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A GENERALIZATION OF THE SPACE OF COMPLETE QUADRICS

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To any homogeneous polynomial h we naturally associate a variety Ω_h which maps birationally onto the graph Γ_h of the gradient map ∇h and which agrees with the space of complete quadrics when h is the determinant of the generic symmetric matrix. We give a sufficient criterion for Ω_h being smooth which applies for example when h is an elementary symmetric polynomial. In this case Ω_h is a smooth toric variety associated to a certain generalized permutohedron. We also give examples when Ω_h is not smooth.

1. Introduction and results

Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d. We will always assume that there is no invertible linear change of coordinates T such that $h(Tx) \in \mathbb{R}[x_1, \dots, x_{k-1}]$. The *gradient map* of h is the rational map

$$\nabla h : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}, x \mapsto [\nabla h(x)] = [\frac{\partial}{\partial x_1} h(x) : \cdots : \frac{\partial}{\partial x_n} h(x)].$$

It is a regular map on the open subset $U \subset \mathbb{P}^{n-1}$ of all points where h does not vanish. Its graph Γ_h is the Zariski closure of all pairs $(x, \nabla h(x))$ in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$

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with $x \in U$. In this note we will study resolutions of singularities of Γ_h for certain h and thereby address Question 43 in [16]:

Question 43. Can we define a generalization of the space of complete quadrics where the role of the symmetric determinant is played by an arbitrary hyperbolic polynomial h? Such a manifold could be a canonical resolution of the graph of the gradient map of h.

One motivation for this question is the study of so-called *hyperbolic exponential families* in [12] where the gradient map plays a prominent role. While the construction we present in this article is motivated by the theory of hyperbolic polynomials, hyperbolicity plays a subordinate role in the rest of the paper. Instead, other properties such as M-convexity will matter.

In the case when $h = \det(X)$ is the determinant of the $n \times n$ generic symmetric matrix X, such a resolution of singularities is given by the *space of complete quadrics*. For any integer $0 < i < n$ and any symmetric matrix $A \in \mathbb{S}^n$ we denote by $\wedge^i A \in \mathbb{S}^{\binom{n}{i}}$ the representing matrix of the linear map $\wedge^i \mathbb{R}^n \to \wedge^i \mathbb{R}^n$ induced by A. Note that $\wedge^i A$ is nonzero if $\det(A) \neq 0$. Now the space of complete quadrics $\Omega_{\det X}$ is the Zariski closure of all tuples $([A], [\wedge^2 A], \dots, [\wedge^{n-1} A])$ in $\mathbb{P}(\mathbb{S}^n) \times \mathbb{P}(\mathbb{S}^{\binom{n}{2}}) \times \cdots \times \mathbb{P}(\mathbb{S}^{\binom{n}{n-2}}) \times \mathbb{P}(\mathbb{S}^n)$ with A invertible. The projection of $\Omega_{\det X}$ onto the first and the last coordinate is a birational map onto $\Gamma_{\det(X)}$. Moreover, it was shown for example in [11] that $\Omega_{\det X}$ is smooth.

In this note we will define a variety Ω_h for an arbitrary homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ together with a regular and birational map to Γ_h which agrees with the space of complete quadrics when $h = \det(X)$ is the determinant of the generic symmetric matrix. Before we give the definition of Ω_h, we recall the definition of a hyperbolic polynomial.

Definition 1.1. A homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ is *hyperbolic* with respect to $e \in \mathbb{R}^n$ if the univariate polynomial $h(te - v) \in \mathbb{R}[t]$ has only real zeros for all $v \in \mathbb{R}^n$. The *hyperbolicity cone* of h at e is

$$\Lambda_e(h) = \{v \in \mathbb{R}^n : h(te - v) \text{ has only nonnegative roots}\}.$$

The prototype of a hyperbolic polynomial is the determinant of the generic symmetric matrix $\det(X)$. Indeed, since a real symmetric matrix has only real eigenvalues, the polynomial $\det(X)$ is hyperbolic with respect to the identity matrix I. The hyperbolicity cone of $\det(X)$ at I is the cone of positive semidefinite matrices.

The entries of $\wedge^{k+1} X$ cut out the variety of symmetric matrices with rank at most k. For a real symmetric matrix A the algebraic and geometric multiplicity of an eigenvalue agree. Thus the rank of A equals to the degree of the univariate polynomial $\det(tI+A)$. In fact the same holds true when we replace I by any positive definite matrix. This shows that we can express the degeneracy locus of the rational map $\mathbb{P}(\mathbb{S}^n) \dashrightarrow \mathbb{P}(\mathbb{S}^{\binom{n}{k+1}})$, $[A] \mapsto [\wedge^{k+1}A]$ in terms of the hyperbolic rank function of $\det(X)$:

Definition 1.2. Let $h \in \mathbb{R}[x_1,\dots,x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. The *hyperbolic rank function* of h is defined as

$$\mathrm{rank}_{h,e} : \mathbb{R}^n \to \mathbb{N},\, v \mapsto \deg(h(e+tv)).$$

It was shown in [3, Lemma 4.4] that $\mathrm{rank}_{h,e} = \mathrm{rank}_{h,a}$ for any $a \in \mathrm{int}(\Lambda_e(h))$. We let $d = \deg(h)$, $0 \le k < d$ and $v \in \mathbb{R}^n$. Then we have $\mathrm{rank}_{h,e}(v) \le d-k-1$ if and only if all kth order partial derivatives $\frac{\partial^k h}{\partial x_{i_1}\cdots\partial x_{i_k}}$ of h vanish in v. Lets denote by $D_1^k,\dots,D_{m_k}^k$ a basis of the span of all kth order partial derivatives of h. We consider the rational map

$$\Delta h : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m_1-1} \times \cdots \times \mathbb{P}^{m_{d-1}-1},$$

$$[x] \mapsto ([D_1^1(x) : \cdots : D_{m_1}^1(x)], \dots, [D_1^{d-1}(x) : \cdots : D_{m_{d-1}}^{d-1}(x)]).$$

We define the variety Ω_h to be the normalisation of the image of this rational map. The projection on the first and the last coordinate gives a birational morphism $\omega_h : \Omega_h \to \Gamma_h$. Moreover, when $h = \det(X)$ is the determinant of the generic symmetric matrix, then $\Omega_{\det(X)}$ is isomorphic to the space of complete quadrics as defined above and thus $\Omega_{\det(X)}$ is smooth in that case.

Another important example for hyperbolic polynomials are the elementary symmetric polynomials.

Theorem 1.1. *Let $\sigma_{d,n}$ be the elementary symmetric polynomial of degree d in n variables. Then $\Omega_{\sigma_{d,n}}$ is a smooth toric variety.*

It is well-known that $\sigma_{d,n}$ is hyperbolic with respect to every point in the positive orthant. Such polynomials are called *stable*. The theory of stable polynomials connects nicely to discrete convex analysis [13]. We denote by $\delta_k \in \mathbb{Z}^n$ the kth unit vector.

Definition 1.3. A nonempty set of integer points $B \subset \mathbb{Z}^n$ is called *M-convex* if for all $x,y \in B$ and every index i with $x_i > y_i$, there exists an index j with $x_j < y_j$ such that $x - \delta_i + \delta_j \in B$ and $y + \delta_i - \delta_j \in B$.

Theorem 1.2 (Theorem 3.2 in [2]). *Let $h \in \mathbb{R}[x_1,\dots,x_n]$ be a homogeneous stable polynomial. Then the support of h is M-convex.*

In Section 5 we will give a sufficient criterion for Ω_h being smooth when the support of h is M-convex. We will apply this criterion for proving Theorem 1.1. However, there are also stable (and thus hyperbolic) polynomials h for which Ω_h is not smooth.

Example 1.4. Consider the polynomial

$$h = w(2x+4y+7z)(4x+2y+7z)$$
$$+x^3+11x^2y+11xy^2+y^3+15x^2z+46xyz+15y^2z+37xz^2+37yz^2+21z^3.$$

One can check that h is stable. Further, using the the computer algebra system `Macaulay2` [9], one checks that Ω_h is not smooth.

2. A simple polymatroid

In this section we prepare the proof of Theorem 1.1. Recall that a *polymatroid* on the ground set $[n] = \{1,\dots,n\}$ is a function $r : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ such that for all $S,T \subset [n]$ we have:

1. $r(S) \leq r(T)$ if $S \subset T$,
2. $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$, and
3. $r(\emptyset) = 0$.

If further $r(\{i\}) \leq 1$ for all $i \in [n]$, then r is called a *matroid*. The second property is usually called *submodularity*. We call the number $d = r([n])$ the *rank* of r. See [17, Chapter 18] for a general reference on the theory of polymatroids.

Example 2.1. Let $h \in \mathbb{R}[x_1,\dots,x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. The function that sends $S \subset [n]$ to $\mathrm{rank}_{h,e}(\sum_{i\in S} \delta_i)$ is a polymatroid [3, Proposition 3.2].

For all $0 \leq k \leq d$ the *kth truncation* r_k is the polymatroid defined by

$$r_k(S) = \min(d-k, r(S))$$

for all $S \subset [n]$. It follows directly from the definition that the sum of polymatroids is again a polymatroid. We define the following polymatroid

$$\bar{r} = r_0 + \ldots + r_d.$$

To every polymatroid r one associates the *independence polytope*

$$P(r) = \{x \in (\mathbb{R}_{\geq 0})^n : \sum_{i\in S} x_i \leq r(S) \text{ for all } S \subset [n]\}.$$

The goal of this section is to show that for every polymatroid r on $[n]$ the polytope $P(\overline{r})$ is simple. A characterization of polymatroids, whose independence polytope is simple, was given in [8, Theorem 2]. The following lemmata will enable us to apply this criterion.

Definition 2.2. Let r be a polymatroid on $[n]$. We say that a subset $S \subset [n]$ is *r-inseparable* if for every two disjoint and nonempty subsets $S_1, S_2 \subset [n]$ with $S = S_1 \cup S_2$ we have $r(S) < r(S_1) + r(S_2)$.

Remark 2.1. If $|S| \leq 1$, then S is r-inseparable for every polymatroid r.

Lemma 2.2. *Let r, r' be polymatroids on $[n]$. If $S \subset [n]$ is r-inseparable, then S is $(r+r')$-inseparable.*

Proof. Assume that S is not $(r+r')$-inseparable. Let $\emptyset \neq S_1, S_2 \subset [n]$ such that S is the disjoint union of S_1 and S_2. If $r(S)+r'(S) \geq r(S_1)+r'(S_1)+r(S_2)+r'(S_2)$, then by submodularity of r' we get $r(S) \geq r(S_1)+r(S_2)$ which shows that S is not r-inseparable. $\square$

Remark 2.3. Let $|S| \geq 2$ and let $x \in [n]$ be a *loop* of r, i.e. $r(\{x\}) = 0$. If $x \in S$, then S is not r-inseparable: $r(S) = r(S \setminus \{x\}) + r(\{x\})$.

Lemma 2.4. *Let $S \subset [n]$ with $|S| \geq 2$ and r a polymatroid on $[n]$. Then S is $\overline{r}$-inseparable if and only if S does not contain a loop of r.*

Proof. We first observe that $x \in [n]$ is a loop of r if and only if x is a loop of all truncations r_k and thus of $\overline{r}$. Now the "only if" direction follows from Remark 2.3. For the "if" direction assume that S does not contain any loop of r. By Lemma 2.2 it suffices to show that S is r_{d-1}-inseparable. This is clear since

$$r_{d-1}(S) = 1 < 2 = r_{d-1}(S_1) + r_{d-1}(S_2)$$

for all nonempty subsets $S_1, S_2 \subset S$. $\square$

Lemma 2.5. *Let r be a polymatroid on $[n]$ of rank d. Let $S, T \subset [n]$ such that*

1. *$S \cap T \neq \emptyset$, $S \not\subset T$, $T \not\subset S$,*
2. *$\overline{r}(S \cap T) < \overline{r}(S)$, $\overline{r}(S \cap T) < \overline{r}(T)$, and*
3. *the sets $S, T, S \cup T$ are $\overline{r}$-inseparable.*

Then $\overline{r}(S \cap T) + \overline{r}(S \cup T) < \overline{r}(S) + \overline{r}(T)$.

Proof. We proceed by induction on d. We first show that for $d \leq 1$ there are no subsets $S,T \subset [n]$ satisfying $(1),(2),(3)$. If $d = 0$, then r and $\overline{r}$ are both the zero function. Thus there are no subsets $S,T \subset [n]$ satisfying (2). If $d = 1$, we still have $r = \overline{r}$. Condition (1) implies that $|S| \geq 2$. Thus (3) and Lemma 2.4 imply that S contains no loop of r. Therefore, we have $r(S) = r(S \cap T) = 1$ contradicting (2).

Now let $d > 1$ and assume that the claim is true for the polymatroid r_1 of rank $d-1$. We assume for the sake of a contradiction that $S,T \subset [n]$ satisfy $(1),(2),(3)$ but $\overline{r}(S\cap T)+\overline{r}(S\cup T) = \overline{r}(S)+\overline{r}(T)$. Again (1) implies that $|S| \geq 2$. So by (3) and Lemma 2.4 the set $S \cup T$ contains no loop of r. Since $d > 1$, this implies that $S \cup T$ contains no loop of r_1 as well. Thus again by Lemma 2.4 the sets $S,T,S\cup T$ are $\overline{r_1}$-inseparable. By submodularity and because $\overline{r} = r + \overline{r_1}$ we have

$$r(S)+r(T) = r(S\cap T)+r(S\cup T) \text{ and } \overline{r_1}(S)+\overline{r_1}(T) = \overline{r_1}(S\cap T)+\overline{r_1}(S\cup T).$$

So by induction hypothesis we have without loss of generality that $\overline{r_1}(S\cap T) = \overline{r_1}(S)$, which implies $r_1(S\cap T) = r_1(S)$, and $r(S\cap T) < r(S)$. Thus we must have $r(S) = d$ and the equation

$$d + r(T) = r(S) + r(T) = r(S\cap T) + r(S\cup T) = r(S\cap T) + d$$

implies that $r(T) = r(S\cap T)$. This in turn shows that $\overline{r}(T) = \overline{r}(S\cap T)$ contradicting (2). □

Lemma 2.6. *Let r be a polymatroid on $[n]$ of rank d. Let $k \geq 2$ and $S_1,\dots,S_k \subset [n]$ nonempty and pairwise disjoint. Let $S \subset [n]$ $\overline{r}$-inseparable with $\cup_{i=1}^k S_i \subset S$ and $\overline{r}(\cup_{i=1}^k S_i) = \overline{r}(S)$. Then $\overline{r}(\cup_{i=1}^k S_i) < \sum_{i=1}^k \overline{r}(S_i)$.*

Proof. We first observe that since $|S| \geq 2$ and S is $\overline{r}$-inseparable, Lemma 2.4 implies that S contains no loop of r. Thus each S_i also contains no loop of r.

We proceed again by induction on d. If $d = 0$, then there every element is a loop contradicting the assumptions. If $d = 1$, then we have

$$\overline{r}(\cup_{i=1}^k S_i) = 1 < 2 \leq k = \sum_{i=1}^k \overline{r}(S_i).$$

Now let $d > 1$. Then because S contains no loop of r, it also contains no loop of r_1 which shows that S is $\overline{r_1}$-inseparable. Further $\cup_{i=1}^k S_i \subset S$ and $\overline{r}(\cup_{i=1}^k S_i) = \overline{r}(S)$ imply that $\overline{r_1}(\cup_{i=1}^k S_i) = \overline{r_1}(S)$. By induction hypothesis we have $\overline{r_1}(\cup_{i=1}^k S_i) < \sum_{i=1}^k \overline{r_1}(S_i)$ which implies the claim because $r = r_0$ is submodular. □

Theorem 2.7. *Let r be a polymatroid on $[n]$. Then the polytope $P(\overline{r})$ is simple.*

Proof. A characterization of simple independence polytopes of polymatroids was given in [8, Theorem 2]. It says that the polytope $P(\bar{r})$ is simple if and only if the conclusion of the two preceding Lemmas 2.5 and 2.6 holds. $\square$

We will be interested in the base polytope of a polymatroid rather than in its independent polytope. If $r : 2^{[n]} \to \mathbb{R}$ is a submodular function, then its *base polytope* $B(r)$ is defined as

$$B(r) = \{x \in (\mathbb{R}_{\geq 0})^n : \sum_{i \in S} x_i \leq r(S) \text{ for all } S \subset [n] \text{ and } \sum_{i=1}^{n} x_i = r([n])\}.$$

Corollary 2.8. *Let r be a polymatroid on $[n]$. Then the polytope $B(\bar{r})$ is simple.*

Proof. Clearly, the base polytope is a face of the independence polytope. Thus the claim follows from Theorem 2.7. $\square$

Remark 2.9. Taking the sum of submodular functions is compatible with taking the Minkowski sum of their base polytopes [13, Theorem 4.23(1)]. Thus if r is a polymatroid on $[n]$ of rank d, then we have that $B(\bar{r}) = B(r_0) + \ldots + B(r_d)$.

We end this section with describing the polytope $B(\bar{r})$ explicitely when r is the rank function of a matroid. We start with the following easy lemma.

Lemma 2.10. *Let $r = r_{\mathcal{M}}$ be the rank function of a matroid $\mathcal{M}$ of rank d on $[n]$. There is a basis B of $\mathcal{M}$ such that for all $i \in [n]$ we have*

$$a_i := r([i]) - r([i-1]) = \begin{cases} 1 & \text{if } i \in B \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $\mathcal{M}$ is a matroid of rank d, we have $a_i \in \{0, 1\}$ and $B = \{i \in [n] : a_i = 1\}$ has cardinality d. Let $k_1 < \cdots < k_d$ the elements of B. We show by induction on m that $I_m := \{k_1, \ldots, k_m\}$ is independent. Assume that I_{m-1} is independent. Since $r([k_m]) = m$, there is an independent subset I of $[k_m]$ of cardinality m. Thus there is an element $e \in I \setminus I_{m-1}$ such that $I_{m-1} \cup \{e\}$ is independent. Since $r([k_m - 1]) = m - 1$, we must have $e = k_m$. $\square$

Proposition 2.11. *Let $r = r_{\mathcal{M}}$ be the rank function of a matroid $\mathcal{M}$ of rank d on $[n]$. The vertices of the polytope $B(\bar{r})$ are exactly those points $v \in \mathbb{R}^n$ whose support is a basis of $\mathcal{M}$ and whose nonzero entries comprise the numbers $1, \ldots, d$.*

Proof. Let v be a vertex of $B(\bar{r})$. Then v is also a vertex of $P(\bar{r})$. Then by [17, §18.4, Theorem 1] there exists an integer $0 \leq k \leq n$ and a bijection $\pi : [n] \to [n]$ such that $v_{\pi(j)} = \bar{r}(\{\pi(1), \ldots, \pi(j)\}) - \bar{r}(\{\pi(1), \ldots, \pi(j-1)\})$ if $j \in$

$[k]$ and $v_{\pi(j)} = 0$ otherwise. Since v lies in $B(\bar{r})$, we can assume without loss of generality that $k = n$. Further, after relabeling, we can assume that π is the identity map. Now let $B = \{k_1, \dots, k_d\}$ with $k_1 < \cdots < k_d$ be the basis of $\mathcal{M}$ as in the preceding lemma. Then we have $v_j = d - m + 1$ if $j = k_m$ and zero if $j \notin B$. This shows that v is of the desired form.

Conversely, take $v \in \mathbb{R}^n$ whose support $\{i_1, \dots, i_d\}$ is a basis of $\mathcal{M}$ such that $v_{i_j} = d - j + 1$ for $j = 1, \dots, d$. Since $v_{i_j} = \bar{r}(\{i_1, \dots, i_j\}) - \bar{r}(\{i_1, \dots, i_{j-1}\})$ for $j = 1, \dots, d$ and all other entries of v are zero, it is a vertex of $P(\bar{r})$ by [17, §18.4, Theorem 1]. One checks that $\sum_{i=1}^n v_i = \bar{r}([n])$, so v is a vertex of $B(\bar{r})$. $\square$

Example 2.3. For instance when $\mathcal{M} = U(2,4)$ is the uniform matroid on 4 elements of rank 2, then $B(r_1)$ is the standard 3-simplex in $\mathbb{R}^4$ and $B(r_0)$ is the octahedron whose vertices are the permutations of $(1,1,0,0)$ (and thus is not simple). The Minkowski sum $B(\bar{r}) = B(r_0) + B(r_1)$ is simple by Corollary 2.8. It is the truncated tetrahedron whose vertices are the permutations of $(2,1,0,0)$.

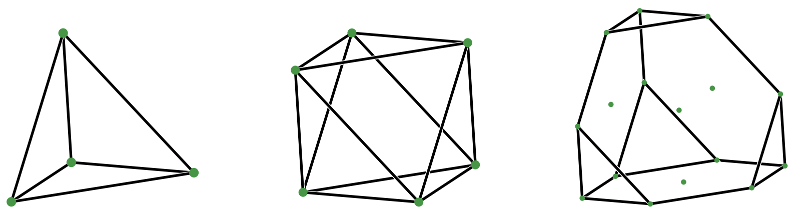

Figure 1: Lattice polytopes from Example 2.3 (left to right): $B(r_0)$, $B(r_1)$ and $B(\bar{r})$. The figures were created using `polymake` [1].

3. Polynomials with M-convex support

Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d and assume that its support $\operatorname{supp}(h) \subset \mathbb{Z}^n$ is M-convex (see Definition 1.3). Recall that the *Newton polytope* $\operatorname{Newt}(h)$ of h is defined as the convex hull of $\operatorname{supp}(h)$ in $\mathbb{R}^n$. The statements in the following theorem are standard in the literature of polymatroids. Proofs can be found for example in [13, §4.4].

Theorem 3.1. *Consider the function $\rho_h : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ defined by*

$$\rho_h(S) = \max\{\sum_{i \in S} \alpha_i : \alpha \in \operatorname{supp}(h)\}$$

for all $S \subset [n]$. Then ρ_h is a polymatroid of rank d, $\operatorname{Newt}(h) = B(\rho_h)$ *and* $\operatorname{supp}(h) = B(\rho_h) \cap \mathbb{Z}^n$.

Remark 3.2. If h is stable, then all coefficients of h have the same sign, see e.g. [3, Lemma 4.3]. This implies that for every $e \in (\mathbb{R}_{>0})^n$ we have that

$$\rho_h(S) = \mathrm{rank}_{h,e}(\sum_{i\in S} \delta_i)$$

as there can be no cancellation of terms.

An intriguing class of polynomials with M-convex support are Lorentzian polynomials.

Definition 3.1. Let $h \in \mathbb{R}[x_1,\ldots,x_n]$ be a homogeneous polynomial of degree d whose support is M-convex and all of whose coefficients are nonnegative. Then h is *Lorentzian* if for every $i_1,\ldots,i_{d-2} \in [n]$ the Hessian of the derivative

$$\frac{\partial^{d-2}}{\partial x_{i_1}\cdots\partial x_{i_{d-2}}} h$$

has at most one positive eigenvalue.

Remark 3.3. Let us clarify the relations between the different classes of polynomials that appeared so far. Let $h \in \mathbb{R}[x_1,\ldots,x_n]$ be a homogeneous polynomial with nonnegative coefficients. Then we have the following implications:

$$\begin{aligned} & h \text{ is stable} \Leftrightarrow h \text{ is hyperbolic w.r.t. every } a \in (\mathbb{R}_{>0})^n \\ \Rightarrow \quad & h \text{ is Lorentzian} \\ \Rightarrow \quad & \mathrm{supp}(h) \text{ is } M\text{-convex} \end{aligned}$$

There are Lorentzian polynomials that are not stable [4, Example 2.3]. Furthermore, not every homogeneous polynomial with M-convex support and nonnegative coefficients is Lorentzian.

Theorem 3.4 (Theorem 3.10 in [4])**.** *A subset $B \subset (\mathbb{Z}_{\geq 0})^n$ is M-convex if and only if there is a Lorentzian polynomial $h \in \mathbb{R}[x_1,\ldots,x_n]$ with $B = \mathrm{supp}(h)$.*

Lemma 3.5 (Corollary 2.11 in [4])**.** *Let $h \in \mathbb{R}[x_1,\ldots,x_n]$ be a Lorentzian polynomial and $e \in (\mathbb{R}_{\geq 0})^n$. The derivative*

$$D_e h = \sum_{i=1}^{n} e_i \frac{\partial h}{\partial x_i}$$

is Lorentzian as well. In particular, the support of $D_e h$ is M-convex.

Lemma 3.6. *If $h \in \mathbb{R}[x_1,\ldots,x_n]$ is Lorentzian of degree d and $e \in (\mathbb{R}_{>0})^n$, then we have for all $0 \leq k \leq d$ that $(\rho_h)_k = \rho_{D_e^k h}$.*

Proof. It suffices to prove the claim in the case $k = 1$ because the general case follows from an iterative application of this case.

Let $S \subset [n]$. Since $\mathrm{D}_e h$ has degree $d-1$, we have $\rho_{\mathrm{D}_e h}(S) \leq d-1$. If $\rho_h(S) = d$, then there is an $\alpha \in \mathrm{supp}(h)$ such that $\sum_{i\in S} \alpha_i = d$. For any $j \in [n]$ with $\alpha_j > 0$ we have $\alpha' = \alpha - \delta_j \in \mathrm{supp}(\mathrm{D}_e h)$ and thus $\rho_{\mathrm{D}_e h}(S) \geq d-1$. Now let $\rho_h(S) < d$ and $\alpha \in \mathrm{supp}(h)$ such that $\sum_{i\in S} \alpha_i = \rho_h(S)$. Since the degree of h is d, there must be an index $j \in [n] \setminus S$ such that $\alpha_j > 0$. We have $\alpha' = \alpha - \delta_j \in \mathrm{supp}(\mathrm{D}_e h)$ and thus $\rho_{\mathrm{D}_e h}(S) \geq \rho_h(S)$. If $\beta \in \mathrm{supp}(\mathrm{D}_e h)$ satisfies $\rho_{\mathrm{D}_e h}(S) = \sum_{i\in S} \beta_i$, then there is a $j \in [n]$ such that $\beta + \delta_j \in \mathrm{supp}(h)$ so $\rho_{\mathrm{D}_e h}(S) \leq \rho_h(S)$. $\square$

The following lemma connects the polymatroid $\overline{\rho_h}$ with the variety Ω_h.

Proposition 3.7. *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be homogeneous of degree d with* $\mathrm{supp}(h)$ *being M-convex. Consider the polymatroid $r = \rho_h$. For each $0 \leq k \leq d$ the set $B(r_k) \cap \mathbb{Z}^n$ agrees with the set B_k of all $\alpha \in \mathbb{Z}^n$ such that the monomial $\prod_{i=1}^n x_i^{\alpha_i}$ is in the support of a kth order partial derivative of h.*

Proof. Both r_k and B_k only depend on the support of h. Thus we can assume without loss of generality that h is Lorentzian by Theorem 3.4. Then for any $e \in (\mathbb{R}_{>0})^n$ we have that B_k is the support of $\mathrm{D}_e^k h$ because h has nonnegative coefficients. Thus B_k is M-convex by Lemma 3.5 and the result follows from Theorem 3.1 and the preceding lemma. $\square$

4. Preliminaries from algebraic and toric geometry

In this section we revisit some notions and results from algebraic geometry that will be used in the final section. Let A be a nonzero $(m+1) \times (n+1)$ matrix. Recall that a rational map $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ of the form $[x] \mapsto [Ax]$ is called a *linear projection* and that the *centre* of π is the linear subspace E of all $[x] \in \mathbb{P}^n$ such that $Ax = 0$. Clearly, the rational map π is regular on $\mathbb{P}^n \setminus E$. Thus if $X \subset \mathbb{P}^n$ is a projective variety with $X \cap E = \emptyset$, the restriction $f = \pi|_X : X \to \pi(X)$ is a morphism. This map f is *finite*, so in particular it has only finite fibers. See [15, §I.5.3] for the definition and proofs. We will use the following standard facts on finite morphisms which follow for example from [10, Lemma 14.8].

Lemma 4.1. *Let $f_i : X_i \to Y_i$, $i = 1, 2$, be finite morphisms of projective varieties. The product map $f_1 \times f_2$ is also finite. If $Y_1 = X_2$, then $f_2 \circ f_1$ is finite. If $Z \subset X_1$ is closed, then $f_1|_Z : Z \to f_1(Z)$ is finite.*

Given a projective variety X, a *normalisation* of X is a normal variety X^ν with a finite birational morphism $\nu : X^\nu \to X$. The normalisation is unique up to isomorphism. In particular, if $Y \to X$ is a finite birational morphism from a smooth variety Y, then Y is the normalisation of X because every smooth variety is normal. See [15, §II.5] for definitions and proofs.

We will especially consider *toric varieties*. The book [6] gives a comprehensive introduction to toric varieties and we will adopt their notation. For example, given a lattice polytope $P \subset \mathbb{R}^n$, we denote by X_P the associated toric variety [6, §2.3]. Similarly, for any finite set $A \subset \mathbb{Z}^n$ of lattice points we denote by $X_A \subset \mathbb{P}^{|A|-1}$ the image of the monomial map whose exponents are given by the elements of A. Note that in general $X_{P\cap\mathbb{Z}^n}$ is not necessarily isomorphic to X_P but when P is a smooth polytope this is the case by [6, Proposition 2.4.4]: A lattice polytope $P \subset \mathbb{R}^n$ is called *smooth* if its associated toric variety X_P is smooth [6, §2.4]. We further have:

Corollary 4.2. *Let r be a polymatroid on $[n]$. Then the polytope $B(\bar{r})$ is a smooth lattice polytope.*

Proof. By the Corollary to [17, §18.4, Theorem 1] the independence polytope $P(\bar{r})$ is a lattice polytope. Since $B(\bar{r})$ is a face of $P(\bar{r})$, it is a lattice polytope as well. Corollary 2.8 states that $B(\bar{r})$ is simple and the Submodularity Theorem[1] [7] states that $B(\bar{r})$ is a so-called *generalized permutohedron*. Now the claim follows from [14, Corollary 3.10] which says that a simple lattice polytope, which is a generalized permutohedron, is smooth. □

Remark 4.3. Let $h \in \mathbb{R}[x_1,\dots,x_n]$ be homogeneous of degree d with $\mathrm{supp}(h)$ being M-convex and let B_k the set of all $\alpha \in \mathbb{Z}^n$ such that the monomial $\prod_{i=1}^n x_i^{\alpha_i}$ is in the support of a kth order partial derivative of h. Then it follows from Proposition 3.7 and Corollary 4.2 that the Minkowski sum $B_1+\ldots+B_{d-1}$ is the set of lattice points in a smooth polytope. In general, if we drop the assumption of M-convexity, this is no longer true. Consider for example $h = a\cdot x_1x_2^2 + b\cdot x_3^3$ with nonzero a,b. Then B_1+B_2 is the set of lattice points in a simple polytope that is not smooth.

5. A sufficient criterion for smoothness

Let $h \in \mathbb{R}[x_1,\dots,x_n]$ be a homogeneous polynomial of degree d whose support is M-convex with nonnegative coefficients and $r = \rho_h$. Recall that we denote by $D_1^k,\dots,D_{m_k}^k$ a basis of the span of all kth order partial derivatives of h. For all $1 \le k < d$ consider the rational map

$$\Delta^k h : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m_k-1},\ [x] \mapsto [D_1^k(x) : \cdots : D_{m_k}^k(x)].$$

[1] The Submodularity Theorem was proved by Edmonds [7] although not stated in the language of generalized permutahedra. An alternative, combinatorial proof is given in [5, Theorem 1.2].

By Proposition 3.7 we can decompose the map $\Delta^k h$ as $\pi_k \circ f_k$ where f_k is the monomial map associated to the polytope $B(r_k)$ (whose image is $X_{B(r_k)\cap\mathbb{Z}^n}$) and π_k the linear projection given by summing the monomials in each D_i^k.

Example 5.1. Let $h = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3$. Then $\Delta^1 h(x)$ equals:

$$[2x_1x_2 + x_2^2 + 2x_1x_3 + x_2x_3 : x_1^2 + 2x_1x_2 + x_1x_3 + 2x_2x_3 : x_1^2 + x_1x_2 + x_2^2].$$

We further have $f_1 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4, [x_1 : x_2 : x_3] \mapsto [x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3]$. We label the coordinates on $\mathbb{P}^4$ by z_{ij} where i and j keep track of the exponent of x_1 and x_2 respectively. The image X of f_1 is cut out by $z_{10}z_{02} - z_{11}z_{01}, z_{11}z_{10} - z_{20}z_{01}$ and $z_{11}^2 - z_{20}z_{02}$. Furthermore, the projection π_1 sends $[z_{20} : z_{11} : z_{10} : z_{02} : z_{01}]$ to

$$[2z_{11} + z_{02} + 2z_{10} + z_{01} : z_{20} + 2z_{11} + z_{10} + 2z_{01} : z_{20} + z_{11} + z_{02}].$$

The centre of π_1 is spanned by $[0 : -1 : 0 : 1 : 1]$ and $[1 : -1 : 1 : 0 : 0]$ and is disjoint from X. Thus π_1 restricts to a finite morphism $X \to \mathbb{P}^2$.

Proposition 5.1. *If the centre of the linear projection π_k is disjoint from $X_{B(r_k)\cap\mathbb{Z}^n}$ for each $1 \leq k < d$, then Ω_h is smooth. More precisely, it is isomorphic to the smooth toric variety $X_{B(\overline{r_1})}$.*

Proof. Let $P = B(\overline{r_1})$. By definition Ω_h is the normalisation of the image $Y \subset \prod_{i=1}^{d-1} \mathbb{P}^{m_i-1}$ of the birational map $\Delta h(x) = (\Delta^1 h(x), \ldots, \Delta^{d-1} h(x))$. Consider the rational map $f : \mathbb{P}^{n-1} \dashrightarrow \prod_{i=1}^{d-1} \mathbb{P}^{|B(r_i)\cap\mathbb{Z}^n|-1}$ given by $f(x) = (f_1(x), \ldots, f_{d-1}(x))$ and let $\pi = \prod_{i=1}^{d-1} \pi_i$. By construction Δh factors as $\pi \circ f$. If we compose f with the Segre embedding of $\prod_{i=1}^{d-1} \mathbb{P}^{|B(r_i)\cap\mathbb{Z}^n|-1}$, we obtain the monomial map associated to the Minkowski sum $\sum_{i=1}^{d-1} B(r_i)$ which is P by Remark 2.9. Thus the image of f can be identified with $X_{P\cap\mathbb{Z}^n}$ and Y is the image of $X_{P\cap\mathbb{Z}^n}$ under the rational map $\pi : \prod_{i=1}^{d-1} \mathbb{P}^{|B(r_i)\cap\mathbb{Z}^n|-1} \dashrightarrow \prod_{i=1}^{d-1} \mathbb{P}^{m_i-1}$. Because P is smooth by Corollary 4.2 the variety $X_{P\cap\mathbb{Z}^n}$ is the smooth toric variety X_P. Letting E_i be the centre of π_i, this rational map π is regular on $U = \prod_{i=1}^{d-1} (\mathbb{P}^{|B(r_i)\cap\mathbb{Z}^n|-1} \setminus E_i)$. Since the projection of X_P on the ith factor $\mathbb{P}^{|B(r_i)\cap\mathbb{Z}^n|-1}$ is $X_{B(r_i)\cap\mathbb{Z}^n}$, which is disjoint from E_i by assumption, it follows that $X_P \subset U$. Thus restricting π gives a surjective morphism $p = \pi|_{X_P} : X_P \to Y$ which is finite since each $\pi_i|_{X_{B(r_i)\cap\mathbb{Z}^n}}$ is finite and by Lemma 4.1. Since $\Delta h = p \circ f$ is birational and f is birational, it follows that p is also birational. Thus X_P is the normalisation of Y. $\square$

Example 5.2. Consider the polynomial

$$h = x^3 + 11x^2y + 11xy^2 + y^3 + 15x^2z + 46xyz + 15y^2z + 37xz^2 + 37yz^2 + 21z^3$$

$$+ w(29x^2 + 90xy + 29y^2 + 150xz + 150yz + 137z^2).$$

One can check that h is stable. Note that h has the same support as the polynomial in Example 1.4 but different coefficients. Using `Macaulay2` [9], one checks that the conditions of Proposition 5.1 are fulfilled and thus Ω_h is smooth. It is the toric variety associated to the triangular frustum whose vertices are obtained by permuting the last three entries of $(0,3,0,0)$ and $(2,1,0,0)$. Here the coordinates correspond to the variables in alphabetic order.

Now let $h = \sigma_{d,n}$ be the elementary symmetric polynomial of degree d.

Lemma 5.2. *The centre of the linear projection π_k is disjoint from $X_{B(r_k)\cap\mathbb{Z}^n}$ for each $1 \le k < d$.*

Proof. The lattice points of $B(r_k)$ are exactly the points $v \in \mathbb{R}^n$ with $d-k$ entries equal to 1 and all other entries 0. Thus $B(r_k)$ is the hypersimplex Δ_{d-k}. We denote $X = X_{\Delta_{d-k}\cap\mathbb{Z}^n} \subset \mathbb{P}^{\binom{n}{d-k}-1}$ and we label the coordinates on $\mathbb{P}^{\binom{n}{d-k}-1}$ by z_S for $S \subset [n]$ of size $d-k$. Every kth order derivative of $\sigma_{d,n}$ is an elementary symmetric polynomial of degree $d-k$ in the variables indexed by some subset $T \subset [n]$ of size $n-k$. Thus the centre E of π_k is the common zero set of all linear forms $L_T = \sum_{S\subset T,\,|S|=d-k} z_S$ for subsets $T \subset [n]$ of size $n-k$. The statement of [12, Lemma 6.4] is that X is disjoint from the common zero set E' of all linear forms $H_i = \sum_{S\subset[n]\setminus i,\,|S|=d-k} z_S$ for $i \in [n]$. We have for all $i \in [n]$:

$$\binom{n+k-1-d}{n-d} \cdot H_i = \sum_{T\subset[n]\setminus\{i\},\,|T|=n-k} L_T.$$

This implies $E \subset E'$ and thus X is also disjoint from E. $\square$

Proof of Theorem 1.1. This follows from Lemma 5.2 and Proposition 5.1. $\square$

Remark 5.3. By Proposition 2.11, we have that $\Omega_{\sigma_{d,n}}$ is the smooth toric variety X_P where P is the convex hull of all permutations of $(1,\dots,d-1,0,\dots,0) \in \mathbb{R}^n$.

Remark 5.4. Fix some M-convex set $S \subset (\mathbb{Z}_{\ge 0})^n$ and let V be the vector space of polynomials h with $\operatorname{supp}(h) \subseteq S$. Note that there is an integer d such that any $h \in V$ is homogeneous of degree d. There are matrices A_h^k whose entries depend linearly on the coefficients of h such that, when $\operatorname{supp}(h) = S$, the centre of π_k is the set of all $[x]$ with $A_h^k x = 0$. Consider the incidence correspondence

$$\Sigma_k = \{([h],[x]) \in \mathbb{P}(V) \times X_{B(r_k)\cap\mathbb{Z}^n} : A_h^k x = 0\}.$$

This is a projective variety. Thus the projection of Σ_k onto the first factor is a closed subvariety Y_k of $\mathbb{P}(V)$. By construction the criterion from Proposition 5.1 applies to a polynomial $h \in V$ with $\operatorname{supp}(h) = S$ if and only if $[h]$ is not contained in any of the Y_k. Therefore, depending on S, either Proposition 5.1 applies to no polynomial with support S, or to a generic such polynomial. We say that S is *torically smoothable* if the latter is the case. The support of the elementary symmetric polynomial $\sigma_{d,n}$ is torically smoothable by Lemma 5.2. Based on experiments we conjecture that S is torically smoothable at least when S contains the support of $\sigma_{d,n}$. This condition is empty when $d > n$.

Remark 5.5. If h has nonnegative coefficients, then we can assume the same for each D_i^k. Then the linear projection π_k is at least regular on the nonnegative part of $X_{B(r_k)}$ as there can be no cancellation of terms. Thus we have at least a regular map on the nonnegative part of $X_{B(\overline{r_1})}$ that maps birationally onto the graph $\Gamma_{h,+}$ of ∇h restricted to the nonnegative orthant. In general, even when h is stable, we cannot expect π_k to be regular on all of $X_{B(r_k)}$. Take for instance the stable polynomial from Example 1.4. In this case $B(r_2)$ is the triangular frustum whose vertices are obtained by permuting the last three entries of $(0,2,0,0)$ and $(1,1,0,0)$. Using `Macaulay2` [9] one checks that the centre of π_2 intersects the toric variety $X_{B(r_2)}$ in a real point of the torus orbit corresponding to the face with vertices $(1,1,0,0),(1,0,1,0)$ and $(1,0,0,1)$.

Remark 5.6. Let $h \in \mathbb{R}[x_1,\dots,x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. In the spirit of the preceding remark one can speculate whether hyperbolicity of h guarantees smoothness of Ω_h at least at some distinguished subset. To make this more precise let $U \subset \mathbb{P}^{n-1}$ be the set of all $[p]$ such that p is in the interior of $\Lambda_e(h)$. Since h does not vanish on U, the gradient map ∇h is regular on U. We can thus consider the subset $C = \omega_h^{-1}(\nabla h(U))$ of Ω_h. We think it is reasonable to ask whether the Euclidean closure of C contains only smooth points of Ω_h.

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REFERENCES

[1] Benjamin Assarf, Ewgenij Gawrilow, Katrin Herr, Michael Joswig, Benjamin Lorenz, Andreas Paffenholz, and Thomas Rehn. Computing convex hulls and counting integer points with `polymake`. *Math. Program. Comput.*, 9(1):1–38, 2017.

[2] Petter Brändén. Polynomials with the half-plane property and matroid theory. *Adv. Math.*, 216(1):302–320, 2007.

[3] Petter Brändén. Obstructions to determinantal representability. *Adv. Math.*, 226(2):1202–1212, 2011.

[4] Petter Brändén and June Huh. Lorentzian polynomials. *Ann. of Math. (2)*, 192(3):821–891, 2020.

[5] Federico Castillo and Fu Liu. Deformation Cones of Nested Braid Fans. *International Mathematics Research Notices*, 2020. rnaa090.

[6] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[7] Jack Edmonds. Submodular functions, matroids, and certain polyhedra. In *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*, pages 69–87. Gordon and Breach, New York, 1970.

[8] Eberhard Girlich and Michail Kovalev. Classification of polyhedral matroids. volume 43, pages 143–159. 1996. Discrete optimization.

[9] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry.

[10] Joe Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course.

[11] Dan Laksov. Completed quadrics and linear maps. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 371–387. Amer. Math. Soc., Providence, RI, 1987.

[12] Mateusz Michałek, Bernd Sturmfels, Caroline Uhler, and Piotr Zwiernik. Exponential varieties. *Proc. Lond. Math. Soc. (3)*, 112(1):27–56, 2016.

[13] Kazuo Murota. *Discrete convex analysis*. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.

[14] Alex Postnikov, Victor Reiner, and Lauren Williams. Faces of generalized permutohedra. *Doc. Math.*, 13:207–273, 2008.

[15] I. R. Shafarevich. *Basic algebraic geometry*. Springer-Verlag, Berlin-New York, study edition, 1977. Translated from the Russian by K. A. Hirsch, Revised printing of Grundlehren der mathematischen Wissenschaften, Vol. 213, 1974.

[16] Bernd Sturmfels. 3264 questions about symmetric matrices. Pdf on: *https://www.orlandomarigliano.com/3264questions.pdf*, 2020.

[17] D. J. A. Welsh. *Matroid theory*. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976. L. M. S. Monographs, No. 8.

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