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DUPLICATE, BERNSTEIN ALGEBRAS AND EVOLUTION ALGEBRAS

A. CONSEIBO - S. SAVADOGO - M. OUATTARA

In this paper, we firstly study a commutative algebra $\mathcal{E}$ over a field F of $Char(F) \neq 2$ that satisfying $\dim(\mathcal{E}^2) = 1$. We show that, such an algebra is an evolution algebra. Afterwards, we pay attention to commutative duplicate of a commutative algebra $\mathcal{E}$. We find necessary and sufficient condition in which the duplicate $D(\mathcal{E})$ is an evolution algebra. And, we finish by studying an evolution algebra that is a Bernstein algebra. We classify that algebras, up to isomorphism, in dimension ≤ 4.

1. Introduction

Given a commutative field F and a finite dimensional algebra $\mathcal{E}$, we say that $\mathcal{E}$ is an *evolution algebra* if it admits a basis $B = \{e_1, \ldots, e_n\}$ such that

$$e_i e_j = 0, \text{ for } 1 \leq i \neq j \leq n \text{ and } e_i^2 = \sum_{k=1}^{n} a_{ik} e_k, \text{ for } 1 \leq i \leq n. \quad (1)$$

Such a basis is called a *natural basis* of $\mathcal{E}$. The matrix $M = (a_{ik})_{1 \leq i,k \leq n}$ is called *the matrix of structural constants* of $\mathcal{E}$ relative to the natural basis B. Evolution algebras are commutative ([15]). The origin and the first study of the evolution algebras date from 1941 with the first formulation due to Etherington ([6,

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Page 34]) of strict self-fertilization in the absence of mutation. Subsequently, Holgate extended Etherington's formulation to study the case of partial self-fertilization ([9]). It is from work of Tian ([14]) that these algebras were popularized and studied under the denomination of evolution algebras.

In section 2, we study n-dimensional commutative algebras $\mathcal{E}$ satisfying $\dim(\mathcal{E}^2) = 1$. We show that such algebras are evolution algebras, then we give a classification in dimension 2, 3 and 4.

In section 3, we exhibit a necessary and sufficient condition for a commutative duplicate of commutative algebra to be an evolution algebra.

In section 4, we characterize the baric algebras that are Bernstein algebras and we give a classification in dimension 2, 3 and 4.

2. Quadratic forms and evolution algebras

In this section, we study finite dimensional commutative algebra $\mathcal{E}$ over a commutative field F of $Char(F) \neq 2$ and satisfying $\dim(\mathcal{E}^2) = 1$.

2.1. Case of dimensions 2 and 3

Example 2.1. Let $\mathcal{E}$ be a commutative 2-dimensional algebra such that $\dim(\mathcal{E}^2) = 1$. Then $\mathcal{E}$ is an evolution algebra.

Proof. Let $\mathcal{E} = Fe_1 \oplus Fe_2$ with $\dim(\mathcal{E}^2) = 1$, i.e. $\mathcal{E}^2 = Fc$ for a certain $c \in \mathcal{E}$. The multiplication table of $\mathcal{E}$ in the basis $\{e_1, e_2\}$ is given by $e_1^2 = \alpha c$, $e_2^2 = \beta c$ and $e_1e_2 = \gamma c$. We set $x = x_1e_1 + x_2e_2 \in \mathcal{E}$ and we have $x^2 = (\alpha x_1^2 + \beta x_2^2 + 2\gamma x_1x_2)c$. For the reduction of the quadratic form $q(x) = \alpha x_1^2 + \beta x_2^2 + 2\gamma x_1x_2$, we distinguish two cases

- $(\alpha, \beta) \neq 0$. Without loss of generality, we assume that $\alpha \neq 0$. Then $x^2 = (\alpha(x_1^2 + \frac{2\gamma}{\alpha}x_1x_2) + \beta x_2^2)c = (\alpha(x_1 + \frac{\gamma}{\alpha}x_2)^2 + (\beta - \frac{\gamma^2}{\alpha})x_2^2)c$. By taking $e_2' = -\frac{\gamma}{\alpha}e_1 + e_2$, we get $e_1e_2' = 0$. Thus, $\mathcal{E}$ is an evolution algebra in the natural basis $\{e_1, e_2'\}$.
- $\alpha = \beta = 0$. We have $x^2 = 2\gamma x_1x_2c = \frac{\gamma}{2}((x_1+x_2)^2 - (x_1-x_2)^2)c$. By setting $e_1' = e_1 + e_2$ and $e_2' = e_1 - e_2$, we have $(e_1+e_2)(e_1-e_2) = 0$. Consequently, $\mathcal{E}$ is an evolution algebra in the natural basis $\{e_1', e_2'\}$.

□

Example 2.2. Let $\mathcal{E}$ be a commutative 3-dimensional algebra such that $\dim(\mathcal{E}^2) = 1$. Then $\mathcal{E}$ is an evolution algebra.

Proof. Let $\mathcal{E} = Fe_1 \oplus Fe_2 \oplus Fe_3$ with $\dim(\mathcal{E}^2) = 1$, i.e. $\mathcal{E}^2 = Fc$ for a certain $c \in \mathcal{E}$. The multiplication table of $\mathcal{E}$ in the basis $\{e_1, e_2, e_3\}$ is given by $e_1^2 = \alpha c$, $e_2^2 = \beta c$, $e_3^2 = \gamma c$, $e_1e_2 = \delta c$, $e_1e_3 = \mu c$ and $e_2e_3 = \lambda c$. Let $x = x_1e_1 + x_2e_2 + x_3e_3 \in \mathcal{E}$, we have $x^2=(\alpha x_1^2+\beta x_2^2+\gamma x_3^2+2\delta x_1x_2+2\mu x_1x_3+2\lambda x_2x_3)c$. For the reduction of the quadratic form $q(x)=(\alpha x_1^2+\beta x_2^2+\gamma x_3^2+2\delta x_1x_2+2\mu x_1x_3+2\lambda x_2x_3)$, we distinguish the following cases

- $(\alpha,\beta,\gamma)\neq 0$. Without loss of generality, we assume that $\alpha\neq 0$. Then

$$\begin{aligned} x^2 &= \left(\alpha\left(x_1^2+2\left(\frac{\delta}{\alpha}x_2+\frac{\mu}{\alpha}x_3\right)x_1\right)+\beta x_2^2+\gamma x_3^2+2\lambda x_2x_3\right)c \\ &= \left(\alpha\left(x_1+\frac{\delta}{\alpha}x_2+\frac{\mu}{\alpha}x_3\right)^2+\left(\beta-\frac{\delta^2}{\alpha}\right)x_2^2+\left(\gamma-\frac{\mu^2}{\alpha}\right)x_3^2+\right. \\ &\quad \left. 2\left(\lambda-\frac{\delta\mu}{\alpha}\right)x_2x_3\right)c \end{aligned}$$

i) $\delta^2-\beta\alpha\neq 0$ or $\mu^2-\gamma\alpha\neq 0$. We can take $\delta^2-\beta\alpha\neq 0$, without loss of generality.

$$\begin{aligned} x^2 &= \left(\alpha\left(x_1+\frac{\delta}{\alpha}x_2+\frac{\mu}{\alpha}x_3\right)^2+\left(\beta-\frac{\delta^2}{\alpha}\right)\left(x_2^2+2\frac{\alpha\lambda-\delta\mu}{\alpha\beta-\delta^2}x_2x_3\right)+\right. \\ &\quad \left.\left(\gamma-\frac{\mu^2}{\alpha}\right)x_3^2\right)c \\ &= \left(\alpha\left(x_1+\frac{\delta}{\alpha}x_2+\frac{\mu}{\alpha}x_3\right)^2+\left(\beta-\frac{\delta^2}{\alpha}\right)\left(x_2+\frac{\alpha\lambda-\delta\mu}{\alpha\beta-\delta^2}x_3\right)^2+\right. \\ &\quad \left.\frac{1}{\alpha}\left(\alpha\gamma-\mu^2-\frac{(\alpha\lambda-\delta\mu)^2}{\alpha\beta-\delta^2}\right)x_3^2\right)c \end{aligned}$$

By setting $e_2'=-\frac{\delta}{\alpha}e_1+e_2$ and $e_3'=\frac{1}{\alpha}\left(\frac{\lambda\delta}{\beta}-\frac{\delta^2\mu}{\alpha\beta}-\mu\right)e_1-\frac{\alpha\lambda-\delta\mu}{\alpha\beta}e_2+e_3$, we get $e_1e_2'=e_1e_3'=e_2'e_3'=0$. So $\mathcal{E}$ is an evolution algebra in the natural basis $\{e_1,e_2',e_3'\}$.

ii) $\delta^2-\beta\alpha=\mu^2-\gamma\alpha=0$. Then

$$\begin{aligned} x^2 &= \left(\alpha\left(x_1+\frac{\delta}{\alpha}x_2+\frac{\mu}{\alpha}x_3\right)^2+2\left(\lambda-\frac{\delta\mu}{\alpha}\right)x_2x_3\right)c \\ &= \left(\alpha\left(x_1+\frac{\delta}{\alpha}x_2+\frac{\mu}{\alpha}x_3\right)^2+\frac{1}{2}\left(\lambda-\frac{\delta\mu}{\alpha}\right)\right. \\ &\quad \left.\left((x_2+x_3)^2-(x_2-x_3)^2\right)\right)c \end{aligned}$$

By taking $e_2'=\frac{\delta+\mu}{2\alpha}e_1+\frac{1}{2}e_2+\frac{1}{2}e_3$ and $e_3'=\frac{\delta-\mu}{2\alpha}e_1+\frac{1}{2}e_2-\frac{1}{2}e_3$, we obtain $e_1e_2'=e_1e_3'=e_2'e_3'=0$. So $\mathcal{E}$ is an evolution algebra in the natural basis $\{e_1,e_2',e_3'\}$.

- $\alpha=\beta=\gamma=0$. Without loss of generality, we can take $\delta\neq 0$. Thus

$$
\begin{aligned}
x^2 &= 2\delta\left(x_1x_2+\frac{\mu}{\delta}x_1x_3+\frac{\lambda}{\delta}x_2x_3\right)c\\
&= 2\delta\left(\left(x_1+\frac{\lambda}{\delta}x_3\right)\left(x_2+\frac{\mu}{\delta}x_3\right)-\frac{\lambda\mu}{\delta^2}x_3^2\right)c\\
&= \left(\frac{\delta}{2}\left(x_1+x_2+\frac{\lambda+\mu}{\delta}x_3\right)^2-\frac{\delta}{2}\left(x_1-x_2+\frac{\lambda-\mu}{\delta}x_3\right)^2-\frac{2\lambda\mu}{\delta}x_3^2\right)c
\end{aligned}
$$

By setting $e'_1=e_1+e_2$, $e'_2=e_1-e_2$ and $e'_3=-\frac{\lambda}{\delta}e_1-\frac{\mu}{\delta}e_2+e_3$, we get $e'_1e'_2=e'_1e'_3=e'_2e'_3=0$. So $\mathcal{E}$ is an evolution algebra in the natural basis $\{e'_1,e'_2,e'_3\}$. □

2.2. General case

Let $(\mathcal{E},b)$ be a bilinear space. A vector $x\neq 0$ of $\mathcal{E}$ is said to be *isotropic* if $b(x,x)=0$. Otherwise x is said to be *anisotropic*. If $(\mathcal{E},b)$ contains an isotropic vector, then $(\mathcal{E},b)$ is also called *isotropic bilinear space*. Otherwise $(\mathcal{E},b)$ is called *anisotropic*. A subspace W of $\mathcal{E}$ is *totally isotropic* if $b(W,W)=0$, i.e. $b(x,y)=0$ for all $x,y\in W$. The *radical* of a symmetric bilinear form $b(x,y)$ is the set of all x such that $b(x,y)=0$, for all $y\in\mathcal{E}$.

Theorem 2.3 ([10, Theorem 4.1, Witt's Decomposition]). *In characteristic $\neq 2$, any quadratic space $(\mathcal{E},q)$ admits orthogonal sum decomposition*

$$\mathcal{E}=\mathcal{E}_t\perp\mathcal{E}_{hyp}\perp\mathcal{E}_{an}, \tag{2}$$

called Witt's decomposition, where $\mathcal{E}_t=rad(q)$ is totally isotropic, $\mathcal{E}_{hyp}=H_1\perp\cdots\perp H_s$ is a hyperbolic space and $\mathcal{E}_{an}$ is an anisotropic space.

Proposition 2.4. *Any finite dimensional commutative algebra $\mathcal{E}$ such that* $\dim(\mathcal{E}^2)=1$ *is an evolution algebra. The natural basis being the orthogonal basis of Witt's decomposition of the induced bilinear form.*

Proof. Let $\mathcal{E}$ be such an algebra. We choose $c\in\mathcal{E}$ such that $\mathcal{E}^2=Fc$. For $x,y\in\mathcal{E}$, $xy=b(x,y)c$ where $b:\mathcal{E}\times\mathcal{E}\to F$ is a non-zero symmetric bilinear form. The corresponding quadratic form $q:\mathcal{E}\to F$ is defined by $x^2=q(x)c$. If another c' is chosen as the generator of $\mathcal{E}^2$, then $c'=\lambda c$, for a certain $\lambda\in F^*$. The corresponding bilinear form b' is $\lambda^{-1}b$. Since q is a quadratic form, Theorem 2.3 tell us, algebra $\mathcal{E}$ admits an orthogonal basis given by Witt's decomposition. It follows that algebra $\mathcal{E}$ is an evolution algebra and the natural basis being the orthogonal basis. □

2.3. Classification

Let $\mathcal{E} = \mathcal{E}_t \perp \mathcal{E}_{\text{hyp}} \perp \mathcal{E}_{\text{an}}$ be Witt's decomposition of the finite dimensional evolution algebra $\mathcal{E}$ satisfying $\dim(\mathcal{E}^2) = 1$ over a commutative field F of $Char(F) \neq 2$. The Proof of Proposition 2.4 tells us, there are a non-zero symmetric bilinear $b : \mathcal{E} \times \mathcal{E} \to F$ and $c \in \mathcal{E}$ such that $\mathcal{E}^2 = Fc$ and $xy = b(x,y)c$ for all $x,y \in \mathcal{E}$. Let $q : \mathcal{E} \to F$ be the corresponding quadratic form of b. We choose a basis $\{u_1,\dots,u_r\}$ of $\mathcal{E}_{\text{an}}$ such that $b(u_i,u_j) = 0$, for $i \neq j$, and $q(u_i) = d_i \neq 0$ $(i = 1,\dots,r)$. Then, we choose a basis $\{x_i,y_i\}$ of H_i such that $b(x_i,y_i) = 0$, $q(x_i) = -q(y_i) = 1$ and finally, we choose a basis $\{v_1,\dots,v_t\}$ of $\mathcal{E}_t = rad(b)$. Since $x^2 = q(x)c$, it follows that $x^3 = q(x)b(x,c)c$, ..., $x^{k+2} = q(x)b(x,c)^k c$. If $\mathcal{E}$ is a nil-algebra, then $b(x,c) = 0$ for all $x \in \mathcal{E}$; in this case $c \in \mathcal{E}_t$. Let us suppose that $\mathcal{E}$ is non-nil. There exists $z \in \mathcal{E}$ such that $b(z,c) \neq 0$. Thus three cases are to be considered.

- c belongs to $\mathcal{E}_t = rad(b)$, i.e. $b(x,c) = 0$ for all $x \in \mathcal{E}$. The multiplication table of $\mathcal{E}$ in the basis $\{u_1,\dots,u_r,v_1,\dots,v_t\}$ is

$$u_i^2 = d_i c \quad (i = 1,\dots,r), \text{ the others products are zero.} \tag{3}$$

- c is isotropic, i.e. $b(c,c) = 0$ and $c^2 = 0$ but $b(z,c) \neq 0$, for some z. So $c \in \mathcal{E}_{\text{hyp}}$ and then there is an i such that $c = x_i + y_i$. Without loss of generality, we can assume that $i = 1$. In this case $\mathcal{E} = \mathcal{E}_{\text{hyp}} \perp \mathcal{E}_{\text{an}}$, where $\mathcal{E}_{\text{hyp}} = H_1$ and the multiplication table of $\mathcal{E}$ in the basis $\{u_1,\dots,u_r,x_1,y_1,v_1,\dots,v_t\}$ is

$$\begin{aligned} u_i^2 &= d_i(x_1+y_1) \quad (i = 1,\dots,r), \quad x_1^2 = -y_1^2 = x_1 + y_1, \\ &\text{the others products are zero.} \end{aligned} \tag{4}$$

- c is anisotropic, i.e. $b(c,c) \neq 0$. We have $c^2 = q(c)c$ and by setting $c' = q(c)^{-1}c$, it follows that $c'^2 = c'$ is a non-zero idempotent. The multiplication table of $\mathcal{E}$ in the basis $\{v_1,\dots,v_t,u_1,\dots,u_r\}$ is

$$u_1^2 = u_1, u_i^2 = d_i u_1 \quad (i = 2,\dots,r), \text{ the others products are zero.} \tag{5}$$

Now, we give a low-dimensional classification of such algebras.

Proposition 2.5. *[4, Theorem 4.1] Any 2-dimensional evolution algebra, over a commutative field F of $Char(F) \neq 2$, satisfying $\dim_F(\mathcal{E}^2) = 1$ is isomorphic to one of the following algebras :*

- $\mathcal{E}_1 : u_1^2 = u_2, u_2^2 = 0.$

- $\mathcal{E}_2 : u_1^2 = -u_2^2 = u_1 + u_2$.
- $\mathcal{E}_3 : u_1^2 = u_1, u_2^2 = 0$.
- $\mathcal{E}_4(\alpha) : u_1^2 = u_1, u_2^2 = \alpha u_1$, *with* $\alpha \in F^*$.

Proposition 2.6. *[3, Theorem 3.5(ii), Table 1] Any* 3*-dimensional evolution algebra, over a commutative field F of $Char(F) \neq 2$, satisfying* $\dim_F(\mathcal{E}^2) = 1$ *is isomorphic to one of the following algebras*

- $\mathcal{E}_1 : u_1^2 = u_1 + u_2, u_2^2 = -(u_1 + u_2), u_3^2 = 0$.
- $\mathcal{E}_2 : u_1^2 = u_1 + u_2, u_2^2 = -(u_1 + u_2), u_3^2 = u_1 + u_2$.
- $\mathcal{E}_3 : u_1^2 = u_3, u_2^2 = 0, u_3^2 = 0$.
- $\mathcal{E}_4(\alpha) : u_1^2 = u_3, u_2^2 = \alpha u_3, u_3^2 = 0$, *with* $\alpha \in F^*$.
- $\mathcal{E}_5 : u_1^2 = u_1, u_2^2 = u_3^2 = 0$.
- $\mathcal{E}_6(\alpha) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = 0$, *with* $\alpha \in F^*$.
- $\mathcal{E}_7(\alpha, \beta) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = \beta u_1$ *with* $\alpha, \beta \in F^*$.

With regard to dimension 4, by varying the dimension of $\mathcal{E}_t$ from 0 to 3 in the equation (2) and taking account the three cases defined above, we have

Proposition 2.7. *Any* 4*-dimensional evolution algebra, over a commutative field F of $Char(F) \neq 2$, satisfying* $\dim_F(\mathcal{E}^2) = 1$ *is isomorphic to one of the following algebras*

- $\mathcal{E}_1 : u_1^2 = v_3, v_1^2 = v_2^2 = v_3^2 = 0$;
- $\mathcal{E}_2 : x_1^2 = -y_1^2 = x_1 + y_1, v_1^2 = v_2^2 = 0$;
- $\mathcal{E}_3 : x_1^2 = -y_1^2 = x_1 + y_1, u_1^2 = x_1 + y_1, v_1^2 = 0$;
- $\mathcal{E}_4(\alpha) : x_1^2 = -y_1^2 = x_1 + y_1, u_1^2 = \alpha(x_1 + y_1), u_2^2 = -\alpha(x_1 + y_1)$;
- $\mathcal{E}_5(\alpha) : u_1^2 = v_2, u_2^2 = \alpha v_2, v_1^2 = v_2^2 = 0$;
- $\mathcal{E}_6 : u_1^2 = u_1, u_2^2 = u_3^2 = u_4^2 = 0$;
- $\mathcal{E}_7(\alpha) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = u_4^2 = 0$;
- $\mathcal{E}_8(\alpha, \beta) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = \beta u_1, u_4^2 = 0$;

- $\mathcal{E}_9(\alpha,\beta,\gamma) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = \beta u_1, u_4^2 = \gamma u_1$;

with $\alpha,\beta,\gamma \in F^*$.

Remark 2.8. If F is an algebraically closed field, in particular if any scalar α of F is a square, i.e. $F = F^2$, the scalars α, β and γ will be replaced by 1.

3. Duplicate and evolution algebras

Let $\mathcal{E}$ be a commutative algebra over a commutative field of $Char(F) \neq 2$, not necessarily associative, nor having an unit element and let $S_F^2(\mathcal{E})$ be a second symmetric power of the F-linear space $\mathcal{E}$. Let I and J be two countable parts. The multiplication $\sum_{i\in I}(x_i.y_i)\sum_{j\in J}(x'_j.y'_j) = \sum_{i\in I} x_iy_i.\sum_{j\in J} x'_jy'_j$, where x_i, y_i, x'_j, y'_j in $\mathcal{E}$ and $x_i.y_i$ denotes the symmetric product of x_i by y_i, defines on $S_F^2(\mathcal{E})$ a commutative F-algebra structure called a *commutative duplicate* of $\mathcal{E}$ [11].

The duplicate will be denoted by $D(\mathcal{E})$. The F-linear map $\mu : D(\mathcal{E}) \to \mathcal{E}^2$ defines by $x.y \mapsto xy$ is an onto F-algebra homomorphism called *Etherington's homomorphism.* We have $D(\mathcal{E})\ker(\mu) = 0$ and $D(\mathcal{E}) = \mathcal{E}^2 \underset{s.d}{\times} \ker(\mu)$ ($s.d.$ for semi-direct) algebras isomorphism. The semi-direct product is given by $(x,x')(y,y') = (xy, \varphi(x,y))$ for all x,y in $\mathcal{E}^2$; x',y' in $\ker(\mu)$ and $\varphi : \mathcal{E}^2 \times \mathcal{E}^2 \to \ker(\mu)$ is a F-bilinear map. We set $N_F(\mathcal{E}) = \ker(\mu)$. If the family $\{e_1,\cdots,e_n\}$ is a basis of $\mathcal{E}$, then $\{e_i.e_j \mid 1 \leq i \leq j \leq n\}$ is a basis of $D(\mathcal{E})$, called the canonical basis of $D(\mathcal{E})$ and $\dim(D(\mathcal{E})) = \frac{n(n+1)}{2}$.

Let $\mathcal{E}$ be an evolution algebra in the natural basis $\{e_1,\cdots,e_n\}$. We suppose that $D(\mathcal{E})$ is an evolution algebra with the canonical basis as the natural basis.

For $i \neq j$, we have $e_ie_j = 0$, i.e. $e_i.e_j \in N_F(\mathcal{E})$. For $i \neq j$, we have $0 = (e_i.e_i)(e_j.e_j) = e_i^2.e_j^2$. Either $e_i^2 = 0$ for all $i \in \{1,\ldots,n\}$, i.e. $\mathcal{E}^2 = 0$, or there exists $i_0 \in \{1,\ldots,n\}$ such that $e_{i_0}^2 \neq 0$ and $e_j^2 = 0$ for all $j \neq i_0$. So either $\mathcal{E}^2 = 0$ or $\mathcal{E}^2 = Fe_{i_0}^2$, i.e. $\dim(\mathcal{E}^2) = 1$. The multiplication table of $D(\mathcal{E})$ in natural basis $\{e_i.e_j \mid 1 \leq i \leq j \leq n\}$ is given by $(e_{i_0}.e_{i_0})^2 = e_{i_0}^2.e_{i_0}^2$, the others products are zero.

The canonical basis of $D(\mathcal{E})$ is not always a natural basis.

Example 3.1. Let $\mathcal{E}_2 : e_1e_1 = e_1, e_2e_2 = e_1$ be an evolution algebra. By taking $e_{ij} := e_i.e_j$, the multiplication table of $D(\mathcal{E}_2)$ in the canonical basis is given by $e_{11}^2 = e_{11}$, $e_{11}e_{22} = e_{11}$, $e_{22}e_{22} = e_{11}$, the others products are zero. Since $e_{11}e_{22} \neq 0$, this basis is not a natural basis. By taking $u = e_{22} - e_{11}$, we get $e_{11}^2 = e_{11}$, $e_{11}e_{12} = e_{11}u = e_{12}e_{12} = e_{12}u = u^2 = 0$. The duplicate algebra is an evolution algebra in the natural basis $\{e_{11}, e_{12}, u\}$.

For z and w in $D(\mathcal{E})$, we notice that the product in $D(\mathcal{E})$ is given by $zw = \mu(z).\mu(w)$. So, if $\mathcal{E}$ is a zero algebra, then for all $z, w \in D(\mathcal{E})$, we have $zw = \mu(z).\mu(w) = 0$ because $\mu(z) = \mu(w) = 0$. Consequently, $D(\mathcal{E})$ is an evolution algebra.

Theorem 3.2. *Let $\mathcal{E}$ be a n-dimensional non zero commutative F-algebra and $D(\mathcal{E})$ its commutative duplicate. Then $D(\mathcal{E})$ is an evolution algebra if and only if* $\dim(\mathcal{E}^2) = 1$.

Proof. Let us suppose that $D(\mathcal{E})$ is an evolution algebra in the natural basis $\{z_1, \dots, z_s\}$, with $s = \frac{n(n+1)}{2}$. For $i \neq j$, the equality $z_i z_j = 0$ is equivalent to $\mu(z_i).\mu(z_j) = 0$. Since $\mathcal{E}^2 \neq \{0\}$, it follows that there exists i_0 such that $\mu(z_{i_0}) \neq 0$. Thus, $\mu(z_j) = 0$ for all $j \neq i_0$, $z_j \in N_F(\mathcal{E}) = \{x \in D(\mathcal{E}) \mid x \cdot D(\mathcal{E}) = 0\} = ann(D(\mathcal{E}))$, where $ann(D(\mathcal{E}))$ is the *annihilator* of $D(\mathcal{E})$. So $\dim(N_F(\mathcal{E})) = s - 1$ and $\dim(\mathcal{E}^2) = 1$.

Conversely, let $\mathcal{E}$ be a commutative F-algebra such that $\dim(\mathcal{E}^2) = 1$. According to Proposition 2.4, such an algebra is an evolution algebra, the natural basis $\{e_1, e_2, \dots, e_n\}$ being that orthogonal. Since $D(\mathcal{E})/N_F(\mathcal{E}) \simeq \mathcal{E}^2$, it follows that $\dim(N_F(\mathcal{E})) = s - 1$. If $e_{i_0}^2 \neq 0$, then $(e_{i_0} \cdot e_{i_0})^2 = e_{i_0}^2 \cdot e_{i_0}^2 \neq 0$, generates $D(\mathcal{E})^2$ and we always deduce from Proposition 2.4 that $D(\mathcal{E})$ is an evolution algebra. □

4. Bernstein Algebra

A finite dimensional commutative algebra $\mathcal{E}$ over a commutative field F is said to be *baric*, if there is nontrivial homomorphism $\omega : \mathcal{E} \longrightarrow F$ of algebras. The baric algebra $(\mathcal{E}, \omega)$ is called *Bernstein algebra* if

$$x^2 x^2 - \omega(x)^2 x^2 = 0, \text{ for all } x \in \mathcal{E}. \tag{6}$$

Bernstein algebras have their origins in genetics ([2]). Holgate was the first to use the language of non-associative algebras to translate Bernstein's problem ([8]).
We defined inductively *plenary powers* of an element $x \in \mathcal{E}$ by :

$$x^{(1)} = x \text{ and } x^{(k+1)} = x^{(k)} x^{(k)}, \quad k \in \mathbb{N},$$

while that of $\mathcal{E}$ is defined by :

$$\mathcal{E}^{(1)} = \mathcal{E} \text{ and } \mathcal{E}^{(k+1)} = \mathcal{E}^{(k)} \mathcal{E}^{(k)}, \quad k \in \mathbb{N}.$$

4.1. Some properties of Bernstein algebras

Let $(\mathcal{E},\omega)$ be a Bernstein algebra over a commutative field F of $Char(F)\neq 2$. The following results are well known ([16]).

1) The homomorphism $\omega:\mathcal{E}\longrightarrow F$ is the unique weight function of $\mathcal{E}$.
2) Algebra $\mathcal{E}$ has at least one non-zero idempotent.
3) For an idempotent e of $\mathcal{E}$, the algebra $\mathcal{E}$ admits the following Peirce decomposition $\mathcal{E}=Fe\oplus U_e\oplus V_e$, where $U_e=\{x\in\mathcal{E}\mid\ ex=\frac{1}{2}x\}$ and $V_e=\{x\in\mathcal{E}\mid\ ex=0\}$. The subspaces U_e and V_e satisfy the relations
$$U_eV_e\subseteq U_e,\ V_e^2\subseteq U_e,\ U_e^2\subseteq V_e \text{ and } U_eV_e^2=0$$
4) The set of idempotents of $\mathcal{E}$ is given by $\mathcal{I}(\mathcal{E})=\{e+\sigma+\sigma^2\mid\ \sigma\in U_e\}$ for any idempotent e of $\mathcal{E}$.
5) Let $e_1=e+\sigma+\sigma^2$, with $\sigma\in U_e$, be another idempotent of $\mathcal{E}$. We have the following relations $U_{e_1}=\{u+\sigma u\mid\ u\in U_e\}$ and $V_{e_1}=\{v-2(\sigma+\sigma^2)v\mid\ v\in V_e\}$. It follows that although the decomposition of the Bernstein algebra depends on the choice of the idempotent e, the dimension of the subspaces U_e and V_e of $\mathcal{E}$ are invariants of $\mathcal{E}$. If $r=\dim U_e$ and $s=\dim V_e$, the pair $(1+r,s)$ is called the type of $\mathcal{E}$. Also $\dim_F(U_e^2)$ and $\dim_F(U_eV_e+V_e^2)$ are invariants of the algebra $\mathcal{E}$.

In ([1]), the authors obtain the identities (7) and (8) by linearizing (6).
$$2x^2(xy)=\omega(xy)x^2+\omega(x^2)(xy) \tag{7}$$
$$4(xz)(xy)+2x^2(zy)=\omega(zy)x^2+2\omega(xy)(xz)+2\omega(xz)(xy)+\omega(x^2)(zy) \tag{8}$$
for all $x,y,z\in\mathcal{E}$ and replacing y by z in (8), we get
$$4(xz)^2+2x^2z^2=\omega(z)^2x^2+4\omega(xz)(xz)+\omega(x^2)z^2 \tag{9}$$
for all $x,z\in\mathcal{E}$.

4.2. Characterization of Bernstein algebras that are evolution algebras

Let F be a commutative field of $Char(F)\neq 2$.

Theorem 4.1 ([13, Corollary 3.1.4]). *A n-dimensional baric evolution algebra $(\mathcal{E},\omega)$ admits a natural basis $\{e_1,e_2,\dots,e_n\}$ such that $\omega(e_1)=1$ and $\omega(e_i)=0$ for $i>1$. Moreover $\mathcal{E}=Fe_1\oplus\ker\omega$ with $e_1\ker\omega=0$.*

We deduce from Theorem 4.1 that the algebra $(\mathcal{E},\omega)$ admits a natural basis $\{e_1,e_2,\dots,e_n\}$ which multiplication table is given by
$$e_1^2=e_1+\sum_{k=2}^{n}a_{1k}e_k,\ e_j^2=\sum_{k=2}^{n}a_{jk}e_k \tag{10}$$

with $\omega(e_1) = 1$, $\omega(e_j) = 0$ and $2 \leq j \leq n$.

In the following, any finite n-dimensional baric evolution algebra will be provided with such a natural basis.

Theorem 4.2 (of characterization)**.** *A n-dimensional baric evolution algebra is a Bernstein algebra $(\mathcal{E}, \omega)$ if and only if the following conditions are satisfying*

i) $(e_1^2)^2 = e_1^2$ *;*

ii) $e_i^2 e_j^2 = 0$*, for* $2 \leq i, j \leq n$ *;*

iii) $e_1^2 e_i^2 = \frac{1}{2} e_i^2$*, for* $2 \leq i \leq n$.

Proof. Let us suppose that algebra $(\mathcal{E}, \omega)$ is a Bernstein algebra. Then
(6) leads to $i)$, we take $x = e_1$.
(9) gives $ii)$, we set $x = e_i$ and $z = e_j$ with $i, j \neq 1$.
(9) gives $iii)$, we take $x = e_1$ and $z = e_i$ with $i \neq 1$.

Conversely, it is assumed that conditions $i)$, $ii)$ and $iii)$ are satisfied. Let $x = \sum_{k=1}^{n} x_k e_k$ be an element of $\mathcal{E}$ with $\omega(x) = x_1$. We have the following equalities $x^2 = \sum_{k=1}^{n} x_k^2 e_k^2 = x_1^2 e_1^2 + \sum_{k=2}^{n} x_k^2 e_k^2$ and $x^2 x^2 = x_1^2 x_1^2 e_1^2 e_1^2 + 2x_1^2 \sum_{k=2}^{n} x_k^2 e_1^2 e_k^2 + \sum_{k,j=2}^{n} x_k^2 x_j^2 e_k^2 e_j^2 = x_1^2 (x_1^2 e_1^2 + \sum_{k=2}^{n} x_k^2 e_k^2) = \omega(x)^2 x^2$. So the baric evolution algebra $(\mathcal{E}, \omega)$ is a Bernstein algebras. □

We see that e_1^2 is a non-zero idempotent of $\mathcal{E}$ and $e_i^2 \in U_{e_1^2}$ for $i \neq 1$. We deduce that $(\ker \omega)^2 \subseteq U_{e_1^2}$.

Proposition 4.3. *If a n-dimensional baric evolution algebra $(\mathcal{E}, \omega)$ is a Bernstein algebra, then*

i) $U_{e_1^2} = \{x \in \ker \omega \mid e_1^2 x = \frac{1}{2} x\} = (\ker \omega)^2$ *and*

ii) $V_{e_1^2} = \{x \in ker\omega \mid e_1^2 x = 0\} = \langle e_i - 2a_{1i} e_i^2 \mid 2 \leq i \leq n \rangle$.

Proof. $i)$ Let us show that $(\ker \omega)^2 = U_{e_1^2}$. Since $(\ker \omega)^2 \subseteq U_{e_1^2}$, it remains to show that $U_{e_1^2} \subseteq (\ker \omega)^2$. Let $x = \sum_{i=2}^{n} x_i e_i \in U_{e_1^2}$,
then $x = 2e_1^2 x = 2\sum_{i=2}^{n} x_i (a_{1i} e_i^2) \in (\ker \omega)^2$. Hence $U_{e_1^2} \subseteq (\ker \omega)^2$ and $U_{e_1^2} = (\ker \omega)^2$.

$ii)$ For $i \in \{2, \ldots, n\}$, we have $e_1^2 (e_i - 2a_{1i} e_i^2) = 0$; so $\langle e_i - 2a_{1i} e_i^2 \mid 2 \leq i \leq n \rangle \subset V_{e_1^2}$. Let $x = \sum_{i=2}^{n} x_i e_i \in V_{e_1^2}$, then $0 = e_1^2 x = \sum_{i=2}^{n} x_i a_{1i} e_i^2$. Thus $x = \sum_{i=2}^{n} x_i (e_i - 2a_{1i} e_i^2)$ and we have $V_{e_1^2} \subset \langle e_i - 2a_{1i} e_i^2 \mid 2 \leq i \leq n \rangle$. We deduce that $V_{e_1^2} = \langle e_i - 2a_{1i} e_i^2 \mid 2 \leq i \leq n \rangle$. □

Remark 4.4. If the baric evolution algebra $(\mathcal{E}, \omega)$ is a Bernstein algebra, then $U_{e_1^2}^2 = (\ker \omega)^{(3)} = (\ker \omega)^2 (\ker \omega)^2 = 0$, i.e. $\mathcal{E}$ is a *exceptional Bernstein* algebra ([7]).

Definition 4.5 ([17])**.** Let $(\mathcal{E},\omega)$ be a $(n+1)$-dimensional Bernstein algebra of type $(r+1,s)$. If $\ker\omega$ is a *zero algebra*, i.e. $(\ker\omega)^2=0$, then the algebra $\mathcal{E}$ is called a *trivial Bernstein algebra* of type $(r+1,s)$.

Remark 4.6. In ([12]), the authors show that an algebra is a Jordan Bernstein algebra if and only if it is a train algebra of rank 3. We deduce that a finite dimensional evolution algebra $(\mathcal{E},\omega)$ is a Jordan Bernstein algebra if and only if $(\ker\omega)^2=0$ ([13, Theorem 3.2.3]). Thus, the only finite dimensional evolution algebras $(\mathcal{E},\omega)$, that are Jordan Bernstein algebras, are evolution algebras, that are trivial Bernstein algebras.

Proposition 4.7. *If a baric evolution algebra $(\mathcal{E},\omega)$ is a 2-dimensional Bernstein algebra, then $\mathcal{E}$ is a trivial Bernstein algebra.*

Proof. Since $\ker\omega=\langle e_2\rangle$, it follows that there are $\alpha\in F$ such that $e_2^2=\alpha e_2$. $0=(\ker\omega)^{(3)}$ leads to $0=e_2^2e_2^2=\alpha^3e_2$; hence $\alpha^3=0$, i.e. $\alpha=0$. We deduce that $(\ker\omega)^2=0$ and the algebra $\mathcal{E}$ is a trivial Bernstein algebra. □

Proposition 4.8. *If a finite-dimensional baric evolution algebra $(\mathcal{E},\omega)$ is a Bernstein algebra, then $U_{e_1^2}V_{e_1^2}$ is an invariant of $\mathcal{E}$. Moreover, if $U_{e_1^2}\neq 0$, then $U_{e_1^2}V_{e_1^2}\neq 0$.*

Proof. Let $e=e_1^2+\sigma+\sigma^2\in\mathcal{I}(\mathcal{E})$, $u=u_1+\sigma u_1\in U_e$ and $v=v_1-2(\sigma+\sigma^2)v_1\in V_e$ with $\sigma,u_1\in U_{e_1^2}$ and $v_1\in V_{e_1^2}$. Since $U_{e_1^2}^2=0$, we have $e=e_1^2+\sigma$, $u=u_1$ and $v=v_1-2\sigma v_1$. We also have $uv=u_1(v_1-2\sigma v_1)=u_1v_1-2u_1(\sigma v_1)=u_1v_1$ because $U_{e_1^2}(U_{e_1^2}V_{e_1^2})\subset U_{e_1^2}^2=0$. So $U_eV_e=U_{e_1^2}V_{e_1^2}$.

We assume that $dim_F(\ker\omega)^2=k\neq 0$. By renumbering the vectors of the family $\{e_2,\dots,e_n\}$, we can assume that the family $\{e_j^2 \mid 2\le j\le k+1\}$ is a basis of $(\ker\omega)^2$. Set $e_j^2=\sum_{t=2}^{k+1}\alpha_{jt}e_t^2$ with $k+2\le j\le n$. If $U_{e_1^2}V_{e_1^2}=0$, then we would have $e_2^2(e_j-2a_{1j}e_j^2)=0$ for $2\le j\le n$. What would result

$$\begin{cases} a_{2j}=0,\ 2\le j\le k+1\\ a_{2j}\alpha_{jt}=0,\ 2+k\le j\le n \text{ and } 2\le t\le k+1.\end{cases}$$

Thus, we would have $\frac{1}{2}e_2^2=e_1^2e_2^2=e_1^2\sum_{j=2}^n a_{2j}e_j=e_1^2\sum_{j=k+2}^n a_{2j}e_j=\sum_{j=k+2}^n a_{2j}a_{1j}e_j^2=\sum_{t=2}^{k+1}\sum_{j=k+2}^n a_{1j}(a_{2j}\alpha_{jt})e_t^2=0$, so, $e_2^2=0$. This would contradict linear independence of the family $\{e_j^2 \mid 2\le j\le k+1\}$. We deduce that $U_{e_1^2}V_{e_1^2}\neq 0$. □

Lemma 4.9. *If a n-dimensional baric evolution algebra $(\mathcal{E},\omega)$ is a Bernstein algebra, then the family $\{e_i^2 \mid 2\le i\le n\}$ is linear dependent.*

Proof. We have $\ker\omega=<e_2,\dots,e_n>$ and $(\ker\omega)^2\subset\ker\omega$. We assume that the family is linear independent. Then $(\ker\omega)^2=\ker\omega$; hence $0=(ker\omega)^{(3)}=$

$(ker\omega)^2 = ker\omega$, this is impossible. We deduce that the family is linear dependent. □

Theorem 4.10. *If a n-dimensional baric evolution algebra* $(\mathcal{E},\omega)$ *(with* $n > 2$*) is a non trivial Bernstein algebra, then* $\dim_F(\ker\omega)^2 \leq \frac{1}{2}(n-1)$.

Proof. We have $0 \neq (\ker\omega)^2 \subsetneq \ker\omega$. We assume that $p = \dim_F(\ker\omega)^2$. By renumbering the basis vectors, we can assume that $(\ker\omega)^2 =< e_2^2,\dots,e_{p+1}^2 >$. Let us show that the family $\{e_2,\dots,e_{p+1},e_2^2,\dots,e_{p+1}^2\}$ is linear independent. Let $(\alpha_k,\beta_k)_{2\leq k\leq p+1} \in F^p \times F^p$ such that

$$\sum_{k=2}^{p+1}(\alpha_k e_k + \beta_k e_k^2) = 0 \tag{11}$$

By multiplying (11) by e_i, we obtain $\alpha_i e_i^2 + \sum_{k=2}^{p+1}\beta_k e_i e_k^2 = \alpha_i e_i^2 + \sum_{k=2}^{p+1}\beta_k a_{ki} e_i^2 = 0$, either

$$\alpha_i + \sum_{k=2}^{p+1}\beta_k a_{ki} = 0, \text{ for all } i \in \{2,\dots,p+1\}. \tag{12}$$

By squaring (11), we get
$\sum_{k=2}^{p+1}\left(\alpha_k^2 e_k^2 + \sum_{j=2}^{p+1} 2\alpha_k\beta_j e_k e_j^2\right) = \sum_{k=2}^{p+1}\alpha_k\left(\alpha_k + 2\sum_{j=2}^{p+1}\beta_j a_{jk}\right)e_k^2 = 0$, either

$$\alpha_i(\alpha_i + 2\sum_{j=2}^{p+1}\beta_j a_{ji}) = 0, \text{ for all } i \in \{2,\dots,p+1\}. \tag{13}$$

By multiplying (12) by $2\alpha_i$ we get

$$\alpha_i(2\alpha_i + 2\sum_{j=2}^{p+1}\beta_j a_{ji}) = 0, \text{ for all } i \in \{2,\dots,p+1\} \tag{14}$$

and by making the difference of (13) and (14), we have $\alpha_i^2 = 0$. This leads to $\alpha_i = 0$, for all $i \in \{2,\dots,p+1\}$. Then (11) tell us that $\beta_i = 0, \forall i \in \{2,\dots,p+1\}$. We deduce that $\dim_F(\ker\omega)^2 \leq \frac{1}{2}(n-1)$. □

Corollary 4.11. *If a finite n-dimensional baric evolution algebra* $(\mathcal{E},\omega)$ *is a Bernstein algebra such that* $\dim(U_{e_1^2}) = p$*, then* $\dim(V_{e_1^2}) \geq p$.

Proof. We assume that $(\ker\omega)^2 = \langle e_2^2,\dots,e_{p+1}^2\rangle$ and let us show that the family $\{e_2 - 2a_{12}e_2^2,\dots,e_{p+1} - 2a_{1,p+1}e_{p+1}^2\}$ is linear independent. Let $(\alpha_k)_{2\leq k\leq p+1} \in F^p$ such that $\sum_{k=2}^{p+1}\alpha_k(e_k - 2a_{1k}e_k^2) = 0$.
We have $\sum_{k=2}^{p+1}\alpha_k(e_k - 2a_{1k}e_k^2) = \sum_{k=2}^{p+1}\alpha_k e_k - 2a_{1k}\alpha_k e_k^2 = 0$. So $\alpha_k = 0$, for all $k \in \{2,\dots,p+1\}$ because $\{e_2,\dots,e_{p+1},e_2^2,\dots,e_{p+1}^2\}$ is linear independent. Consequently, the family $\{e_2 - 2a_{12}e_2^2,\dots,e_{p+1} - 2a_{1,p+1}e_{p+1}^2\}$ is linear independent and $\dim(V_{e_1^2}) \geq\ p$. □

4.3. Classification

Let $(\mathcal{E},\omega)$ be a Bernstein algebra that is evolution algebra in natural basis $\{e_1,e_2,\dots,e_n\}$ such that $e_1^2 = e_1 + \sum_{k=2}^n a_{1k}e_k$ and $e_j^2 = \sum_{k=2}^n a_{jk}e_k$. For $(a_{12},\dots,a_{1n}) = 0$, we have $e_1^2 = e_1$, i.e. e_1 is a non-zero idempotent of $\mathcal{E}$ and $e_1 \ker\omega = 0$ leads to $\mathcal{E}$ is of type $(1,n-1)$, constant Bernstein algebra.

4.3.1. Three-dimensional Classification

Theorem 4.12. *Let $(\mathcal{E},\omega)$ be an evolution algebra that is a 3-dimensional non trivial Bernstein algebra with canonical basis $\{e,u,v\}$. Then, the algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_0$: $e^2 = e$, $eu = \frac{1}{2}u$, $uv = u$, the others products are zero.*

Proof. Let $(\mathcal{E},\omega)$ be a 3-dimensional non trivial Bernstein algebra that is evolution algebra in the natural basis $\{e_1,e_2,e_3\}$. The multiplication table of $\mathcal{E}$ in the natural basis is given by $e_1^2 = e_1 + a_{12}e_2 + a_{13}e_3$, $e_2^2 = a_{22}e_2 + a_{23}e_3$ and $e_3^2 = a_{32}e_2 + a_{33}e_3$ with $\omega(e_1) = 1$ and $\omega(e_2) = \omega(e_3) = 0$. We have $(\ker\omega)^2 \neq 0$ and $1 \leq \dim(\ker\omega)^2 \leq \frac{1}{2}(3-1) = 1$. So $\dim(\ker\omega)^2 = 1$ and we set $(\ker\omega)^2 = Fe_2^2$. Then the vector $e_2 - 2a_{12}e_2^2$ is a non-zero vector of $V_{e_1^2}$ and we set $e = e_1^2$, $u = e_2^2$, $v = e_2 - 2a_{12}e_2^2$. The multiplication table of $\mathcal{E}$ in the canonical basis $\{e,u,v\}$ is $e^2 = e$, $eu = \frac{1}{2}u$, $uv = e_2^2(e_2 - 2a_{12}e_2^2) = a_{22}u$ and $v^2 = (e_2 - 2a_{12}e_2^2)^2 = e_2^2 - 4a_{12}e_2^2e_2 = (1-4a_{12}a_{22})u$. Since the algebra $\mathcal{E}$ is a non trivial Bernstein algebra, it follows that $U_{e_1^2}V_{e_1^2} \neq 0$. Consequently, $a_{22} \neq 0$. Let us find a canonical basis $\{e',u',v'\}$ of $\mathcal{E}$ such that $u'v' = u'$ and $v'^2 = 0$. We set $e' = e + au$, $u' = bu$ and $v' = cv - 2au(cv) = c(v - 2aa_{22}u)$ with $b,c \in F^*$. We have $u' = u'v' = bcuv = a_{22}bcu = a_{22}cu'$ leads to $c = a_{22}^{-1}$ and $0 = v'^2 = c^2(v - 2aa_{22}u)^2 = a_{22}^{-2}(v^2 - 4aa_{22}uv) = a_{22}^{-2}((1-4a_{12}a_{22}) - 4aa_{22}^2)u = a_{22}^{-2}b^{-1}((1-4a_{12}a_{22}) - 4aa_{22}^2)u'$ implies $0 = (1-4a_{12}a_{22}) - 4aa_{22}^2$, i.e. $a = a_{22}^{-2}(\frac{1}{4} - a_{12}a_{22})$. We can take $b = 1$ and we have $e' = e + a_{22}^{-2}(\frac{1}{4} - a_{12}a_{22})u$, $u' = u$ and $v' = a_{22}^{-1}(v - a_{22}^{-1}(\frac{1}{2} - 2a_{12}a_{22})u)$. We deduce that algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_0$. □

4.3.2. Four-dimensional Classification

Theorem 4.13. *Let $(\mathcal{E},\omega)$ be an evolution algebra that is 4-dimensional non trivial Bernstein algebra with canonical basis $\{e,u,v,w\}$. Then, $\mathcal{E}$ is isomorphic to one and only one of the following algebras $\mathcal{E}_1$: $uv = u$, $e^2 = e$, $eu = \frac{1}{2}u$; $\mathcal{E}_2$: $uv = uw = vw = u$, $e^2 = e$, $eu = \frac{1}{2}u$ and the others products are zero.*

The proof of the theorem uses the lemma below which follows from [5, Proof of Theorem, page 1435].

Lemma 4.14. *Let $\mathcal{E}$ be a 4-dimensional Bernstein algebra with a canonical basis $\{e,u,v,w\}$ such that $e^2 = e$, $eu = \frac{1}{2}u$, $uv = u$, $v^2 = \gamma u$, $w^2 = \lambda u$, $vw = \mu u$ and the others products are zero.*
- *If $\lambda = \mu = 0$, then the algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_1$.*
- *If $\lambda \neq 0$, then the algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_2$.*

Where the algebras $\mathcal{E}_1$ and $\mathcal{E}_2$ are defined in Theorem 4.13.

Proof of Theorem 4.13. Let $(\mathcal{E},\omega)$ be a 4-dimensional non trivial Bernstein algebra that is evolution algebra with the natural basis $\{e_1,e_2,e_3,e_4\}$. The multiplication table of $\mathcal{E}$ in the natural basis is given by $e_1^2 = e_1 + \sum_{k=2}^4 a_{1k}e_k$, $e_j^2 = \sum_{k=2}^4 a_{jk}e_k$ with $\omega(e_1) = 1$ and $\omega(e_j) = 0$ where $2 \leq j \leq 4$. We have $(\ker\omega)^2 \neq 0$ and $1 \leq \dim(\ker\omega)^2 \leq \frac{1}{2}(4-1) = 1.5$. So $\dim(\ker\omega)^2 = 1$ and we set $(\ker\omega)^2 = Fe_2^2$. Then $e_2 - 2a_{12}e_2^2$ is a non-zero vector of $V_{e_1^2}$ and there are scalars α_3, α_4 such that $e_3^2 = \alpha_3 e_2^2$ and $e_4^2 = \alpha_4 e_2^2$. We assume that $a_{23} = a_{24} = 0$, then the equality $0 = e_2^2e_2^2 = a_{22}^2e_2^2$ leads to $a_{22} = 0$. Thus $e_2^2 = 0$, this is impossible and we deduce that $(a_{23},a_{24}) \neq 0$. Since $V_{e_1^2}$ is generated by $(e_2-2a_{12}e_2^2),(e_3-2a_{13}e_3^2),(e_4-2a_{14}e_4^2)$, let us show that $\{e_2-2a_{12}e_2^2, e_3 - 2a_{13}e_3^2\}$ or $\{e_2-2a_{12}e_2^2, e_4-2a_{14}e_4^2\}$ is a basis of $V_{e_1^2}$. For this reason, consider the scalars α, β and γ such that $0 = \alpha(e_2-2a_{12}e_2^2)+\beta(e_3-2a_{13}e_3^2)+\gamma(e_4-2a_{14}e_4^2)$. Since $\alpha(e_2-2a_{12}e_2^2)+\beta(e_3-2a_{13}e_3^2)+\gamma(e_4-2a_{14}e_4^2) = (\alpha - 2(\alpha a_{12}+\beta a_{13}\alpha_3+\gamma a_{14}\alpha_4)a_{22})e_2 + (\beta - 2(\alpha a_{12}+\beta a_{13}\alpha_3\gamma a_{14}\alpha_4)a_{23})e_3 + (\gamma - 2(\alpha a_{12}+\beta a_{13}\alpha_3+\gamma a_{14}\alpha_4)a_{24})e_4$, it follows that the equality
$0 = \alpha(e_2-2a_{12}e_2)+\beta(e_3-2a_{13}e_3^2)+\gamma(e_4-2a_{14}e_4^2)$ gives
$(\alpha - 2(\alpha a_{12}+\beta a_{13}\alpha_3+\gamma a_{14}\alpha_4)a_{22}) = (\beta - 2(\alpha a_{12}+\beta a_{13}\alpha_3+\gamma a_{14}\alpha_4)a_{23}) = (\gamma - 2(\alpha a_{12}+\beta a_{13}\alpha_3+\gamma a_{14}\alpha_4)a_{24}) = 0$.

If $a_{23} \neq 0$, for $\beta = 0$, we have $\alpha a_{12}+\beta a_{13}\alpha_3+\gamma a_{14}\alpha_4 = 0$ because $a_{23} \neq 0$. Hence $\alpha = \gamma = 0$ and $\{e_2-2a_{12}e_2^2, e_4-2a_{14}e_4^2\}$ is a basis of $V_{e_1^2}$.

If $a_{24} \neq 0$, for $\gamma = 0$, we have $\alpha = \beta = 0$ and we similarly conclude that $\{e_2-2a_{12}e_2^2, e_3-2a_{13}e_3^2\}$ is a basis of $V_{e_1^2}$.

1) The multiplication table of $\mathcal{E}$ in the canonical basis $\{e_1^2, e_2^2, e_2-2a_{12}e_2^2, e_3 - 2a_{13}e_3^2\}$ is given by $e_1^2e_1^2 = e_1^2$, $e_1^2e_2^2 = \frac{1}{2}e_2^2$, $e_2^2(e_2-2a_{12}e_2^2) = e_2^2e_2 = a_{22}e_2^2$, $e_2^2(e_3-2a_{13}e_3^2) = e_2^2e_3 = a_{23}e_3^2 = a_{23}\alpha_3e_2^2$, $(e_2-2a_{12}e_2^2)^2 = e_2^2 - 4a_{12}e_2^2e_2 = (1-4a_{12}a_{22})e_2^2$, $(e_2-2a_{12}e_2^2)(e_3-2a_{13}e_3^2) =$ $(e_2-2a_{12}e_2^2)(e_3-2a_{13}\alpha_3e_2^2) = -2a_{13}\alpha_3e_2^2e_2 - 2a_{12}e_2^2e_3 = -2a_{13}\alpha_3e_2^2e_2 - 2a_{12}a_{23}e_3^2 = -2\alpha_3(a_{13}a_{22}+a_{12}a_{23})e_2^2$, $(e_3-2a_{13}e_3^2)^2 = (e_3-2a_{13}\alpha_3e_2^2)^2 = e_3^2 - 4a_{13}\alpha_3e_2^2e_3 = e_3^2 - 4a_{13}a_{23}\alpha_3e_3^2 = \alpha_3(1-4a_{13}a_{23}\alpha_3)e_2^2$ and the others products are zero. Since algebra $\mathcal{E}$ is a non trivial Bernstein algebra, we have $U_{e_1^2}V_{e_1^2} \neq 0$. Consequently, $(a_{22}, a_{23}\alpha_3) \neq 0$ and we set $e = e_1^2$, $u = e_2^2$. We distinguish the following three cases

   a) $a_{22} = 0$, then $a_{23}\alpha_3 \neq 0$ and we set $v = a_{23}^{-1}\alpha_3^{-1}(e_3-2a_{13}e_3^2)$, $w = e_2 -$

$2a_{12}e_2^2$. The multiplication table of $\mathcal{E}$ in the canonical basis $\{e,u,v,w\}$ is $e^2=e$, $eu=\frac{1}{2}u$, $uv=u$, $v^2=a_{23}^{-2}\alpha_3^{-1}(1-4a_{13}a_{23}\alpha_3)u$, $vw=-2a_{12}u$, $w^2=u$ and the others products are zero. We deduce from Lemma 4.14 that the algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_2$.

b) $\alpha_3 a_{23}=0$, then $a_{22}\neq 0$, we set $v=a_{22}^{-1}(e_2-2a_{12}e_2^2)$ and $w=e_3-2a_{13}e_3^2$. The multiplication table of $\mathcal{E}$ in the canonical basis $\{e,u,v,w\}$ is given by $e^2=e$, $eu=\frac{1}{2}u$, $uv=u$, $v^2=a_{22}^{-2}(1-4a_{12}a_{22})u$, $vw=-2a_{13}\alpha_3 u$, $w^2=\alpha_3 u$ and the others products are zero. Lemma 4.14 tells us that, for $\alpha_3=0$ we get algebra $\mathcal{E}_1$ and for $\alpha_3\neq 0$, algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_2$.

c) $a_{22}a_{23}\alpha_3\neq 0$, then we set $v=a_{22}^{-1}(e_2-2a_{12}e_2^2)$ and $w=(e_3-2a_{13}e_3^2)-a_{22}^{-1}a_{23}\alpha_3(e_2-2a_{12}e_2^2)$. The multiplication table of $\mathcal{E}$ in the canonical basis $\{e,u,v,w\}$ is $e^2=e$, $eu=\frac{1}{2}u$, $uv=u$, $v^2=a_{22}^{-2}(1-4a_{12}a_{22})u$, $vw=a_{22}^{-1}(e_2-2a_{12}e_2^2)(e_3-2a_{13}e_3^2)-a_{22}^{-2}a_{23}\alpha_3(e_2-2a_{12}e_2^2)^2=$ $(-2\alpha_3 a_{22}^{-1}(a_{13}a_{22}+a_{12}a_{23})-a_{22}^{-2}a_{23}\alpha_3(1-4a_{12}a_{22}))u=$ $-a_{22}^{-2}a_{23}\alpha_3(2a_{13}a_{22}^2a_{23}^{-1}-2a_{12}a_{22}+1)u$, $w^2=(e_3-2a_{13}e_3^2)^2+$ $a_{22}^{-2}a_{23}^2\alpha_3^2(e_2-2a_{12}e_2^2)^2-2a_{22}^{-1}a_{23}\alpha_3(e_3-2a_{13}e_3^2)(e_2-2a_{12}e_2^2)=$ $(\alpha_3(1-4a_{13}a_{23}\alpha_3)+a_{22}^{-2}a_{23}^2\alpha_3^2(1-4a_{12}a_{22})+$ $4a_{22}^{-1}a_{23}\alpha_3^2(a_{13}a_{22}+a_{12}a_{23}))u=\alpha_3(1+a_{22}^{-2}a_{23}^2\alpha_3)u$ and the others products are zero.

- For $1+a_{22}^{-2}a_{23}^2\alpha_3\neq 0$, algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_2$.
- For $1+a_{22}^{-2}a_{23}^2\alpha_3=0$, $w^2=\alpha_3(1+a_{22}^{-2}a_{23}^2\alpha_3)u=0$. We have $0=e_2^2e_2^2=a_{22}^2e_2^2+a_{23}^2e_3^2+a_{24}^2e_4^2=(a_{22}^2+a_{23}^2\alpha_3+a_{24}^2\alpha_4)e_2^2=$ $a_{22}^2(1+a_{22}^{-2}a_{23}^2\alpha_3+a_{22}^{-2}a_{24}^2\alpha_4)e_2^2=a_{24}^2\alpha_4e_2^2$ leads to $\alpha_4=0$ because $a_{24}\neq 0$. So $e_4^2=0$ and we have $\frac{1}{2}e_2^2=e_1^2e_2^2=a_{12}a_{22}e_2^2+a_{13}a_{23}e_3^2=$ $(a_{12}a_{22}-a_{13}a_{22}^2a_{23}^{-1})e_2^2$ gives $a_{12}a_{22}-a_{13}a_{22}^2a_{23}^{-1}=\frac{1}{2}$. Therefore $vw=$ $-a_{22}^{-2}a_{23}\alpha_3(2a_{13}a_{22}^2a_{23}^{-1}-2a_{12}a_{22}+1)u=0$ and we deduce that algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_1$.

2) The multiplication table of $\mathcal{E}$ in the canonical basis $\{e_1^2,e_2^2,e_2-2a_{12}e_2^2,e_4-2a_{14}e_4^2\}$ is given by $e_1^2e_1^2=e_1^2$, $e_1^2e_2^2=\frac{1}{2}e_2^2$, $e_2^2(e_2-2a_{12}e_2^2)=e_2^2e_2=a_{22}e_2^2$, $e_2^2(e_4-2a_{14}e_4^2)=e_2^2e_4=a_{24}\alpha_4e_2^2$, $(e_2-2a_{12}e_2^2)^2=e_2^2-4a_{12}e_2^2e_2=(1-4a_{12}a_{22})e_2^2$, $(e_2-2a_{12}e_2^2)(e_4-2a_{14}e_4^2)=(e_2-2a_{12}e_2^2)(e_4-2a_{14}\alpha_4e_2^2)=$ $-2a_{14}\alpha_4e_2^2e_2-2a_{12}e_2^2e_4=-2a_{14}\alpha_4e_2^2e_2-2a_{12}a_{24}e_4^2=$ $-2\alpha_4(a_{14}a_{22}+a_{12}a_{24})e_2^2$, $(e_4-2a_{14}e_4^2)^2=(e_4-2a_{14}\alpha_4e_2^2)^2=$ $e_4^2-4a_{14}\alpha_4e_2^2e_4=e_4^2-4a_{14}a_{24}\alpha_4e_4^2=\alpha_4(1-4a_{14}a_{24}\alpha_4)e_2^2$ and the others products are zero. We obtain the multiplication table of the algebra defined in 1). We deduce that, for $a_{22}=0$ or for $\alpha_4\neq 0$ and $a_{24}=0$ or for $a_{22}a_{24}\alpha_4(1+a_{22}^{-2}a_{24}^2\alpha_4)\neq 0$, algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_2$. It is isomorphic to algebra $\mathcal{E}_1$ for $\alpha_4=0$ or for $a_{22}a_{24}\alpha_4\neq 0$ and $1+a_{22}^{-2}a_{24}^2\alpha_4=0$.

□

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A. CONSEIBO
Université Norbert Zongo, BP 376 Koudougou, Burkina Faso
e-mail: andreconsebo@yahoo.fr

S. SAVADOGO
Université Norbert Zongo, BP 376 Koudougou, Burkina Faso
e-mail: sara01souley@yahoo.fr

M. OUATTARA
Université Joseph KI-ZERBO, 03 BP 7021 Ouagadougou 03, Burkina Faso
e-mail: ouatt_ken@yahoo.fr