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Tangent conic sections to the graph of a function and their derivatives

Secciones cónicas tangentes a la gráfica de una función y sus derivadas

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Abstract The geometric problem of the tangent line to the graph of a function at a point P is studied within the derivative concept. The geometric problem of the tangent can be extended to other tangent curves and not only to the simplest curve (the tangent straight line). The tangent line is a specific case of a tangent conic section since a line is a degenerate conic section. Therefore a tangent conic section is a more general solution as this contains also the geometric problem of the tangent line. Here, the tangent conic sections to the graph of a function are determined. The tangent conic section contains the point P as its vertex whose tangent line is equal to the tangent line of the function at that point P. The second-degree equation of a tangent conic section is an implicitly defined function. So, the parametric equations are used to obtain the graphs and the derivatives easily. Tangent conic sections to a given point P of a function are calculated in illustrative examples.
Keywords derivative, limit situation, parametric curve, tangent conic sections, tangent curve.

Resumen El problema geométrico de la recta tangente a la gráfica de una función en un punto P se estudia en el concepto de derivada. El problema geométrico de la tangente se puede extender a otras curvas tangentes y no solo a la curva más simple (la recta tangente). La línea tangente es un caso específico de una sección cónica tangente ya que una línea es una sección cónica degenerada. Por lo tanto, una sección cónica tangente es una solución más general, ya que también contiene el problema geométrico de la recta tangente. En este artículo, las secciones cónicas tangentes a la gráfica de una función son determinadas. La sección cónica tangente contiene el punto P como su vértice, cuya línea tangente es igual a la línea tangente de la función en ese punto P. La ecuación de segundo grado de una sección cónica tangente es una función definida implícitamente. Entonces, las ecuaciones paramétricas son usadas para obtener las gráficas y las derivadas fácilmente. Las secciones cónicas

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tangentes a un punto P dado de una función se calculan en ejemplos ilustrativos.
Palabras Claves curva paramétrica, curva tangente, derivada, secciones cónicas tangentes, situación límite.

1 Introduction

The conic sections were described by the Greeks. The Greek Apollonius obtained the conic sections cutting a double cone by a plane. If the plane passes through the vertex, the intersection is a degenerate conic. If the slicing plane is parallel to a side of the cone, a parabola is obtained. If the plane slices a cone at right angle with respect to the cone axis, a circle is generated. If the plane slices a cone with an oblique angle to the cone axis, an ellipse is obtained. If the plane is parallel to the cone axis, a hyperbola is generated. Conic sections contributed to the development of analytic geometry and calculus. In the 17th century, Galileo determined that the projectile path is parabolic and Kepler discovered that the planets follow elliptical orbits. Analytic geometry combines algebra and geometry and is based on the work of Descartes and Fermat. Geometric problems can be solved algebraically and algebraic problems can be solved geometrically. The geometric problem of conic sections can be reduced to an algebraic problem. Conic sections can be expressed as second-degree equations. Newton and Leibniz used the analytic geometry to develop calculus (Boyer, 2004; Das, 2013; Eddy, 2017; Larson, Hostetler, & Edwards, 2002; Newcomb, 2010; Serdarushich, 2015; Strang, 2010; Swokowski & Cole, 2008; Weir, Hass, & Thomas, 2010).

The first solution to the tangent line problem is attributed to Isaac Newton and Gottfried Leibniz. To determine the tangent line to the graph of a function is necessary the limit concept. The slope of the tangent line at a point P can be approximated using a secant line through P and other point on the curve. Approximations to the slope of the tangent line can be determined choosing a second point close to P. The limit of the secant line slope when the second point tends to P is the slope of the tangent line to the graph of the function at the point P. In this case, a tangent straight line to the graph of a function can be determined (Larson et al., 2002; Leithold, 1995; Palmer & Krathwohl, 2018; Tan, 2011).

A tangent line is the simplest curve. The geometric problem of the tangent line to the graph of a function can be extended to other tangent curves. The geometric problem of tangent curves as the conic sections to the graph of a function can also be solved. The tangent line to the graph of a function is a specific case of a tangent conic section since a line is a degenerate conic section. Therefore a tangent conic section is a more general solution as this contains also the geometric problem of the tangent line. Concepts related to the tangent line to the graph of a function can be extended using tangent conic sections, e.g. tangent conic sections can be used to approximate function values instead of tangent lines (Stewart, 2015; Swokowski, 1979). Under certain circumstances the approximation with tangent conic sections can be more accurate. The geometrical interpretation of some circle equations in the complex plane can be done using their equivalent real-valued function and their tangent conic sections.

A descriptive study was conducted integrating known concepts to obtain the tangent conic sections to the graph of a function and their derivatives. The content can be used as a pedagogical tool in the areas of science, engineering and mathematics since it shows an interdisciplinary approach combining general topics of analytic geometry, functional analysis and differential calculus. The article opens the possibility for new investigations that explore the applications of the tangent conic sections.

The results are applied in illustrative examples. The second-degree equations and the parametric equations for the first and second tangent conic sections (parabola, ellipse and hyperbola) to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$ are determined for different values of the parameters (focal distance p -tangent parabola-, semi-major axis a and semi-minor axis b -tangent ellipse or hyperbola-). The tangent and perpendicular lines to the graph of the function $f(x)$ at the point P are also shown as reference. The parametric equations for the tangent conic sections are used to generate the tangent conic sections graphs and derivatives. The limit of the tangent parabola when the focal length tends to infinite is calculated. The tangent line is obtained.

2 Tangent conic section to the graph of a function

The notation $y = \tilde{f}(x)$ is used for the tangent conic section to the graph of a function $y = f(x)$. Let $f(x)$ be a continuous and differentiable function, the tangent conic section to the graph of a function $y = f(x)$ at a point $P(x_0, f(x_0))$ is determined as the conic section $y = \tilde{f}(x)$ containing the point P as its vertex whose tangent line is equal to the tangent line of the function $y = f(x)$ at that point P. Thus, from the definition:

$$\begin{gathered}\tilde{f} \text{ is a conic section and its vertex is at the point } P(x_0, f(x_0)),\\ \tilde{f}(x_0) = f(x_0) \text{ and}\\ \tilde{f}'(x_0) = f'(x_0).\end{gathered}$$

3 Determination of the tangent conic sections to the graph of a function

The point P with coordinates $(x_0, f(x_0))$ and the point Q with coordinates $(x_0 + \triangle x, f(x_0 + \triangle x))$ are located on the graph of f. The line containing P and Q is a secant line with slope:

$$m_{\overline{PQ}} = \frac{\triangle y}{\triangle x} = \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} \tag{1}$$

The midpoint M of the line segment with endpoints $P(x_0, f(x_0))$ and $Q(x_0 + \triangle x, f(x_0 + \triangle x))$ is given by $M\left(\frac{2x_0+\triangle x}{2}, \frac{f(x_0)+f(x_0+\triangle x)}{2}\right)$.

The coordinate system $x'y'$ is translated. The translated axes remain parallel to the original cartesian coordinate system. The positive directions of the translated axes x' and y' are the same as the positive x and y directions. The origin O' of the translated coordinate system $x'y'$ is the midpoint of the line segment and has coordinates $M\left(\frac{2x_0+\triangle x}{2}, \frac{f(x_0)+f(x_0+\triangle x)}{2}\right)$ in the original system (Figure 1a). Any point has coordinates (x', y') with respect to the translated system and (x, y) in the original coordinate system, where:

$$x' = x - \frac{2x_0 + \triangle x}{2} \tag{2}$$

$$y' = y - \frac{f(x_0) + f(x_0 + \triangle x)}{2} \tag{3}$$

The translated coordinate system is then rotated so that the x''-axis is aligned to the secant line between the points P and Q. The coordinate system $x''y''$ is rotated with the angle θ relative to the coordinate system $x'y'$ (Figure 1b).

The angle θ can be represented as one of the angles of a right triangle (Figure 1c).

The triangle's hypotenuse is calculated using the Pythagorean theorem. The values of the sine and cosine can be determined from the right triangle.

$$\sin\theta = \frac{f(x_0 + \triangle x) - f(x_0)}{\sqrt{\triangle x^2 + [f(x_0 + \triangle x) - f(x_0)]^2}} = \frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{4}$$

$$\cos\theta = \frac{\triangle x}{\sqrt{\triangle x^2 + [f(x_0 + \triangle x) - f(x_0)]^2}} = \frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{5}$$

The rotation of axes formulas can be found in Leithold (1995). The coordinates in the rotated $x''\ y''$ and translated $x'\ y'$ systems are related by:

$$x'' = x'\cos\theta + y'\sin\theta \tag{6}$$

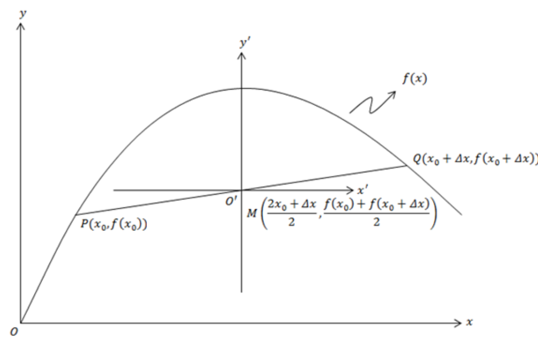
$$y'' = y'\cos\theta - x'\sin\theta \tag{7}$$

Using the values of the trigonometric functions of the angle θ (eq. 4 and 5) in the eq. 6 and 7.

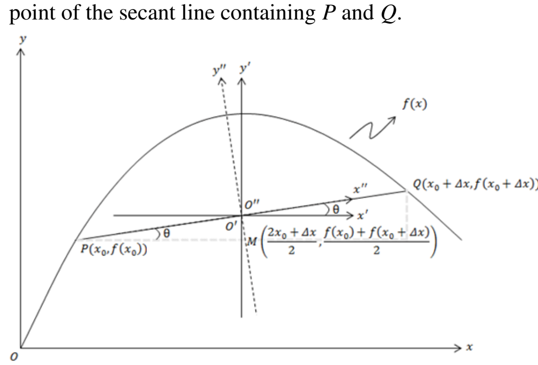
$$x'' = x'\frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} + y'\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{8}$$

$$y'' = y'\frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} - x'\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{9}$$

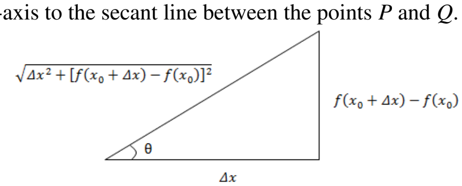
The coordinates in the rotated $x''\ y''$ and the original xy systems are related by (replacing eq. 2 and 3 in eq. 8 and 9):



(a) Translation of the coordinate system $x'y'$ to the midpoint of the secant line containing P and Q.



(b) Rotation of the coordinate system x'' y'' to align the x''-axis to the secant line between the points P and Q.



(c) The angle θ as one of the angles of a right triangle.

Figure 1: Translation and rotation of the coordinate system $x''y''$ to align the x''-axis to the secant line between the points P and Q.

Source: Own creation

$$x'' = \left(x - \frac{2x_0 + \triangle x}{2}\right)\frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x - f(x_0))}{\triangle x}\right]^2}} + \left(y - \frac{f(x_0) + f(x_0 + \triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{10}$$

$$y'' = \left(y - \frac{f(x_0) + f(x_0 + \triangle x)}{2}\right)\frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} - \left(x - \frac{2x_0 + \triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{11}$$

The y''-axis is aligned to the line perpendicular to the line segment with endpoints P and Q that intersects at the midpoint M since the y''-axis is perpendicular to the x''-axis. The slope of a line perpendicular to the line containing P and Q is its negative reciprocal:

$$m_\perp = \frac{\triangle x}{\triangle y} = -\frac{\triangle x}{f(x_0 + \triangle x) - f(x_0)} \tag{12}$$

The point-slope equation of a line is used to determine the perpendicular line to the segment $\overline{PQ}$ at the midpoint M.

$$y - \frac{f(x_0) + f(x_0 + \triangle x)}{2} = -\frac{\triangle x}{f(x_0 + \triangle x) - f(x_0)}\left(x - \frac{2x_0 + \triangle x}{2}\right) \tag{13}$$

Operating with the eq. 13:

$$y = -\frac{\triangle x}{f(x_0 + \triangle x) - f(x_0)}x + \left(\frac{\triangle x}{f(x_0 + \triangle x) - f(x_0)}\right)\left(\frac{2x_0 + \triangle x}{2}\right) + \frac{f(x_0) + f(x_0 + \triangle x)}{2} \tag{14}$$

Once the axes have been rotated, the conic sections in the coordinate system x'' y'' that pass through the origin O'' (and O') are established.

Parabola

The equations and parametric equations of the conic sections in Cartesian coordinates can be found in Swokowski y Cole (2008). A simple equation for a parabola is obtained if its vertex is placed at the origin O'', the focus is the point $F(0, p)$ (with respect to the coordinate system x'' y'') (Figure 2a) and the directrix has the equation $y'' = -p$ and is parallel to the x''-axis.

$$x''^2 = 4py'' \tag{15}$$

Obtaining y'' from the parabola equation (eq. 15):

$$y'' = \frac{1}{4p}x''^2; p \neq 0 \tag{16}$$

The derivative of the parabolic function is calculated (eq. 16):

$$\frac{dy''}{dx''} = \frac{1}{2p}x'' \tag{17}$$

If $x'' = 0$ in eq. 17:

$$\frac{dy''}{dx''} = 0 \tag{18}$$

The origin $O''(0,0)$ belongs to the graph of the parabola. The function $y'' = \frac{1}{4p}x''^2$ has a horizontal tangent line at the origin O'' and it is aligned to the x''-axis (and to the secant line between the points P and Q).

The equation of the parabola is expressed in terms of the original coordinate system xy (replacing eq. 10 and 11 in the eq.16):

$$\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}=$$

$$\frac{1}{4p}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2 \tag{19}$$

Since $y'' = g(x'')$, the following parametric equations of the parabola are obtained:

$$x'' = t \tag{20}$$

And $y'' = g(t)$:

$$y'' = \frac{1}{4p}t^2 \tag{21}$$

Where $-\infty < t < \infty$

Using the relation between the coordinates in the rotated $x''y''$ and the original xy systems (replacing eq. 10 and 11 in the eq. 20 and 21):

$$t=\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{22}$$

$$\frac{1}{4p}t^2=\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{23}$$

An alternative parabola can be obtained from the description of the tangent conic section to the graph of a function. In the first parabola, the rotation was done by an angle θ counterclockwise. If the coordinate system $x'''y'''$ is rotated π radians counterclockwise a second parabola can be obtained (Figure 2b).

The equation of the parabola in the coordinate system $x'''y'''$ is given by:

$$x'''^2 = 4py''' \tag{24}$$

The coordinates in the rotated $x'''y'''$ and the $x''y''$ systems are related by:

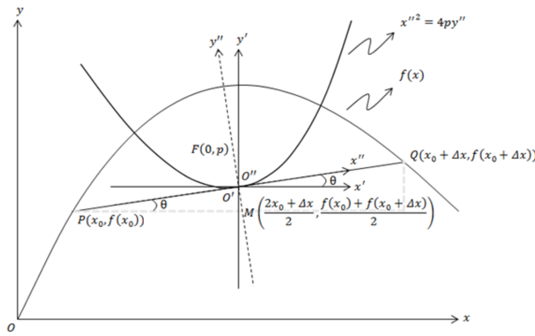
$$x''' = x''\cos(\pi) + y''\sin(\pi) = -x'' \tag{25}$$

$$y''' = y''\cos(\pi) - x''\sin(\pi) = -y'' \tag{26}$$

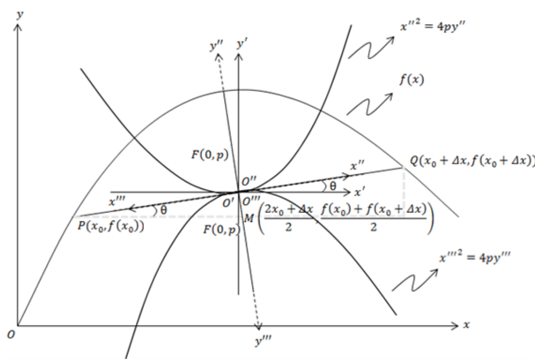
Replacing eq. 25 and 26 in the eq. 24:

$$(-x'')^2 = -4py'' \tag{27}$$

Obtaining y'' from the parabola equation (eq.27):



(a) Parabola $x''^2 = 4py''$ with its vertex placed at the origin O'' and focus at $F(0, p)$ (with respect to the coordinate system $x''y''$).



(b) Parabola $x'''^2 = 4py'''$ with its vertex placed at the origin O''' and focus at $F(0, p)$ (with respect to the coordinate system $x'''y'''$).

Figure 2: Parabolas $x''^2 = 4py''$ and $x'''^2 = 4py'''$.

Source: Own creation

$$y'' = -\frac{1}{4p}x''^2 \;; \quad p \neq 0 \tag{28}$$

The equation of the parabola is expressed in terms of the original coordinate system xy (replacing eq.10 and 11 in the eq.28):

$$\left(y - \frac{f(x_0) + f(x_0 + \triangle x)}{2}\right) \frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x) - f(x_0)}{\triangle x}\right]^2}} - \left(x - \frac{2x_0 + \triangle x}{2}\right) \frac{\frac{f(x_0+\triangle x) - f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x) - f(x_0)}{\triangle x}\right]^2}} =$$

$$-\frac{1}{4p}\left\{\left(x - \frac{2x_0 + \triangle x}{2}\right) \frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x) - f(x_0)}{\triangle x}\right]^2}} + \left(y - \frac{f(x_0) + f(x_0 + \triangle x)}{2}\right) \frac{\frac{f(x_0+\triangle x) - f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x) - f(x_0)}{\triangle x}\right]^2}}\right\}^2 \tag{29}$$

In this case, the parametric equations of the parabola are:

$$x''' = t \tag{30}$$

$$y''' = \frac{1}{4p}t^2 \tag{31}$$

Where $-\infty < t < \infty$

Using the relation between the coordinates in the rotated $x'''y'''$ and the $x''y''$ systems (replacing eq.25 and 26 in the eq.30 and 31):

$$x'' = -t \tag{32}$$

$$y'' = \frac{1}{4p}t^2 \tag{33}$$

Using the relation between the coordinates in the $x''y''$ and the original xy systems (replacing eq. 10 and 11 in the eq. 32 and 33):

$$-t = \left(x - \frac{2x_0 + \triangle x}{2}\right)\frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} + \left(y - \frac{f(x_0) + f(x_0 + \triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{34}$$

$$-\frac{1}{4p}t^2 = \left(y - \frac{f(x_0) + f(x_0 + \triangle x)}{2}\right)\frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} - \left(x - \frac{2x_0 + \triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{35}$$

Ellipse

In order to obtain a simple equation for an ellipse, the focus is placed on the x''-axis at the point $F(c, 0)$ (with respect to the coordinate system $x''y''$) and the directrix has the equation $x'' = d$ and is parallel to the y''-axis.

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1\ ; \quad a \geq b > 0 \tag{36}$$

If the ellipse with a horizontal major axis $\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1$ is translated along the y''-axis, the co-vertex can be placed at the origin O''. By interchanging x'' and y'' the ellipse with a vertical major axis and focus located on the y''-axis at the point $F(0, c)$ (with respect to the coordinate system $x''y''$) (Figure 3a) is obtained:

$$\frac{y''^2}{a^2} + \frac{x''^2}{b^2} = 1\ ; \quad a \geq b > 0 \tag{37}$$

If this ellipse is translated along the y''-axis at the point $(0, a)$, the vertex can be placed at the origin O'' (Figure 3b). The ellipse equation in the coordinate system $x'''y'''$ is given by:

$$\frac{y'''^2}{a^2} + \frac{x'''^2}{b^2} = 1\ ; \quad a \geq b > 0 \tag{38}$$

The relation between the coordinates (x''', y''') and the system (x'', y'') is given by:

$$x''' = x'' \tag{39}$$

$$y''' = y'' - a \tag{40}$$

Replacing in the ellipse equation (replacing eq.39 and 40 in the eq. 38):

$$\frac{(y'' - a)^2}{a^2} + \frac{x''^2}{b^2} = 1\ ; \quad a \geq b > 0 \tag{41}$$

Obtaining y'' from the ellipse equation (eq. 41):

$$(y'' - a)^2 = a^2 \left(1 - \frac{x''^2}{b^2}\right) \tag{42}$$

Operating with the result[1]:

$$y'' - a = a\sqrt{\frac{b^2 - x''^2}{b^2}} \tag{43}$$

$$y'' = a\left(1 + \frac{\sqrt{b^2 - x''^2}}{b}\right) \tag{44}$$

The derivative of the function of the ellipse is calculated (eq. 44):

$$\frac{dy''}{dx''} = \frac{a}{b}\left(b^2 - x''^2\right)^{-\frac{1}{2}}(-2x'') \tag{45}$$

Operating with the result:

$$\frac{dy''}{dx''} = -\frac{2ax''}{b\sqrt{b^2 - x''^2}} \tag{46}$$

If $x'' = 0$ in the eq.46:

$$\frac{dy''}{dx''} = 0 \tag{47}$$

Now, the origin $O''(0, 0)$ belongs to the graph of the ellipse. The function $y'' = a\left(1 + \frac{\sqrt{b^2 - x''^2}}{b}\right)$ has a horizontal tangent line at the origin O'' and it is aligned to the x''-axis (and to the secant line between the points P and Q).

Simplifying the ellipse expression (eq.41):

$$\frac{y''^2 - 2ay'' + a^2}{a^2} + \frac{x''^2}{b^2} = 1 \tag{48}$$

$$\frac{y''^2}{a^2} - \frac{2y''}{a} + \frac{x''^2}{b^2} = 0 \tag{49}$$

The equation of the ellipse is expressed in terms of the original coordinate system xy (replacing eq. 10 and 11 in the eq. 49):

$$\frac{1}{a^2}\left\{\left(y - \frac{f(x_0) + f(x_0 + \triangle x)}{2}\right)\frac{1}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} - \left(x - \frac{2x_0 + \triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1 + \left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2$$

[1] The calculation is done with the positive root $a\sqrt{\frac{b^2-x''2}{b^2}}$. With the negative root $-a\sqrt{\frac{b^2-x''2}{b^2}}$, the $\frac{dy''}{dx''}$ is also 0 at $x'' = 0$.

$$-\frac{2}{a}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}$$

$$+\frac{1}{b^2}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2=0 \quad (50)$$

The parametric equations of the ellipse are given by:

$$y''' = a\cos(t) \quad (51)$$

$$x''' = b\sin(t) \quad (52)$$

Where $0 \leq t \leq 2\pi$

Using the relation between the coordinates in the translated $x'''y'''$ and the rotated $x''y''$ systems (replacing eq.39 and 40 in the eq. 51 and 52):

$$y'' - a = a\cos(t) \quad (53)$$

$$x'' = b\sin(t) \quad (54)$$

Using the relation between the coordinates in the rotated $x''y''$ and the original xy systems (replacing eq. 10 and 11 in the eq. 53 and 54):

$$a[\cos(t)+1]=\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \quad (55)$$

$$b\sin(t)=\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \quad (56)$$

If the final ellipse (coordinate system $x'''y'''$) is rotated π radians respect to the coordinate system $x''y''$ a second ellipse can be obtained (Figure 3c).

The equation of the ellipse in the coordinate system $x^{iv}y^{iv}$ is given by:

$$\frac{(y^{iv}-a)^2}{a^2}+\frac{x^{iv2}}{b^2}=1\,; \quad a \geq b > 0 \quad (57)$$

The coordinates in the rotated $x^{iv}y^{iv}$ and the $x''y''$ systems are related by:

$$x^{iv} = x''\cos(\pi) + y''\sin(\pi) = -x'' \quad (58)$$

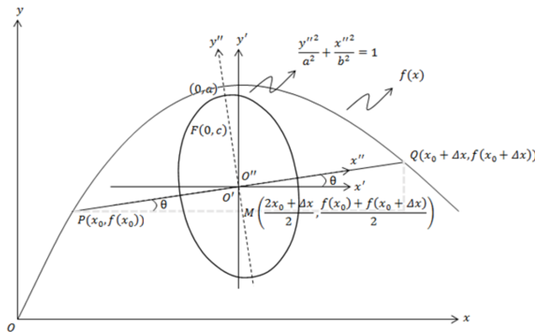
$$y^{iv} = y''\cos(\pi) - x''\sin(\pi) = -y'' \quad (59)$$

Replacing eq. 58 and 59 in the eq. 57:

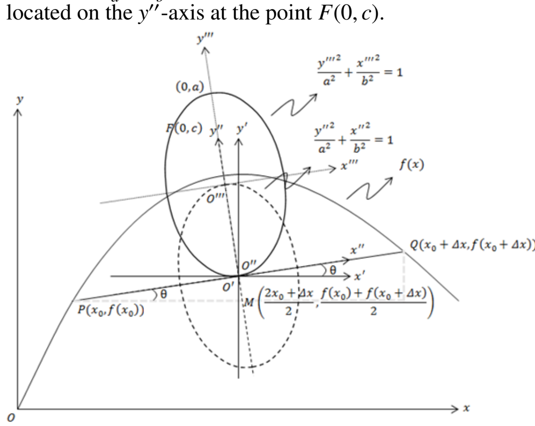
$$\frac{(-y''-a)^2}{a^2}+\frac{(-x'')^2}{b^2}=1\,; \quad a \geq b > 0 \quad (60)$$

Simplifying the ellipse expression (eq.60):

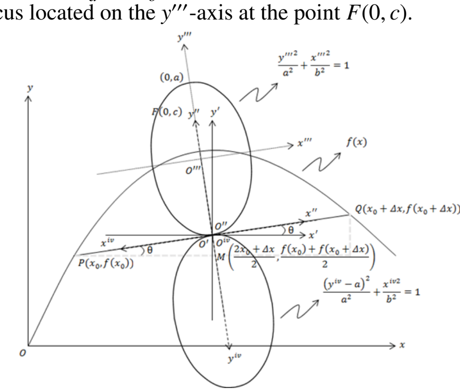
$$\frac{y''^2-2a(-y'')+a^2}{a^2}+\frac{x''^2}{b^2}=1 \quad (61)$$



(a) Ellipse $\frac{y''^2}{a^2}+\frac{x''^2}{b^2}=1$ with a vertical major axis and focus located on the y''-axis at the point $F(0,c)$.



(b) Ellipse $\frac{y'''^2}{a^2}+\frac{x'''^2}{b^2}=1$ with a vertical major axis and focus located on the y'''-axis at the point $F(0,c)$.



(c) Ellipse $\frac{(y^{iv}-a)^2}{a^2}+\frac{x^{iv2}}{b^2}=1$ with a vertical major axis located on the y^{iv}-axis.

Figure 3: Ellipses $\frac{y''^2}{a^2}+\frac{x''^2}{b^2}=1$, $\frac{y'''^2}{a^2}+\frac{x'''^2}{b^2}=1$ and $\frac{(y^{iv}-a)^2}{a^2}+\frac{x^{iv2}}{b^2}=1$.

Source: Own creation

$$\frac{y''^2}{a^2}+\frac{2y''}{a}+\frac{x''^2}{b^2}=0 \tag{62}$$

The equation of the ellipse is expressed in terms of the original coordinate system xy (replacing eq. 10 and 11 in the eq.62):

$$\frac{1}{a^2}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2$$
$$+\frac{2}{a}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}$$
$$+\frac{1}{b^2}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2=0 \tag{63}$$

In this case, the parametric equations of the ellipse are:

$$y^{iv}-a=a\cos(t) \tag{64}$$

$$y^{iv}=a\cos(t)+a \tag{65}$$

$$x^{iv}=b\sin(t) \tag{66}$$

Where $0\le t\le 2\pi$

Using the relation between the coordinates in the rotated $x^{iv}y^{iv}$ and the $x''y''$ systems (replacing eq. 58 and 59 in the eq. 65 and 66):

$$y''=-a\left[\cos(t)+1\right] \tag{67}$$

$$x''=-b\sin(t) \tag{68}$$

Using the relation between the coordinates in the rotated $x''y''$ and the original xy systems (replacing eq.10 and 11 in the eq. 67 and 68):

$$-a\left[\cos(t)+1\right]=\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{69}$$

$$-b\sin(t)=\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{70}$$

Hyperbola

In order to obtain a simple equation for a hyperbola, the focus is placed on the x''-axis at the point $F(c,0)$ (with respect to the coordinate system $x''y''$) and the directrix has the equation $x''=d$ and is parallel to the y''-axis.

$$\frac{x''^2}{a^2} - \frac{y''^2}{b^2} = 1 \tag{71}$$

If the hyperbola with a horizontal major axis $\frac{x''^2}{a^2} - \frac{y''^2}{b^2} = 1$ is translated along the y''-axis, none of the vertices of the hyperbola can't be placed at the origin O''. By interchanging x'' and y'' the hyperbola with a vertical major axis and focus located on the y''-axis at the point $F(0, c)$ (with respect to the coordinate system $x''y''$) (Figure 4a) is obtained:

$$\frac{y''^2}{a^2} - \frac{x''^2}{b^2} = 1 \tag{72}$$

If this hyperbola is translated along the $y''-$axis at the point $(0, -a)$, the vertex can be placed at the origin O'' (Figure 4b). The hyperbola equation in the coordinate system $x'''y'''$ is given by:

$$\frac{y'''^2}{a^2} - \frac{x'''^2}{b^2} = 1 \tag{73}$$

The relation between the coordinates (x''', y''') and the system (x'', y'') is given by:

$$x''' = x'' \tag{74}$$

$$y''' = y'' + a \tag{75}$$

Replacing in the hyperbola equation (replacing eq.74 and 75 in the eq.73):

$$\frac{(y'' + a)^2}{a^2} - \frac{x''^2}{b^2} = 1 \tag{76}$$

Obtaining y'' from the hyperbola equation (eq. 76):

$$(y'' + a)^2 = a^2\left(1 + \frac{x''^2}{b^2}\right) \tag{77}$$

Operating with the result:[2]

$$y'' + a = a\sqrt{\frac{b^2 + x''^2}{b^2}} \tag{78}$$

$$y'' = a\left(\frac{\sqrt{b^2 + x''^2}}{b^2} - 1\right) \tag{79}$$

The derivative of the function of the hyperbola is calculated (eq. 79):

$$\frac{dy''}{dx''} = \frac{a}{b}\left(b^2 + x''^2\right)^{-\frac{1}{2}}(2x'') \tag{80}$$

$$\frac{dy''}{dx''} = \frac{2ax''}{b\sqrt{b^2 + x''^2}} \tag{81}$$

If $x'' = 0$ in eq.81:

$$\frac{dy''}{dx''} = 0 \tag{82}$$

[2] The calculation is done with the positive root $a\sqrt{\frac{b^2+x''^2}{b^2}}$. With the negative root $-a\sqrt{\frac{b^2+x''^2}{b^2}}$, the $\frac{dy''}{dx''}$ is also 0 at $x'' = 0$.

Now, the origin $O''(0,0)$ belongs to the graph of the hyperbola. The function $y'' = a\left(\frac{\sqrt{b^2+x''^2}}{b} - 1\right)$ has a horizontal tangent line at the origin O'' and it is aligned to the $x''-$axis (and to the secant line between the points P and Q).

Simplifying the hyperbola expression (eq. 76):

$$\frac{y''^2 + 2ay'' + a^2}{a^2} - \frac{x''^2}{b^2} = 1 \tag{83}$$

$$\frac{y''^2}{a^2} + \frac{2y''}{a} - \frac{x''^2}{b^2} = 0 \tag{84}$$

The equation of the hyperbola is expressed in terms of the original coordinate system xy (replacing eq. 10 and 11 in the eq.84):

$$\begin{aligned}
&\frac{1}{a^2}\left\{\left(y - \frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} - \left(x - \frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2 \\
&+\frac{2}{a}\left\{\left(y - \frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} - \left(x - \frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \\
&-\frac{1}{b^2}\left\{\left(x - \frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} + \left(y - \frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2 = 0
\end{aligned} \tag{85}$$

The parametric equations of the hyperbola are given by:

$$y''' = a\sec(t) \tag{86}$$

$$x''' = b\tan(t) \tag{87}$$

Where $-^{\pi}/_{2} \leq t \leq {}^{3\pi}/_{2},\ t \neq {}^{\pi}/_{2}$

Using the relation between the coordinates in the translated $x'''y'''$ and the $x''y''$ systems (replacing eq. 74 and 75 in the eq.86 and 87):

$$y'' + a = a\sec(t) \tag{88}$$

$$x'' = b\tan(t) \tag{89}$$

Using the relation between the coordinates in the rotated $x''y''$ and the original xy systems (replacing eq.10 and 11 in the eq.88 and 89):

$$a[\sec(t)-1] = \left(y - \frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} - \left(x - \frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{90}$$

$$b\tan(t) = \left(x - \frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} + \left(y - \frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \tag{91}$$

If the final hyperbola (coordinate system $x'''y'''$) is rotated π radians respect to the coordinate system $x''y''$ a second hyperbola can be obtained (Figure 4c).

The equation of the hyperbola in the coordinate system $x^{iv}y^{iv}$ is given by:

$$\frac{(y^{iv}+a)^2}{a^2}-\frac{x^{iv2}}{b^2}=1 \tag{92}$$

The coordinates in the rotated $x^{iv}y^{iv}$ and the $x''y''$ systems are related by:

$$x^{iv}=x''\cos(\pi)+y''\sin(\pi)=-x'' \tag{93}$$

$$y^{iv}=y''\cos(\pi)-x''\sin(\pi)=-y'' \tag{94}$$

Replacing eq. 93 and 94 in the eq. 92:

$$\frac{(-y''+a)^2}{a^2}-\frac{(-x'')^2}{b^2}=1 \tag{95}$$

Simplifying the hyperbola expression (eq.95):

$$\frac{(-y''^2)+2a(-y'')+a^2}{a^2}-\frac{x''^2}{b^2}=1 \tag{96}$$

$$\frac{y''^2}{a^2}-\frac{2y''}{a}-\frac{x''^2}{b^2}=0 \tag{97}$$

The equation of the hyperbola is expressed in terms of the original coordinate system xy (replacing eq.10 and 11 in the eq. 97):

$$\begin{aligned}&\frac{1}{a^2}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2\\&-\frac{2}{a}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}\\&-\frac{1}{b^2}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2=0\end{aligned} \tag{98}$$

In this case, the parametric equations of the hyperbola are:

$$y^{iv}+a=a\sec(t) \tag{99}$$

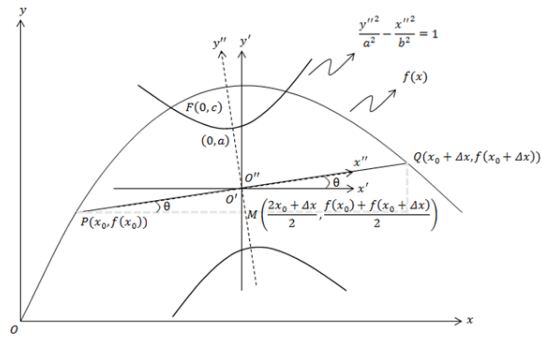
$$x^{iv}=b\tan(t) \tag{100}$$

Where $-^{\pi}/_2 \leq t \leq {}^{3\pi}/_2,\ t \neq {}^{\pi}/_2$

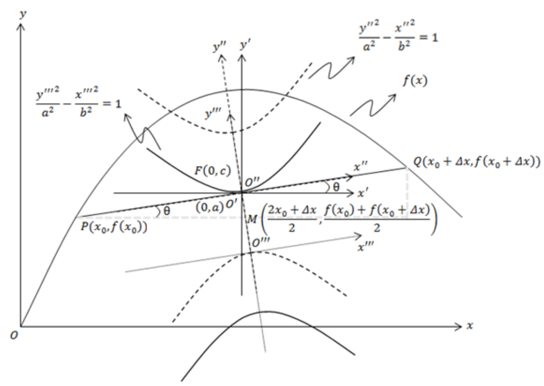
Using the relation between the coordinates in the rotated $x^{iv}y^{iv}$ and the $x''y''$ systems (replacing eq. 93 and 94 in the eq. 99 and 100):

$$y''=-a[\sec(t)-1] \tag{101}$$

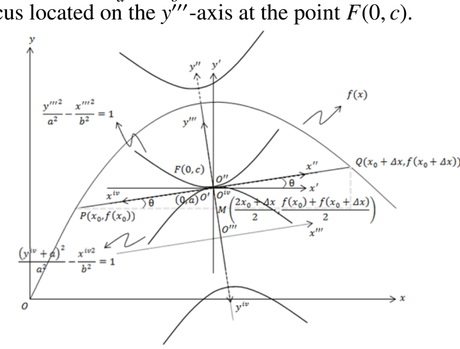
$$x''=-b\tan(t) \tag{102}$$



(a) Hyperbola $\frac{y''^2}{a^2} - \frac{x''^2}{b^2} = 1$ with a vertical major axis and focus located on the y''-axis at the point $F(0, c)$.



(b) Hyperbola $\frac{y'''^2}{a^2} - \frac{x'''^2}{b^2} = 1$ with a vertical major axis and focus located on the y'''-axis at the point $F(0, c)$.



(c) Hyperbola $\frac{(y^{iv}+a)^2}{a^2} - \frac{x^{iv2}}{b^2} = 1$ with a vertical major axis located on the y^{iv}−axis.

Figure 4: Hyperbolas $\frac{y''^2}{a^2} - \frac{x''^2}{b^2} = 1$, $\frac{y'''^2}{a^2} - \frac{x'''^2}{b^2} = 1$ and $\frac{(y^{iv}+a)^2}{a^2} - \frac{x^{iv2}}{b^2} = 1$.

Source: Own creation

Using the relation between the coordinates in the rotated $x''y''$ and the original xy systems (replacing eq. 10 and 11 in the eq. 101 and 102):

$$-a\left[\sec(t)-1\right]=\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \quad (103)$$

$$-b\tan(t)=\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} \quad (104)$$

The limit as $\triangle x \to 0$ is considered, the limit position of the secant line slope (eq. 1) is the tangent line slope at the point $P(x_0, f(x_0))$:

$$\lim_{\triangle x\to 0} m_{\overline{PQ}}=\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}=\lim_{\triangle x\to 0}\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]=f'(x_0) \quad (105)$$

The limiting value for the midpoint $M\left(\frac{2x_0+\triangle x}{2}, \frac{f(x_0)+f(x_0+\triangle x)}{2}\right)$ of the line segment with endpoints $P(x_0, f(x_0))$ and $Q(x_0+\triangle x, f(x_0+\triangle x))$ as $\triangle x \to 0$ is given by:

$$M\left(\lim_{\triangle x\to 0}\left[\frac{2x_0+\triangle x}{2}\right], \lim_{\triangle x\to 0}\left[\frac{f(x_0)+f(x_0+\triangle x)}{2}\right]\right) \quad (106)$$

$$M(x_0, f(x_0)) \quad (107)$$

The limit for the coordinate system translation (eq. 2 and 3):

$$\lim_{\triangle x\to 0} x'=\lim_{\triangle x\to 0}\left(x-\frac{2x_0+\triangle x}{2}\right) \quad (108)$$

$$x'=x-x_0 \quad (109)$$

$$\lim_{\triangle x\to 0} y'=\lim_{\triangle x\to 0}\left[y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right] \quad (110)$$

$$y'=y-f(x_0) \quad (111)$$

The limit for the values of the sine and cosine in the coordinate system rotation (eq. 4 and 5):

$$\lim_{\triangle x\to 0}\sin\theta=\lim_{\triangle x\to 0}\left\{\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \quad (112)$$

$$\sin\theta=\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \quad (113)$$

$$\lim_{\triangle x\to 0}\cos\theta=\lim_{\triangle x\to 0}\left\{\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \quad (114)$$

$$\cos\theta=\frac{1}{\sqrt{1+[f'(x_0)]^2}} \quad (115)$$

The limit for the relation between the coordinates in the rotated $x''y''$ and the original xy systems (eq. 10 and 11):

$$\lim_{\triangle x\to 0} x'' = \lim_{\triangle x\to 0}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \tag{116}$$

$$x'' = (x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{117}$$

$$\lim_{\triangle x\to 0} y'' = \lim_{\triangle x\to 0}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \tag{118}$$

$$y'' = [y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{119}$$

The limit for the equation of the perpendicular line to the segment $\overline{PQ}$ at the midpoint M (eq. 14):

$$\lim_{\triangle x\to 0} y = \lim_{\triangle x\to 0}\left\{-\frac{\triangle x}{f(x_0+\triangle x)-f(x_0)}x+\left(\frac{\triangle x}{f(x_0+\triangle x)-f(x_0)}\right)\left(\frac{2x_0+\triangle x}{2}\right)+\frac{f(x_0)+f(x_0+\triangle x)}{2}\right\} \tag{120}$$

$$y = -\frac{1}{f'(x_0)}x+\left[\frac{x_0}{f'(x_0)}+f(x_0)\right] \tag{121}$$

This is the equation of the perpendicular line to the tangent line at the point P. The Figures 2a, 2b, 3b, 3c, 4b, 4c show the secant line containing P and Q. When $\triangle x\to 0$, Q moves along the graph of the function f towards P. The limit position of this secant line is the tangent line to the graph of the function f at the point $P(x_0, f(x_0))$. Simultaneously, the limit of the midpoint M of the line segment with endpoints P and Q is the point $P(x_0, f(x_0))$ and the limiting conic section is tangent to the graph of the function f at the point P (Figures 5 - 7).

Parabola

The tangent parabola to the graph of the function f at the point P is the limiting parabola that passes through the origin located at the midpoint M of the line segment with endpoints P and Q as $\triangle x\to 0$ and the tangent line of the parabola is aligned to the tangent line of the function f at the point P (Figure 5).

The equation 122 is obtained taking the limit $\triangle x\to 0$ in the expression of the parabola (eq. 19):

$$\lim_{\triangle x\to 0}\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} =$$

$$\lim_{\triangle x\to 0}\frac{1}{4p}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2 \tag{122}$$

Thus:

$$[y - f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}} - (x - x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} =$$

$$\frac{1}{4p}\left\{(x - x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}} + [y - f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2 \tag{123}$$

The general second-degree equation is obtained:

$$\frac{1}{4p\{1+[f'(x_0)]^2\}}x^2 + \frac{f'(x_0)}{2p\{1+[f'(x_0)]^2\}}xy + \frac{[f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}}y^2$$
$$+\left\{\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} - \frac{x_0 + f(x_0)f'(x_0)}{2p\{1+[f'(x_0)]^2\}}\right\}x - \left\{\frac{f'(x_0)[x_0 + f(x_0)f'(x_0)]}{2p\{1+[f'(x_0)]^2\}} + \frac{1}{\sqrt{1+[f'(x_0)]^2}}\right\}y$$
$$+\frac{[x_0 + f(x_0)f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}} - \frac{f'(x_0)x_0 - f(x_0)}{\sqrt{1+[f'(x_0)]^2}} = 0 \tag{124}$$

Taking the limit $p \to \infty$ of the general second-degree equation of the tangent parabola (eq. 124):

$$\lim_{p\to\infty}\left\{\frac{1}{4p\{1+[f'(x_0)]^2\}}x^2 + \frac{f'(x_0)}{2p\{1+[f'(x_0)]^2\}}xy + \frac{[f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}}y^2\right.$$
$$+\left\{\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} - \frac{x_0 + f(x_0)f'(x_0)}{2p\{1+[f'(x_0)]^2\}}\right\}x - \left\{\frac{f'(x_0)[x_0 + f(x_0)f'(x_0)]}{2p\{1+[f'(x_0)]^2\}} + \frac{1}{\sqrt{1+[f'(x_0)]^2}}\right\}y$$
$$\left.+\frac{[x_0 + f(x_0)f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}} - \frac{f'(x_0)x_0 - f(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\} = \lim_{p\to\infty} 0 \tag{125}$$

Thus:

$$\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}x - \frac{1}{\sqrt{1+[f'(x_0)]^2}}y - \frac{f'(x_0)x_0 - f(x_0)}{\sqrt{1+[f'(x_0)]^2}} = 0 \tag{126}$$

Operating with the result (eq. 126):

$$y = f'(x_0)x - f'(x_0)x_0 + f(x_0) \tag{127}$$

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{128}$$

The tangent parabola as $p \to \infty$ is the tangent line to the graph of the function f at the point $P(x_0, f(x_0))$.

The equations 129 and 130 are obtained taking the limit $\triangle x \to 0$ in the parametric equations (eq. 22 and 23):

$$\lim_{\triangle x\to 0} t = \lim_{\triangle x\to 0}\left\{\left(x - \frac{2x_0 + \triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} + \left(y - \frac{f(x_0) + f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \tag{129}$$

$$\lim_{\triangle x\to 0}\frac{1}{4p}t^2 = \lim_{\triangle x\to 0}\left\{\left(y - \frac{f(x_0) + f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}} - \left(x - \frac{2x_0 + \triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \tag{130}$$

Thus:

$$t=(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{131}$$

$$\frac{1}{4p}t^2=[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{132}$$

Multiplying eq. 131 by $f'(x_0)$:

$$f'(x_0)t=(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{[f'(x_0)]^2}{\sqrt{1+[f'(x_0)]^2}} \tag{133}$$

Adding eq. 132 and 133:

$$\frac{1}{4p}t^2+f'(x_0)t=\frac{[y-f(x_0)]}{\sqrt{1+[f'(x_0)]^2}}\{1+[f'(x_0)]^2\} \tag{134}$$

Operating with the result (eq. 134):

$$y=\frac{\frac{1}{4p}t^2+f'(x_0)t}{\sqrt{1+[f'(x_0)]^2}}+f(x_0) \tag{135}$$

Replacing eq. 135 in eq. 131:

$$t=(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+\frac{\frac{1}{4p}t^2+f'(x_0)t}{\sqrt{1+[f'(x_0)]^2}}\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{136}$$

Operating with the result (eq. 136):

$$x=\sqrt{1+[f'(x_0)]^2}\left\{t-\frac{f'(x_0)\left[\frac{1}{4p}t^2+f'(x_0)t\right]}{1+[f'(x_0)]^2}\right\}+x_0 \tag{137}$$

Taking the limit $p\to\infty$ in the parametric equations (eq. 135 and 137):

$$\lim_{p\to\infty} y=\lim_{p\to\infty}\left\{\frac{\frac{1}{4p}t^2+f'(x_0)t}{\sqrt{1+[f'(x_0)]^2}}+f(x_0)\right\} \tag{138}$$

Thus:

$$y=\frac{f'(x_0)t}{\sqrt{1+[f'(x_0)]^2}}+f(x_0) \tag{139}$$

$$\lim_{p\to\infty} x=\lim_{p\to\infty}\left\{\sqrt{1+[f'(x_0)]^2}\left\{t-\frac{f'(x_0)\left[\frac{1}{4p}t^2+f'(x_0)t\right]}{1+[f'(x_0)]^2}\right\}+x_0\right\} \tag{140}$$

Thus:

$$x=\sqrt{1+[f'(x_0)]^2}\left\{t-\frac{[f'(x_0)]^2t}{1+[f'(x_0)]^2}\right\}+x_0 \tag{141}$$

Operating with the result (eq. 141):

$$x=\frac{t}{\sqrt{1+[f'(x_0)]^2}}+x_0 \tag{142}$$

Eliminating the parameter (t is obtained from the eq. 142):

$$(x-x_0)\sqrt{1+[f'(x_0)]^2}=t \tag{143}$$

Replacing t (eq. 143) in the eq. 139:

$$y=\frac{f'(x_0)(x-x_0)\sqrt{1+[f'(x_0)]^2}}{\sqrt{1+[f'(x_0)]^2}}+f(x_0) \tag{144}$$

Operating with the result (eq. 144):

$$y-f(x_0)=f'(x_0)(x-x_0) \tag{145}$$

The tangent parabola as $p\to\infty$ is also the tangent line to the graph of the function f at the point $P(x_0,f(x_0))$ using the parametric equations.

The equation 146 is obtained taking the limit $\triangle x\to\infty$ of the second tangent parabola expression (eq. 29) (Figure 5b).

$$\lim_{\triangle x\to 0}\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}$$

$$\lim_{\triangle x\to 0}-\frac{1}{4p}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2 \tag{146}$$

Thus:

$$[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}=$$
$$-\frac{1}{4p}\left\{(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2 \tag{147}$$

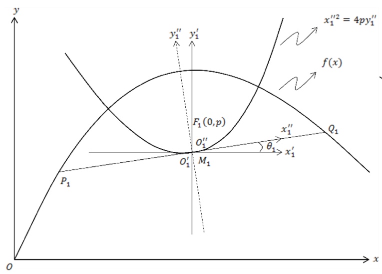
The general second-degree equation is obtained:

$$-\frac{1}{4p\{1+[f'(x_0)]^2\}}x^2-\frac{f'(x_0)}{2p\{1+[f'(x_0)]^2\}}xy-\frac{[f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}}y^2$$
$$+\left\{\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}+\frac{x_0+f(x_0)f'(x_0)}{2p\{1+[f'(x_0)]^2\}}\right\}x+\left\{\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{2p\{1+[f'(x_0)]^2\}}-\frac{1}{\sqrt{1+[f'(x_0)]^2}}\right\}y$$
$$-\frac{[x_0+f(x_0)f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}}-\frac{f'(x_0)x_0-f(x_0)}{\sqrt{1+[f'(x_0)]^2}}=0 \tag{148}$$

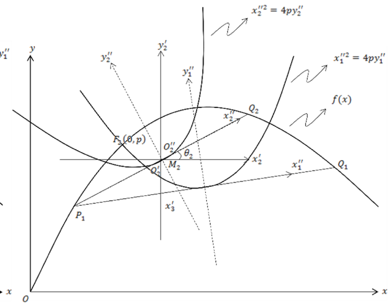
The equations 149 and 150 are obtained taking the limit $\triangle x\to 0$ in the parametric equations (eq. 34 and 35):

$$\lim_{\triangle x\to 0}-t=\lim_{\triangle x\to 0}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \tag{149}$$

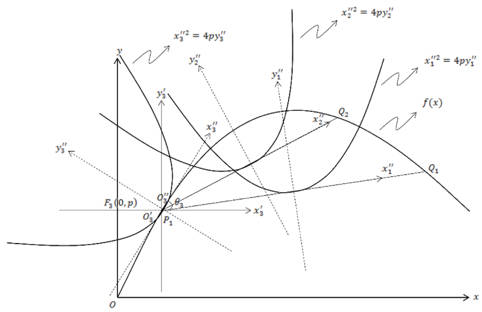
$$\lim_{\triangle x\to 0}-\frac{1}{4p}t^2=\lim_{\triangle x\to 0}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \tag{150}$$



(a. 1)

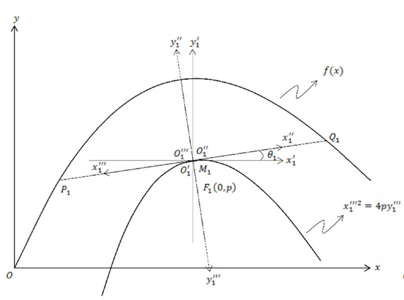


(a. 2)

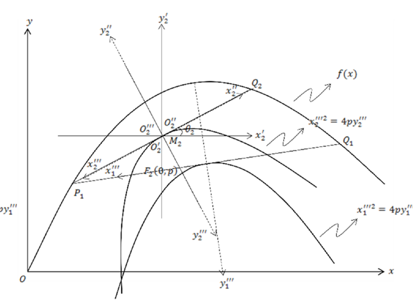


(a. 3)

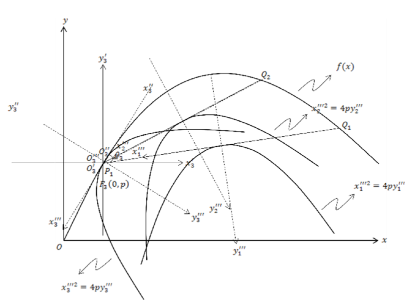
(a) First limiting parabola tangent to the graph of the function f at the point P.



(b. 1)



(b. 2)



(b. 3)

(b) Second limiting parabola tangent to the graph of the function f at the point P.

Figure 5: First and second limiting parabola tangent to the graph of the function f at the point P.

Source: Own creation

Thus:

$$-t=(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{151}$$

$$-\frac{1}{4p}t^2=[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{152}$$

Solving for x and y in the eq. 151 and 152:

$$y=\frac{-\frac{1}{4p}t^2-f'(x_0)t}{\sqrt{1+[f'(x_0)]^2}}+f(x_0) \tag{153}$$

$$x=\sqrt{1+[f'(x_0)]^2}\left\{\frac{f'(x_0)\left[\frac{1}{4p}t^2+f'(x_0)t\right]}{1+[f'(x_0)]^2}-t\right\}+x_0 \tag{154}$$

The tangent line to the graph of the function f at the point $P(x_0, f(x_0))$ is also obtained with the equations of the second tangent parabola as $p \to \infty$.

Ellipse

The tangent ellipse to the graph of the function f at the point P is the limiting ellipse that passes through the origin located at the midpoint M of the line segment with endpoints P and Q as $\triangle x \to 0$ and the tangent line of the ellipse is aligned to the tangent line of the function f at the point P (Figure 6).

The equation 155 is obtained taking the limit $\triangle x \to 0$ in the expression of the tangent ellipse (eq. 50):

$$\lim_{\triangle x\to 0}\frac{1}{a^2}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2$$

$$-\frac{2}{a}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}$$

$$+\frac{1}{b^2}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2=\lim_{\triangle x\to 0}0 \tag{155}$$

Thus:

$$\frac{1}{a^2}\left\{[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2$$

$$-\frac{2}{a}\left\{[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}$$

$$+\frac{1}{b^2}\left\{(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2=0 \tag{156}$$

The general second-degree equation is obtained:

$$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{[f'(x_0)]^2}{a^2}+\frac{1}{b^2}\right\}x^2+\frac{2f'(x_0)}{1+[f'(x_0)]^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)xy+\frac{1}{1+[f'(x_0)]^2}\left\{\frac{1}{a^2}+\frac{[f'(x_0)]^2}{b^2}\right\}y^2$$
$$+\left\{\frac{2}{1+[f'(x_0)]^2}\left[\frac{f'(x_0)[f(x_0)-x_0f'(x_0)]}{a^2}-\frac{x_0+f(x_0)f'(x_0)}{b^2}\right]+\frac{2f'(x_0)}{a\sqrt{1+[f'(x_0)]^2}}\right\}x$$
$$+\left\{-\frac{2}{1+[f'(x_0)]^2}\left[\frac{f(x_0)-x_0f'(x_0)}{a^2}+\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{b^2}\right]-\frac{2}{a\sqrt{1+[f'(x_0)]^2}}\right\}y$$
$$+\left\{\frac{1}{1+[f'(x_0)]^2}\left[\frac{[f(x_0)-x_0f'(x_0)]^2}{a^2}+\frac{[x_0+f(x_0)f'(x_0)]^2}{b^2}\right]-\frac{2[f'(x_0)x_0-f(x_0)]}{a\sqrt{1+[f'(x_0)]^2}}\right\}=0 \tag{157}$$

The equations 158 and 159 are obtained taking the limit $\triangle x\to 0$ in the parametric equations (eq. 55 and 56):

$$\lim_{\triangle x\to 0}\{a[\cos(t)+1]\}=\lim_{\triangle x\to 0}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \tag{158}$$

$$\lim_{\triangle x\to 0}\{b\sin(t)\}=\lim_{\triangle x\to 0}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \tag{159}$$

Thus:

$$a[\cos(t)+1]=[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{160}$$

$$b\sin(t)=(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{161}$$

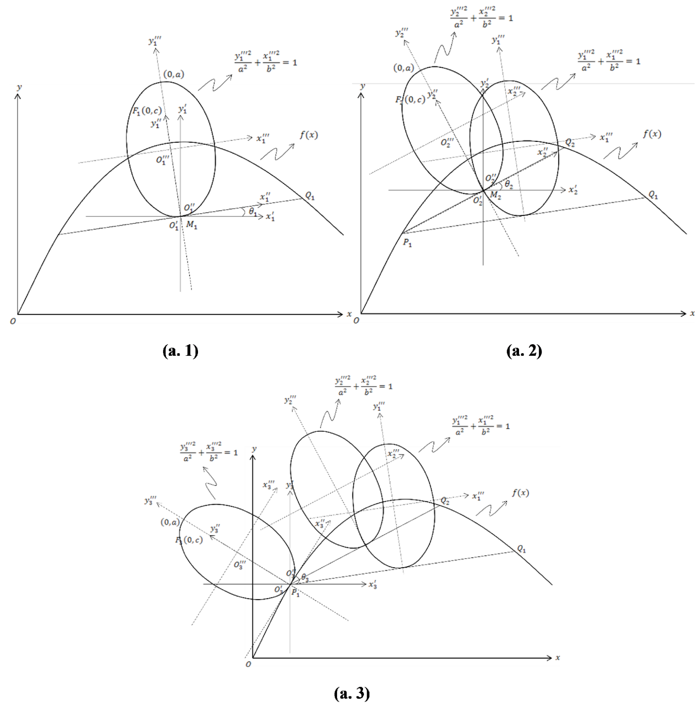
Solving for x and y in the eq. 160 and 161:

$$y=\frac{f'(x_0)b\sin(t)+a[\cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}}+f(x_0) \tag{162}$$

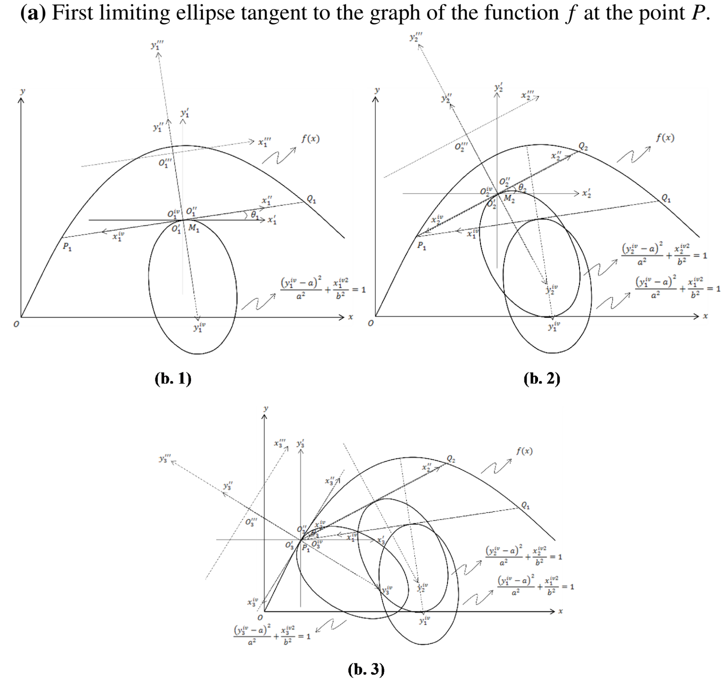
$$x=\frac{b\sin(t)-f'(x_0)a[\cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}}+x_0 \tag{163}$$

The equation 164 is obtained taking the limit $\triangle x\to 0$ of the second tangent ellipse expression (eq. 63) (Figure 6b).

$$\lim_{\triangle x\to 0}\frac{1}{a^2}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2$$
$$+\frac{2}{a}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}$$



(a) First limiting ellipse tangent to the graph of the function f at the point P.



(b) Second limiting ellipse tangent to the graph of the function f at the point P.

Figure 6: First and second limiting ellipse tangent to the graph of the function f at the point P.

Source: Own creation

$$+\frac{1}{b^2}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2=\lim_{\triangle x\to 0} 0 \quad (164)$$

Thus:

$$\frac{1}{a^2}\left\{[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2$$
$$+\frac{2}{a}\left\{[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}$$
$$+\frac{1}{b^2}\left\{(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2=0 \quad (165)$$

The general second-degree equation is obtained:

$$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{[f'(x_0)]^2}{a^2}+\frac{1}{b^2}\right\}x^2+\frac{2f'(x_0)}{1+[f'(x_0)]^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)xy+\frac{1}{1+[f'(x_0)]^2}\left\{\frac{1}{a^2}+\frac{[f'(x_0)]^2}{b^2}\right\}y^2$$
$$+\left\{\frac{2}{1+[f'(x_0)]^2}\left[\frac{f'(x_0)[f(x_0)-x_0f'(x_0)]}{a^2}-\frac{x_0+f(x_0)f'(x_0)}{b^2}\right]-\frac{2f'(x_0)}{a\sqrt{1+[f'(x_0)]^2}}\right\}x$$
$$+\left\{-\frac{2}{1+[f'(x_0)]^2}\left[\frac{f(x_0)-x_0f'(x_0)}{a^2}-\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{b^2}\right]+\frac{2}{a\sqrt{1+[f'(x_0)]^2}}\right\}y$$
$$+\left\{\frac{1}{1+[f'(x_0)]^2}\left[\frac{[f(x_0)-x_0f'(x_0)]^2}{a^2}+\frac{[x_0+f(x_0)f'(x_0)]^2}{b^2}\right]+\frac{2[f'(x_0)x_0-f(x_0)]}{a\sqrt{1+[f'(x_0)]^2}}\right\}=0 \quad (166)$$

The equations 167 and 168 are obtained taking the limit $\triangle x\to 0$ in the parametric equations (eq. 69 and 70):

$$\lim_{\triangle x\to 0}\{-a[\cos(t)+1]\}=\lim_{\triangle x\to 0}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \quad (167)$$

$$\lim_{\triangle x\to 0}\{-b\sin(t)\}=\lim_{\triangle x\to 0}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \quad (168)$$

Thus:

$$-a[\cos(t)+1]=[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \quad (169)$$

$$-b\sin(t)=(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \quad (170)$$

Solving for x and y in the eq.169 and 170:

$$y=\frac{-f'(x_0)b\sin(t)-a[\cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}}+f(x_0) \quad (171)$$

$$x=\frac{-b\sin(t)+f'(x_0)a[\cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}}+x_0 \quad (172)$$

Hyperbola

The tangent hyperbola to the graph of the function f at the point P is the limiting hyperbola that passes through the origin located at the midpoint M of the line segment with endpoints P and Q as $\triangle x \to 0$ and the tangent line of the hyperbola is aligned to the tangent line of the function f at the point P (Figure 7).

The equation 173 is obtained taking the limit $\triangle x \to 0$ in the expression of the tangent hyperbola (eq. 85):

$$\lim_{\triangle x\to 0} \frac{1}{a^2}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2$$
$$+\frac{2}{a}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}$$
$$-\frac{1}{b^2}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2=\lim_{\triangle x\to 0} 0 \quad (173)$$

Thus:

$$\frac{1}{a^2}\left\{[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2$$
$$+\frac{2}{a}\left\{[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}$$
$$-\frac{1}{b^2}\left\{(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2=0 \quad (174)$$

The general second-degree equation is obtained:

$$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{[f'(x_0)]^2}{a^2}-\frac{1}{b^2}\right\}x^2-\frac{2f'(x_0)}{1+[f'(x_0)]^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)xy+\frac{1}{1+[f'(x_0)]^2}\left\{\frac{1}{a^2}-\frac{[f'(x_0)]^2}{b^2}\right\}y^2$$
$$+\left\{\frac{2}{1+[f'(x_0)]^2}\left[\frac{f'(x_0)[f(x_0)-x_0f'(x_0)]}{a^2}+\frac{x_0+f(x_0)f'(x_0)}{b^2}\right]-\frac{2f'(x_0)}{a\sqrt{1+[f'(x_0)]^2}}\right\}x$$
$$+\left\{-\frac{2}{1+[f'(x_0)]^2}\left[\frac{f(x_0)-x_0f'(x_0)}{a^2}-\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{b^2}\right]+\frac{2}{a\sqrt{1+[f'(x_0)]^2}}\right\}y$$
$$+\left\{\frac{1}{1+[f'(x_0)]^2}\left[\frac{[f(x_0)-x_0f'(x_0)]^2}{a^2}-\frac{[x_0+f(x_0)f'(x_0)]^2}{b^2}\right]+\frac{2[f'(x_0)x_0-f(x_0)]}{a\sqrt{1+[f'(x_0)]^2}}\right\}=0 \quad (175)$$

The equations 176 and 177 are obtained taking the limit $\triangle x \to 0$ in the parametric equations (eq. 90 and 91):

$$\lim_{\triangle x\to 0}\{a[\sec(t)-1]\}=\lim_{\triangle x\to 0}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \quad (176)$$

$$\lim_{\triangle x\to 0}\{b\tan(t)\} = \lim_{\triangle x\to 0}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \tag{177}$$

Thus:

$$a[\sec(t)-1] = [y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}} - (x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{178}$$

$$b\tan(t) = (x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}} + [y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \tag{179}$$

Solving for x and y in the eq. 178 and 179:

$$y = \frac{f'(x_0)b\tan(t)+a[\sec(t)-1]}{\sqrt{1+[f'(x_0)]^2}} + f(x_0) \tag{180}$$

$$x = \frac{b\tan(t)-f'(x_0)a[\sec(t)-1]}{\sqrt{1+[f'(x_0)]^2}} + x_0 \tag{181}$$

The equation 182 is obtained taking the limit $\triangle x \to 0$ of the second tangent hyperbola expression (eq. 98) (Figure 7b):

$$\lim_{\triangle x\to 0}\frac{1}{a^2}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2$$

$$-\frac{2}{a}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}$$

$$-\frac{1}{b^2}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\}^2 = \lim_{\triangle x\to 0} 0 \tag{182}$$

Thus:

$$\frac{1}{a^2}\left\{[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}} - (x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2$$

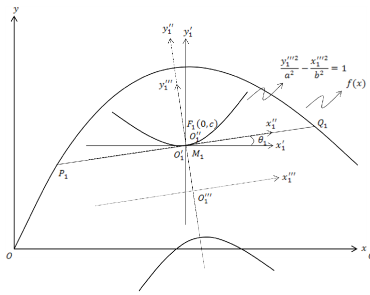
$$-\frac{2}{a}\left\{[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}} - (x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}$$

$$-\frac{1}{b^2}\left\{(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}} + [y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2 = 0 \tag{183}$$

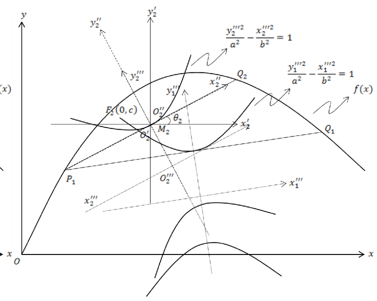
The general second-degree equation is obtained:

$$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{[f'(x_0)]^2}{a^2}-\frac{1}{b^2}\right\}x^2 - \frac{2f'(x_0)}{1+[f'(x_0)]^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)xy + \frac{1}{1+[f'(x_0)]^2}\left\{\frac{1}{a^2}-\frac{[f'(x_0)]^2}{b^2}\right\}y^2$$

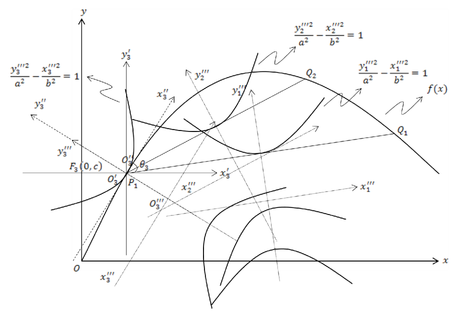
$$+\left\{\frac{2}{1+[f'(x_0)]^2}\left[\frac{f'(x_0)[f(x_0)-x_0f'(x_0)]}{a^2}+\frac{x_0+f(x_0)f'(x_0)}{b^2}\right]+\frac{2f'(x_0)}{a\sqrt{1+[f'(x_0)]^2}}\right\}x$$



(a. 1)

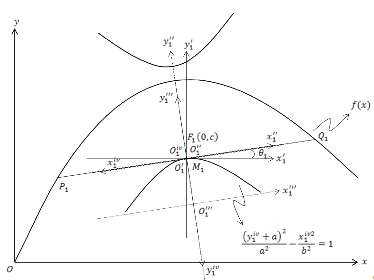


(a. 2)

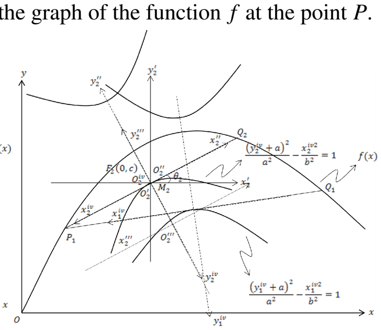


(a. 3)

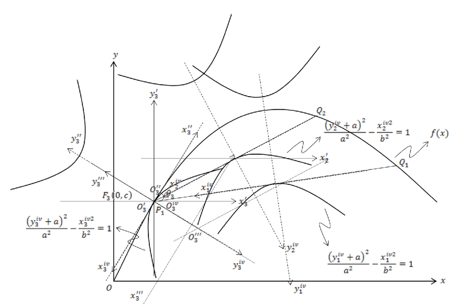
(a) First limiting hyperbola tangent to the graph of the function f at the point P.



(b. 1)



(b. 2)



(b. 3)

(b) Second limiting hyperbola tangent to the graph of the function f at the point P.

Figure 7: First and second limiting hyperbola tangent to the graph of the function f at the point P.

Source: Own creation

$$+\left\{-\frac{2}{1+[f'(x_0)]^2}\left[\frac{f(x_0)-x_0f'(x_0)}{a^2}-\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{b^2}\right]-\frac{2}{a\sqrt{1+[f'(x_0)]^2}}\right\}y$$

$$+\left\{\frac{1}{1+[f'(x_0)]^2}\left[\frac{[f(x_0)-x_0f'(x_0)]^2}{a^2}-\frac{[x_0+f(x_0)f'(x_0)]^2}{b^2}\right]-\frac{2[f'(x_0)x_0-f(x_0)]}{a\sqrt{1+[f'(x_0)]^2}}\right\}=0 \quad (184)$$

The equations 185 and 186 are obtained taking the limit $\triangle x \to 0$ in the parametric equations (eq. 103 and 104):

$$\lim_{\triangle x\to 0}\{-a[\sec(t)-1]\}=\lim_{\triangle x\to 0}\left\{\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}-\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \quad (185)$$

$$\lim_{\triangle x\to 0}\{-b\tan(t)\}=\lim_{\triangle x\to 0}\left\{\left(x-\frac{2x_0+\triangle x}{2}\right)\frac{1}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}+\left(y-\frac{f(x_0)+f(x_0+\triangle x)}{2}\right)\frac{\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}}{\sqrt{1+\left[\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\right]^2}}\right\} \quad (186)$$

Thus:

$$-a[\sec(t)-1]=[y-f(x_0)]\frac{1}{\sqrt{1+[f'(x_0)]^2}}-(x-x_0)\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \quad (187)$$

$$-b\tan(t)=(x-x_0)\frac{1}{\sqrt{1+[f'(x_0)]^2}}+[y-f(x_0)]\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}} \quad (188)$$

Solving for x and y in the eq. 187 and 188:

$$y=\frac{-f'(x_0)b\tan(t)-a[\sec(t)-1]}{\sqrt{1+[f'(x_0)]^2}}+f(x_0) \quad (189)$$

$$x=\frac{-b\tan(t)+f'(x_0)a[\sec(t)-1]}{\sqrt{1+[f'(x_0)]^2}}+x_0 \quad (190)$$

Equations of conic sections can be written using the general second-degree equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. The coefficients of the second-degree equation for the first and second tangent conic sections to the graph of a function $y=f(x)$ at a point $P(x_0, f(x_0))$ are listed in the Table A1 and Table A2 (Appendix A). The parametric equations are listed in the Table B1 (Appendix B).

The tangent conic sections to the graph of a function were obtained using the translation of the initial coordinate system to the midpoint of the line segment with endpoints P and Q (Figure 1a) but the coordinate system could have been translated to any point $R(x_r, y_r)$ on the line segment $\overline{PQ}$. Any point $R(x_r, y_r)$ is given by the parametric equations (parametric equations can be found in Swokowski y Cole (2008)):

$$x_r=tx_0+(1-t)(x_0+\triangle x) \quad (191)$$

$$y_r=tf(x_0)+(1-t)[f(x_0+\triangle x)] \quad (192)$$

Where $t\in[0,1]$

If $t={}^1/_2$, the point $R(x_r, y_r)$ is the midpoint $M\left(\frac{2x_0+\triangle x}{2}, \frac{f(x_0)+f(x_0+\triangle x)}{2}\right)$ of the line segment $\overline{PQ}$. The limiting value for the point $R(x_r, y_r)$ as $\triangle x\to 0$ is given by:

$$\lim_{\triangle x\to 0}x_r=\lim_{\triangle x\to 0}[tx_0+(1-t)(x_0+\triangle x)] \quad (193)$$

$$x_r = x_0 \tag{194}$$

$$\lim_{\triangle x \to 0} y_r = \lim_{\triangle x \to 0} \{t f(x_0) + (1-t)[f(x_0 + \triangle x]\} \tag{195}$$

$$y_r = f(x_0) \tag{196}$$

The limiting value for any point $R(x_r, y_r)$ as $\triangle x \to 0$ is $R(x_0, f(x_0))$. Thus, the limiting value and the subsequent results are the same taking the midpoint or any point of the line segment $\overline{PQ}$.

4 The derivatives of the tangent conic sections to the graph of a function

The second-degree equation of a conic section is an implicitly defined function. The parametric equations are used to obtain the derivatives. The derivative of the function defined by parametric equations is given by (the derivative of a parametrically defined curve can be found in Leithold (1995)):

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0 \tag{197}$$

The expressions for the derivatives of the tangent conic sections are determined:

Parabola

The derivatives of the parametric equations (eq. 135 and 137) for the first tangent parabola are given by:

$$\frac{dy}{dt} = \frac{\frac{1}{2p}t + f'(x_0)}{\sqrt{1 + [f'(x_0)]^2}} \tag{198}$$

$$\frac{dx}{dt} = \sqrt{1 + [f'(x_0)]^2} - \frac{f'(x_0)\left[\frac{1}{2p}t + f'(x_0)\right]}{\sqrt{1 + [f'(x_0)]^2}} \tag{199}$$

Determining $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{\frac{1}{2p}t + f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}}{\frac{1+[f'(x_0)]^2 - f'(x_0)\left[\frac{1}{2p}t + f'(x_0)\right]}{\sqrt{1+[f'(x_0)]^2}}} \tag{200}$$

Operating with the result:

$$\frac{dy}{dx} = \frac{\frac{1}{2p}t + f'(x_0)}{1 - \frac{f'(x_0)}{2p}t} \tag{201}$$

$$\frac{dy}{dx} = \frac{\frac{t + 2pf'(x_0)}{2p}}{\frac{2p - f'(x_0)t}{2p}} \tag{202}$$

$$\frac{dy}{dx} = \frac{t + 2pf'(x_0)}{2p - f'(x_0)t}; \quad t \neq \frac{2p}{f'(x_0)} \tag{203}$$

The limit $p \to \infty$ of the derivative is determined:

$$\lim_{p\to\infty} \frac{dy}{dx} = \lim_{p\to\infty} \frac{t + 2pf'(x_0)}{2p - f'(x_0)t} = \lim_{p\to\infty} \frac{\frac{t}{p} + 2f'(x_0)}{2 - \frac{f'(x_0)t}{p}} = f'(x_0) \tag{204}$$

The limit of the first tangent parabola derivative as $p \to \infty$ is $f'(x_0)$ (the tangent line slope at the point $P(x_0, f(x_0))$).

Ellipse

The derivatives of the parametric equations (eq. 162 and 163) for the first tangent ellipse are given by:

$$\frac{dy}{dt} = \frac{f'(x_0)b\cos(t) - a\sin(t)}{\sqrt{1 + [f'(x_0)]^2}} \tag{205}$$

$$\frac{dx}{dt} = \frac{b\cos(t) + f'(x_0)a\sin(t)}{\sqrt{1 + [f'(x_0)]^2}} \tag{206}$$

Determining $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{f'(x_0)b\cos(t) - a\sin(t)}{\sqrt{1+[f'(x_0)]^2}}}{\frac{b\cos(t) + f'(x_0)a\sin(t)}{\sqrt{1+[f'(x_0)]^2}}} \tag{207}$$

Operating with the result:

$$\frac{dy}{dx} = \frac{f'(x_0)b\cos(t) - a\sin(t)}{b\cos(t) + f'(x_0)a\sin(t)}; \quad \sin^2 t \neq \frac{b^2}{a^2[f'(x_0)]^2 + b^2} \tag{208}$$

Hyperbola

The derivatives of the parametric equations (eq. 180 and 181) for the first tangent hyperbola are given by:

$$\frac{dy}{dt} = \frac{f'(x_0)b\sec^2(t) + a\sec(t)\tan(t)}{\sqrt{1 + [f'(x_0)]^2}} = \frac{\sec(t)[f'(x_0)b\sec(t) + a\tan(t)]}{\sqrt{1 + [f'(x_0)]^2}} \tag{209}$$

$$\frac{dx}{dt} = \frac{b\sec^2(t) - f'(x_0)a\sec(t)\tan(t)}{\sqrt{1 + [f'(x_0)]^2}} = \frac{\sec(t)[b\sec(t) - f'(x_0)a\tan(t)]}{\sqrt{1 + [f'(x_0)]^2}} \tag{210}$$

Determining $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{\sec(t)[f'(x_0)b\sec(t)+a\tan(t)]}{\sqrt{1+[f'(x_0)]^2}}}{\frac{\sec(t)[b\sec(t)-f'(x_0)a\tan(t)]}{\sqrt{1+[f'(x_0)]^2}}} \tag{211}$$

Operating with the result:

$$\frac{dy}{dx} = \frac{f'(x_0)b\sec(t) + a\tan(t)}{b\sec(t) - f'(x_0)a\tan(t)}; \quad \tan^2(t) \neq \frac{b^2}{a^2[f'(x_0)]^2 - b^2} \tag{212}$$

The derivatives for the second tangent conic sections are the same as for the first tangent conic sections. The derivatives for the first and second tangent conic sections to the graph of a function f at the point $P(x_0, f(x_0))$ are listed in the Table C1 (Appendix C). Illustrative examples are shown in the Appendix D (page 40) .

5 Conclusions

The geometric problem of the tangent straight line can be extended to other tangent curves. The tangent conic sections, namely tangent parabola, ellipse and hyperbola are obtained. The traditional tangent straight line is a specific case of a tangent conic section. The limit of the tangent parabola when the focal length tends to infinite is a tangent line. The tangent conic section to the graph of a function at a point P is determined as the conic section containing the point P as its vertex whose tangent line is equal to the tangent line of the function at that point P. There are two alternatives that satisfy the previous definition, they are called here the first and second tangent conic section. Graphically, the first tangent conic section situates “upward” and the second situates “downward” relative to the tangent straight line. The parametric equations are used to obtain the derivatives easily and they are useful in the tangent conic sections plotting since the second-degree equation of the tangent conic sections is an implicitly defined function. The results of the second-degree equations and parametric equations for the first and second tangent conic sections to the graph of a function f at the point $P(x_0, f(x_0))$ as well as their derivatives are summarized. Examples of the tangent parabola, ellipse and hyperbola to the graph of a function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$ are calculated.

6 Bibliography

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Appendix A. Coefficients of the second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Table A 1: Coefficients of the second-degree equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ for the first tangent conic sections to the graph of a function f at the point $P(x_0, f(x_0))$

	Parabola	Ellipse	Hyperbola
A	$\frac{1}{4p\{1+[f'(x_0)]^2\}}$	$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{[f'(x_0)]^2}{a^2}+\frac{1}{b^2}\right\}$	$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{[f'(x_0)]^2}{a^2}-\frac{1}{b^2}\right\}$
B	$\frac{f'(x_0)}{2p\{1+[f'(x_0)]^2\}}$	$\frac{2f'(x_0)}{1+[f'(x_0)]^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)$	$-\frac{2f'(x_0)}{1+[f'(x_0)]^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$
C	$\frac{[f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}}$	$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{1}{a^2}+\frac{[f'(x_0)]^2}{b^2}\right\}$	$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{1}{a^2}-\frac{[f'(x_0)]^2}{b^2}\right\}$
D	$\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}-\frac{x_0+f(x_0)f'(x_0)}{2p\{1+[f'(x_0)]^2\}}$	$\frac{2}{1+[f'(x_0)]^2}\left[\frac{f'(x_0)[f(x_0)-x_0f'(x_0)]}{a^2}-\frac{x_0+f(x_0)f'(x_0)}{b^2}\right]+\frac{2f'(x_0)}{a\sqrt{1+[f'(x_0)]^2}}$	$\frac{2}{1+[f'(x_0)]^2}\left[\frac{f'(x_0)[f(x_0)-x_0f'(x_0)]}{a^2}+\frac{x_0+f(x_0)f'(x_0)}{b^2}\right]-\frac{2f'(x_0)}{a\sqrt{1+[f'(x_0)]^2}}$
E	$-\left\{\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{2p\{1+[f'(x_0)]^2\}}+\frac{1}{\sqrt{1+[f'(x_0)]^2}}\right\}$	$-\frac{2}{1+[f'(x_0)]^2}\left[\frac{f(x_0)-x_0f'(x_0)}{a^2}+\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{b^2}\right]-\frac{2}{a\sqrt{1+[f'(x_0)]^2}}$	$-\frac{2}{1+[f'(x_0)]^2}\left[\frac{f(x_0)-x_0f'(x_0)}{a^2}-\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{b^2}\right]+\frac{2}{a\sqrt{1+[f'(x_0)]^2}}$
F	$\frac{[x_0+f(x_0)f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}}-\frac{f'(x_0)x_0-f(x_0)}{\sqrt{1+[f'(x_0)]^2}}$	$\frac{1}{1+[f'(x_0)]^2}\left[\frac{[f(x_0)-x_0f'(x_0)]^2}{a^2}+\frac{[x_0+f(x_0)f'(x_0)]^2}{b^2}\right]-\frac{2[f'(x_0)x_0-f(x_0)]}{a\sqrt{1+[f'(x_0)]^2}}$	$\frac{1}{1+[f'(x_0)]^2}\left[\frac{[f(x_0)-x_0f'(x_0)]^2}{a^2}-\frac{[x_0+f(x_0)f'(x_0)]^2}{b^2}\right]+\frac{2[f'(x_0)x_0-f(x_0)]}{a\sqrt{1+[f'(x_0)]^2}}$

Source: Own creation

Table A 2: Coefficients of the second-degree equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ for the second tangent conic sections to the graph of a function f at the point $P(x_0, f(x_0))$

	Parabola	Ellipse	Hyperbola
A	$-\frac{1}{4p\{1+[f'(x_0)]^2\}}$	$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{[f'(x_0)]^2}{a^2}+\frac{1}{b^2}\right\}$	$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{[f'(x_0)]^2}{a^2}-\frac{1}{b^2}\right\}$
B	$-\frac{f'(x_0)}{2p\{1+[f'(x_0)]^2\}}$	$\frac{2f'(x_0)}{1+[f'(x_0)]^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)$	$-\frac{2f'(x_0)}{1+[f'(x_0)]^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$
C	$-\frac{[f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}}$	$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{1}{a^2}+\frac{[f'(x_0)]^2}{b^2}\right\}$	$\frac{1}{1+[f'(x_0)]^2}\left\{\frac{1}{a^2}-\frac{[f'(x_0)]^2}{b^2}\right\}$
D	$\frac{f'(x_0)}{\sqrt{1+[f'(x_0)]^2}}+\frac{x_0+f(x_0)f'(x_0)}{2p\{1+[f'(x_0)]^2\}}$	$\frac{2}{1+[f'(x_0)]^2}\left[\frac{f'(x_0)[f(x_0)-x_0f'(x_0)]}{a^2}-\frac{x_0+f(x_0)f'(x_0)}{b^2}\right]-\frac{2f'(x_0)}{a\sqrt{1+[f'(x_0)]^2}}$	$\frac{2}{1+[f'(x_0)]^2}\left[\frac{f'(x_0)[f(x_0)-x_0f'(x_0)]}{a^2}+\frac{x_0+f(x_0)f'(x_0)}{b^2}\right]+\frac{2f'(x_0)}{a\sqrt{1+[f'(x_0)]^2}}$
E	$-\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{2p\{1+[f'(x_0)]^2\}}-\frac{1}{\sqrt{1+[f'(x_0)]^2}}$	$-\frac{2}{1+[f'(x_0)]^2}\left[\frac{f(x_0)-x_0f'(x_0)}{a^2}+\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{b^2}\right]+\frac{2}{a\sqrt{1+[f'(x_0)]^2}}$	$-\frac{2}{1+[f'(x_0)]^2}\left[\frac{f(x_0)-x_0f'(x_0)}{a^2}-\frac{f'(x_0)[x_0+f(x_0)f'(x_0)]}{b^2}\right]-\frac{2}{a\sqrt{1+[f'(x_0)]^2}}$
F	$-\frac{[x_0+f(x_0)f'(x_0)]^2}{4p\{1+[f'(x_0)]^2\}}-\frac{f'(x_0)x_0-f(x_0)}{\sqrt{1+[f'(x_0)]^2}}$	$\frac{1}{1+[f'(x_0)]^2}\left[\frac{[f(x_0)-x_0f'(x_0)]^2}{a^2}+\frac{[x_0+f(x_0)f'(x_0)]^2}{b^2}\right]+\frac{2[f'(x_0)x_0-f(x_0)]}{a\sqrt{1+[f'(x_0)]^2}}$	$\frac{1}{1+[f'(x_0)]^2}\left[\frac{[f(x_0)-x_0f'(x_0)]^2}{a^2}-\frac{[x_0+f(x_0)f'(x_0)]^2}{b^2}\right]-\frac{2[f'(x_0)x_0-f(x_0)]}{a\sqrt{1+[f'(x_0)]^2}}$

Source: Own creation

Appendix B. Parametric equations for the first and second tangent conic sections to the graph of a function f at the point $P(x_0, f(x_0))$

Table B 1

	Parabola	Ellipse	Hyperbola
t	$-\infty < t < \infty$	$0 \leq t \leq 2\pi$	$-\pi/2 \leq t \leq 3\pi/2,\ t \neq \pi/2$
		First tangent conic section	
$x(t)$	$\sqrt{1+[f'(x_0)]^2}\left\{t - \frac{f'(x_0)\left[\frac{1}{4p}t^2 + f'(x_0)t\right]}{1+[f'(x_0)]^2}\right\} + x_0$	$\frac{b\sin(t) - f'(x_0)a[cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}} + x_0$	$\frac{b\tan(t) - f'(x_0)a[\sec(t)-1]}{\sqrt{1+[f'(x_0)]^2}} + x_0$
$y(t)$	$\frac{\frac{1}{4p}t^2 + f'(x_0)t}{\sqrt{1+[f'(x_0)]^2}} + f(x_0)$	$\frac{f'(x_0)b\sin(t) + a[\cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}} + f(x_0)$	$\frac{f'(x_0)b\tan(t) + a[\sec(t)-1]}{\sqrt{1+[f'(x_0)]^2}} + f(x_0)$
		Second tangent conic section	
$x(t)$	$\sqrt{1+[f'(x_0)]^2}\left\{\frac{f'(x_0)\left[\frac{1}{4p}t^2 + f'(x_0)t\right]}{1+[f'(x_0)]^2} - t\right\} + x_0$	$\frac{-b\sin(t) + f'(x_0)a[cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}} + x_0$	$\frac{-b\tan(t) + f'(x_0)a[\sec(t)-1]}{\sqrt{1+[f'(x_0)]^2}} + x_0$
$y(t)$	$\frac{-\frac{1}{4p}t^2 - f'(x_0)t}{\sqrt{1+[f'(x_0)]^2}} + f(x_0)$	$\frac{-f'(x_0)b\sin(t) - a[\cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}} + f(x_0)$	$\frac{-f'(x_0)b\tan(t) - a[\sec(t)-1]}{\sqrt{1+[f'(x_0)]^2}} + f(x_0)$

Source: Own creation

Appendix C. Derivatives for the first and second tangent conic sections to the graph of a function f at the point $P(x_0, f(x_0))$

Table C 1

	$\frac{dy}{dx}$	t
Parabola	$\frac{t+2pf'(x_0)}{2p-f'(x_0)t}$	$-\infty < t < \infty$ $t \neq \frac{2p}{f'(x_0)}$
Ellipse	$\frac{f'(x_0)b\cos(t)-a\sin(t)}{b\cos(t)+f'(x_0)a\sin(t)}$	$0 \leq t \leq 2\pi$ $\sin^2(t) \neq \frac{b^2}{a^2[f'(x_0)]^2+b^2}$
Hyperbola	$\frac{f'(x_0)b\sec(t)+a\tan(t)}{b\sec(t)-f'(x_0)a\tan(t)}$	$-\pi/2 \leq t \leq 3\pi/2, t \neq \pi/2$ $\tan^2(t) \neq \frac{b^2}{a^2[f'(x_0)]^2-b^2}$

Source: Own creation

Appendix D. Illustrative examples

Note: the following values were rounded (where applicable).

Find the equations of the tangent conic sections to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$.

$$x_0 = 1$$

$$f(x_0) = 2$$

The derivative of f is:

$$f'(x) = 2 + 2x - 3x^2$$

f is differentiable on the interval $(-\infty, +\infty)$ and therefore continuous on $(-\infty, +\infty)$.

The derivative of f at x_0 is:

$$f'(1) = 2 + 2(1) - 3(1)^2 = 1$$

$$f'(x_0) = 1$$

The tangent line to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$ is given by:

$$y - 2 = 1(x - 1)$$

$$y = x + 1$$

The equation of the perpendicular line to the tangent line at the point $P(1, 2)$ is given by:

$$y = -x + [1 + 2]$$

$$y = -x + 3$$

Parabola

The general second-degree equation for the first tangent parabola (Table A1) is given by:

$$\frac{1}{4p\{1+1\}}x^2 + \frac{1}{2p\{1+1\}}xy + \frac{1}{4p\{1+1\}}y^2 + \left\{\frac{1}{\sqrt{2}} - \frac{1+(2)(1)}{2p\{1+1\}}\right\}x - \left\{\frac{1[1+(2)(1)]}{2p\{1+1\}} + \frac{1}{\sqrt{2}}\right\}y + \frac{[1+(2)(1)]^2}{4p\{1+1\}} - \frac{(1)(1)-(2)}{\sqrt{2}} = 0$$

$$\frac{1}{8p}x^2 - \frac{1}{4p}xy + \frac{1}{8p}y^2 + \left\{\frac{1}{\sqrt{2}} - \frac{3}{4p}\right\}x - \left\{\frac{3}{4p} + \frac{1}{\sqrt{2}}\right\}y + \frac{9}{8p} + \frac{1}{\sqrt{2}} = 0$$

The second-degree equation is determined using the values of 1/8, 1/4, 1 and ∞ for p:

If $p = 1/8$:

$$x^2 - 2xy + y^2 + \left\{\frac{\sqrt{2}}{2} - 6\right\} x - \left\{6 + \frac{\sqrt{2}}{2}\right\} y + 9 + \frac{\sqrt{2}}{2} = 0$$

If $p = 1/4$:

$$\frac{1}{2}x^2 - xy + \frac{1}{2}y^2 + \left\{\frac{\sqrt{2}}{2} - 3\right\} x - \left\{3 + \frac{\sqrt{2}}{2}\right\} y + \frac{9}{2} + \frac{\sqrt{2}}{2} = 0$$

If $p = 1$:

$$\frac{1}{8}x^2 - \frac{1}{4}xy + \frac{1}{8}y^2 + \left\{\frac{\sqrt{2}}{2} - \frac{3}{4}\right\} x - \left\{\frac{3}{4} + \frac{\sqrt{2}}{2}\right\} y + \frac{9}{8} + \frac{\sqrt{2}}{2} = 0$$

Taking the limit $p \to \infty$ in the second-degree equation:

$$\lim_{p\to\infty} \left\{\frac{1}{8p}x^2 - \frac{1}{4p}xy + \frac{1}{8p}y^2 + \left\{\frac{1}{\sqrt{2}} - \frac{3}{4p}\right\} x - \left\{\frac{3}{4p} + \frac{1}{\sqrt{2}}\right\} y + \frac{9}{8p} + \frac{1}{\sqrt{2}}\right\} = \lim_{p\to\infty} 0$$

$$\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}} = 0$$

$$x - y + 1 = 0$$

$$y = x + 1$$

The parametric equations for the first tangent parabola (Table B1 - Appendix B) are given by:

$$y = \frac{\frac{1}{4p}t^2 + t}{\sqrt{2}} + 2$$

$$x = \sqrt{2}\left\{t - \frac{\frac{1}{4p}t^2 + t}{2}\right\} + 1$$

$$x = \frac{-\frac{1}{4p}t^2 + t}{\sqrt{2}} + 1$$

Where $-\infty < t < \infty$

The parametric equations are determined using the values of 1/8, 1/4, 1 and ∞ for p:

If $p = 1/8$:

$$y = \frac{2t^2 + t}{\sqrt{2}} + 2$$

$$x = \frac{-2t^2 + t}{\sqrt{2}} + 1$$

If $p = 1/4$:

$$y = \frac{t^2 + t}{\sqrt{2}} + 2$$

$$x = \frac{-t^2 + t}{\sqrt{2}} + 1$$

If $p = 1$:

$$y = \frac{\frac{1}{4}t^2 + t}{\sqrt{2}} + 2$$

$$x = \frac{-\frac{1}{4}t^2 + t}{\sqrt{2}} + 1$$

Taking the limit $p \to \infty$ in the parametric equations of the parabola:

$$\lim_{p\to\infty} y = \lim_{p\to\infty}\left(\frac{\frac{1}{4p}t^2 + t}{\sqrt{2}} + 2\right)$$

$$\lim_{p\to\infty} x = \lim_{p\to\infty}\left(\frac{-\frac{1}{4p}t^2 + t}{\sqrt{2}} + 1\right)$$

$$y = \frac{1}{\sqrt{2}}t + 2$$

$$x = \frac{1}{\sqrt{2}}t + 1$$

Eliminating the parameter:

$$t = \sqrt{2}(x - 1)$$

$$y = \frac{1}{\sqrt{2}}\sqrt{2}(x - 1) + 2$$

$$y = x + 1$$

The first tangent parabolas to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$ with different values of p are plotted using the parametric equations (Figure D1a).

If $p = 1/8$ and $t = 1/2$:

$$y = \frac{2\left(\frac{1}{2}\right)^2 + \frac{1}{2}}{\sqrt{2}} + 2$$

$$y = \frac{1}{\sqrt{2}} + 2 = \frac{1 + 2\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2} + 4}{2} = \frac{\sqrt{2}}{2} + 2$$

$$x = \frac{-2\left(\frac{1}{2}\right)^2 + \frac{1}{2}}{\sqrt{2}} + 1$$

$$x = 1$$

The derivative of the first tangent parabola (Table C1 - Appendix C) with $p = 1/8$ at the point $\left(1, \frac{\sqrt{2}}{2} + 2\right)$ is given by:

$$\frac{dy}{dx} = \frac{\frac{1}{2} + \frac{1}{4}(1)}{\frac{1}{4} - (1)\frac{1}{2}} = \frac{\frac{3}{4}}{-\frac{1}{4}} = -3$$

The tangent line to the tangent parabola with $p = 1/8$ at the point $\left(1, \frac{\sqrt{2}}{2} + 2\right)$ is given by (Figure D1b):

$$y - \left(\frac{\sqrt{2}}{2} + 2\right) = -3(x - 1)$$

$$y = -3x + 5 + \frac{\sqrt{2}}{2}$$

The general second-degree equation for the second tangent parabola (Table A2) is given by:

$$-\frac{1}{4p\{1+1\}}x^2 - \frac{1}{2p\{1+1\}}xy - \frac{1}{4p\{1+1\}}y^2 + \left\{\frac{1}{\sqrt{2}} + \frac{1 + (2)(1)}{2p\{1+1\}}\right\}x + \left\{\frac{1[1 + (2)(1)]}{2p\{1+1\}} - \frac{1}{\sqrt{2}}\right\}y - \frac{[1 + (2)(1)]^2}{4p\{1+1\}} - \frac{(1)(1) - (2)}{\sqrt{2}} = 0$$

$$-\frac{1}{8p}x^2 - \frac{1}{4p}xy - \frac{1}{8p}y^2 + \left\{\frac{1}{\sqrt{2}} + \frac{3}{4p}\right\}x + \left\{\frac{3}{4p} - \frac{1}{\sqrt{2}}\right\}y - \frac{9}{8p} + \frac{1}{\sqrt{2}} = 0$$

The second-degree equation is determined using $p = 1/8$:

$$-x^2 - 2xy - y^2 + \left\{\frac{\sqrt{2}}{2} + 6\right\}x + \left\{6 - \frac{\sqrt{2}}{2}\right\}y - 9 + \frac{\sqrt{2}}{2} = 0$$

The parametric equations of the second tangent parabola (Table B1 - Appendix B) are given by:

$$y = \frac{-\frac{1}{4p}t^2 - t}{\sqrt{2}} + 2$$

$$x = \sqrt{2}\left\{\frac{\frac{1}{4p}t^2 + t}{2} - t\right\} + 1$$

$$x = \frac{\frac{1}{4p}t^2 - t}{\sqrt{2}} + 1$$

Where $-\infty < t < \infty$
The parametric equations are determined using $p = 1/8$:

$$y = \frac{-2t^2 - t}{\sqrt{2}} + 2$$

$$x = \frac{2t^2 - t}{\sqrt{2}} + 1$$

If $p = 1/8$ and $t = 1/2$:

$$y = \frac{-2\left(\frac{1}{2}\right)^2 - \frac{1}{2}}{\sqrt{2}} + 2 = -\frac{1}{\sqrt{2}} + 2 = \frac{-1 + 2\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2} + 4}{2} = -\frac{\sqrt{2}}{2} + 2$$

$$x = \frac{2\left(\frac{1}{2}\right)^2 - \frac{1}{2}}{\sqrt{2}} + 1 = 1$$

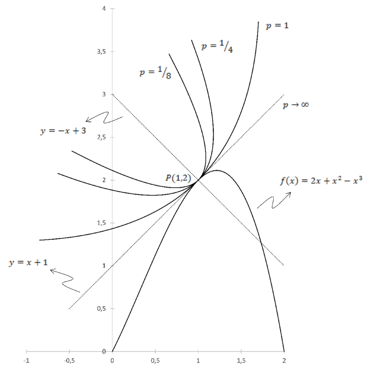
The derivative of the second tangent parabola with $p = 1/8$ at the point $\left(1, -\frac{\sqrt{2}}{2} + 2\right)$ is given by (Table C1 - Appendix C):

$$\frac{dy}{dx} = \frac{\frac{1}{2} + \frac{1}{4}(1)}{\frac{1}{4} - (1)\frac{1}{2}} = \frac{\frac{3}{4}}{-\frac{1}{4}} = -3$$

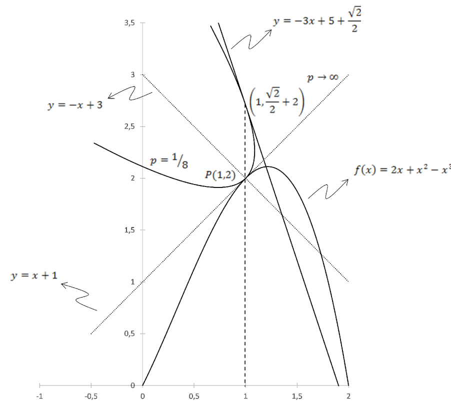
The tangent line to the second tangent parabola with $p = 1/8$ at the point $\left(1, -\frac{\sqrt{2}}{2} + 2\right)$ is given by (Figure D1c):

$$y - \left(-\frac{\sqrt{2}}{2} + 2\right) = -3(x - 1)$$

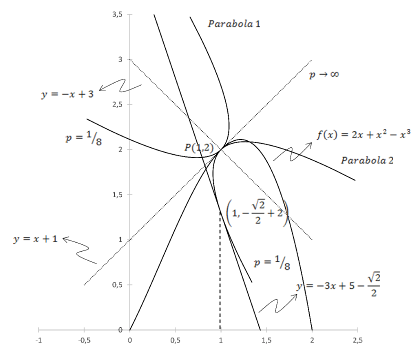
$$y = -3x + 5 - \frac{\sqrt{2}}{2}$$



(a) First tangent parabolas to the graph of the function $f(x) = 2x+x^2-x^3$ at the point $P(1,2)$ with different values of p.



(b) Tangent line to the first tangent parabola with $p = 1/8$ at the point $(1, \frac{\sqrt{2}}{2} + 2)$.



(c) Tangent line to the second tangent parabola with $p = 1/8$ at the point $\left(1, -\frac{\sqrt{2}}{2} + 2\right)$.

Figure D 1: Tangent parabolas to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1,2)$.

Source: Own creation

Ellipse

The general second-degree equation for the first tangent ellipse (Table A1) is given by:

$$\frac{1}{2}\left\{\frac{1}{a^2}+\frac{1}{b^2}\right\}x^2+\left(\frac{1}{b^2}-\frac{1}{a^2}\right)xy+\frac{1}{2}\left\{\frac{1}{a^2}+\frac{1}{b^2}\right\}y^2+\left\{\left[\frac{1}{a^2}-\frac{3}{b^2}\right]+\frac{2}{a\sqrt{2}}\right\}x$$
$$+\left\{-\left[\frac{1}{a^2}+\frac{3}{b^2}\right]-\frac{2}{a\sqrt{2}}\right\}y+\left\{\frac{1}{2}\left[\frac{1}{a^2}+\frac{9}{b^2}\right]+\frac{2}{a\sqrt{2}}\right\}=0$$

The second-degree equation is determined using the values of $a = 1, b = 1/2; a = 1/2, b = 1/4$ and $a = b = 1$:

If $a = 1, b = 1/2$:

$$\frac{5}{2}x^2 + 3xy + \frac{5}{2}y^2 + \left[-11 + \sqrt{2}\right]x + \left[-13 - \sqrt{2}\right]y + \left[\frac{37}{2} + \sqrt{2}\right] = 0$$

If $a = 1/2, b = 1/4$:

$$10x^2 + 12xy + 10y^2 + \left\{-44 + 2\sqrt{2}\right\}x + \left\{-52 - 2\sqrt{2}\right\}y + \left\{74 + 2\sqrt{2}\right\} = 0$$

If $a = b = 1$:

$$x^2 + y^2 + \left[-2 + \sqrt{2}\right]x + \left[-4 - \sqrt{2}\right]y + \left[5 + \sqrt{2}\right] = 0$$

The parametric equations for the first tangent ellipse (Table B1 - Appendix B) are given by:

$$y = \frac{b\sin(t) + a[\cos(t) + 1]}{\sqrt{2}} + 2$$

$$x = \frac{b\sin(t) - a[\cos(t) + 1]}{\sqrt{2}} + 1$$

Where $0 \leq t \leq 2\pi$.

The parametric equations are determined using the values of $a = 1, b = 1/2; a = 1/2, b = 1/4$ and $a = b = 1$:

If $a = 1, b = 1/2$:

$$y = \frac{1/2\sin(t) + \cos(t) + 1}{\sqrt{2}} + 2$$

$$x = \frac{1/2\sin(t) - \cos(t) - 1}{\sqrt{2}} + 1$$

If $a = 1/2, b = 1/4$:

$$y = \frac{\frac{\sin(t)}{4} + \frac{[\cos(t)+1]}{2}}{\sqrt{2}} + 2$$

$$x = \frac{\frac{\sin(t)}{4} - \frac{[\cos(t)+1]}{2}}{\sqrt{2}} + 1$$

If $a = b = 1$:

$$y = \frac{\sin(t) + \cos(t) + 1}{\sqrt{2}} + 2$$

$$x = \frac{\sin(t) - \cos(t) - 1}{\sqrt{2}} + 1$$

The first tangent ellipses to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$ with different values of a and b are plotted using the parametric equations (Figure D2a).

If $a = 1, b = 1/2$ and $t = \pi/4$:

$$y = \frac{1/2 \sin(\pi/4) + \cos(\pi/4) + 1}{\sqrt{2}} + 2$$

$y = 3{,}457$

$$x = \frac{1/2 \sin(\pi/4) - \cos(\pi/4) - 1}{\sqrt{2}} + 1$$

$x = 0{,}043$

The derivative of the first tangent ellipse (Table C1- Appendix C) with $a = 1, b = 1/2$ at the point (0,043 , 3,457) is given by:

$$\frac{dy}{dx} = \frac{1/2 \cos(\pi/4) - \sin(\pi/4)}{1/2 \cos(\pi/4) + \sin(\pi/4)} = -0{,}333$$

The tangent line to the first tangent ellipse with $a = 1, b = 1/2$ at the point (0,043 , 3,457) is given by (Figure D2b):

$$y - 3{,}457 = -0{,}333(x - 0{,}043)$$

$$y = -0{,}333x + 3{,}471$$

The general second-degree equation for the second tangent ellipse (Table A2) is given by:

$$\frac{1}{2}\left\{\frac{1}{a^2} + \frac{1}{b^2}\right\}x^2 + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)xy + \frac{1}{2}\left\{\frac{1}{a^2} + \frac{1}{b^2}\right\}y^2 + \left\{\left[\frac{1}{a^2} - \frac{3}{b^2}\right] - \frac{2}{a\sqrt{2}}\right\}x$$
$$+ \left\{-\left[\frac{1}{a^2} + \frac{3}{b^2}\right] + \frac{2}{a\sqrt{2}}\right\}y + \left\{\frac{1}{2}\left[\frac{1}{a^2} + \frac{9}{b^2}\right] - \frac{2}{a\sqrt{2}}\right\} = 0$$

The second-degree equation is determined using the values of $a = 1$ and $b = 1/2$:

$$\frac{5}{2}x^2 + 3xy + \frac{5}{2}y^2 + \left[-11 - \sqrt{2}\right]x + \left[-13 + \sqrt{2}\right]y + \left[\frac{37}{2} - \sqrt{2}\right] = 0$$

The parametric equations of the second tangent ellipse (Table B1 - Appendix B) are given by:

$$y = \frac{-b\sin(t) - a[\cos(t) + 1]}{\sqrt{2}} + 2$$

$$x = \frac{-b\sin(t) + a[\cos(t) + 1]}{\sqrt{2}} + 1$$

Where $0 \leq t \leq 2\pi$.

The parametric equations are determined using the values of $a = 1$ and $b = 1/2$:

$$y = \frac{-1/2\sin(t) - \cos(t) - 1}{\sqrt{2}} + 2$$

$$x = \frac{-1/2\sin(t) + \cos(t) + 1}{\sqrt{2}} + 1$$

If $a = 1, b = 1/2$ and $t = \pi/4$:

$$y = \frac{-1/2\sin(\pi/4) - \cos(\pi/4) - 1}{\sqrt{2}} + 2$$

$y = 0{,}543$

$$x = \frac{-1/2\sin(\pi/4) + \cos(\pi/4) + 1}{\sqrt{2}} + 1$$

$x = 1{,}957$

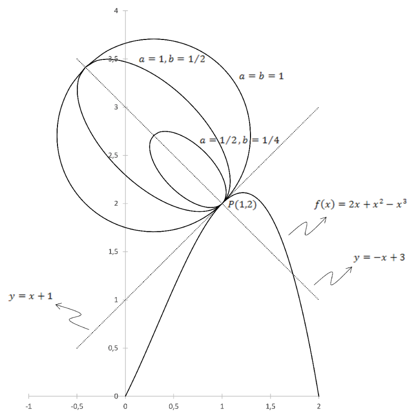
The derivative of the second tangent ellipse using the values of $a = 1$ and $b = 1/2$ at the point (1,957 , 0,543) is given by (Table C1 -Appendix C):

$$\frac{dy}{dx} = \frac{1/2\cos(\pi/4) - \sin(\pi/4)}{1/2\cos(\pi/4) + \sin(\pi/4)} = -0{,}333$$

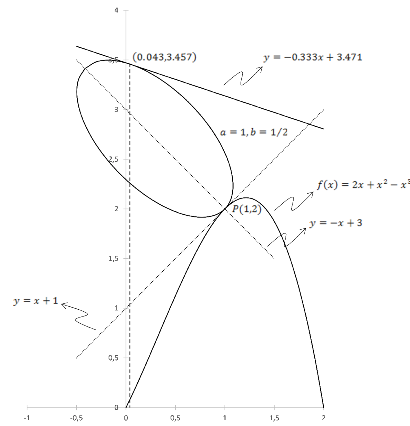
The tangent line to the second tangent ellipse with $a = 1, b = 1/2$ at the point (1,957 , 0,543) is given by (Figure D2c):

$y - 0{,}543 = -0{,}333(x - 1{,}957)$

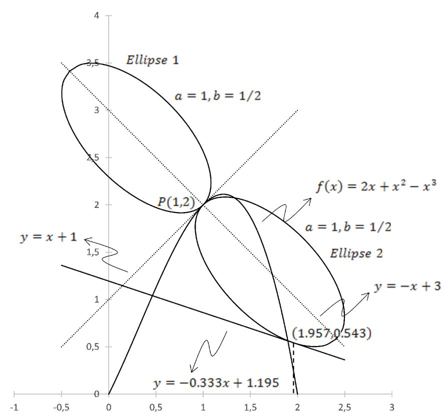
$y = -0{,}333x + 1{,}195$



(a) First tangent ellipses to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$ with different values of a and b.



(b) Tangent line to the first tangent ellipse with $a = 1, b = 1/2$ at the point (0,043 , 3,457).



(c) Tangent line to the second tangent ellipse with $a = 1, b = 1/2$ at the point (1,957 , 0,543).

Figure D 2: Tangent ellipses to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$.

Source: Own creation

Hyperbola

The general second-degree equation for the first tangent hyperbola (Table A1) is given by:

$$\frac{1}{2}\left\{\frac{1}{a^2}-\frac{1}{b^2}\right\}x^2-\left(\frac{1}{a^2}+\frac{1}{b^2}\right)xy+\frac{1}{2}\left\{\frac{1}{a^2}-\frac{1}{b^2}\right\}y^2+\left\{\left[\frac{1}{a^2}+\frac{3}{b^2}\right]-\frac{2}{a\sqrt{2}}\right\}x$$
$$+\left\{-\left[\frac{1}{a^2}-\frac{3}{b^2}\right]+\frac{2}{a\sqrt{2}}\right\}y+\left\{\frac{1}{2}\left[\frac{1}{a^2}-\frac{9}{b^2}\right]-\frac{2}{a\sqrt{2}}\right\}=0$$

The second-degree equation is determined using the values of $a = 1, b = 1/2; a = 1/2, b = 1/4$ and $a = b = 1$:

If $a = 1, b = 1/2$:

$$-\frac{3}{2}x^2-5xy-\frac{3}{2}y^2+\left\{13-\sqrt{2}\right\}x+\left\{11+\sqrt{2}\right\}y+\left\{-\frac{35}{2}-\sqrt{2}\right\}=0$$

If $a = 1/2, b = 1/4$:

$$-6x^2-20xy-6y^2+\left\{52-2\sqrt{2}\right\}x+\left\{44+2\sqrt{2}\right\}y+\left\{-70-2\sqrt{2}\right\}=0$$

If $a = b = 1$:

$$-2xy+\left\{4-\sqrt{2}\right\}x+\left\{2+\sqrt{2}\right\}y+\left\{-4-\sqrt{2}\right\}=0$$

The parametric equations of the hyperbola (Table B1 - Appendix B) are given by:

$$y=\frac{b\tan(t)+a[\sec(t)-1]}{\sqrt{2}}+2$$
$$x=\frac{b\tan(t)-a[\sec(t)-1]}{\sqrt{2}}+1$$

Where $-\pi/2 \le t \le 3\pi/2, t \ne \pi/2$.

The parametric equations are determined using the values of $a = 1, b = 1/2; a = 1/2, b = 1/4$ and $a = b = 1$:

If $a = 1, b = 1/2$:

$$y=\frac{1/2\tan(t)+\sec(t)-1}{\sqrt{2}}+2$$
$$x=\frac{1/2\tan(t)-\sec(t)+1}{\sqrt{2}}+1$$

If $a = 1/2, b = 1/4$:

$$y=\frac{\frac{\tan(t)}{4}+\frac{\sec(t)-1}{2}}{\sqrt{2}}+2$$
$$x=\frac{\frac{\tan(t)}{4}-\frac{\sec(t)-1}{2}}{\sqrt{2}}+1$$

If $a = b = 1$:

$$y=\frac{\tan(t)+\sec(t)-1}{\sqrt{2}}+2$$

$$x = \frac{\tan(t) - \sec(t) + 1}{\sqrt{2}} + 1$$

The first tangent hyperbolas to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$ with different values of a and b are plotted using the parametric equations (Figure D3a).

If $a = 1, b = 1/2$ and $t = \pi/4$:

$$y = \frac{1/2\tan(\pi/4) + \sec(\pi/4) - 1}{\sqrt{2}} + 2$$

$y = 2{,}646$

$$x = \frac{1/2\tan(\pi/4) - \sec(\pi/4) + 1}{\sqrt{2}} + 1$$

$x = 1{,}061$

The derivative of the first tangent hyperbola (Table C1 - Appendix C) with $a = 1, b = 1/2$ at the point (1,061 , 2,646) is given by:

$$\frac{dy}{dx} = \frac{1/2\sec(\pi/4) + \tan(\pi/4)}{1/2\sec(\pi/4) - \tan(\pi/4)} = -5{,}828$$

The tangent line to the tangent hyperbola with $a = 1, b = 1/2$ at the point (1,061 , 2,646) is given by (Figure D3b):

$$y - 2{,}646 = -5{,}828(x - 1{,}061)$$

$$y = -5{,}828x + 8{,}830$$

The general second-degree equation for the second tangent hyperbola (Table A2) is given by:

$$\frac{1}{2}\left\{\frac{1}{a^2} - \frac{1}{b^2}\right\}x^2 - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)xy + \frac{1}{2}\left\{\frac{1}{a^2} - \frac{1}{b^2}\right\}y^2 + \left\{\left[\frac{1}{a^2} + \frac{3}{b^2}\right] + \frac{2}{a\sqrt{2}}\right\}x$$
$$+\left\{-\left[\frac{1}{a^2} - \frac{3}{b^2}\right] - \frac{2}{a\sqrt{2}}\right\}y + \left\{\frac{1}{2}\left[\frac{1}{a^2} - \frac{9}{b^2}\right] + \frac{2}{a\sqrt{2}}\right\} = 0$$

The second-degree equation is determined using the values of $a = 1$ and $b = 1/2$:

$$-\frac{3}{2}x^2 - 5xy - \frac{3}{2}y^2 + \left\{13 + \sqrt{2}\right\}x + \left\{11 - \sqrt{2}\right\}y + \left\{-\frac{35}{2} + \sqrt{2}\right\} = 0$$

The parametric equations of the second tangent hyperbola (Table B1 - Appendix B) are given by:

$$y = \frac{-b\tan(t) - a[\sec(t) - 1]}{\sqrt{2}} + 2$$

$$x = \frac{-b\tan(t) + a[\sec(t) - 1]}{\sqrt{2}} + 1$$

Where $-\pi/2 \leq t \leq 3\pi/2, t \neq \pi/2$.

The parametric equations are determined using the values of $a = 1$ and $b = 1/2$:

$$y = \frac{-1/2\tan(t) - \sec(t) + 1}{\sqrt{2}} + 2$$

$$x = \frac{-1/2\tan(t) + \sec(t) - 1}{\sqrt{2}} + 1$$

If $a = 1, b = 1/2$ and $t = \pi/4$:

$$y = \frac{-1/2\tan(\pi/4) - \sec(\pi/4) + 1}{\sqrt{2}} + 2$$

$y = 1{,}354$

$$y = \frac{-1/2\tan(\pi/4) + \sec(\pi/4) - 1}{\sqrt{2}} + 1$$

$x = 0{,}939$

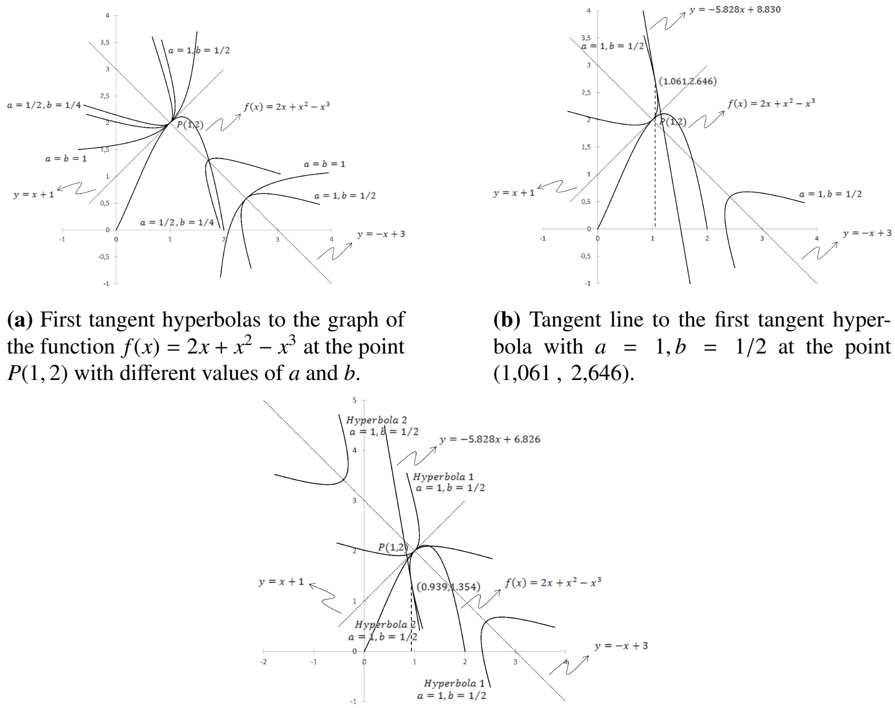
The derivative of the second tangent hyperbola with $a = 1, b = 1/2$ at the point (0,939 , 1,354) is given by (Table C1 - Appendix C):

$$\frac{dy}{dx} = \frac{1/2\sec(\pi/4) + \tan(\pi/4)}{1/2\sec(\pi/4) - \tan(\pi/4)} = -5{,}828$$

The tangent line to the second tangent hyperbola with $a = 1, b = 1/2$ at the point (0,939 , 1,354) is given by (Figure D3c):

$y - 1{,}354 = -5{,}828(x - 0{,}939)$

$y = -5{,}828x + 6{,}826$



(a) First tangent hyperbolas to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$ with different values of a and b.

(b) Tangent line to the first tangent hyperbola with $a = 1, b = 1/2$ at the point (1,061 , 2,646).

(c) Tangent line to the second tangent hyperbola with $a = 1, b = 1/2$ at the point (0,939 , 1,354)

Figure D 3: Tangent hyperbolas to the graph of the function $f(x) = 2x + x^2 - x^3$ at the point $P(1, 2)$.

Source: Own creation