \rightleftarrows \boldsymbol{Flow}$ which is the left Quillen adjoint in a Quillen pair $(L \dashv N_L) : \boldsymbol{PrSh}(\square) \rightleftarrows \boldsymbol{Flow}$ (note that we are essentially only assuming that everything in $I = \{\partial\square[n] \hookrightarrow \square[n]\}_{n=0}^{\infty} \cup \{\square[0] \sqcup \square[0] \to \square[0]\}$ is sent to a cofibration and that the images of $\square[n]$ are weakly equivalent to $\{0 < 1\}^n$ for any $n$). Then this adjunction cannot be a Quillen equivalence.*

Next, we introduce a new model category, **sSemiCat**, analogous to one of the above frameworks, namely that of flows. This model category has a more combinatorial flavor than the category of flows, while yielding the same homotopy theory.

**Proposition 1.2.2.** *The model structure on $\boldsymbol{Flow}$ may be left induced via the adjunction $(|-| \dashv Sing) : \boldsymbol{sSemiCat} \rightleftarrows \boldsymbol{Flow}$ (constant on objects and acting via realization/singular set on Hom objects). This upgrades $(|-| \dashv Sing)$ into a Quillen pair.*

**Theorem 1.2.3.** *The Quillen adjunction $(|-| \dashv Sing) : \boldsymbol{sSemiCat} \rightleftarrows \boldsymbol{Flow}$ is a Quillen equivalence.*

The chapter ends by introducing a slight enlargement of the category of cubical sets, in which we treat each pointed tree as a basic "interval," and showing that the classical homotopy theory of spaces can be recovered by endowing this category with an appropriate model structure (in other words showing that it is a test category in the sense of Grothendieck and Cisinski **Cis**). We denote this category $\boxed{\mathcal{T}}$.

**Theorem 1.2.4.** $\boxed{\mathcal{T}}$ *is a test category.*

## 1.3 Motivic Measures

The third and final chapter can in some ways be seen as a return to some of the topics relevant to the initial chapter. In this chapter, our central object of study is a derived motivic measure. In other words, a map of K-theory spectra whose source is the K-theory spectrum $K(\mathbf{Var}_S)$ of varieties over some $S$.

In particular, we show a way to take an abstract six functors formalism **Khan** and output such a K-theory spectrum with values in the K-theory of constructible objects $\mathbb{D}_{\text{cons}}(S)$ over $S$.

**Theorem 1.3.1.** *Suppose that $\mathbb{D}$ satisfies one of the following two sets of conditions:*

- *The four functors preserve constructible objects when given input a seperable morphism of finite type (note that compactness is trivially preserved by tensor)*
- *The six functors preserve constructible objects over Noetherian quasi-excellent schemes of finite dimension with respect to morphisms of finite type (in other words, for any finite type morphism $f : X \to S$ with target Noetherian quasi-excellent of finite dimension, the four functors preserve constructible objects, while tensor and Hom preserve constructible objects over $S$)*

*Then, given a scheme $S$ (assumed to be Noetherian quasi-excellent of finite dimension if $\mathbb{D}$ satisfies the second condition in particular), there is a weakly W-exact functor*

$$M^c_{\mathbb{D}(S)} : \boldsymbol{Var}_S \to \mathbb{D}_{cons}(S)$$

*sending each variety (smooth or otherwise) $(X \xrightarrow{f} S) \in \boldsymbol{Var}_S$ to $M^c_{\mathbb{D}(S)}(X) := f_* f^! \mathbf{1}_S$.*

**Corollary 1.3.2.** *Suppose $\mathbb{D}$ sastisfies one of the two conditions of the above theorem. Then, given a scheme $S$ (assumed to be Noetherian quasi-excellent of finite dimension if $\mathbb{D}$ satisfies the second set of conditions), one obtains a map of K-theory spectra*

$$K(M^c_{\mathbb{D}(S)}) : K(\boldsymbol{Var}_S) \to K(\mathbb{D}_{cons}(S)).$$

We then discuss the Gillet-Soulé motivic measure, and show how in the case of Beilinson motives, $\mathbb{D} = \mathbf{DM}_B$, the functor on K-theory $K(\mathbf{Var}_S) \to K(\mathbf{DM}^c_B(S))$ obtained from the procedure outlined above yields a lifting of the Gillet-Soulé motivic measure:

**Theorem 1.3.3.** *Considering a perfect base field $k$ and rational coefficients, the map*

$$K(M^c_k) : K(\boldsymbol{Var}_k) \to K(\boldsymbol{DM}^c_B(k))$$

*yields a derived lift of the Gillet-Soulé motivic measure.*

## 1.4 General Overview

In all of the chapters outlined above, the main commonality is an attempt to understand how different fields can be affected by the importation of homotopical thinking. This may be from the use of techniques from K-theory to extract more meaningful geometrizations of existing objects, such as in the first chapter. It may be from trying to recast problems of computer science into an already rich abstract homotopical framework, as in the second chapter. Or it may be trying to probe rigid structures such as six functors formalisms by casting them into the well-described apparatus of K-theory, as in the third chapter. There are many ways that applying homotopical reasoning to diverse areas of mathematics can yield new and different insights into classical problems and definitions, and we feel that we have only just begun to plumb the depths of this new homotopical world.

*Chapter 2*

# BOST-CONNES AND $\mathbb{F}_1$: GROTHENDIECK RINGS, SPECTRA, NORI MOTIVES

This material is all joint work with Matilde Marcolli and Yuri Manin, who have graciously allowed me to incorporate it into my thesis.

# ABSTRACT

We construct geometric lifts of the Bost-Connes algebra to Grothendieck rings and to the associated assembler categories and spectra, as well as to certain categories of Nori motives. These categorifications are related to the integral Bost-Connes algebra via suitable Euler characteristic type maps and zeta functions, and in the motivic case via fiber functors. We also discuss aspects of $\mathbb{F}_1$-geometry, in the framework of torifications, that fit into this general setting.

## 2.1 Introduction and summary

This survey/research paper interweaves many different strands that recently became visible in the fabric of algebraic geometry, arithmetics, (higher) category theory, quantum statistics, homotopical "brave new algebra" etc., see especially A. Connes and C. Consani **CoCo2 CC16** A. Huber, St. Müller–Stach **HuM-S17** etc.

In this sense, our present paper can be considered as a continuation and further extension of **ManMar2** and we will be relying on much of the work in that paper for details and examples. The motivational starting point in **ManMar2** came from the interpretation given in **CCM** of the Bost-Connes quantum statistical mechanical system, and in particular the integral Bost-Connes algebra, as a form of $\mathbb{F}_1$-structure, or "geometry below S$pec(\mathbb{Z})$." The main theme of **ManMar2** is an exploration of how this structure manifests itself beyond the usual constructions of $\mathbb{F}_1$-structures on certain classes of varieties over $\mathbb{Z}$. In particular, the results of **ManMar2** focus on lifts of the integral Bost-Connes algebra to certain Grothendieck rings and to associated homotopy-theoretic spectra obtained via assembler categories, and also on another form of $\mathbb{F}_1$-structures arising through quasi-unipotent Morse-Smale dynamical systems.

The main difference between the present paper and **ManMar2** consists in a change of the categorical environment: the unifying vision we already considered in **ManMar2** was provided by I. Zakharevich's notions of assemblers and scissors congruences: cf. **Zak1 Zak2 Zak3** and **CaWoZa** . In this paper, we continue to use the formalism of assemblers and the associated spectra, but we complement it with categories of Nori motives, **HuM-S17** .

As in **ManMar2** we focus primarily on various geometrizations of the Bost-Connes algebra(s). Some of these constructions take place in Grothendieck rings, like the previous cases considered in **ManMar2** and are aimed at lifting the Bost-Connes endomorphisms to the level of homotopy theoretic spectra through the use of Zakharevich's formalism of assembler categories. We focus on the case of relative Grothendieck rings, endowed with appropriate equivariant Euler characteristic maps. We consider the case of varieties that admits torifications, for which we introduce zeta functions based on the counting of points over $\mathbb{F}_1$ and over extensions $\mathbb{F}_{1^m}$. We present a more general construction of Bost-Connes type systems associated to exponentiable motivic measures and the associated zeta functions with values in Witt rings, obtained using a lift of the Bost-Connes algebra to Witt rings via Frobenius and Verschiebung maps.

We consider lifts of the Bost-Connes algebra to Nori motives. We use a (slightly generalized) version of Nori motives, which may be of independent interest in view of possible versions of equivariant periods. In this categorical setting we show that the fiber functor from Nori motives maps to a categorification of the Bost-Connes algebra previously constructed by Tabuada and the third author, compatibly with the functors realizing the Bost-Connes structure.

**Structure of the paper and main results**

Below we will briefly describe the content of the subsequent sections, and the main results of the paper, with pointers to the specific statements where these are proved.

**Bost-Connes systems and relative equivariant Grothendieck rings**

In § 2.2, we show the existence of a lift of the Bost-Connes structure to the relative equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, extending similar results previously obtained in **ManMar2** for the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$. The main result in this part of the paper is Theorem 2.2.11, where the existence of this lift is proved. The rest of the section covers the preliminary results needed for this main result.

In particular, we first introduce the integral Bost-Connes algebra in §2.2, in the form in which it was introduced in **CCM** . We recall in §2.2 and 2.2 the relative and the equivariant relative Grothendieck rings, and in §2.2 the associated equivariant Euler characteristic map.

In §2.2 we recall from **ManMar2** the geometric form of the Verschiebung map that is used in the lifting of the Bost-Connes structure to varieties with suitable $\hat{\mathbb{Z}}$-actions. In §2.2 we introduce the Bost-Connes maps $\sigma_n$ and $\tilde{\rho}_n$ on classes in $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, and Proposition 2.2.6 shows the way they transform the varieties and the base scheme with their respective $\hat{\mathbb{Z}}$-action.

In §2.2 we recall from **CCM** the prime decomposition of the integral Bost-Connes algebra, which for a finite set of primes $F$ separates out an $F$-part and an $F$-coprime part of the algebra. We then show in §2.2, and in particular Proposition 2.2.8, that, given a scheme $S$ with a good effectively finite action of $\hat{\mathbb{Z}}$, there is an associated finite set of primes $F$ such that the $F$-coprime part of the Bost-Connes algebra lifts to endomorphisms of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$.

Finally in §2.2 we show how to lift the full Bost-Connes algebra to homomorphisms between Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$ where the scheme $S$ and the action $\alpha$ are also transformed by the Bost-Connes map. By an analysis of the structure of periodic points in Lemma 2.2.9 we show the compatibility with the equivariant Euler characteristic, so we can them prove the main result in Theorem 2.2.11, showing that the equivariant Euler characteristic intertwines the Bost-Connes maps $\sigma_n$ and $\tilde{\rho}_n$ on the $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$ rings with the original $\sigma_n$ and $\tilde{\rho}_n$ maps of the integral Bost-Connes algebra.

**Bost-Connes systems on assembler categories and spectra**

In §2.3 we further lift the Bost-Connes structure obtained at the level of Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ to assembler categories underlying these Grothendieck rings and to the homotopy-theoretic spectra defined by these categories. Again this extends to the equivariant relative case results that were obtained in **ManMar2** for the non-relative setting. The main result in this part of the paper is Theorem 2.3.15, where it is shown that the maps $\sigma_n$ and $\tilde{\rho}_n$ on the Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$ constructed in the previous section lift to functors of the underlying assembler categories, that induce these maps on $K_0$.

In §2.3 we recall the formalism of assembler categories of **Zak1** underlying scissor congruence relations and Grothendieck rings. In §2.3 we review Segal's Γ-spaces formalism and how one obtains the homotopy-theoretic spectrum associated to an assembler. In §2.3 we then lift this formalism by endowing the main relevant objects with an action of a finite cyclic group, with appropriate compatibility conditions. It is this further structure that provides a framework for the respective lifts of the Bost-Connes algebras, as in the cases discussed in **ManMar2** and in the ones we will be discussing in the following sections. We give here a very general definition of Bost-Connes systems in categories, based on endofunctors of subcategories of the automorphism category. In the applications considered in this paper we will be using only the special case where the automorphisms are determined by an effectively finite action of $\hat{\mathbb{Z}}$, but we introduce the more general framework in anticipation of other possible applications.

In §2.3 we construct the assembler underlying the equivariant relative Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ and we prove the main result in Theorem 2.3.15 on the lift of the Bost-Connes structure to functors of these assemblers.

**Bost-Connes systems on Grothendieck rings and assemblers of torified varieties**

In §2.4 we consider the approach to $\mathbb{F}_1$-geometry via torifications of varieties over $\mathbb{Z}$, introduced in **LoLo** . The main results of this part of the paper are Proposition 2.4.2 and Proposition 2.4.5, where we construct assembler categories of torified varieties and we show the existence of a lift of the Bost-Connes algebra to these categories.

In §2.4 we recall the notion of torified varieties from **LoLo** and the different versions of morphisms of torified varieties from **ManMar** , and we construct Grothendieck rings of torified varieties for each flavor of morphisms. In §2.4 we introduce a relative version of these Grothendieck rings of torified varieties. In §2.4 we describe $\mathbb{Q}/\mathbb{Z}$ and $\hat{\mathbb{Z}}$-actions on torifications.

In §2.4 we construct the assembler categories underlying these relative Grothendieck rings and in §2.4 we prove the first main result of this section by constructing the lift of the Bost-Connes structure.

**Torified varieties, $\mathbb{F}_1$-points, and zeta functions**

This section continues the theme of torified varieties from the previous section but with main focus on some associated zeta functions. We consider two different kinds of zeta function: $\mathbb{F}_1$-zeta functions that count $\mathbb{F}_1$-points of torified varieties, in an appropriate sense that is discussed in §2.5, and dynamical zeta functions associated to endomorphisms of torified varieties that are compatible with the torification. The use of dynamical zeta functions is motivated by a proposal made in **ManMar2** for a notion of $\mathbb{F}_1$-structures based on dynamical systems that induce quasi-unipotent maps in homology.

The two main results of this section are Proposition 2.5.4 and Proposition 2.5.8, where we show that the $\mathbb{F}_1$-zeta function, respectively the dynamical zeta function, determine exponentiable motivic measures from the Grothendieck rings of torified varieties introduced in the previous section to the ring $W(\mathbb{Z})$ of Witt vectors.

We introduce in §2.5 and §2.5 the counting of $\mathbb{F}_1$-points of a torified variety and its relation to the Grothendieck class. We in show in §2.5 how the Bialynicki–Birula decomposition can be used to determine torifications and we give in §2.5 some explicit examples of computations of Grothendieck classes in simple cases that have physical significance in the context of BPS counting in string theory.

In §2.5 we introduce the $\mathbb{F}_1$-zeta function and we prove Proposition 2.5.4. In §2.5 we explain how the $\mathbb{F}_1$-zeta function can be obtained from the Hasse–Weil zeta function.

In §2.5 we consider torified varieties with dynamical systems compatible with the torification and the associated Lefschetz and Artin–Mazur dynamical zeta functions. We recall the definition and main properties of these zeta functions in §2.5 and we prove in Proposition 2.5.8 in §2.5.

**Spectrification of Witt vectors and lifts of zeta functions**

In the constructions described in §§ 3 and 4 of **ManMar2** and in §§ 2.2–2.6 of the present paper we obtain lifts of the integral Bost-Connes algebra to various assembler categories and associated spectra, starting from a ring homomorphism (motivic measure) from the relevant Grothendieck ring to the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ of the integral Bost-Connes algebra, that is equivariant with respect to the maps $\sigma_n$ and $\tilde{\rho}_n$ of (2.2.4) and (2.2.5) of the Bost-Connes algebra and the maps (also denoted by $\sigma_n$ and $\tilde{\rho}_n$) on the Grothendieck ring induced by a Bost-Connes system on the corresponding assembler category. The motivic measure provides in this way a map that lifts the Bost-Connes structure.

This part of the paper considers then a more general class of zeta functions $\zeta_\mu$ associated to exponentiable motivic measures $\mu : K_0(\mathcal{V}) \to R$ with values in a commutative ring $R$, that admit a factorization into linear factors in the subring $W_0(R)$ of the Witt ring $W(R)$.

Our main results in this section are Proposition 2.6.9, showing that these zeta functions lift to the level of assemblers and spectra, and Proposition 2.6.14, which shows that the Frobenius and Verschiebung maps on the endomorphism category lift, through the lift of the zeta function, to a Bost-Connes system on the assembler category of the Grothendieck ring of varieties $K_0(\mathcal{V})$.

The main step toward establishing the main results of this section is the construction in §2.6 and §2.6 of a spectrification of the ring $W_0(R)$. This is obtained using its description in terms of the $K_0$ of the endomorphism category $\mathcal{E}_R$ and of $R$, and the formalism of Segal Gamma-spaces. The spectrification we use here is not the same as the spectrification of the ring of Witt vectors introduced in **Hess** . The lifting of Bost-Connes systems via motivic measures is discussed in §2.6, where Proposition 2.6.14 is proved.

We also consider again in §2.6 the setting on dynamical $\mathbb{F}_1$-structures proposed in **ManMar2** with a pair $(X, f)$ of a variety and an endomorphism that induces a quasi-unipotent map in homology, and we associate to these data the operator-theoretic spectrum of the quasi-unipotent map, seen as an element in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. This determines a spectral map $\sigma : K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ with the properties of an Euler characteristic.

Another main result in this section is Proposition 2.6.16, showing that this spectral Euler characteristic lifts to a functor from the assembler category underlying $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ to the Tannakian category $\mathrm{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ that categorifies the ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, and that the resulting functor lifts the Bost-Connes structure on $\mathrm{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ described in **MaTa** to a Bost-Connes structure on the assembler of $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$.

**Bost-Connes systems in categories of Nori motives**

When we replace the formalism of assembler categories and homotopy theoretic spectra underlying the Grothendieck rings with geometric diagrams and associated Tannakian categories of Nori motives, with the same notion of categorical Bost-Connes systems introduced in Definitions 2.3.9 and 2.3.11, we can lift the Euler characteristic type motivic measures to the level of categorifications, where, as in the previous section, the categorification of the Bost-Connes algebra is the one introduced in **MaTa** given by a Tannakian category of $\mathbb{Q}/\mathbb{Z}$-graded vector spaces endowed with Frobenius and Verschiebung endofunctors.

In §2.7 in this paper we construct Bost-Connes systems in categories of Nori motives. The main result of this part of the paper is Theorem 2.7.7, which shows that there is a categorical Bost-Connes system on a category of equivariant Nori motives, and that the fiber functor to the categorification of the Bost-Connes algebra constructed in **MaTa** intertwines the respective Bost-Connes endofunctors.

In §2.7 and §2.7 we review the construction of Nori motives from diagrams and their representations. In §2.7 we construct a category of equivariant Nori motives. In §2.7 we describe the endofunctors of this category that implement the Bost-Connes structure and we prove the main result in Theorem 2.7.7. In §2.7 we generalize this result to the relative case, using Arapura's motivic sheaves version of Nori motives.

Finally, in §2.7 we consider Nori diagrams associated to assemblers and we formulate the question of their "universal cohomological representations." This is a contemporary embodiment of the primordial Grothendieck's dream that motives constitute a universal cohomology theory of algebraic varieties.

### 2.2 Bost-Connes systems in Grothendieck rings

In **ManMar2** it was shown that the integral Bost-Connes algebra of **CCM** admits lifts to certain Grothendieck rings, via corresponding equivariant Euler characteristic maps. The cases analyzed in **ManMar2** included the cases of the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ and the equivariant Konsevich–Tschinkel Burnside ring $\mathrm{B}urn^{\hat{\mathbb{Z}}}(\mathbb{K})$. We treat here, in a similar way, the case of the relative equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. This case is more delicate than the other cases considered in **ManMar2** because when the Bost-Connes maps act on the classes in $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ they also change the base scheme $S$ with its $\hat{\mathbb{Z}}$-action.

The main result in this section is the existence of a lifting of the Bost-Connes structure to $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, proved in Theorem 2.2.11.

We first review the definition of the integral Bost-Connes algebra in §2.2 and the equivariant relative Grothendieck ring in §2.2. The rest of the section then develops the intermediate steps leading to the proof of the main results of Theorem 2.2.11.

#### Bost-Connes algebra

The Bost-Connes algebra was introduced in **BC** as a quantum statistical mechanical system that exhibits the Riemann zeta function as partition function, the generators of the cyclotomic extensions of $\mathbb{Q}$ as values of zero-temperature KMS equilibrium states on arithmetic elements in the algebra, and the abelianized Galois group $\hat{\mathbb{Z}}^* \simeq \mathrm{G}al(\bar{\mathbb{Q}}/\mathbb{Q})^{ab}$ as group of quantum symmetries. In particular, the arithmetic subalgebra of the Bost-Connes system is given by the semigroup crossed product

$$\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N} \tag{2.2.1}$$

of the multiplicative semigroup $\mathbb{N}$ of positive integers acting on the group algebra of the group $\mathbb{Q}/\mathbb{Z}$.

The additive group $\mathbb{Q}/\mathbb{Z}$ can be identified with the multiplicative group $\nu^*$ of *roots of unity embedded into* $\mathbb{C}^*$: namely, $r \in \mathbb{Q}/\mathbb{Z}$ corresponds to $e(r) := \exp(2\pi i\, r)$. More generally, the choice of the embedding can be modified by an arbitrary choice of an element in $\hat{\mathbb{Z}}^* = \mathrm{H}om(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, as is usually done in representations of the Bost-Connes algebra, see **BC** . Thus, we will use here interchangeably the notation $\zeta$ or $r$ for elements of $\mathbb{Q}/\mathbb{Z}$ assuming a choice of embedding as above. The group algebra $\mathbb{Q}[\nu^*]$ consists of formal finite linear combinations $\sum_{a_\zeta\in\mathbb{Q}} a_\zeta \zeta$ of roots of unity $\zeta \in \nu^*$. Formality means here that the sum is *not* related to the additive structure of $\mathbb{C}$.

The action of the semigroup $\mathbb{N}$ on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ that defines the crossed product (2.2.1) is given by the endomorphisms

$$\rho_n(\sum a_\zeta \zeta) := \sum a_\zeta \, \frac{1}{n} \sum_{\zeta'^n=\zeta} \zeta'. \tag{2.2.2}$$

Equivalently, the algebra (2.2.1) is generated by elements $e(r)$ with the relations $e(0) = 1$, $e(r + r') = e(r)e(r')$, and elements $\mu_n$ and $\mu_n^*$ satisfying the relations

$$\begin{gathered} \mu_n^*\mu_n = 1, \ \forall n; \qquad \mu_n\mu_n^* = \pi_n, \ \forall n \qquad \text{with } \pi_n = \tfrac{1}{n}\textstyle\sum_{nr=0} e(r); \\ \mu_{nm} = \mu_n\mu_m, \ \forall n, m; \quad \mu_{nm}^* = \mu_n^*\mu_m^*, \ \forall n, m; \quad \mu_n^*\mu_m = \mu_m\mu_n^* \text{ if } (n, m) = 1. \end{gathered} \tag{2.2.3}$$

The semigroup action (2.2.2) is then equivalently written as $\rho_n(a) = \mu_n \, a \, \mu_n^*$, for all $a = \sum a_\zeta \zeta$ in $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$. The element $\pi_n \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ is an idempotent, hence the generators $\mu_n$ are isometries but not unitaries. See **BC** and §3 of **CoMa** for a detailed discussion of the Bost-Connes system and the role of the arithmetic subalgebra (2.2.1).

In **CCM** an integral model of the Bost-Connes algebra was constructed in order to develop a model of $\mathbb{F}_1$-geometry in which the Bost-Connes system encodes the extensions $\mathbb{F}_{1^m}$, in the sense of **KapSmi** of the "field with one element" $\mathbb{F}_1$.

The integral Bost-Connes algebra is obtained by considering the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, which we can again implicitly identify with $\mathbb{Z}[\nu^*]$ for a choice of embedding $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}$ as roots of unity.

Define its *ring endomorphisms* $\sigma_n$:

$$\sigma_n(\sum a_\zeta \zeta) := \sum a_\zeta \zeta^n. \tag{2.2.4}$$

Define *additive maps* $\tilde{\rho}_n$: $\mathbb{Z}[\nu^*] \to \mathbb{Z}[\nu^*]$:

$$\tilde{\rho}_n(\sum a_\zeta \zeta) := \sum a_\zeta \sum_{\zeta'^n=\zeta} \zeta'. \tag{2.2.5}$$

The maps $\sigma_n$ and $\tilde{\rho}_n$ satisfy the relations

$$\sigma_n \circ \tilde{\rho}_n = n \, \mathrm{id}, \quad \tilde{\rho}_n \circ \sigma_n = n \, \pi_n. \tag{2.2.6}$$

The integral Bost-Connes algebra is then defined as the algebra generated by the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and generators $\tilde{\mu}_n$ and $\mu_n^*$ with the relations

$$\begin{aligned}&\tilde{\mu}_n\, a\, \mu_n^* = \tilde{\rho}_n(a),\ \forall n; \quad \mu_n^*\, a = \sigma_n(a)\, \mu_n^*,\ \forall n; \qquad a\, \tilde{\mu}_n = \tilde{\mu}_n\, \sigma_n(a),\ \forall n;\\ &\tilde{\mu}_{nm} = \tilde{\mu}_n\tilde{\mu}_m,\ \forall n,m; \quad \mu_{nm}^* = \mu_n^*\mu_m^*,\ \forall n,m; \quad \tilde{\mu}_n\mu_m^* = \mu_m^*\tilde{\mu}_n \text{ if } (n,m)=1.\end{aligned} \tag{2.2.7}$$

where the relations in the first line hold for all $a = \sum a_\zeta \zeta \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, with $\sigma_n$ and $\tilde{\rho}_n$ as in (2.2.4) and (2.2.5).

The maps $\tilde{\rho}_n$ of the integral Bost-Connes algebra and the semigroup action $\rho_n$ in the rational Bost-Connes algebra (2.2.1) are related by

$$\rho_n = \frac{1}{n}\tilde{\rho}_n$$

with $\tilde{\rho}_n$ defined as in (2.2.5).

**Relative Grothendieck ring**

We describe here a variant of construction of **ManMar2** where we work with relative Grothendieck rings and with an Euler characteristic with values in a Grothendieck ring of locally constant sheaves. We show that this relative setting provides ways of lifting to the level of Grothendieck classes certain subalgebras of the integral Bost-Connes algebras associated to the choice of a finite set of non-archimedean places.

**Definition 2.2.1.** *The relative Grothendieck ring $K_0(\mathcal{V}_S)$ of varieties over a base variety $S$ over a field $\mathbb{K}$ is generated by the isomorphism classes of data $f : X \to S$ of a variety $X$ over $S$ with the relations*

$$[f : X \to S] = [f|_Y : Y \to S] + [f|_{X\smallsetminus Y} : X \smallsetminus Y \to S]$$

*as in* (2.7.1) *for a closed embedding $Y \hookrightarrow X$ of varieties over $S$. The product is given by the fibered product $X \times_S Y$. We will write $[X]_S$ as shorthand notation for the class $[f : X \to S]$ in $K_0(\mathcal{V}_S)$.*

A morphism $\phi : S \to S'$ induces a base change ring homomorphism $\phi^* : K_0(\mathcal{V}_{S'}) \to K_0(\mathcal{V}_S)$ and a direct image map $\phi_* : K_0(\mathcal{V}_S) \to K_0(\mathcal{V}_{S'})$ which is a group homomorphisms and a morphism of $K_0(\mathcal{V}_{S'})$-modules, but not a ring homomorphism. The class $[\phi : S \to S']$ as an element in $K_0(\mathcal{V}_{S'})$ is the image of $1 \in K_0(\mathcal{V}_S)$ under $\phi_*$.

When $S = \mathrm{S}pec(\mathbb{K})$ one recovers the ordinary Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$.

**Equivariant relative Grothendieck ring**

Let $X$ be a variety with a good action $\alpha : G \times X \to X$ by a finite group $G$ and $X'$ a variety with a good action $\alpha'$ by $G'$. As morphisms we then consider pairs $(\phi, \varphi)$ of a morphism $\phi : X \to X'$ and a group homomorphism $\varphi : G \to G'$ such that $\phi(\alpha(g, x)) = \alpha'(\varphi(g), \phi(x))$, for all $g \in G$ and $x \in X$. Thus, isomorphisms of varieties with good $G$-actions are pairs of an isomorphism $\phi : X \to X'$ of varieties and a group automorphism $\varphi \in \mathrm{A}ut(G)$ with the compatibility condition as above.

Given a base variety (or scheme) $S$ with a given good action $\alpha_S$ of a finite group $G$, and varieties $X, X'$ over $S$, with good $G$-actions $\alpha_X, \alpha_{X'}$ and $G$-equivariant maps $f : X \to S$ and $f' : X' \to S$, we consider morphisms given by a triple $(\phi, \varphi, \phi_S)$ of a morphism $\phi : X \to X'$, a group homomorphism $\varphi : G \to G$ with the compatibility as above, and an endomorphism $\phi_S : S \to S$ such that $f' \circ \phi = \phi_S \circ f$. Then these maps also satisfy $\phi_S(\alpha_S(g, f(x)) = \alpha_S(\varphi(g), \phi_S(f(x)))$.

**Definition 2.2.2.** *The relative equivariant Grothendieck ring $K_0^G(\mathcal{V}_S)$ is obtained as follows. Consider the abelian group generated by isomorphism classes $[f : X \to S]$ of varieties over $S$ with compatible good $G$-actions, with respect to isomorphisms $(\phi, \varphi, \phi_S)$ as above, with the inclusion-exclusion relations generated by equivariant embeddings with compatible $G$-equivariant maps*

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr, "f|_Y"'] & X \arrow[d, "f"] & X \smallsetminus Y \arrow[l, hook'] \arrow[dl, "f|_{X \smallsetminus Y}"] \\ & S & \end{tikzcd} \tag{2.2.8}$$

*and isomorphisms. This means that we have $[f : X \to S] = [f_Y : Y \to S] + [f_{X \smallsetminus Y} : X \smallsetminus Y \to S]$ if there are isomorphisms $(\phi_Y, \varphi_Y, \phi_{S,Y})$ and $(\phi_{X \smallsetminus Y}, \varphi_{X \smallsetminus Y}, \phi_{S,X \smallsetminus Y})$, such that the diagram commutes*

$$\begin{tikzcd} Y \arrow[r, "\phi_Y"] \arrow[d, "f_Y"] & Y \arrow[r, hook] \arrow[dr, "f|_Y"'] & X \arrow[d, "f"] & X \smallsetminus Y \arrow[l, hook'] \arrow[dl, "f|_{X \smallsetminus Y}"] & X \smallsetminus Y \arrow[l, "\phi'_{X \smallsetminus Y}"'] \arrow[d, "f_{X \smallsetminus Y}"'] \\ S \arrow[rr, "\phi_{S,Y}"'] & & S & & S \arrow[ll, "\phi_{S,X \smallsetminus Y}"] \end{tikzcd} \tag{2.2.9}$$

*The product $[f : X \to S] \cdot [f' : X' \to S]$ given by $[\tilde{f} : X \times_S X' \to S]$ with $\tilde{f} = f \circ \pi_X = f' \circ \pi_{X'}$ is well defined on isomorphism classes, with the diagonal action $\tilde{\alpha}(g, (x, x')) = (\alpha_X(g, x), \alpha_{X'}(g, x'))$ satisfying $f(\alpha_X(g, x)) = \alpha_S(g, f(x)) = \alpha_S(g, f'(x')) = f'(\alpha_{X'}(g, x'))$.*

We will use the following terminology for the $\hat{\mathbb{Z}}$-actions we consider.

**Definition 2.2.3.** *A good effectively finite action of $\hat{\mathbb{Z}}$ on a variety $X$ is a good action that factors through an action of some quotient $\mathbb{Z}/N\mathbb{Z}$. We will write $\mathbb{Z}/N\mathbb{Z}$-effectively finite when we need to explicitly keep track of the level $N$.*

In the case of the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ considered in **ManMar2** we can then also consider a relative version $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, with $S$ a variety with a good effectively finite $\hat{\mathbb{Z}}$-action as above. We consider the Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ given by the isomorphism classes of $S$-varieties $f : X \to S$ with good effectively finite $\hat{\mathbb{Z}}$-actions with respect to which $f$ is equivariant, with the notion of isomorphism described above. The product is given by the fibered product over $S$ with the diagonal $\hat{\mathbb{Z}}$-action. The inclusion-exclusion relations are as in (2.7.1), where $Y \hookrightarrow X$ and $X \smallsetminus Y \hookrightarrow X$ are equivariant embeddings with compatible $\hat{\mathbb{Z}}$-equivariant maps as in (2.2.9).

**Equivariant Euler characteristic**

There is an Euler characteristic map given by a ring homomorphism

$$\chi_S^{\hat{\mathbb{Z}}} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S) \tag{2.2.10}$$

to the Grothendieck ring of constructible sheaves over $S$ with $\hat{\mathbb{Z}}$-action, **GuZa Looij MaxSch Verdier** .

**Lemma 2.2.4.** *Let $S$ be a variety with a good $\mathbb{Z}/N\mathbb{Z}$-effectively finite $\hat{\mathbb{Z}}$-action. Given a constructible sheaf $[\mathcal{F}]$ in $K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$, let $\mathcal{F}|_{S^g}$ denote the restrictions to the fixed point sets $S^g$, for $g \in \mathbb{Z}/N\mathbb{Z}$. These determine classes in $K_0(\mathbb{Q}_{S^g}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. One obtains in this way a map*

$$\chi : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to \bigoplus_{g \in \mathbb{Z}/N\mathbb{Z}} K_0(\mathbb{Q}_{S^g}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]. \tag{2.2.11}$$

*Proof.* The $\hat{\mathbb{Z}}$ action on $S$ factors through some $\mathbb{Z}/N\mathbb{Z}$, hence the fixed point sets are given by $S^g$ for $g \in \mathbb{Z}/N\mathbb{Z}$. Given a constructible sheaves $\mathcal{F}$ over $S$ with $\hat{\mathbb{Z}}$-action, consider the restrictions $\mathcal{F}|_{S^g}$. The subgroup $\langle g \rangle$ generated by $g$ acts trivially on $S^g$, hence for each $s \in S^g$ it acts on the stalk $\mathcal{F}_s$. Thus, these restrictions define classes $[\mathcal{F}|_{S^g}] \in K_0(\mathbb{Q}_{S^g}) \otimes R(\langle g \rangle) \subset K_0(\mathbb{Q}_{S^g}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. By precomposing with the Euler characteristic (2.2.10) one then obtains the map (2.2.11). $\square$

We will also consider the map $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0(\mathbb{Q}_{S^G}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ given by the Euler characteristic followed by restriction of sheaves to the fixed point set $S^G$ of the group action.

**Fixed points and delocalized homology**

Equivariant characteristic classes from constructible sheaves to delocalized homology are constructed in **MaxSch** .

For a variety $S$ with a good action by a finite group $G$, and a (generalized) homology theory $H$, the associated delocalized equivariant theory is given by

$$H^G(S) = (\oplus_{g\in G} H(S^g))^G$$

where the disjoint union $\sqcup_g S^g$ of the fixed point sets $S^g$ has an induced $G$-action $h : S^g \to S^{hgh^{-1}}$. In the case of an abelian group we have $H^G(S) = (\oplus_{g\in G} H(S^g))^G$.

As an observation, we can see explicitly the relation of delocalized homology to the integral Bost-Connes algebra, by considering the following cases (see Remark 2.2.12). Let $S$ be a variety with a good $\mathbb{Z}/N\mathbb{Z}$-effectively finite $\hat{\mathbb{Z}}$-action. If $S$ has the trivial $\mathbb{Z}/N\mathbb{Z}$-action we have $H^{\mathbb{Z}/N\mathbb{Z}}(S) = H(S) \otimes \mathbb{Z}[\mathbb{Z}/N\mathbb{Z}]$. In particular, if $S$ is just a point, then this is $\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}]$. More generally, there is a morphism

$$\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}] \times H^{\mathbb{Z}/N\mathbb{Z}}(S) \to H^{\mathbb{Z}/N\mathbb{Z}}(S)$$

induced by $H^{\mathbb{Z}/N\mathbb{Z}}(pt) \times H^{\mathbb{Z}/N\mathbb{Z}}(S) \to H^{\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}}(pt \times S) \to H^{\mathbb{Z}/N\mathbb{Z}}(S)$ with the restriction to the diagonal subgroup as the last map.

**The geometric Verschiebung action**

We recall here how to construct the geometric Verschiebung action used in **ManMar2** to lift the Bost-Connes maps to the level of Grothendieck rings. This has the effect of transforming an action of $\hat{\mathbb{Z}}$ on $X$ that factors through some $\mathbb{Z}/N\mathbb{Z}$ into an action of $\hat{\mathbb{Z}}$ on $X \times Z_n$, with $Z_n = \{1, \dots, n\}$, that factors through $\mathbb{Z}/Nn\mathbb{Z}$. For $x \in X$, let $\underline{x} = (x, a_i)_{a_i \in Z_n} = (x_i)_{i=1}^n$ be the subset $\{x\} \times Z_n$. For $\zeta_N$ a primmitive $N$-th root of unity, we write in matrix form

$$V_n(\zeta_{Nn}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha(\zeta_N) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

so that we can write

$$V_n(\zeta_{Nn}) \cdot \underline{x} = \begin{cases} (x, a_{i+1}) & i = 1, \dots, n-1 \\ (\alpha(\zeta_N) \cdot x, a_1) & i = n \end{cases} \tag{2.2.12}$$

which satisfies $V_n(\zeta_{Nn})^n = \alpha(\zeta_N) \times \mathrm{I}d_{Z_n}$. The resulting action $\Phi_n(\alpha)$ of $\hat{\mathbb{Z}}$ on $X \times Z_n$ that factors through $\mathbb{Z}/Nn\mathbb{Z}$ is specified by setting

$$\Phi_n(\alpha)(\zeta_{Nn}) \cdot (x, a) = (V_n(\alpha(\zeta_N)) \cdot \underline{x})_a. \tag{2.2.13}$$

**Lifting the Bost-Connes endomorphisms**

Consider a base scheme $S$ with a good effectively finite action of $\hat{\mathbb{Z}}$. Let $f : X \to S$ be a variety over $S$ with a good effectively finite $\hat{\mathbb{Z}}$ action such that the map is $\hat{\mathbb{Z}}$-equivariant. We denote by $\alpha_S : \hat{\mathbb{Z}} \times S \to S$ the action on $S$ and by $\alpha_X : \hat{\mathbb{Z}} \times X \to X$ the action on $X$. We write the equivariant relative Grothendieck ring as $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$ to explicitly remember the fixed (up to isomorphisms as in §2.2) action on $S$.

**Definition 2.2.5.** *Let $(S, \alpha_S)$ be a scheme with a good effectively finite action of $\hat{\mathbb{Z}}$. Let $Z_n = Spec(\mathbb{Q}^n)$ and let $\Phi_n(\alpha_S)$ be the action of $\hat{\mathbb{Z}}$ on $S \times Z_n$ as in* (2.2.12) *and* (2.2.13). *Given a class $[f : (X, \alpha_X) \to (S, \alpha_S)]$ in $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$, with $\alpha_X$ the compatible $\hat{\mathbb{Z}}$-action on $X$, let*

$$\sigma_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f : (X, \alpha_X \circ \sigma_n) \to (S, \alpha_S \circ \sigma_n)] \tag{2.2.14}$$

$$\tilde{\rho}_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f \times id : (X \times Z_n, \Phi_n(\alpha_X)) \to (S \times Z_n, \Phi_n(\alpha_S))]. \tag{2.2.15}$$

**Proposition 2.2.6.** *For all $n \in \mathbb{N}$ the $\sigma_n$ defined in* (2.2.14) *are ring homomorphisms*

$$\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S \circ \sigma_n)}) \tag{2.2.16}$$

*and the $\tilde{\rho}_n$ defined in* (2.2.15) *are group homomorphisms*

$$\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \Phi_n(\alpha_S))}), \tag{2.2.17}$$

*with compositions satisfying*

$$\tilde{\rho}_n \circ \sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S \times Z_n, \alpha_S \times \alpha_n)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$$

$$\sigma_n \circ \tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)^{\oplus n}}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}),$$

*with $\sigma_n \circ \tilde{\rho}_n = n\, id$ and $\tilde{\rho}_n \circ \sigma_n$ is the product by $(Z_n, \alpha_n)$.*

*Proof.* Consider the $\sigma_n$ defined in (2.2.14). Since the group $\hat{\mathbb{Z}}$ is commutative and so is its endomorphism ring, these transformations $\sigma_n$ respect isomorphism classes since for an isomorphism $(\phi, \varphi, \phi_S)$ the actions satisfy

$$\phi_X(\alpha_X(\sigma_n(g), x)) = \alpha'_X(\varphi(\sigma_n(g)), \phi(x)) = \alpha'_X(\sigma_n(\varphi(g)), \phi(x)),$$

and similarly for the actions $\alpha_S, \alpha'_S$, so that $(\phi, \varphi, \phi_S)$ is also an isomorphism of the images under $\sigma_n$. Similarly, the $\tilde{\rho}_n$ defined in (2.2.15) are well defined on the isomorphism classes.

As in **ManMar2** we see that $\sigma_n \circ \tilde{\rho}_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f : (X, \alpha_X) \to (S, \alpha_S)]^{\oplus n}$ and $\tilde{\rho}_n \circ \sigma_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f \times id : (X \times Z_n, \alpha_X \times \alpha_n) \to (S \times Z_n, \alpha_S \times \alpha_n)]$. The Grothendieck groups $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n, \alpha_S\times\alpha_n)})$ and $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)^{\oplus n}})$ map to $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$ via the morphism induced by composition with the natural maps of the respective base varieties to $(S, \alpha_S)$. □

The fact that the ring homomorphisms (2.2.16) and (2.2.17) determine a lift of the ring endomorphism $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and group homomorphisms $\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ of the integral Bost-Connes algebra is discussed in Proposition 2.2.8 and §2.2.

We know from **ManMar2** that the integral Bost-Connes algebra lifts to the equivariant Grothendieck ring $K^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{Q}})$ via maps $\sigma_n$ and $\tilde{\rho}_n$ that, respectively, precompose the action with the Bost-Connes endomorphism $\sigma_n$ and apply a geometric form of the Verschiebung map. The main difference with the relative case considered here lies in the fact that the lifts to the equivariant relative Grothendieck rings given by the maps (2.2.16) and (2.2.17) need to transform in a compatible way the actions on both $X$ and $S$.

**Remark 2.2.7.** *Because the maps $\sigma_n$ and $\tilde{\rho}_n$ of* (2.2.16) *and* (2.2.17) *simultaneously modify the action on the varieties and on the base scheme S, they do not give endomorphisms of the same $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$. However, given $(S, \alpha_S)$, it is possible to identify a subalgebra of the integral Bost-Connes algebra that lift to endomorphisms of a corresponding subring of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$, using the notion of "prime decomposition" of the Bost-Connes algebra. We discuss this more carefully in §2.2.*

**Prime decomposition of the Bost-Connes algebra**

As in **CCM** for each prime $p$, we can decompose the group $\mathbb{Q}/\mathbb{Z}$ into a product $\mathbb{Q}_p/\mathbb{Z}_p \times (\mathbb{Q}/\mathbb{Z})^{(p)}$, where $\mathbb{Q}_p/\mathbb{Z}_p$ is the Prüfer group, namely the subgroup of elements of $\mathbb{Q}/\mathbb{Z}$ where the denominator is a power of $p$, isomorphic to $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, while $(\mathbb{Q}/\mathbb{Z})^{(p)}$ consists of the elements with denominator prime to $p$.

Similarly, given a finite set $F$ of primes, we can decompose $\mathbb{Q}/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})_F \times (\mathbb{Q}/\mathbb{Z})^F$, where the first term $(\mathbb{Q}/\mathbb{Z})_F$ is identified with fractions in $\mathbb{Q}/\mathbb{Z}$ whose denominator has prime factor decomposition consisting only of primes in $F$, while elements in $(\mathbb{Q}/\mathbb{Z})^F$ have denominators prime to all $p \in F$. The group ring decomposes accordingly as $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F] \otimes \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^F]$.

The subsemigroup $\mathbb{N}_F \subset \mathbb{N}$ generated multiplicatively by the primes $p \in F$ acts on $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[(\mathbb{Q}/\mathbb{Z})_F]$ by endomorphisms

$$\rho_n(e(r)) = \frac{1}{n} \sum_{nr'=r} e(r'), \quad n \in \mathbb{N}_F, \; r \in (\mathbb{Q}/\mathbb{Z})_F.$$

The corresponding morphisms $\sigma_n(e(r)) = e(nr)$ and maps $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$ act on $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F]$ and we can consider the associated algebra $\mathcal{A}_{\mathbb{Z},F}$ generated by $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F]$ and $\tilde{\mu}_n$, $\mu_n^*$ with $n \in \mathbb{N}_F$, with the relations

$$\tilde{\mu}_{nm} = \tilde{\mu}_n \tilde{\mu}_m, \quad \mu_{nm}^* = \mu_n^* \mu_m^*, \quad \mu_n^* \tilde{\mu}_n = n, \quad \tilde{\mu}_n \mu_m^* = \mu_m^* \tilde{\mu}_n, \tag{2.2.18}$$

where the first two relations hold for arbitrary $n, m \in \mathbb{N}$, the third for arbitrary $n \in \mathbb{N}$ and the forth for $n, m \in \mathbb{N}$ satisfying $(n, m) = 1$, and the relations

$$x\tilde{\mu}_n = \tilde{\mu}_n \sigma_n(x) \quad \mu_n^* x = \sigma_n(x) \mu_n^*, \quad \tilde{\mu}_n x \mu_n^* = \tilde{\rho}_n(x), \tag{2.2.19}$$

for any $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, where $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$, and with

$$\mathcal{A}_{\mathbb{Z},F} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[(\mathbb{Q}/\mathbb{Z})_F] \rtimes \mathbb{N}_F.$$

We refer to $\mathcal{A}_{\mathbb{Z},F}$ as the $F$-part of the integral Bost-Connes algebra.

The decomposition $\mathbb{N} = \mathbb{N}_F \times \mathbb{N}^{(F)}$, where $\mathbb{N}^{(F)}$ is generated by all primes $p \notin F$, gives also an algebra $\mathcal{A}_{\mathbb{Z}}^{(F)}$ generated by $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^F]$ and the $\tilde{\mu}_n$ and $\mu_n^*$ as in (2.2.19) with $p \notin F$ with

$$\mathcal{A}_{\mathbb{Z}}^{(F)} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[(\mathbb{Q}/\mathbb{Z})^F] \rtimes \mathbb{N}^{(F)}.$$

We refer to $\mathcal{A}_{\mathbb{Z}}^{(F)}$ as the $F$-coprime part of the integral Bost-Connes algebra.

**Lifting the $F_N$-coprime Bost-Connes algebra**

Let $F = F_N$ be the set of prime factors of $N$ and let $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^F]$ denote, as before, the part of the group ring of $\mathbb{Q}/\mathbb{Z}$ involving only denominators relatively prime to $N$. The semigroup $\mathbb{N}^{(F)}$ is generated by primes $p \nmid N$ and we consider the morphisms $\sigma_n(e(r)) = e(nr)$ and maps $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$ with $n \in \mathbb{N}^{(F)}$ and $r \in (\mathbb{Q}/\mathbb{Z})^F$ as discussed above.

**Proposition 2.2.8.** *Let $S$ be a base scheme with a good $\mathbb{Z}/N\mathbb{Z}$-effectively finite action of $\hat{\mathbb{Z}}$. Let $\mathcal{Z}_{n,S}$ be defined as $\mathcal{Z}_{n,S} = S \times Z_n$, with $Z_n = Spec(\mathbb{Q}^n)$, with the action $\Phi_n(\alpha_S)$ obtained as in* (2.2.12) *and* (2.2.13)*. The endomorphisms $\sigma_n : \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}] \to \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ with $n \in \mathbb{N}^{(F_N)}$ of the $F_N$-coprime part of the integral Bost-Connes algebra lift to endomorphisms $\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, as in* (2.2.14)*, which define a semigroup action of the multiplicative group $\mathbb{N}^{(F_N)}$ on the Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. The maps $\tilde{\rho}_n$, for $n \in \mathbb{N}^{(F_N)}$, lift to group homomorphisms $\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, as in* (2.2.15)*, so that $\sigma_n \circ \tilde{\rho}_n[f : X \to S] = [f : X \to S]^{\oplus n}$ and $\tilde{\rho}_n \circ \sigma_n[f : X \to S] = [f : X \to S] \cdot \mathcal{Z}_{n,S}$.*

*Proof.* Given the base variety $S$ with a good $\mathbb{Z}/N\mathbb{Z}$-effectively finite $\hat{\mathbb{Z}}$-action, let $F = F_N$ denote the set of prime factors of $N$. Let $X$ be a variety over $S$, with a $\hat{\mathbb{Z}}$-equivariant map $f : (X, \alpha_X) \to (S, \alpha_S)$, where we explicitly write the actions, satisfying $f(\alpha_X(\zeta, x)) = \alpha_S(\zeta, f(x))$. For $(N, n) = 1$, the maps $\sigma_n : [f : (X, \alpha_X) \to (S, \alpha_S)] = [f : (X, \alpha_X \circ \sigma_n) \to (S, \alpha_S \circ \sigma_n)]$, as in (2.2.14), satisfy $(S, \alpha_S \circ \sigma_n) \simeq (S, \alpha_S)$ with the notion of isomorphism discussed in §2.2, since $\zeta \mapsto \sigma_n(\zeta)$ is an automorphism of $\mathbb{Z}/N\mathbb{Z}$. Thus, the maps $\sigma_n$, for $n \in \mathbb{N}^{(F_N)}$ determine a semigroup action of $\mathbb{N}^{(F_N)}$ by endomorphisms of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$.

Consider then $(\mathcal{Z}_{n,N}, \Phi_n(\alpha_S))$ as above, which we write equivalently as $\tilde{\rho}_n(S, \alpha_S)$ where $\tilde{\rho}_n$ is the lift of the Bost-Connes map to $K^{\hat{\mathbb{Z}}}(\mathcal{V})$ as in Proposition 3.5 of **ManMar2** . We know that $\tilde{\rho}_n \circ \sigma_n[S, \alpha_S] = [S, \alpha_S] \cdot [Z_n, \alpha_n]$ in $K^{\hat{\mathbb{Z}}}(\mathcal{V})$. Since for $(n, N) = 1$ we have $(S, \alpha_S \circ \sigma_n) \simeq (S, \alpha_S)$, this gives $(\mathcal{Z}_{n,N}, \Phi_n(\alpha_S)) \simeq (S \times Z_n, \alpha_S \times \gamma_n)$. Then setting $\tilde{\rho}_n(f : X \to S) = (\tilde{f} : X \times_S \mathcal{Z}_{n,S} \to S)$ with $\tilde{f} = f \circ \pi_X$ gives $X \times_S \mathcal{Z}_{n,S} \simeq X \times Z_n$, and the composition properties for $\tilde{\rho}_n \circ \sigma_n$ and $\sigma_n \circ \tilde{\rho}_n$ are satisfied.

Given a class $[f : X \to S]$, let $[\mathcal{F}_{X,S}]$ be the class in $K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$ of the constructible sheaf given by the Euler characteristic (2.2.10) of $[f : X \to S]$. Let $[\mathcal{F}_{X,S}|_{S^{\mathbb{Z}/N\mathbb{Z}}}]$ be the resulting class in $K_0(S^{\mathbb{Z}/N\mathbb{Z}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ obtained by restriction to the fixed point set $S^{\mathbb{Z}/N\mathbb{Z}}$ with the element in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ specifying the representation of $\hat{\mathbb{Z}}$ on the stalks of the sheaf $\mathcal{F}_{X,S}|_{S^{\mathbb{Z}/N\mathbb{Z}}}$. For $(N, n) = 1$, the action of $\sigma_n$ by automorphisms of $\mathbb{Z}/N\mathbb{Z}$ wih the resulting action by isomorphisms of $S$ induces an action by isomorphisms on the $K_0(S^{\mathbb{Z}/N\mathbb{Z}})$ part and the usual Bost-Connes action on $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. The restriction of the semigroup action of $\mathbb{N}^{(F_N)}$ to the subring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ is then the image of the action of the maps $\sigma_n$ and $\tilde{\rho}_n$ on the preimage of this subring under the morphism $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0(\mathbb{Q}_{S^G}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. $\square$

While this construction captures a lift of the $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ part of the Bost-Connes algebra with the semigroup action of $\mathbb{N}^{(F_N)}$, the fact that the endomorphisms $\sigma_n$ acting on the roots of unity in $\mathbb{Z}/N\mathbb{Z}$ are automorphisms when $(N, n) = 1$ loses some of the interesting structure of the Bost-Connes algebra, which stems from the partial invertibility of these morphisms. Thus, one also wants to recover the structure of the complementary part of the Bost-Connes algebra with the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_{F_N}]$ and the semigroup $\mathbb{N}_{F_N}$.

**Lifting the full Bost-Connes algebra**

Unlike the $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ part of the Bost-Connes algebra described above, when one considers the full Bost-Connes algebra, including the $F_N$-part, the lift to the Grothendieck ring no longer consists of endomorphisms of a fixed $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$, but is given as in Proposition 2.2.6 by homomorphisms as in (2.2.14), (2.2.16) and (2.2.15), and (2.2.17),

$$\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S\circ\sigma_n)}),$$

$$\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n,\Phi_n(\alpha_S))}).$$

For $G$ a finite abelian group with a good action $\alpha : G \times S \to S$ on a variety $S$, let $(S, \alpha)_k^G = \{s \in S \,:\, \alpha(g^k, s) = s,\ \forall g \in G\}$ denote the set of periodic points of period $k$, with $(S, \alpha)_1^G = (S, \alpha)^G$ the set of fixed points. We always have $(S, \alpha)_k^G \subseteq (S, \alpha)_{km}^G$ for all $m \in \mathbb{N}$, hence in particular a copy of the fixed point set $(S, \alpha)^G$ is contained in all $(S, \alpha)_k^G$. For $G = \mathbb{Z}/N\mathbb{Z}$, with $\zeta_N$ a primitive $N$-th root of unity generator, the set of $k$-periodic points is given by $(S, \alpha)_k^{\mathbb{Z}/N\mathbb{Z}} = \{s \in S \,:\, \alpha(\zeta_N^k, s) = s\}$.

**Lemma 2.2.9.** *The sets of periodic points satisfy* $(S, \alpha \circ \sigma_n)_k^G = (S, \alpha)_{nk}^G$. *The sets* $(S \times Z_n, \Phi_n(\alpha))_k^G$ *can be non-empty only when* $n|k$ *with* $(S \times Z_n, \Phi_n(\alpha))_k^G = ((S, \alpha)_{k/n}^G)^n$.

*Proof.* Under the action $\alpha \circ \sigma_n$ the periodicity condition means $\alpha \circ \sigma_n(\zeta^k, s) = \alpha(\zeta^{nk}, s) = s$ for all $\zeta \in G$ hence the identification $(S, \alpha \circ \sigma_n)_k^G = (S, \alpha)_{nk}^G$. In the case of the geometric Verschiebung action $\Phi_n(\alpha)$ on $S \times Z_n$, the $k$-periodicity condition $\Phi_n(\alpha)(\zeta^k, (s, z)) = (s, z)$ implies that $n|k$ for the $k$-periodicity in the $z \in Z_n$ variable and that $\alpha(\zeta^{k/n}, s) = s$. □

The identification $(S, \alpha\circ\sigma_n)_k^G = (S, \alpha)_{nk}^G$ implies the inclusion $(S, \alpha)_k^G \subseteq (S, \alpha\circ\sigma_n)_k^G$ and in particular the inclusion of the fixed point sets $(S, \alpha)^G \subseteq (S, \alpha\circ\sigma_n)^G$. Similarly,

$(S \times Z_n, \Phi_n(\alpha))^G \subseteq ((S,\alpha)^G)^n$. Since these inclusions will in general be strict, due to the fact that the endomorphisms $\sigma_n$ are not automorphisms, one cannot simply use the map given by the equivariant Euler characteristic followed by the restriction to the fixed point set

$$K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0(\mathbb{Q}_{S^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

to lift the Bost-Connes endomorphisms to the maps (2.2.16) and (2.2.17) of Proposition 2.2.6. However, a simple variant of the same idea, where we consider sets of periodic points, gives the lift of the full Bost-Connes algebra to the equivariant relative Grothendieck rings $K_0^G(\mathcal{V}_{(S,\alpha)})$.

Consider the equivariant Euler characteristic map followed by the restrictions to the sets of periodic points

$$K_0^G(\mathcal{V}_{(S,\alpha)}) \xrightarrow{\chi_S^G} K_0^G(\mathbb{Q}_{(S,\alpha)}) \to \bigoplus_{k\geq 1} K_0^G(\mathbb{Q}_{(S,\alpha)_k^G}). \tag{2.2.20}$$

Also, for a given $n \in \mathbb{N}$, consider the same map composed with the projection to the summands with $n|k$

$$\chi_{S,n}^G : K_0^G(\mathcal{V}_{(S,\alpha)}) \xrightarrow{\chi_S^G} K_0^G(\mathbb{Q}_{(S,\alpha)}) \to \bigoplus_{k\geq 1\,:\, n|k} K_0^G(\mathbb{Q}_{(S,\alpha)_k^G}). \tag{2.2.21}$$

For simplicity we consider the case where the fixed point set and periodic points sets of the action $(S,\alpha)$ are all finite sets.

**Definition 2.2.10.** *Let $(S,\alpha)$ be a variety with a good effectively finite $\hat{\mathbb{Z}}$-action. Consider data $(A_{(S,\alpha),n}, f_{(S,\alpha),n})$ and $(B_{(S,\alpha)}, h_{(S,\alpha)})$ of a family of rings $A_{(S,\alpha),n}$ with $n \in \mathbb{N}$ and $B_{(S,\alpha)}$ and ring homomorphisms $f_{(S,\alpha),n} : K_0^G(\mathcal{V}_{(S,\alpha)}) \to A_{(S,\alpha),n} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and $h_{(S,\alpha)} : K_0^G(\mathcal{V}_{(S,\alpha)}) \to B_{(S,\alpha)} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. The maps $f_{(S,\alpha),n}$ and $h_{(S,\alpha)}$ are said to intertwine the Bost-Connes structure if there are ring isomorphisms $J_n : A_{(S,\alpha),n} \to B_{(S,\alpha\circ\sigma_n)}$ and isomorphisms of abelian groups $\tilde{J}_n : B_{(S,\alpha)} \to A_{(S\times Z_n,\Phi_n(\alpha))}$, such that the following holds.*

1. *There is a commutative diagram of ring homomorphisms*

$$\begin{array}{ccc} K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) & \xrightarrow{f_{(S,\alpha),n}} & A_{(S,\alpha),n} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ \Big\downarrow{\scriptstyle \sigma_n} & & \Big\downarrow{\scriptstyle J_n \otimes \sigma_n} \\ K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha\circ\sigma_n)}) & \xrightarrow{h_{(S,\alpha)}} & B_{(S,\alpha\circ\sigma_n)} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \end{array}$$

*where the maps* $\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha\circ\sigma_n)})$ *are as in* (2.2.16) *and the maps* $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ *are the endomorphisms of the integral Bost-Connes algebra.*

2. *There is a commutative diagram of group homomorphisms*

$$\begin{tikzcd}
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \arrow[r, "h_{(S,\alpha)}"] \arrow[d, "\tilde{\rho}_n"] & B_{(S,\alpha)} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \arrow[d, "\tilde{J}_n \otimes \tilde{\rho}_n"] \\
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n,\Phi_n(\alpha))}) \arrow[r, "f_{(S\times Z_n,\Phi_n(\alpha)),n}"] & A_{(S\times Z_n,\Phi_n(\alpha)} \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]
\end{tikzcd}$$

*where the maps* $\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n,\Phi_n(\alpha))})$ *are as in* (2.2.17) *and the* $\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ *are the maps* (2.2.5) *of the integral Bost-Connes algebra.*

**Theorem 2.2.11.** *Let* $(S,\alpha)$ *be a variety with a good effectively finite* $\hat{\mathbb{Z}}$*-action, such that the set* $(S,\alpha)_k^{\hat{\mathbb{Z}}}$ *of* $k$*-periodic points for this action is finite, for all* $k \geq 1$*. Then the maps* (2.2.20) *and* (2.2.21) *intertwine the Bost-Connes structure in the sense of Definition 2.2.10.*

*Proof.* Under the assumptions that all the $(S,\alpha)_k^G$ for $k \geq 0$ are finite sets, we can identify the target of the map with $\oplus_k K_0(\mathbb{Q}_{(S,\alpha)_k^G}) \otimes R(G)$. In the case of varieties with good effectively finite $\hat{\mathbb{Z}}$ actions, we obtain in this way ring homomorphisms

$$\chi^{\hat{\mathbb{Z}}}_{(S,\alpha)} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \to \bigoplus_{k\geq 1} K_0(\mathbb{Q}_{(S,\alpha)_k^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

$$\chi^{\hat{\mathbb{Z}}}_{(S,\alpha),n} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \to \bigoplus_{k\geq 1\,:\,n|k} K_0(\mathbb{Q}_{(S,\alpha)_k^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}].$$

These maps fit in the following commutative diagrams of ring homomorphisms

$$\begin{tikzcd}
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \arrow[r, "\chi^{\hat{\mathbb{Z}}}_{(S,\alpha),n}"] \arrow[d, "\sigma_n"] & \bigoplus_{n|k} K_0(\mathbb{Q}_{(S,\alpha)_k^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \arrow[d, "J_n \otimes \sigma_n"] \\
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha\circ\sigma_n)}) \arrow[r, "\bar{\chi}^{\hat{\mathbb{Z}}}_{(S,\alpha\circ\sigma_n)}"'] & \bigoplus_{\ell} K_0(\mathbb{Q}_{(S,\alpha\circ\sigma_n)_\ell^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]
\end{tikzcd}$$

where the map $(J_n)_{k,\ell}$ is non-trivial for $k = \ell n$ and identifies $K_0(\mathbb{Q}_{(S,\alpha)_\ell^{\hat{\mathbb{Z}}}})$ with $K_0(\mathbb{Q}_{(S,\alpha\circ\sigma_n)_k^{\hat{\mathbb{Z}}}})$, while the maps $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ are the Bost-Connes endomorphisms. Similarly, we obtain commutative diagrams of group homomorphisms

$$\begin{tikzcd}
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \arrow[r, "\chi^{\hat{\mathbb{Z}}}_{(S,\alpha)}"] \arrow[d, "\tilde{\rho}_n"] & \bigoplus_\ell K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_\ell}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \arrow[d, "\tilde{J}_n \otimes \tilde{\rho}_n"] \\
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n,\Phi_n(\alpha))}) \arrow[r, "\chi^{\hat{\mathbb{Z}}}_{(S\times Z_n,\Phi_n(\alpha)),n}"'] & \bigoplus_{n|k} K_0(\mathbb{Q}_{(S\times Z_n,\Phi_n(\alpha))^{\hat{\mathbb{Z}}}_k}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]
\end{tikzcd}$$

where $(\tilde{J}_n)_{\ell,k}$ is non-trivial for $k = \ell n$ and maps $K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_k})$ to $K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_k})^{\oplus n}$ and identifies the latter with $K_0(\mathbb{Q}_{(S\times Z_n,\Phi_n(\alpha))^{\hat{\mathbb{Z}}}_\ell})$. □

**Remark 2.2.12.** *A similar argument can be given using a map obtained by composing the equivariant Euler characteristic considered here with values in $K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$ with equivariant characteristic classes from constructible sheaves to delocalized equivariant homology as in* **MaxSch** *see §2.2.*

## 2.3 From Grothendieck Rings to Spectra

In this section we show that the Bost-Connes structure can be lifted further from the level of the relative Grothendieck ring to the level of spectra, using the assembler category construction of **Zak1** .

The results of this section are a natural continuation of the results in **ManMar2** . The general theme considered there consisted of the following steps:

- The construction of appropriate equivariant Euler characteristic maps from certain $\hat{\mathbb{Z}}$-equivariant Grothendieck rings to the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$.
- These Euler characteristic maps were then used to lift the Bost-Connes operations $\sigma_n$ and $\tilde{\rho}_n$ from $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ to corresponding operations in the equivariant Grothendieck ring.
- The construction of assembler categories with $K_0$ given by the equivariant Grothendieck ring.
- The construction of endofunctors $\sigma_n$ and $\tilde{\rho}_n$ of these assembler categories that induce the Bost-Connes structure in the Grothendieck ring.
- Induced maps of spectra are obtained from these endofunctors through the Gamma-space construction that associated a spectrum to an assembler category.

The construction of Bost-Connes operations $\sigma_n$ and $\tilde{\rho}_n$ on the equivariant Grothendieck rings was generalized in the previous section to the case of relative Grothendieck rings. This section deals with the corresponding generalization of the remaining steps.

We start this section by a brief survey in §2.3 of Zakharevich's formalism of *assemblers* which axiomatizes the "scissors congruence" relations (2.7.1).

A general framework for categorical Bost-Connes systems is introduced in §2.3 and §2.3 in terms of subcategories of the automorphism category (in our examples encoding the $\hat{\mathbb{Z}}$-actions) and endofunctors $\sigma_n$ and $\tilde{\rho}_n$ implementing the Bost-Connes structure.

In §2.3 we construct an assembler category for the equivariant relative Grothendieck ring, and we prove the main result of this section, Theorem 2.3.15, on the lifting of the Bost-Connes structure to this assembler category.

**Assemblers**

Below we will recall the basics of a general formalism for scissors congruence relations applicable in algebraic geometric contexts defined by I. Zakharevich in **Zak1** and **Zak2** . The abstract form of scissors congruences consists of categorical data called *assemblers*, which in turn determine a homotopy-theoretic *spectrum*, whose homotopy groups embody scissors congruence relations. This formalism is applied in **Zak3** in the framework, producing an assembler and a spectrum whose $\pi_0$ recovers the Grothendieck ring of varieties. This is used to obtain a characterization of the kernel of multiplication by the Lefschetz motive, which provides a general explanation for the observations of **Bor14 Mart16** on the fact that the Lefschetz motive is a zero divisor in the Grothendieck ring of varieties.

Consider a (small) category $C$ and an object $X$ in $C$.

**Definition 2.3.1.** *A sieve $\mathcal{S}$ over $X$ in $C$ is a family of morphisms $f_i : X'_i \to X$ (also called "objects over $X$") satisfying the following conditions:*

*a) Any isomorphism with target $X$ belongs to $\mathcal{S}$ (as a family with one element).*

*b) If a morphism $X' \to X$ belongs to $\mathcal{S}$, then its precomposition with any other morphism in $C$ with target $X'$*

$$X'' \to X' \to X$$

*also belongs to $\mathcal{S}$.*

It follows that composition of any two morphisms in $\mathcal{S}$ composable in $C$ itself belongs to $\mathcal{S}$ so that any sieve is a category in its own right.

**Definition 2.3.2.** *A Grothendieck topology on a category $C$ consists of the assignment of a collection of sieves $\mathcal{J}(X)$ given for all objects $X$ in $C$, with the following properties:*

- *a) The total overcategory $C/X$ of morphisms with target $X$ is a member of the collection $\mathcal{J}(X)$.*
- *b) The pullback of any sieve in $\mathcal{J}(X)$ under a morphism $f : Y \to X$ exists and is a sieve in $\mathcal{J}(Y)$. Here pullback of a sieve is defined as the family of pullbacks of its objects, $X' \to X$, whereas pullback of such an object w.r.t. $Y \to X$ is defined as $pr_Y : Y \times_X X' \to Y$.*
- *c) Given $C' \in \mathcal{J}(X)$ and a sieve $\mathcal{T}$ in $C/X$, if for every $f : Y \to X$ in $C'$ the pullback $f^*\mathcal{T}$ is in $\mathcal{J}(Y)$ then $\mathcal{T}$ is in $\mathcal{J}(X)$.*

For more details, see **KSch06** Chapters 16 and 17, or **HuM-S17** pp. 20–22.

Let $C$ be a category with a Grothendieck topology. Zakharevich's notion of an assembler category is then defined as follows.

**Definition 2.3.3.** *A collection of morphisms $\{f_i : X_i \to X\}_{i\in I}$ in $C$ is a covering family if the full subcategory of $C/X$ that contains all the morphisms of $C$ that factor through the $f_i$,*

$$\{g : Y \to X \mid \exists i \in I \;\; h : Y \to X_i \;\; \text{such that } f_i \circ h = g\},$$

*belongs to the sieve collection $\mathcal{J}(X)$.*

In a category $C$ with an initial object $\emptyset$ two morphisms $f : Y \to X$ and $g : W \to X$ are called *disjoint* if the pullback $Y \times_X W$ exists and is equal to $\emptyset$. A collection $\{f_i : X_i \to X\}_{i\in I}$ in $C$ is disjoint if $f_i$ and $f_j$ are disjoint for all $i \neq j \in I$.

**Definition 2.3.4.** *An assembler category $\mathcal{C}$ is a small category endowed with a Grothendieck topology, which has an initial object $\emptyset$ (with the empty family as covering family), and where all morphisms are monomorphisms, with the property that any two finite disjoint covering families of $X$ in $\mathcal{C}$ have a common refinement that is also a finite disjoint covering family.*

A morphism of assemblers is a functor $F : \mathcal{C} \to \mathcal{C}'$ that is continuous for the Grothendieck topologies and preserves the initial object and the disjointness property, that is, if two morphisms are disjoint in $\mathcal{C}$ their images are disjoint in $\mathcal{C}'$.

For $X$ a finite set, the coproduct of assemblers $\bigvee_{x\in X} \mathcal{C}_x$ is a category whose objects are the initial object $\emptyset$ and all the non-initial objects of the assemblers $\mathcal{C}_x$. Morphisms of non-initial objects are induced by those of $\mathcal{C}_x$.

Consider a pair $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ is an assembler category, and $\mathcal{D}$ is a sieve in $\mathcal{C}$.

One has then an associated assembler $\mathcal{C} \smallsetminus \mathcal{D}$ defined as the full subcategory of $\mathcal{C}$ containing all the objects that are not initial objects of $\mathcal{D}$. The assembler structure on $\mathcal{C} \smallsetminus \mathcal{D}$ is determined by taking as covering families in $\mathcal{C} \smallsetminus \mathcal{D}$ those collections $\{f_i : X_i \to X\}_{i\in I}$ with $X_i, X$ objects in $\mathcal{C} \smallsetminus \mathcal{D}$ that can be completed to a covering family in $\mathcal{C}$, namely such that there exists $\{f_j : X_j \to X\}_{j\in J}$ with $X_j$ in $\mathcal{D}$ such that $\{f_i : X_i \to X\}_{i\in I} \cup \{f_j : X_j \to X\}_{j\in J}$ is a covering family in $\mathcal{C}$.

Moreover, there is a morphism of assemblers $\mathcal{C} \to \mathcal{C} \smallsetminus \mathcal{D}$ that maps objects of $\mathcal{D}$ to $\emptyset$ and objects of $\mathcal{C} \smallsetminus \mathcal{D}$ to themselves and morphisms with source in $\mathcal{C} \smallsetminus \mathcal{D}$ to themselves and morphisms with source in $\mathcal{D}$ to the unique morphism to the same target with source $\emptyset$. The data $(\mathcal{C}, \mathcal{D}, \mathcal{C} \smallsetminus \mathcal{D})$ are called the abstract scissors congruences.

The construction of $\Gamma$-spaces, which we review more in detail in §2.3, then provides the homotopy theoretic spectra associated to assembler categories as in **Zak1** . This construction of assembler categories and spectra provides the formalism we use here and in the previous paper **ManMar2** to lift Bost-Connes type algebras to the level of Grothendieck rings and spectra.

### From categories to $\Gamma$-spaces and spectra

The Segal construction **Segal** associates a $\Gamma$-space (hence a spectrum) to a category $\mathcal{C}$ with a zero object and a categorical sum. Let $\Gamma^0$ be the category of finite pointed sets with objects $\underline{n} = \{0, 1, \ldots, n\}$ and morphisms $f : \underline{n} \to \underline{m}$ the functions with

$f(0) = 0$. Let $\Delta_*$ denote the category of pointed simplicial sets. The construction can be generalized to symmetric monoidal categories, **Thoma** . The associated $\Gamma$-space $F_C : \Gamma^0 \to \Delta_*$ is constructed as follows. First assign to a finite pointed set $X$ the category $P(X)$ with objects all the pointed subsets of $X$ with morphisms given by inclusions. A functor $\Phi_X : P(X) \to C$ is summing if it maps $\emptyset \in P(X)$ to the zero object of $C$ and given $S, S' \in P(X)$ with $S \cap S' = \{\star\}$ the base point of $X$, the morphism $\Phi_X(S) \oplus \Phi_X(S') \to \Phi_X(S \cup S')$ is an isomorphism. Let $\Sigma_C(X)$ be the category whose objects are the summing functors $\Phi_X$ with morphisms the natural transformations that are isomorphisms on all objects of $P(X)$. Setting

$$\Sigma_C(f)(\Phi_X)(S) = \Phi_X(f^{-1}(S)),$$

for a morphisms $f : X \to Y$ of pointed sets and $S \in P(Y)$ gives a functor $\Sigma_C : \Gamma^0 \to \mathcal{C}at$ to the category of small categories. Composing with the nerve $\mathcal{N}$ gives a functor

$$F_C = \mathcal{N} \circ \Sigma_C : \Gamma^0 \to \Delta_*$$

which is the $\Gamma$-space associated to the category $C$. The functor $F_C : \Gamma^0 \to \Delta_*$ obtained in this way is extended to an endofunctor $F_C : \Delta_* \to \Delta_*$ via the coend

$$F_C(K) = \int^{\underline{n}} K^n \wedge F_C(\underline{n}).$$

One obtains the spectrum $\mathbb{X} = F_C(\mathbb{S})$ associated to the $\Gamma$-space $F_C$ by setting $\mathbb{X}_n = F_C(S^n)$ with maps $S^1 \wedge F_C(S^n) \to F_C(S^{n+1})$. The construction is functorial in $C$, with respect to functors preserving sums and the zero object.

When $C$ is the category of finite sets, $F_C(\mathbb{S})$ is the sphere spectrum $\mathbb{S}$, and when $C = \mathcal{P}_R$ is the category of finite projective modules over a commutative ring $R$, the spectrum $F_{\mathcal{P}_R}(\mathbb{S}) = K(R)$ is the $K$-theory spectrum of $R$.

The Segal construction determines a functor from the category of small symmetric monoidal categories to the category of $-1$-connected spectra. It is shown in **Thoma** that this functor determines an equivalence of categories between the stable homotopy category of $-1$-connected spectra and a localization of the category of small symmetric monoidal categories, obtained by inverting morphisms sent to weak homotopy equivalences by the functor.

Given an assembler category $C$, one considers a category $\mathcal{W}(C)$ with objects $\{A_i\}_{i \in I}$ given by collections of non-initial objects $A_i$ in $C$ indexed by finite sets and morphisms $\phi : \{A_i\}_{i \in I} \to \{B_j\}_{j \in J}$ consisting of a map of finite sets $f : I \to J$ and morphisms $\phi_i : A_i \to B_{f(i)}$ that form disjoint covering families $\{\phi_i \,|\, i \in f^{-1}(j)\}$, for all $j \in J$. One then obtains a $\Gamma$-space as the functor that assigns to a finite pointed set $(X, x_0)$ the simplicial set $\mathcal{N}\mathcal{W}(X \wedge C)$, the nerve of the category $\mathcal{W}(X \wedge C)$ where $X \wedge C$ is the assembler $X \wedge C = \bigvee_{x \in X \smallsetminus \{x_0\}} C$. The spectrum associated to the assembler $C$ is the spectrum defined by this $\Gamma$ space, namely $X_n = \mathcal{N}\mathcal{W}(S^n \wedge C)$.

For another occurrence of $\Gamma$-spaces in the context of $\mathbb{F}_1$-geometry, see **CC16** .

### Automorphism category and enhanced assemblers

We describe in this and the next subsection a general formalism of "enhanced assemblers" underlying all the explicit cases of Bost-Connes structures in Grothendieck rings discussed in **ManMar2** and in some of the later sections of this paper.

We first recall the definition of the automorphism category.

**Definition 2.3.5.** *The automorphism category $Aut(C)$ of a category $C$ is given by:*

*(i) Objects of $Aut(C)$ are pairs $\hat{X} = (X, v_X)$ where $X \in Obj(C)$ and $v_X : X \to X$ is an automorphism of $X$.*

*(ii) Morphisms $\hat{f} : (X, v_X) \to (Y, v_Y)$ in $Aut(C)$ are morphisms $f : X \to Y$ such that $f \circ v_X = v_Y \circ f : X \to Y$ in $C$.*

*(iii) The forgetful functor sends $\hat{X}$ to $X$ and $\hat{f}$ to $f$.*

We use here a standard categorical notation according to which, say, $f \circ v_X$ is the precomposition of $f$ with $v_X$.

More generally, we will consider subcategories $\hat{C}$ of the automorphism category A$ut(C)$ that only use objects of a particular subcategory of $C$ rather than the full $C$ and only certain automorphisms. This will be stated clearly in the specific cases we discuss later.

Thus, we can make the following general definition. In the following we will be especially interested in the case where the chosen subcategory is determined by a group action, see Remark 2.3.7.

**Definition 2.3.6.** *Let $C$ be a category. We will call here an enhancement of $C$ a pair consisting of a choice of a subcategory $\hat{C}$ of the automorphism category $Aut(C)$ and the forgetful functor $\hat{C} \to C$, where objects $(X, v_X)$ of $\hat{C}$ have automorphisms $v_X : X \to X$ of finite order, .*

The main idea here is that a subcategory category $\hat{C}$ of the automorphism category of $C$ is where the endofunctors defining the lifts of the Bost-Connes structure are defined, as we make more precise in Definitions 2.3.9 and 2.3.11 .

**Remark 2.3.7.** *In the cases considered in* **ManMar2** *and in this paper, the subcategory of $\hat{C}$ of $Aut(C)$ is usually determined by a finite group action, so that elements of $\hat{C}$ are of the form $(X, \alpha_X(g))$ with $\alpha_X : G \times X \to X$ the group action. However, one expects other interesting examples that are not necessarily given by group actions, hence it is worth considering this more general formulation.*

**Remark 2.3.8.** *Assume that $C$ is endowed with a structure of assembler. Then a series of constructions presented in §§ 3 and 4 of* **ManMar2** *and in §§ 2.2–2.6 of this paper, and restricted there to various categories of schemes, show in fact how this structure of assembler can be lifted from $C$ to $\hat{C}$.*

In particular the Bost-Connes type structures we are investigating can be formulated broadly in this setting of enhanced assemblers as follows.

**Bost-Connes systems on categories**

Let $\hat{C}$ be an enhancement of a category $C$, in the sense of Definition 2.3.6.

**Definition 2.3.9.** *We assume here that $C$ is an additive (symmetric) monoidal category and that the enhancement $\hat{C}$ is compatible with this structure. A Bost-Connes system in an enhancement $\hat{C}$ of $C$ consists of two families of endofunctors $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{\tilde{\rho}_n\}_{n\in\mathbb{N}}$ of $\hat{C}$ with the following properties:*

1. *The functors $\sigma_n$ are compatible with both the additive and the (symmetric) monoidal structure, while the functors $\tilde{\rho}_n$ are functors of additive categories.*
2. *For all $n, m \in \mathbb{N}$ these endofunctors satisfy*
$$\sigma_{nm} = \sigma_n \circ \sigma_m, \quad \tilde{\rho}_{nm} = \tilde{\rho}_n \circ \tilde{\rho}_m.$$
3. *The compositions satisfy*
$$\sigma_n \circ \tilde{\rho}_n(X, v_X) = (X, v_X)^{\oplus n} \quad \text{and} \quad \tilde{\rho}_n \circ \sigma_n(X, v_X) = (X, v_X) \otimes (Z_n, v_n), \tag{2.3.1}$$
*for an object $(Z_n, v_n)$ in $\hat{C}$ that depends on $n$ but not on $(X, v_X)$, and similarly on morphisms.*

*Here $\oplus$ refers to the additive structure of $C$ and $\otimes$ to the monoidal structure.*

**Remark 2.3.10.** *In all the explicit cases considered in* **ManMar2** *and in this paper, the endofunctors $\sigma_n$ and $\tilde{\rho}_n$ of Definition 2.3.9 have the form*

$$\sigma_n(X, v_X) = (X, v_X \circ \sigma_n) \quad \textit{and} \quad \tilde{\rho}_n(X, v_X) = (X \times Z_n, \Phi_n(v_X)),$$

*where the endomorphism $v_X$ is the action of a generator of some finite cyclic group $\mathbb{Z}/N\mathbb{Z}$ quotient of $\hat{\mathbb{Z}}$ and the action satisfies $v_X \circ \sigma_n(\zeta, x) = v_X(\sigma_n(\zeta), x)$, where $\sigma_n(\zeta) = \zeta^n$ is the Bost-Connes map of* (2.2.4)*, while the action $\Phi_n(v_X)$ on $X \times Z_n$ is a geometric form of the Verschiebung, as will be discussed more explicitly in §2.2. The object $(Z_n, v_n)$ in Definition 2.3.9 plays the role of the element $n\pi_n$ in the integral Bost-Connes algebra and the relations* (2.3.1) *play the role of the relations* (2.2.6).

This definition covers the main examples considered in §§ 3 and 4 of **ManMar2** obtained using the assembler categories associated to the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ of varieties with a good $\hat{\mathbb{Z}}$-action factoring through some finite cyclic quotient and to the equivariant version $\mathrm{B}urn^{\hat{\mathbb{Z}}}$ of the Kontsevich–Tschinkel Burnside ring. This same definition also accounts for the construction we will discuss in §2.4 of this paper, based on assembler categories associated to torified varieties (see Remark 2.4.6).

The more general formulation given in Definition 2.3.9 is motivated by the fact that one expects other significant examples of categorical Bost-Connes structures where the choice of the subcategory $\hat{C}$ of the automorphism category $\mathrm{A}ut(C)$ is not determined by the action of a cyclic group as in the cases discussed here. Such more general classes of categorical Bost-Connes systems are not discussed in the present paper, but they are a motivation for future work, for which we just set the general framework in this section.

A generalization of Definition 2.3.9 is needed when considering relative cases, in particular the lift to assemblers of the construction presented in §2.2 for relative equivariant Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. The reason why we need the following modification of Definition 2.3.9 is the fact that, in the relative setting, the base scheme $S$ itself has its enhancement structure (the group action, in the specific examples) modified by the endofunctors implementing the Bost-Connes structure and this needs to be taken into account. We will see this additional structure more explicitly applied in §2.3, in the specific case where the automorphisms are determined by a group action (see Remark 2.3.16).

**Definition 2.3.11.** *Let $\hat{\mathcal{I}}$ be an enhancement of an additive (symmetric) monoidal category $\mathcal{I}$ as above, endowed with a Bost-Connes system given by endofunctors $\{\sigma_n^{\mathcal{I}}\}$ and $\{\tilde{\rho}_n^{\mathcal{I}}\}$ of $\hat{\mathcal{I}}$ as in Definition 2.3.9, with $\alpha_n$ the object in $\hat{\mathcal{I}}$ with $\tilde{\rho}_n \circ \sigma_n(\alpha) = \alpha \otimes \alpha_n$. Let $\{\hat{C}_\alpha\}_{\alpha\in\hat{\mathcal{I}}}$ be a collection of enhancements of additive (symmetric) monoidal categories $C_\alpha$, indexed by the objects of the auxiliary category $\hat{\mathcal{I}}$, endowed with functors $f_n : \hat{C}_{\alpha^{\oplus n}} \to \hat{C}_\alpha$ and $h_n : \hat{C}_{\alpha\times\beta} \to \hat{C}_\alpha$. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{\tilde{\rho}_n\}_{n\in\mathbb{N}}$ be two collections of functors*

$$\sigma_n : \hat{C}_\alpha \to \hat{C}_{\sigma_n^{\mathcal{I}}(\alpha)} \quad \text{and} \quad \tilde{\rho}_n : \hat{C}_\alpha \to \hat{C}_{\tilde{\rho}_n^{\mathcal{I}}(\alpha)}$$

*satisfying the properties:*

1. *The functors $\sigma_n$ are compatible with both the additive and the (symmetric) monoidal structure, while the functors $\tilde{\rho}_n$ are functors of additive categories.*
2. *For all $n, m \in \mathbb{N}$ these functors satisfy*

$$\sigma_{nm} = \sigma_n \circ \sigma_m, \quad \tilde{\rho}_{nm} = \tilde{\rho}_n \circ \tilde{\rho}_m.$$

3. *The compositions*

$$\sigma_n \circ \tilde{\rho}_n : \hat{C}_\alpha \to \hat{C}_{\alpha^{\oplus n}} \quad \text{and} \quad \tilde{\rho}_n \circ \sigma_n : \hat{C}_\alpha \to \hat{C}_{\alpha\otimes\alpha_n}$$

*satisfy*

$$\begin{aligned} f_n \circ \sigma_n \circ \tilde{\rho}_n(X, \nu_X)_\alpha &= (X, \nu_X)_\alpha^{\oplus n} \quad \text{and} \\ h_n \circ \tilde{\rho}_n \circ \sigma_n(X, \nu_X)_\alpha &= (X, \nu_X)_\alpha \otimes (Z_n, \nu_n)_\alpha, \end{aligned} \tag{2.3.2}$$

*for an object $(Z_n, \nu_n)_\alpha$ in $\hat{C}_\alpha$ that depends on $n$ and $\alpha$, but not on $(X, \nu_X)$.*

We will first focus on the case of assembler categories, as those were at the basis of our constructions of Bost-Connes systems in **ManMar2** , but we will also consider in §2.7 a different categorical setting that will allow us to identify analogous structures at a motivic level, following the formalism of geometric diagrams and Nori motives.

**Assemblers for the relative Grothendieck ring**

We consider the relative Grothendieck ring $K_0(\mathcal{V}_S)$ of varieties over a base variety $S$ over a field $\mathbb{K}$, as in Definition 2.2.1.

An assembler $\mathcal{C}_S$ such that the associated spectrum $K(\mathcal{C}_S)$ has $K_0(\mathcal{C}_S) = \pi_0 K(\mathcal{C}_S)$ given by the relative Grothendieck ring $K_0(\mathcal{V}_S)$ can be obtained as a slight modification of the construction given in **Zak3** for the ordinary Grothendieck ring $K_0(\mathcal{V}_\mathbb{K})$.

**Definition 2.3.12.** *The assembler $C_S$ for the relative Grothendieck ring $K_0(\mathcal{V}_S)$ has objects $f : X \to S$ that are varieties over $S$ and morphisms that are locally closed embeddings of varieties over $S$.*

**Lemma 2.3.13.** *The category $C_S$ of Definition 2.3.12 is indeed as assembler, with the Grothendieck topology on $C_S$ is generated by the covering families*

$$\{Y \hookrightarrow X, X \smallsetminus Y \hookrightarrow X\}$$

*with compatible maps* (2.2.8)

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr, "f|_Y"'] & X \arrow[d, "f"] & X \smallsetminus Y \arrow[l, hook'] \arrow[dl, "f|_{X \smallsetminus Y}"] \\ & S & \end{tikzcd} \tag{2.3.3}$$

*Proof.* The argument is the same as in **Zak1 Zak3** and in **ManMar2** . In this setting finite disjoint covering families are maps

$$\begin{tikzcd} X_i \arrow[r, hook] \arrow[dr, "f_i"'] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$

where $X_i = Y_i \smallsetminus Y_{i-1}$ with commutative diagrams

$$\begin{tikzcd} Y_0 \arrow[r, hook] \arrow[drrr, "f_0"'] & Y_1 \arrow[r, hook] \arrow[drr, "f_1"] & \cdots \arrow[r, hook] & Y_n = X \arrow[d, "f"] \\ & & & S \end{tikzcd}$$

The category has pullbacks, hence as shown in **Zak1** (Remark after Definition 2.4) this suffices to obtain that any two finite disjoint covering families have a common refinement. Morphisms are embeddings compatible with the structure maps as in (2.3.3) hence in particular monomorphisms. Theorem 2.3 of **Zak1** then shows that the spectrum $K(C_S)$ associated to this assembler category has $\pi_0 K(C_S) = K_0(\mathcal{V}_S)$. ◻

In a similar way we obtain an assembler category and spectrum for the equivariant version $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. The argument is as in the previous case and in Lemma 4.5.1 of **ManMar2** using the inclusion-exclusion relations (2.2.9).

**Corollary 2.3.14.** *An assembler category $C^{\hat{\mathbb{Z}}}_{(S,\alpha)}$ for $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$ is constructed as in Lemma 2.3.13 with objects the $\hat{\mathbb{Z}}$-equivariant $f : X \to S$, morphisms given by $\hat{\mathbb{Z}}$-equivariant locally closed embeddings of varieties over $S$ and with Grothendieck topology generated by the covering families given by $\hat{\mathbb{Z}}$-equivariant maps as in* (2.2.8) *and* (2.2.9).

As in Proposition 4.2 of **ManMar2** we show that the lifting of the integral Bost-Connes algebra obtained in Proposition 2.2.6 and Theorem 2.2.11 further lifts to functors of the associated assembler categories, with the $\sigma_n$ compatible with the monoidal structure, but not the $\tilde{\rho}_n$.

**Theorem 2.3.15.** *The maps $\sigma_n : (f : (X, \alpha_X) \to (S, \alpha)) \mapsto (f : (X, \alpha_X \circ \sigma_n) \to (S, \alpha \circ \sigma_n))$ and $\tilde{\rho}_n : (f : (X, \alpha_X) \to (S, \alpha)) \mapsto (f \times id : (X \times Z_n, \Phi_n(\alpha_X)) \to (S \times Z_n, \Phi_n(\alpha)))$ define functors of the assembler categories $\sigma_n : C^{\hat{\mathbb{Z}}}_{(S,\alpha)} \to C^{\hat{\mathbb{Z}}}_{(S,\alpha\circ\sigma_n)}$ and $\tilde{\rho}_n : C^{\hat{\mathbb{Z}}}_{(S,\alpha)} \to C^{\hat{\mathbb{Z}}}_{(S\times Z_n,\Phi_n(\alpha))}$. The functors $\sigma_n$ are compatible with the monoidal structure.*

*Proof.* The functors $\sigma_n$ defined as above on objects are compatibly defined on morphisms by assigning to a locally closed embedding

$$\sigma_n : \quad \begin{tikzcd} (Y,\alpha_Y) \arrow[r, "j"] \arrow[dr, "f_Y"'] & (X,\alpha_X) \arrow[d, "f_X"] \\ & (S,\alpha) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} (Y,\alpha_Y\circ\sigma_n) \arrow[r, "j"] \arrow[dr, "f_Y"'] & (X,\alpha_X\circ\sigma_n) \arrow[d, "f_X"] \\ & (S,\alpha\circ\sigma_n) \end{tikzcd}$$

Similarly, we define the $\tilde{\rho}_n$ on morphisms by

$$\tilde{\rho}_n : \quad \begin{tikzcd} (Y,\alpha_Y) \arrow[r, "j"] \arrow[dr, "f_Y"'] & (X,\alpha_X) \arrow[d, "f_X"] \\ & (S,\alpha) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} (Y\times Z_n,\Phi_n(\alpha_Y)) \arrow[r, "j"] \arrow[dr, "f_Y"'] & (X\times Z_n,\Phi_n(\alpha_X)) \arrow[d, "f_X"] \\ & (S\times Z_n,\Phi_n(\alpha)) \end{tikzcd}$$

The functors $\sigma_n$ are compatible with the monoidal structure since $\sigma_n(X, \alpha_X) \times \sigma_n(X', \alpha_{X'}) = (X \times X', (\alpha \times \alpha') \circ \Delta \circ \sigma_n) = \sigma_n((X, \alpha_X) \times (X', \alpha_{X'}))$. □

The functor of assembler categories determines an induced map of spectra and in turn an induced map of homotopy groups. By construction the induced maps on the $\pi_0$ homotopy agree with the maps (2.2.16) and (2.2.17) of Proposition 2.2.6.

**Remark 2.3.16.** *We can associate to the assembler category $\mathcal{C}^{\hat{\mathbb{Z}}}_{(S,\alpha)}$ of Corollary 2.3.14 with the endofunctors $\sigma_n$ and $\tilde{\rho}_n$ a categorical Bost-Connes structure in the sence of Definition 2.3.11, where the objects are $f : X \to S$ as above with the automorphisms given by elements $g \in \hat{\mathbb{Z}}$ acting on $f : X \to S$ through the action by $\alpha_X(g)$ on $X$ and by $\alpha_S(g)$ on $S$, intertwined by the equivariant map $f$.*

## 2.4 Torifications, $\mathbb{F}_1$-points, zeta functions, and spectra

In this section we relate the point of view developed in **ManMar2** with lifts of the Bost-Connes system to Grothendieck rings and spectra, to the approach to $\mathbb{F}_1$-geometry based on torifications. This was first introduced in **LoLo** . Weaker forms of torification were also considered in **ManMar** which allow for the development of a form of $\mathbb{F}_1$-geometry suitable for the treatment of certain classical moduli spaces.

The approach we follow here, in order the relate the case of torified geometry with the Bost-Connes systems on Grothendieck rings, assemblers, and spectra discussed in **ManMar2** is based on the following simple setting. Instead of working with the equivariant Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ and $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, where one assumes the varieties have a good effectively finite $\hat{\mathbb{Z}}$-action, we consider here a variant that connects to the torifications point of view on $\mathbb{F}_1$-geometry of **LoLo** . We replace varieties with $\mathbb{Z}/N\mathbb{Z}$-effectively finite $\hat{\mathbb{Z}}$-actions with varieties with a $\mathbb{Q}/\mathbb{Z}$-action induced by a torification, where the group schemes $\mathfrak{m}_n$ of $n$-th roots of unity, given by the kernels

$$1 \to \mathfrak{m}_n \to \mathbb{G}_m \xrightarrow{\lambda \mapsto \lambda^n} \mathbb{G}_m \to 1$$

determine a diagonal embedding in each torus and an action by multiplication. This is a very restrictive class of varieties, because the existence of a torification on a variety implies that the Grothendieck class is a sum of classes of tori with non-negative coefficients. The resulting construction will be more restrictive than the one considered in **ManMar2** . We will see, however, that one can still see in this context several interesting phenomena, especially in connection with the “dynamical" approach to $\mathbb{F}_1$-geometry proposed in **ManMar2** .

### Torifications

A torification of an algebraic variety $X$ defined over $\mathbb{Z}$ is a decomposition $X = \sqcup_{i \in \mathcal{I}} T_i$ into algebraic tori $T_i = \mathbb{G}_m^{d_i}$. Weaker to stronger forms of torification **ManMar** include

1. *torification of the Grothendieck class*: $[X] = \sum_{i\in\mathcal{I}}(\mathbb{L}-1)^{d_i}$ with $\mathbb{L}$ the Lefschetz motive;

2. *geometric torification*: $X = \sqcup_{i\in\mathcal{I}} T_i$ with $T_i = \mathbb{G}_m^{d_i}$;

3. *affine torification*: the existence of an affine covering compatible with the geometric torification, **LoLo** ;

4. *regular torification*: the closure of each torus in the geometric torification is also a union of tori of the torification, **LoLo** .

Similarly, there are different possibilities when one considers morphisms of torified varieties, see **ManMar** . In view of describing associated Grothendieck rings, we review the different notions of morphisms. The Grothendieck classes are then defined with respect to the corresponding type of isomorphism.

A torified morphism of geometric torifications in the sense of **LoLo** between torified varieties $f : (X,T) \to (Y,T')$ is a morphism $f : X \to Y$ of varieties together with a map $h : I \to J$ of the indexing sets of the torifications $X = \sqcup_{i\in I} T_i$ and $Y = \sqcup_{j\in J} T'_j$ such that the restriction of $f$ to tori $T_i$ is a morphism of algebraic groups $f_i : T_i \to T'_{h(i)}$. There are then three different classes of morphisms of torified varieties that were introduced in **ManMar** : strong, ordinary, and weak morphisms. To describe them, one first defines strong, ordinary, and weak equivalences of torifications, and one then uses these to define the respective class of morphisms.

Let $T$ and $T'$ be two geometric torifications of a variety $X$.

1. The torifications $(X,T)$ and $(X,T')$ are *strongly equivalent* if the identity map on $X$ is a torified morphism as above.

2. The torifications $(X,T)$ and $(X,T')$ are *ordinarily equivalent* if there exists an automorphism $\phi : X \to X$ that is a torified morphism.

3. The torifications $(X,T)$ and $(X,T')$ are *weakly equivalent* if $X$ has two decompositions $X = \cup_i X_i$ and $X = \cup_j X'_j$ into a disjoint union of subvarieties, compatible with the torifications, such that there are isomorphisms of varieties $\phi_i : X_i \to X'_{j(i)}$ that are torified.

In the weak case, a “decomposition compatible with torifications" means that the intersections $T_i \cap X_j$ of the tori of $T$ with the pieces of the decomposition (when non-empty) are tori of the torification of $X_j$, and similarly for $T'_i \cap X'_j$. In general weakly equivalent torification are not ordinarily equivalent because the maps $\phi_i$ need not glue together to define a single map $\phi$ on all of $X$.

We then have the following classes of morphisms of torified varieties from **ManMar**

1. *Strong morphisms*: these are torified morphisms in the sense of **LoLo** namely morphisms that restrict to morphisms of tori of the respective torifications.

2. *Ordinary morphisms*: an ordinary morphism of torified varieties $(X,T)$ and $(Y,T')$ is a morphism $f : X \to Y$ such that becomes a torified morphism after composing with strong isomorphisms, that is, $\phi_Y \circ f \circ \phi_X : (X,T) \to (Y,T')$ is a strong morphism of torified varieties, for some isomorphisms $\phi_X : X \to X$ and $\phi_Y : Y \to Y$. In other words, if we denote by $T_\phi$ and $T'_\phi$ the torifications such that $\phi_X : (X,T) \to (X,T_\phi)$ and $\phi_Y : (Y,T'_\phi) \to (Y,T')$ are torified, then $f : (X,T_\phi) \to (Y,T'_\phi)$ is torified.

3. *Weak morphisms*: the torified varieties $(X,T)$ and $(Y,T')$ admit decompositions $X = \sqcup_i X_i$ and $Y = \sqcup_j Y_i$, compatible with the torifications, such that there exist ordinary morphisms $f_i : (X_i, T_i) \to (Y_{f(i)}, T'_{f(i)})$ of these subvarieties.

Note that the strong isomorphisms $\phi_X : (X,T) \to (X,T_\phi)$ and $\phi_Y : (Y,T'_\phi) \to (Y,T')$ used in the definition of ordinary morphisms are ordinary equivalences of the torifications $T$ and $T_\phi$, respectively $T'$ and $T'_\phi$.

Given these notions of morphisms, we can correspondingly construct Grothendieck rings $K_0(\mathcal{T})^s$, $K_0(\mathcal{T})^o$, and $K_0(\mathcal{T})^w$ in the following way.

As an abelian group $K_0(\mathcal{T})^s$ is generated by isomorphism classes $[X,T]_s$ of pairs of a torifiable variety $X$ and a torification $T$ modulo strong isomorphisms, with the inclusion-exclusion relations $[X,T]_s = [Y,T_Y]_s + [X \smallsetminus Y, T_{X\smallsetminus Y}]_s$ whenever $(Y,T_Y) \hookrightarrow (X,T)$ is a strong morphism (that is, the inclusion of $Y$ in $X$ is compatible with the torification: $Y$ is a union of tori of the torification of $X$) and $(Y,T_Y)$ is a *complemented subvariety* in $(X,T)$, which means that the complement $X \smallsetminus Y$ is also a union of tori of the torification so that the inclusion of $(X \smallsetminus Y, T_{X\smallsetminus Y})$ in $(X,T)$ is also a strong morphism. This complemented condition is very strong. Indeed, one can see that, for example, there are in general very few complemented points in a torified variety. The product operation is $[X,T]_s \cdot [Y,T']_s = [X \times Y, T \times T']_s$ with the torification $T \times T'$ given by the product tori $T_{ij} = T_i \times T'_j = \mathbb{G}_m^{d_i+d_j}$.

The abelian group $K_0(\mathcal{T})^o$ is generated by isomorphism classes $[X]_o$ varieties that admit a torification with respect to ordinary isomorphisms, with the inclusion-exclusion relations $[X]_o = [Y]_o + [X \setminus Y]_o$ whenever the inclusions $Y \hookrightarrow X$ and $X \setminus Y \hookrightarrow X$ are ordinary morphisms. The product is the class of the Cartesian product $[X]_o \cdot [Y]_o = [X \times Y]_o$.

The abelian group $K_0(\mathcal{T})^w$ is generated by the isomorphism classes $[X]_w$ of torifiable varieties $X$ with respect to weak morphisms, with the inclusion-exclusion relations $[X]_w = [Y]_w + [X \setminus Y]_w$ whenever the inclusions $Y \hookrightarrow X$ and $X \setminus Y \hookrightarrow X$ are weak morphisms. The product structure is again given by $[X]_w \cdot [Y]_w = [X \times Y]_w$.

The reader can consult the explicit examples given in **ManMar** to see how these notions (and the resulting Grothendieck rings) can be different. For example, as mentioned in §2.2 of **ManMar** , consider the variety $X = \mathbb{P}^1 \times \mathbb{P}^1$ and consider on it two torifications $T$ and $T'$, where $T$ is the standard torification given by the decomposition of each $\mathbb{P}^1$ into cells $\mathbb{A}^0 \cup \mathbb{A}^1$, with the cell $\mathbb{A}^1$ torified as $\mathbb{A}^0 \cup \mathbb{G}_m$, while $T'$ is the torification where in the big cell $\mathbb{A}^2$ of $\mathbb{P}^1 \times \mathbb{P}^1$ we take a torification of the diagonal $\mathbb{A}^1$ and a torification of the complement of the diagonal, and we use the same torification of the lower dimensional cells as in $T$. These two torifications are related by a weak isomorphism, hence the elements $(\mathbb{P}^1 \times \mathbb{P}^1, T)$ and $(\mathbb{P}^1 \times \mathbb{P}^1, T')$ define the same class in $K_0(\mathcal{T})^w$, but they are not related by an ordinary isomorphism so they define different classes in $K_0(\mathcal{T})^o$.

Note however that, in all these cases, the Grothendieck classes $[X]_a$ with $a = s, o, w$ have the form $[X]_a = \sum_{n \geq 0} a_n \mathbb{T}^n$ with $a_n \in \mathbb{Z}_+$ and $\mathbb{T}^n = [\mathbb{G}_m^n]$.

In the following, whenever we simply write $a = s, o, w$ without specifying one of the three choices of morphisms, it means that the stated property holds for all of these choices.

**Relative case**

In a similar way, we can construct relative Grothendieck rings $K_S(\mathcal{T})^a$ with $a = s, o, w$ where in the strong case $S = (S, T_S)$ is a choice of a variety with an assigned torification, with $K_S(\mathcal{T})^s$ generated as an abelian group by isomorphisms classes $[f : (X, T) \to (S, T_S)]$ where $f$ is a strong morphism of torified varieties and the isomorphism class is taken with respect to strong isomorphisms $\phi, \phi_S$ such that the diagram commutes

$$\begin{array}{ccc} (X,T) & \xrightarrow{\phi} & (X',T') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (S,T_S) & \xrightarrow{\phi_S} & (S,T_S) \end{array}$$

with inclusion-exclusion relations

$$[f:(X,T)\to(S,T_S)] =$$

$$[f|_{(Y,T_Y)}:(Y,T_Y)\to(S,T_S)] + [f|_{(X\smallsetminus Y,T_{X\smallsetminus Y})}:(X\smallsetminus Y,T_{X\smallsetminus Y})\to(S,T_S)]$$

where $\iota_Y : (Y,T_Y) \hookrightarrow (X,T)$ is a strong morphism and $(Y,T_Y)$ is complemented with $\iota_{X\smallsetminus Y} : (X\smallsetminus Y, T_{X\smallsetminus Y}) \hookrightarrow (X,T)$ also a strong morphism, and both these inclusions are compatible with the map $f:(X,T)\to(S,T_S)$, so that $f_Y = f\circ\iota_Y$ and $f|_{(X\smallsetminus Y,T_{X\smallsetminus Y})} = f\circ\iota_{X\smallsetminus Y}$ are strong morphisms. The construction for ordinary and weak morphism is similar, with the appropriate changes in the definition.

**Group actions**

In order to operate on Grothendieck classes with Bost-Connes type endomorphisms, we introduce appropriate group actions.

Torified varieties carry natural $\mathbb{Q}/\mathbb{Z}$ actions, since the roots of unity embed diagonally in each torus of the torification and act on it by multiplication. However, we will also be interested in considering good effectively finite $\hat{\mathbb{Z}}$-actions, in the sense already discussed in **ManMar2** , that is, actions of $\hat{\mathbb{Z}}$ as in Definition 2.2.3.

**Remark 2.4.1.** *The main reason for working with $\hat{\mathbb{Z}}$-actions rather than with $\mathbb{Q}/\mathbb{Z}$ actions lies in the fact that, in the construction of the geometric Vershiebung action discussed in §2.2, we need to be able to describe the cyclic permutation action of $\mathbb{Z}/n\mathbb{Z}$ on the finite set $Z_n$ as an action factoring through $\mathbb{Z}/n\mathbb{Z}$. This cannot be done in the case of $\mathbb{Q}/\mathbb{Z}$-actions because there are no nontrivial group homomorphisms $\mathbb{Q}/\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}$ since $\mathbb{Q}/\mathbb{Z}$ is infinitely divisible.*

In the case of the natural $\mathbb{Q}/\mathbb{Z}$-actions on torifications, we consider objects of the form $(X,T,\alpha)$ where $X$ is a torifiable variety, $T$ a choice of a torification, and $\alpha : \mathbb{Q}/\mathbb{Z}\times X \to X$ an action of $\mathbb{Q}/\mathbb{Z}$ determined by an embedding of $\mathbb{Q}/\mathbb{Z}$ as roots of unity in $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$, which act on each torus $T_i = \mathbb{G}_m^{k_i}$ diagonally by multiplication. An embedding of $\mathbb{Q}/\mathbb{Z}$ in $\mathbb{G}_m$ is determined by an invertible element in $\mathrm{H}om(\mathbb{Q}/\mathbb{Z},\mathbb{G}_m) = \hat{\mathbb{Z}}$, hence the action $\alpha$ is uniquely determined by the torification $T$ and by the choice of an element in $\hat{\mathbb{Z}}^*$.

The corresponding morphisms are, respectively, strong, ordinary, or weak morphisms of torified varieties compatible with the $\mathbb{Q}/\mathbb{Z}$-actions, in the sense that the resulting torified morphism (after composing with isomorphisms or with local isomorphisms in the ordinary and weak case) are $\mathbb{Q}/\mathbb{Z}$-equivariant. We can then proceed as above and obtain equivariant Grothendieck rings $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^s$, $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^o$, and $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^w$ of torified varieties.

In the case of good $\mathbb{Z}/N\mathbb{Z}$-effectively finite $\hat{\mathbb{Z}}$-actions, the setting is essentially the same. We consider objects of the form $(X, T, \alpha)$ where $X$ is a torifiable variety, $T$ a choice of a torification, and $\alpha : \mathbb{Z}/N\mathbb{Z} \times X \to X$ is given by the action of the $N$-th roots of unity on the tori $T_i = \mathbb{G}_m^{k_i}$ by multiplication. Thus, a good $\hat{\mathbb{Z}}$-action is determined by $T$, by the choice of an embedding of roots of unity in $\mathbb{G}_m$ (an element of $\hat{\mathbb{Z}}^*$) as above, and by the choice of $N \in \mathbb{N}$ that determines which subgroup of roots of unity is acting.

This choice of good $\mathbb{Z}/N\mathbb{Z}$-effectively finite $\hat{\mathbb{Z}}$-actions, with strong, ordinary, or weak morphisms whose associated torified morphisms are $\mathbb{Z}/N\mathbb{Z}$-equivariant, determine equivariant Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^s$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^o$, and $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^w$ of torified varieties with good effectively finite $\hat{\mathbb{Z}}$-actions.

**Assembler and spectrum of torified varieties**

As in the previous cases of $K_0^{\hat{Z}}(\mathcal{V})$ of **ManMar2** and in the case of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ discussed above, we consider the Grothendieck rings $K_0(\mathcal{T})^s$, $K_0(\mathcal{T})^o$, and $K_0(\mathcal{T})^w$ and their corresponding equivariant versions $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^s$, $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^o$, $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^w$, and $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^s$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^o$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^w$ from the point of view of assemblers and spectra developed in **Zak1** , **Zak2** , and **Zak3** .

**Proposition 2.4.2.** *For $a = s, o, w$, the category $\mathcal{C}^a_{\mathcal{T}}$ has objects that are pairs $(X, T)$ of a torifiable variety and a torification, with morphisms the locally closed embeddings that are, respectively, strong, ordinary, or weak morphisms of torified varieties. The Grothendieck topology is generated by the covering families*

$$\{(Y, T_Y) \hookrightarrow (X, T_X), (X \smallsetminus Y, T_{X \smallsetminus Y}) \hookrightarrow (X, T_X)\} \tag{2.4.1}$$

*where both embeddings are strong, ordinary, or weak morphisms, respectively. The category $\mathcal{C}^a_{\mathcal{T}}$ is an assembler with spectrum $K(\mathcal{C}^a_{\mathcal{T}})$ satisfying $\pi_0 K(\mathcal{C}^a_{\mathcal{T}}) = K_0(\mathcal{T})^a$. Similarly, for $G = \mathbb{Q}/\mathbb{Z}$ or $G = \hat{\mathbb{Z}}$ let $\mathcal{C}^{G,a}_{\mathcal{T}}$ be the category with objects $(X, T, \alpha)$ given by a torifiable variety $X$ with a torification $T$ and a $G$-action $\alpha$ of the kind discussed in §2.4 and morphisms the locally closed embeddings that are $G$-equivariant strong,* *ordinary, or weak morphisms. The Grothendieck topology is generated by covering families* (2.4.1) *with* $G$*-equivariant embeddings. The category* $C_{\mathcal{T}}^{G,a}$ *is also an assembler, whose associated spectrum* $K(C_{\mathcal{T}}^{G,a})$ *satisfies* $\pi_0 K(C_{\mathcal{T}}^{G,a}) = K_0^G(\mathcal{T})^a$.

*Proof.* The argument is again as in **Zak1** , see Lemma 2.3.13. We check that the category admits pullbacks. In the strong case, if $(Y, T_Y)$ and $(Y', T_{Y'})$ are objects with morphisms $f : (Y, T_Y) \hookrightarrow (X, T_X)$ and $f' : (Y', T_{Y'}) \hookrightarrow (X, T_X)$ given by embeddings that are strong morphisms of torified varieties. This means that the tori of the torification $T_Y$ are restrictions to $Y$ of tori of the torification $T_X$ of $X$. Thus, both $Y$ and $Y'$ are unions of subcollections of tori of $T_X$. Their intersection $Y \cap Y'$ will then also inherit a torification consisting of a subcollection of tori of $T_X$, and the resulting embedding $(Y \cap Y', T_{Y \cap Y'}) \hookrightarrow (X, T_X)$ is a strong morphism of torified varieties. In the ordinary case, we consider embeddings $f : Y \hookrightarrow X$ and $f' : Y' \hookrightarrow X$ that are ordinary morphisms of torified varieties, which means that, for isomorphisms $\phi_X$, $\phi'_X$, $\phi_Y$, $\phi_{Y'}$, the compositions

$$\phi_X \circ f \circ \phi_Y : (Y, T_Y) \hookrightarrow (X, T_X), \qquad \phi'_X \circ f' \circ \phi_{Y'} : (Y', T_{Y'}) \hookrightarrow (X, T_X)$$

are (strong) torified morphisms. Thus, the tori of the torifications $T_Y$ and $T_{Y'}$ are subcollections of tori of $X$, under the embeddings $\phi_X \circ f \circ \phi_Y$ and $\phi'_X \circ f' \circ \phi_{Y'}$. The intersection $\phi_X \circ f \circ \phi_Y(Y) \cap \phi'_X \circ f' \circ \phi_{Y'}(Y') \subset X$ is isomorphic to a copy of $Y \cap Y'$ and has an induced torification $T_{Y \cap Y'}$ by a subcollection of tori of $T_X$. The embedding of $Y \cap Y'$ in $X$ with this image is an ordinary morphism with respect to this torification. The weak case is constructed similarly to the ordinary case on the pieces of the decomposition. The equivariant cases are constructed analogously, as discussed in the case of equivariant Grothendieck rings of varieties in **ManMar2** . □

**Corollary 2.4.3.** *There are inclusions of assemblers* $C_{\mathcal{T}}^s \hookrightarrow C_{\mathcal{T}}^o \hookrightarrow C_{\mathcal{T}}^w$ *that induce maps of* $K$*-theory, in particular* $K_0(\mathcal{T})^s \to K_0(\mathcal{T})^o$ *and* $K_0(\mathcal{T})^o \to K_0(\mathcal{T})^w$. *Similarly, for the* $G$*-equivariant cases of Proposition 2.4.2.*

*Proof.* Since for morphisms, strong implies ordinary and ordinary implies weak, one obtains inclusions of assemblers as stated. □

**Lifting of the Bost-Connes system for torifications**

We consider here lifts of the integral Bost-Connes algebra to the Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^s$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^o$, and $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^w$ and to the assemblers and spectra $K^{\hat{\mathbb{Z}}}(C^s_{\mathcal{T}})$, $K^{\hat{\mathbb{Z}}}(C^o_{\mathcal{T}})$, and $K^{\hat{\mathbb{Z}}}(C^w_{\mathcal{T}})$.

**Definition 2.4.4.** *We regard the zero-dimensional variety $Z_n$ as a torified variety with the torification consisting of n zero dimensional tori and with a good $\hat{\mathbb{Z}}$ action factoring through $\mathbb{Z}/n\mathbb{Z}$ that cyclically permutes the points of $Z_n$. We write $(Z_n, T_0, \gamma)$ for this object. For $(X, T, \alpha)$ a triple of a torifiable variety $X$, a given torification $T$, and an effectively finite action $\alpha$ of $\hat{\mathbb{Z}}$, we then set, for all $n \in \mathbb{N}$,*

$$\sigma_n(X, T, \alpha) = (X, T, \alpha \circ \sigma_n) \quad \textit{and} \quad \tilde{\rho}_n(X, T, \alpha) = (X \times Z_n, \sqcup_{a \in Z_n} T, \Phi_n(\alpha))\,. \quad (2.4.2)$$

**Proposition 2.4.5.** *The $\sigma_n$ and $\tilde{\rho}_n$ defined as in* (2.4.2) *determine endofunctors of the assembler categories $C^{\hat{\mathbb{Z}},a}_{\mathcal{T}}$ that induce, respectively, ring homomorphisms $\sigma_n : K^{\hat{\mathbb{Z}}}(C^a_{\mathcal{T}}) \to K^{\hat{\mathbb{Z}}}(C^a_{\mathcal{T}})$ and group homomorphisms $\tilde{\rho}_n : K^{\hat{\mathbb{Z}}}(C^a_{\mathcal{T}}) \to K^{\hat{\mathbb{Z}}}(C^a_{\mathcal{T}})$ with the Bost-Connes relations*

$$\tilde{\rho}_n \circ \sigma_n(X, T, \alpha) = (X, T, \alpha) \times (Z_n, T_0, \gamma) \qquad \sigma_n \circ \tilde{\rho}_n(X, T, \alpha) = (X, T, \alpha)^{\oplus n}.$$

*Proof.* The proof is completely analogous to the case discussed in Theorem 2.3.15 and to the similar cases discussed in **ManMar2** . ◻

**Remark 2.4.6.** *The $\sigma_n$ and $\tilde{\rho}_n$ defined as in* (2.4.2) *determine a categorical Bost-Connes system as in Definition 2.3.9, where the objects are pairs $(X, T)$ and the automorphisms are elements $g \in \hat{\mathbb{Z}}$ acting through the effectively finite action $\alpha(g)$.*

**Remark 2.4.7.** *Bost-Connes type quantum statistical mechanical systems associated to individual toric varieties (and more generally to varieties admitting torifications) were constructed in* **JinMar** *. Here instead of Bost-Connes endomorphisms of individual varieties we are interested in a Bost-Connes system over the entire Grothendieck ring and its associated spectrum.*

**Remark 2.4.8.** *Variants of the construction above can be obtained by considering the multivariable versions of the Bost-Connes system discussed in* **Mar** *with actions of subsemigroups of $M_N(\mathbb{Z})^+$ on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes N}$, that is, subalgebras of the crossed product algebra*

$$\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes N} \rtimes_\rho M_N(\mathbb{Z})^+$$

*generated by $e(\underline{r})$ and $\mu_\alpha$, $\mu_\alpha^*$ with*

$$\rho_\alpha(e(\underline{r})) = \mu_\alpha e(\underline{r})\mu_\alpha^* = \frac{1}{\det\alpha}\sum_{\alpha(\underline{s})=\underline{r}} e(\underline{s})$$

$$\sigma_\alpha(e(\underline{r})) = \mu_\alpha^* e(\underline{r})\mu_\alpha = e(\alpha(\underline{r})).$$

*The relevance of this more general setting to $\mathbb{F}_1$-geometries lies in a result of Borger and de Smit* **BorgerdeSmit** *showing that every torsion free finite rank $\Lambda$-ring embeds in some $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^{\otimes N}$ with the action of $\mathbb{N}$ determined by the $\Lambda$-ring structure compatible with the diagonal subsemigroup of $M_N(\mathbb{Z})^+$.*

## 2.5 Torified varieties and zeta functions

We discuss in this section the connection between the dynamical point of view on $\mathbb{F}_1$-geometry proposed in **ManMar2** and the point of view based on torifications.

We first discuss in §2.5 and §2.5 the notion of $\mathbb{F}_1$-points of a torified variety and its relation to the torification of the Grothendieck class, with some explicit examples. We then introduce the $\mathbb{F}_1$-zeta function in §2.5 and we show its main properties in Proposition 2.5.4, while in §2.5 we explain the relation between the $\mathbb{F}_1$-zeta function and the Hasse–Weil zeta function.

In §2.5 we consider the point of view on $\mathbb{F}_1$-structures proposed in **ManMar2** based on dynamical systems inducing quasi-uniponent endomorphisms on homology, in the particular case of torified varieties with dynamical systems compatible with the torification. We focus on the associated dynamical zeta functions, the Lefschetz zeta function and the Artin–Mazur zeta function, whose properties we recall in §2.5. We then prove in Proposition 2.5.8 that the resulting dynamical zeta function have similar properties to the $\mathbb{F}_1$-zeta function in the sense that both define exponentiable motivic measures from the Grothendieck rings of torified varieties to the Witt ring.

### Counting $\mathbb{F}_1$-points

Assuming that a variety $X$ over $\mathbb{Z}$ admits an $\mathbb{F}_1$-structure, regarded here as one of several possible forms of torified structure recalled above, **LoLo** , **ManMar** , the number of points of $X$ over $\mathbb{F}_1$ is computed as the $q \to 1$ limit of the counting function $N_X(q)$ of points over $\mathbb{F}_q$ of the mod $p$ reduction of $X$, for $q$ a power of $p$. Any form of torified structure in particular implies that the variety is polynomially countable, hence that the counting function $N_X(q)$ is a polynomial in $q$ with $\mathbb{Z}$-coefficients. The limit $\lim_{q\to 1} N_X(q)$, possibly normalized by a power of $q-1$, is interpreted as the number of $\mathbb{F}_1$-points of $X$, see **Soule** . Similarly, one can define "extensions" $\mathbb{F}_{1^m}$ of $\mathbb{F}_1$, in the sense of **KapSmi** (see also **CCM**). These corresponds to actions of the groups $\mathfrak{m}_m$ of $m$-th roots of unity. In terms of a torified structure, the points over $\mathbb{F}_{1^m}$ count $m$-th roots of unity in each torus of the decomposition. In terms of the counting function $N_X(q)$ the counting of points of $X$ over the extension $\mathbb{F}_{1^m}$ is obtained as the value $N_X(m+1)$, see Theorem 4.10 of **CoCo** and Theorem 1 of **Deit**). Summarizing, we have the following.

**Lemma 2.5.1.** *Let $X$ be a variety over $\mathbb{Z}$ with torified Grothendieck class*

$$[X] = \sum_{i=0}^{N} a_i \mathbb{T}^i \tag{2.5.1}$$

*with coefficients $a_i \in \mathbb{Z}_+$ and $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1$. Then the number of points over $\mathbb{F}_{1^m}$ of $X$ is given by*

$$\#X(\mathbb{F}_{1^m}) = \sum_{i=0}^{N} a_i \, m^i. \tag{2.5.2}$$

*In particular, $\#X(\mathbb{F}_1) = a_0 = \chi(X)$ the Euler characteristic.*

**Bialynicki-Birula decompositions and torified geometries**

As shown in **Bano** and **Brosnan** , the motive of a smooth projective variety with action of the multiplicative group admits a decomposition, obtained via the method of Bialynicki-Birula, **BiBi1** , **BiBi2** , **BiBi3** . We recall the result here, in a particular case which gives rise to examples of torified varieties.

**Lemma 2.5.2.** *Let $X$ be a smooth projective $k$-variety $X$ endowed with a $\mathbb{G}_m$ action such that the fixed point locus $X^{\mathbb{G}_m}$ admits a torification of the Grothendieck class. Then $X$ also admits a torification of the Grothendieck class. Consider the filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset \emptyset$ with affine fibrations $\phi_i : X_i \smallsetminus X_{i-1} \to Z_i$ over the components $X^{\mathbb{G}_m} = \sqcup_i Z_i$, associated to the Bialynicki-Birula decomposition. If the fixed point locus $X^{\mathbb{G}_m}$ admits a geometric torification such that the restrictions of the fibrations $\phi_i$ to the individual tori of the torification of $Z_i$ are trivializable, then $X$ also admits a geometric torification.*

*Proof.* The Bialynicki-Birula decomposition, **BiBi1** , **BiBi2** , **BiBi3** , see also **Hessel** , shows that a smooth projective $k$-variety $X$ endowed with a $\mathbb{G}_m$ action has smooth closed fixed point locus $X^{\mathbb{G}_m}$ which decomposes into a finite union of components $X^{\mathbb{G}_m} = \sqcup_i Z_i$, of dimensions $\dim Z_i$ the dimension of $TX_z^{\mathbb{G}_m}$ at $z \in Z_i$.

The variety $X$ has a filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset \emptyset$ with affine fibrations $\phi_i : X_i \smallsetminus X_{i-1} \to Z_i$ of relative dimension $d_i$ equal to the dimension of the positive eigenspace of the $\mathbb{G}_m$-action on the tangent space of $X$ at points of $Z_i$. The scheme $X_i \smallsetminus X_{i-1}$ is identified with $\{x \in X \,:\, \lim_{t\to 0} tx \in Z_i\}$ under the $\mathbb{G}_m$-action $t : x \mapsto tx$, with $\phi_i(x) = \lim_{t\to 0} tx$. As shown in **Brosnan** , the object $M(X)$ in the category of correspondences $\mathcal{C}orr_k$ with integral coefficients (and in the category of Chow motives) decomposes as

$$M(X) = \bigoplus_i M(Z_i)(d_i), \tag{2.5.3}$$

where $M(Z_i)$ are the motives of the components of the fixed point set and $M(Z_i)(d_i)$ are Tate twists. The class in the Grothendieck ring $K_0(\mathcal{V}_k)$ decomposes then as

$$[X] = \sum_i [Z_i]\mathbb{L}^{d_i}. \tag{2.5.4}$$

It is then immediate that, if the components $Z_i$ admit a geometric torification (respectively, a torification of the Grothendieck class) then the variety $X$ also does. If $Z_i = \cup_{j=1}^{n_i} T_{ij}$ with $T_{ij} = \mathbb{G}_m^{a_{ij}}$ or, respectively $[Z_i] = \sum_{j=1}^{n_i}(\mathbb{L}-1)^{a_{ij}}$, then $X = \cup_{i=0}^{n}(X_i \smallsetminus X_{i-1}) = \cup_{i=0}^{n}\mathcal{F}^{d_i}(Z_i)$, where $\mathcal{F}^{d_i}(Z_i)$ denotes the total space of the affine fibration $\phi_i : X_i \smallsetminus X_{i-1} \to Z_i$ with fibers $\mathbb{A}^{d_i}$. The Grothendieck class is then torified by

$$[X] = \sum_{i=1}^{n}\sum_{j=1}^{n_i} \mathbb{T}^{a_{ij}}(1 + \sum_{k=1}^{d_i}\binom{d_i}{k}\mathbb{T}^k),$$

with $\mathbb{T} = \mathbb{L} - 1$ the class of the multiplicative group $\mathbb{T} = [\mathbb{G}_m]$, and where the affine spaces are torified by

$$\mathbb{L}^n - 1 = \sum_{k=1}^{n}\binom{n}{k}\mathbb{T}^k.$$

If the restriction of the fibration $\mathcal{F}^{d_i}(Z_i)$ to the individual tori $T_{ij}$ of the torification of $Z_i$ is trivial, then it can be torified by a products $T_{ij} \times T_k$ of the torus $T_{ij}$ and the tori $T_k$ of a torification of the fiber affine space $\mathbb{A}^{d_i}$. This determines a a geometric torification of the affine fibrations $\mathcal{F}^{d_i}(Z_i)$, hence of $X$. $\square$

**An example of torified varieties**

A physically significant example of torified varieties of the type described in Lemma 2.5.2 arises in the context of BPS state counting of **CKK** . Refined BPS state counting computes the multiplicities of BPS particles with charges in a lattice ($K$-theory changes of even $D$-branes) for assigned spin quantum numbers of a $Spin(4) = SU(2) \times SU(2)$ representation, see **CKK** , **ChoiMai** , and **DreMai** .

We mention here the following explicit example from **ChoiMai** , namely the case of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1)$ of Gieseker semi-stable shaved on $\mathbb{P}^2$ with Hilbert polynomial equal to $4m+1$. In this case, it is proved in **ChoiMai** that $\mathcal{M}_{\mathbb{P}^2}(4,1)$ has a torus action of $\mathbb{G}_M^2$ for which the fixed point locus consists of 180 isolated points and 6 components isomorphic to $\mathbb{P}^1$. The Grothendieck class, obtained through the Bialynicki-Birula decomposition **ChoiMai** is given by

$$[\mathcal{M}_{\mathbb{P}^2}(4,1)] = 1 + 2\mathbb{L} + 6\mathbb{L}^2 + 10\mathbb{L}^3 + 14\mathbb{L}^4 + 15\mathbb{L}^5$$

$$+16\mathbb{L}^6 + 16\mathbb{L}^7 + 16\mathbb{L}^8 + 16\mathbb{L}^9 + 16\mathbb{L}^{10} + 16\mathbb{L}^{11}$$

$$+15\mathbb{L}^{12} + 14\mathbb{L}^{13} + 10\mathbb{L}^{14} + 6\mathbb{L}^{15} + 2\mathbb{L}^{16} + \mathbb{L}^{17}.$$

Note that, for a smooth projective variety with Grothendieck class that is a polynomial in the Lefschetz motive $\mathbb{L}$, the Poincaré polynomial and the Grothendieck class are related by replacing $x^2$ with $\mathbb{L}$, since the variety is Hodge–Tate. In torified form the above gives

$$[\mathcal{M}_{\mathbb{P}^2}(4,1)] = \mathbb{T}^{17} + 19\,\mathbb{T}^{16} + 174\,\mathbb{T}^{15} + 1020\,\mathbb{T}^{14} + 4284\,\mathbb{T}^{13} + 13665\,\mathbb{T}^{12} + 34230\,\mathbb{T}^{11}$$

$$+68678\,\mathbb{T}^{10} + 111606\,\mathbb{T}^9 + 147653\,\mathbb{T}^8 + 159082\,\mathbb{T}^7 + 139008\,\mathbb{T}^6$$

$$+97643\,\mathbb{T}^5 + 54320\,\mathbb{T}^4 + 23370\,\mathbb{T}^3 + 7468\,\mathbb{T}^2 + 1632\,\mathbb{T} + 192,$$

where $192 = \chi(\mathcal{M}_{\mathbb{P}^2}(4,1))$ is the Euler characteristics, which is also the number of points over $\mathbb{F}_1$. The number of points over $\mathbb{F}_{1^m}$ gives 864045 for $m = 1$ (the number of tori in the torification), 383699680 for $m = 2$ (roots of unity of order two), 36177267945 for $m = 3$ (roots of unity of order three), etc.

In this example, the Euler characteristic $\chi(\mathcal{M}_{\mathbb{P}^2}(4,1))$, which can also be seen as the number of $\mathbb{F}_1$-points, is interpreted physically as determining the BPS counting. It is natural to ask whether the counting of $\mathbb{F}_{1^m}$-points, which corresponds to the counting of roots of unity in the tori of the torification, can also carry physically significant information.

Other examples of torified varieties relevant to physics can be found in the context of quantum field theory, see **BejMar** and **Mu** .

**BPS counting and the virtual motive**

The formulation of the refined BPS counting given in **CKK** can be summarized as follows. The virtual motive $[X]_{\mathrm{v}ir} = \mathbb{L}^{-n/2}[X]$, with $n = \dim(X)$, is a class in the ring of motivic weights $K_0(\mathcal{V})[\mathbb{L}^{-1/2}]$, see **BeBrSz** . When $X$ admits a $\mathbb{G}_m$ action and a Bialynicki-Birula decomposition as discussed in the previous section, where all the components $Z_i$ of the fixed point locus of the $\mathbb{G}_m$-action have Tate classes $[Z_i] = \sum_j c_{ij}\mathbb{L}^{b_{ij}} \in K_0(\mathcal{V})$, with $c_{ij} \in \mathbb{Z}$ and $b_{ij} \in \mathbb{Z}_+$, the virtual motive $[X]_{\mathrm{v}ir}$ is a Laurent polynomial in the square root $\mathbb{L}^{1/2}$ of the Lefschetz motive,

$$[X]_{\mathrm{v}ir} = \sum_{i,j} c_{ij}\, \mathbb{L}^{b_{ij}+d_i-1/2}, \tag{2.5.5}$$

where, as before, $d_i$ is the dimension of the positive eigenspace of the $\mathbb{G}_m$-action on the tangent space of $X$ at points of $Z_i$. In applications to BPS counting, one considers the virtual motive of a moduli space $M$ that admits a perfect obstruction theory, so that it has virtual dimension zero and an associated invariant $\#_{\mathrm{v}ir}M$ which is computed by a virtual index

$$\#_{\mathrm{v}ir}M = \chi_{\mathrm{v}ir}(M, K^{1/2}_{M,\mathrm{v}ir}) = \chi(M, K^{1/2}_{M,\mathrm{v}ir} \otimes O_{M,\mathrm{v}ir}),$$

where $O_{M,\mathrm{v}ir}$ is the virtual structure sheaf and $K^{1/2}_{M,\mathrm{v}ir}$ is a square root of the virtual canonical bundle, see **FaGo** .

**The formal square root of the Leftschetz motive**

The formal square root $\mathbb{L}^{1/2}$ of the Leftschetz motive that occurs in (2.5.5) as Grothendieck class can be introduced, at the level of the category of motives, as shown in §3.4 of **KoSo** using the Tannakian formalism, **Deligne** . Let $C = \mathrm{N}um^{\dagger}_{\mathbb{Q}}$ be the Tannakian category of pure motives with the numerical equivalence relation and the Koszul sign rule twist † in the tensor structure, with motivic Galois group $G = \mathrm{G}al(C)$. The inclusion of the Tate motives (with motivic Galois group $\mathbb{G}_m$) determines a group homomorphism $t : G \to \mathbb{G}_m$, which satisfies $t \circ w = 2$ with the weight homomorphism $w : \mathbb{G}_m \to G$ (see §5 of **DeMi**). The category $C(\mathbb{Q}(\frac{1}{2}))$ obtained by adjoining a square root of the Tate motive to $C$ is then obtained as the Tannakian category whose Galois group is the fibered product

$$G^{(2)} = \{(g, \lambda) \in G \times \mathbb{G}_m \,:\, t(g) = \lambda^2\}.$$

The construction of square roots of Tate motives described in **KoSo** was generalized in **LoMa** to arbitrary $n$-th roots of Tate motives, obtained via the same Tannakian construction, with the category $C(\mathbb{Q}(\frac{1}{n}))$ obtained by adjoining an $n$-th root of the Tate motive determined by its Tannakian Galois group

$$G^{(n)} = \{(g,\lambda) \in G \times \mathbb{G}_m \,:\, t(g) = \sigma_n(\lambda)\},$$

with $\sigma_n : \mathbb{G}_m \to \mathbb{G}_m$, $\sigma_n(\lambda) = \lambda^n$. The category $\hat{C}$ obtained by adjoining to $C = \mathrm{N}um^{\dagger}_{\mathbb{Q}}$ arbitrary roots of the Tate motives is the Tannakian category with Galois group $\hat{G} = \varprojlim_n G^{(n)}$. The category $\hat{C}$ has an action of $\mathbb{Q}^*_+$ by automorphisms induced by the endomorphisms $\sigma_n$ of $\mathbb{G}_m$. These roots of Tate motives give rise to a good formalism of $\mathbb{F}_\zeta$-geometry, with $\zeta$ a root of unity, lying "below" $\mathbb{F}_1$-geometry and expressed at the motivic level in terms of a Habiro ring type object associated to the Grothendieck ring of orbit categories of $\hat{C}$, see **LoMa** .

**Counting $\mathbb{F}_1$-points and zeta function**

For a variety $X$ over $\mathbb{Z}$ that is polynomially countable (that is, the counting functions $N_X(q) = \#X_p(\mathbb{F}_q)$ with $X_p$ the mod $p$ reduction is a polynomial in $q$ with $\mathbb{Z}$ coefficients) the counting of points over the "extensions" $\mathbb{F}_{1^m}$ (in the sense of **KapSmi**) can be obtained as the values $N_X(m+1)$ (see Theorem 4.10 of **CoCo** and Theorem 1 of **Deit**). As we discussed earlier, in the case of a torified variety, with Grothendieck class $[X] = \sum_{i\geq 0} a_i \mathbb{T}^i$ with $a_i \in \mathbb{Z}_+$, this corresponds to the counting given in (2.5.2). This is the counting of the number of $m$-th roots of unity in each torus $\mathbb{T}^i = [\mathbb{G}_m^i]$ of the torification.

For a variety $X$ over a finite field $\mathbb{F}_q$ the Hasse–Weil zeta function is given, in logarithmic form by

$$\log Z_{\mathbb{F}_q}(X,t) = \sum_{m\geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m} t^m. \tag{2.5.6}$$

In the case of torified varieties, there is an analogous zeta function over $\mathbb{F}_1$. We think of this $\mathbb{F}_1$-zeta function as defined on torified Grothendieck classes, $Z_{\mathbb{F}_1}([X],t)$. In the case of geometric torifications, we can regard it as a function of the variety and the torification, $Z_{\mathbb{F}_1}((X,T),t)$. For simplicity of notation, we will simply write $Z_{\mathbb{F}_1}(X,t)$ by analogy to the Hasse–Weil zeta function, with

$$\log Z_{\mathbb{F}_1}(X,t) := \sum_{m\geq 1} \frac{\#X(\mathbb{F}_{1^m})}{m} t^m. \tag{2.5.7}$$

**Lemma 2.5.3.** *Let $X$ be a variety over $\mathbb{Z}$ with a torified Grothendieck class $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$ with $a_k \in \mathbb{Z}_+$. Then the $\mathbb{F}_1$-zeta function is given by*

$$\log Z_{\mathbb{F}_1}(X,t) = \sum_{k=0}^{N} a_k \, Li_{1-k}(t), \tag{2.5.8}$$

*where $Li_s(t)$ is the polylogarithm function with $Li_1(t) = -\log(1-t)$ and for $k \geq 1$*

$$Li_{1-k}(t) = (t\frac{d}{dt})^{k-1}\frac{t}{1-t}.$$

*Proof.* For $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$ with $a_k \in \mathbb{Z}_+$ as above, we can consider a similar zeta function based on the counting of $\mathbb{F}_{1^m}$-points described above. Using (2.5.2), we obtain an expression of the form

$$\log Z_{\mathbb{F}_1}(X,t) = \sum_{m\geq 1} \frac{\#X(\mathbb{F}_{1^m})}{m} t^m = \sum_{k=0}^{N} a_k \sum_{m\geq 1} m^{k-1} t^m = \sum_{k=0}^{N} a_k \,\mathrm{L}i_{1-k}(t),$$

given by a linear combination of polylogarithm functions $\mathrm{L}i_s(t)$ at integer values $s \leq 1$. □

Such polylogarithm functions can be expressed explicitly in the form $\mathrm{L}i_1(t) = -\log(1-t)$ and for $k \geq 1$

$$\mathrm{L}i_{1-k}(t) = (t\frac{d}{dt})^{k-1}\frac{t}{1-t} = \sum_{\ell=0}^{k-1} \ell!\, S(k,\ell+1) \left(\frac{t}{1-t}\right)^{\ell+1},$$

with $S(k,r)$ the Stirling numbers of the second kind

$$S(k,r) = \frac{1}{r!}\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} j^k.$$

As in the case of the Hasse–Weil zeta function over $\mathbb{F}_q$ (see **Ram**), the $\mathbb{F}_1$-zeta function gives an exponentiable motivic measure.

**Proposition 2.5.4.** *The $\mathbb{F}_1$-zeta function is an exponentiable motivic measure, that is, a ring homomorphism $Z_{\mathbb{F}_1} : K_0(\mathcal{T})^a \to W(\mathbb{Z})$ from the Grothendieck ring of torified varieties (with either $a = w, o, s$) to the Witt ring.*

*Proof.* Clearly with respect to addition in the Grothendieck ring of torified varieties we have $[X] + [X'] = \sum_{i\geq 0} a_i \mathbb{T}^i + \sum_{j\geq 0} a'_j \mathbb{T}^j = \sum_{k\geq 0} b_k \mathbb{T}^k$ with $b_k = a_k + a'_k$, hence

$$\log Z_{\mathbb{F}_1}([X]+[X'],t) = \sum_{k=0}^{N} b_k \,\mathrm{L}i_{1-k}(t) = \log Z_{\mathbb{F}_1}([X],t) + \log Z_{\mathbb{F}_1}([X'],t).$$

The behavior with respect to products $[X]\cdot[Y]$ in the Grothendieck ring of torified varieties can be analyzed as in **Ram** for the Hasse–Weil zeta function. We view the $\mathbb{F}_1$-zeta function

$$Z_{\mathbb{F}_1}(X,t) = \exp(\sum_{k=0}^{N} a_k \,\mathrm{L}i_{1-k}(t))$$

as the element in the Witt ring $W(\mathbb{Z})$ with ghost components $\#X(\mathbb{F}_{1^m}) = \sum_{k=0}^N m^k$, by writing the ghost map $gh : W(\mathbb{Z}) \to \mathbb{Z}^{\mathbb{N}}$ as

$$gh : Z(t) = \exp(\sum_{m\geq 1} \frac{N_m}{m} t^m) \mapsto t\frac{d}{dt} \log Z(t) = \sum_{m\geq 1} N_m t^m \mapsto (N_m)_{m\geq 1}.$$

The ghost map is an injective ring homomorphism. Thus, it suffices to see that on the ghost components $N_m(X \times Y) = N_m(X) \cdot N_m(Y)$. If $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$ and $[Y] = \sum_{\ell\geq 0} b_\ell \mathbb{T}^\ell$ then $[X \times Y] = \sum_{n\geq 0} \sum_{k+\ell=n} a_k b_\ell \mathbb{T}^n$ and $N_m(X \times Y) = \sum_{n\geq 0} \sum_{k+\ell=n} a_k b_\ell m^n = N_m(X) \cdot N_m(Y)$. □

**Relation to the Hasse–Weil zeta function**

We discuss here the relation between the $\mathbb{F}_1$-zeta function $Z_{\mathbb{F}_1}(X, t)$ introduced in (2.5.7) above, for a variety $X$ over $\mathbb{Z}$ with torified Grothendieck class $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$, and the Hesse–Weil zeta function $Z_{\mathbb{F}_q}(X, t)$, defined as in (2.5.6).

**Definition 2.5.5.** *Consider the following elements in the Witt ring $W(\mathbb{Z})$, for $k \geq 0$:*

$$Z_{0,k,q}(t) := \exp(\sum_{m\geq 1} (q-1)^k \frac{t^m}{m}) = \frac{1}{(1-t)^{(q-1)^k}} \tag{2.5.9}$$

$$Z_{1,k,q}(t) := \exp(\sum_{m\geq 1} (\#\mathbb{P}^{m-1}(\mathbb{F}_q))^k \frac{t^m}{m}) = \exp(\sum_{m\geq 1} (\sum_{\ell=0}^{m-1} q^\ell)^k \frac{t^m}{m}). \tag{2.5.10}$$

**Lemma 2.5.6.** *Let $Z_{\mathbb{F}_q}(\mathbb{T}^k, t)$ be the Hasse-Weil zeta function of a torus $\mathbb{T}^k$. The function $Z_{0,k,q}(t)$ of* (2.5.9) *divides $Z_{\mathbb{F}_q}(\mathbb{T}^k, t)$ in the Witt ring with quotient the function $Z_{1,k,q}(t)$ of* (2.5.10).

*Proof.* Given elements $Q = Q(t)$ and $P = P(t)$ in the Witt ring $W(\mathbb{Z})$, we have that $Q$ divides $P$ iff the ghost components $q_m$ of $Q$ in $\mathbb{Z}^{\mathbb{N}}$ divide the corresponding ghost components $p_m$ of $P$. There is then an element $S = S(t)$ in $W(\mathbb{Z})$, with ghost components $s_m = p_m/q_m$, such that the Witt product gives $S \star_W Q = P$. The $m$-th ghost components of $Z_{\mathbb{F}_q}(\mathbb{T}^k, t)$ is $(q^m - 1)^k = \#\mathbb{T}^k(\mathbb{F}_{q^m})$, and we have $(q^m - 1)^k/(q-1)^k = (1 + q + \cdots + q^{m-1})^k$. □

Given elements $Q, P \in W(\mathbb{Z})$ such that $Q|P$ as above, we write $S = P/_W Q$ for the resulting element $S \in W(\mathbb{Z})$ with $S \star_W Q = P$.

The $\mathbb{F}_1$-zeta function of (2.5.7) is obtained from the Hasse–Weil zeta function of (2.5.6) in the following way.

**Proposition 2.5.7.** *Let $X$ be a variety $X$ over $\mathbb{Z}$ with torified Grothendieck class $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$. The $\mathbb{F}_1$-zeta function is given by*

$$Z_{\mathbb{F}_1}(X,t) = \lim_{q\to 1} {}^W\sum_{k\geq 0} \left( Z_{\mathbb{F}_q}(\mathbb{T}^k,t) /_W \; Z_{0,k,q}(t) \right)^{a_k} = \lim_{q\to 1} {}^W\sum_{k\geq 0} Z_{1,k,q}(t)^{a_k}, \tag{2.5.11}$$

*while the Hasse–Weil zeta function is given by*

$$Z_{\mathbb{F}_q}(X,t) = {}^W\sum_{k\geq 0} Z_{\mathbb{F}_q}(\mathbb{T}^k,t)^{a_k}, \tag{2.5.12}$$

*where ${}^W\sum$ denotes the sum in the Witt ring.*

*Proof.* For the Hasse–Weil zeta function we have

$$\begin{aligned} Z_{\mathbb{F}_q}(X,t) &= \exp\Big(\sum_{m\geq 1} \#X(\mathbb{F}_{q^m}) \frac{t^m}{m}\Big) = \exp\Big(\sum_{k\geq 0} a_k \sum_{m\geq 1} (q^m - 1)^k \frac{t^m}{m}\Big) \\ &= \prod_{k\geq 0} \exp(a_k \log Z_{\mathbb{F}_q}(\mathbb{T}^k,t)), \end{aligned}$$

hence we get (2.5.12). To obtain the $\mathbb{F}_1$-zeta function we then use Lemma 2.5.6 and the fact that $(q^m-1)^k/(q-1)^k = (1+q+\cdots+q^{m-1})^k$, with $\lim_{q\to 1}(1+q+\cdots+q^{m-1})^k = m^k$. ◻

### Dynamical zeta functions

The dynamical approach to $\mathbb{F}_1$-structures proposed in **ManMar2** is based on the existence of an endomorphism $f : X \to X$ that induces a quasi-unipotent morphism $f_*$ on the homology $H_*(X,\mathbb{Z})$. In particular, this means that the map $f_*$ has eigenvalues that are roots of unity.

In the case of a variety $X$ endowed with a torification $X = \sqcup_i T^{d_i}$, one can consider in particular endomorphisms $f : X \to X$ that preserve the torification and that restrict to endomorphisms of each torus $T^{d_i}$.

We recall the definition and main properties of the relevant dynamical zeta functions, which we will consider in Proposition 2.5.8.

### Properties of dynamical zeta functions

In general to a self-map $f : X \to X$, one can associate the dynamical Artin–Mazur zeta function and the homological Lefschetz zeta function. A particular class of maps with the property that they induce quasi-unipotent morphisms in homology is given by the Morse–Smale diffeomorphisms of smooth manifolds, see **ShuSul** . These are diffeomorphisms characterized by the properties that the set of nonwandering points is finite and hyperbolic, consisting of a finite number of periodic points, and for any pair of these points $x, y$ the stable and unstable manifolds $W^s(x)$ and $W^u(y)$ intersect transversely. Morse–Smale diffeomorphisms are structurally stable among all diffeomorphisms (see **Franks** and **ShuSul** ).

The Lefschetz zeta function

$$\zeta_{\mathcal{L},f}(t) = \exp\left(\sum_{m\geq 1} \frac{L(f^m)}{m} t^m\right), \tag{2.5.13}$$

with $L(f^m)$ the Lefschetz number of the $m$-th iterate $f^m$,

$$L(f^m) = \sum_{k\geq 0} (-1)^k \mathrm{T}r((f^m)_* \mid H_k(X, \mathbb{Q})),$$

which for a function with finitely many fixed points is also equal to

$$L(f^m) = \sum_{x\in \mathrm{F}ix(f^m)} \mathcal{I}(f^m, x),$$

with $\mathcal{I}(f^m, x)$ the index of the fixed point. This is a rational function of the form

$$\zeta_{\mathcal{L},f}(t) = \prod_k \det(1 - t\, f_* | H_k(X, \mathbb{Q}))^{(-1)^{k+1}}.$$

In the case of a map $f$ with finitely many periodic points, all hyperbolic, the Lefschetz zeta function can be equivalently written (see **Franks**) as the rational function

$$\zeta_{\mathcal{L},f}(t) = \prod_\gamma (1 - \Delta_\gamma t^{p(\gamma)})^{(-1)^{u(\gamma)+1}},$$

with the product over periodic orbits $\gamma$ with least period $p(\gamma)$ and with $u(\gamma) = \dim E^u_x$ for $x \in \gamma$, the dimension of the span of eigenvectors of $Df_x^{p(\gamma)} : T_x M \to T_x M$ with eigenvalues $\lambda$ with $|\lambda| > 1$, and $\Delta_\gamma = \pm 1$ according to whether $Df_x^{p(\gamma)}$ is orientation preserving or reversing. The relation comes from the identity $\mathcal{I}(f^m, x) = (-1)^{u(\gamma)} \Delta_\gamma$. The Artin–Mazur zeta function is given by

$$\zeta_{AM,f}(t) = \exp\left(\sum_{m\geq 1} \frac{\#\mathrm{F}ix(f^m)}{m} t^m\right). \tag{2.5.14}$$

The case of Morse–Smale diffeomorphisms can be treated as in **Franks2** to obtain rationality and a description in terms of the homological zeta functions.

In the setting of real tori $\mathbb{R}^d/\mathbb{Z}^d$, one can considers the case of a toral endomorphism specified by a matrix $M \in M_d(\mathbb{Z})$. In the hyperbolic case, the counting of isolated fixed points of $M^m$ is given by $|\det(1-M^m)|$ and the dynamical Artin–Mazur zeta function is expressible in terms of the Lefschetz zeta function, associated to the signed counting of fixed points, through the fact that the Lefschetz zeta function agrees with the zeta function

$$\zeta_M(t) = \exp(\sum_{n\geq 1} \frac{t^n}{n} a_n), \quad \text{with} \quad a_n = \det(1-M^n), \tag{2.5.15}$$

where $a_n = \det(1-M^n)$ is a signed fixed point counting. The general relation between the zeta functions for the signed $\det(1-M^n)$ and for $|\det(1-M^m)|$ is shown in **Baake** for arbitrary toral endomorphisms, with $M \in M_d(\mathbb{Z})$.

In the case of complex algebraic tori $T^d = \mathbb{G}_m^d(\mathbb{C})$, one can similarly consider the endomorphisms action of the semigroup of matrices $M \in M_d(\mathbb{Z})^+$ by the linear action on $\mathbb{C}^d$ preserving $\mathbb{Z}^d$ and the exponential map $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1$ so that, for $M = (m_{ab})$ and $\lambda_a = \exp(2\pi i u_a)$, with the action given by

$$\lambda = (\lambda_a) \mapsto M(\lambda) = \exp(2\pi i \sum_b m_{ab} u_b).$$

The subgroup $SL_n(\mathbb{Z}) \subset M_n(\mathbb{Z})^+$ acts by automorphisms. These generalize the Bost-Connes endomorphisms $\sigma_n : \mathbb{G}_m \to \mathbb{G}_m$, which correspond to the ring homomorphisms of $\mathbb{Z}[t,t^{-1}]$ given by $\sigma_n : P(t) \mapsto P(t^n)$ and determine multivariable versions of the Bost-Connes algebra, see **Mar** . We can consider in this way maps of complex algebraic tori $T^d_{\mathbb{C}} = \mathbb{G}_m^d(\mathbb{C})$ that induce maps of the real tori obtained as the subgroup $T^d_{\mathbb{R}} = U(1)^d \subset \mathbb{G}_m^d(\mathbb{C})$, and associate to these maps the Lefschetz and Artin–Mazur zeta functions of the induced map of real tori.

### Torifications and dynamical zeta functions

In the case of a variety with a torification, we consider endomorphisms $f : X \to X$ that preserves the tori of the torification and restricts to each torus $T^{d_i}$ to a diffeomorphism $f_i : T^{d_i}_{\mathbb{R}} \to T^{d_i}_{\mathbb{R}}$. In particular, we consider toral endomorphism with a matrix $M_i \in M_{d_i}(\mathbb{Z})$, we can associate to the pair $(X, f)$ a zeta function of the form

$$\zeta_{\mathcal{L},f}(X,t) = \prod_i \zeta_{\mathcal{L},f_i}(t), \qquad \zeta_{AM,f}(X,t) = \prod_i \zeta_{AM,f_i}(t). \tag{2.5.16}$$

**Proposition 2.5.8.** *The zeta functions* (2.5.13) *and* (2.5.14) *define exponentiable motivic measures on the Grothendieck ring* $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ *of §6 of* **ManMar2** *with values in the Witt ring* $W(\mathbb{Z})$*. The zeta functions* (2.5.16) *define exponentiable motivic measures on the Grothendieck ring* $K_0(\mathcal{T})^a$ *of torified varieties with values in* $W(\mathbb{Z})$*.*

*Proof.* The Grothendieck ring $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ considered in §6 of **ManMar2** consists of pairs $(X, f)$ of a complex quasi-projective variety and an automorphism $f : X \to X$ that induces a quasi-uniponent map $f_*$ in homology. The addition is simply given by the disjoint union, and both the counting of periodic points $\#\mathrm{F}ix(f^m)$ and the Lefschetz numbers $L(f^m)$ behave additively under disjoint unions. Thus, the zeta functions $\zeta_{\mathcal{L},f}(t)$ and $\zeta_{AM,f}(t)$, seen as elements in the Witt ring $W(\mathbb{Z})$ add

$$\zeta_{\mathcal{L},f_1 \sqcup f_2}(t) = \exp\left(\sum_{m\geq 1} \frac{L((f_1 \sqcup f_2)^m)}{m} t^m\right) =$$

$$\exp\left(\sum_{m\geq 1} \frac{L(f_1^m)}{m} t^m\right) \cdot \exp\left(\sum_{m\geq 1} \frac{L(f_2^m)}{m} t^m\right) = \zeta_{\mathcal{L},f_1}(t) +_W \zeta_{\mathcal{L},f_2}(t)$$

and similarly for $\zeta_{AM,f_1\sqcup f_2}(t) = \zeta_{AM,f_1}(t) +_W \zeta_{AM,f_2}(t)$. The product is given by the Cartesian product $(X_1, f_1) \times (X_2, f_2)$. Since $\mathrm{F}ix((f_1 \times f_2)^m) = \mathrm{F}ix(f_1^m) \times \mathrm{F}ix(f_2^m)$ and the same holds for Lefschetz numbers since

$$L((f_1 \times f_2)^m) = \sum_{k\geq 0} \sum_{\ell+r=k} (-1)^{\ell+r} \mathrm{T}r((f_1^m)_* \otimes (f_2^m)_* \,|\, H_\ell(X_1, \mathbb{Q}) \otimes H_r(X_2, \mathbb{Q}))$$

which gives $L(f_1^m) \cdot L(f_2^m)$. Thus, we can use as in Proposition 2.5.4 the fact that the ghost map $gh : W(\mathbb{Z}) \to \mathbb{Z}^{\mathbb{N}}$

$$gh : \exp(\sum_{m\geq 1} \frac{N_m}{m} t^m) \mapsto \sum_{m\geq 1} N_m t^m \mapsto (N_m)_{m\geq 1}$$

is an injective ring homomorphism to obtain the multiplicative property. The case of the torified varieties and the zeta functions (2.5.16) is analogous, combining the additive and multiplicative behavior of the fixed point counting and the Lefschetz numbers on each torus and of the decomposition into tori as in Proposition 2.5.4. □

In the case of quasi-unipotent maps of tori, the Lefschetz zeta function can be computed completely explicitly. Indeed, it is shown in **Berr1 Berr2** that, for a quasi-unipotent self map $f : T^n_{\mathbb{R}} \to T^n_{\mathbb{R}}$, the Lefschetz zeta function has an explicit form that is completely determined by the map on the first homology. Under the quasi-unipotent assumption, all the eigenvalues of the induced map on $H_1$ are roots of unity, hence the characteristic polynomial $\det(1 - t\, f_* | H_1(X))$ is a product of cyclotomic polynomials $\Phi_{m_1}(t) \cdots \Phi_{m_N}(t)$ where

$$\Phi_m(t) = \prod_{d|m} (1 - t^d)^{\mu(m/d)},$$

with $\mu(n)$ the Möbius function. It is shown in **Berr2** that the Lefschetz zeta function has the form

$$\zeta_{\mathcal{L},f}(t) = \prod_{d|m} (1 - t^d)^{-s_d}, \tag{2.5.17}$$

where $m = \mathrm{l}cm\{m_1, \ldots, m_N\}$ and

$$s_d = \frac{1}{d} \sum_{k|d} F_k \, \mu(d/k)$$

$$F_k = \prod_{i=1}^{N} (\Phi_{m_i/(k,m_i)}(1))^{\varphi(m_i)/\varphi(m_i/(k,m_i))}$$

where the Euler function

$$\varphi(m) = m \prod_{p|m,\, p \text{ prime}} (1 - p^{-1})$$

is the degree of $\Phi_m(t)$.

**Remark 2.5.9.** *The properties of dynamical Artin–Mazur zeta functions change significantly when, instead of considering varieties over $\mathbb{C}$ one considers varieties in positive characteristic,* **Bridy BysCor** *. The prototype model of this phenomenon is illustrated by considering the Bost-Connes endomorphisms $\sigma_n : \lambda \mapsto \lambda^n$ of $\mathbb{G}_m(\bar{\mathbb{F}}_p)$. In this case, the dynamical zeta function of $\sigma_n$ is rational or transcendental depending on whether p divides n (Theorem 1.2 and 1.3 and §3 of* **Bridy** *and Theorem 1 of* **Bridy2***). Similar phenomena in the more general case of endomorphisms of abelian varieties in positive characteristic have been investigated in* **BysCor** *. In the positive characteristic setting, where one is considering the characteristic p version of the Bost-Connes system of* **CCM** *, one should then replace the dynamical zeta function by the tame zeta function considered in* **BysCor** *.*

## 2.6 Spectra and zeta functions

We have already discussed in §2.5 and §2.5 zeta functions arising from certain counting functions that define ring homomorphisms from suitable Grothendieck rings to the Witt ring $W(\mathbb{Z})$. We consider here a more general setting of exponentiable motivic measures.

A motivic measure is a ring homomorphism $\mu : K_0(\mathcal{V}) \to R$, from the Grothendieck ring of varieties $K_0(\mathcal{V})$ to a commutative ring $R$. Examples include the counting measure, for varieties defined over finite fields, which counts the number of algebraic points over $\mathbb{F}_q$, the topological Euler characteristic or the Hodge–Deligne polynomials for complex algebraic varieties.

The Kapranov motivic zeta function **Kapr** is defined as $\zeta(X,t) = \sum_{n=0}^{\infty}[S^n(X)]t^n$, where $S^n(X) = X^n/S_n$ are the symmetric products of $X$ and $[S^n(X)]$ are the classes in $K_0(\mathcal{V})$. Similarly, the zeta function of a motivic measure is defined as

$$\zeta_\mu(X,t) = \sum_{n=0}^{\infty} \mu(S^n(X))\, t^n. \tag{2.6.1}$$

It is viewed as an element in the Witt ring $W(R)$. The addition in $K_0(\mathcal{V})$ is mapped by the zeta function to the addition in $W(R)$, which is the usual product of the power series,

$$\zeta_\mu(X \sqcup Y, t) = \zeta_\mu(X,t) \cdot \zeta_\mu(Y,t) = \zeta_\mu(X,t) +_{W(R)} \zeta_\mu(Y,t). \tag{2.6.2}$$

The motivic measure $\mu : K_0(\mathcal{V}) \to R$ is said to be exponentiable (see **Ram RamTab**) if the zeta function (2.6.1) defines a ring homomorphism

$$\zeta_\mu : K_0(\mathcal{V}) \to W(R),$$

that is, if in addition to (2.6.2) one also has

$$\zeta_\mu(X \times Y, t) = \zeta_\mu(X,t) \star_{W(R)} \zeta_\mu(Y,t). \tag{2.6.3}$$

We investigate here how to lift the zeta functions of exponentiable motivic measures to the level of spectra. To this purpose, we first investigate how to construct a spectrum whose $\pi_0$ is a dense subring $W_0(R)$ of the Witt ring $W(R)$ and then we consider how to lift the ring homomorphisms given by zeta functions $\zeta_\mu$ of exponentiable measures with a rationality and a factorization condition.

**The Endomorphism Category**

Let $R$ be a commutative ring. We denote by $\mathcal{E}_R$ the endomorphism category of $R$, which is defined as follows (see **Alm1 Alm2 DreSie**).

**Definition 2.6.1.** *The category $\mathcal{E}_R$ has objects given by the pairs $(E, f)$ of a finite projective module $E$ over $R$ and an endomorphism $f \in End_R(E)$, and morphisms given by morphisms $\phi : E \to E'$ of finite projective modules that commute with the endomorphisms, $f' \circ \phi = \phi \circ f$. The endomorphism category has direct sum $(E, f) \oplus (E', f') = (E \oplus E', f \oplus f')$ and tensor product $(E, f) \otimes (E', f') = (E \otimes E', f \otimes f')$.*

The category of finite projective modules over $R$ is identified with the subcategory corresponding to the objects $(E, 0)$ with trivial endomorphism.

An exact sequence in $\mathcal{E}_R$ is a sequence of objects and morphisms in $\mathcal{E}_R$ which is exact as a sequence of finite projective modules over $R$ (forgetting the endomorphisms). This determines a collection of admissible short exact sequence (and of admissible monomorphisms and epimorphisms). The endomorphism category $\mathcal{E}_R$ is then an exact category, hence it has an associated $K$-theory defined via the Quillen $Q$-construction, **Quillen** . This assigns to the exact category $\mathcal{E}_R$ the category $Q\mathcal{E}_R$ with the same objects and morphisms $\mathrm{H}om_{Q\mathcal{E}_R}((E, f), (E', f'))$ given by diagrams

$$\begin{tikzcd} & (E'', f'') \arrow[dl] \arrow[dr] & \\ (E, f) & & (E', f'), \end{tikzcd}$$

where the first arrow is an admissible epimorphism and the second an admissible monomorphism, with composition given by pullback. By the Quillen construction $K$-theory of $\mathcal{E}_R$ is then $K_{n-1}(\mathcal{E}_R) = \pi_n(\mathcal{N}(Q\mathcal{E}_R))$, with $\mathcal{N}(Q\mathcal{E}_R)$ the nerve of $Q\mathcal{E}_R$.

The forgetful functor $(E, f) \mapsto E$ induces a map on $K$-theory

$$K_n(\mathcal{E}_R) \to K_n(\mathcal{P}_R) = K_n(R),$$

which is a split surjection. Let

$$\mathcal{E}_n(R) := \mathrm{K}er(K_n(\mathcal{E}_R) \to K_n(R)).$$

In the case of $K_0$, an explicit description is given by the following, **Alm1** , **Alm2** . Let $K_0(\mathcal{E}_R)$ denote the $K_0$ of the endomorphism category $\mathcal{E}_R$. It is a ring with the product structure induced by the tensor product. It is proved in **Alm1** , **Alm2** that the quotient

$$W_0(R) = K_0(\mathcal{E}_R)/K_0(R) \tag{2.6.4}$$

embeds as a dense subring of the big Witt ring $W(R)$ via the map

$$L : (E, f) \mapsto \det(1 - t\, M(f))^{-1}, \tag{2.6.5}$$

with $M(f)$ the matrix associated to $f \in \mathrm{E}nd_R(E)$, where $\det(1-t\, M(f))^{-1}$ is viewed as an element in $\Lambda(R) = 1 + tR[[t]]$. As a subring $W_0(R) \hookrightarrow W(R)$ of the big Witt ring, $W_0(R)$ consists of the rational Witt vectors

$$W_0(R) = \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} \mid a_i, b_i \in R,\ n, m \geq 0 \right\}.$$

Equivalently, one can consider the ring $\mathcal{R} = (1+tR[t])^{-1}R[t]$ and identify the above with $1+t\mathcal{R}$, where the multiplication in $1+t\mathcal{R}$ corresponds to the addition in the Witt ring, and the Witt product is determined by the identity $(1-at)\star(1-bt) = (1-abt)$.

This description of Witt rings in terms of endomorphism categories was applied to investigate the arithmetic structures of the Bost-Connes quantum statistical mechanical system, see **CoCo** , **MaRe** , **MaTa** .

This relation between the Grothendieck ring and Witt vectors was extended to the higher $K$-theory in **Gray** where an explicit description for the kernels $\mathcal{E}_n(R)$ is obtained, by showing that

$$\mathcal{E}_{n-1}(R) = \mathrm{C}oker(K_n(R) \to K_n(\mathcal{R})),$$

where $\mathcal{R} = (1 + tR[t])^{-1}R[t]$ and $K_n(R) \to K_n(\mathcal{R})$ is a split injection. The identification above is obtained in **Gray** by showing that there is an exact sequence

$$0 \to K_n(R) \to K_n(\mathcal{R}) \to K_{n-1}(\mathcal{E}_R) \to K_{n-1}(R) \to 0. \tag{2.6.6}$$

The identification (2.6.4) for $K_0$ is then recovered as the case with $n = 0$ that gives an identification $\mathcal{E}_0(R) \simeq 1 + t\mathcal{R}$.

**Spectrum of the Endomorphism Category and Witt vectors**

Let $\mathcal{P}_R$ denote the category of finite projective modules over a commutative ring $R$ with unit. Also let $\mathcal{E}_R$ be the endomorphism category recalled in §2.3. By the Segal construction described in §2.3, we obtain associated $\Gamma$-spaces $F_{\mathcal{P}_R}$ and $F_{\mathcal{E}_R}$ and spectra $F_{\mathcal{P}_R}(\mathbb{S}) = K(R)$, the $K$-theory spectrum of $R$, and $F_{\mathcal{E}_R}(\mathbb{S})$, the spectrum of the endomorphism category.

We obtain in the following way a functorial "spectrification" of the Witt ring $W_0(R)$, namely a spectrum $\mathbb{W}(R)$ with $\pi_0\mathbb{W}(R) = W_0(R)$.

**Definition 2.6.2.** *For a commutative ring $R$, with $\mathcal{P}_R$ the category of finite projective modules and $\mathcal{E}_R$ the category of endomorphisms, the spectrum $\mathbb{W}(R)$ is defined as the cofiber $\mathbb{W}(R) := F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})$ obtained from the $\Gamma$-spaces $F_{\mathcal{P}_R} : \Gamma^0 \to \Delta_*$ and $F_{\mathcal{E}_R} : \Gamma^0 \to \Delta_*$ associated to the categories $\mathcal{P}_R$ and $\mathcal{E}_R$.*

**Lemma 2.6.3.** *For a commutative ring $R$, the inclusion of the category $\mathcal{P}_R$ of finite projective modules as the subcategory of the endomorphism category $\mathcal{E}_R$ determines a long exact sequence*

$$\cdots \to \pi_n(F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_{n-1}(F_{\mathcal{P}_R}(\mathbb{S})) \to \cdots$$

$$\cdots \to \pi_0(F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S}))$$

*of the homotopy groups of the spectra $F_{\mathcal{P}_R}(\mathbb{S})$, $F_{\mathcal{E}_R}(\mathbb{S})$ with cofiber $\mathbb{W}(R)$ as in Definition 2.6.2. The spectrum $\mathbb{W}(R)$ satisfies $\pi_0\mathbb{W}(R) = W_0(R)$.*

*Proof.* The functoriality of the Segal construction implies that the inclusion of $\mathcal{P}_R$ as the subcategory of $\mathcal{E}_R$ given by objects $(E, 0)$ with trivial endomorphism determines a map of $\Gamma$-spaces $F_{\mathcal{P}_R} \to F_{\mathcal{E}_R}$, which is a natural transformation of the functors $F_{\mathcal{P}_R} : \Gamma^0 \to \Delta_*$ and $F_{\mathcal{E}_R} : \Gamma^0 \to \Delta_*$. After passing to endofunctors $F_{\mathcal{P}_R} : \Delta_* \to \Delta_*$ and $F_{\mathcal{E}_R} : \Delta_* \to \Delta_*$ we obtain a map of spectra $K(R) \to F_{\mathcal{E}_R}(\mathbb{S})$, induced by the inclusion of $\mathcal{P}_R$ as subcategory of $\mathcal{E}_R$. The category $\Delta_*$ of simplicial sets has products and equalizers, hence pullbacks. Thus, given two functors $F, F' : \Gamma^0 \to \Delta_*$, a natural transformation $\alpha : F \to F'$ is mono if and only if for all objects $X \in \Gamma^0$ the morphism $\alpha_X : F(X) \to F'(X)$ is a monomorphism in $\Delta_*$. An embedding $C \hookrightarrow C'$ determines by composition an embedding $\Sigma_C(X) \hookrightarrow \Sigma_{C'}(X)$ of the categories of summing functors, for each object $X \in \Gamma^0$. This gives a monomorphism $F_C(X) = \mathcal{N}\Sigma_C(X) \to F_{C'}(X) = \mathcal{N}\Sigma_{C'}(X)$, hence a monomorphism $F_C \to F_{C'}$ of $\Gamma$-spaces. Arguing as in Lemma 1.3 of **Schwede** we then obtain from such a map $F_C \to F_{C'}$ of $\Gamma$-spaces a long exact sequence of homotopy groups of the associated spectra

$$\cdots \to \pi_n(F_C(\mathbb{S})) \to \pi_n(F_{C'}(\mathbb{S})) \to \pi_n(F_{C'}(\mathbb{S})/F_C(\mathbb{S})) \to \pi_{n-1}(F_C(\mathbb{S})) \to \cdots$$

$$\cdots \to \pi_0(F_C(\mathbb{S})) \to \pi_0(F_{C'}(\mathbb{S})) \to \pi_0(F_{C'}(\mathbb{S})/F_C(\mathbb{S})),$$

where $F_{C'}(\mathbb{S})/F_C(\mathbb{S})$ is the cofiber. When applied to the subcategory $\mathcal{P}_R \hookrightarrow \mathcal{E}_R$ this gives the long exact sequence

$$\cdots \to \pi_n(F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_{n-1}(F_{\mathcal{P}_R}(\mathbb{S})) \to \cdots$$

$$\cdots \to \pi_0(F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})).$$

Here we have $\pi_n(F_{\mathcal{P}_R}(\mathbb{S})) = K_n(R)$. Moreover, by construction we have $\pi_0(F_{\mathcal{E}_R}(\mathbb{S})) = K_0(\mathcal{E}_R)$ so that we identify

$$\pi_0(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) = W_0(R) = K_0(\mathcal{E}_R)/K_0(R).$$

Thus, the spectrum $\mathbb{W}(R) := F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})$ given by the cofiber of $F_{\mathcal{P}_R}(\mathbb{S}) \to F_{\mathcal{E}_R}(\mathbb{S})$ provides a spectrum whose zeroth homotopy group is the Witt ring $W_0(R)$. $\square$

The forgetful functor $\mathcal{E}_R \to \mathcal{P}_R$ also induces a corresponding map of $\Gamma$-spaces $F_{\mathcal{E}_R} \to F_{\mathcal{P}_R}$. Moreover, one can also construct a spectrum with $\pi_0$ equal to $W_0(R)$ using the characterization given in **Gray** that we recalled above, in terms of the map on $K$-theory (and on $K$-theory spectra) $K(R) \to K(\mathcal{R})$ with $\mathcal{R} = (1 + rR[t])^{-1}R[t]$. One can obtain in this way a reformulation in terms of spectra of the result of **Gray** . However, for our purposes here, it is preferable to work with the spectrum constructed in Lemma 2.6.3.

We give a variant of Lemma 2.6.3 that will be useful in the following. We denote by $\mathcal{P}_R^{\pm}$ and $\mathcal{E}_R^{\pm}$, respectively, the categories of $\mathbb{Z}/2\mathbb{Z}$-graded finite projective $R$-modules and the $\mathbb{Z}/2\mathbb{Z}$-graded endomorphism category with objects given by pairs $\{(E_+, f_+), (E_-, f_-)\}$, which we write simply as $(E_\pm, f_\pm)$ and with morphisms $\phi : E_\pm \to E'_\pm$ of $\mathbb{Z}/2\mathbb{Z}$-graded finite projective modules that commute with $f_\pm$. The sum in $\mathcal{E}_R^{\pm}$ is given by

$$(E_\pm, f_\pm) \oplus (E'_\pm, f'_\pm) = ((E_+ \oplus E'_+, E_- \oplus E'_-), (f_+ \oplus f'_+, f_- \oplus f'_-))$$

while the tensor product $(E_\pm, f_\pm) \otimes (E'_\pm, f'_\pm)$ is given by

$$((E_+ \otimes E'_+ \oplus E_- \otimes E'_-, f_+ \otimes f'_+ \oplus f_- \otimes f'_-), (E_+ \otimes E'_- \oplus E_- \otimes E'_+, f_+ \otimes f'_- \oplus f_- \otimes f'_+)).$$

Again we consider $\mathcal{P}_R^{\pm}$ as a subcategory of $\mathcal{E}_R^{\pm}$ with trivial endomorphisms.

**Lemma 2.6.4.** *The map* $\delta : K_0(\mathcal{E}_R^{\pm}) \to K_0(\mathcal{E}_R)$ *given by* $[E_\pm, f_\pm] \mapsto [E_+, f_+] - [E_-, f_-]$ *is a ring homomorphism and it descends to a ring homomorphism*

$$K_0(\mathcal{E}_R^{\pm})/K_0(\mathcal{P}_R^{\pm}) \to K_0(\mathcal{E}_R)/K_0(R) \simeq W_0(R).$$

*Proof.* The map is clearly compatible with sums. Compatibility with product also holds since $[E_\pm, f_\pm]\cdot[E'_\pm, f'_\pm] \mapsto ([E_+, f_+]-[E_-, f_-])\cdot([E'_+, f'_+]-[E'_-, f'_-])$. Moreover, it maps $K_0(\mathcal{P}_R^{\pm})$ to $K_0(\mathcal{P}_R)$. $\square$

As before, the categories $\mathcal{P}^{\pm}_R$ and $\mathcal{E}^{\pm}_R$ have associated $\Gamma$-spaces $F_{\mathcal{P}^{\pm}_R} : \Gamma^0 \to \Delta_*$ and $F_{\mathcal{E}^{\pm}_R} : \Gamma^0 \to \Delta_*$ and spectra $F_{\mathcal{P}^{\pm}_R}(\mathbb{S})$ and $F_{\mathcal{E}^{\pm}_R}(\mathbb{S})$. The following result follows as in Lemma 2.6.3.

**Lemma 2.6.5.** *The inclusion of $\mathcal{P}^{\pm}_R$ as a subcategory of $\mathcal{E}^{\pm}_R$ induces a long exact sequence*

$$\cdots \to \pi_n(F_{\mathcal{P}^{\pm}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}^{\pm}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}^{\pm}_R}(\mathbb{S})/F_{\mathcal{P}^{\pm}_R}(\mathbb{S})) \to \pi_{n-1}(F_{\mathcal{P}^{\pm}_R}(\mathbb{S})) \to \cdots$$

$$\cdots \to \pi_0(F_{\mathcal{P}^{\pm}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}^{\pm}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}^{\pm}_R}(\mathbb{S})/F_{\mathcal{P}^{\pm}_R}(\mathbb{S}))$$

*of the homotopy groups of the spectra $F_{\mathcal{P}^{\pm}_R}(\mathbb{S})$ and $F_{\mathcal{E}^{\pm}_R}(\mathbb{S})$, which at the level of $\pi_0$ gives $K_0(\mathcal{P}^{\pm}_R) \to K_0(\mathcal{E}^{\pm}_R) \to K_0(\mathcal{E}^{\pm}_R)/K_0(\mathcal{P}^{\pm}_R)$.*

We denote by $\mathbb{W}^{\pm}(R) = F_{\mathcal{E}^{\pm}_R}(\mathbb{S})/F_{\mathcal{P}^{\pm}_R}(\mathbb{S})$ the cofiber of $F_{\mathcal{P}^{\pm}_R}(\mathbb{S}) \to F_{\mathcal{E}^{\pm}_R}(\mathbb{S})$.

**Remark 2.6.6.** *It is important to point out that our treatment of Witt vectors and their spectrification, as presented in this section, differs from the one in* **Hess** *(see especially Theorem 2.2.9 and equation (2.2.11) in that paper), and in* **Camp** *. Nonetheless, the circle action on THH that is used to obtain the spectrum TR is closely related to the Bost-Connes structure investigated in the present paper. A more direct relation between Bost-Connes structures and topological Hochschild and cyclic homology will also relate naturally to the point of view on $\mathbb{F}_1$-geometry developed in* **CC16** *. We will leave this topic for future work.*

### Exponentiable measures and maps of Γ-spaces

The problem of lifting to the level of spectra the Hasse–Weil zeta function associated to the counting motivic measure for varieties over finite fields was discussed in **CaWoZa** . We consider here a very similar setting and procedure, where we want to lift a zeta function $\zeta_\mu : K_0(\mathcal{V}) \to W(R)$ associated to an exponentiable motivic measure to the level of spectra. To this purpose, we make some assumptions of rationality and the existence of a factorization for our zeta functions of exponentiable motivic measures. We then consider the spectrum $K(\mathcal{V})$ of **Zak1 Zak3** with $\pi_0 K(\mathcal{V}) = K_0(\mathcal{V})$ and a spectrum, obtained from a $\Gamma$-space, associated to the subring $W_0(R)$ of the big Witt ring $W(R)$.

**Definition 2.6.7.** *A motivic measure, that is, a ring homomorphism $\mu : K_0(\mathcal{V}) \to R$ of the Grothendieck ring of varieties to a commutative ring R, is called factorizable if it satisfies the following three properties:*

1. ***e**xponentiability: the associated zeta function $\zeta_\mu(X,t)$ is a ring homomorphism $\zeta_\mu : K_0(\mathcal{V}) \to W(R)$ to the Witt ring of $R$;*

2. ***r**ationality: the homomorphism $\zeta_\mu$ factors through the inclusion of the subring $W_0(R)$ of the Witt ring, $\zeta_\mu : K_0(\mathcal{V}) \to W_0(R) \hookrightarrow W(R)$;*

3. ***f**actorization: the rational functions $\zeta_\mu(X,t)$ admit a factorization into linear factors*
$$\zeta_\mu(X,t) = \frac{\prod_i (1-\alpha_i t)}{\prod_j (1-\beta_j t)} = \zeta_{\mu,+}(X,t) \,-_W\, \zeta_{\mu,-}(X,t)$$
*where $\zeta_{\mu,+}(X,t) = \prod_j (1-\beta_j t)^{-1}$ and $\zeta_{\mu,-}(X,t) = \prod_i (1-\alpha_i t)^{-1}$ and $-_W$ is the difference in the Witt ring, that is the ratio of the two polynomials.*

**Lemma 2.6.8.** *A factorizable motivic measure $\mu : K_0(\mathcal{V}) \to R$, as in Definition 2.6.7, determines a functor $\Phi_\mu : \mathcal{C}_\mathcal{V} \to \mathcal{E}_R^\pm$ where $\mathcal{C}_\mathcal{V}$ is the assembler category encoding the scissor-congruence relations of the Grothendieck ring $K_0(\mathcal{V})$ and $\mathcal{E}_R^\pm$ is the $\mathbb{Z}/2\mathbb{Z}$-graded endomorphism category.*

*Proof.* The objects of $\mathcal{C}_\mathcal{V}$ are varieties $X$ and the morphisms are locally closed embeddings, **Zak1 Zak3** . To an object $X$ we assign an object of $\mathcal{E}_R$ obtained in the following way. Consider a factorization

$$\zeta_\mu(X,t) = \frac{\prod_{i=1}^n (1-\alpha_i t)}{\prod_{j=1}^m (1-\beta_j t)} = \zeta_{\mu,+}(X,t) \,-_W\, \zeta_{\mu,-}(X,t)$$

as above of the zeta function of $X$. Let $E_+^{X,\mu} = R^{\oplus m}$ and $E_-^{X,\mu} = R^{\oplus n}$ with endomorphisms $f_\pm^{X,\mu}$ respectively given in matrix form by $M(f_+^{X,\mu}) = \mathrm{d}iag(\beta_j)_{j=1}^m$ and $M(f_-^{X,\mu}) = \mathrm{d}iag(\alpha_i)_{i=1}^n$. The pair $(E_\pm^{X,\mu}, f_\pm^{X,\mu})$ is an object of the endomorphism category $\mathcal{E}_R^\pm$. Given an embedding $Y \hookrightarrow X$, the zeta function satisfies

$$\zeta_\mu(X,t) = \zeta_\mu(Y,t) \cdot \zeta_\mu(X \smallsetminus Y, t) = \zeta_\mu(Y,t) \,+_W\, \zeta_\mu(X \smallsetminus Y, t).$$

Using the factorizations of each term, this gives

$$(E_\pm^{X,\mu}, f_\pm^{X,\mu}) = (E_\pm^{Y,\mu}, f_\pm^{Y,\mu}) \oplus (E_\pm^{X\smallsetminus Y,\mu}, f_\pm^{X\smallsetminus Y,\mu}),$$

hence a morphism in $\mathcal{E}_R^\pm$ given by the canonical morphism to the direct sum

$$(E_\pm^{Y,\mu}, f_\pm^{Y,\mu}) \to (E_\pm^{X,\mu}, f_\pm^{X,\mu}).$$

□

**Proposition 2.6.9.** *The functor $\Phi_\mu : \mathcal{C}_{\mathcal{V}} \to \mathcal{E}_R^\pm$ of Lemma 2.6.8 induces a map of $\Gamma$-spaces and of the associated spectra $\Phi_\mu : K(\mathcal{V}) \to F_{\mathcal{E}_R^\pm}(\mathbb{S})$. The induced maps on the homotopy groups has the property that the composition*

$$K_0(\mathcal{V}) \stackrel{\Phi_\mu}{\to} K_0(\mathcal{E}_R^\pm) \stackrel{\delta}{\to} K_0(\mathcal{E}_R) \to K_0(\mathcal{E}_R)/K_0(R) = W_0(R) \tag{2.6.7}$$

*with $\delta$ as in Lemma 2.6.4, is given by the zeta function $\zeta_\mu : K_0(\mathcal{V}) \to W_0(R)$.*

*Proof.* The $\Gamma$-space associated to the assembler category $\mathcal{C}_{\mathcal{V}}$ is obtained in the following way, **Zak1 Zak3** . One first associates to the assembler category $\mathcal{C}_{\mathcal{V}}$ another category $\mathcal{W}(\mathcal{C}_{\mathcal{V}})$ whose objects are finite collections $\{X_i\}_{i\in I}$ of non-initial objects of $\mathcal{C}_{\mathcal{V}}$ with morphisms $\varphi = (f, f_i) : \{X_i\}_{i\in I} \to \{X'_j\}_{j\in J}$ given by a map of the indexing sets $f : I \to J$ and morphisms $f_i : X_i \to X'_{f(i)}$ in $\mathcal{C}_{\mathcal{V}}$, such that, for every fixed $j \in J$ the collection $\{f_i : X_i \to X'_j : i \in f^{-1}(j)\}$ is a disjoint covering family of the assembler $\mathcal{C}_{\mathcal{V}}$. This means, in the case of the assembler $\mathcal{C}_{\mathcal{V}}$ underlying the Grothendieck ring of varieties, that the $f_i$ are closed embeddings of the varieties $X_i$ in the given $X'_j$ with disjoint images. We first show that the functor $\Phi_\mu : \mathcal{C}_{\mathcal{V}} \to \mathcal{E}_R^\pm$ of Lemma 2.6.8 extends to a functor (for which we still use the same notation) $\Phi_\mu : \mathcal{W}(\mathcal{C}_{\mathcal{V}}) \to \mathcal{E}_R^\pm$. We define $\Phi_\mu(\{X_i\}_{i\in I}) = \oplus_{i\in I}\Phi_\mu(X_i) = \oplus_{i\in I}(E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu})$. Given a covering family $\{f_i : X_i \to X'_j : i \in f^{-1}(j)\}$ as above, each morphism $f_i : X_i \to X'_j$ determines a morphism $\Phi_\mu(f_i) : (E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu}) \to (E_\pm^{X'_j,\mu}, f_\pm^{X'_j,\mu})$ given by the canonical morphism to the direct sum $(E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu}) \to (E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu}) \oplus (E_\pm^{X'_j\smallsetminus X_i,\mu}, f_\pm^{X'_j\smallsetminus X_i,\mu})$. This determines a morphism $\Phi_\mu(\varphi) : \oplus_{i\in I}(E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu}) \to \oplus_{j\in J}(E_\pm^{X'_j,\mu}, f_\pm^{X'_j,\mu})$. We then show that the functor $\Phi_\mu : \mathcal{W}(\mathcal{C}_{\mathcal{V}}) \to \mathcal{E}_R^\pm$ constructed in this way determines a map of the associated $\Gamma$-spaces. The $\Gamma$-space associated to $\mathcal{W}(\mathcal{C}_{\mathcal{V}})$ is constructed in **Zak1 Zak3** as the functor that assigns to a finite pointed set $S \in \Gamma^0$ the simplicial set given by the nerve $\mathcal{N}\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}})$, where the coproduct of assemblers $S \wedge \mathcal{C}_{\mathcal{V}} = \bigvee_{s\in S\smallsetminus\{s_0\}} \mathcal{C}_{\mathcal{V}}$ has an initial object and a copy of the non-initial objects of $\mathcal{C}_{\mathcal{V}}$ for each point $s \in S \smallsetminus \{s_0\}$ and morphisms induced by those of $\mathcal{C}_{\mathcal{V}}$. This means that we can regard objects of $\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}})$ as collections $\{X_{s,i}\}_{i\in I}$, for some $s \in S \smallsetminus \{s_0\}$ and morphisms $\varphi_s = (f_s, f_{s,i}) : \{X_{s,i}\}_{i\in I} \to \{X'_{s,j}\}_{j\in J}$ as above. In order to obtain a map of $\Gamma$-spaces between $F_{\mathcal{V}} : S \mapsto \mathcal{N}\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}})$ and $F_{\mathcal{E}_R^\pm} : S \mapsto \mathcal{N}\Sigma_{\mathcal{E}_R^\pm}(S)$, we construct a functor $\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}}) \to \Sigma_{\mathcal{E}_R^\pm}(S)$ from the category $\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}})$ described above to the category of summing functors $\Sigma_{\mathcal{E}_R^\pm}(S)$. To an object $X_{S,I} := \{X_{s,i}\}_{i\in I}$ in $\mathcal{W}(S \wedge \mathcal{C}_{\mathcal{V}})$ we associate a functor $\Phi_{X_{S,I}} : \mathcal{P}(S) \to \mathcal{E}_R^\pm$ that maps a subset $A_+ = \{s_0\} \sqcup A \in \mathcal{P}(X)$ to $\Phi_{X_{S,I}}(A_+) = \oplus_{a\in A}\Phi_\mu(\{X_{a,i}\}_{i\in I})$ where $\Phi_\mu : \mathcal{W}(\mathcal{C}_{\mathcal{V}}) \to \mathcal{E}_R^\pm$

is the functor constructed above. It is a summing functor since $\Phi_{X_{S,I}}(A_+ \cup B_+) = \Phi_{X_{S,I}}(A_+) \oplus \Phi_{X_{S,I}}(B_+)$ for $A_+ \cap B_+ = \{s_0\}$. This induces a map of simplicial sets $\mathcal{NW}(S \wedge C_{\mathcal{V}}) \to \mathcal{N}\Sigma_{\mathcal{E}_R^\pm}(S)$ which determines a natural transformation of the functors $F_{\mathcal{V}} : S \mapsto \mathcal{NW}(S \wedge C_{\mathcal{V}})$ and $F_{\mathcal{E}_R^\pm} : S \mapsto \mathcal{N}\Sigma_{\mathcal{E}_R^\pm}(S)$. This map of $\Gamma$-spaces in turn determines a map of the associated spectra and an induced map of their homotopy groups. It remains to check that the induced map at the level of $\pi_0$ agrees with the expected map of Grothendieck rings $K_0(\mathcal{V}) \to K_0(\mathcal{E}_R^\pm)$, hence with the zeta function when further mapped to $K_0(\mathcal{E}_R)$ and to the quotient $K_0(\mathcal{E}_R)/K_0(R)$. This is the case since by construction the induced map $\pi_0 K(\mathcal{V}) = K_0(\mathcal{V}) \to K_0(\mathcal{E}_R^\pm) = \pi_0 F_{\mathcal{E}_R^\pm}(\mathbb{S})$ is given by the assignment $[X] \mapsto [E_\pm^{X,\mu}, f_\pm^{X,\mu}]$. □

**Corollary 2.6.10.** *The map of Grothendieck rings given by the composition* (2.6.7) *also lifts to a map of spectra.*

*Proof.* It is possible to realize the map $\delta : K_0(\mathcal{E}_R^\pm) \to K_0(\mathcal{E}_R)$ of Lemma 2.6.4 at the level of spectra. The $K$-theory spectrum of an abelian category $\mathcal{A}$ is weakly equivalent to the $K$-theory spectrum of the category of bounded chain complexes over $\mathcal{A}$. In fact, this holds more generally for $\mathcal{A}$ an exact category closed under kernels. Thus, in the case of the category $\mathcal{E}_R$, there is a weak equivalence $K(\mathrm{C}h^\flat(\mathcal{E}_R)) \tilde{\to} K(\mathcal{E}_R)$ which descends on the level $\pi_0$ to the map $K_0(\mathrm{C}h^\flat(\mathcal{E}_R)) \tilde{\to} K_0(\mathcal{E}_R)$ given by $[E^\cdot, f^\cdot] \mapsto \sum_k (-1)^k [E^k, f^k]$. To an object $(E^\pm, f^\pm)$ of $\mathcal{E}_R^\pm$ we can assign a chain complex in $\mathrm{C}h^\flat(\mathcal{E}_R)$ of the form $0 \to (E^-, f^-) \xrightarrow{0} (E^+, f^+) \to 0$, where $(E^+, f^+)$ sits in degree 0. This descends on the level of K-theory to a map $K(\mathcal{E}_R^\pm) \to K(\mathrm{C}h^\flat(\mathcal{E}_R))$, which at the level of $\pi_0$ gives the map $[E^\pm, f^\pm] \mapsto [E^+, f^+] - [E^-, f^-]$. The functor $\mathcal{E}_R^\pm \to \mathrm{C}h^\flat(\mathcal{E}_R)$ used here does not respect tensor products, although the induced map $\delta : K_0(\mathcal{E}_R^\pm) \to K_0(\mathcal{E}_R)$ at the level of $K_0$ is compatible with products. Thus, the composition (2.6.7) can also be lifted at the level of spectra. □

It should be noted that the construction of a derived motivic zeta function outlined above is not the first to appear in the literature. In **CaWoZa** , the authors describe a derived motivic measure $\zeta : K(\mathcal{V}_k) \to K(\mathrm{Rep}_{cts}(\mathrm{Gal}(k^s/k); \mathbb{Z}_\ell))$ from the Grothendieck spectrum of varieties to the $K$-theory spectrum of the category of continuous $\ell$-adic Galois representations. This map corresponds to the assignment $X \mapsto H^*_{\mathrm{et},c}(X \times_k k^s, \mathbb{Z}_\ell)$. In particular, they show that when $k = \mathbb{F}_q$ for $\ell$ coprime to $q$, on the level of $\pi_0$, $\zeta$ corresponds to the Hasse-Weil zeta function. They then use $\zeta$ to prove that $K_1(\mathcal{V}_{\mathbb{F}_q})$ is not only nontrivial, but contains interesting algebro-geometric data.

Essentially, the approach in **CaWoZa** was to start with a Weil Cohomology theory (in this case, $\ell$-adic cohomology) and then to construct a derived motivic measure realizing on the level of $K$-theory the assignment to a variety $X$ of its corresponding cohomology groups. The methods used in the case of $\ell$-adic cohomology may not immediately generalize to other Weil cohomology theories. This method has yielded deep insight into the world of algebraic geometry. Our approach here, in contrast, is to take an interesting class of motivic measures, namely Kapranov motivic zeta functions (exponentiable motivic measures, **Kapr Ram RamTab**), and to determine reasonable conditions under which such a motivic measure can be derived directly. This method still needs to be studied further to yield additional insights into what it captures about the geometry of varieties.

**Bost-Connes type systems via motivic measures**

The lifting of the integral Bost-Connes algebra to various Grothendieck rings, their assembler categories, and the associated spectra, that we discussed in **ManMar2** and in the earlier sections of this paper, can be viewed as an instance of a more general kind of operation. As discussed in **CoCo2** there is a close relation between the endomorphisms $\sigma_n$ and the maps $\tilde{\rho}_n$ of the integral Bost-Connes algebra and the operation of Frobenius and Verschiebung in the Witt ring. Thus, we can formulate a more general form of the question investigated above, of lifting of the integral Bost-Connes algebra to a Grothendieck ring through an Euler characteristic map, in terms of lifting the Frobenius and Verschiebung operations of a Witt ring to a Grothendieck ring through the zeta function $\zeta_\mu$ of an exponentiable motivic measure. A prototype example of this more general setting is provided by the Hasse–Weil zeta function $Z : K_0(\mathcal{V}_{\mathbb{F}_q}) \to W(\mathbb{Z})$, which has the properties that the action of the Frobenius $F_n$ on the Witt ring $W(\mathbb{Z})$ corresponds to passing to a field extension, $F_n Z(X_{\mathbb{F}_q}, t) = Z(X_{\mathbb{F}_{q^n}}, t)$ and the action of the Verschiebung $V_n$ on the Witt ring $W(\mathbb{Z})$ is related to the Weil restriction of scalars from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$ (see **Ram** for a precise statement).

Recall that, if one denotes by $[a]$ the elements $[a] = (1-at)^{-1}$ in the Witt ring $W(R)$, for $a \in R$, then the Frobenius ring homomorphisms $F_n : W(R) \to W(R)$ of the Witt ring are determined by $F_n([a]) = [a^n]$ and the Verschiebung group homomorphisms $V_n : W(R) \to W(R)$ are defined on an arbitrary $P(t) \in W(R)$ as $F_n : P(t) \mapsto P(t^n)$.

These operations satisfy an analog of the Bost-Connes relations

$$F_n \circ F_m = F_{nm}, \ V_n \circ V_m = V_{nm}, \ \ F_n \circ V_n = n \cdot id, \ \ F_n \circ V_m = V_m F_n \text{ if } (n,m) = 1. \quad (2.6.8)$$

These correspond, respectively, to the semigroup structure of the $\sigma_n$ and $\tilde{\rho}_n$ of the integral Bost-Connes algebra and the relations $\sigma_n \circ \tilde{\rho}_n = n \cdot \mathrm{i}d$, while the last relation is determined in the Bost-Connes case by the commutation of the generators $\tilde{\mu}_n$ and $\mu_m^*$ for $(n,m) = 1$.

**Definition 2.6.11.** *A factorizable motivic measure $\mu : K_0(\mathcal{V}) \to R$, in the sense of Definition 2.6.7, is of* Bost-Connes type *if there is a lift to $K_0(\mathcal{V})$ of the Frobenius $F_n$ and Verschiebung $V_n$ of the Witt ring $W(R)$ to $K_0(\mathcal{V})$ such that the diagrams commute*

$$\begin{array}{ccc} K_0(\mathcal{V}) & \xrightarrow{\zeta_\mu} & W(R) \\ \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle F_n} \\ K_0(\mathcal{V}) & \xrightarrow{\zeta_\mu} & W(R) \end{array} \qquad \begin{array}{ccc} K_0(\mathcal{V}) & \xrightarrow{\zeta_\mu} & W(R) \\ \downarrow{\scriptstyle \tilde{\rho}_n} & & \downarrow{\scriptstyle V_n} \\ K_0(\mathcal{V}) & \xrightarrow{\zeta_\mu} & W(R) \end{array}$$

*Such a motivic measure $\mu : K_0(\mathcal{V}) \to R$ is of* homotopic Bost-Connes type *if the maps $\sigma_n$ and $\tilde{\rho}_n$ in the diagrams above also lift to endofunctors of the assembler category $C_{\mathcal{V}}$ of the Grothendieck ring $K_0(\mathcal{V})$ with the endofunctors $\sigma_n$ compatible with the monoidal structure.*

**Definition 2.6.12.** *The Frobenius and Verschiebung on the category $\mathcal{E}_R^{\pm}$ are defined as the endofunctors $F_n(E, f) = (E, f^n)$ and $V_n(E_\pm, f_\pm) = (E_\pm^{\oplus n}, V_n(f_\pm))$ with $V_n(f)$ defined by*

$$V_n : (E, f) \mapsto (E^{\oplus n}, V_n(f)), \quad V_n(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & f \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \qquad (2.6.9)$$

It is worth noting that the endofunctors of Definition 2.6.12 are akin to those used in the definitions of topological cyclic and topological restriction homology, **HesMa** .

**Lemma 2.6.13.** *The Frobenius and Verschiebung $F_n$ and $V_n$ of Definition 2.6.12 are endofunctors of the category $\mathcal{E}_R^{\pm}$ with the property that the maps they induce on $W_0(R) = K_0(\mathcal{E}_R)/K_0(R)$ agree with the restrictions to $W_0(R) \subset W(R)$ of the Frobenius and Verschiebung maps. These endofunctors determine natural transformations (still denoted $F_n$ and $V_n$) of the $\Gamma$-space $F_{\mathcal{E}_R^{\pm}} : \Gamma^0 \to \Delta_*$.*

*Proof.* The homomorphism $K_0(\mathcal{E}_R) \to W_0(R)$ given by

$$(E, f) \mapsto L(E, f) = \det(1 - tM(f))^{-1}$$

sends the pair $(R, f_a)$ with $f_a$ acting on $R$ as multiplication by $a \in R$ to the element $[a] = (1 - at)^{-1}$ in the Witt ring. The action of the Frobenius $F_n([a]) = [a^n]$ is induced from the Frobenius $F_n(E, f) = (E, f^n)$ which is an endofunctor of $\mathcal{E}_R$. This extends to a compatible endofunctor of $\mathcal{E}_R^{\pm}$ by $F_n(E_\pm, f_\pm) = (E_\pm, f_\pm^n)$. Similarly, the Verschiebung map that sends $\det(1 - tM(f))^{-1} \mapsto \det(1 - t^n M(f))^{-1}$ is induced from the Verschiebung on $\mathcal{E}_R$ given by (2.6.9), since we have $L(E^{\oplus n}, V_n(f)) = \det(1 - t^n M(f))^{-1}$, with compatible endofunctors $V_n(E_\pm, f_\pm) = (E_\pm^{\oplus n}, V_n(f_\pm))$ on $\mathcal{E}_R^{\pm}$. The Frobeniius and Verschiebung on $\mathcal{E}_R^{\pm}$ induce natural transformations of the $\Gamma$-space $F_{\mathcal{E}_R^{\pm}} : \Gamma^0 \to \Delta_*$ by composition of the summing functors $\Phi : \mathcal{P}(X) \to \mathcal{E}_R^{\pm}$ in $\Sigma_{\mathcal{E}_R^{\pm}}(X)$ with the endofunctors $F_n$ and $V_n$ of $\mathcal{E}_R^{\pm}$. □

**Proposition 2.6.14.** *Let $\mu : K_0(\mathcal{V}) \to R$ be a factorizable motivic measure, as in Definition 2.6.7, that is of homotopical Bost-Connes type. Then the endofunctors $\sigma_n$ and $\tilde{\rho}_n$ of the assembler category $C_{\mathcal{V}}$ determine natural transformations (still denoted by $\sigma_n$ and $\tilde{\rho}_n$) of the associated $\Gamma$-space $F_{\mathcal{V}} : \Gamma^0 \to \Delta_*$ that fit in the commutative diagrams*

$$\begin{tikzcd} F_{\mathcal{V}} \arrow[r, "\Phi_\mu"] \arrow[d, "\sigma_n"] & F_{\mathcal{E}_R^{\pm}} \arrow[d, "F_n"] \\ F_{\mathcal{V}} \arrow[r, "\Phi_\mu"] & F_{\mathcal{E}_R^{\pm}} \end{tikzcd} \qquad \begin{tikzcd} F_{\mathcal{V}} \arrow[r, "\Phi_\mu"] \arrow[d, "\tilde{\rho}_n"] & F_{\mathcal{E}_R^{\pm}} \arrow[d, "V_n"] \\ F_{\mathcal{V}} \arrow[r, "\Phi_\mu"] & F_{\mathcal{E}_R^{\pm}} \end{tikzcd}$$

*where $\Phi_\mu : F_{\mathcal{V}} \to F_{\mathcal{E}_R^{\pm}}$ is the natural transformation of $\Gamma$-spaces of* (2.6.9) *and $F_n$ and $V_n$ are the natural transformations of Lemma 2.6.13.*

*Proof.* The natural transformation $\Phi_\nu$ is determined as in Proposition 2.6.9 by the functor $\Phi_\mu : C_{\mathcal{V}} \to \mathcal{E}_R^{\pm}$ that assigns $\Phi_\mu : X \mapsto (E_\pm^X, f_\pm^X)$ constructed as in Lemma 2.6.8. Suppose we have endofunctors $\sigma_n$ and $\tilde{\rho}_n$ of the assembler category $C_{\mathcal{V}}$ that induce maps $\sigma_n$ and $\tilde{\rho}_n$ on $K_0(\mathcal{V})$ that lift the Frobenius and Verschiebung maps of $W(R)$ through the zeta function $\zeta_\mu : K_0(\mathcal{V}) \to W(R)$. This means that $\zeta_\mu(\sigma_n(X), t) = F_n \zeta_\mu(X, t)$ and $\zeta_\mu(\tilde{\rho}_n(X), t) = V_n \zeta_\mu(X, t) = \zeta_\mu(X, t^n)$. By Lemma 2.6.13, we have $F_n \zeta_\mu(X, t) = L(F_n(E_\pm^X, f_\pm^X)) = L(E_\pm^X, (f_\pm^X)^n)$ and $V_n \zeta_\mu(X, t) = L(V_n(E_\pm^X, f_\pm^X)) = L((E_\pm^X)^{\oplus n}, V_n(f_\pm^X))$. This shows the compatibilities of the natural transformations in the diagrams above. □

**Spectra and spectra**

We apply a construction similar to the one discussed in the previous subsections to the case of the map $(X, f) \mapsto \sum_{\lambda \in Spec(f_*)} m_\lambda \lambda$ that assigns to a variety over $\mathbb{C}$ with a quasi-unipotent map the spectrum of the induced map $f_*$ in homology, seen as an element in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, as in §6 of **ManMar2** .

In this section the term spectrum will appear both in its homotopy theoretic sense and in its operator sense. Indeed, we consider here a lift to the level of spectra (in the homotopy theoretic sense) of the construction described in §6 of **ManMar2** based on the spectrum (in the operator sense) Euler characteristic.

We consider here a setting as in **EbGuZa GuZa** where $(X, f)$ is a pair of a variety over $\mathbb{C}$ and an endomorphism $f : X \to X$ such that the induced map $f_*$ on $H_*(X, \mathbb{Z})$ has spectrum consisting of roots of unity. As discussed in **ManMar2** and in a related form in **EbGuZa** the spectrum determines a ring homomorphism (an Euler characteristic)

$$\sigma : K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \tag{2.6.10}$$

where $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ denotes the Grothendieck ring of pairs $(X, f)$ with the operations defined by the disjoint union and the Cartesian product. It is shown in **ManMar2** that one can lift the operations $\sigma_n$ and $\tilde{\rho}_n$ of the integral Bost-Connes algebra from $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ to $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ via the “spectral Euler characteristic" (2.6.10), and that the operations can further be lifted from $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ to a (homotopy theoretic) spectrum with $\pi_0$ equal to $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ via the assembler category construction of **Zak1** .

In the next sub section we discuss how to lift the right hand side of (2.6.10), namely the original Bost-Connes algebra $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ with the operations $\sigma_n$ and $\tilde{\rho}_n$ to the level of a homotopy theoretic spectrum, so that the spectral Euler characteristic (2.6.10) becomes induced by a map of spectra.

**Bost-Connes Tannakian categorification and lifting of the spectral Euler characteristic**

To construct a categorification of the map (2.6.10) compatible with the Bost-Connes structure, we use the lift of the left-hand side of (2.6.10) to an assembler category, as in Proposition 6.6 of **ManMar2** , while for the right-hand side of (2.6.10) we use the categorification of Bost-Connes system constructed in **MaTa** .

We begin by recalling the categorification of the Bost-Connes algebra of **MaTa** . Let $\mathrm{V}ect_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ be the category of pairs $(W, \oplus_{r \in \mathbb{Q}/\mathbb{Z}} \bar{W}_r)$ with $W$ a finite dimensional $\mathbb{Q}$-

vector space and $\oplus_r \bar{W}_r$ a $\mathbb{Q}/\mathbb{Z}$-graded vector space with $\bar{W} = W \otimes \bar{\mathbb{Q}}$. This is a neutral Tannakian category with fiber functor the forgetful functor $\omega : \mathcal{V}ect^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q}) \to \mathcal{V}ect(\mathbb{Q})$ and with $\mathcal{A}ut^{\otimes}(\omega) = \mathcal{S}pec(\bar{\mathbb{Q}}[\mathbb{Q}/\mathbb{Z}]^G)$ and $G = \mathcal{G}al(\bar{\mathbb{Q}}/\mathbb{Q})$, see Theorem 3.2 of **MaTa** . The category $\mathcal{V}ect^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ is endowed with additive symmetric monoidal functors $\sigma_n(W) = W$ and $\overline{\sigma_n(W)}_r = \oplus_{r' : \sigma_n(r')=r} \bar{W}_{r'}$ if $r$ is in the range of $\sigma_n$ and zero otherwise and additive functors $\tilde{\rho}_n(W) = W^{\oplus n}$ and $\overline{\tilde{\rho}_n(W)}_r = \bar{W}_{\sigma_n(r)}$ satisfying $\sigma_n \circ \tilde{\rho}_n = n \cdot id$ that induce the Bost-Connes maps on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$.

As shown in Theorem 3.18 of **MaTa** this category can be equivalently described as a category of automorphisms $\mathcal{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ with objects pairs $(W, \phi)$ of a $\mathbb{Q}$-vector space $V$ and a $G$-equivariant diagonalizable automorphism of $\bar{W}$ with eigenvalues that are roots of unity (seen as elements in $\mathbb{Q}/\mathbb{Z}$). There is an equivalence of categories between $\mathcal{V}ect^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ and $\mathcal{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ under which the functors $\sigma_n$ and $\tilde{\rho}_n$ correspond, respectively, to the Frobenius and Verschiebung on $\mathcal{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$, given by

$$F_n : (W, \phi) \mapsto (W, \phi^n), \quad V_n : (W, \phi) \mapsto (W^{\oplus n}, V_n(\phi)), \tag{2.6.11}$$

with

$$V_n(\phi) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \phi \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{2.6.12}$$

The equivalence is realized by mapping $(W, \phi) \mapsto (W, \oplus_r \bar{W}_r)$ where $\bar{W}_r$ are the eigenspaces of $\phi$ with eigenvalue $r \in \mathbb{Q}/\mathbb{Z}$.

**Remark 2.6.15.** *Conceptually, the first description of the categorification in terms of the Tannakian category $\mathcal{V}ect^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ is closer to the integral Bost-Connes algebra as introduced in* **CCM** *, while its equivalent description in terms of $\mathcal{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ is closer to the reinterpretation of the Bost-Connes algebra in terms of Frobenius and Verschiebung operators, as in* **CoCo2** *. Since we have introduced here the Bost-Connes algebra in the form of* **CCM** *, we are recalling both of these descriptions of the categorification, even though in the following we will be using only the one in terms of $\mathcal{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$.*

**Proposition 2.6.16.** *Let $C^{\mathbb{Z}}_{\mathbb{C}}$ be the assembler category underlying $K^{\mathbb{Z}}_0(\mathcal{V}_{\mathbb{C}})$, as in Proposition 6.6 of* **ManMar2** *. The assignment $\Phi(X, f) = (H_*(X, \mathbb{Q}), \oplus_r E_r(f_*))$, where $E_r(f_*)$ is the eigenspace with eigenvalue $r \in \mathbb{Q}/\mathbb{Z}$, determines a functor $\Phi$ :*

*$\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}} \to Aut_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ that lifts the Frobenius and Vershiebung functors on $Aut_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ to the endofunctors $\sigma_n$ and $\tilde{\rho}_n$ of $\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}}$ implementing the Bost-Connes structure.*

*Proof.* We can construct the functor from the assembler category $\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}}$ of §6 of **ManMar2** underlying $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ to $\mathrm{A}ut_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ by following along the lines of Lemma 2.6.8 and Proposition 2.6.9, where we assign $\Phi(X, f) = (H_*(X, \mathbb{Q}), \oplus_r E_r(f_*))$ where $E_r(f_*)$ is the eigenspace with eigenvalue $r \in \mathbb{Q}/\mathbb{Z}$. The Bost-Connes algebra then lifts to the Frobenius and Vershiebung functors on $\mathrm{A}ut_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ and the latter lift to geometric Frobenius and Verschiebung operations on the pairs $(X, f)$ mapping to $(X, f^n)$ and to $(X \times Z_n, \Phi_n(f))$. ◻

This point of view, that replaces the Bost-Connes algebra with it categorification in terms of the Tannakian category $\mathrm{A}ut_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ as in **MaTa** will also be useful in § 2.7, where we reformulate our categorical setting, by passing from Grothendieck rings, assemblers and spectra, to Tannakian categories of Nori motives, and we compare in Lemma 2.7.4 and Theorem 2.7.7 the categorification of the Bost-Connes algebra obtained via Nori motives with the one of **MaTa** recalled here.

### 2.7 Bost-Connes systems in categories of Nori motives

We introduce in this section a motivic framework, with Bost-Connes type systems that on Tannakian categories of motives. The main result in this part of the paper will be Theorem 2.7.7, showing the existence of a fiber functor from the Tannakian category of Nori motives with good effectively finite $\hat{\mathbb{Z}}$-action to the Tannakian category $\mathrm{A}ut_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ that lifts the Bost-Connes system given by Frobenius and Verschiebung on the target category to a Bost-Connes system on Nori motives. Proposition 2.7.9 then extends this Bost-Connes structure to the relative case of motivic sheaves.

This is a natural generalization of the approach to Grothendieck rings via assemblers, which can be extended in an interesting way to the domain of motives, namely, Nori motives.

Roughly speaking, the theory of Nori motives starts with lifting the relations

$$[f : X \to S] = [f|_Y : Y \to S] + [f|_{X \smallsetminus Y} : X \smallsetminus Y \to S] \tag{2.7.1}$$

of (relative) Grothendieck rings $K_0(\mathcal{V}_S)$ to the level of "diagrams," which intuitively can be imagined as "categories without multiplication of morphisms."

### Nori diagrams

More precisely, (cf. Definition 7.1.1 of **HuM-S17** p. 137), we have the following definitions.

**Definition 2.7.1.** *A diagram (also called a quiver) D is a family consisting of a set of vertices $V(D)$ and a set of oriented edges, $E(D)$. Each edge e either connects two different vertices, going, say, from a vertex $\partial_{out}e = v_1$ to a vertex $\partial_{in}e = v_2$, or else is "an identity," starting and ending with one and the same vertex $v$. We will consider only diagrams with one identity for each vertex.*

Diagrams can be considered as objects of a category, with obvious morphisms.

**Definition 2.7.2.** *Each small category $C$ can be considered as a diagram $D(C)$, with $V(D(C)) = Ob\, C$, $E(D(C)) = Mor\, C$, so that each morphism $X \to Y$ "is" an oriented edge from $X$ to $Y$. More generally, a representation $T$ of a diagram $D$ in a (small) category $C$ is a morphism of directed graphs $T : D \to D(C)$.*

Notice that a considerably more general treatment of graphs with markings, including diagrams etc. in the operadic environment, can be found in **MaBo07** . We do not use it here, although it might be highly relevant.

### From geometric diagrams to Nori motives

We recall the main idea in the construction of Nori motives from geometric diagrams. For more details, see **HuM-S17** pp. 140–144.

1. *Start* with the following data:

   a) a diagram $D$;

   b) a noetherian commutative ring with unit $R$ and the category of finitely generated $R$–modules $R$-Mod;

   c) a representation $T$ of $D$ in $R$-Mod, in the sense of Definition 2.7.2.

2. *Produce* from them the category $C(D,T)$ defined in the following way:

   d1) If $D$ is finite, then $C(D,T)$ is the category of finitely generated $R$–modules equipped with an $R$–linear action of $End(T)$.

d2) If $D$ is infinite, first consider its all finite subdiagrams $F$.

d3) For each $F$ construct $C(F, T|_F)$ as in d1). Then apply the following limiting procedure:

$$C(D,T) := colim_{F \subseteq D \text{ finite}}\ C(F, T|_F)$$

Thus, the category $C(D,T)$ has the following structure:

- Objects of $C(D,T)$ will be all objects of the categories $C(F, T|_F)$. If $F \subset F'$, then each object $X_F$ of $C(F, T|_F)$ can be canonically extended to an object of $C(F', T|_{F'})$.
- Morphisms from $X$ to $Y$ in $C(D,T)$ will be defined as colimits over $F$ of morphisms from $X_F$ to $Y_F$ with respect to these extensions.

d4) The fact that $C(D,T)$ has a functor to $R$-Mod follows directly from the definition and the finite case.

3. *The result* is called *the diagram category* $C(D,T)$.

   It is an $R$–linear abelian category which is endowed with $R$–linear faithful exact forgetful functor

$$f_T\, :\; C(D,T) \to R\text{-Mod}.$$

**Universal diagram category**

The following results explain why abstract diagram categories play a central role in the formalism of Nori motives: they formalize the Grothendieck intuition of motives as objects of the universal cohomology theory.

**Theorem 2.7.3. HuM-S17**

*(i) Any representation $T\, :\; D \to R$-Mod can be presented as post-composition of the forgetful functor $f_T$ with an appropriate representation $\tilde{T}\, :\; D \to C(D,T)$:*

$$T = f_T \circ \tilde{T}$$

*with the following universal property:*

*Given any $R$–linear abelian category $A$ with a representation $F\, :\; D \to A$ and $R$–linear faithful exact functor $f\, :\; A \to R$-Mod with $T = f \circ F$, it factorizes* *through a faithful exact functor $L(F) : C(D,T) \to A$ compatibly with the decomposition*

$$T = f_T \circ \tilde{T}.$$

(ii) *The functor $L(F)$ is unique up to unique isomorphism of exact additive functors.*

For proofs, cf. **HuM-S17** pp. 140–141 and p. 167.

### Nori geometric diagrams

If we start not with an abstract category but with a "geometric" category $C$ (in the sense that its objects are spaces/varieties/schemes, possibly endowed with additional structures), in which one can define morphisms of closed embeddings $Y \hookrightarrow X$ (or $Y \subset X$) and morphisms of complements to closed embeddings $X \setminus Y \to X$, we can define the Nori diagram of *effective pairs* $D(C)$ in the following way (see **HuM-S17** pp. 207–208).

a) One vertex of $D(C)$ is a triple $(X, Y, i)$ where $Y \hookrightarrow X$ is a closed embedding, and $i$ is an integer.

b) Besides obvious identities, there are edges of two types.

b1) Let $(X, Y)$ and $(X', Y')$ be two pairs of closed embeddings. Every morphism $f : X \to X'$ such that $f(Y) \subset Y'$ produces functoriality edges $f^*$ (or rather $(f^*, i)$) going from $(X', Y', i)$ to $(X, Y, i)$.

b2) Let $(Z \subset Y \subset X)$ be a stair of closed embeddings. Then it defines coboundary edges $\partial$ from $(Y, Z, i)$ to $(X, Y, i + 1)$.

### (Co)homological representatons of Nori geometric diagrams

If we start not just from the initial category of spaces $C$, but rather from a pair $(C, H)$ where $H$ is a cohomology theory, then assuming reasonable properties of this pair, we can define the respective representation $T_H$ of $D(C)$ that we will call a *(co)homological representation of* $D(C)$.

For a survey of such pairs $(C, H)$ that were studied in the context of Grothendieck's motives, see **HuM-S17** pp. 31–133. The relevant cohomology theories include, in particular, singular cohomology, and algebraic and holomorphic de Rham cohomologies.

Below we will consider the basic example of cohomological representations of Nori diagrams that leads to Nori motives.

### Effective Nori motives

We follow **HuM-S17** pp. 207–208. Take as a category $C$, the starting object in the definition of Nori geometric diagrams above, the category $\mathcal{V}_k$ of varieties $X$ defined over a subfield $k \subset \mathbb{C}$.

We can then define the Nori diagram $D(C)$ as above. This diagram will be denoted $\mathrm{P}airs^{eff}$ from now on,

$$\mathrm{P}airs^{eff} = D(\mathcal{V}_k).$$

The category of effective mixed Nori motives is the diagram category $C(\mathrm{P}airs^{eff}, H^*)$ where $H^i(X, \mathbb{Z})$ is the respective singular cohomology of the analytic space $X^{an}$ (cf. **HuM-S17** pp. 31–34 and further on).

It turns out (see **HuM-S17** Proposition 9.1.2. p. 208) that the map

$$H^* : \mathrm{P}airs^{eff} \to \mathbb{Z}\text{-Mod}$$

sending $(X, Y, i)$ to the relative singular cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$, naturally extends to a representation of the respective Nori diagram in the category of finitely generated abelian groups $\mathbb{Z}$-Mod.

### Category of equivariant Nori motives

We now introduce the specific category of Nori motives that we will be using for the construction of the associated Bost-Connes system.

Let $D(\mathcal{V})$ the Nori geometric diagrams associated to the category $\mathcal{V}$ of varieties over $\mathbb{Q}$, constructed as described in §2.7.

As in **ManMar2** and in § 2.2 of this paper, we consider here the category $\mathcal{V}^{\hat{\mathbb{Z}}}$ of varieties $X$ with a good effectively finite action of $\hat{\mathbb{Z}}$. We can view $\mathcal{V}^{\hat{\mathbb{Z}}}$ as an enhancement $\hat{\mathcal{V}}$ of the category $\mathcal{V}$, in the sense described in §2.3.

Define the Nori diagram of *effective pairs* $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ as we recalled earlier in §2.7:

a) One vertex of $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ is a triple $((X, \alpha_X), (Y, \alpha_Y), i)$, of varieties $X$ and $Y$ with good effectively finite $\hat{\mathbb{Z}}$ actions, $\alpha_X : \hat{\mathbb{Z}} \times X \to X$ and $\alpha_Y : \hat{\mathbb{Z}} \times Y \to Y$, and an integer $i$, together with a closed embedding $j : Y \hookrightarrow X$ that is equivariant with respect to the $\hat{\mathbb{Z}}$ actions. For brevity, we will denote such a triple $(\hat{X}, \hat{Y}, i)$ and call it a closed embedding in the enhancement $\hat{\mathcal{V}}$.

b) Identity edges, functoriality edges, and coboundary edges are obvious enhancements of the respective edges defined in §2.7, with the requirement that all these maps are $\hat{\mathbb{Z}}$-equivariant.

b1) Let $(\hat{X}, \hat{Y})$ and $(\hat{X}', \hat{Y}')$ be two pairs of closed embeddings in $\hat{\mathcal{V}}$. Every morphism $f : X \to X'$ such that $f(Y) \subset Y'$ and $f \circ \alpha_X = \alpha_{X'} \circ f$ produces functoriality edges $f^*$ (or rather $(f^*, i)$) going from $((X', \alpha_{X'}), (Y', \alpha_{Y'|}), i)$ to $(X, Y, i)$.

b2) Let $(Z \subset Y \subset X)$ be a stair of closed embeddings compatible with enhancements (equivariant with respect tot the $\hat{\mathbb{Z}}$-actions). Then it defines coboundary edges $\partial$

$$((Y, \alpha_Y), (Z, \alpha_Z), i) \to ((X, \alpha_X), (Y, \alpha_Y), i + 1).$$

We have thus defined the Nori geometric diagram of enhanced effective pairs, which we denote equivalently by $D(\hat{\mathcal{V}})$ or $D(\mathcal{V}^{\hat{\mathbb{Z}}})$.

Notice that forgetting in this diagram all enhancements, we obtain the map $D(\hat{\mathcal{V}}) \to D(\mathcal{V})$ which is *injective* both on vertices and edges.

**Bost-Connes system on Nori motives**

We now construct a Bost-Connes system on a category of Nori motives obtained from the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ described above, which lifts to the level of motives the categorification of the Bost-Connes algebra constructed in **MaTa** .

As we recalled in §2.6, we can describe the categorification of the Bost-Connes algebra of **MaTa** in terms of the Tannakian category $\mathrm{V}ec^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ with suitable functors $\sigma_n$ and $\tilde{\rho}_n$ constructed as in Theorem 3.7 of **MaTa** or in terms of an equivalent Tannakian category $\mathrm{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ endowed with Frobenius and Verschiebung functors. We are going to use here the second description.

**Lemma 2.7.4.** *The assignment $T : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ determines a representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \to Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ of the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ constructed above.*

*Proof.* As discussed in the previous subsection, we view elements $((X, \alpha_X), (Y, \alpha_Y), i)$ of $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ in terms of an enhancement $\hat{\mathcal{V}}$ of the category $\mathcal{V}$ defined as in §2.3, by choosing a primitive root of unity that generates the cyclic group $\mathbb{Z}/N\mathbb{Z}$, so that the actions $\alpha_X$ and $\alpha_Y$ are determined by self maps $v_X$ and $v_Y$ as in §2.3. We identify the element above with $((X, v_X), (Y, v_Y), i)$, which we also denoted by $(\hat{X}, \hat{Y}, i)$ in the previous subsection. Since the embedding $Y \hookrightarrow X$ is $\hat{\mathbb{Z}}$-equivariant, the map $v_Y$ is the restriction to $Y$ of the map $v_X$ under this embedding. We denote by $\phi^i$ the induced map on the cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$. The eigenspaces of $\phi^i$ are the subspaces of the decomposition of $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ according to characters of $\hat{\mathbb{Z}}$, that is, elements in $\mathrm{H}om(\hat{\mathbb{Z}}, \mathbb{C}^*) = \nu^* \simeq \mathbb{Q}/\mathbb{Z}$. Thus, we obtain an object $(H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q}), \phi^i)$ in the category $\mathrm{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$. Edges in the diagram are $\hat{\mathbb{Z}}$-equivariant maps so they induce morphisms between the corresponding objects in the category $\mathrm{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$. □

One can also see in a similar way that the fiber functor $T : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ determines an object in the category $\mathrm{V}ec^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$. Indeed, the pair $(X, Y)$ with $Y \subset X$ is endowed with compatible good effectively finite $\hat{\mathbb{Z}}$-actions $\alpha_X$ and $\alpha_Y$, hence the singular cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ carries a resulting $\hat{\mathbb{Z}}$-representation. Thus, the vector space $H^i(X(\mathbb{C}), Y(\mathbb{C}), \bar{\mathbb{Q}})$ can be decomposed into eigenspaces of this representations according to characters $\chi \in \mathrm{H}om(\hat{\mathbb{Z}}, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$. Thus, we obtain a decomposition of $H^i(X(\mathbb{C}), Y(\mathbb{C}), \bar{\mathbb{Q}}) = \oplus_{r \in \mathbb{Q}/\mathbb{Z}} \bar{V}_r$ as a $\mathbb{Q}/\mathbb{Z}$-graded vector space. We choose to work with the category $\mathrm{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ because the Bost-Connes structure is more directly expressed in terms of Frobenius and Verschiebung, which will make the lifting of this structure to the resulting category of Nori motives more immediately transparent, as we discuss below.

The representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \to \mathrm{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ replaces, at this motivic level, our previous use in **ManMar2** of the equivariant Euler characteristics $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ (see **Looij**) as a way to lift the Bost-Connes algebra. We proceed in the following way to obtain the Bost-Connes structure in this setting.

**Definition 2.7.5.** *Let $D$ be a diagram, endowed with a representation $T : D \to Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$, and let $C(D, T)$ be the associated diagram category, obtained as in §2.7, with the induced functor $\tilde{T} : C(D, T) \to Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$. We say that the functor $\tilde{T}$ intertwines the Bost-Connes structure, if there are endofunctors $\sigma_n$ and $\tilde{\rho}_n$ of $C(D, T)$ (where the $\sigma_n$ but not the $\tilde{\rho}_n$ are compatible with the tensor product structure) such* *that the following diagrams commute,*

$$\begin{tikzcd} \mathcal{C}(D,T) \arrow[r, "\tilde{T}"] \arrow[d, "\sigma_n"] & Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q}) \arrow[d, "F_n"] \\ \mathcal{C}(D,T) \arrow[r, "\tilde{T}"] & Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(D,T) \arrow[r, "\tilde{T}"] & Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q}) \\ \mathcal{C}(D,T) \arrow[r, "\tilde{T}"] \arrow[u, "\tilde{\rho}_n"] & Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q}) \arrow[u, "V_n"'] \end{tikzcd}$$

*where on the right-hand-side of the diagrams, the $F_n$ and $V_n$ are the Frobenius and Verschiebung on $Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$, defined as in* (2.6.11) *and* (2.6.12)*.*

**Definition 2.7.6.** *For $((X,\alpha_X),(Y,\alpha_Y),i)$ in the category $\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}),T)$ of Nori motives associated to the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ define*

$$\sigma_n : ((X,\alpha_X),(Y,\alpha_Y),i) \mapsto ((X,\alpha_X\circ\sigma_n),(Y,\alpha_Y\circ\sigma_n),i) \tag{2.7.2}$$

$$\tilde{\rho}_n : ((X,\alpha_X),(Y,\alpha_Y),i) \mapsto (X\times Z_n,\Phi_n(\alpha_X),(Y\times Z_n,\Phi_n(\alpha_Y)),i), \tag{2.7.3}$$

*where $Z_n = Spec(\mathbb{Q}^n)$ and $\Phi_n(\alpha)$ is the geometric Verschiebung defined as in §2.2.*

**Theorem 2.7.7.** *The $\sigma_n$ and $\tilde{\rho}_n$ of* (2.7.2) *and* (2.7.3) *determine a Bost-Connes system on the category $\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}),T)$ of Nori motives associated to the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$. The representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \to Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ constructed above has the property that the induced functor*

$$\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}),T) \to Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$$

*intertwines the endofunctors $\sigma_n$ and $\tilde{\rho}_n$ of the Bost-Connes system on $\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}),T)$ and the Frobenius $F_n$ and Verschiebung $V_n$ of the Bost-Connes structure on $Aut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$.*

*Proof.* Consider the mappings $\sigma_n$ and $\tilde{\rho}_n$ defined in (2.7.2) and (2.7.3), The effect of the transformation $\sigma_n$, when written in terms of the data $((X,v_X),(Y,v_Y),i)$ is to send $v_X \mapsto v_X^n$ and $v_Y \mapsto v_Y^n$, hence it induces the Frobenius map $F_n$ acting on $(H^i(X(\mathbb{C}),Y(\mathbb{C}),\mathbb{Q}),\phi^i)$ in $\mathrm{A}ut^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$. Similarly, we have $T(X\times Z_n,\Phi_n(\alpha_X),(Y\times Z_n,\Phi_n(\alpha_Y)),i) = H^i(X\times Z_n, Y\times Z_n,\mathbb{Q})$ where by the relative version of the Künneth formula we have $(H^i(X(\mathbb{C})\times Z_n(\mathbb{C}),Y(\mathbb{C})\times Z_n(\mathbb{C}),\mathbb{Q}) \simeq H^i(X(\mathbb{C}),Y(\mathbb{C}),\mathbb{Q})^{\oplus n}$ with the induced map $V_n(\phi^i)$. The maps $\sigma_n$ and $\tilde{\rho}_n$ defined as above determine self maps of the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$. By Lemma 7.2.6 of **HuM-S17** given a map $F : D_1 \to D_2$ of diagrams and a representation $T : D_2 \to R$-Mod, there is an $R$-linear exact functor $\mathcal{F} : C(D_1, T \circ F) \to C(D_2, T)$ such that the following diagram commutes:

$$\begin{tikzcd}
D_1 \arrow[rr, "F"] \arrow[d] & & D_2 \arrow[d] \\
C(D_1, T \circ F) \arrow[rr, "\mathcal{F}"] \arrow[dr] & & C(D_2, T) \arrow[dl] \\
& R\text{-Mod} &
\end{tikzcd}$$

We still denote by $\sigma_n$ and $\tilde{\rho}_n$ the endofunctors induced in this way on $C(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$. To check the compatibility of the $\sigma_n$ functors with the monoidal structure, we use the fact that for Nori motives the product structure is constructed using "good pairs" (see §9.2.1 of **HuM-S17**), that is, elements $(X, Y, i)$ with the property that $H^j(X, Y, \mathbb{Z}) = 0$ for $j \neq i$. For such elements the product is given by $(X, Y, i) \times (X', Y', j) = (X \times X', X \times Y' \cup Y \times X', i + j)$. The diagram category $C(\mathrm{G}ood^{eff}, T)$ obtained by replacing effective pairs $\mathrm{P}airs^{eff}$ with good effective pairs $\mathrm{G}ood^{eff}$ is equivalent to $C(\mathrm{P}airs^{eff}, T)$ (Theorem 9.2.22 of **HuM-S17**), hence the tensor structure defined in this way on $C(\mathrm{G}ood^{eff}, T)$ determines the tensor structure of $C(\mathrm{P}airs^{eff}, T)$ and on the resulting category of Nori motives, see §9.3 of **HuM-S17** . Thus, to check the compatibility of the functors $\sigma_n$ with the tensor structure, it suffices to see that on a product of good pairs, where indeed we have

$$\sigma_n((X, \alpha_X), (Y, \alpha_Y), i) \times \sigma_n((X', \alpha'_X), (Y', \alpha'_Y), j) =$$

$$((X \times X', (\alpha_X \times \alpha'_X) \circ \Delta \circ \sigma_n), ((X \times Y', (\alpha_X \times \alpha'_Y) \circ \Delta \circ \sigma_n) \cup (Y \times X', (\alpha_Y \times \alpha'_X) \circ \Delta \circ \sigma_n)), i + j)$$

$$= \sigma_n(((X, \alpha_X), (Y, \alpha_Y), i) \times ((X', \alpha'_X), (Y', \alpha'_Y), j)).$$

The functors $\tilde{\rho}_n$ are not compatible with the tensor product structure, as expected. $\square$

**Remark 2.7.8.** *In* **MaTa** *, a motivic interpretation of the categorification of the Bost-Connes algebra is given by identifying the Tannakian category* $Vec^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ *with a limit of orbit categories of Tate motives. Here we presented a different motivic categorification of the Bost-Connes algebra by lifting the Bost-Connes structure to the level of the category of Nori motives. In* **MaTa** *a motivic Bost-Connes structure was also constructed using the category of motives over finite fields and the larger class of Weil numbers replacing the roots of unity of the Bost-Connes system.*

**Motivic sheaves and the relative case**

The argument presented in Theorem 2.7.7 lifting the Bost-Connes structure to the category of Nori motives, which provides a Tannakian category version of the list to Grothendieck rings via the equivariant Euler characteristics $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, can also be generalized to the relative setting, where we considered the Euler characteristic

$$\chi_S^{\hat{\mathbb{Z}}} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$$

with values in the Grothendieck ring of constructible sheaves, discussed in §2.2 of this paper. The categorical setting of Nori motives that is appropriate for this relative case is the Nori category of motivic sheaves introduced in **Arapura** .

We recall here briefly the construction of the category of motivic sheaves of **Arapura** and we show that the Bost-Connes structure on the category of Nori motives described in Theorem 2.7.7 extends to this relative setting.

Consider pairs $(X \to S, Y)$ of varieties over a base $S$ with $Y \subset X$ endowed with the restriction $f_Y : Y \to S$. Morphisms $f : (X \to S, Y) \to (X' \to S, Y')$ are morphisms of varieties $h : X \to X'$ satisfying the commutativity of

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & S & \end{tikzcd}$$

and such that $h(Y) \subset Y'$. As before, we consider varieties endowed with good effectively finite $\hat{\mathbb{Z}}$-action. We denote by $(S, \alpha)$ the base with its good effectively finite $\hat{\mathbb{Z}}$-action and by $((X\alpha_X) \to (S, \alpha), (Y, \alpha_Y))$ the pairs as above where we assume that the map $f : X \to S$ and the inclusion $Y \hookrightarrow X$ are $\hat{\mathbb{Z}}$-equivariant.

Following **Arapura** a diagram $D(\mathcal{V}_S)$ is obtained by considering as vertices elements of the form $(X \to S, Y, i, w)$ with $(X \to S, Y)$ a pair as above, $i \in \mathbb{N}$ and $w \in \mathbb{Z}$. The edges are given by the three types of edges:

1. geometric morphisms $h : (X \to S, Y) \to (X' \to S, Y')$ as above determine edges $h^* : (X' \to S, Y', i, w) \to (X \to S, Y, i, w)$;
2. connecting morphisms $\partial : (Y \to S, Z, i, w) \to (X \to S, Y, i + 1, w)$ for a chain of inclusions $Z \subset Y \subset X$;
3. twisted projections: $(X, Y, i, w) \to (X \times \mathbb{P}^1, Y \times \mathbb{P}^1 \cup X \times \{0\}, i + 2, w + 1)$.

For consistency with our previous notation we have here written the morphisms in the contravariant (cohomological) way rather than in the covariant (homological) way used in §3.3 of **Arapura** .

Note that in the previous section, following **HuM-S17** we described the effective Nori motives as $\mathcal{MN}^{eff} = C(\mathrm{P}airs^{eff}, T)$, with the category of Nori motives $\mathcal{MN}$ being then obtained as the localization of $\mathcal{MN}^{eff}$ at $(\mathbb{G}_m, \{1\}, 1)$ (inverting the Lefschetz motive). Here in the setting of **Arapura** the Tate motives are accounted for in the diagram construction by the presence of the twist $w$ and the last class of edges.

Given $f : X \to S$ and a sheaf $\mathcal{F}$ on $X$ one has $H^i_S(X; \mathcal{F}) = R^i f_* \mathcal{F}$. In the case of a pair $(f : X \to S, Y)$, let $j : X \smallsetminus Y \hookrightarrow X$ be the inclusion and consider $H^i_S(X, Y; \mathcal{F}) = R^i f_* j_! \mathcal{F}|_{X \smallsetminus Y}$. The diagram representation $T$ in this case maps $T(X \to S, Y, i, w) = H^i_S(X, Y, \mathcal{F})(w)$ to the (Tate twisted) constructible sheaf $H^i_S(X, Y; \mathcal{F})$. It is shown in **Arapura** that the Nori formalism of geometric diagrams applies to this setting and gives rise to a Tannakian category of motivic sheaves $\mathcal{MN}_S$. In particular one considers the case where $\mathcal{F}$ is constant with $\mathcal{F} = \mathbb{Q}$, so that the diagram representation $T : D(\mathcal{V}_S) \to \mathbb{Q}_S$ and the induced functor on $\mathcal{MN}_S$ replace at the motivic level the Euler characteristic map on the relative Grothendieck ring $K_0(\mathcal{V}_S) \to K_0(\mathbb{Q}_S)$.

As in the previous cases, we consider an enhancement of this category of motivic sheaves, in the sense of §2.3, by introducing good effectively finite $\hat{\mathbb{Z}}$-actions. We modify the construction of **Arapura** in the following way.

We consider a diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha)})$ where the vertices are elements

$$((X, \alpha_X) \to (S, \alpha), (Y, \alpha_Y), i, w)$$

so that the maps $f : X \to S$ and the inclusion $Y \hookrightarrow X$ are $\hat{\mathbb{Z}}$-equivariant, and with morphisms as above, where all the maps are required to be compatible with the $\hat{\mathbb{Z}}$-actions. One obtains by the same procedure as in **Arapura** a category of equivariant motivic sheaves $\mathcal{MN}^{\hat{\mathbb{Z}}}_S$. The representation above maps $D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha)})$ to $\hat{\mathbb{Z}}$-equivariant constructible sheaves over $(S, \alpha)$. Then the same argument we used in §2.2 at the level of Grothendieck rings, assemblers, and spectra applies to this setting and gives the following result.

**Proposition 2.7.9.** *The maps of diagrams*

$$\sigma_n : D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha)}) \to D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha \circ \sigma_n)})$$

$$\tilde{\rho}_n : D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha)}) \to D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S\times Z_n, \Phi_n(\alpha))})$$

*given by*

$$\sigma_n((X,\alpha_X) \to (S,\alpha),(Y,\alpha_Y),i,w) = ((X,\alpha_X \circ \sigma_n) \to (S,\alpha \circ \sigma_n),(Y,\alpha_Y \circ \sigma_n),i,w)$$

$$\tilde{\rho}_n((X,\alpha_X) \to (S,\alpha),(Y,\alpha_Y),i,w) =$$

$$((X \times Z_n, \Phi_n(\alpha_X)) \to (S \times Z_n, \Phi_n(\alpha)),(Y \times Z_n, \Phi_n(\alpha_Y)),i,w)$$

*determine functors of the resulting category of motivic sheaves $\mathcal{MN}^{\hat{\mathbb{Z}}}_S$ such that $\sigma_n \circ \tilde{\rho}_n = n\,id$ and $\tilde{\rho}_n \circ \sigma_n$ is a product with $(Z_n, \alpha_n)$. Thus, one obtains on the category $\mathcal{MN}^{\hat{\mathbb{Z}}}_S$ a Bost-Connes system as in Definition 2.3.11.*

*Proof.* The argument is as in Proposition 2.2.6, using again, as in Theorem 2.7.7 the fact that maps of diagrams induce functors of the resulting categories of Nori motives. □

**Nori geometric diagrams for assemblers, and a challenge**

We conclude this section on Bost-Connes systems and Nori motives by formulating a question about Nori diagrams and assembler categories.

According to the Nori formalism as it is presented in **HuM-S17** we must start with a "geometric" category $C$ of spaces/varieties/schemes, possibly endowed with additional structures, in which one can define morphisms of closed embeddings $Y \hookrightarrow X$ (or $Y \subset X$) and morphisms of complements to closed embeddings $X \smallsetminus Y \to X$. Then the Nori diagram of *effective pairs* $D(C)$ is defined as in **HuM-S17** pp. 207–208, see §2.7.

In the current context, *objects* of our category $C$ will be *assemblers* $\mathcal{C}$ (of course, described in terms of a category of lower level). In particular, each such $\mathcal{C}$ is endowed with a Grothendieck topology.

A *vertex* of the Nori diagram $D(C)$ will be a triple $(\mathcal{C}, \mathcal{C} \setminus \mathcal{D}, i)$ where its first two terms are taken from an abstract scissors congruence in $C$, and $i$ is an integer. Intuitively, this means that we we are considering the canonical embedding $\mathcal{C} \setminus \mathcal{D} \hookrightarrow \mathcal{C}$ as an analog of closed embedding. This intuition makes translation of the remaining components of Nori's diagrams obvious, except for one: *what is the geometric meaning of the integer $i$ in $(\mathcal{C}, \mathcal{C} \setminus \mathcal{D}, i)$ ?*

The answer in the general context of assemblers, seemingly, was not yet suggested, and already in the algebraic-geometric contexts is non-obvious and non-trivial. Briefly, $i$ translates to the level of Nori geometric diagrams the *weight filtration* of various cohomology theories (cf. **HuM-S17** 10.2.2, pp. 238–241), and the existence of such translation and its structure are encoded in several versions of *Nori's Basic Lemma* independently and earlier discovered by A. Beilinson and K. Vilonen (cf. **HuM-S17** 2.5, pp. 45–59).

The most transparent and least technical version of the Basic Lemma (**HuM-S17** Theorem 2.5.2 , p. 46) shows that in algebraic geometry the existence of weight filtration is based upon special properties of *affine schemes.* As we will see in the last section, lifts of Bost-Connes algebras to the level of cohomology based upon the techniques of *enhancement* also require a definition of *affine* assemblers. Since we do not know its combinatorial version, the enhancements that we can study now, force us to return to algebraic geometry.

This challenge suggests to think about other possible geometric contexts in which dimensions/weights of the relevant objects may take, say, $p$–adic values (as in the theory of $p$–adic weights of automorphic forms inaugurated by J. P. Serre), or rational values (as it happens in some corners of "geometries below $\mathrm{S}pec\,\mathbb{Z}$"), or even real values (as in various fractal geometries).

*Can one transfer the scissors congruences imagery there?*

See, for example, the formalism of Farey semi–intervals as base of $\infty$–adic topology.

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*Chapter 3*

# COMPARISON BETWEEN DIFFERENT TOPOLOGICAL MODELS OF CONCURRENCY

# ABSTRACT

In this note, we provide an explicit non-Quillen equivalence between the category of precubical sets and Gaucher's category of flows via a class of "realization functors" (with mild assumptions on the cofibrations of the category of precubical sets). In addition, we demonstrate a Quillen equivalence between simplicial semicategories and flows before proving that simplicial semicategories satisfy many of the same properties as flows. Finally, we introduce the category of boxed symmetric trees, presheaves on which may provide a slightly more flexible setting for concurrent computing than (pre)cubical sets, before showing that when endowed with degeneracies, the aforementioned presheaf category is a test category (although not strict test).

## 3.1 Introduction

Over the years, numerous models for concurrent computing have been proposed, each with unique advantages and disadvantages. One hope is that at least some of these models might be equivalent in a suitably weak sense, so that in choosing to work with one model over another, one is not really making a choice at all (with respect to all relevant data). Unfortunately, this does not seem to be the case in general.

The two existing models that we consider in this article are precubical sets (so-called higher dimensional automata, c.f. **FGHMR**) and flows (first investigated by Gaucher in **Gau1**). Precubical sets are a relatively classical model of concurrent computing. In a given cube, each of the different directions corresponds to different concurrently executing computations. One can think of each direction in a standard n-cube as coding for a factor of "done-ness" which meets with all other possible executions at the other end of the longest diagonal. More complicated computations are merely built out of these basic units. On the other hand, the category of flows basically consists of topologically enriched semicategories (categories without identity) and semifunctors between them (continuous on Hom spaces). Each object of a given flow can be seen as a different state that a computation can be in, and each morphism of that flow can be seen as an execution path between different states. The reason for topologizing the spaces of morphisms is to allow us to have a notion of (computational) equivalence between different execution paths, and to allow us to differentiate between different ways of moving between paths between execution paths, and so on.

The structure of our article is as follows. In Section 2, we provide a rapid refresher on many of the central notions of the theory of model categories. This includes the basics on model structures and homotopy categories, as well as Quillen adjunction and Quillen equivalence. We also introduce Cofibrantly Generated and Combinatorial model categories, which provide most of the basic setting in which we work throughout the article (every model structure we define will end up being combinatorial). The one more advanced topic we consider will be that of left-induction of model structure, whereby one pulls back a model structure along a left adjoint. This seemingly obvious notion is actually rather non-trivial, as several technical conditions must be satisfied for left-induction to succeed (thankfully, all but one of these may be elided by virtue of the fact that our model structures are combinatorial).

In Section 3, we begin by introducing the model category of flows. After discussing a few basic properties of this category (largely citing Gaucher's work itself for the proofs), we provide the definition (also due to Gaucher) of a class of functors from the category of precubical sets to flows called realization functors. These functors can basically be thought of as semicategorical (and topological) analogues of the realization functor from simplicial sets to simplicially enriched categories. Analogously, these realizations also admit right adjoint nerve functors. This is all a lead-up to our first theorem of the paper, namely:

**Theorem 3.1.1.** *Suppose that there is a model structure on the category of precubical sets along with a realization functor* $L : \boldsymbol{PrSh}(\square) \rightleftarrows \boldsymbol{Flow}$ *which is the left Quillen adjoint in a Quillen pair* $(L \dashv N_L) : \boldsymbol{PrSh}(\square) \rightleftarrows \boldsymbol{Flow}$ *(note that we are essentially only assuming that everything in* $I = \{\partial\square[n] \hookrightarrow \square[n]\}_{n=0}^{\infty} \cup \{\square[0] \sqcup \square[0] \to \square[0]\}$ *is sent to a cofibration and that the images of* $\square[n]$ *are weakly equivalent to* $\{0 < 1\}^n$ *for any n). Then this adjunction cannot be a Quillen equivalence.*

In spite of this, there is a reasonable combinatorial model for flows, provided by simplicial semicategories (with simplicial semifunctors as morphisms). We begin the section by introducing a few key definitions, and then set up a geometric realization/nerve adjunction pair (essentially just applying the geometric realization/singular space adjunction between simplicial sets and topological spaces on the Homs of simplicial semicategories or flows). This allows the model structure to be left induced.

**Proposition 3.1.2.** *The model structure on* $\boldsymbol{Flow}$ *may be left induced via the adjunction* $(|-| \dashv Sing) : \boldsymbol{sSemiCat} \rightleftarrows \boldsymbol{Flow}$ *(constant on objects and acting via realization/singular set on Hom objects). This upgrades* $(|-| \dashv Sing)$ *into a Quillen pair.*

This then brings us to our second main theorem of the section.

**Theorem 3.1.3.** *The Quillen adjunction* $(|-| \dashv Sing) : \boldsymbol{sSemiCat} \rightleftarrows \boldsymbol{Flow}$ *is a Quillen equivalence.*

We then show that there is an equivalent model structure on **sSemiCat** that has a nice set of generating cofibrations. Afterwards, we go on to demonstrate that **sSemiCat** has several properties in common with the category of flows (indeed, many of the proofs become even simpler in **sSemiCat**), including a way of defining a stronger notion homotopy equivalence.

In Section 3, we change gears, and define a small category called the category of boxed-symmetric trees, denoted $\boxed{\mathcal{T}}$. One can think of this category as an analogue of the category of cubes, where we allow any rooted tree (either all directed away from the root or all directed towards) as a "basic interval." Presheaves on this allow us a bit more flexibility as models of concurrency, as our basic "cubes" in this case correspond to performing flowchart computations in any of several concurrent directions. As a quick check, we prove that $\boxed{\mathcal{T}}$ is a test category in the sense of Grothendieck.

**Theorem 3.1.4.** $\boxed{\mathcal{T}}$ *is a test category.*

**Convention:** In this note, **Top** will refer to a convenient category of topological spaces (such as compactly generated weak Hausdorff spaces or $\Delta$-generated spaces), i.e., a full replete subcategory of the category of all topological spaces which is cartesian-closed, bicomplete, and which contains all CW complexes. Topological space will be used to mean a member of this convenient category.

### 3.2 Model Categories

The following section is devoted to introducing enough of the basic definitions of the theory of model categories (a certain type of category equipped with additional structure that provides a "good setting for homotopy theory" originally defined by Dan Quillen in **Qui**) to understand the following sections. In particular, this should not be thought of as a comprehensive introduction to the theory of model categories, and there will be several glaring omissions even in the very basics. For a good introduction to the extremely rich theory of model categories, we refer the interested reader to **Hir** .

Classical sources of motivation for model categories come from considering topological spaces/simplicial sets up to weak homotopy equivalence, and the Gabriel-Zisman localization of a category. Given a category $C$ and any class of morphisms $\mathcal{W} \subset \mathrm{Mor}(C)$, we may form its *Gabriel-Zisman localization* $C[\mathcal{W}^{-1}]$ by formally inverting the morphisms in $\mathcal{W}$. In general, this is extremely poorly behaved. For example, if one begins with a locally small category, its localization at a class of morphisms need not be locally small in general (and, in fact, often isn't). As will be noted later, model categories provide one setting in which the localization can be controlled (in the sense that it will be equivalent to a category with a much simpler description—one which, thankfully, is always locally small when one starts with a locally small category).

### Model Structures, Model Categories, and Homotopy Categories

**Definition 3.2.1.** *Suppose we have a category $C$. A* model structure *on $C$ consists of three classes of maps,* weak equivalences $\mathcal{W}_C$, cofibrations $\boldsymbol{cof}_C$, *and* fibrations $\boldsymbol{fib}_C$ *(we will suppress the subscripts if the context is clear) satisfying the following axioms:*

- *(2-out-of-3 axiom) Given morphisms $f, g \in \boldsymbol{Mor}(C)$ such that $g \circ f$ is defined, if any two of $f$, $g$, and $g \circ f$ are in $\mathcal{W}_C$, then so is the third.*
- *(retract axiom) If $f$ is a retract of $g$, and $g$ is a weak equivalence, fibration, or cofibration, then so is $f$.*
- *(lifting axiom) Suppose we have the commutative diagram of solid arrows*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}.$$

  *Then the dotted arrow exists and results in a commutative diagram if*

  1. *$i \in \boldsymbol{cof}_C$ and $p \in \boldsymbol{fib}_C \cap \mathcal{W}_C$,*
  2. *$i \in \boldsymbol{cof}_C \cap \mathcal{W}_C$ and $p \in \boldsymbol{fib}_C$.*

- *(factorization axiom) There are two functorial factorizations of every morphism $f \in \boldsymbol{Mor}(C)$.*

  1. *$f = qi$, where $q \in \boldsymbol{fib}_C \cap \mathcal{W}_C$ and $i \in \boldsymbol{cof}_C$,*
  2. *$f = pj$, where $p \in \boldsymbol{fib}_C$ and $j \in \boldsymbol{cof}_C \cap \mathcal{W}_C$.*

*Elements of $\boldsymbol{fib}_C \cap \mathcal{W}_C$ are known as* trivial fibrations *and elements of $\boldsymbol{cof}_C \cap \mathcal{W}_C$ are known as* trivial cofibrations

**Definition 3.2.2.** *A bicomplete category $C$ equipped with a model structure is known as a* model category.

Now, we wish to demonstrate that homotopy categories have well-behaved localizations with respect to their classes of weak equivalences. This will involve several definitions.

**Definition 3.2.3.** *Given a model category $\mathcal{M}$, and object $X \in \mathcal{M}$ will be known as* fibrant *if the unique map $X \to *$ is a fibration. Analagously, $X$ will be known as* cofibrant *if the unique map $\emptyset \to X$ is a cofibration.*

Note that we have canonical functors $\mathcal{M} \to \mathcal{M}^{\Delta^1}$ given by $X \mapsto (X \to *)$ and $X \mapsto (\emptyset \to X)$. By our assumptions above, we have a functorial factorization of $X \to *$ into $X \to X^{fib} \to *$, where $X \to X^{fib}$ is a trivial cofibration and $X^{fib} \to *$ is a fibration. Analogously, we have a functorial factorization of $\emptyset \to X$ into $\emptyset \to X^{cof} \to X$, where $\emptyset \to X^{cof}$ is a cofibration and $X^{cof} \to X$ is a trivial fibration.

**Definition 3.2.4.** *The endofunctors $(-)^{fib}$ and $(-)^{cof}$ of $\mathcal{M}$ we implicitly defined in the preceding paragraph are known as* fibrant replacement *and* cofibrant replacement*, respectively.*

Recalling that our model category admits small coproducts, for any object $X \in \mathcal{M}$, we may define the fold map $X \sqcup X \to X$. By functorial factorization once again, this allows us to define a *good cylinder object* for $X$ by factoring $X \sqcup X \to X$ as

$$X \sqcup X \xrightarrow{i_0 \sqcup i_1} \mathrm{Cyl}(X) \xrightarrow{p} X,$$

where $i_0 \sqcup i_1$ is a cofibration and $p$ is a trivial fibration.

**Definition 3.2.5.** *Two morphisms $f, g : X \to Y$ are known as* left homotopy equivalent *if there exists a map $H : Cyl(X) \to Y$ such that the compositions $H \circ i_0 = f$ and $H \circ i_1 = g$.*

As one might suspect from the name, left homotopy equivalence generates an equivalence relation on the Hom-sets of $\mathcal{M}$. In fact, we have the following definition and theorem.

**Definition 3.2.6.** *Given a model category $\mathcal{M}$, one defines its* homotopy category *$Ho(\mathcal{M})$ as follows:*

- *The objects of $Ho(\mathcal{M})$ are the objects of $\mathcal{M}$ which are both fibrant and cofibrant,*
- *Given any two $X, Y \in Ho(\mathcal{M})$, one has that $Hom_{Ho(\mathcal{M})}(X, Y)$ is the quotient of $Hom_{\mathcal{M}}(X, Y)$ by left homotopy equivalence.*

**Theorem 3.2.7.** *For any model category $\mathcal{M}$, its homotopy category $Ho(\mathcal{M})$ is equivalent to its localization $\mathcal{M}[\mathcal{W}_{\mathcal{M}}^{-1}]$ by its class of weak equivalences.*

*Proof.* This is **Hir** Theorem 8.3.9. □

Indeed, if our starting model category $\mathcal{M}$ was locally small, $\mathrm{Ho}(\mathcal{M})$ is a locally small model for $\mathcal{M}[\mathcal{W}_{\mathcal{M}}^{-1}]$.

**Quillen Adjunction and Quillen Equivalence**

Now that we have defined the notion of a model category, it would be helpful to have some way of comparing the model structures on these categories. Note that arguably the most important form of comparison between two categories (equipped with no extra structure) is the data of an adjunction between the two. With that in mind, we have the following definition.

**Definition 3.2.8.** *Given two model categories $\mathcal{M}$ and $\mathcal{N}$, a* Quillen adjunction *or* Quillen pair *between $\mathcal{M}$ and $\mathcal{N}$ is the data of an adjunction $(L \dashv R) : \mathcal{M} \rightleftarrows \mathcal{N}$ between the two such that one of the following equivalent conditions is satisfied:*

1. *L preserves cofibrations and trivial cofibrations;*
2. *R preserves fibrations and trivial fibrations;*
3. *L preserves cofibrations and R preserves fibrations;*
4. *L preserves trivial cofibrations and R preserves trivial fibrations.*

One particularly important aspect of the notion of a Quillen adjunction is that it induces an adjunction on the level of homotopy theory. In other words,

**Theorem 3.2.9.** *Given a Quillen adjunction $(L \dashv R) : \mathcal{M} \rightleftarrows \mathcal{N}$, it induces an adjunction $(\mathbb{L} \dashv \mathbb{R}) : Ho(\mathcal{M}) \rightleftarrows Ho(\mathcal{N})$ between homotopy categories.*

*Proof.* This can be found in **Hir** (several propositions in Section 8.5). □

This gives us a particularly well-structured way of comparing two homotopy theories. Of course, the nicest form of adjunction between categories is a categorical equivalence. It will be especially useful to us to import the notion of equivalence into this weaker setting.

**Definition 3.2.10.** *Given a Quillen adjunction* $(L \dashv R) : \mathcal{M} \rightleftarrows \mathcal{N}$*, we say that it is a* Quillen equivalence *if it descends to an equivalence of categories on the level of homotopy theory. In particular, this holds if it satisfies one of the following conditions:*

- *The induced adjunction* $(\mathbb{L} \dashv \mathbb{R}) : Ho(\mathcal{M}) \rightleftarrows Ho(\mathcal{N})$ *between homotopy categories is an equivalence of categories;*
- *For any cofibrant object* $X \in \mathcal{M}$ *and any fibrant object* $Y \in \mathcal{N}$*, a map* $LX \to Y$ *is a weak equivalence in* $\mathcal{N}$ *if and only if the corresponding map* $X \to RY$ *under the adjunction is a weak equivalence in* $\mathcal{M}$*;*
- *Both of the following two conditions hold:*
  1. *For every cofibrant object* $X \in \mathcal{M}$*, the composition* $X \to R(L(X)) \to R(L(X)^{fib})$ *(known as the derived adjunction unit) is a weak equivalence*
  2. *For every fibrant object* $Y \in \mathcal{N}$*, the composition* $L(R(Y)^{cof}) \to L(R(Y)) \to Y$ *(known as the derived adjunction counit) is a weak equivalence.*

In particular, this last characterization will be important to us in demonstrating the non-equivalence between precubical sets and flows. Another pair of characterizations (dual to one another) will be particularly useful to us going forward as well.

**Theorem 3.2.11.** *Consider the Quillen Pair* $(L \dashv R) : \mathcal{M} \rightleftarrows \mathcal{N}$*.*

- *If* $L$ *creates weak equivalences (i.e.,* $f \in Mor(\mathcal{M})$ *is a weak equivalence if and only if* $L(f)$ *is), then* $(L \dashv R)$ *is a Quillen equivalence if and only if for every fibrant object* $Y \in \mathcal{N}$*, the adjunction counit* $\epsilon : L(R(Y)) \to Y$ *is a weak equivalence.*
- *If* $R$ *creates weak equivalences (i.e.,* $f \in Mor(\mathcal{N})$ *is a weak equivalence if and only if* $R(f)$ *is), then* $(L \dashv R)$ *is a Quillen equivalence if and only if for every cofibrant object* $X \in \mathcal{M}$*, the adjunction unit* $\epsilon : X \to R(L(X))$ *is a weak equivalence.*

*Proof.* A proof of the first statement (the second is essentially dual) can be found in **ErIl** (Lemma 3.3). □

**Cofibrantly Generated Model Categories**

A cofibrantly generated model category is a nice type of model category generated mostly by small data. We will use these as an entry point into combinatorial model categories, which we use to define a model structure on precubical sets satisfying certain properties.

**Definition 3.2.12.** *Let $C$ be a cocomplete category and take $S \subset \textbf{Mor}(C)$. We define:*

- ***llp***$(S)$ *to be the class of morphisms which has the left lifting property with respect to all morphisms in $S$.*
- ***rlp***$(S)$ *to be the class of morphisms which has the right lifting property with respect to all morphisms in $S$.*
- ***cell***$(S)$ *to be the class of transfinite compositions of elements of $S$.*
- ***cof***$(S) :=$ ***llp***$($***rlp***$(S))$.

**Definition 3.2.13.** *A model category $C$ is* cofibrantly generated *if there are small sets of morphisms $I, J \subset \textbf{Mor}(C)$ such that*

- *$I$ and $J$ admit the small object argument.*
- ***cof***$(I)$ *is precisely the class of cofibrations of $C$*
- ***cof***$(J)$ *is precisely the class of trivial cofibrations of $C$*

One very important proposition for cofibrantly generated model categories is the following.

**Proposition 3.2.14.** *Given a cofibrantly generated model category $C$, one has*

- ***cof***$(I)$ *is also the class of retracts of elements of* **cell**$(I)$.
- ***cof***$(J)$ *is also the class of retracts of elements of* **cell**$(J)$.
- ***rlp***$(I)$ *is precisely the class of trivial fibrations.*
- ***rlp***$(J)$ *is precisely the class of fibrations.*

*Proof.* Found in **Hir** Chapter 11 (combines several propositions). □

Finally, before we move on to discussing combinatorial model categories, we will simply state a particularly important theorem due to Daniel Kan, which allows us to produce a cofibrantly generated model structure from the data of $I$ and $J$ given an appropriate set of weak equivalences.

**Theorem 3.2.15.** *Let $\mathcal{C}$ be a bicomplete category and $\mathcal{W} \subset \boldsymbol{Mor}(\mathcal{C})$ closed under retracts and satisfying the 2-out-of-3 property. If $I$ and $J$ are sets of morphisms of $\mathcal{C}$ such that*

- *Both $I$ and $J$ admit the small object argument;*
- $\boldsymbol{cof}(J) \subseteq \boldsymbol{cof}(I) \cap \mathcal{W}$*;*
- $\boldsymbol{rlp}(I) \subseteq \boldsymbol{rlp}(J) \cap \mathcal{W}$*;*
- *One of $\boldsymbol{cof}(I) \cap \mathcal{W} \subseteq \boldsymbol{cof}(J)$ and $\boldsymbol{rlp}(J) \cap \mathcal{W} \subseteq \boldsymbol{rlp}(I)$ holds.*

*Then $\mathcal{C}$ is a cofibrantly generated model category with weak equivalences specified by $\mathcal{W}$, with $I$ a set of generating cofibrations, and $J$ a set of generating trivial cofibrations.*

*Proof.* This is **Hir** Theorem 11.3.1. □

**Combinatorial Model categories**

In this section we will discuss a particularly nice type of model structure, generated by an extremely minimal amount of data, but with very good properties. In particular, we will see that all one needs is a class of weak equivalences and a set of generating cofibrations satisfying certain properties.

**Definition 3.2.16.** *A model category $\mathcal{C}$ is* combinatorial *if it is a cofibrantly generated model category which is locally presentable as a category.*

That's it. But this seemingly simple class of model categories admit several extremely powerful classification theorems. We will discuss only Jeff Smith's theorem here, but there is another important classification result due to Daniel Dugger.

**Theorem 3.2.17.** *(Jeff Smith's Theorem)*

*Suppose that one has the data of*

- *a locally presentable category $\mathcal{C}$,*

- *a class of morphisms $W \subset Mor(C)$ such that the subcategory of the arrow category of $C$ it defines, $\boldsymbol{Arr}_W(C) \subset \boldsymbol{Arr}(C)$ is an accessibly embedded accessible full subcategory,*

- *a small set $I \subset Mor(C)$ of morphisms in $C$,*

*such that*

- *$\mathcal{W}$ satisfies the 2-out-of-3 property,*

- *$inj(I) \subset \mathcal{W}$,*

- *$cof(I) \cap \mathcal{W}$ is closed under pushout and transfinite composition.*

*Then we have that $C$ is a combinatorial (and hence cofibrantly generated) model category with*

- *weak equivalences $\mathcal{W}$,*

- *cofibrations $cof(I)$.*

*Finally, all combinatorial model structures arise in this way.*

*Proof.* Can be found in **Bar** (Proposition 1.7) and **Bek** . □

This theorem essentially gives us a minimal recipe for concocting model categories, and extremely well-behaved ones at that.

**Left-induced model structures**

Now, the last topic we will discuss in the theory of model categories is that of induced model structures, specifically left-induced model structures (we will forego discussion of right-induced model structures, as they are irrelevant to our current topic).

**Definition 3.2.18.** *Let $C$ be a bicomplete category and let $\mathcal{M}$ be a model category. Furthermore, suppose there is an adjunction of the form*

$$(L \dashv R) : C \rightleftarrows \mathcal{M}$$

*running between them. The* left-induced model structure *on $C$, if it exists, has*

1. *weak equivalences given by those morphisms which map to weak equivalences in $\mathcal{M}$ under $L$ (i.e. $L^{-1}\mathcal{W}$),*

2. *cofibrations given by those morphisms which map to cofibrations in $\mathcal{M}$ under $L$ (i.e. $L^{-1}\boldsymbol{cof}_{\mathcal{M}}$)),*

3. *fibrations determined by the other two classes of morphisms.*

We now cite an important theorem **HKRS** which determines conditions (known as acyclicity conditions) under which a left-induced model structure exists. We will actually state a corollary of the original theorem from **HKRS** as it is all we will need at the moment.

**Theorem 3.2.19.** *Suppose that $C$ is a bicomplete category and $\mathcal{M}$ is a combinatorial model category and that there is an adjunction of the form*

$$(L \dashv R) : C \rightleftarrows \mathcal{M}$$

*running between them. Then the left-induced model structure on $C$ exists if and only if*

$$\boldsymbol{rlp}(L^{-1}\boldsymbol{cof}_{\mathcal{M}}) \subset L^{-1}\mathcal{W}_{\mathcal{M}}.$$

*Proof.* This is a specialization of **HKRS** Proposition 2.1.4 to the situation of a combinatorial model category. □

### 3.3 Flows, Precubical Sets, and Simplicial Semicategories

The category of flows is a model for the theory of concurrency. As we will see shortly, the category of flows may equivalently be thought of as the category of small topologically enriched semicategories. A basic heuristic for understanding the relation between flows and concurrency is that the objects of flows correspond to possible states that a computation can be in, whereas morphism spaces represent all the possible ways of getting from one state to another. The topology simply allows us to compare the relation between different ways of getting between states. We want to be able to say when two execution paths between states are equivalent for the purposes of our computation, and to be able to specify precisely *how* they are equivalent, thus justifying the usage not only of spaces, but more general spaces than those corresponding to mere 1-types.

**Definition 3.3.1.** *A* flow $X := (X^0, \mathbb{P}X, s, t, *)$ *is a quintuple consisting of a discrete set* $X^0$*, a locally compact space* $\mathbb{P}X$ *called the* path space*,* source *and* target *continuous maps* $s, t : \mathbb{P}X \to X^0$*, and a continuous and associative* path concatenation *operation*

$$* : \mathbb{P}X \times_{s,t} \mathbb{P}X = \{(x, y) \in \mathbb{P}X^2 | t(x) = s(y)\} \to \mathbb{P}X.$$

*We will abuse notation and write* $X$ *both for the quintuple and its* total space $\mathbb{P}X \sqcup X^0$.

*A morphism of flows* $f : X \to Y$ *consists of a set map* $f^0 : X^0 \to Y^0$ *and a map of topological spaces* $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ *(we abuse notation and use* $f$ *as a stand-in for both) such that* $f(s(x)) = s(f(x))$*,* $f(t(x)) = t(f(x))$*, and* $f(x * y) = f(x) * f(y)$ *for all* $x, y \in \mathbb{P}X$*.*

Together flows and maps of flows form the category **Flow** of flows.

For any flow $X$, given any $\alpha, \beta \in X^0$, one may describe the path space from $\alpha$ to $\beta$ as $\mathbb{P}_{\alpha,\beta}X := \{x \in X | s(x) = \alpha \text{ and } t(x) = \beta\}$ equipped with the subspace topology (or the kaonization thereof if needed).

Furthermore, there is a functor Glob : **Top** $\to$ **Flow** which assigns to each topological space $X$ its globe $\mathrm{Glob}(X)$, a flow such that $\mathrm{Glob}(X)^0 = \{0, 1\}$, $\mathbb{P}\mathrm{Glob}(X) = X$, $s = 0$, and $t = 1$. Given some string $X_1, ..., X_n$ of topological spaces, we may "concatenate" their globes to define a new flow

$$\mathrm{Glob}(X_1) * \cdots * \mathrm{Glob}(X_n)$$

which is the flow that you get from identifying the target of the $i$th globe with the source of the $(i + 1)$th globe. In other words, if we label the flow above as $Y$, we get that $Y^0 = \{0, 1, ..., n\}$ and that $\mathbb{P}Y = X_1 \sqcup \cdots \sqcup X_n$ such that $s|_{X_i} = i - 1$ and $t|_{X_i} = i$ for $i = 1 \cdots n$.

**Theorem 3.3.2.** ***Flow*** *is a bicomplete, topologically enriched category.*

*Proof.* Combines Theorem 4.17 and Notation 4.14 in **Gau1** (in that order). □

We will elide in what follows several details for the sake of brevity, and mostly refer the reader to **Gau1** and **Gau4** to fill in any remaining details on the fundamental theory of Flows.

**The Homotopy Theory of Flows**

**Definition 3.3.3.** *Take two morphisms of flows $f, g : X \to Y$. Then $f$ and $g$ are referred to as* S-homotopic*, denoted $f \sim_S g$ if, considering $Hom_{\boldsymbol{Flow}}(X, Y)$ equipped with its enriched structure as a topological space, there exists a morphism*

$$h \in Hom_{\boldsymbol{Top}}([0, 1], Hom_{\boldsymbol{Flow}}(X, Y))$$

*such that $h(0) = f$ and $h(1) = g$.*

*Two flows $X$ and $Y$ are referred to as* S-homotopy equivalent *if there exists a morphism $f : X \to Y$ and a morphism $g : Y \to X$ such that $g \circ f \sim_S id_X$ and $f \circ g \sim_S id_Y$.*

There is an equivalent definition of two maps being S-homotopic that more directly parallels the first definition in topological spaces, but requires more machinery to set up. Furthermore, our primary interest will be in weak S-homotopy equivalence.

In what follows, we take $I_+^{gl} = \{\text{Glob}(S^{n-1}) \hookrightarrow \text{Glob}(D^n)\}_{n=0}^{\infty} \cup \{\emptyset \hookrightarrow *, * \sqcup * \to *\}$, where we follow the convention $S^{-1} = \emptyset$ and where we have $*$ be the flow consisting of a single object and the empty space of morphisms.

**Definition 3.3.4.** *A map $f : X \to Y$ of flows is a* weak S-homotopy equivalence *if the map $f^0 : X^0 \to Y^0$ of zero-skeleta is a bijection, and the map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ is a weak-homotopy equivalence of topological spaces. We denote the class of all weak S-homotopy equivalences by $\mathcal{W}_S$.*

**Theorem 3.3.5.** *There is a combinatorial model structure on the category of flows such that $I_+^{gl}$ is the generating set of cofibrations and $\mathcal{W}_S$ is the class of weak equivalences. Furthermore, with respect to this model structure, all objects are fibrant.*

*Proof.* This is Proposition 18.1 in **Gau1** . □

**Geometric Realization of Precubical Sets and Homotopy Coherent Nerve of Flows**

Another commonly discussed model for concurrent computing is that of precubical sets. These are presheaves on the category of cubes □, which may be thought of as the subcategory of **Cat** given by taking all strictly non-decreasing maps between the posets $\{0 \leq 1\}$ (including symmetry maps for the time being).

Note that there is a natural inclusion of posets into flows, where you consider the thin semicategory of the poset under strict inequality. In other words, we have an inclusion functor **PoSet** $\hookrightarrow$ **Flow**. In particular, for any cube $\square[n] = \{0 < 1\}^n$, we may consider it a flow in a natural way. Now, on cubes $\square[n]$, we define their realization $|\square[n]|$ to be the cofibrant replacement of their inclusion into **Flow**. Consider the category of precubical sets $\mathbf{PrSh}(\square)$. In other words, functors from the category of cubes, with morphisms only given by face operators, into sets. There is a geometric realization functor from the category of precubical sets to flows

$$|-| : \mathbf{PrSh}(\square) \to \mathbf{Flow} \text{ such that } K \mapsto \varinjlim_{\square\downarrow K} |\square[n]|$$

Its right adjoint is the homotopy coherent nerve of a flow, given by

$$N : \mathbf{Flow} \to \mathbf{PrSh}(\square) \text{ such that } X \mapsto (\square[n] \mapsto \mathrm{H}om_{\mathbf{Flow}}(|\square[n]|, X))$$

We will prove this statement now

**Theorem 3.3.6.** *The geometric realization and the homotopy coherent nerve functors form an adjunction*

$$\begin{tikzcd} \boldsymbol{PrSh}(\square) \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \boldsymbol{Flow} \arrow[l, bend left, "N"] \end{tikzcd}.$$

*Proof.* We want to prove that for all flows $X$ and all precubical sets $K$ that we have a natural isomorphism

$$\mathrm{H}om_{\mathbf{Flow}}(|K|, X) \cong \mathrm{H}om_{\mathbf{PrSh}(\square)}(K, N(X)).$$

We do this via

$$\begin{aligned} \mathrm{H}om_{\mathbf{Flow}}(|K|, X) &= \mathrm{H}om_{\mathbf{Flow}}(\varinjlim_{\square\downarrow K} |\square[n]|, X) \\ &\cong \varprojlim_{\square\downarrow K} \mathrm{H}om_{\mathbf{Flow}}(|\square[n]|, X) \\ &\cong \varprojlim_{\square\downarrow K} N(X)_n \\ &\cong \varprojlim_{\square\downarrow K} \mathrm{H}om_{\mathbf{PrSh}(\square)}(\square[n], N(X)) \\ &\cong \mathrm{H}om_{\mathbf{PrSh}(\square)}(\varinjlim_{\square\downarrow K} \square[n], N(X)) \\ &\cong \mathrm{H}om_{\mathbf{PrSh}(\square)}(K, N(X)). \end{aligned}$$

□

Now that we have our adjunction as above, we must try to find a model structure on precubical sets that upgrades the above adjunction to a Quillen pair.

That said, there is a slightly modified notion of geometric realization that has very slightly nicer properties. First, one should note that

**Theorem 3.3.7.** *For any $n \geq 0$, one has that $\mathbb{P}_{0\cdots 0,1\cdots 1}|\partial\square[n]|$ is homotopic to $S^{n-1}$ and one has that the commutative square*

$$\begin{tikzcd} Glob(S^{n-1}) \arrow[r] \arrow[d, hook] & \mid \partial\square[n+1] \mid \arrow[d, hook] \\ Glob(D^n) \arrow[r] & \mid \square[n+1] \mid \end{tikzcd}$$

*is a homotopy pushout.*

*Proof.* Found in **Gau2** as Proposition 4.2.2. □

Now, this naturally makes one wonder if there is some related functor which might render this homotopy pushout into an actual pushout. Indeed, this is the case. We have the following theorem/definition of Gaucher.

**Theorem 3.3.8.** *There exists a colimit-preserving functor $gl$ : $\boldsymbol{PrSh}(\square) \to \boldsymbol{Flow}$ such that for all n, one has a pushout square*

$$\begin{tikzcd} Glob(S^{n-1}) \arrow[r] \arrow[d, hook] & gl(\partial\square[n+1]) \arrow[d, hook] \\ Glob(D^n) \arrow[r] & gl(\square[n+1]) \end{tikzcd} .$$

*In particular, this means the maps $gl(\partial\square[n] \hookrightarrow \square[n])$ are all cofibrations. Furthermore, there exist natural transformations $\mu : gl \to |-|$ and $\nu : |-| \to gl$ which specialize to natural S-homotopy equivalences, and indeed are mutually naturally S-homotopy inverses, for every precubical set $K$. Finally, there exists a weak S-homotopy equivalence of cocubical flows $gl(\square[*]) \to \{0 < 1\}^*$. We refer to $gl$ as the* globular realization functor.

*Proof.* Found in **Gau2** as Theorem 4.2.4. □

In what follows, we will often use the globular realization for its particularly nice properties, although any realization will do. What do we mean by any realization?

**Theorem 3.3.9.** *Suppose one has an object-wise weak equivalence of cocubical flows $X \to \{0 < 1\}^*$ such that $\hat{X}(\partial\square[n]) \to \hat{X}(\square[n])$ is a cofibration for all n (where $\hat{X}$ is the extension of X to a functor from precubical sets to flows). Then there exist natural transformations $\mu^X : gl \to \hat{X}$ and $\nu^X : \hat{X} \to gl$ which specialize to natural S-homotopy equivalences, and indeed are mutually naturally S-homotopy inverses, for every precubical set K. Furthermore the diagram*

$$\begin{tikzcd} Glob(S^{n-1}) \arrow[r] \arrow[d, hook] & \hat{X}(\partial\square[n+1]) \arrow[d, hook] \\ Glob(D^n) \arrow[r] & \hat{X}(\square[n+1]) \end{tikzcd}$$

*is a homotopy pushout square.*

*Proof.* Found in **Gau2** as Theorem 4.2.6. □

Note that $gl$ has a right adjoint $N_{gl}$ : **Flow** $\to$ **PrSh**$(\square)$ defined and verified analogously to that of $|-|$. Namely, we have that $N_{gl}(X)_n = \mathrm{H}om_{\mathbf{Flow}}(gl(\square[n]), X)$ for all $n$. We will call this the *globular nerve* of a flow.

Indeed, for any realization $\hat{X}$, one has a corresponding nerve $N_X$ defined analogously.

**Lemma 3.3.10.** *Fix a realization functor $L$ : **PrSh**$(\square) \to$ **Flow**. Then for any precubical set K, $L(K)$ is a cofibrant flow.*

*Proof.* This proof is adapted from **Gau3** Proposition 7.5. We will prove this by induction on $n$-skeleta. Consider an arbitrary precubical set $K$. Recall that for $L$ to be a realization as defined above, one must have that $L(\partial\square[n] \hookrightarrow \square[n])$ is a cofibration for all $n$. In particular, we know that $L(\square[0]) = *$ and that $L(\emptyset \hookrightarrow \square[0]) = \emptyset \hookrightarrow *$ is a cofibration. In particular, the fact that cofibations are closed under pushout implies that $* \to * \sqcup *$ is a cofibration, and by transfinite composition, that $\sqcup_\alpha *$ is cofibrant for any cardinal $\alpha$. In particular, this shows that $L(K_{\leq 0}) = L(K)^0 = \sqcup_{K_0} *$ is a cofibrant flow. Now, suppose that $L(K_{\leq n-1})$ is cofibrant. Then, as cofibrations are composed under transfinite composition, it suffices to prove that $L(K_{\leq n-1}) \to L(K_{\leq n})$ is a cofibration. Given that $L(\partial\square[n]) \to L(\square[n])$ is a cofibration, the disjoint union of any number of copies of this morphism can be shown to be as well. Now, note that one has the pushout diagram

$$\begin{tikzcd}
\sqcup_{\alpha\in K_n}\partial\square[n] \arrow[r] \arrow[d, hook] & K_{\leq n-1} \arrow[d, hook] \\
\sqcup_{\alpha\in K_n}\square[n] \arrow[r] & K_{\leq n}.
\end{tikzcd}$$

As colimits commute with colimits, and $L$ is defined by way of colimits, we have the pushout square of flows

$$\begin{tikzcd}
\sqcup_{\alpha\in K_n}L(\partial\square[n]) \arrow[r] \arrow[d, hook] & L(K_{\leq n-1}) \arrow[d, hook] \\
\sqcup_{\alpha\in K_n}L(\square[n]) \arrow[r] & L(K_{\leq n}).
\end{tikzcd}$$

Thus, one has that $L(K_{\leq n-1}) \hookrightarrow L(K_{\leq n})$ is a cofibration, as cofibrations are closed under pushout. Furthermore, as they are closed under transfinite compositions as well, we see that $L(K)$ must be cofibrant. $\square$

**Lemma 3.3.11.** *A flow $X$ is synchronized (is bijective on constant states) if and only if it has the right-lifting property with respect to the set $\{\emptyset \hookrightarrow *, * \sqcup * \to *\}$.*

*Proof.* This is found in **Gau1** (Proposition 16.2), but we prove an analogous result via identical means in the following subsection. $\square$

We denote a hopeful set of generating cofibrations $I = \{\partial\square[n] \hookrightarrow \square[n]\}_{n=0}^{\infty} \cup \{\square[0] \sqcup \square[0] \to \square[0]\}$, where $\partial\square[0] = \emptyset$. However, we come to a deeply unfortunate fact. While we have adjunctions between **PrSh**($\square$) and **Flow** given by any one of the realization functors discussed above, this adjunction is not a Quillen equivalence as one might hope.

**Theorem 3.3.12.** *Suppose that there is a model structure on the category of precubical sets along with a realization functor $L$ : **PrSh**$(\square) \rightleftarrows$ **Flow** which is the left Quillen adjoint in a Quillen pair $(L \dashv N_L)$ : **PrSh**$(\square) \rightleftarrows$ **Flow** (note that we are essentially only assuming that everything in $I$ is sent to a cofibration and that the images of $\square[n]$ are weakly equivalent to $\{0 < 1\}^n$ for any $n$). Then this adjunction cannot be a Quillen equivalence.*

*Proof.* Recall that given Quillen pair $(L \dashv R) : \mathcal{C} \rightleftarrows \mathcal{D}$, if $(L \dashv R)$ is a Quillen equivalence, then on every fibrant object $d \in \mathcal{D}$, the derived counit of the adjunction (composition of cofibrant replacement with the adjunction counit) $L(R(d)^{cof}) \to$

$L(R(d)) \xrightarrow{\varepsilon_d} d$ is a weak equivalence. Now, in our case, note that every $X \in \mathbf{Flow}$ is fibrant **Gau1** . Furthermore, note that $\partial\square[n] \hookrightarrow \square[n]$ is a cofibration, and every $K \in \mathbf{PrSh}(\square)$ may be built up as the transfinite composition of pushouts of such maps, starting from copies of $\emptyset \hookrightarrow \square[0]$, so every $K \in \mathbf{PrSh}(\square)$ is cofibrant. Thus, $(L \dashv N_L) : \mathbf{PrSh}(\square) \rightleftarrows \mathbf{Flow}$ is a Quillen equivalence if and only if for all $X \in \mathbf{Flow}$, one has that the natural map $L(N_L(X)) \to X$ is a weak equivalence. Now, consider $\mathrm{Glob}(Y)$ for some $Y \in \mathbf{Top}$. Note first that the only $n$ for which one has any maps $L(\square[n]) \to \mathrm{Glob}(Y)$ are $n = 0$ and $n = 1$. This is because for any $n > 1$, one has that there are simply too many distinct vertices of $L(\square[n])$, and one would have to map a nonempty space to the empty space, which is impossible. Furthermore, there are exactly two maps $L(\square[0]) \to \mathrm{Glob}(Y)$, one which singles out 0 and one which singles out 1. Finally, noting that $L(\square[1]) = \mathrm{Glob}(Z)$ for some contractible space $Z$, we see that each map $L(\square[1]) \to \mathrm{Glob}(Y)$ singles out a different point of $\mathrm{H}om_{\mathbf{Top}}(Z, Y)$. Putting all of this data together, we get that

$$N_L(\mathrm{Glob}(Y))_n = \begin{cases} \{0,1\} & n = 0 \\ \mathrm{H}om_{\mathbf{Top}}(Z,Y)_{disc} & n = 1 \\ \emptyset & n > 1 \end{cases}.$$

Now, since no higher gluing data is specified, this implies that $L(N_L(\mathrm{Glob}(Y))) = \mathrm{Glob}(\mathrm{H}om_{\mathbf{Top}}(Z,Y)_{disc})$. Noting further that our counit in this case is a map $L(N_L(\mathrm{Glob}(Y))) \to \mathrm{Glob}(Y)$, this tells us that our counit in this case essentially amounts to a map from a discrete space to $Y$. This can only be an equivalence in the case that $Y$ was weakly equivalent to a discrete space to begin with. Thus, taking $Y = S^2$, for example, we have a non-equivalence. □

As an aside, what we have actually demonstrated above can be used to say something stronger. In particular, even if we replaced the weak equivalences on flows with equivalences which induce weak equivalences on path spaces and equivalences of semicategories on homotopy categories (in analogy with the ∞-categorical model structure on topologically enriched categories), this problem would still persist, and we could not get a Quillen equivalence between this ∞-semicategorical model structure and precubical sets. Similarly, it would not work with semisimplicial sets and this new model structure either. The unusual thing here is that on the level of cubical/simplicial sets and small topologically enriched categories, this would be precisely the Quillen equivalence between the appropriate Joyal-type model structure and the $\infty$-categorical model structure on topologically enriched categories. Thus, the procedure of "forgetting about identities" on either side of the equivalence results in something strictly less well-behaved, and indeed, in non-Quillen equivalent model categories. One, in fact, needs identities to make one of the core equivalences of the theory of $\infty$-categories work at all.

### Quillen Equivalence between Simplicial Semicategories and Flows

**Remark 3.3.13.** *In what follows, we will be dealing with several different simplicially enriched categories and at several points it will be necessary to differentiate between their sets of morphisms and their simplicial sets of morphisms. To limit confusion, we will introduce the following piece of notation. Suppose* $C$ *is a simplicially enriched category with* $X, Y \in C$. *We will let* $Hom_C(X, Y)$ *be the set of morphisms from* $X$ *to* $Y$ *and let* $C(X, Y)$ *be the corresponding simplicial set (in other words,* $Hom_C(X, Y) := C(X, Y)_0$*).*

**Definition 3.3.14.** *A* simplicial semicategory *is a semicategory* $K$ *enriched in simplicial sets. Simplicial semicategories and simplicial semifunctors between them form a category* ***sSemiCat***.

We will devote this section to proving the existence of a Quillen equivalence between **sSemiCat** and **Flow**. First, we have the following.

**Proposition 3.3.15.** *There exists an adjunction* $(|-| \dashv Sing) : \textbf{\textit{sSemiCat}} \rightleftarrows \textbf{\textit{Flow}}$.

*Proof.* Let us begin by defining $|-| : \textbf{sSemiCat} \to \textbf{Flow}$. Given any $K \in \textbf{sSemiCat}$, one may define $|K| \in \textbf{Flow}$ by taking the same object set, and defining for any two $x, y \in \mathrm{ob}(K)$, $\mathbb{P}_{x,y}|K| := |\mathrm{Map}_K(x, y)|$. This definition also yields a corresponding definition for functors which satisfies all the appropriate associativity conditions. Conversely, we may define $Sing : \textbf{Flow} \to \textbf{sSemiCat}$ as follows. For any $X \in \textbf{Flow}$, we define $Sing(X)$ by taking the same object set, and defining for any two $x, y \in X^0$, $\mathrm{Map}_{Sing(X)}(x, y) = Sing(\mathbb{P}_{x,y}X)$. This analogously also defines a functor, and these two functors are clearly adjoints of one another. □

From now on, we will employ the notation $\mathbb{P}^\Delta$ for our mapping simplicial sets in **sSemiCat** in analogy with the path spaces in **Flow**.

**Definition 3.3.16.** *There exists an analogous globe functor* $Glob^\Delta : \textbf{\textit{sSet}} \to \textbf{\textit{sSemiCat}}$ *to the one in the case of* ***Flow***.

*The* generating cofibrations *of* ***sSemiCat*** *are*

$$I_{Simp} := \{Glob^{\Delta}(\partial\Delta^n \hookrightarrow \Delta^n)\}_{n=0}^{\infty} \cup \{\emptyset \hookrightarrow *, * \sqcup * \to *\}$$

Before we move on to our main propositions and theorems, we must prove the following lemma.

**Lemma 3.3.17.** *A map of semisimplicial categories $f : K \to L$ is synchronized (induces a bijection on object sets) if and only if it has the right lifting property with respect to $\{\emptyset \hookrightarrow *, * \sqcup * \to *\}$.*

*Proof.* Let us consider the diagrams

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook, "i"'] & K \arrow[d, "f"] \\ * \arrow[r, "\alpha"'] \arrow[ur, dashed, "\exists\tilde{\alpha}"] & L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} * \sqcup * \arrow[r, "\beta"] \arrow[d, "p"'] & K \arrow[d, "f"] \\ * \arrow[r, "\gamma"'] \arrow[ur, dashed, "\exists\tilde{\gamma}"] & L \end{tikzcd} .$$

The first diagram is simply saying that for all $\alpha \in \mathrm{H}om_{\mathbf{sSemiCat}}(*, L) \cong L^0$, there exists $\tilde{\alpha} \in \mathrm{H}om_{\mathbf{sSemiCat}}(*, K) \cong K^0$ such that $f \circ \tilde{\alpha} = \alpha$. In other words, the map $f^0 : K^0 \to L^0$ is surjective.

Now, let's unravel what the second diagram is saying. Labeling the two objects in $* \sqcup *$ as $a$ and $b$ for brevity, the dashed arrow always exists if and only if for any $\beta : * \sqcup * \to K$ and $\gamma : * \to L$ such that $f \circ \beta = \gamma \circ p$ (in other words, such that $f(\beta(a)) = f(\beta(b)) = \gamma(*)$), there exists a $\tilde{\gamma} : * \to K$ such that

$$\beta(a) = \tilde{\gamma}(p(a)) = \tilde{\gamma}(c) = \tilde{\gamma}(p(b)) = \beta(b).$$

This, in turn holds if and only if $f^0 : K^0 \to L^0$ is injective.

□

Now, we are ready to prove the main results of this section.

Before we can prove that **sSemiCat** is a model category equipped with a left-induced model structure, we must prove that it is bicomplete.

**Theorem 3.3.18.** ***sSemiCat*** *is bicomplete.*

*Proof.* This proof is analogous to the proof that **Flow** is bicomplete (found in **Gau1**), and essentially follows from the bicompleteness of **sSet**. Note first and foremost that for a set $A$ and a simplicial semicategory $K$, letting $F : \mathbf{Set} \to \mathbf{sSemiCat}$ be the functor sending a set to the simplicial semicategory with that set as objects and no morphisms,

$$\mathrm{H}om_{\mathbf{Set}}(A, K^0) \cong \mathrm{H}om_{\mathbf{sSemiCat}}(FA, K).$$

Thus, the object set functor is a right adjoint, and so if limits exist in **sSemiCat**, we know exactly what their object sets must look like. Indeed, given a (small) diagram $K_{(-)} : I \to \mathbf{sSemiCat}$, we can define its limit $\varprojlim_i K_i$ as follows:

- $(\varprojlim_i K_i)^0 = \varprojlim_i (K_i^0)$.
- Given any two $\alpha, \beta \in (\varprojlim_i K_i)^0$, taking $\alpha_i$ and $\beta_i$ to be their images in $K_i$ for $i \in I$, define
$$\mathbb{P}^{\Delta}_{\alpha\beta}(\varprojlim_i K_i) = \varprojlim_i \mathbb{P}^{\Delta}_{\alpha_i\beta_i} K_i.$$
- Given any three $\alpha, \beta, \gamma \in (\varprojlim_i K_i)^0$, define
$$* : \mathbb{P}^{\Delta}_{\alpha\beta}(\varprojlim_i K_i) \times \mathbb{P}^{\Delta}_{\beta\gamma}(\varprojlim_i K_i) \to \mathbb{P}^{\Delta}_{\alpha\gamma}(\varprojlim_i K_i)$$
as the limit over all $i \in I$ of the $i$th level compositions
$$*_i : \mathbb{P}^{\Delta}_{\alpha_i\beta_i} K_i \times \mathbb{P}^{\Delta}_{\beta_i\gamma_i} K_i \to \mathbb{P}^{\Delta}_{\alpha_i\gamma_i} K_i.$$

Taken altogether, this is sufficient to define limits of simplicial semicategories. For colimits, a slightly subtler argument is needed. In particular, given that **sSemiCat** is complete, we prove that colimits exist by appealing to Freyd's Adjoint Functor Theorem. In particular, let $\Delta_I : \mathbf{sSemiCat} \to \mathbf{sSemiCat}^I$ denote the constant diagram functor from simplicial semicategories to the category of $I$-shaped diagrams in simplicial semicategories (we assume, of course, that $I$ is small). Clearly, $\Delta_I$ commutes with limits (and is thus continuous). We now wish to show that $\Delta_I$ satisfies the solution set condition. Take a diagram $D \in \mathbf{sSemiCat}^I$. As in Gaucher, we note that all morphisms $f : D \to \Delta_I K$ for $K \in \mathbf{sSemiCat}$ form a proper class of solutions, so let's try to pair this down to a set. Now, consider the cardinal $\kappa = \aleph_0 \cdot \sum_{i \in I} \#(D_i^0 \sqcup \mathbb{P}^{\Delta} D_i)$ (where here we have the cardinalities of the underlying sets). Now, choose a representative for every isomorphism class of simplicial semicategories whose underlying object and morphism sets have cardinality less than or equal to $2^{\kappa}$, and let $\mathcal{A}$ be the set of all these representatives. Then $\bigcup_{A \in \mathcal{A}} \mathrm{H}om_{\mathbf{sSemiCat}^I}(D, \Delta_I A)$ forms a set of solutions, which may be proved as follows. Consider an arbitrary natural transformation $\alpha : D \to \Delta_I K$ for some $K \in$ **sSemiCat**. Then for every $i \in I$, we get a map $\alpha_i : D_i \to K$. Thus, we obtain a sub-simplicical semicategory $\langle \bigcup_{i \in I} \alpha_i(D_i) \rangle \subset K$ generated by the images of the $D_i$ with overall cardinality less than or equal to $2^\kappa$. Thus, $\langle \bigcup_{i \in I} \alpha_i(D_i) \rangle \cong A$ for some $A \in \mathcal{A}$, and our morphism must factor through some map $\beta : D \to \Delta_I A$ for $A \in \mathcal{A}$. □

Now we can continue on with left-induction of the model structure.

**Proposition 3.3.19.** *The model structure on* ***Flow*** *may be left-induced via the adjunction introduced in the previous proposition. This upgrades* $(|-| \dashv Sing)$ : ***sSemiCat*** $\rightleftarrows$ ***Flow*** *into a Quillen pair.*

*Proof.* Define $\mathcal{W}_{\mathbf{sSemiCat}}$ to consist of all those morphisms that become weak equivalences under realization. Recall that **Flow** is a combinatorial model category, and hence the model structure on **Flow** may be left-induced along $(|-| \dashv Sing)$ : **sSemiCat** $\rightleftarrows$ **Flow** if and only if $|\mathbf{rlp}(|-|^{-1}(\mathbf{cof_{Flow}}))| \subset \mathcal{W}_{\mathbf{Flow}}$. Note that by definition, for all $n$, $|\mathrm{Glob}^\Delta(\partial\Delta^n \hookrightarrow \Delta^n)| \cong \mathrm{Glob}(\partial D^n \hookrightarrow D^n)$. Now, this implies that $I_{Simp} \subset |-|^{-1}(\mathbf{cof_{Flow}})$, which in turn yields that $\mathbf{rlp}(I_{Simp}) \supset \mathbf{rlp}(|-|^{-1}(\mathbf{cof_{Flow}}))$. Thus, if we can show that $|rlp(I_{Simp})| \subset \mathcal{W}_{\mathbf{Flow}}$, then we are done. Suppose that we have a map $(f : K \to L) \in \mathbf{rlp}(I_{Simp})$. Then, first of all, because $f$ has the right lifting property with respect to $\{\emptyset \hookrightarrow *, * \sqcup * \to *\}$, it is a synchronized morphism, or in other words, $\mathrm{ob}(f) : \mathrm{ob}(K) \to \mathrm{ob}(L)$ is a bijection of sets. Furthermore, for all $n$, and for all commutative squares

$$\begin{tikzcd} \mathrm{Glob}^\Delta(\partial\Delta^n) \arrow[r] \arrow[d, hook] & K \arrow[d, "f"] \\ \mathrm{Glob}^\Delta(\Delta^n) \arrow[r] \arrow[ur, dashed] & L \end{tikzcd},$$

the dashed arrow exists. Note that this last condition holds if and only if $\mathbb{P}^\Delta f : \mathbb{P}^\Delta K \to \mathbb{P}^\Delta L$ is a trivial Kan fibration of simplicial sets. However, this implies that $|\mathbb{P}^\Delta f| = \mathbb{P}|f|$ is a trivial Serre fibration of topological spaces. This, plus the fact that $|f|^0 : |K|^0 \to |L|^0$ is a bijection of sets, yields that $|f|$ is a trivial fibration of flows, and hence that $|f| \in \mathcal{W}_{\mathbf{Flow}}$. Thus, we have shown that $|\mathbf{rlp}(I_{Simp})| \subset \mathcal{W}_{\mathbf{Flow}}$, and hence that the left-induced model structure on **sSemiCat** exists. □

Furthermore this Quillen pair is actually a Quillen equivalence.

**Theorem 3.3.20.** *The Quillen adjunction* $(|-| \dashv Sing)$ : ***sSemiCat*** $\rightleftarrows$ ***Flow*** *is a Quillen equivalence.*

*Proof.* To start with, note that since our model structure on **sSemiCat** is left induced, the left-adjoint realization functor creates weak equivalences. This implies that we only need to show that for every fibrant $X \in$ **Flow** (in other words, any flow), the counit of the adjunction $\epsilon : |Sing(X)| \to X$ is a weak equivalence. However, this holds due to the same result for the Quillen adjunction between simplicial sets and topological spaces. □

Thus, we actually have a nice combinatorial model for Flows, much akin to that for spaces. Furthermore, we have the following results.

**Theorem 3.3.21.** *There is a combinatorial model structure on* ***sSemiCat*** *which has as its generating set of cofibrations $I_{Simp}$ Quillen equivalent to the structure constructed above via the identity.*

*Proof.* First, we briefly show using Jeff Smith's theorem that we have a valid combinatorial model category structure. Note that $\mathbf{rlp}(I_{Simp}) \subset \mathcal{W}_{\mathbf{sSemiCat}}$, that $\mathcal{W}_{\mathbf{sSemiCat}}$ satisfies 2-out-of-3, and that $\mathcal{W}_{\mathbf{sSemiCat}}$ and $\mathbf{cof}(I_{Simp})$ are both closed under pushout and transfinite composition. Thus, there is a valid combinatorial model category structure on **sSemiCat** with $I_{Simp}$ as its generating cofibrations, and $\mathcal{W}_{\mathbf{sSemiCat}}$ as its weak equivalences.

We now prove that it is Quillen equivalent to the model category structure from above. Note first that by definition the identity functor is self-adjoint. Now, observe that since $I_{Simp} \subset | - |^{-1}(\mathbf{cof}_{\mathbf{Flow}})$, one has that $\mathbf{rlp}(| - |^{-1}(\mathbf{cof}_{\mathbf{Flow}})) \subset \mathbf{rlp}(I_{Simp})$ and furthermore that $\mathbf{cof}(I_{Simp}) \subset | - |^{-1}(\mathbf{cof}_{\mathbf{Flow}})$. Thus, by definition, taking the identity as a left adjoint, it takes cofibrations into cofibrations, and taking the identity as a right adjoint, it takes fibrations into fibrations. Thus, the identity functor on **sSemiCat** is a Quillen self-adjunction between the two model structures we have discussed thus far. Since these two model structures have precisely the same weak equivalences, it is automatic that the adjunction unit between cofibrant objects is a weak equivalence. Indeed, it is the identity. Thus, the identity functor forms a Quillen equivalence with itself between these two model structures on **sSemiCat**. □

**Remark 3.3.22.** *In fact, this same technique yields a proof for demonstrating the equivalence of simplicial categories and topological categories found in* **Ili** . *The one difference is in showing that $|\boldsymbol{rlp}(| - |^{-1}\boldsymbol{cof}_{\boldsymbol{Cat(Top)}})| \subset \mathcal{W}_{\boldsymbol{Cat(Top)}}$, but this is a relatively simple alteration.*

**A Few Properties of Simplicial Semicategories**

As was hinted at in the previous section, simplicial semicategories admit a number of definitions analogous to those found in **Flow**, for example globes, path simplicial sets, and so on. We will simply use the same notation in what follows.

Note that **sSemiCat** obeys the same formal properties as **Flow**. That said, there are many situations in which simplicial semicategories are particularly well-behaved. For example, they have an extremely natural notion of simplicially enriched Hom-set, and many of the same theorems admit a shorter proof, with several conditions being removed as opposed to their counterparts in flows. In particular, see the proofs of proposition 3.25 and theorems 3.26 through 3.28 below.

**Definition 3.3.23.** *Given a simplicial set $S$ and $K \in$ **sSemiCat**, we define $\{S, K\} \in$ **sSemiCat** as follows:*

- $\{S, K\}^0 := K^0$.
- *For any two $\alpha, \beta \in K^0$, $\mathbb{P}^{\Delta}_{\alpha\beta}\{S, K\} := \boldsymbol{sSet}(S, \mathbb{P}^{\Delta}_{\alpha\beta}K)$.*
- *For any three $\alpha, \beta, \gamma \in K^0$, the composition law is the composite*

$$* : \mathbb{P}^{\Delta}_{\alpha\beta}\{S, K\} \times \mathbb{P}^{\Delta}_{\beta\gamma}\{S, K\} \cong \boldsymbol{sSet}(S, \mathbb{P}^{\Delta}_{\alpha\beta}K \times \mathbb{P}^{\Delta}_{\beta\gamma}K) \to \boldsymbol{sSet}(S, \mathbb{P}^{\Delta}_{\alpha\gamma}K) = \mathbb{P}^{\Delta}_{\alpha\gamma}\{S, K\}.$$

**Theorem 3.3.24.** *The assignment $\{-, -\} :$ **sSet** $\times$ **sSemiCat** $\to$ **sSemiCat** is contravariantly functorial in the first argument and covariantly functorial in the second argument. Furthermore, one has the natural isomorphisms $\{S, \varprojlim_i K_i\} \cong \varprojlim_i \{S, K_i\}$ and $\{\varinjlim_i S_i, K\} \cong \varprojlim_i \{S_i, K\}$. Finally, given any two $S, T \in$ **sSet**, one has for all $K \in$ **sSemiCat** that $\{S \times T, K\} \cong \{S, \{T, K\}\}$.*

*Proof.* The functoriality is clear, the behavior with respect to limits in both arguments follows from the behavior of internal homs in **sSet** with respect to limits in both arguments, and the last condition follows from the adjunction in **sSet** between internal hom and cartesian product. □

This pairing actually yields the following theorem:

**Proposition 3.3.25.** ***sSemiCat** is simplicially enriched, and the assignment*

$$\boldsymbol{sSemiCat}^{op} \times \boldsymbol{sSemiCat} \to \boldsymbol{sSet}$$

*given by $(K, L) \mapsto$ **sSemiCat**$(K, L)$ is functorial.*

*Proof.* We adapt Joyal's discussion of the enrichement of simplicial categories found in **Joy** to the setting of simplicial semicategories. We first show that for any $n \in \mathbb{N}$, the functor $\{\Delta^n, -\} : \mathbf{sSemiCat} \to \mathbf{sSemiCat}$ described above defines a monad. To show this, we will employ the contravariant functoriality of the first argument of $\{-,-\}$. Consider the evident morphism $\Delta^n \to \Delta^0$. For any $K \in \mathbf{sSemiCat}$, this provides us with a unit map $K \to \{\Delta^n, K\}$ upon noting the natural isomorphism $\{\Delta^0, K\} \cong K$. Our multiplication map arises from the diagonal $\Delta^n \hookrightarrow \Delta^n \times \Delta^n$ via the composition

$$\{\Delta^n, \{\Delta^n, K\}\} \to \{\Delta^n \times \Delta^n, K\} \to \{\Delta^n, K\}.$$

Now for any $K, L \in \mathbf{sSemiCat}$, let us define $\mathbf{sSemiCat}(K, L)$ via the assignment $\mathbf{sSemiCat}(K, L)_n = \mathrm{H}om_{\mathbf{sSemiCat}}(K, \{\Delta^n, L\})$. We can define composition for $K, L, M \in \mathbf{sSemiCat}$ as a Kleisli multiplication

$$\mathbf{sSemiCat}(K, L)_n \times \mathbf{sSemiCat}(L, M)_n \to \mathbf{sSemiCat}(K, M)_n.$$

Explicitly, this is the following composition, where we omit Hom subscripts for brevity:

$$\begin{aligned}
&\mathbf{sSemiCat}(K, L)_n \times \mathbf{sSemiCat}(L, M)_n = \mathrm{H}om(K, \{\Delta^n, L\}) \times \mathrm{H}om(L, \{\Delta^n, M\}) \\
&\to \mathrm{H}om(K, \{\Delta^n, L\}) \times \mathrm{H}om(\{\Delta^n, L\}, \{\Delta^n, \{\Delta^n, M\}\}) \longrightarrow \mathrm{H}om(K, \{\Delta^n, \{\Delta^n, M\}\}) \\
&\to \mathrm{H}om(K, \{\Delta^n, M\}) = \mathbf{sSemiCat}(K, M)_n.
\end{aligned}$$

Taken together, this determines the simplicial set $\mathbf{sSemiCat}(K, L)$, and the composition law $\mathbf{sSemiCat}(K, L) \times \mathbf{sSemiCat}(L, M) \to \mathbf{sSemiCat}(K, M)$. Given this definition, functoriality is immediate. □

Moreover, one has the following theorem.

**Theorem 3.3.26.** *The functor $\{S, -\} : \boldsymbol{sSemiCat} \to \boldsymbol{sSemiCat}$ has a left adjoint denoted by $S \boxtimes (-)$. This defines a bifunctor $(-) \boxtimes (-)$. Furthermore, this has the following properties:*

- *There is a natural isomorphism of simplicial semicategories given by*

$$S \boxtimes (\varinjlim_i K_i) \cong \varinjlim (S \boxtimes K_i).$$

- $\Delta^0 \boxtimes K \cong K$.
- $S \boxtimes Glob(T) \cong Glob(S \times T)$.
- *There is a natural bijection* $(S \boxtimes K)^0 \cong K^0$.
- $(S \times T) \boxtimes K \cong S \boxtimes (T \boxtimes K)$.

*Proof.* We begin by proving the existence of a left adjoint at all. First, note that for any simplicial set $S$, one has that $\{S, -\}$ commutes with small limits. Thus, we need only verify the solution set condition. Begin by choosing a simplicial semicategory $K$. As in the proof that **sSemiCat** is cocomplete, we start by analyzing the class of solutions $f : K \longrightarrow \{S, L\}$ for all $L \in$ **sSemiCat** and all $f \in \mathrm{H}om_{\mathbf{sSemiCat}}(K, \{S, L\})$. Now, consider the cardinal $\kappa = \#S \cdot (\#K_0 + \#\mathbb{P}^{\Delta}K)$. By definition, if $K$ is nonempty, $\kappa \geq \aleph_0$, because the underlying set of a simplicial set is nonempty in a countably infinite number of degrees. Now, choose a representative of every isomorphism class of simplicial semicategories whose object and morphism simplicial sets have underlying set cardinality less than or equal to $2^\kappa$, and denote the set of all these representatives by $\mathcal{A}$. Now, we verify that $\bigcup_{A \in \mathcal{A}} \mathrm{H}om_{\mathbf{sSemiCat}}(K, \{S, A\})$ form a set of solutions. Consider an arbitrary $f : K \longrightarrow \{S, M\}$ for some simplicial semicategory $M$. Now, we let $N \subset M$ be the subsimplicial semicategory generated by elements of the form $f(K)(S)$. We know by definition that $\#N \leq 2^\kappa$. Thus, in particular, $\#\{S, N\} \leq \#S \cdot 2^\kappa = 2^\kappa$ (given that $\kappa \geq \aleph_0$), and hence, $\{S, N\} \cong \{S, A\}$ for some $A \in \mathcal{A}$. Thus, our initial morphism factors through our solution set, and we have a well-defined left adjoint, which we denote by $S \boxtimes (-)$.

- Note that for all $L \in$ **sSemiCat**, one has that

$$\begin{aligned}
\mathrm{H}om_{\mathbf{sSemiCat}}(S \boxtimes (\varinjlim_i K_i), L) &\cong \mathrm{H}om_{\mathbf{sSemiCat}}(\varinjlim_i K_i, \{S, L\}) \\
&\cong \varprojlim_i \mathrm{H}om_{\mathbf{sSemiCat}}(K_i, \{S, L\}) \\
&\cong \varprojlim_i \mathrm{H}om_{\mathbf{sSemiCat}}(S \boxtimes K_i, L) \\
&\cong \mathrm{H}om_{\mathbf{sSemiCat}}(\varinjlim_i (S \boxtimes K_i), L),
\end{aligned}$$

  which implies a natural isomorphism.

- Similarly to the above, one has

$$\mathrm{H}om_{\mathbf{sSemiCat}}(\Delta^0 \boxtimes K, L) \cong \mathrm{H}om_{\mathbf{sSemiCat}}(K, \{\Delta^0, L\}) \cong \mathrm{H}om_{\mathbf{sSemiCat}}(K, L).$$

- This arises from the adjunction $\mathrm{H}om_{\mathbf{sSet}}(K \times L, M) \cong \mathrm{H}om_{\mathbf{sSet}}(K, \mathbf{sSet}(L, M))$ on the level of simplicial sets.

- Denoting by $*$ the simplicial semicategory with one object and no morphisms, note that by definition, $K^0 \cong \mathrm{H}om_{\mathbf{sSemiCat}}(*, K)$, and considered as a simplicial semicategory, $K^0 \cong \sqcup_{* \to K} *$. Thus, as $S \boxtimes (-)$ commutes with colimits, we only need to show that $S \boxtimes * \cong *$. This follows from

$$\mathrm{H}om_{\mathbf{sSemiCat}}(S \boxtimes *, L) \cong \mathrm{H}om_{\mathbf{sSemiCat}}(*, \{S, L\}) \cong \mathrm{H}om_{\mathbf{sSemiCat}}(*, L),$$

  since one always has $\{S, L\}^0 \cong L^0$ by construction.

- Finally, one has that

$$\begin{aligned} \mathrm{H}om_{\mathbf{sSemiCat}}(S \boxtimes (T \boxtimes K), L) &\cong \mathrm{H}om_{\mathbf{sSemiCat}}(T \boxtimes K, \{S, L\}) \\ &\cong \mathrm{H}om_{\mathbf{sSemiCat}}(K, \{T, \{S, L\}\}) \\ &\cong \mathrm{H}om_{\mathbf{sSemiCat}}(K, \{T \times S, L\}) \\ &\cong \mathrm{H}om_{\mathbf{sSemiCat}}((S \times T) \boxtimes K, L). \end{aligned}$$

$\square$

**Theorem 3.3.27.** *For all $S \in \mathbf{sSet}$ and all $K, L \in \mathbf{sSemiCat}$, one has that*

$$Hom_{\mathbf{sSet}}(S, \mathbf{sSemiCat}(K, L)) \cong Hom_{\mathbf{sSemiCat}}(K, \{S, L\}) \cong Hom_{\mathbf{sSemiCat}}(S \boxtimes K, L).$$

*Proof.* Given that the second equivalence in the theorem follows from the mere fact of having an adjunction, we can simply focus on the first equivalence. Note that one may write any simplicial set $S$ naturally as a colimit $S \cong \varinjlim_{\Delta \downarrow S} \Delta^n$. This, in turn, yields the following:

$$\begin{aligned}
\mathrm{H}om_{\mathbf{sSet}}(S, \mathbf{sSemiCat}(K, L)) &\cong \mathrm{H}om_{\mathbf{sSet}}(\varinjlim_{\Delta\downarrow S} \Delta^n, \mathbf{sSemiCat}(K, L)) \\
&\cong \varprojlim_{\Delta\downarrow S} \mathrm{H}om_{\mathbf{sSet}}(\Delta^n, \mathbf{sSemiCat}(K, L)) \\
&\cong \varprojlim_{\Delta\downarrow S} \mathbf{sSemiCat}(K, L)_n \\
&= \varprojlim_{\Delta\downarrow S} \mathrm{H}om_{\mathbf{sSemiCat}}(K, \{\Delta^n, L\}) \\
&\cong \mathrm{H}om_{\mathbf{sSemiCat}}(K, \varprojlim_{\Delta\downarrow S}\{\Delta^n, L\}) \\
&\cong \mathrm{H}om_{\mathbf{sSemiCat}}(K, \{\varinjlim_{\Delta\downarrow S} \Delta^n, L\}) \\
&\cong \mathrm{H}om_{\mathbf{sSemiCat}}(K, \{S, L\}).
\end{aligned}$$

$\square$

**Theorem 3.3.28.** *The enriched Hom simplicial sets in **sSemiCat** behave "as one would expect" with respect to limits and colimits. Namely:*

- $\boldsymbol{sSemiCat}(\varinjlim_i K_i, L) \cong \varprojlim_i \boldsymbol{sSemiCat}(K_i, L)$ *for any colimit.*
- $\boldsymbol{sSemiCat}(K, \varprojlim_i L_i) \cong \varprojlim_i \boldsymbol{sSemiCat}(K, L_i)$ *for any limit.*

*Proof.* We will only prove the first statement, as the second may be proven analogously. Note that by Yoneda, $\mathbf{sSemiCat}(K, L)_n \cong \mathrm{H}om_{\mathbf{sSet}}(\Delta^n, \mathbf{sSemiCat}(K, L))$ and by the above, $\mathrm{H}om_{\mathbf{sSet}}(\Delta^n, \mathbf{sSemiCat}(K, L)) \cong \mathrm{H}om_{\mathbf{sSemiCat}}(\Delta^n \boxtimes K, L)$, Now, note that

$$
\begin{aligned}
\mathbf{sSemiCat}(\varinjlim_i K_i, L)_n &\cong \mathrm{H}om_{\mathbf{sSet}}(\Delta^n, \mathbf{sSemiCat}(\varinjlim_i K_i, L)) \\
&\cong \mathrm{H}om_{\mathbf{sSemiCat}}(\Delta^n \boxtimes (\varinjlim_i K_i), L) \\
&\cong \mathrm{H}om_{\mathbf{sSemiCat}}(\varinjlim_i (\Delta^n \boxtimes K_i), L) \\
&\cong \varprojlim_i \mathrm{H}om_{\mathbf{sSemiCat}}(\Delta^n \boxtimes K_i, L) \\
&\cong \varprojlim_i \mathrm{H}om_{\mathbf{sSet}}(\Delta^n, \mathbf{sSemiCat}(K_i, L)) \\
&\cong \mathrm{H}om_{\mathbf{sSet}}(\Delta^n, \varprojlim_i \mathbf{sSemiCat}(K_i, L)) \\
&\cong (\varprojlim_i \mathbf{sSemiCat}(K_i, L))_n,
\end{aligned}
$$

where equivalence of the unenriched homs in this string results from abstract nonsense. Now, finally, note that for any $S \in \mathbf{sSet}$, one has that $S \cong \varinjlim_{\Delta\downarrow S} \Delta^n$. Thus, making use of the above equivalences, we have

$$
\begin{aligned}
\mathrm{H}om_{\mathbf{sSet}}(S, \mathbf{sSemiCat}(\varinjlim_i K_i, L)) &\cong \mathrm{H}om_{\mathbf{sSet}}(\varinjlim_{\Delta\downarrow S} \Delta^n, \mathbf{sSemiCat}(\varinjlim_i K_i, L)) \\
&\cong \varprojlim_{\Delta\downarrow S} \mathrm{H}om_{\mathbf{sSet}}(\Delta^n, \mathbf{sSemiCat}(\varinjlim_i K_i, L)) \\
&\cong \varprojlim_{\Delta\downarrow S} \mathrm{H}om_{\mathbf{sSet}}(\Delta^n, \varprojlim_i \mathbf{sSemiCat}(K_i, L)) \\
&\cong \mathrm{H}om_{\mathbf{sSet}}(\varinjlim_{\Delta\downarrow S} \Delta^n, \varprojlim_i \mathbf{sSemiCat}(K_i, L)) \\
&\cong \mathrm{H}om_{\mathbf{sSet}}(S, \varprojlim_i \mathbf{sSemiCat}(K_i, L)).
\end{aligned}
$$

Thus, we have that there is a natural isomorphism between the simplicial sets $\mathbf{sSemiCat}(\varinjlim_i K_i, L)$ and $\varprojlim_i \mathbf{sSemiCat}(K_i, L)$. □

We now end our discussion with an important definition and a theorem.

**Definition 3.3.29.** *Two simplicial semifunctors $f, g : K \to L$ are simplicial $S$-homotopy equivalent if there exists $H \in Hom_{\mathbf{sSemiCat}}(\Delta^1 \boxtimes K, L)$ such that $H|_0 = f$ and $H|_1 = g$. Equivalently, if there exists $h \in Hom_{\mathbf{sSet}}(\Delta^1, \mathbf{sSemiCat}(K, L))$ such that $h(0) = f$ and $h(1) = g$.*

Finally, we note the following theorem.

**Theorem 3.3.30.** *The functor* $\Delta^1 \boxtimes (-)$ *is a cylinder functor when equipped with the natural transformations* $e_0 : \{0\} \boxtimes (-) \to \Delta^1 \boxtimes (-)$ *and* $e_1 : \{1\} \boxtimes (-) \to \Delta^1 \boxtimes (-)$ *and the natural transformation* $p : \Delta^1 \boxtimes (-) \to \{0\} \boxtimes (-)$ *induced by projection.*

*Proof.* This is relatively clear. One merely notes for $i \in \{0, 1\}$ that $p \circ e_i$ is naturally equivalent to the identity transformation. □

## 3.4 Tree-like Flows and Boxed tree sets

Now, analyzing the cubical nerve above, it has a natural extension to the study of branching concurrent processes. In the following section, we will introduce two homotopy-coherent operations between the category of flows, and the category of (pre-)boxed tree sets and a cubical/boxed tree analogue. We denote the category of finite symmetric trees considered as posets by $\mathcal{T}$. Finite trees with only injective morphisms between them will be denoted by $\mathcal{T}_{inj}$. Now, let us note that since we have the obvious inclusion of the category of posets into flows, we have an inclusion of $\mathcal{T}_{inj}$ into **Flow**. Now, taking the cofibrant replacement of everything, we obtain a "geometric realization" of every finite tree $T$, which we denote $|T|_{\mathcal{T}}$. Using these realizations, we can cook up homotopy coherent nerve objects between flows and pre-tree-sets.

**Remark 3.4.1.** *These tree sets are* not *dendroidal sets, as this category* $\mathcal{T}$ *is not the category* $\Omega$ *of Moerdijk and Weiss.*

Our realization is given as follows. For all $K \in \mathbf{PrSh}(\mathcal{T}_{inj})$, we obtain a flow given by $|K|_{\mathcal{T}} := \varinjlim_{\mathcal{T}_{inj} \downarrow K} |T|_{\mathcal{T}}$. Similarly, our nerve is given for all flows by $N(X)_T := \mathrm{H}om_{\mathbf{Flow}}(|T|_{\mathcal{T}}, X)$. These functors form an adjunction

$$\begin{tikzcd} \mathbf{PrSh}(\mathcal{T}_{inj}) \arrow[r, bend left, "|-|_{\mathcal{T}}"] \arrow[r, phantom, "\perp"] & \mathbf{Flow} \arrow[l, bend left, "N_{\mathcal{T}}"] \end{tikzcd}.$$

This is proved much in the same way as the adjunction before is.

Now, we can define a category which will aid us in our attempts to understand the interactions between concurrent and sequential processes.

**The Category of Boxed Symmetric Trees**

**Definition 3.4.2.** *The category of* boxed symmetric trees $\boxed{\mathcal{T}}$ *is the category whose objects consist of n-tuples of elements of $\mathcal{T}$ for varying $n \in \mathbb{N}$. Morphisms are generated by the following types of arrow:*

*i. For any n-tuple $T_1 \times \cdots \times T_n \in \boxed{\mathcal{T}}$, and for any $\{f_i : T_i \to T_i'\} \in T_i \downarrow \mathcal{T}$ for any $i \in \{0, ..., n\}$, the map*

$$f_i : T_1 \times \cdots \times T_i \times \cdots \times T_n \to T_0 \times \cdots \times T_i' \times \cdots \times T_n$$

*given by applying the map $\sigma_i$ to the ith coordinate and leaving the others unchanged.*

*ii. For any n-tuple $T_1 \times \cdots \times T_n \in \boxed{\mathcal{T}}$, any $\varepsilon \in \{0, 1\}$, and any $i \in \{1, ..., n+1\}$, one has*

$$\partial_i^\varepsilon : T_1 \times \cdots \times T_i \times \cdots \times T_n \to T_1 \times \cdots \times T_{i-1} \times [1] \times T_i \times \cdots \times T_n$$

*via $\partial_i^\varepsilon(a_1, ..., a_n) = (a_1, ..., a_{i-1}, \varepsilon, a_i, ..., a_n)$.*

*iii. For any n-tuple $T_1 \times \cdots \times T_n \in \boxed{\mathcal{T}}$ and any $i \in \{0, ..., n\}$, one has*

$$s_i : T_1 \times \cdots \times T_n \to T_1 \times \cdots \times \hat{T}_i \times \cdots \times T_n$$

*given by omitting the ith coordinate.*

*iv. For all $\sigma \in \Sigma_n$ we obtain the obvious map*

$$\sigma : T_1 \times \cdots \times T_n \to T_{\sigma(1)} \times \cdots \times T_{\sigma(n)}$$

*permuting the different factors.*

*We define $\boxed{\mathcal{T}}_{inj}$ to be the subcategory of $\boxed{\mathcal{T}}$ defined by taking as generators only the injective morphisms described above (i.e. only the injective tree morphisms in i and morphisms in iii).*

In more informal language, $\boxed{\mathcal{T}}$ consists of cubes where we allow as the sides not just the standard interval, but in fact all finite trees as our intervals. Furthermore, we prune our trees and grow branches.

When describing what simplicial nerves describe in the categorical or homotopy coherent categorical setting, we see that they correspond to chaining together composable arrows, in the setting above, they would correspond to running computations, not concurrently, but sequentially.

Cubical nerves, on the other hand, model processes running concurrently, where each independent direction corresponds to a different operation being run at the same time.

What we hope to achieve with $\boxed{\mathcal{T}}$ is to describe chained concurrent processes, possibly each with their own "flowchart" allowing for branched procedures in each of the factors. In what follows, we briefly ponder the categories of $\boxed{\mathcal{T}}$-sets and pre $\boxed{\mathcal{T}}$-sets, before trying to understand their geometric realization into flows and the homotopy coherent nerve back.

We define the presheaf categories **PrSh**($\boxed{\mathcal{T}}$) and **PrSh**($\boxed{\mathcal{T}}_{inj}$) to be the categories of $\boxed{\mathcal{T}}$-sets and pre $\boxed{\mathcal{T}}$-sets respectively.

We may also consider a slightly larger category of shapes, which we can call $\mathfrak{T}$. We may define this as the full subcategory of **FinPoSet** generated by the objects $\Pi_i T_i$ as in $\boxed{\mathcal{T}}$. This, in particular has "connection-like" morphisms built into it, among other things. It allows for a slightly wider set of computational interpretations than $\boxed{\mathcal{T}}$, as illustrated by the following idea. Given any $T_1 \times T_2 \in \mathfrak{T}$ consisting of the product of two trees, one has a morphism $T_1 \times T_2 \to \{0, 1\}$ in $\mathfrak{T}$ given by $(s, t) \mapsto 0$ if $s$ and $t$ are both the root, and $(s, t) \mapsto 1$ if else, which corresponds roughly to checking if both computations involved have initialized or not.

### The category of Boxed Trees is a Test Category

We briefly recall the notion of a test category before demonstrating that the category of boxed trees is a test category. Test categories were first introduced by Grothendieck in **Gro** in order to come up with reasonable combinatorial models for spaces (a particularly nice introduction can be found in**Cis**). In particular, a test category can be thought of as a small category with the property that all homotopy types may be modeled by presheaves on it. This is done in the following manner.

Recall that **Cat** is the category of small categories. Let us define $\mathcal{W}_\infty$ to be the class of "weak equivalences of categories." Namely, these are functors which become weak homotopy equivalences under the nerve functor into simplicial sets (in other words, the $\infty$-groupoidifications of these categories are equivalent). Note that while certainly equivalences of categories are weak equivalences in this manner, it is a much wider class of functors, including any functor which is a left or right adjoint, among others (this is shown by noting that natural transformations are mapped via the nerve construction to simplicial homotopies, which ensures that the unit and counit map to a homotopy equivalence). An important theorem is that the localization of **Cat** by $\mathcal{W}_\infty$ is equivalent to the standard homotopy category of CW complexes/simplicial sets (in fact, there is a model structure on **Cat** due to Thomason **Tho** which realizes this equivalence as a Quillen equivalence).

Consider a small category $C$. Note that there is a natural adjunction

$$(|-|_C \dashv N_C) : \mathbf{PrSh}(C) \rightleftarrows \mathbf{Cat}$$

defined in one direction by taking for every $C \in \mathbf{PrSh}(C)$, $|C|_C = C \downarrow C$, and in the other direction by taking $N_C(\mathcal{D})_c = \mathrm{H}om_{\mathbf{Cat}}(C \downarrow c, \mathcal{D})$.

**Definition 3.4.3.** *We may define* weak test categories *as those small categories $C$ for which the counit of the adjunction above $|N_C(\mathcal{D})|_C \to \mathcal{D}$ is always a weak equivalence.*

Now, we may further analyze the adjunction $(h \dashv N) : \mathbf{sSet} \rightleftarrows \mathbf{Cat}$. Note that we have $N(|-|_C) \dashv N_C \circ h$.

**Definition 3.4.4.** *If this composite adjunction may be upgraded to a Quillen equivalence with a model structure on **PrSh**$(C)$ whose cofibrations are monomorphisms, we then say that $C$ is a* test category *(This was noted to be equivalent to the more technical definition given below in* **ArCiMo***). In other words, test categories are precisely those which provide a good combinatorial model of spaces upon taking presheaves.*

There are numerous equivalent classifications of (weak) test categories which provide concrete criteria which may be checked (sacrificing brevity and ease of understanding for an actual ability to perform calculations). The following can be found in **Cis** .

**Proposition 3.4.5.** *A category $C$ is a* test category *if and only if the following conditions hold:*

1. *$C$ is aspherical (i.e. $N(C)$ is a contractible simplicial set.*

2. *One of these equivalent conditions hold*

   a) *$C$ is a local test category (for every object $c \in C$, the overcategory $C/c$ is a weak test category, which in turn means that.*

   b) *The subobject classifier $L_C$ in **PrSh**$(C)$ is locally aspherical.*

   c) *There exists a locally aspherical separating interval in **PrSh**$(C)$.*

We will now provide some of the necessary definitions from the proposition above.

**Definition 3.4.6.** *Given* $\textbf{\textit{PrSh}}(C)$ *as above, an* interval *in* $C$ *is a triple* $(I, d_0, d_1)$*, where* $I \in \textbf{\textit{PrSh}}(C)$ *and* $d_i : *_C \to I$ *for* $i = 0, 1$*, where* $*_C$ *is the terminal object of* $\textbf{\textit{PrSh}}(C)$*. This interval is called* aspherical *if* $|I|_C$ *is weakly equivalent to the terminal category (in other words, I is aspherical as a homotopy type), and is called* separating *if the equalizer of the double arrow* $(d_0, d_1)$ *is the empty presheaf on* $C$*.*

To prove that $\boxed{\mathcal{T}}$ is a test category, it suffices to prove that $\boxed{\mathcal{T}}$ is both aspherical and a local test category. Let's work this out more concretely.

**Lemma 3.4.7.** *Any small category* $C$ *with a terminal object is acyclic.*

*Proof.* Let $C$ be a small category which has a terminal object $e \in C$. There exists a natural transformation from the identity functor to the constant functor at $e$ whose components are the unique maps to $e$. Furthermore, upon taking the nerve of $C$, this natural transformation becomes a homotopy from $N(C)$ to a point, yielding the result. □

Now, due to the above, we obtain that $\boxed{\mathcal{T}}$ is acyclic, since $[0]$ is a terminal object for $\boxed{\mathcal{T}}$.

**Definition 3.4.8.** *Given a small category* $C$*, a* functorial precylinder *is a triple* $(I, \partial^0, \partial^1)$ *such that* $I : C \to C$ *is an endofunctor of* $C$ *and* $\partial^i$ *are natural transformations from the identity functor on* $C$ *to* $I$*.*

*An* augmentation *of* $(I, \partial^0, \partial^1)$ *is a collection of morphisms* $\sigma_a : I(a) \to a$ *for all objects* $a \in C$ *such that for* $i = 0, 1$ *and* $a \in C$*, one has that* $\sigma_a \circ \partial^i_a = id_a$*. A precylinder which may be equipped with an augmentation is called* augmented*.*

**Proposition 3.4.9.** *Fix a small category* $C$*, a functor* $i : C \to \textbf{Cat}$*, and a functorial augmented precylinder* $(I, \partial^0, \partial^1)$*. Suppose the following conditions are satisfied:*

- *For all* $a \in C$*, the functor* $i \circ \partial^0_a : i(a) \to iI(a)$ *is an open immersion (i.e. there exists an isomorphism between* $i(a)$ *and a sieve* $U_a$ *of the category* $iI(a)$*), and the functor* $i \circ \partial^1_a : i(a) \to iI(a)$ *factorizes through the complementary cosieve of* $U_a$*, which we denote by* $F_a = iI(a) - U_a$*.*
- *For all morphisms* $\alpha : a \to a'$ *in* $C$*, one has* $iI(\alpha)(F_a) \subset F_{a'}$*.*
- *For all* $a \in C$*, the category* $i(a)$ *has a final object.*

*Then $i$ is a local test functor and $C$ is therefore a local test category.*

*Proof.* This is proven in **Cis** lemma 8.4.12. □

In particular, this implies the following lemma.

**Lemma 3.4.10.** *The category $\boxed{\mathcal{T}}$ is a local test category.*

*Proof.* We take for simplicity all elements of $\mathcal{T}$ to be directed towards the root. There is a natural embedding of $\boxed{\mathcal{T}}$ into **Cat**, which we will label $i : \boxed{\mathcal{T}} \hookrightarrow$ **Cat**, in which we take the elements of $\boxed{\mathcal{T}}$ and map them to their corresponding poset categories. Note that for every $\Pi T_i \in \boxed{\mathcal{T}}$, one has that $i(\Pi T_i)$ has a final object by convention. Thus, the last condition of the above lemma is satisfied, and we need only concern ourselves with the first two.

Now, we will introduce our augmented functorial precylinder, which we will denote $(I, \partial^0, \partial^1)$. In particular, $I = [1] \times (-)$ and $\partial^i$ is the inclusion of either of the endpoint copies of what we start with via

$$(-) \xrightarrow{\sim} [0] \times (-) \rightrightarrows [1] \times (-).$$

Now, let us go through the remaining two points. First, note that $\Pi T_i \xhookrightarrow{\partial^0_{\Pi T_i}} [1] \times \Pi T_i$ is a sieve and that $\Pi T_i \xhookrightarrow{\partial^1_{\Pi T_i}} [1] \times \Pi T_i$ is its complementary cosieve. Thus, the first point of the above proposition is automatically satisfied. The last point follows from noting that if we have $f : \Pi S_j \to \Pi T_i$, the following square commutes:

$$\begin{tikzcd} \Pi S_j \arrow[r, "f"] \arrow[d, hook, "\partial^1_{\Pi S_j}"] & \Pi T_i \arrow[d, hook, "\partial^1_{\Pi T_i}"] \\ {[1] \times \Pi S_j} \arrow[r, "id_{[1]} \times f"] & {[1] \times \Pi T_i} \end{tikzcd} .$$

Thus, we have that $\boxed{\mathcal{T}}$ is a local test category.

□

Thus we have

**Theorem 3.4.11.** *$\boxed{\mathcal{T}}$ is a test category.*

*Proof.* Since $\boxed{\mathcal{T}}$ is a local test category and acyclic, it is a test category. □

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*Chapter 4*

# FROM SIX FUNCTORS FORMALISMS TO DERIVED MOTIVIC MEASURES

# ABSTRACT

In this paper, we generally describe a method of taking an abstract six functors formalism in the sense of Khan or Cisinski-Déglise, and outputting a derived motivic measure in the sense of Campbell-Wolfson-Zakharevich. In particular, we use this framework to define a lifting of the Gillet-Soulé motivic measure.

Conventions: Throughout this paper, $k$ will always refer to a perfect field, and $R$ will refer to an arbitrary commutative ring. Furthermore, given a base scheme $S$, the category $\mathbf{Var}_S$ of varieties over $S$ will simply be the category of finite type separated schemes over $S$ (we do not require our varieties to be reduced). Importantly, unless otherwise stated, our ambient $\infty$-cosmos **RieVer** will be that of Kan complex-enriched categories, the so-called fibrant $S$-categories of Toën-Vezzozi **ToeVez2** . This is because we will be working with commutative diagrams directly for much of the work, and it is easier to prove commutativity in a model with strict horizontal composition. That said, we may invoke comparison between the K-theory of $S$-categories and that of Quasicategories developed in **BGT** .

## 4.1 Introduction

This paper was born out of a desire to construct a lift of the Gillet-Soulé motivic measure. While originally more limited in scope, its current iteration synthesizes a number of different recent developments in algebraic geometry and homotopy theory with an eye towards constructing meaningful derived motivic measures from six functors formalism.

One of the unusual features of this construction is that we are using something quite rare—namely, a six functors formalism (a highly structured object) to define a map of K-theory spectra (much less structure ultimately). That said, neither object is particularly approachable in and of itself, and our hope is that each can be used to study the other.

We begin the paper with a brief exploration of the three main types of categories we will require for the constructions that follow. These are Waldhausen categories, a now classical type of relative category used as a particularly general setting of K-theory; SW-categories, a more recently developed analogue of Waldhausen categories used to describe and categorify cutting and pasting data; and stable $\infty$-categories, a recent $\infty$-categorical enrichment of the notion of triangulated category which simplifies the definition considerably and makes many constructions (such as gluing and taking cones) functorial that are simply not on the level of triangulated categories. Note that we will not discuss the formalisms of model categories or DG/spectral categories in this work because they are only briefly touched upon. The motivated reader is referred to the phenomenal book by Hovey **Hov** for model categories and the wonderful introduction by Toën **Toe** for DG categories (spectral categories are a focal point of the groundbreaking paper **BGT**).

We will then briefly look at how to extract K-theory from each of these types of categories. The one novel aspect of this section will be to demonstrate how to meaningfully extend the definition of weakly W-exact functor **Cam** from the original setting (in which one compares an SW-category to a Waldhausen category) to a comparison between SW-categories and pointed, finitely homotopy cocomplete $\infty$-categories (as before incarnated as fibrant $S$-categories):

**Theorem 4.1.1.** *Given a weakly W-exact functor $F : C \to \mathcal{A}$, where $C$ is an SW-category and $\mathcal{A}$ is a pointed finitely homotopy cocomplete $\infty$-category, it may be composed with the Yoneda embedding $\mathcal{A} \to \mathcal{M}(\mathcal{A})$ to obtain a weakly W-exact functor $\iota \circ F : C \to \mathcal{M}(\mathcal{A})$ to the Waldhausen $\mathcal{M}(\mathcal{A})$ which yields a map on K-theory $K(C) \to K(\mathcal{A})$. This construction is functorial in exact functors both of SW-categories and of pointed finitely homotopy cocomplete $\infty$-categories.*

The interesting thing here is that in some sense, this is not even an extension of the definition, as given a particularly nice $\infty$-category $\mathcal{A}$ (in a sense to be defined in a later section) that embeds into a simplicial model category, we may functorially assign it a Waldhausen category $\mathcal{M}(\mathcal{A})$ such that $K(\mathcal{A}) \simeq K(\mathcal{M}(\mathcal{A})) \simeq K(N^{hc}(\mathcal{A}^{cf}))$, where the last term is the K-theory of the homotopy coherent nerve of the cofibrant-fibrant objects in $\mathcal{A}$, which is the corresponding quasicategory under the homotopy coherent nerve functor. It just so happens that in all cases we consider, the necessary niceness conditions will be met.

From here, we go on to describe the $\infty$-categorical six functors formalism outlined in Khan **Khan** alternately titled the theory of $(*, \#, \otimes)$-formalisms satisfying the Voevodsky criteria or the theory of motivic $\infty$-categories. Upon introducing the basic setup, we go on to show that for a motivic $\infty$-category $\mathbb{D}$ valued over $\mathcal{S}$, where $\mathcal{S}$ is some appropriate subcategory of schemes, given any excellent geometrically unibranch scheme $S \in \mathcal{S}$ and a variety $f : X \to S$ over $S$, the assignment $X \mapsto f_* f^!(\mathbf{1}_S)$, where $\mathbf{1}_S$ is the tensor unit $\mathbb{D}(S)$, upgrades to a weakly W-exact functor in the following way.

**Theorem 4.1.2.** *Suppose that $\mathbb{D}$ satisfies one of the following two sets of conditions:*

- *The four functors preserve constructible objects when given input a seperable morphism of finite type (note that compactness is trivially preserved by tensor)*
- *The six functors preserve constructible objects over Noetherian quasi-excellent schemes of finite dimension with respect to morphisms of finite type (in other*

*words, for any finite type morphism $f : X \to S$ with target Noetherian quasi-excellent of finite dimension, the four functors preserve constructible objects, while tensor and Hom preserve constructible objects over S)*

*Then, given a scheme S (assumed to be Noetherian quasi-excellent of finite dimension if $\mathbb{D}$ satisfies the second condition in particular), there is a weakly W-exact functor*

$$M^c_{\mathbb{D}(S)} : \boldsymbol{Var}_S \to \mathbb{D}_{cons}(S)$$

*sending each variety (smooth or otherwise) $(X \xrightarrow{f} S) \in \boldsymbol{Var}_S$ to $M^c_{\mathbb{D}(S)}(X) := f_* f^! \mathbf{1}_S$.*

This theorem forms the core of the paper, as it allows us to conclude the existence of a derived motivic measure:

**Corollary 4.1.3.** *Suppose $\mathbb{D}$ sastisfies one of the two conditions of the above theorem. Then, given a scheme S (assumed to be Noetherian quasi-excellent of finite dimension if $\mathbb{D}$ satisfies the second set of conditions), one obtains a map of K-theory spectra*

$$K(M^c_{\mathbb{D}(S)}) : K(\boldsymbol{Var}_S) \to K(\mathbb{D}_{cons}(S)).$$

From here, we apply our construction to the motivic $\infty$-category of Beilinson motives $\mathbf{DM}_B$, and show that the construction restricts to a spectral lift of the Gillet-Soulé motivic measure, thus demonstrating our stated intent.

**Theorem 4.1.4.** *Considering a perfect base field k and rational coefficients, the map*

$$K(M^c_k) : K(\boldsymbol{Var}_k) \to K(\boldsymbol{DM}^c_B(k))$$

*yields a derived lift of the Gillet-Soulé motivic measure.*

This paper concludes with a brief discussion of another approach to lifting the Gillet-Soulé motivic measure (which is almost surely equivalent), before mentioning some future directions this work might take.

## 4.2 Waldhausen Categories, SW-Categories, Stable $\infty$-Categories and the K-Theory of Varieties

### Relevant Categorical Definitions and Examples

In this section, we will outline the different notions of 1 or $\infty$-categories that are needed in this paper, as well as elucidate how to extract K-theory from each of them.

**Definition 4.2.1.** *A* Waldhausen category $(C, \boldsymbol{cof}, \mathcal{W})$ *consists of the data of a category* $C$ *with two distinguished classes of morphisms:* cofibrations $\boldsymbol{cof}$ *(elements of which are denoted* $\hookrightarrow$*) and* weak equivalences $\mathcal{W}$ *(elements of which are denoted* $\xrightarrow{\sim}$*) which are required to satisfy the following axioms:*

1. *All isomorphisms are cofibrations*
2. $C$ *has a zero object, and for any zero object* $0 \in C$ *and any* $X \in C$*, the map* $0 \to X$ *is a cofibration*
3. *Cofibrations are stable under pushout, so given any cofibration* $X \hookrightarrow Y$ *and any morphism* $X \to Y$*, the map* $Y \to Y \cup_X Z$ *is a cofibration*
4. *All isomorphisms are weak equivalences*
5. *Weak equivalences are closed under composition and hence form a subcategory*
6. *Given a commutative diagram of the form*

$$\begin{tikzcd} Z \arrow[d, "\sim"] & X \arrow[l] \arrow[r, hook] \arrow[d, "\sim"] & Y \arrow[d, "\sim"] \\ Z' & X' \arrow[l] \arrow[r, hook] & Y' \end{tikzcd}$$

*where the vertical arrows are weak equivalences and the horizontal arrows of the right square are cofibrations, one has that the induced map*

$$Y \cup_X Z \xrightarrow{\sim} Y' \cup_{X'} Z'$$

*is a weak equivalence as well.*

There is also a natural notion of functor between Waldhausen categories.

**Definition 4.2.2.** *A functor* $F : C \to C'$ *between two Waldhausen categories is* exact *if it preserves cofibrations, weak equivalences, and finite (homotopy) colimits. The category of Waldhausen categories and exact functors will be denoted Wald.*

While Waldhausen categories provided one of the earliest settings for the algebraic K-theory of categories (preceded only by exact categories), in recent years, several other (often related) frameworks have been used. One of the most important (for this and many other reasons) is that of stable quasicategories.

**Definition 4.2.3.** *Suppose that one has a pointed quasicategory $C$. Given any morphism $X \to Y$ in $C$, its* kernel *and* cokernel *are defined, if they exist, by the homotopy cartesian and cocartesian squares*

$$\begin{tikzcd} Ker f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Coker f \end{tikzcd}$$

*respectively.*

*In general, an arbitrary commutative square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

*is called a* triangle. *If it is cartesian, it is called an* exact triangle *and if it is cocartesian, it is called a* coexact triangle.

**Definition 4.2.4.** *A quasicategory $C$ is* stable *if it satisfies the following conditions:*

- *$C$ is pointed (i.e. has a zero object)*
- *Every morphism in $C$ has a kernel and a cokernel*
- *Every exact triangle is coexact and every coexact triangle is exact.*

*An* exact functor $F : \mathcal{A} \to \mathcal{B}$ *between stable quasicategories is one which preserves finite colimits.*

This extremely simple definition belies its incredible depth. In particular, the homotopy category of a stable quasicategory is a triangulated category, and essentially every important example of triangulated categories arises as such a homotopy category. We will not describe the theory of stable quasicategories here (a comprehensive guide is that of **Lur2**).

Before continuing, we note the following proposition/definition

**Proposition 4.2.5.** *There is a Quillen equivalence between the model category of simplicial sets equipped with the Joyal model structure and the model category of simplicial categories equipped with the Bergner model structure:*

$$(\mathfrak{C} \dashv N^{hc}) : \boldsymbol{sSet}_{Joyal} \rightleftarrows \boldsymbol{sSetCat}_{Bergner}.$$

*The right adjoint is known as the* homotopy coherent nerve *and takes Kan-enriched categories to quasicategories. Furthermore, it fits into a strictly commutative triangle*

$$\begin{tikzcd} \textbf{\textit{Cat}} \arrow[rr, "N"] \arrow[dr, hook] & & \textbf{\textit{sSet}} \\ & \textbf{\textit{sSetCat}} \arrow[ur, "N^{hc}"] & \end{tikzcd} .$$

*In other words, the classical nerve of a category may be factored into the inclusion of categories into simplicial categories as simplicial categories with discrete Hom simplicial sets followed by the homotopy coherent nerve.*

*Proof.* See **Ber** section 7.4. □

This allows us a direct comparison between our chosen model of ∞-categories and the one which is most often used.

**Definition 4.2.6.** *A stable ∞-category (incarnated as a fibrant S-category) will be one which maps under the homotopy coherent nerve to a stable quasicategory. It should be noted that these will be precisely the S-categories for which the above three axioms hold, but the word homotopy is inserted in front of the word (co)cartesian in the definition of (co)exact triangles.*

*An* exact functor $F : \mathcal{A} \to \mathcal{B}$ *between two stable ∞-categories will be one which preserves finite homotopy colimits.*

The last class of categories we will look at are so-called SW-categories and their precursors. These categories were introduced as a way of categorifying the notion of subtraction present in settings such as decomposing varieties into open and closed complementary subsets and decomposition of polytopes. In particular, they allow us a method of lifting universal Euler characteristics to the level of spectra, and will be extremely important later in the paper. To define SW-categories, we must first define a few prerequisite categorical structures along the way. It should also be noted that there are other categorical frameworks that deal with the concept of subtraction and scissors congruence —most notably the notion of an assembler category first introduced by Inna Zakharevich in **Zakh** . Assembler categories are more natural to define, but are slightly less easy to map out of (into targets such as Waldhausen categories or stable ∞-categories, for example). For this latter reason, our preferred setting is that of SW-categories. It should also be noted that for our purposes, subtractive categories are really enough, as our "weak equivalences" will merely be isomorphisms.

**Definition 4.2.7.** *A* pre-subtractive category $C$ *is a category equipped with two wide subcategories* $\boldsymbol{cof}(C)$ *and* $\boldsymbol{comp}(C)$ *referred to as* cofibrations *and* complements *(morphisms in* $\boldsymbol{cof}(C)$ *are denoted by* $\hookrightarrow$ *and morphisms in* $\boldsymbol{comp}(C)$ *are denoted by* $\overset{\circ}{\to}$*) and equipped with a subclass* $\boldsymbol{sub}(C)$ *of diagrams of the form* $Z \hookrightarrow X \overset{\circ}{\leftarrow} Y$ *referred to as* subtraction sequences*. These are all required to satisfy the following axioms:*

- *$C$ has an initial object (often referred to via $\emptyset$ in practice)*
- *$C$ has finite coproducts, and for every $X, Y \in C$, one has that $X \to X \coprod Y$ is both a cofibration and a complement*
- *Pullbacks along cofibrations and complements exist and are cofibrations and complements respectively*
- *For every $X, Y \in C$, one has $X \hookrightarrow X \coprod Y \overset{\circ}{\leftarrow} Y \in \boldsymbol{sub}(C)$*
- *Every cofibration $X \hookrightarrow Y$ participates in a subtraction sequence $Z \hookrightarrow X \overset{\circ}{\leftarrow} Y$ which is unique up to unique isomorphism. We denote this unique $Y$ by $X - Z$. The same statement is true in reverse for every complement*
- *Any cartesian square of cofibrations*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\ Y \arrow[r, hook] & W \end{tikzcd}$$

*can be completed into a diagram of the form*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] & X - Z \arrow[l, "\circ"'] \arrow[d, hook] \\ Y \arrow[r, hook] & W & W - Y \arrow[l, "\circ"'] \\ Y - Z \arrow[u, "\circ"] \arrow[r, hook] & W - X \arrow[u, "\circ"] & S \arrow[l, "\circ"'] \arrow[u, "\circ"] \end{tikzcd}$$

*where $S := (W - X) - (Y - Z) \cong (W - Y) - (X - Z)$ in such a way that every row and column will be a subtraction sequence, the bottom row and rightmost column are uniquely determined once choices of the complement are made,* *and the bottom-right square will also be cartesian. The dual statement holds for cartesian squares of complements*

- *Subtraction is stable under pullback, or in other words, given any subtraction sequence* $Z \hookrightarrow X \overset{\circ}{\leftarrow} Y$ *and any morphism* $W \to X$ *in* $C$*, one has that* $Z \times_X W \hookrightarrow W \overset{\circ}{\leftarrow} Y \times_X W$ *is also a subtraction sequence*

We will often merely refer to a pre-subtractive category as $C$ instead of specifying all of the attendant data. The most important examples of pre-subtractive categories for us are actually subtractive categories, as discussed immediately below.

**Definition 4.2.8.** *A* subtractive category *is a pre-subtractive category* $C$ *which additionally satisfies the following axioms:*

- *The pushout of a diagram in which both legs are cofibrations exists and satisfies base-change (the created maps are also cofibrations). Furthermore, cocartesian diagrams of this form are cartesian as well*

- *Given a cartesian diagram of cofibrations*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\ Y \arrow[r, hook] & W \end{tikzcd},$$

  *the natural map* $X \coprod_Z Y \hookrightarrow W$ *must also be a cofibration*

- *Given a diagram of the form*

$$\begin{tikzcd} X' \arrow[d, hook] & W' \arrow[l, hook'] \arrow[r, hook] \arrow[d, hook] & Y' \arrow[d, hook] \\ X & W \arrow[l, hook'] \arrow[r, hook] & Y \\ X'' \arrow[u, "\circ"] & W'' \arrow[l, hook'] \arrow[r, hook] \arrow[u, "\circ"] & Y'' \arrow[u, "\circ"] \end{tikzcd}$$

  *in which all the columns are subtraction sequences and all of the horizontal morphisms are cofibrations, the sequence* $X' \coprod_{W'} Y' \hookrightarrow W \coprod_W Y \overset{\circ}{\leftarrow} X'' \coprod_{W''} Y''$ *is a subtraction sequence*

In particular, it is proven in **Cam** proposition 3.28 that given a base scheme $S$, the categories $\mathbf{Var}_S$ and $\mathbf{Sch}_S$ of $S$-varieties and $S$-schemes, respectively, have the structure of subtractive categories with cofibrations being closed immersions, complements being open immersions, and subtractions given by decomposition of varieties into complementary closed and open subvarieties.

**Definition 4.2.9.** *An* exact functor $F : C \to C'$ *between subtractive categories is a functor which satisfies the following properties:*

- *$F$ preserves zero objects*
- *$F$ preserves subtraction sequences*
- *$F$ preserves cocartesian squares*

*The category of subtractive categories and exact functors is denoted SubCat.*

As one can imagine, the inclusion functor $\mathbf{Var}_S \hookrightarrow \mathbf{Sch}_S$ is in fact an exact functor of subtractve categories.

Now, we have all that is needed for the current paper, but if we happen to want a notion of weak equivalence that "plays nicely" with our other categories, we may enlarge our definition somewhat.

**Definition 4.2.10.** *An* SW-category *is a subtractive category $C$ equipped with an additional distinguished wide subcategory $wC$ of* weak equivalences *(arrows in which are denoted $\xrightarrow{\sim}$) subject to the following conditions:*

- *$wC$ contains all isomorphisms*
- *Given a commutative diagram of the form*

$$\begin{tikzcd} X' \arrow[d, "\sim"] & W' \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\sim"] & Y' \arrow[d, "\sim"] \\ X & W \arrow[l, hook'] \arrow[r, hook] & Y \end{tikzcd}$$

  *in which horizontal arrows are cofibrations and vertical arrows are weak equivalences, one has that the resulting map $X' \coprod_{W'} Y' \xrightarrow{\sim} X \coprod_W Y$ is a weak equivalence*

- *Weak equivalences respect subtraction, or in other words, given any commutative square*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\sim"] & X \arrow[d, "\sim"] \\ W \arrow[r, hook] & Y \end{tikzcd}$$

*we may complete it into a commutative diagram*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\sim"] & X \arrow[d, "\sim"] & X - Z \arrow[l, "\circ"'] \arrow[d, "\sim"] \\ W \arrow[r, hook] & Y & Y - W \arrow[l, "\circ"'] \end{tikzcd}.$$

**Definition 4.2.11.** *A functor $F : C \to C'$ of SW-categories is* exact *if it is exact as a functor of subtractive categories and preserves weak equivalences. The category of SW-categories and weak equivalences will be denoted SW-Cat.*

If it seems thus far that we have been specifying a good deal more data than in the Waldhausen case, you are correct. In truth, cutting and pasting/subtraction data does not play particularly nicely with homotopy, and several strong axioms are needed to ensure that we can say anything at all. Luckily, in spite of the strength of the axioms introduced, the most important examples, namely $\mathbf{Var}_S$ and $\mathbf{Sch}_S$, satisfy these strong axioms, allowing us to define K-theory, among other things.

**Maps from Waldhausen and SW 1-categories into stable $\infty$-categories**

In the current subsection, we will only develop as much of the comparison theory as is needed for this paper. All of this can be developed more generally, as has already been done in the case of Waldhausen $\infty$-categories by Clark Barwick. That said, it ought to be possible to replace the 1-categorical theory of SW-categories with an $\infty$-categorical analogue, although we do not do this here. Indeed, this generality would be unnecessary for our current setting, as the sub-$\infty$-category of derived prestacks generated by algebraic varieties is discrete (or in other words, forms a 1-category). All our 1-categories will be implicitly included into $\infty$-categories (in other words, we are making use of the inclusion of 1-categories into simplicial categories as the simplicial categories with discrete H*om* simplicial sets). Furthermore, for this subsection in particular, we will make the assumption that all of our $\infty$-categories are small as a precaution to prevent swindles and ensure everything is well defined.

Before we begin discussing maps between the different types of $\infty$-categories we will be using, let us say a little bit about how to extract K-theory from each one.

**Definition 4.2.12.** *Define $Ar[n]$ to be the full subcategory of $[n] \times [n]$ consisting of $(i, j) \in [n] \times [n]$ with $i \leq j$ and $\tilde{A}r[n]$ to be the full subcategory of $[n]^{op} \times [n]$ consisting of $(i, j) \in [n]^{op} \times [n]$ with $i \leq j$. The former is used in the construction of Waldhausen's K-theory and the K-theory of pointed $\infty$-categories (specifically,* *stable $\infty$-categories), while the latter is used in the construction of K-theory of SW-categories.*

Waldhausen categories were of course first defined as a general setting for algebraic K-theory. Waldhausen's $S_\bullet$-construction is quite well-known at this point, so we will omit the basic variation here (see, for example, **Wei** or **Wald**). That said, we will make relatively heavy use of a variation known as the $S'_\bullet$-construction which is defined for Waldhausen categories which admit a functorial factorization of any morphism into a cofibration followed by a weak equivalence. We recall briefly that, in such a Waldhausen category, a homotopy cocartesian square is weakly equivalent via a zigzag of equivalences to a pushout square with one leg a cofibration and a weak cofibration is a map equivalent to a cofibration by a zigzag of weak equivalences.

**Definition 4.2.13.** *We begin by defining $S'_nC$ to be the full subcategory of $Fun(Ar[n], C)$ spanned by functors $F : Ar[n] \to C$ such that*

- *$F(i,i)$ is a zero object for all $i$ between $0$ and $n$*
- *$F(i,j) \to F(i,k)$ is a weak cofibration for any $i \le j \le k$*
- *Whenever $i < j < k$, one has that the diagram*

$$\begin{array}{ccc} F(i,j) & \longrightarrow & F(i,k) \\ \downarrow & & \downarrow \\ F(j,j) & \longrightarrow & F(j,k) \end{array}$$

  *is homotopy cocartesian*

*These categories bundle together to form a simplicial object in categories $S'_\bullet C$. This construction is functorial in exact functors.*

Each $S'_nC$ may naturally be given the structure of a Waldhausen category itself, so we may define a multisimplicial category $(S'_\bullet)^nC$ by iterating the construction at every simplicial level $n$ times. We may then define the K-theory spectrum $K(C)$ of this Waldhausen category with $n$-th space $|w(S'_\bullet)^nC|$. We do not use a different notation for this K-theory space, as it coincides with the standard Waldhausen K-theory in the case we described before.

We will also define K-theory for general pointed finitely cocomplete quasicategories. Given any pointed finitely cocomplete quasicategory $\mathcal{A}$, one can define K-theory in the following way.

**Definition 4.2.14.** *We begin by defining $Gap([n], \mathcal{A})$ to be the full subcategory of $Fun(Ar[n], \mathcal{A})$ spanned by functors $F : Ar[n] \to \mathcal{A}$ such that*

- *$F(i, i)$ is a zero object for all $i$ between $0$ and $n$*
- *Whenever $i < j < k$, one has that the diagram*

$$\begin{array}{ccc} F(i,j) & \longrightarrow & F(i,k) \\ \downarrow & & \downarrow \\ F(j,j) & \longrightarrow & F(j,k) \end{array}$$

  *is cocartesian*

Note that $\mathrm{Gap}([n], \mathcal{A})$ is always pointed and finitely cocomplete via **BGT** and when $\mathcal{A}$ is stable, it is as well.

**Definition 4.2.15.** *Given $\mathcal{A}$ pointed and finitely cocomplete (resp. stable), we define the simplicial pointed finitely cocomplete (resp. stable) quasicategory $S^{\infty}_{\bullet}\mathcal{A}$ via $S^{\infty}_{n}\mathcal{A} := Gap([n], \mathcal{A})$ on objects, with face maps given by the composition of the arrows into and out of the objects on the ith row and column and degeneracy maps given by inserting a copy of the identity in position i on rows and columns. In particular, denoting by $\iota$ the largest internal $\infty$-groupoid functor, this also yields a simplicial $\infty$-groupoid $\iota S^{\infty}_{\bullet}\mathcal{A}$.*

Note that we can actually iterate the construction above to give a multisimplicial pointed finitely cocomplete (resp. stable) $\infty$-category $(S^{\infty}_{\bullet})^n\mathcal{A}$ and corresponding multisimplicial $\infty$-groupoid $\iota(S^{\infty}_{\bullet})^n\mathcal{A}$.

**Definition 4.2.16.** *Given a pointed finitely cocomplete (resp. stable) $\infty$-category $\mathcal{A}$, we define its* K-theory spectrum $K(\mathcal{A})$ *as (the fibrant-cofibrant replacement of what) follows. For every n, the nth space in the spectrum is $|\iota(S^{\infty}_{\bullet})^n\mathcal{A}|$. Noting that $Gap([0], \mathcal{A})$ is contractible and that $Gap([1], \mathcal{A})$ is naturally equivalent to $\mathcal{A}$, we obtain a natural map $S^1 \wedge |\iota\mathcal{A}| \to |S^{\infty}_{\bullet}\mathcal{A}|$, which induces all of our connecting maps.*

*This construction is natural in exact functors.*

When restricted to small stable $\infty$-categories equipped with the Lurie tensor product, the above construction defines a symmetric monoidal functor to the category of spectra. In particular, this implies given a symmetric monoidal small $\infty$-category $\mathcal{A}$, that the K-theory spectrum $K(\mathcal{A})$ is a $\mathbb{E}_\infty$-ring spectrum.

We mostly introduce the notion of K-theory of quasicategories as a "standard" K-theory of $\infty$-categories to compare against, simply because this K-theory has had many nice properties proved about it already. Before this, we introduce the K-theory of SW-categories and describe how to compare it to the K-theory of $\infty$-categories (incarnated as $S$-categories), which we also introduce.

**Definition 4.2.17.** *Given any pointed finitely cocomplete $\infty$-category $\mathcal{A}$ (considered as a Kan-enriched category), one may consider its category of pointed simplicial presheaves $\mathcal{P}(\mathcal{A})$ equipped with the projective model structure. From here, one can left Bousfeld localize this category to form the simplicial model category $\mathcal{P}_{ex}(\mathcal{A})$ whose local objects are those functors $\mathcal{A}^{op}$ to simplicial sets which commute with finite (homotopy) colimits. Finally, we can construct a Waldhausen subcategory $\mathcal{M}(\mathcal{A})$ of $\mathcal{P}_{ex}(\mathcal{A})$ which consists of those objects which are cofibrant and weakly equivalent to representable presheaves. From here, we may define* the K-theory spectrum of $\mathcal{A}$ *to be the the Waldhausen K-theory $K(\mathcal{A}) := K(\mathcal{M}(\mathcal{A}))$.*

It is this definition of K-theory of $\infty$-categories that we will compare with the K-theory of SW-categories. Note that if one has a morphism $\mathcal{A} \to \mathcal{B}$ of pointed, finitely homotopy cocomplete $\infty$-categories which commutes with the relevant structures (a so-called weakly exact functor), one obtains an exact morphism $\mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ of Waldhausen categories by restricting the left Kan extension of our original morphism. In this way, we obtain a functor from pointed finitely homotopy cocomplete $\infty$-categories and weakly exact functors to (strongly saturated) Waldhausen categories and exact functors. This, in turn, may be composed with K-theory, which shows that the above definition of K-theory is functorial (**Cis** page 544).

This will allow us an easier way of defining SW-exact functors to $\infty$-categories without actually needing to enlarge our definition all that much.

Before we continue, let us begin by comparing the above two constructions on nice $\infty$-categories to ensure they coincide.

**Theorem 4.2.18.** *Given a simplicial model category $C$ and a small full subcategory $\mathcal{A} \subseteq C$ whose objects are all cofibrant, which admits all homotopy colimits, and whose underlying category inherits the structure of a Waldhausen category from* *the model structure on $C$, one has equivalences*

$$K(\mathcal{A}) \simeq K(\mathcal{M}(\mathcal{A})) \simeq K(N^{hc}(\mathcal{A}^{cf})),$$

*where $\mathcal{A}^{cf}$ is the full subcategory of $\mathcal{A}$ generated by objects which are both cofibrant and fibrant in $C$.*

*Proof.* This combines **BGT** Theorems 7.8 and 7.11 and Corollary 7.12. □

In particular, this will be true for all of our categories of interest, as any stable $S$-category embeds into a stable simplicial model category (see **ToeVez** page 789 paragraph 1).

**Definition 4.2.19.** *Suppose that $C$ is an SW-category. In analogy with the Waldhausen case, we may define a simplicial object in categories $\tilde{S}_\bullet C$ as follows. We start by defining it as a simplicial set. For each n, define $\tilde{S}_n C$ to the full subcategory of Fun($\tilde{A}r[n], C$) on the functors $F : \tilde{A}r[n] \to C$ satisfying the following conditions:*

- *$F(i,i)$ is initial for any $i$ between $0$ and $n$*
- *Whenever $j < k$, one has $F(i,j) \to F(i,k)$ is in **cof**$(C)$*
- *Whenever $i < j < k$ one has that*

  $$F(i,j) \to F(i,k) \leftarrow F(j,k)$$

  *is a subtraction sequence.*

*The face maps $d_k : \tilde{S}_n C \to \tilde{S}_{n-1} C$ are given by deleting the kth row and column of the requisite functors from $\tilde{A}r[n]$, and composing the appropriate morphisms to yield a functor from $\tilde{A}r[n-1]$ into $C$. The degeneracy maps are given by inserting identity maps into the ith row and column. Furthermore, each $\tilde{S}_n C$ naturally has the structure of an SW-category (we can check whether a morphism of diagrams is a cofibration, complement, or weak equivalence object-wise), so we can iterate this construction to an n-ary simplicial structure $\tilde{S}^n_\bullet C$.*

**Definition 4.2.20.** *Given an SW-category $C$, one can define its* K-theory spectrum *$K(C)$ as (the fibrant-cofibrant repacement of what) follows. Define the nth space of the spectrum to be $|w\tilde{S}^n_\bullet C|$. By* **Cam** *corollary 4.17, one gets a natural map $|wC| \to \Omega|w\tilde{S}_\bullet C|$ which, while not an equivalence, becomes one on higher levels, so we actually have a spectrum.*

As an example, $|w\tilde{S}_\bullet C| \xrightarrow{\sim} \Omega|w\tilde{S}_\bullet\tilde{S}_\bullet C|$. The spectrum $K(C)$ also has the natural structure of a $\mathbb{E}_\infty$-ring spectrum whenever $C$ is equipped with a symmetric monoidal structure compatible with subtraction (**Cam** theorem 5.14). This construction defines a functor from SW-categories equipped with exact functors of SW-categories to spectra.

**Definition 4.2.21.** *Let $C$ be an SW-category and $\mathcal{A}$ be a pointed homotopy cocomplete $\infty$-category. A* weakly W-exact functor $F : C \to \mathcal{A}$ *actually consists of a triple $(F_!, F^!, F_w)$ of functors such that*

- *$F_!$ is a functor $F_! : \boldsymbol{cof}(C) \to \mathcal{A}$. We abbreviate $F_!(i)$ to $i_!$ for all cofibrations $i$*
- *$F^!$ is a functor $F^! : \boldsymbol{comp}(C)^{op} \to \mathcal{A}$. We abbreviate $F^!(j)$ to $j^!$ for all complement maps $j$*
- *$F_w$ is a functor $F_w : wC \to \iota(\mathcal{A})$. We abbreviate $F_w(f)$ to $f_w$ for all weak equivalences*
- *For all objects $X \in C$, one has $F_!(X) = F^!(X) = F_w(X) =: F(X)$*
- *Given a cartesian square in $C$*

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "i"] & Z \arrow[d, "i'"] \\ Y \arrow[r, hook, "j'"] & W \end{tikzcd}$$

  *with horizontal morphisms cofibrations and vertical morphisms complement maps, one gets a commutative diagram*

$$\begin{tikzcd} F(X) \arrow[r, "j_!"] & F(Z) \\ F(Y) \arrow[u, "i^!"] \arrow[r, "j'_!"] & F(W) \arrow[u, "i'^!"] \end{tikzcd}$$

- *For all subtraction sequences $Z \overset{i}{\hookrightarrow} X \overset{j}{\underset{\circ}{\leftarrow}} U$, one gets a homotopy cocartesian square*

$$\begin{tikzcd} F(Z) \arrow[r, "i_!"] \arrow[d] & F(X) \arrow[d, "j^!"] \\ 0 \arrow[r] & F(U) \end{tikzcd}$$

  *in $\mathcal{A}$*

- *For all commutative squares*

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "g"] & Z \arrow[d, "g'"] \\ Y \arrow[r, hook, "f'"] & W \end{tikzcd}$$

  *with vertical morphisms weak equivalences and horizontal morphisms cofibrations, one gets a commutative square*

$$\begin{tikzcd} F(X) \arrow[r, "f_!"] \arrow[d, "g_w"] & F(Z) \arrow[d, "g'_w"] \\ F(Y) \arrow[r, "f'_!"] & F(W) \end{tikzcd}$$

  *in $\mathcal{A}$. One gets an analogous diagram if one replaces cofibrations with complements*

Our reason for introducing such a map is that such a map induces a contractible space of maps of K-theory spectra.

**Remark 4.2.22.** *Note that this is precisely the same as the definition of a weakly W-exact functor where the target is a Waldhausen category (***CWZ** *definition 2.17). If $F : C \to \mathcal{A}$ is a weakly W-exact functor with target a Waldhausen category, it induces a map $K(\mathcal{F}) : K(C) \to K(\mathcal{A})$ (***CWZ** *proposition 2.19). We make use of this in the following theorem.*

**Theorem 4.2.23.** *Given a weakly W-exact functor $F : C \to \mathcal{A}$, it may be composed with the Yoneda embedding $\mathcal{A} \to \mathcal{M}(\mathcal{A})$ to obtain a weakly W-exact functor $\iota \circ F : C \to \mathcal{M}(\mathcal{A})$ to the good Waldhausen $\mathcal{M}(\mathcal{A})$ which yields a map on K-theory $K(C) \to K(\mathcal{A})$. This construction is functorial in exact functors both of SW-categories and of pointed finitely homotopy cocomplete $\infty$-categories.*

*Proof.* Let us begin by noting that $\mathcal{M}(\mathcal{A})$ has a natural enrichment as a simplicial category which makes the restriction of the Yoneda embedding $\iota : \mathcal{A} \to \mathcal{M}(\mathcal{A})$ into a functor of simplicial categories (and indeed one which preserves finite homotopy colimits by construction). As such, the corresponding map $\iota \circ F$ remains weakly W-exact. Furthermore, by construction, it must descend to a weakly W-exact map on the level of the underlying Waldhausen category $\mathcal{M}(\mathcal{A})$. Thus, one has a natural map on K-theory $K(C) \to K(\mathcal{A})$ induced from $\iota \circ F$. Furthermore, one has that functoriality in weakly W-exact functors and weakly exact functors arises from the functoriality of the corresponding constructions on their respective K-theories. □

## 4.3 The Six Functors Formalism and Derived Motivic Measures

### A Brief Description of the Generalized Six Functors Formalism

In this section, we will discuss some of the basic aspects of the $\infty$-categorical theory of six functors formalisms described by Adeel Khan in **Khan** . We will not go into detail regarding the proofs, and will only supply the details necessary for the sections that follow. It should be noted that none of this section is novel. Any interested reader is urged to peruse the stellar paper by Khan on the subject. Before we begin, we will detail a few conventions.

For the rest of the section $\mathcal{S}$ will refer to a category of classical Noetherian schemes over some fixed Noetherian base. While Khan's original formalism works more generally for derived algebraic spaces, for the moment, we are only concerned with a more limited setup. Note that these stronger conditions imply, among other things, that $\mathcal{S}$ is actually a 1-category (a discrete $\infty$-category). Furthermore, we will assume that $\mathcal{S}$ is such that for all $S \in \mathcal{S}$, one has

- $U \in \mathcal{S}$ for every quasicompact open subspace $U \subseteq S$
- $Z \in \mathcal{S}$ for every closed subspace $Z \subseteq S$
- $\mathbb{P}(\mathcal{E}) \in \mathcal{S}$ for every finite locally free sheaf $\mathcal{E}$ on $S$

We further fix a class of so-called *admissible* morphisms in $\mathcal{S}$, which we mandate to contain all open immersions and all projections $X \times \mathbb{P}^n \to X$, to be closed under composition and base change, and to satisfy 2-out-of-3. We denote by $\mathcal{A} \subseteq \mathcal{S}$ the subcategory of admissible morphisms.

If we consider a presheaf of $\infty$-categories $\mathbb{D}^*$ on $\mathcal{S}$, we will use the notation $\mathbb{D}(S) := \mathbb{D}^*(S)$ for every $S \in \mathcal{S}$. Furthermore, for every morphism $f : X \to Y \in \mathcal{S}$, we will denote by

$$f^* : \mathbb{D}(Y) \to \mathbb{D}(X)$$

the inverse image $\mathbb{D}^*(f)$ of $f$.

If, furthermore, $\mathbb{D}^*$ takes values in presentable $\infty$-categories and colimit-preserving functors, we refer to $\mathbb{D}$ as a *presheaf of presentable $\infty$-categories*. Note that this, in particular, means that $f^*$ must admit a right adjoint, which we denote by $f_*$ and call the direct image of $f$.

Finally, if $\mathbb{D}^*$ further factors through symmetric monoidal presentable $\infty$-categories (presentable symmetric monoidal $\infty$-categories for which the tensor product commutes with colimits in each variable), we refer to $\mathbb{D}^*$ as a *presheaf of symmetric monoidal presentable $\infty$-categories*. Given any $S \in \mathcal{S}$, we will let $\otimes : \mathbb{D}(S)\times\mathbb{D}(S) \to \mathbb{D}(S)$ denote the monoidal product and $\mathbf{1}_S$ denote the monoidal unit. Now, since $\otimes$ commutes with colimits in each variable, it admits a right-adjoint internal hom bifunctor $\underline{Hom} : \mathbb{D}(S)^{op} \times \mathbb{D}(S) \to \mathbb{D}(S)$. From now on, when we talk about this last scenario, we will simply omit the upper $*$ from the notation unless we want to make it clear that we are referring to $\mathbb{D}$ as presheaf with respect to $*$ morphisms.

**Definition 4.3.1.** *A* premotivic $\infty$-category *or* $(*, \#, \otimes)$-formalism *on* $(\mathcal{S}, \mathcal{A})$ *is a presheaf of symmetric monoidal presentable $\infty$-categories $\mathbb{D}$ on $\mathcal{S}$ which satisfies the following additional properties*

- *For every morphism $f : T \to S$ in $\mathcal{A}$, the inverse image functor admits a left-adjoint $f_\# : \mathbb{D}(T) \to \mathbb{D}(S)$ which we call the* #-direct image
- *$f_\# : \mathbb{D}(T) \to \mathbb{D}(S)$ is a morphism of $\mathbb{D}(S)$-modules, where we note that $\mathbb{D}(T)$ has a natural $\mathbb{D}(S)$-module structure via the symmetric monoidal functor $f^*$. In other words, $\mathbb{D}$ satisfies the* projection formula *with respect to #-direct images*
- *$\mathbb{D}$ satisfies* admissible base change *for #-direct image. In other words, given any cartesian diagram of the form*

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & S' \arrow[d, "p"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

*with p and q admissible, then there is a natural equivalence of functors*

$$Ex^*_\# : q_\# g^* \xrightarrow{\sim} f^* p_\#$$

- *Given any finite family $S_\alpha$ in $\mathcal{S}$, the induced functor*

$$\mathbb{D}(\amalg_\alpha S_\alpha) \to \Pi_\alpha \mathbb{D}(S_\alpha)$$

*from inclusion is an equivalence. In other words, $\mathbb{D}$ satisfies additivity*

*Generally speaking, when talking about premotivic $\infty$-categories on $\mathcal{S}$, if admissible morphisms are not specified as part of the data, then we are making the assumption that the admissible morphisms are simply the smooth morphisms in $\mathcal{S}$ (or $\mathcal{A} = Sm$).*

**Remark 4.3.2.** *Note that a $(*, \#, \otimes)$-formalism is the same as a premotivic $\infty$-category in the sense of Cisinski-Déglise (hence the conflation of the two terms above), so we may directly import the notions of premotivic morphism and premotivic adjunction to this setting (indeed, we note that premotivic stable-symmetric monoidal model categories are a direct realization of our current situation in more classical language when specified to the stable case).*

We will delay discussion of examples until the following section.

**Definition 4.3.3.** *Let $\mathbb{D}$ be a $(*, \#, \otimes)$-formalism on $(\mathcal{S}, \mathcal{A})$, and take $S \in \mathcal{S}$ and a finite locally free sheaf $\mathcal{E}$ on $S$. Denote the total space $\mathbb{V}_S(\mathcal{E})$ by $E$, and let $p : E \to S$ be the projection and $s : S \to E$ be the zero section. Supposing that $p$ is admissible, define the* Thom twist $\langle \mathcal{E} \rangle$ *to be the endofunctor on $\mathbb{D}(S)$ given by*

$$\mathcal{F} \mapsto \mathcal{F}\langle \mathcal{E} \rangle := p_\# s_*(\mathcal{F}).$$

**Definition 4.3.4.** *A $(*, \#, \otimes)$-formalism on $(\mathcal{S}, \mathcal{A})$* satisfies the Voevodsky conditions *if it satisfies the following conditions*

- Homotopy invariance: *for every $S \in \mathcal{S}$ and every vector bundle $p : E \to S$ on $S$, the unit map*
$$id \to p_* p^*$$
*is an equivalence*

- Localization: *for every decomposition of a variety $X$ into a closed subvariety and a complementary open*
$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$
*in $\mathcal{S}$, $i_*$ is a fully faithful functor with essential image spanned by objects in the kernel of $j^*$*

- Thom stability: *for every $S \in \mathcal{S}$ and every locally free sheaf $\mathcal{E}$ on $S$, the Thom twist endofunctor $\langle \mathcal{E} \rangle$ is an equivalence on $\mathbb{D}(S)$*

*We will also use the term* motivic category over $(\mathcal{S}, \mathcal{A})$ *to describe a $(*, \#, \otimes)$-formalism over $(\mathcal{S}, \mathcal{A})$ which satisfies the Voevodsky conditions, since the Voevodsky conditions are equivalent (in the triangulated case) to the conditions of Cisinski-Déglise under which a premotivic category defines a motivic category.*

**Remark 4.3.5.** *If $\mathbb{D}$ is a $(*, \#, \otimes)$-formalism on $\mathcal{S}$ satisfying the Voevodsky conditions, then the $\infty$-categories $\mathbb{D}$ are stable.*

Note that the features we have already discussed automatically imply some of the most characteristic features of the notion of a six functors formalism, namely, the exceptional operations.

**Theorem 4.3.6.** *Given any finite type morphism $f : \mathbb{D}(X) \to \mathbb{D}(Y)$, there exists an adjunction*

$$(f_! \dashv f^!) : \mathbb{D}(X) \rightleftarrows \mathbb{D}(Y)$$

*and a natural transformation $\alpha_f : f_! \to f_*$ satisfying the following conditions:*

- *There are canonical equivalences $f_! \cong f_\#$ and $f^! \cong f^*$ if $f$ is an open immersion*
- *$\alpha_f$ is an equivalence if $f$ is a proper morphism*
- *The functor $f_!$ satisfies base change, or in other words, given a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

  *in $\mathcal{S}$, the natural transformations*

$$Ex^*_! : v^* f_! \to g_! u^* \text{ and } Ex^!_* : u_* g^! \to f^! v_*$$

  *are equivalences*

- *The functor $f_!$ satisfies the projection formula. In other words, $f_!$ is a morphism of $\mathbb{D}(Y)$-modules, where $\mathbb{D}(X)$ is regarded as a $\mathbb{D}(Y)$ module via the symmetric monoidal functor $f^*$. Furthermore, the canonical morphisms*

$$\mathcal{F} \otimes f_!(\mathcal{G}) \to f_!(f^*(\mathcal{F}) \otimes \mathcal{G}),$$
$$\underline{Hom}(f^*(\mathcal{F}), f^!(\mathcal{F}')) \to f^!(\underline{Hom}(\mathcal{F}, \mathcal{F}')),$$
$$f_*(\underline{Hom}(\mathcal{F}, f^!(\mathcal{G}))) \to \underline{Hom}(f_!(\mathcal{F}), \mathcal{G})$$

  *are equivalences for all $\mathcal{F}, \mathcal{F}' \in \mathbb{D}(X)$ and $\mathcal{G} \in \mathbb{D}(Y)$*

*Proof.* This is **Khan** theorem 2.34. ◻

In fact, if $f$ is étale, then the natural map $f^! \to f^*$ is an equivalence.

Before we continue, at this point we should introduce a bit of notation. Suppose that $f : X \to Y$ is a morphism of finite type in $\mathcal{S}$. By the above, we have adjunctions $f^* \dashv f_*$ and $f_! \dashv f^!$. We will use the notation

- $\eta_f^* : \mathrm{i}d \to f_* f^*$ and $\epsilon_f^* : f^* f_* \to \mathrm{i}d$ for the unit and counit of the first adjunction
- $\eta_f^! : \mathrm{i}d \to f^! f_!$ and $\epsilon_f^! : f_! f^! \to \mathrm{i}d$ for the unit and counit of the second adjunction

If, in addition, $f$ happens to be smooth, we have a third adjunction $f_\# \dashv f^*$, and we will denote its unit and counit by

$$\eta_f^\# : \mathrm{i}d \to f^* f_\# \text{ and } \epsilon_f^\# : f_\# f^* \to \mathrm{i}d$$

if needed.

**Definition 4.3.7.** *A premotivic category $\mathbb{D}$ on $(\mathcal{S}, \mathcal{A})$ is* compactly generated *if*

- *For every $S \in \mathcal{S}$, the $\infty$-category $\mathbb{D}(S)$ is compactly generated*
- *For every morphism $f : T \to S$ in $\mathcal{S}$, the inverse image functor $f^* : \mathbb{D}(S) \to \mathbb{D}(T)$ is a contact functor (preserves compact objects)*

**Definition 4.3.8.** *Given a premotivic category $\mathbb{D}$ over $(\mathcal{S}, \mathcal{A})$, we refer to $\mathbb{D}$ as* continuous *if for every cofiltered system of affine schemes $(S_\alpha)_\alpha$ in $\mathcal{S}$ with limit $S$, the canonical functor*

$$\varprojlim_{\alpha} \mathbb{D}(S_\alpha) \to \mathbb{D}(S)$$

*is an equivalence.*

In practice, essentially all of the (pre)motivic categories we encounter will be compactly generated, and many will be continuous as well. There will be more on compact (and in particular constructible) generation towards the end of the section.

**Theorem 4.3.9.** *Suppose that $\mathbb{D}$ is a motivic category over $(\mathcal{S}, \mathcal{A})$ and that one has a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner"] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

*in $\mathcal{S}$. Then we have the following different forms of base change in addition to those discussed before (presented in their most general forms):*

- Proper base change: *If $f$ is a proper morphism, then there is a canonical equivalence*
$$Ex^*_* : p^* f_* \to g_* q^*$$
*of functors $\mathbb{D}(X) \to \mathbb{D}(Y')$*

- Smooth-proper base change: *If $f$ is a proper morphism and $p$ and $q$ are smooth, then there is a canonical equivalence*
$$Ex_{\#*} : p_\# g_* \to f_* q_\#$$
*of functors $\mathbb{D}(X') \to \mathbb{D}(Y)$*

- Finite type-smooth base change: *If $f$ is of finite type and $p$ and $q$ are smooth, then there is a natural equivalence*
$$Ex^{*!} : q^* f^! \to g^! p^*$$
*of functors $\mathbb{D}(Y) \to \mathbb{D}(X')$*

- Finite type-proper base change: *If $f$ is of finite type and $p$ is proper, then there is a natural equivalence*
$$Ex_{!*} : f_! q_* \to p_* g_!$$
*of functors $\mathbb{D}(X') \to \mathbb{D}(Y)$*

*Proof.* This consists of various statements in **Khan** theorem 2.24, corollary 2.37, and corollary 2.39. □

Really the crux of many of the proofs in the following section will be the various forms of base change we have discussed.

**Remark 4.3.10.** *In what follows, noting that Tate twists generally commute with all of the six operations, we abuse notation by doing things such as writing $f\langle\mathcal{E}\rangle$ instead of $\langle f^*\mathcal{E}\rangle \circ f^*$, etc.*

**Theorem 4.3.11.** *Consider $S \in \mathcal{S}$ and two smooth $S$-schemes $p : X \to S$ and $q : Y \to S$. Then one has the following two results:*

- Relative purity: *If $X$ and $Y$ are connected via a closed immersion $i : X \hookrightarrow Y$ over $S$, then there is a canonical isomorphism*

$$q_{\#}i_{*} \simeq p_{\#}\langle \mathcal{N}_{X/Y} \rangle,$$

*with $\mathcal{N}_{X/Y}$ the conormal sheaf of $i$*

- *If $f : X \to Y$ is an unrammified morphism over $S$, then there is a canonical isomorphism*

$$f^! q^* \simeq p^* \langle \mathcal{L}_{X/Y} \rangle$$

*where $\mathcal{L}_{X/Y}$ is the relative cotangent complex of $f$*

*Proof.* This is **Khan** theorem 2.25 and 2.43. □

In particular, these can be used to conclude two important results.

**Theorem 4.3.12.** Atiyah duality: *If $f : X \to Y$ in $\mathcal{S}$ is smooth and proper, then one has a canonical morphism of functors*

$$\epsilon_f : f_{\#}\langle \mathcal{L}_f \rangle \to f_*,$$

*where $\mathcal{L}_f$ is the cotangent complex of $f$.*

*Proof.* This is **Khan** theorem 2.24 (iii). □

This theorem is very interesting, as it implies, among other things, that the left and right adjoints of $f^*$ are related by a Thom twist when $f$ is smooth and proper.

**Theorem 4.3.13.** Purity: *If $\mathbb{D}$ is a motivic category over $\mathcal{S}$, then for any smooth morphism $f : X \to Y$, one has a canonical equivalence*

$$pur_f : f^! \to f^*\langle \mathcal{L}_f \rangle$$

*of functors $\mathbb{D}(Y) \to \mathbb{D}(X)$, thus generalizing our previous result on étale morphisms from before.*

*Proof.* This is **Khan** theorem 2.44. □

We will also make very heavy use of the following consequence of localization.

**Proposition 4.3.14.** *Let $\mathbb{D}$ be a premotivic category on $\mathcal{S}$, and suppose that one has a closed immersion $i$ with open complement $j$ of the form*

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U.$$

*Then the functors $j_*$ and $j_\#$ are fully faithful and furthermore one has $j^* i_* \simeq 0$ and $i^* j_\# \simeq 0$. If, furthermore $\mathbb{D}$ satisfies the localization property, then one has the canonical cofiber sequences*

$$j_\# j^* \to id \to i_* i^* \text{ and } i_* i^! \to id \to j_* j^*$$

*(there is a way to define $i^!$ in this case without assuming that $\mathbb{D}$ is motivic, simply as the homotopy fiber of $i^* \to i^* j_* j^*$). If, in fact, $\mathbb{D}$ happens to be motivic, then the above cofiber sequences are equivalent to*

$$j_! j^! \to id \to i_* i^* \text{ and } i_! i^! \to id \to j_* j^*.$$

*Proof.* This is **Khan** remark 2.9 □

We now come to a very important definition. The subcategory of constructible objects will be what allows us to extract meaningful K-theory from our six functors formalism and not fall prey to swindles.

**Definition 4.3.15.** *Consider a motivic category $\mathbb{D}$ over $\mathcal{S}$ and a scheme $S \in \mathcal{S}$. For any $\mathcal{F} \in \mathbb{D}(S)$, we say that $\mathcal{F}$ is* constructible *if it satisfies one of the following equivalent conditions:*

- *$\mathcal{F}$ lies in the thick subcategory generated by $f_\# f^*(\mathbf{1}_S)\langle -n \rangle \simeq f_! f^!(\mathbf{1}_S)\langle -n \rangle$ with $f : X \to S$ a smooth morphism of finite presentation and $n\mathbb{Z}_{\geq 0}$*
- *$\mathcal{F}$ lies in the thick subcategory generated by $f_\# f^*(\mathbf{1}_S)\langle -n \rangle \simeq f_! f^!(\mathbf{1}_S)\langle -n \rangle$ with $f : X \to S$ a smooth morphism of finite presentation with $X$ affine and $n\mathbb{Z}_{\geq 0}$*

*The* subcategory of constructible objects over *$S$ is denoted by $\mathbb{D}_{cons}(S) \subseteq \mathbb{D}(S)$.*

Since it is this subcategory of constructible objects that we will need to extract meaningful K-theory in what follows, we will list here several of its important properties.

**Definition 4.3.16.** *We say that a motivic category $\mathbb{D}$ is constructibly generated if it is compactly generated and every constructible object is compact. It then follows that the compactness is equivalent to constructibility (***Khan** *definition 2.5.5).*

**Proposition 4.3.17.** *The property of constructibility in $\mathbb{D}$ is stable under*

- *Tensor product with any constructible object*
- *Inverse image along any morphism in $\mathcal{S}$*
- *#-direct image along any finitely presented smooth morphism in $\mathcal{S}$*
- *Thom twist by a perfect complex*
- *Exceptional direct image along any finite type morphism in $\mathcal{S}$*

*Proof.* The first three statements comprise **Khan** proposition 2.57, the third comprises **Khan** proposition 2.60, and the fourth comprises **Khan** theorem 2.61. □

This implies that $\mathbb{D}^*_{\text{cons}}$ is a presheaf of symmetric monoidal essentially small $\infty$-categories on $\mathcal{S}$.

**Corollary 4.3.18.** *If $i : Z \hookrightarrow X$ is a closed immersion, the property of constructibility in $\mathbb{D}(X)$ is stable under the endofunctor $i_*i^*$.*

*Proof.* This is **Khan** corollary 2.58. □

**Some Examples of Motivic $\infty$-Categories**

Perhaps the canonical example of a motivic $\infty$-category over $\mathcal{S}$ is the stable motivic homotopy category **SH**. We will go through the definition here. We will fix a subcategory $\mathcal{S}$ of schemes over some base $B$ and an admissible subcategory $\mathcal{A}$ of $\mathbf{Sch}_B$ for the rest of this section. We require that $\mathcal{A}$ satisfy the following properties:

- $\mathcal{A}$ is an essentially small full subcategory of $\mathbf{Sch}_B$
- For every $S \in \mathcal{S}$, one has that the over category $\mathcal{A}_{/S}$ contains $S$
- If $X \in \mathcal{A}_{/S}$ and $Y$ is étale and of finite presentation over $S$, then $Y \in \mathcal{A}_{/S}$
- If $X \in \mathcal{A}_{/S}$, then $X \times \mathbb{A}^1 \in \mathcal{A}_{/S}$
- If $X, Y \in \mathcal{A}_{/S}$, then $X \times_S Y \in \mathcal{A}_{/S}$

Note that we do not necessarily require $\mathcal{A} \subseteq \mathcal{S}$ in this section (in fact, our typical choice for $\mathcal{A}$ will merely be $\mathbf{Sm}_B$).

We construct the stable motivic homotopy category in several steps. The first order of business is to construct the *unstable motivic homotopy category* $\mathbf{H}(\mathcal{A}_{/S})$ for any $S \in \mathcal{S}$.

**Definition 4.3.19.** *Given any $S \in \mathcal{S}$, an $\mathcal{A}$-*fibered space over *$S$ is a presheaf on $\mathcal{A}_{/S}$ with values in a suitable category of spaces $\mathbf{S}$, such as Kan complexes. We denote the $\infty$-category of such presheaves as $\boldsymbol{PrSh_S}(\mathcal{A}_{/S})$.*

- *An element $\mathcal{F} \in \boldsymbol{PrSh_S}(\mathcal{A}_{/S})$* satisfies Nisnevich descent *if it satisfies Čech descent with respect to the restriction of the Nisnevich topology restricted to $\mathcal{A}_{/S}$. In other words, for all $X \in \mathcal{A}_{/S}$ (suppressing structure morphism) and all Nisnevich coverings $\{U_\alpha \to X\}_\alpha$, the diagram*

$$\mathcal{F}(X) \to \prod_{\alpha} \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\beta,\gamma} \mathcal{F}(U_\beta \times_X U_\gamma) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots$$

  *exibits $\mathcal{F}(X)$ as its homotopy limit*

- *An element $\mathcal{F} \in \boldsymbol{PrSh_S}(\mathcal{A}_{/S})$* is $\mathbb{A}^1$-homotopy invariant *if for any $X \in \mathcal{A}_{/S}$, the natural map*

$$\mathcal{F}(X) \to \mathcal{F}(X \times_S \mathbb{A}^1)$$

  *is an equivalence*

*An element $\mathcal{F} \in \boldsymbol{PrSh_S}(\mathcal{A}_{/S})$ is referred to as* motivic *if it satisfies Nisnevich descent and $\mathbb{A}^1$-homotopy invariance. The category of motivic $\mathcal{A}$-fibered spaces is referred to as the* unstable motivic homotopy category over S with respect to *$\mathcal{A}$ and is denoted $\boldsymbol{H}(\mathcal{A}_S)$. It may equally be seen as the successive left Bousfeld localization of $\boldsymbol{PrSh_S}(\mathcal{A}_{/S})$ at the classes of morphisms $\mathcal{F}(X) \to Tot(\{U_\alpha\}, \mathcal{F})$ for any $\mathcal{F} \in \boldsymbol{PrSh_S}(\mathcal{A}_{/S})$, any $X \in \mathcal{A}_{/S}$, and any Nisnevich cover $\{U_\alpha \to X\}_\alpha$ (here $Tot(\{U_\alpha\}, \mathcal{F})$ is the totalization of the Čech complex of $\mathcal{F}$ at our Nisnevich cover) and $\mathcal{F}(X) \to \mathcal{F}(X \times_S \mathbb{A}^1)$ for any $\mathcal{F} \in \boldsymbol{PrSh_S}(\mathcal{A}_{/S})$ and any $X \in \mathcal{A}_{/S}$. This classification is often more useful when working with $\boldsymbol{H}(\mathcal{A}_{/S})$.*

We will let $\mathbf{H}(S)$ denote the above category when $\mathcal{A}_{/S} = \mathbf{Sm}_S$, or in other words $\mathcal{A} = \mathbf{Sm}_B$ for our base scheme $B$.

It should be noted that $\mathbf{H}(\mathcal{A}_{/S})$ is symmetric monoidal via the cartesian product.

Now, we are almost at our desired definition. In particular, we let $\mathbf{H}_\bullet(\mathcal{A}_{/S})$ refer to the pointed objects in $\mathbf{H}(\mathcal{A}_{/S})$. The corresponding symmetric monoidal structure on $\mathbf{H}_\bullet(\mathcal{A}_{/S})$ is induced by $\wedge$. The forgetful functor $\mathbf{H}_\bullet(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{A}_{/S})$ has a symmetric monoidal left adjoint given by taking a free disjoint baseboint $\mathcal{F} \mapsto \mathcal{F}_+$.

**Definition 4.3.20.** *Given a vector bundle $\mathcal{E}$ on $S$ with total space $E \to S$, we define the* Thom space $Th_S(\mathcal{E})$ *to be the cofiber of the inclusion $E \setminus S \hookrightarrow E$. Note that this is naturally an element of $\boldsymbol{H}_\bullet(\mathcal{A}_{/S})$ and that it must be compact.*

Denoting $\mathbb{T}_S := \mathrm{Th}_S(\mathcal{O}_S) = \mathbb{A}^1/(\mathbb{A}^1 - S)$, we obtain suspension endofunctor

$$\Sigma_{\mathbb{T}} := \mathbb{T}_S \wedge (-) : \mathbf{H}_\bullet(\mathcal{A}_{/S}) \to \mathbf{H}_\bullet(\mathcal{A}_{/S}).$$

This has a right adjoint loop space functor

$$\Omega_{\mathbb{T}} : \mathbf{H}_\bullet(\mathcal{A}_{/S}) \to \mathbf{H}_\bullet(\mathcal{A}_{/S}).$$

Now, we are ready to introduce our main definition.

**Definition 4.3.21.** *Given $S \in \mathcal{S}$, we may define the* motivic stable homotopy category over $S$ relative to $\mathcal{A}$, *denoted $\boldsymbol{SH}(\mathcal{A}_{/S})$, to be the cofiltered homotopy colimit of*

$$\boldsymbol{H}_\bullet(\mathcal{A}_{/S}) \stackrel{\Sigma_{\mathbb{T}}}{\to} \boldsymbol{H}_\bullet(\mathcal{A}_{/S}) \stackrel{\Sigma_{\mathbb{T}}}{\to} \cdots$$

*taken in the $(\infty, 2)$-category of presentable $\infty$-categories with left-adjoint functors or equivalently as the filtered homotopy colimit of*

$$\cdots \stackrel{\Omega_{\mathbb{T}}}{\to} \boldsymbol{H}_\bullet(\mathcal{A}_{/S}) \stackrel{\Omega_{\mathbb{T}}}{\to} \boldsymbol{H}_\bullet(\mathcal{A}_{/S})$$

*in either the $(\infty, 2)$-category of presentable $\infty$-categories with right adjoint functors, or just in the $(\infty, 2)$-category of $\infty$-categories. An element of $\boldsymbol{SH}(\mathcal{A}_{/S})$ will be referred to as an* $\mathcal{A}$-fibered spectrum over $S$.

This motivic stable homotopy category, originally defined by Voevodsky and Morel, has many nice properties, among them that it satisfies a six functors formalism. When $\mathcal{A}$ is not specified, as above, we assume that we are dealing with the subcategory of all smooth morphisms in $\mathcal{S}$, and merely use the notation $\mathbf{SH}(S)$, referring to it as *the motivic stable homotopy category over $S$*.

It should be noted that $\mathbf{SH}(\mathcal{A}_{/S})$ has a natural symmetric monoidal structure, denoted $\otimes$, with the unit denoted $\mathbf{1}_S$.

**Theorem 4.3.22.** *The motivic stable homotopy category defines a presheaf of stable $\infty$-categories $\boldsymbol{SH}^* : \mathcal{S}^{op} \to \boldsymbol{Cat}_\infty^{stab}$. This presheaf naturally descends to the structure of a motivic $\infty$-category.*

*Proof.* This is essentially the entire first half of **Khan** . □

For a deeper dive into the understanding of the $\infty$-categorical structure of the stable motivic homotopy category, the reader is encouraged to consult Marc Hoyois' phenomenal paper **Hoy** as well as **Khan** .

An important category related to the motivic stable homotopy category is its rationalization, denoted $\mathbf{SH}_{\mathbb{Q}}(S) := \mathbf{SH}(S) \otimes \mathcal{D}(\mathbb{Q})$, where the tensor product is taken in the $\infty$-category of stable $\infty$-categories. In addition to $\mathbf{SH}_{\mathbb{Q}}$ being a motivic $\infty$-category, it also satisfies several other nice properties that will be relevant for us in our upcoming examples. Among these nice properties are the following.

**Theorem 4.3.23.** *The rationalization of the motivic stable homotopy category $\boldsymbol{SH}_{\mathbb{Q}}$ satisfies the following properties:*

- Absolute Purity: *for any morphism $f : X \to S$ of Noetherian schemes which is factorizable through a closed immersion and a smooth morphism, one obtains an equivalence*
$$\mathbf{1}_X\langle d\rangle \simeq f^!(\mathbf{1}_S)$$
*in $\boldsymbol{SH}_{\mathbb{Q}}(X)$, where $d = rank(T_{X/S})$ is the virtual dimension of $f$*

- Finiteness: *over quasi-excellent scheme, the six functors preserve constructibility*

- Duality: *for every separated morphism of finite type $f : X \to S$ with $S$ quasi-excellent and regular, one has that $f^!(\mathbf{1}_S)$ is a dualizing object in $\boldsymbol{SH}_{\mathbb{Q}}(X)$*

*Proof.* This is **DFJK** corollary C parts II-IV. □

In addition, $\mathbf{SH}_{\mathbb{Q}} \simeq \mathbf{D}_{\mathbb{A}^1}(-, \mathbb{Q})$, where $\mathbf{D}_{\mathbb{A}^1}(S, \mathbb{Q})$ is the same stabilization applied to $\mathbb{A}^1$-local complexes of sheaves of $\mathbb{Q}$-vector spaces satisfying Nisnevich descent.

One of our most important examples will be the category of Beilinson motives $\mathbf{DM}_B$, which is defined as the subcategory of modules over the motivic ring spectrum $\mathbf{H}_B$ for varying $S$. This will in particular be our primary example, and discussion of it will occupy much of a later section. It should be noted that $\mathbf{DM}_B$ satisfies the same nice properties discussed above.

Before one gets the idea that all motivic categories comprise "categories of motives" or "categories of motivic spaces" of some form, let us introduce two related examples decidedly less motivic in nature.

**Definition 4.3.24.** *Given some scheme $S \in \mathcal{S}$, we define $\boldsymbol{D}_{\acute{e}t}(S, \mathbb{Z}/l\mathbb{Z})$ to be the stable derived $\infty$-category of étale sheaves valued in $\mathbb{Z}/l\mathbb{Z}$ for some $l$ coprime to the exponential characteristic of our base scheme. Further define $\mathbf{D}(S, \mathbb{Z}_l)$ to be the stable derived $\infty$-category of $l$-adic sheaves on $S$.*

It has been known since SGA4 and the work of Deligne **Del** that these two categories are motivic at least on the level of triangulated homotopy categories, and more recent work has demonstrated that both of them define motivic $\infty$-categories as well (see, for example, **GaiRoz** or **GaiLur**).

Note that one example we have NOT listed is that of Voevodsky motives. It is still open whether or not Voevodsky's derived (stable $\infty-$)category of motives $\mathbf{DM}(S, \mathbb{Q})$ over a scheme $S$ defines a motivic category in the sense we have described above. It is certainly true if one restricts to particular choices of $\mathcal{S}$. For example, as we will discuss later, for any excellent geometrically unibranch base scheme $S$, one has that $\mathbf{DM}(S, \mathbb{Q}) \simeq \mathbf{DM}_B(S)$. That said, it remains an open problem whether or not Voevodsky's category of motives satisfies a six functors formalism more generally.

### Constructing Derived Motivic Measures from Six Functors Formalisms

The purpose of this section is to demonstrate the existence of a procedure for converting generalized six functors formalisms into derived motivic measures. To do this, we will begin by proving identities about the four functors, and ultimately evaluate everything at the tensor unit over some base, yielding a weakly W-exact map that descends to our desired motivic measure.

**Proposition 4.3.25.** *Given any commutative triangle*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

*of varieties, one has that the triangles*

$$\begin{tikzcd} h_! h^! \arrow[r] \arrow[dr] & g_! g^! \arrow[d] \\ & id \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} id \arrow[r] \arrow[dr] & g_* g^* \arrow[d] \\ & h_* h^* \end{tikzcd}$$

*commute. The dual triangles for (co)units of the appropriate adjunctions also commute.*

*Proof.* This statement is basically a specialization to the setting of the six functors formalism of the fact that ∞-categorical adjunctions compose (cf. **RieVer**). □

**Corollary 4.3.26.** *Given a composition of closed immersions of $S$-schemes*

$$\begin{tikzcd} W \arrow[r, hook, "i"] \arrow[dr, "g"'] & Z \arrow[r, hook, "j"] \arrow[d, "f"] & X \arrow[dl, "h"] \\ & S & \end{tikzcd},$$

*And a composition of open immersions of $S$-schemes*

$$\begin{tikzcd} U \arrow[r, hook, "k"] \arrow[dr, "u"'] & V \arrow[r, hook, "l"] \arrow[d, "s"] & Y \arrow[dl, "t"] \\ & S & \end{tikzcd},$$

*one has the commutative triangles*

$$\begin{tikzcd} g_* g^! \arrow[r] \arrow[dr] & f_* f^! \arrow[d] \\ & h_* h^! \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} t_* t^! \arrow[r] \arrow[dr] & s_* s^! \arrow[d] \\ & u_* u^! \end{tikzcd}$$

*Proof.* This follows pretty directly from the preceding lemma. □

**Lemma 4.3.27.** *Suppose one has an adjunction of cospans of ∞-categories*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, shift left, bend left, "\psi"] & U \arrow[d, shift left, bend left, "\beta"] & V \arrow[l, "f"'] \arrow[d, shift left, bend left, "\gamma"] \\ Z \arrow[r, "i"] \arrow[u, shift left, bend left, "\phi"] \arrow[u, phantom, "\dashv"] & X \arrow[u, shift left, bend left, "\alpha"] \arrow[u, phantom, "\dashv"] & Y \arrow[l, "h"'] \arrow[u, shift left, bend left, "\delta"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

*(note that the above diagram is not commutative as such; rather, it is commutative if one direction of vertical morphisms is omitted, but is presented above to highlight*

*the adjointness of the different vertical morphisms). Then the induced morphisms on pullbacks in both directions form an adjunction*

$$\begin{tikzcd} Y \times_X Z \arrow[r, bend left, "\delta \times_\alpha \phi"] \arrow[r, phantom, "\perp"] & V \times_U W \arrow[l, bend left, "\gamma \times_\beta \psi"] \end{tikzcd} .$$

*Proof.* The crux of this proof will be to define the units and counits for our hypothesized adjunction and to demonstrate that they satisfy the triangle identities. Let us take $\eta, \eta', \eta''$ and $\epsilon, \epsilon', \epsilon''$ to be the unit and counit of the middle, right, and left adjunctions, respectively. Note that we get the isomorphisms

$$(\delta \times_\beta \psi) \circ (\gamma \times_\alpha \phi) \simeq (\delta \circ \gamma) \times_{(\beta \circ \alpha)} (\psi \circ \phi)$$

and

$$(\gamma \times_\alpha \phi) \circ (\delta \times_\beta \psi) \simeq (\gamma \circ \delta) \times_{(\alpha \circ \beta)} (\phi \circ \psi)$$

due to the natural commutativity of diagrams such as this one:

$$\begin{tikzcd}
Y \times_X Z \arrow[rr, "\delta \times_\alpha \phi"] \arrow[dr, "p_1"] \arrow[dd, "p_2"] & & V \times_U W \arrow[rr, "\gamma \times_\beta \psi"] \arrow[dr, "p_1"] \arrow[dd, "p_2"] & & Y \times_X Z \arrow[dr, "p_1"] \arrow[dd, "p_2"] & \\
& Y \arrow[rr, "\delta"] \arrow[dd, "h"] & & V \arrow[rr, "\gamma"] \arrow[dd, "f"] & & Y \arrow[dd, "h"] \\
Z \arrow[rr, "\phi"] \arrow[dr, "i"] & & W \arrow[rr, "\psi"] \arrow[dr, "g"] & & Z \arrow[dr, "i"] & \\
& X \arrow[rr, "\alpha"] & & U \arrow[rr, "\beta"] & & X
\end{tikzcd},$$

and the result of what the values of composition along the long diagonal must be. As a result, we obtain a natural unit and counit given by bringing $\eta' \times_\eta \eta''$ and $\epsilon' \times_\epsilon \epsilon''$ back along the equivalence. All that remains is to show that these two satisfy the triangle identities, but this just follows directly from the triangle identities for each of the units and counits of the adjunctions in the cospan above. ◻

**Lemma 4.3.28.** *Suppose one has an adjunction of cospans of $\infty$-categories*

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, bend left, "r"] \arrow[d, phantom, "\dashv"] & A \arrow[d, bend left, "p"] \arrow[d, phantom, "\dashv"] & B \arrow[l, "f"'] \arrow[d, bend left, "q"] \arrow[d, phantom, "\dashv"] \\
F \arrow[r, "i"] \arrow[u, bend left, "u"] & D \arrow[u, bend left, "s"] & E \arrow[l, "h"'] \arrow[u, bend left, "t"]
\end{tikzcd}$$

*(as before, the above diagram is not commutative as such; rather, it is commutative if one direction of vertical morphisms is omitted, but is presented above to highlight*

*the adjointness of the different vertical morphisms). Then the induced morphisms on comma ∞-categories in both directions form an adjunction*

$$
\begin{tikzcd}
Hom_D(h,i) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Hom_A(f,g) \arrow[l, bend left]
\end{tikzcd} .
$$

*Proof.* The adjunction of cospans in the lemma description yields a related adjunction of cospans

$$
\begin{tikzcd}
C \times B \arrow[r, "g\times f"] \arrow[d, bend left, "r\times q"] \arrow[d, phantom, "\dashv"] & A \times A \arrow[d, bend left, "p\times p"] \arrow[d, phantom, "\dashv"] & A^{\Delta[1]} \arrow[l, "{(p_1,p_0)}"'] \arrow[d, bend left, "p^{\Delta[1]}"] \arrow[d, phantom, "\dashv"] \\
F \times E \arrow[r, "i\times h"] \arrow[u, bend left, "u\times t"] & D \times D \arrow[u, bend left, "s\times s"] & D^{\Delta[1]} \arrow[l, "{(p_1,p_0)}"'] \arrow[u, bend left, "s^{\Delta[1]}"]
\end{tikzcd}.
$$

Now, note that by definition, one has that the pullbacks of these cospans are the appropriate comma ∞-categories, so we obtain our desired adjunction. □

**Proposition 4.3.29.** *Suppose one has a cartesian square of schemes of the form*

$$
\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "i'"] \\
Z \arrow[r, hook, "j'"] & W
\end{tikzcd},
$$

*with $i$ and $i'$ closed immersions and $j$ and $j'$ open immersions. Then one has that the triangles*

$$
\begin{tikzcd}
i'_! i'^! j'_* j'^* \arrow[rr, no head, "\sim"] \arrow[dr, "\epsilon^!_{i'}"'] & & j'_* i_! i^! j'^* \arrow[dl, "\epsilon^!_i"] \\
& j'_* j'^* &
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
& i'_! i'^! \arrow[dl, "\eta^*_j"'] \arrow[dr, "\eta^*_{j'}"] & \\
i'_! j_* j^* i'^! \arrow[rr, no head, "\sim"] & & j'_* j'^* i'_! i'^!
\end{tikzcd}
$$

*homotopy commute.*

*Proof.* This proposition can be proven by showing that it is true for each relevant component of the units and counits involved. We start with the shriek units. Note that given any $\mathcal{F} \in \mathbb{D}(W)$, we have the following adjunction of cospans of ∞-categories

$$
\begin{tikzcd}
\mathbb{D}(X) \arrow[r, "i_! \simeq i_*"] \arrow[d, bend left, "j_*"] \arrow[d, phantom, "\dashv"] & \mathbb{D}(Z) \arrow[d, bend left, "j'_*"] \arrow[d, phantom, "\dashv"] & \mathbf{1} \arrow[l, "j'^*\mathcal{F}"'] \arrow[d, equal] \\
\mathbb{D}(Y) \arrow[r, "i'_! \simeq i'_*"] \arrow[u, bend left, "j^*"] & \mathbb{D}(W) \arrow[u, bend left, "j'^*"] & \mathbf{1} \arrow[l, "j'_* j'^* \mathcal{F}"']
\end{tikzcd},
$$

where the commutativity of the upwards-oriented lefthand square is the result of base change, and the commutativity of the upwards-oriented righthand square is the result of the triangle equivalence (the downwards-oriented squares are more immediately commutative). This implies that the induced functor $\mathcal{H}om_{\mathbb{D}(Z)}(i_!, j'^*\mathcal{F}) \to \mathcal{H}om_{\mathbb{D}(W)}(i'_!, j'_* j'^*\mathcal{F})$ is a right adjoint, and thus preserves limits. In particular, it preserves terminal objects, which shows that for any $\mathcal{F} \in \mathbb{D}(W)$, the first triangle commutes, and thus the triangle commutes in the category of functors. The second proof is dual to the first. □

**Lemma 4.3.30.** *Suppose that we have $\infty$-functors of the form $F, G : \mathcal{B} \to C$ and $H, I : \mathcal{A} \to \mathcal{B}$ equipped with natural transformations $F \xrightarrow{\alpha} H$ and $G \xrightarrow{\beta} I$. Then we have the commutative square*

$$\begin{tikzcd} F \circ G \arrow[r] \arrow[d] & F \circ I \arrow[d] \\ H \circ G \arrow[r] & H \circ I \end{tikzcd} .$$

*Proof.* This is just a restatement of the fact that 2-morphisms compose horizontally in the $(\infty, 2)$-category of $\infty$-categories. □

**Lemma 4.3.31.** *Suppose we are in the situation of the last lemma, except that $\mathcal{A} = \mathcal{B} = C$, and all of our $\infty$-functors are $\infty$-autofunctors. Let us further assume that all four of the endofunctors homotopy commute in a way that is compatible in the homotopy category. Then we have a commutative cube*

$$\begin{tikzcd} F \circ G \arrow[rr] \arrow[dr] \arrow[dd] & & F \circ I \arrow[dr] \arrow[dd] & \\ & H \circ G \arrow[rr] \arrow[dd] & & H \circ I \arrow[dd] \\ G \circ F \arrow[rr] \arrow[dr] & & I \circ F \arrow[dr] & \\ & G \circ H \arrow[rr] & & I \circ H \end{tikzcd} ,$$

*where the vertical morphisms are simply the homotopy-compatible morphisms. Furthermore, considered as an $\infty$-morphism of commutative squares, the diagram above is invertible.*

*Proof.* This proof will make heavy use of the fact that given an $\infty$-category $C$, the functor $\mathbf{Ho}(C^{\Delta[1]}) \to \mathbf{Ho}(C)^{\Delta[1]}$ is surjective on objects, full, and conservative

(**RieVer** 3.1.1). In particular, this further implies that for any $n$, $\mathbf{Ho}(C^{\Delta[1]^n}) \to \mathbf{Ho}(C)^{\Delta[1]^n}$ is surjective on objects, full, and conservative as well.

Now, consider the category of cubes $C^{\Delta[1]^3} \simeq (C^{\Delta[1]})^{\Delta[1]^2} \simeq (C^{\Delta[1]^2})^{\Delta[1]}$. The primary characterization of cubes that we will make use of is as arrows in commutative squares, or the middle term in the above string of equivalences. By our assumptions, we see that the cube written in the lemma description above commutes when considered in the homotopy category $(\mathbf{Ho}(C)^{\Delta[1]^2})^{\Delta[1]}$. Since the canonical functor $\mathbf{Ho}(C^{\Delta[1]^3}) \to \mathbf{Ho}(C)^{\Delta[1]^3}$ is surjective on objects, one has that there must exist a commutative diagram $D \in \mathbf{Ho}(C^{\Delta[1]^3})$ (in other words, an object of $C^{\Delta[1]^3}$) filling the commutative cube, which settles the first part of the lemma.

Let $J$ denote the interval groupoid and consider an inclusion $\Delta[1] \hookrightarrow J$. This induces for any $\infty$-category $C$ a map $C^J \to C^{\Delta[1]}$ whose essential image is the subcategory of equivalences in $C$. Furthermore, the map from $C^J$ onto its essential image is essentially surjective, as a morphism in an $\infty$-category is an equivalence if and only if it can be lifted to a map from $J$ to $C$. (Indeed, it is surjective on objects, in spite of this not being a homotopy-invariant notion.) In particular, on 1-categories, this is the inclusion of a subcategory since inverses to equivalences are uniquely defined in 1-categories. Now, we have the commutative square

$$\begin{tikzcd} \mathbf{Ho}(C^{\Delta[1]}) \arrow[r] & \mathbf{Ho}(C)^{\Delta[1]} \\ \mathbf{Ho}(C^J) \arrow[r] \arrow[u] & \mathbf{Ho}(C)^J \arrow[u] \end{tikzcd}.$$

The bottom morphism is conservative by [R-V]. We wish to show that it is surjective on objects as well. Note that both the left and right vertical morphism are surjective on objects. Note that the vertical morphisms precisely describe those morphisms, either in the underlying $\infty$-category or in the homotopy category, which are invertible. Furthermore, the map restricted to these subcategories must also be surjective on objects. Hence, one must have that

$$\mathbf{Ho}(C^J) \to \mathbf{Ho}(C)^J$$

is surjective on objects as well. Now, we note that there is a string of morphisms

$$\mathbf{Ho}((C^J)^{\Delta[1]^2}) \to \mathbf{Ho}(C^J)^{\Delta[1]^2} \to (\mathbf{Ho}(C)^J)^{\Delta[1]^2}$$

which are surjective on objects. Hence, it must be the case that our cube above admits a filling to $(C^J)^{\Delta[1]^2}$. Furthermore, since $(C^J)^{\Delta[1]^2} \simeq (C^{\Delta[1]^2})^J$, our cube fills to an equivalence of commutative squares in $C$. Finally, this filling must be unique up to a contractible choice since all of this is done ambiently in the $(\infty, 2)$-category of $\infty$-categories. □

**Proposition 4.3.32.** *Suppose one has a cartesian square of schemes of the form*

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "i'"] \\ Z \arrow[r, hook, "j'"] & W \end{tikzcd},$$

*with $i$ and $i'$ closed immersions and $j$ and $j'$ open immersions. Then the square*

$$\begin{tikzcd} j'_* i_! i^! j'^* \simeq i'_! j_* j^* i'^! \arrow[d, "\epsilon^!_i"'] & i'_! i'^! \arrow[l, "\eta^*_j"'] \arrow[d, "\epsilon^!_{i'}"] \\ j'_* j'^* & id \arrow[l, "\eta^*_{j'}"] \end{tikzcd}$$

*in $\mathbb{D}(W)$ commutes.*

*Proof.* Let us begin by noting that due to the various forms of base change, we have the following string of equivalences in $\mathbb{D}(W)$ which we can employ:

$$j'_* i_! i^! j'^* \simeq i'_! j_* j^* i'^! \simeq i'_! i'^! j'_* j'^* \simeq j'_* j'^* i'_! i'^!.$$

This, combined with the commutativity of the triangles of proposition 3.29 allows us to boil the proof down to noting the commutativity of the squares

$$\begin{tikzcd} i'_! i'^! j'_* j'^* \arrow[d, "\epsilon^!_{i'}"'] & i'_! i'^! \arrow[l, "\eta^*_j"'] \arrow[d, "\epsilon^!_{i'}"] \\ j'_* j'^* & \mathrm{i}d \arrow[l, "\eta^*_{j'}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} j'_* j'^* i'_! i'^! \arrow[d, "\epsilon^!_i"'] & i'_! i'^! \arrow[l, "\eta^*_j"'] \arrow[d, "\epsilon^!_{i'}"] \\ j'_* j'^* & \mathrm{i}d \arrow[l, "\eta^*_{j'}"] \end{tikzcd}$$

by lemma 3.30 and then demonstrating that these two squares are connected by isomorphisms which yield a commutative cube. This last step is precisely what we are left with at this point. Indeed, we have the cube

$$\begin{tikzcd} i'_! i'^! \circ \mathrm{i}d \arrow[rr] \arrow[dr] \arrow[dd] & & i'_! i'^! \circ j'_* j'^* \arrow[dr] \arrow[dd] & \\ & id \circ \mathrm{i}d \arrow[rr] \arrow[dd, equal] & & id \circ j'_* j'^* \arrow[dd] \\ id \circ i'_! i'^! \arrow[rr] \arrow[dr] & & j'_* j'^* \circ i'_! i'^! \arrow[dr] & \\ & id \circ \mathrm{i}d \arrow[rr] & & j'_* j'^* \circ \mathrm{i}d \end{tikzcd},$$

which must be (homotopy) commutative by the preceding lemma. ◻

**Corollary 4.3.33.** *Given a cartesian square of $S$-schemes*

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dd, hook, "i"'] \arrow[dr, "f_X"] & & Y \arrow[dd, hook, "i'"] \arrow[dl, "f_Y"'] \\ & S & \\ Z \arrow[rr, "j'"] \arrow[ur, "f_Z"] & & W \arrow[ul, "f_W"'] \end{tikzcd}$$

*where the horizontal morphisms are open immersions and the vertical morphisms are closed immersions, one has that the square*

$$\begin{tikzcd} f_{X*}f_X^! & f_{Y*}f_Y^! \arrow[l, "\eta_j^*"'] \arrow[d, "\epsilon_{i'}^!"] \\ f_{Z*}f_Z^! \arrow[from=u, "\epsilon_i^!"'] & f_{W*}f_W^! \arrow[l, "\eta_{j'}^*"] \end{tikzcd}$$

*in $\mathbb{D}(S)$ commutes.*

*Proof.* This is simply the commutative square of the preceding lemma after precomposition with $f_W^!$ and postcomposition with $f_{W*}$. ◻

**Proposition 4.3.34.** *Given a commutative diagram of finite type $S$-schemes*

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] & U \arrow[l, "j"', "\circ"] \arrow[dl, "h"] \\ & S & \end{tikzcd},$$

*where $i$ is a closed immersion, and $j$ is its complementary open immersion, one obtains a natural cofiber sequence*

$$g_*g^! \to f_*f^! \to h_*h^!.$$

*Proof.* Consider a commutative diagram of finite type $S$-schemes (where the base $S$ is also of finite type)

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] & U \arrow[l, "j"', "\circ"] \arrow[dl, "h"] \\ & S & \end{tikzcd},$$

where $i$ is a closed immersion and $j$ is its complementary open immersion. Note that since $\mathbb{D}$ satisfies the six functors formalism, in particular, it satisfies the localization property. Thus, considering $Z \overset{i}{\hookrightarrow} X \underset{\circ}{\overset{j}{\leftarrow}} U$ as absolute finite type schemes, one has the natural cofiber sequence $i_! i^! \to \mathrm{i}d \to j_* j^*$.

Now, since $i$ is a closed immersion, it is in particular proper, so we have that $i_! \simeq i_*$. Furthermore, since $j$ is an open immersion, it is smooth, so one has that $j^* \simeq \Sigma^{\Omega_j} \circ j^!$. Since $j$ is étale, $\Omega_j = 0$, so $\Sigma^{\Omega_j}$ is equivalent to the identity, and we are left with $j^* \simeq j^!$.

Composing with the appropriate direction of the above equivalences, we get the cofiber sequence

$$i_* i^! \to \mathrm{i}d \to j_* j^!.$$

Now, precomposing with $f^!$, we obtain a cofiber sequence

$$i_* i^! f^! \to f^! \to j_* j^! f^!.$$

Since $f_*$ preserves finite limits and any (co)cartesian square in a stable $\infty$-category is bicartesian, we have that postcomposition with $f_*$ yields a cofiber sequence

$$f_* i_* i^! f^! \to f_* f^! \to f_* j_* j^! f^!.$$

Finally, noting that $f_* i_* i^! f^! \simeq (f \circ i)_* (f \circ i)^! = g_* g^!$ and that $f_* j_* j^! f^! \simeq (f \circ j)_* (f \circ j)^! = h_* h^!$, this reduces to the cofiber sequence

$$g_* g^! \to f_* f^! \to h_* h^!.$$

□

**Theorem 4.3.35.** *Suppose that $\mathbb{D}$ is constructibly generated and satisfies one of the following two sets of conditions:*

- *The four functors preserve constructible objects when given input a seperable morphism of finite type (note that compactness is trivially preserved by tensor)*
- *The six functors preserve constructible objects over Noetherian quasi-excellent schemes of finite dimension with respect to morphisms of finite type (in other words, for any finite type morphism $f : X \to S$ with target Noetherian quasi-excellent of finite dimension, the four functors preserve constructible objects, while tensor and Hom preserve constructible objects over $S$)*

*Then, given a scheme $S$ (assumed to be Noetherian quasi-excellent of finite dimension if $\mathbb{D}$ satisfies the second condition in particular), there is a weakly W-exact functor*

$$M^c_{\mathbb{D}(S)} : \boldsymbol{Var}_S \to \mathbb{D}_{cons}(S)$$

*sending each variety (smooth or otherwise) $(X \xrightarrow{f} S) \in \boldsymbol{Var}_S$ to $M^c_{\mathbb{D}(S)}(X) := f_* f^! \mathbf{1}_S$.*

*Proof.* Assembling the various lemmae and propositions that we have proven above, we are equipped to show the following:

- We have a covariant functor $M^c_{\mathbb{D}(S)} : \mathbf{cof(Var}_S) \to \mathbb{D}_{\mathrm{cons}}(S)$ given by sending objects $X$ to $M^c_{\mathbb{D}(S)}(X)$, and morphisms $Z \overset{i}{\hookrightarrow} X$ to $M^c_{\mathbb{D}(S)}(Z) \xrightarrow{\epsilon^!_i(\mathbf{1}_S)} M^c_{\mathbb{D}(S)}(X)$, where functoriality comes from evaluating the left triangle in the statement of corollary 3.26 at $\mathbf{1}_S$
- We have a contravariant functor $M^c_{\mathbb{D}(S)} : \mathbf{comp}(\mathbf{Var}_S) \to \mathbb{D}_{\mathrm{cons}}(S)$ given by sending objects $X$ to $M^c_{\mathbb{D}(S)}(X)$, and morphisms $U \overset{j}{\hookrightarrow} X$ to $M^c_{\mathbb{D}(S)}(X) \xrightarrow{\eta^*_j(\mathbf{1}_S)} M^c_{\mathbb{D}(S)}(U)$, where functoriality comes from evaluating the right triangle in the statement of corollary 3.26 at $\mathbf{1}_S$
- Since weak equivalences in $\mathbf{Var}_S$ are simply isomorphisms, and isomorphisms are in particular closed immersions, we obtain our functor $M^c_{\mathbb{D}(S)} : w(\mathbf{Var}_S) \to \mathbb{D}_{\mathrm{cons}}(S)$ by restricting the one we already have on closed immersions. Note that isomorphisms must map to weak equivalences in $\mathbb{D}_{\mathrm{cons}}(S)$
- Consider a cartesian square of $S$-motives

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f_X"] \arrow[dd, hook, "i"'] & & Y \arrow[dl, "f_Y"'] \arrow[dd, hook, "i'"] \\ & S & \\ Z \arrow[ur, "f_Z"] \arrow[rr, "j'"'] & & W \arrow[ul, "f_W"'] \end{tikzcd}$$

where the horizontal morphisms are open immersions and the vertical morphisms are closed immersions. By corollary 3.33, we have the commutative square

$$\begin{tikzcd} f_{X*}f^!_X & f_{Y*}f^!_Y \arrow[l, "\eta^*_j"'] \arrow[d, "\epsilon^!_{i'}"] \\ f_{Z*}f^!_Z \arrow[u, phantom] & f_{W*}f^!_W \arrow[l, "\eta^*_{j'}"] \end{tikzcd} ,$$

Which, upon evaluation at $\mathbf{1}_S$, in turn yields the commutative square

$$\begin{tikzcd} M^c_{\mathbb{D}(S)}(X) \arrow[d, "\epsilon_i^!"'] & M^c_{\mathbb{D}(S)}(Y) \arrow[l, "\eta_j^*"'] \arrow[d, "\epsilon_{i'}^!"] \\ M^c_{\mathbb{D}(S)}(Z) & M^c_{\mathbb{D}(S)}(W) \arrow[l, "\eta_{j'}^*"] \end{tikzcd}$$

- Given a closed/open decomposition

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] & U \arrow[l, "j"', "\circ"] \arrow[dl, "h"] \\ & S & \end{tikzcd}$$

of an $S$-variety $X$, composing the natural cofiber sequence

$$g_*g^! \to f_*f^! \to h_*h^!$$

of proposition 3.34 with $\mathbf{1}_S$ yields the cofiber sequence

$$M^c_{\mathbb{D}(S)}(Z) \to M^c_{\mathbb{D}(S)}(X) \to M^c_{\mathbb{D}(S)}(U)$$

- Of the last two conditions we need to check, the one on cofibrations follows from the fact that the two compositions around the diagram of $S$-varieties

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dd, hook, "i"'] \arrow[dr, "f_X"] & & Y \arrow[dd, hook, "i'"] \arrow[dl, "f_Y"'] \\ & S & \\ Z \arrow[rr, "\sim"'] \arrow[ur, "f_Z"] & & W \arrow[ul, "f_W"'] \end{tikzcd}$$

(where the vertical morphisms are closed immersions and the horizonal morphisms are isomorphisms) coincide, and by the left triangle of corollary 3.26. The condition on complements is a special case of the fourth bullet point of this proof, as all isomorphisms are necessarily cofibrations.

□

**Corollary 4.3.36.** *Suppose $\mathbb{D}$ sastisfies one of the two conditions of the above theorem. Then, given a scheme $S$ (assumed to be Noetherian quasi-excellent of finite dimension if $\mathbb{D}$ satisfies the second set of conditions), one obtains a map of K-theory spectra*

$$K(M^c_{\mathbb{D}(S)}) : K(\boldsymbol{Var}_S) \to K(\mathbb{D}_{cons}(S)).$$

*Proof.* This follows immediately from the existence of the weakly W-exact functor above. □

Ultimately, the reason we wanted to ensure that we landed in constructible objects above was simply to avoid swindles that would be permitted by mapping into an essentially large category. The fact that our category is essentially small ensures that our K-theory is nontrivial. Note that the only condition specified above is that the four functors preserve constructibility. While it is certainly preferable that $\mathbb{D}$ be compactly or (better yet) constructibly generated, it is not strictly speaking needed to have a well-defined W-exact functor/map on K-theory of the type above.

## 4.4 Lifting the Gillet-Soulé Motivic Measure

**Preamble on the Gillet-Soulé Motivic Measure and Some Work of Bondarko**

We begin this section with a brief overview of the classical Gillet-Soulé motivic measure, as well as a slight generalization due to Gillet and Soulé, before briefly describing a few theorems of Bondarko that allow for an alternate characterization of the Gillet-Soulé motivic measure in terms of Voevodsky motives. This will then be used in the following sections to construct a derived motivic measure lifting the Gillet-Soulé motivic measure by passing through Voevodsky motives.

Given a field $k$ which satisfies resolution of singularities and weak-factorization, we have the following theorem due to Bittner (see **Bitt** theorem 3.1 or **MNP** theorem 9.1.2).

**Theorem 4.4.1.** *If $k$ is a field which admits resolution of singularities and weak-factorization, then $K_0(\boldsymbol{Var}_k)$ may be presented by the isomorphism classes $[X]$ of smooth projective varieties over $k$ subject to the relations*

- $[\emptyset] = 0$
- $[X] - [Z] = [Y] - [E]$ *where $Z \subseteq X$ is a closed subvariety, $Y = Bl_Z(X)$, and $E$ is the exceptional divisor*

As a result, letting $h : \mathbf{SmProj}_k \to \mathrm{Chow}(k, \mathbb{Q})$ be the natural map from smooth projective varieties to Chow motives, one can make the following definition:

**Definition 4.4.2.** *If $k$ satisfies resolution of singularities and weak factorization, then the map*

$$\chi_{gs} : K_0(\boldsymbol{Var}_k) \to K_0(Chow(k, \mathbb{Q}))$$

*given by* $\chi_{gs}([X]) := [h(X)]$ *is referred to as the* Gillet-Soulé motivic measure.

Now, this definition is really also a proposition (**MNP** proposition 9.1.3), as it is nontrivial that one must have $[h(X)] - [h(Z)] = [h(Y)] - [h(E)]$ as above. That said, we suppress the proof here for brevity.

This motivic measure can actually be redefined in such a way as to employ the standard generators and relations. Recall that given any pseudo-abelian category $\mathfrak{A}$, it is always possible not only to define the category $\mathbf{CH}^\flat\mathfrak{A}$ of bounded chain complexes, but also its chain homotopy category $\mathrm{Hot}^\flat\mathfrak{A}$. In particular, one has that

$$K_0(\mathrm{Hot}^\flat\mathfrak{A}) \cong K_0(\mathfrak{A})$$

via the Euler characteristic map $[A_\bullet] \mapsto \sum_i (-1)^i [A_i]$ via **GilSou** lemma 3. Using the fact that, in particular, $K_0(\mathrm{Chow}(k, \mathbb{Q}) \simeq K_0(\mathrm{Hot}^\flat\mathrm{Chow}(k, \mathbb{Q}))$, we can immediately come up with a refinement of the Gillet-Soulé motivic measure as well as a categorification which anticipates the weakly W-exact functors we will construct below. Note the following definition/theorem.

**Theorem 4.4.3.** *Given any arbitrary* $X \in \boldsymbol{Var}_k$*, we may construct a complex* $W(X) \in \mathit{Hot}^\flat\mathit{Chow}(k, \mathbb{Q})$ *which refines the Gillet-Soulé motivic measure in the sense that* $[W(X)] \mapsto [h(X)] \in K_0(\boldsymbol{Var}_k)$ *whenever* $X$ *is smooth and projective. This is referred to as the* weight complex. *The assignment* $X \mapsto W(X)$ *satisfies the following functoriality properties:*

- *A proper map* $f : X \to Y$ *induces a map* $f^* : W(Y) \to W(X)$ *and for any two composable proper maps* $f$ *and* $g$*, one has that* $(f \circ g)^* = g^* \circ f^*$*. In other words, there is a contravariant functor from varieties equipped with proper morphisms to complexes of Chow motives up to homotopy*
- *An open immersion* $j : U \xrightarrow{\circ} X$ *induces a map* $j_* : W(U) \to W(X)$ *which is covariantly functorial in open immersions analogously to the above*
- *For any* $X, Y \in \boldsymbol{Var}_k$*, one has that* $W(X \times Y) \cong W(X) \otimes W(Y)$
- *Given any closed-open decomposition* $Z \overset{i}{\hookrightarrow} X \overset{j}{\underset{\circ}{\leftarrow}} U$ *of a* $k$*-variety* $X$*, one obtains the exact triangle* $W(U) \xrightarrow{j_i} W(X) \xrightarrow{i^*} W(Z) \to W(U)[1]$ *in* $\mathit{Hot}^\flat\mathit{Chow}(k, \mathbb{Q})$

*Proof.* This is **GilSou** theorem 2 (also **MNP** theorem 9.2.1). □

Given the definition of the Grothendieck group of a triangulated category, we note that the last property of the weight complex functors provides a categorical lift of the cutting-and-pasting property enjoyed by the Gillet-Soulé motivic measure. We will not go into the explicit construction of $W(X)$ here, as it will serve more as a bridge between Voevodsky motives and Chow motives and will not be used in and of itself. That said, for those who are interested in the construction, we recommend the phenomenal paper by Gillet and Soulé **GilSou** and the book by Murre **MNP** .

Note that this construction does not work for us as such, given that the maps constructed "run in the wrong direction," among other things. That said, this can be rectified by simply reversing the arrows and working with homological Chow motives as is described in **Bon** remark 6.5.2 instead of cohomological ones. This will not alter the underlying Grothendieck ring, so from this point forward, we will work exclusively with the homological grading. The paper by Bondarko cited above is very deep and has much more to offer than simply what is taken from it here; in what follows, we will only describe the bare minimum of what we need. Most importantly:

**Theorem 4.4.4.** *There exists a conservative functor $t : DM_{gm}^{eff}(k, \mathbb{Q}) \to Hot^{\flat}Chow^{eff}(k, \mathbb{Q})$ called the* weight complex functor *(for reasons we shall see shortly) which induces an isomorphism of Grothendieck rings $K_0(DM_{gm}^{eff}(k, \mathbb{Q})) \xrightarrow{\sim} K_0(Hot^{\flat}Chow^{eff}(k, \mathbb{Q}))$. This descends on $K_0$ to an isomorphism $K_0(DM_{gm}(k, \mathbb{Q})) \xrightarrow{\sim} K_0(Hot^{\flat}Chow(k, \mathbb{Q}))$.*

*Proof.* This is **Bon** proposition 6.3.1 combined with remark 6.3.2. □

**Theorem 4.4.5.** *For any $X \in$ **Var**$_k$, we have that $t(M^c(X)) \cong W(X)$, and that this assignment is functorial when we restrict to the category of varieties equipped with proper maps.*

*Proof.* This is essentially **Bon** proposition 6.6.2. Our notation difference from the aforementioned proposition is explicable via remark 6.3.2 (2) of the same paper. □

Thus, not only is $K_0(\mathrm{DM}_{gm}(k, \mathbb{Q})) \cong K_0(\mathrm{Chow}(k, \mathbb{Q}))$, but also the corresponding classes of $M^c(X)$ and $W(X)$ always coincide. Indeed, we have reduced our task to lifting a stable or DG-enhancement of the assignment $X \mapsto M^c(X)$ to a weakly W-exact functor, as doing so will provide a lift of the Gillet-Soulé motivic measure. Two different approaches are given in the section below.

**Approach via Six Functors**

For the entirety of this section, all schemes will be based over a perfect field $k$, and we will be working exclusively with rational coefficients (more general coefficients will perhaps be addressed in forthcoming work).

Consider the motivic spectrum $KGL_S \in \mathbf{SH}(S)$ defined to be the representative of algebraic K-theory in $\mathbf{SH}(S)$. It should be noted that for any morphism of schemes $f : X \to Y$, one has that $f^*KGL_Y \simeq KGL_X$. Furthermore, its rationalization $KGL_{S,\mathbb{Q}} := KGL_S \otimes \mathbb{Q}$ breaks up as the sum

$$KGL_{S,\mathbb{Q}} \simeq \bigoplus_{i \in \mathbb{Z}} KGL_S^{(i)}$$

in a way that is compatible with base change.

**Definition 4.4.6.** *We define the Beilinson motivic cohomology over S to be*

$$H_{B,S} := KGL_S^{(0)}$$

*and define the stable ∞-category of Beilinson motives over S to be*

$$\boldsymbol{DM}_B(S) := Mod_{H_{B,S}}.$$

**Theorem 4.4.7.** *$\boldsymbol{DM}_B$ admits a constructibly generated six functors formalism. In particular, $\boldsymbol{DM}_B^*$ defines a $(*, \#, \otimes)$-formalism in the sense of Khan, satisfies the Voevodsky conditions, and is constructibly generated.*

*Proof.* This is part of **RicSch** synopsis 2.1.1 based on prior work by Cisinski and Déglise in **CisDeg** and Ayoub in **Ayo** . □

**Theorem 4.4.8.** *The motivic category of Beilinson motives $\boldsymbol{DM}_B$ satisfies the following properties:*

- Absolute Purity: *for any morphism $f : X \to S$ of Noetherian schemes which is factorizable through a closed immersion and a smooth morphism, one obtains an equivalence*

  $$\mathbf{1}_X\langle d\rangle \simeq f^!(\mathbf{1}_S)$$

  *in $\boldsymbol{SH}_{\mathbb{Q}}(X)$, where $d = rank(T_{X/S})$ is the virtual dimension of $f$*

- Finiteness: *over quasi-excellent scheme, the six functors preserve constructibility*

- Duality: *for every separated morphism of finite type $f : X \to S$ with $S$ quasi-excellent and regular, one has that $f^!(\mathbf{1}_S)$ is a dualizing object in $\boldsymbol{SH}_{\mathbb{Q}}(X)$*

*Proof.* This is **DFJK** theorem A parts II-IV under the equivalence found in part V of the same theorem (based on prior work by Cisinski and Déglise in **CisDeg**). □

We also obtain the following corollary.

**Corollary 4.4.9.** *Given $f : X \to Y$ separated of finite type with $X$ and $Y$ quasi-excellent, the involutive antiequivalence $D_X := Hom_X(-, f^!\mathbf{1}_Y)$ on the category $\boldsymbol{DM}_B(X)$ descends to one on the category of compact/constructible objects $\boldsymbol{DM}^c_B(X)$.*

*Proof.* This combines the second and third statements of the preceding theorem. □

**Proposition 4.4.10.** *The involution $D_{(-)}$ intertwines the four functors in the following way. Where appropriately defined, if one has a morphism $g : S \to T$, one has that*

$$D_T \circ g_! \simeq g_* \circ D_S \text{ and } g^* \circ D_T \simeq D_S \circ g^!.$$

*Proof.* This is **RicSch** synopsis 2.1.1 part viii. □

The reason that we mostly use the above approach is that it has an extremely rigid structure and many nice properties, including but not limited to absolute purity. As an example, the above theorems yield the following central result as a corollary.

**Corollary 4.4.11.** *Given any Noetherian, quasi-excellent scheme $S$ of finite dimension, one obtains a weakly W-exact functor*

$$M^c_S : \boldsymbol{Var}_S \to \boldsymbol{DM}^c_B(S)$$

*which descends on K-theory to a map of K-theory spectra*

$$K(M^c_S) : K(\boldsymbol{Var}_S) \to K(\boldsymbol{DM}^c_B(S)).$$

*Proof.* This follows directly from the above theorem. □

In particular, this yields a generalized derived Gillet-Soulé motivic measure, as we shall see shortly. Before we can say this with certainty, we must take a detour through the DG category of Voevodsky motives.

**Corollary 4.4.12.** *Over any excellent, geometrically unibranch scheme S, the one obtains an adjunction*

$$\begin{tikzcd} SH_{\mathbb{Q}}(S) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & DM(S,\mathbb{Q}) \arrow[l, bend left] \end{tikzcd}$$

*on homotopy categories where the left adjoint arises as sheafification with respect to transfers and the right adjoint arises as a forgetful functor (forgetting transfers). This descends to an equivalence*

$$\begin{tikzcd} DM_B(S) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & DM(S,\mathbb{Q}) \arrow[l, bend left] \end{tikzcd}$$

*which further restricts to an equivalence*

$$\begin{tikzcd} DM^c_B(S) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & DM_{gm}(S,\mathbb{Q}) \arrow[l, bend left] \end{tikzcd}$$

*on the level of compact/constructible objects.*

*Proof.* This is **CisDeg** theorem 16.1.4 coupled with noting that geometric motives are simply the compact objects in Voevodsky's big category of motives.

**Remark 4.4.13.** *These adjunctions upgrade to premotivic adjunctions of stable ∞-categories of coefficients, and proof of this fact will be included in the final version of this paper.*

□

**Proposition 4.4.14.** *Specializing the above equivalence to the case of $S = Spec(k)$, one has that for all $k$-schemes $f : X \to Spec(k)$, considering $\mathbf{1}_k \in DM_B(k)$ one has*

$$f_* f^!(\mathbf{1}_k) \mapsto M^c(X),$$

*the compactly supported motive of X (thus justifying our notation above).*

*Proof.* Note first of all that via **CisDeg** theorem 16.1.4 and **CisDeg2** corollary 5.9, the sheafification functors

$$\mathrm{DM}_B(k) \to \mathrm{DM}(k,\mathbb{Q}) \to \mathrm{DM}_{cdh}(k,\mathbb{Q})$$

are all equivalences of symmetric monoidal triangulated categories. Furthermore, the composite and intermediate sheafifications commute on compact objects with the six functors by **CisDeg** theorem 4.29. Finally, for any $k$-variety $X$ with structure morphism $f : X \to \mathrm{S}peck$, one has that $f_* f^!(\mathbf{1}_k) \cong M^c(X)$ in $\mathrm{DM}_{cdh}(k, \mathbb{Q})$ via **CisDeg2** proposition 8.10. Since the image of $f_* f^!(\mathbf{1}_k)$ considered in $\mathrm{DM}_B(k)$ and $M^c(X)$ considered in $\mathrm{DM}(k, \mathbb{Q})$ coincide in $\mathrm{DM}_{cdh}(k, \mathbb{Q})$, one must have that $f_* f^!(\mathbf{1}_k)$ maps under sheafification to an object isomorphic to $M^c(X)$ in $\mathrm{DM}(k, \mathbb{Q})$. □

**Theorem 4.4.15.** *Considering a perfect base field k and rational coefficients, the map*

$$K(M_k^c) : K(\boldsymbol{Var}_k) \to K(\boldsymbol{DM}_B^c(k))$$

*yields a derived lift of the Gillet-Soulé motivic measure.*

*Proof.* Recall that over rational coefficients, for Bondarko's weight map on the level of triangulated categories $t_{\mathbb{Q}} : DM_{gm}(k, \mathbb{Q}) \to K^{\flat}\mathrm{Chow}(k, \mathbb{Q})$ maps the compactly supported motive $M_k^c(X)$ to the weight complex $W(X)$ for any finite type $k$-scheme $X$. Furthermore, $t_{\mathbb{Q}}$ induces an isomorophism on the level of Grothendieck Groups. Noting that under the isomorphism $K_0(K^{\flat}\mathrm{Chow}(k, \mathbb{Q})) \cong K_0(\mathrm{Chow}(k, \mathbb{Q}))$, the image of the Gillet-Soulé motivic measure is precisely $[W(X)]$. Thus, further noting that the Grothendieck group of a stable $\infty$-category is the same as the grothendieck group of its (triangulated) homotopy category, stringing all of our equivalences together, we have that the Gillet-Soulé motivic measure can be factored as

$$K_0(\mathbf{Var}_k) \overset{K_0(M_k^c)}{\to} K_0(\mathbf{DM}_B^c(k)) \cong K_0(DM_{gm}(k, \mathbb{Q})) \cong K_0(\mathrm{Hot}^{\flat}\mathrm{Chow}(k, \mathbb{Q})) \cong K_0(\mathrm{Chow}(k, \mathbb{Q})),$$

which shows that $K(M_k^c) : K(\mathbf{Var}_k) \to K(\mathbf{DM}_B^c(k))$ is a derived lift of the Gillet-Soulé motivic measure. □

**Remark 4.4.16.** *Let us briefly work in the model of quasicategories and suppose the existence of a motivic t-structure on $DM_{gm}(k, \mathbb{Q})$* **Bei** *. This, in particular, may be lifted to a t-structure on $\boldsymbol{DM}_B^c(k)$ via the above equivalence. Now, Barwick's Theorem of the Heart states that for any bounded t-structure on a stable $\infty$-category $\mathcal{A}$, one necessarily obtains an equivalence $K(\mathcal{A}) \simeq K(\mathcal{A}^{\heartsuit})$, where the latter K-theory is taken as an exact $\infty$-category. This exact K-theory coincides with the classical K-theory of an abelian category if $\mathcal{A}^{\heartsuit}$ is actually just the nerve of an abelian category. Thus, if one can show that the properties of the hypothetical motivic t-structure on $DM_{gm}(k, \mathbb{Q})$ imply that the lift to $\boldsymbol{DM}_B^c(k)$ is accessible (in the sense of Lurie* **Lur2** *definition 1.4.4.12) and bounded upon lifting (it is known to be bounded on the triangulated homotopy category), then $K(\boldsymbol{DM}_B^c(k))$ must in fact be the K-theory of the hypothetical abelian category of mixed motives.*

**Some Other Approaches**

In spite of the machinery which we have built above, there are several other potential approaches that one can take to the problem of constructing a derived lift to the Gillet-Soulé motivic measure. One of them, using the language of cofibration categories and topological triangulated categories, **Schw** , is very likely equivalent to the approach above. The other, making use of recent work by Cisinski and Bunke **BunCis** on the K-theory of additive categories, is almost surely not. We will discuss the former method, but forego discussion of the latter, and simply state that it exists.

Given any pretriangulated DG-category $C$, one can construct a cofibration category (which is, in this case, a Waldhausen category) lying between it and $\mathbf{Ho}(C)$ as follows. One defines the *cycle category* of $C$, denoted $\mathcal{Z}(C)$, to be the additive category with the same objects as $C$, but for any $X, Y \in \mathrm{Ob}(C)$, $\mathrm{H}om_{\mathcal{Z}(C)}(X,Y) = \mathrm{K}er(\mathrm{H}om_C(X,Y)^0 \xrightarrow{d} \mathrm{H}om_C(X,Y)^1)$ (the "closed morphisms" of degree 0 in $\mathrm{H}om_C(X,Y)$) with the composition induced by that of $C$.

$\mathcal{Z}(C)$ can be made into a stable cofibration category (which happens to be a Waldhausen category) in the following way:

- Weak equivalences are those morphisms which become equivalences in $\mathbf{Ho}(C)$
- Cofibrations are those morphisms $f \in \mathrm{H}om_{\mathcal{Z}(C)}(X,Y)$ for $X, Y \in \mathrm{Ob}(C)$ such that for any $Z \in \mathrm{Ob}(C)$, one has that the induced morphism

  $$\mathrm{H}om_C(f, Z) : \mathrm{H}om_C(Y, Z) \to \mathrm{H}om_C(X, Z)$$

  is a surjection

The proof that $\mathcal{Z}(C)$ equipped with these weak equivalences and cofibrations is a stable cofibration category is **Schw** proposition 3.2.

Recall further from **Schw** remark 1.3 that if $C$ is a stable cofibration category, a triangle $X \to Y \to Z \to X[1]$ in $\mathbf{Ho}(C)$ is distinguished if and only if it is connected by a zigzag of weak equivalences to a cofiber sequence in $C$. This will be used in the following theorem.

**Theorem 4.4.17.** *There is a weakly W-exact functor $M^c$ : $\boldsymbol{Var}_S \to \mathcal{Z}(\mathcal{DM}_{gm}(S,R))$ sending each variety (smooth or otherwise) $X \in \boldsymbol{Var}_S$ to its compactly supported motive $M^c(X)$.*

*Proof.* This proof will proceed in several stages. First, we need to construct the three functors that will form the basis of the rest of the proof:

- Recall that the cofibrations in the SW-category $\mathbf{Var}_S$ are merely the closed immersions. In particular, closed immersions are proper, so for any closed immersion $f : X \hookrightarrow Y$ in $\mathbf{Var}_S$, one obtains the map $f_* : M^c(X) \to M^c(Y)$ functorially. This yields the data of a functor $\mathbf{cof(Var}_S) \to \mathcal{Z}(\mathcal{DM}_{gm}(S, R))$
- Recall that complement maps in the SW-category $\mathbf{Var}_S$ are the open immersions. Since open immersions are smooth, for any open immersion $f : X \overset{\circ}{\to} Y$, one obtains the map $f^* : M^c(Y) \to M^c(X)$ contravariantly and functorially. This yields the data of a functor $\mathbf{comp}(\mathbf{Var}_S)^c \to \mathcal{Z}(\mathcal{DM}_{gm}(S, R))$
- Since weak equivalences in the SW-category $\mathbf{Var}_S$ are merely isomorphisms, and are thus closed immersions, the lower star construction yields a functorial mapping into $\mathcal{Z}(\mathcal{DM}_{gm}(S, R))$. This map is furthermore in $w\mathcal{Z}(\mathcal{DM}_{gm}(S, R))$ in this case because isomorphisms map functorially to isomorphisms, and all isomorphisms are in $w\mathcal{Z}(\mathcal{DM}_{gm}(S, R))$

Now we can merely verify that all the appropriate properties are satisfied.

- This is an application of **SusVoe** proposition 3.6.5. In particular, given any cartesian square

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i'} \\ Y & \xrightarrow{j'} & W \end{array},$$

  of schemes with horizontal arrows closed immersions and vertical arrows open immersions, one obtains a commutative diagram

$$\begin{array}{ccc} R^c[X] & \xhookrightarrow{j_*} & R^c[Z] \\ {\scriptstyle i^*}\big\uparrow & & {\scriptstyle i'^*}\big\uparrow \\ R^c[Y] & \xhookrightarrow{j'_*} & R^c[W] \end{array}$$

  on the level of Nisnevich sheaves with transfers and hence a commutative diagram

$$\begin{array}{ccc} M^c(X) & \xhookrightarrow{j_*} & M^c(Z) \\ {\scriptstyle i^*}\big\uparrow & & {\scriptstyle i'^*}\big\uparrow \\ M^c(Y) & \xhookrightarrow{j'_*} & M^c(W) \end{array}$$

on the level of motives

- As was proven in **BeiVol** (see equation 6.9.1), given a closed immersion $i : Z \hookrightarrow X$ with complementary open immersion $j : U = X - Z \overset{\circ}{\to} X$, one has a distinguished triangle $M^c(Z) \overset{i_*}{\to} M^c(X) \overset{j^*}{\to} M^c(U) \to M^c(Z)[1]$ in $DM_{gm}(S, R)$. By **Schw** remark 1.3, that this triangle is distinguished in the homotopy category implies that $M^c(Z) \overset{i_*}{\to} M^c(X) \overset{j^*}{\to} M^c(U)$ is weakly equivalent via a zigzag of weak equivalences to a cofiber sequence. In other words, noting that $M^c(\emptyset) \simeq 0$, we have a homotopy cocartesian square

$$\begin{tikzcd} M^c(Z) \arrow[r, "i_*"] \arrow[d] & M^c(X) \arrow[d, "j^*"] \\ M^c(\emptyset) \arrow[r] & M^c(U) \end{tikzcd}$$

- Given any commutative diagram of schemes

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "g"] & Z \arrow[d, "g'"] \\ Y \arrow[r, hook, "f'"] & W \end{tikzcd}$$

with vertical morphisms isomorphisms and horizontal morphisms closed immersions, noting that the direct image is a functorial assignment, one gets a commutative square

$$\begin{tikzcd} M^c(X) \arrow[r, hook, "f_*"] \arrow[d, "g_*"] & M^c(Z) \arrow[d, "g'_*"] \\ M^c(Y) \arrow[r, hook, "f'_*"] & M^c(W) \end{tikzcd} .$$

If one instead assumes that the horizontal morphisms are open immersions, one may note that any commutative square with vertical morphisms given by isomorphisms is cartesian, and the result then follows from the result for the first bullet point

□

## 4.5 Conclusion

The procedure we have outlined above allows us to go from abstract six functors formalisms to derived motivic measures. Now that we have the method, and an application of it, there are many more questions that need answering, and many more possible directions in which to take this work.

First and foremost, one might seek to upgrade many of the other categories defined by Cisinski and Déglise and upgrade as well the many premotivic adjunctions we have already discussed on the level of triangulated categories to their respective $\infty$-categorical analogues.

In addition, one might try to extend Khan's notion of $\infty$-categorical six functors formalism to related geometric categories. For example, there are many reasons why one might wish to have a six functors formalism not just for derived algebraic spaces, but more generally for derived stacks, and perhaps even more general categories of pre-stacks. Our interest, however, is mainly in defining these formalisms for geometric categories equipped with a group action (perhaps necessarily factoring through finite quotient). For example, given a profinite group $G$, one might want to have the ability to define a notion of six functors formalism that incorporates $G$-actions, so as to cook up a notion of $G$-equivariant motivic categories generalizing the work of Hoyois on the equivariant stable motivic homotopy category **Hoy** . This would allow one to further generalize much of the work found in **LMM** for instance.

Another immediate direction in which to take this work would be to compare it to that found in **CWZ** . In it, Campbell, Wolfson, and Zakharevich define a spectral lifting of the Hasse-Weil zeta function making use of compactly supported $l$-adic cohomology. One has reason to strongly suspect that, using $l$-adic sheaves as our target, one might obtain the same map of K-theory spectra up to homotopy.

More generally, this approach seems to allow one to spectrally lift many of the classical motivic measures, especially those related to categories of sheaves. All of this requires further investigation. This will perhaps be carried out in a later version of this paper, or in a sequel.

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