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MINIMAL PRIME IDEALS OF SKEW HURWITZ SERIES RINGS

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Let R be a ring with an endomorphism α. In this paper we obtain necessary and sufficient conditions on R and α such that the skew Hurwitz series ring (HR,α) is a 2-primal ring. In particular, it is proved that, under suitable conditions, (HR,α) is 2-primal if and only if for every minimal prime ideal $\mathcal{P}^*$ in (HR,α) there exists a minimal prime ideal P of R such that P is completely prime and $\mathcal{P}^* = (HP,\alpha)$ if and only if $\mathbf{P}((HR,\alpha)) = (H(nil(R)),\alpha)$ if and only if R is 2-primal and $nil((HR,\alpha)) = (H(nil(R)),\alpha)$ if and only if every minimal α-prime ideal of R is completely prime.

1. Introduction

The prime radical of a ring R and the set of all nilpotent elements in R are denoted by $\mathbf{P}(R)$ and $nil(R)$, respectively. Recall that $\mathbf{P}(R)$ is the set of all strongly nilpotent elements of R. A ring R is called *2-primal* if $\mathbf{P}(R) = nil(R)$. It is obvious that commutative rings and *reduced rings* (i.e., rings without nonzero nilpotent elements) are 2-primal. The term 2-primal was originated by Birkenmeier, Heatherly and Lee [1] in the context of left near rings. Hirano, using the term an *N-ring* for what is called a 2-primal ring, showed in [7] that an N-ring

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R is strongly π-regular if and only if the $n \times n$ full matrix ring over R is strongly π-regular for $n = 1, 2, \ldots$. Also Sun [33] introduced a condition called *weakly symmetric*, which is equivalent to the 2-primal condition for rings. Part of the attraction of 2-primal rings lies in the structure of their prime ideals. Shin in [32, Proposition 1.11] showed that a ring R is 2-primal if and only if every minimal prime ideal P of R is completely prime (i.e. R/P is a domain). He also proved that the minimal-prime spectrum of a 2-primal ring is a Hausdorff space with a basis of closed-and-open sets [32, Proposition 4.7] (for further information on 2-primal rings, see [20], [21] and the references therein).

Birkenmeier, Heatherly, and Lee proved in [1, Proposition 2.6] that the 2-primal condition is inherited by ordinary polynomial extensions. Nevertheless, a power series ring over a 2-primal ring need not be 2-primal, as examples in [16, Example 1.1] show. Lee, Huh and Kim showed in [16, Proposition 1.2] that if R is 2-primal and $\mathbf{P}(R)$ is nilpotent, then the formal power series ring $R[[X]]$ over R is also 2-primal, where X is any set of commuting indeterminates over R. Also, in [23, Theorem 2.17], the authors obtained partial characterizations for the skew Hurwitz series ring (HR, α) to be 2-primal. Furthermore, Nasr-Isfahani [22] studied Öre extensions of 2-primal rings.

Rings of formal power series have been of interest and have had important applications in many areas, one of which has been differential algebra. In an earlier paper by Keigher [8], the ring of Hurwitz series, a variant of the ring of formal power series, was considered, and some of its properties, especially its categorical properties, were studied. In the papers [9], [10] Keigher demonstrated that the ring of Hurwitz series has many interesting applications in differential algebra and in the discussion about weak normalization. Its product, a product of sequences using binomial coefficients, was studied in papers by Fleiss [4] and Taft [34]. While there are many studies of these rings over a commutative ring, very little is known about them over a noncommutative ring. The ring-theoretical properties of skew Hurwitz series rings have been investigated by many authors (see [8], [9], [10], [11], [18], [23], [24], [25], [26], [27] and [28]).

In the present paper we study Hurwitz series over a noncommutative ring with identity, examine its structure and properties. Motivated by results in [19], [22], [29], [30], [36], and [37], we investigate the 2-primal ring on skew Hurwitz series ring (HR, α), where R is a ring equipped with an endomorphism α (the definition of the ring (HR, α) will be recalled in Section 2). In particular, we obtain necessary and sufficient conditions on R and α such that the skew Hurwitz series ring (HR, α) is a 2-primal ring. Furthermore, it is proved that, under suitable conditions, (HR, α) is 2-primal if and only if for every minimal prime ideal $\mathcal{P}^*$ in (HR, α) there exists a minimal prime ideal P of R such that P is completely prime and $\mathcal{P}^* = (HP, \alpha)$ if and only if $\mathbf{P}((HR,\alpha)) = (H(nil(R)),\alpha)$ if and only if $\mathcal{P}_\alpha(R) = nil(R)$ if and only if R is 2-primal and $nil((HR,\alpha)) = (H(nil(R)),\alpha)$ if and only if every minimal α-prime ideal of R is completely prime.

2. Preliminaries

Throughout this paper, R denotes an associative ring with unity and $\alpha : R \to R$ is an endomorphism such that $\alpha(1) = 1$. The ring (HR,α) of *skew Hurwitz series over a ring R* is defined as follows: the elements of (HR,α) are functions $f : \mathbb{N} \to R$, where $\mathbb{N}$ is the set of integers greater or equal than zero. The operation of addition in (HR,α) is componentwise and the operation of multiplication is defined, for every $f, g \in (HR,\alpha)$, by:

$$fg(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\alpha^k(g(n-k)) \text{ for each } n \in \mathbb{N},$$

where $\binom{n}{k}$ is the binomial coefficient. In the case where the endomorphism α is the identity, we write HR instead of (HR,α). If one identifies a skew formal power series $\sum_{n=0}^{\infty} a_n x^n \in R[[x;\alpha]]$ with the function f such that $f(n) = a_n$, then multiplication in (HR,α) is similar to the usual product of skew formal power series, except that binomial coefficients appear in each term in the product introduced above. To each $r \in R$ and $n \in \mathbb{N}$, we associate elements $h_r, h_n^{'} \in (HR,\alpha)$ defined by

$$h_r(x) = \begin{cases} r & x = 0 \\ 0 & x \neq 0, \end{cases} \quad h_n^{'}(x) = \begin{cases} 1 & x = n \\ 0 & x \neq n. \end{cases}$$

It is clear that $r \mapsto h_r$ is a ring embedding of R into (HR,α) and also (HR,α) is a ring with identity h_1. For every nonempty subset X of R, we set:

$$(HX,\alpha) = \{f \in (HR,\alpha) \mid f(n) \in X \cup \{0\} \text{ for every } n \in \mathbb{N}\}.$$

In [24, Proposition 2.1], it has been shown that for any ring R that contains the field of rational numbers Q and α is a Q-algebra homomorphism of R, then the rings (HR,α) and $R[[x;\alpha]]$ are isomorphic. To avoid repetitions of known results for skew power series ring $R[[x;\alpha]]$, we assume that R is a ring which does not contain the field of rational numbers throughout this paper.

Let R be a ring and α an endomorphism of R. Also, let I be a (left or right) ideal of R. Then I is said to be an *α-(left or right) ideal* if $\alpha(I) \subseteq I$. Also, I is said to be *α-invariant* if $\alpha(I) = I$. To characterize skew Hurwitz series rings that are 2-primal, we will need the following result, which plays a key role in the paper and will be used repeatedly in the sequel.

Proposition 2.1. *Let R be a ring, α an endomorphism of R. Then:*

(a) *If I is a right ideal of R, then (HI,α) is a right ideal of (HR,α).*

(b) *If I is an ideal of R, then (HI,α) is an ideal of (HR,α) if and only if I is an α-ideal.*

(c) *If I and J are α-ideals of R, then $(HI,\alpha)(HJ,\alpha) \subseteq (H(IJ),\alpha)$.*

Proof. (a) Clearly, $0 \in (HI,\alpha)$, so $(HI,\alpha) \neq \emptyset$. Let $f,g \in (HI,\alpha)$ and $h \in (HR,\alpha)$. Then $f(n),g(n) \in I$ for any $n \in \mathbb{N}$ and thus for any $n \in \mathbb{N}$ we have

$$(f+g)(n) = f(n)+g(n) \in I$$

and

$$(fh)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\alpha^k(h(n-k)) \in I.$$

Hence $f+h, fh \in (HI,\alpha)$, and (a) follows.

(b) Let I be an ideal of R. Assume that (HI,α) is an ideal of (HR,α) and let $a \in I$. Then $h_a \in (HI,\alpha)$, and since (HI,α) is an ideal, also $h_1^{'}h_a \in (HI,\alpha)$. Hence $\alpha(a) = (h_1^{'}h_a)(1) \in I$, which shows that $\alpha(I) \subseteq I$. Thus I is an α-ideal.

To complete the proof of (b), assume that I is an α-ideal and let $f \in (HI,\alpha)$. Then for any $k,m \in \mathbb{N}$ we have $\alpha^k(f(m)) \in \alpha^k(I) \subseteq I$. Hence for any $h \in (HR,\alpha)$ and $n \in \mathbb{N}$ we obtain

$$(hf)(n) = \sum_{k=0}^{n} \binom{n}{k} h(k)\alpha^k(f(n-k)) \in I.$$

Thus $hf \in (HI,\alpha)$, which together with (a) shows that (HI,α) is an ideal of (HR,α).

(c) By a straightforward calculation, we can prove the part (c). □

According to Krempa [12], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$, for any $a \in R$. A ring R is said to be α-*rigid* if there exists a rigid endomorphism α of R. In the proof of Lemma 2.5 we will need the following characterization of rigid endomorphisms, given in [12, Lemma 3.2].

Lemma 2.2. *[12, Lemma 3.2] Let R be a reduced ring and let α be an endomorphism of R. Then the following conditions are equivalent:*

(1) *α is a rigid endomorphism of R;*

(2) *for every minimal prime ideal P of R we have $\alpha^{-1}(P) \subseteq P$;*

(3) *α is a monomorphism preserving every minimal prime ideal of R.*

In [5], the authors introduced α-compatible rings and studied their properties. Recall that an endomorphism α of a ring R is called to be *compatible* if for each $a,b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. A ring R is said to be *α-compatible* if there exists a compatible endomorphism α of R.

Basic properties of rigid and compatible endomorphisms, proved by Hashemi and Moussavi in [5, Lemmas 2.1 and 2.2] are summarized here:

Lemma 2.3. *Let α be an endomorphism of a ring R. Then:*

(i) *if α is compatible, then α is injective;*

(ii) *α is compatible if and only if for all $a,b \in R$, $\alpha(a)b = 0 \Leftrightarrow ab = 0$;*

(iii) *the following conditions are equivalent:*

- (1) *α is rigid;*
- (2) *α is compatible and R is reduced;*
- (3) *for every $a \in R$, $\alpha(a)a = 0$ implies that $a = 0$.*

An α-ideal $P \neq R$ is an *α-prime ideal* if for any α-ideals I and J such that $IJ \subseteq P$, we have either $I \subseteq P$ or $J \subseteq P$. It is easy to show that an α-ideal I of R is α-prime if and only if for any $a,b \in R\setminus I$ there exist $n,m \in \mathbb{N}$ and $r \in R$ such that $\alpha^n(a)r\alpha^m(b) \in R\setminus I$ (see [14, Proposition 1.10(a)]. Similarly as [14], we define the *α-prime radical* (in other words, *α-lower nil radical*) of R (denoted by $\mathcal{P}_\alpha(R)$) as the intersection of all α-prime ideals of R.

Lemma 2.4. *Let R be a ring, α an endomorphism of R. Assume that R is α-rigid.*

(i) *If P is a minimal α-prime ideal of R then P is completely prime.*

(ii) *An ideal P of R is a minimal prime ideal if and only if P is a minimal α-prime ideal.*

Proof. (i). We adapt the proof of [13, Lemma 12.6]. Suppose $M := R\setminus P$, and let X be the multiplicative monoid generated by $\{\alpha^n(a) : n \in \mathbb{N}, a \in M\}$. We claim that $0 \notin X$. Assume on the contrary that there exist $x_i \in X$ such that $x_1x_2\cdots x_n = 0$, with n minimal. Clearly, $n \geq 2$. Now from Lemma 2.3 it follows that there exist $a_1,a_2,\ldots,a_n \in M$ such that $a_1a_2\cdots a_n = 0$. Since R is α-rigid

and $(a_nRa_1\cdots a_{n-1})^2=0$, we have $a_nRa_1\cdots a_{n-1}=0$. But P is an α-prime ideal of R, and thus by [14, Proposition 1.10(a)] there exist element $s,t\in\mathbb{N}$ and $r\in R$ such that $a=\alpha^s(a_n)r\alpha^t(a_1)\in M$. Since $a_nra_1a_2\cdots a_{n-1}=0$ and R is α-rigid, we obtain $\alpha^s(a_n)r\alpha^t(a_1)a_2\cdots a_{n-1}=aa_2\cdots a_{n-1}=0$ which is a contradiction to the minimality of n. Therefore $0\notin X$.
From [14, Proposition 1.10(b)] we can enlarge $\{0\}$ to an α-prime ideal $P^{'}$ of R disjoint from X. But P is a minimal α-prime ideal of R and so we must have $P=P^{'}$. Then $M=X$ and so M is closed under multiplication. Thus R/P is a domain. Therefore P is a completely prime ideal of R.
(ii). Suppose that P is a minimal prime ideal of R. Clearly, R is reduced by Lemma 2.3 (iii). Hence P is an α-ideal by Lemma 2.2 and thus, being a prime ideal, P is obviously an α-prime ideal. On the contrary, suppose that P is not a minimal α-prime ideal. By Zorn's Lemma, there exists a minimal α-prime ideal Q such that $Q\subset P$. Part (i) implies that Q is a completely prime ideal, this contradicts the minimality of P. Conversely, assume that P is a minimal α-prime ideal of R. Then P is a completely prime ideal by the part (i). If P is not a minimal prime ideal, then there exists a minimal prime ideal Q such that $Q\subset P$. Then the same argument as in the first part of the proof of (ii) shows that Q is an α-prime ideal, a contradiction. Thus P is a minimal prime ideal of R, and the proof is complete. □

Lemma 2.5. *Let R be a 2-primal and α-compatible ring, where α is an endomorphism of R. Then:*

(1) *Every minimal prime ideal of R is an α-ideal.*

(2) *For every minimal prime ideal P of R we have $\alpha^{-1}(P)\subseteq P$.*

(3) *An ideal P of R is a minimal prime ideal if and only if P is a minimal α-prime ideal.*

Proof. The hypothesis implies that $\mathbf{P}(R)=nil(R)$, and so the ring $\overline{R}:=R/\mathbf{P}(R)$ is a reduced ring. Since $\mathbf{P}(R)=nil(R)$ and obviously $\alpha(nil(R))\subseteq nil(R)$, it follows that $\mathbf{P}(R)$ is an α-ideal. We prove that $\overline{R}$ is $\overline{\alpha}$-rigid, where $\overline{\alpha}:\overline{R}\to\overline{R}$ is the induced endomorphism (i.e., $\overline{\alpha}(r+\mathbf{P}(R))=\alpha(r)+\mathbf{P}(R)$ for any $r\in R$). To prove this, consider any $\overline{a}\in\overline{R}$ with $\overline{a}\cdot\overline{\alpha}(\overline{a})=\overline{0}$. So $a\alpha(a)\in\mathbf{P}(R)$. Therefore $a^2\in nil(R)$, by Lemma 2.3 (ii) and hence $\overline{a}=\overline{0}$. Thus $\overline{R}$ is an $\overline{\alpha}$-rigid ring.
Now, suppose that P is a minimal prime ideal of R. Then $\overline{P}$ is a minimal prime ideal of $\overline{R}$. From Lemma 2.2, we have $\overline{\alpha}(\overline{P}),\overline{\alpha}^{-1}(\overline{P})\subseteq\overline{P}$. This implies that, $\alpha(P)$ and $\alpha^{-1}(P)$ are contained in P proving (1) and (2).

(3). By an easy computation and using Lemma 2.4 (ii), it follows that an ideal P is a minimal prime ideal of R if and only if $\overline{P}$ is a minimal prime ideal of

$\overline{R}$ if and only if $\overline{P}$ is a minimal $\overline{\alpha}$-prime ideal of $\overline{R}$ if and only if P is a minimal α-prime ideal of R, and the proof is complete. $\square$

3. Main Results

In this section we will characterize 2-primal rings of a skew Hurwitz series (HR, α) under various assumptions on R (see Theorems 3.5, 3.13 and Corollary 3.14). We will need the following proposition in the proof of Theorem 3.5.

Proposition 3.1. *Let R be a 2-primal and α-compatible ring that is torsion free as a $\mathbb{Z}$-module, where α is an endomorphism of R. If P is a minimal prime ideal in R, then (HP, α) is a completely prime ideal of (HR, α).*

Proof. Lemma 2.5 (1) implies that P is an α-ideal. Thus $J = (HP, \alpha)$ is an ideal of $T = (HR, \alpha)$, by Proposition 2.1. It follows from Lemma 2.5 that $\alpha(P) \cup \alpha^{-1}(P) \subseteq P$. Hence by defining $\overline{\alpha}(r+P) = \alpha(r) + P$ for all $r \in R$, we obtain a endomorphism $\overline{\alpha} : R/P \to R/P$ such that $\overline{\alpha}$ is injective. Let $\pi : R \to R/P = \overline{R}$ be the natural projection. By defining $\Psi(h) = \pi \circ h$ for every $h \in T$, we obtain an isomorphism $\Psi : T/J \cong (H\overline{R}, \overline{\alpha})$. Since R is 2-primal, it follows from [32, Proposition 1.11] that $\overline{R}$ is a domain. Hence by [23, Proposition 2.3], $(H\overline{R}, \overline{\alpha})$ is a domain. This means that J is a completely prime ideal of T, and we are done. $\square$

To characterize skew Hurwitz series rings that are 2-primal, we will need the following results, which play a key role in the sequel. We denote by $char(R)$ the characteristic of a ring R.

Theorem 3.2. *[23, Theorem 2.6] Let R be a ring that is torsion free as a $\mathbb{Z}$-module and α an endomorphism of R. If R is α-compatible and $nil(R)$ is a nilpotent ideal, then $f \in nil((HR, \alpha))$ if and only if $f(n) \in nil(R)$ for all $n \in \mathbb{N}$.*

Theorem 3.3. *[23, Theorem 2.17] Let R be a ring such that $char(R/\boldsymbol{P}(R)) = 0$ and α an endomorphism of R. If R is α-compatible and $\boldsymbol{P}(R)$ is a nilpotent ideal, then (HR, α) is 2-primal if and only if R is 2-primal.*

By combining Theorems 3.2 and 3.3, we compute the prime radical of a skew Hurwitz series ring (HR, α) in the case where R is 2-primal, $\mathbf{P}(R)$ is nilpotent. We will use this result to prove Theorem 3.5.

Theorem 3.4. *Let R be a 2-primal ring such that $char(R/\boldsymbol{P}(R)) = 0$ and α an endomorphism of R. If R is α-compatible and $\boldsymbol{P}(R)$ is nilpotent, then:*

$$\boldsymbol{P}((HR, \alpha)) = (H(\boldsymbol{P}(R)), \alpha).$$

The following theorem provides a characterization of 2-primal skew Hurwitz series ring (HR,α) in the case where R is α-compatible and $\mathbf{P}(R)$ is nilpotent. This theorem discusses the relationship between minimal prime ideals in R and the ones in the skew Hurwitz series ring (HR,α).

Theorem 3.5. *Let R be a ring such that $char(R/\boldsymbol{P}(R)) = 0$ and α an endomorphism of R. If R is α-compatible and $\boldsymbol{P}(R)$ is a nilpotent ideal, then the following are equivalent:*

(1) *R is 2-primal;*

(2) *(HR,α) is 2-primal;*

(3) *R is 2-primal and $nil((HR,\alpha)) = (H(nil(R)),\alpha)$;*

(4) *for every minimal prime ideal $\mathcal{P}^*$ in (HR,α) there exists a minimal prime ideal P of R such that P is completely prime and $\mathcal{P}^* = (HP,\alpha)$.*

Proof. We set $T = (HR,\alpha)$.
$(1) \Leftrightarrow (2)$. The result follows from Theorem 3.3 and Lemma 2.3 (ii).
$(1) \Leftrightarrow (3)$. This result is a direct consequence of Theorem 3.2.
$(1) \Rightarrow (4)$. Let $\mathcal{P}^*$ be a minimal prime ideal of T. Put $\widehat{T} := T/\mathbf{P}(T)$. Clearly, $\widehat{\mathcal{P}^*}$ is a minimal prime ideal in $\widehat{T}$. From Theorem 3.4 we conclude that $\mathbf{P}(T) = (H(\mathbf{P}(R)),\alpha)$. Now, let $\pi : R \to R/\mathbf{P}(R) = \overline{R}$ be the canonical map. We have an automorphism $\overline{\alpha} : \overline{R} \to \overline{R}$ given by $\overline{\alpha}(\overline{x}) = \overline{\alpha(x)}$ for each $\overline{x} \in \overline{R}$. The surjective ring endomorphism $T \to (H\overline{R},\overline{\alpha})$ given by $f \mapsto \pi \circ f$ induces an isomorphism $\Phi : \widehat{T} \cong (H\overline{R},\overline{\alpha})$. Then the same method as in the proof of Lemma 2.5 shows that $\overline{R}$ is $\overline{\alpha}$-rigid. Thus, by [23, Corollary 2.4], the ring $(H\overline{R},\overline{\alpha})$ is reduced. It is clear that $\Phi(\widehat{\mathcal{P}^*})$ is a minimal prime ideal in $(H\overline{R},\overline{\alpha})$. Then $\Phi(\widehat{\mathcal{P}^*})$ is completely prime, since $(H\overline{R},\overline{\alpha})$ is reduced. Put $Q = \overline{R} \cap \Phi(\widehat{\mathcal{P}^*})$. It is easy to see that Q is an ideal of $\overline{R}$. If $a,b \in Q$ satisfy $ab \in Q$, then $ab \in \Phi(\widehat{\mathcal{P}^*})$, and so $a \in \Phi(\widehat{\mathcal{P}^*})$ or $b \in \Phi(\widehat{\mathcal{P}^*})$. This implies that Q is a completely prime ideal of $\overline{R}$. Thus, there exists a minimal prime ideal $\overline{P}$ of $\overline{R}$ such that $\overline{P} \subseteq Q$ and $\overline{P}$ is completely prime. We claim that $\Phi(\widehat{\mathcal{P}^*}) = (H\overline{P},\overline{\alpha})$. In fact, applying Proposition 3.1, we infer that $(H\overline{P},\overline{\alpha})$ is a completely prime ideal of $(H\overline{R},\overline{\alpha})$. On the other hand, since $\overline{P} \subseteq \Phi(\widehat{\mathcal{P}^*})$, it is easy to see that $(H\overline{P},\overline{\alpha}) \subseteq \Phi(\widehat{\mathcal{P}^*})$, and thus $\Phi(\widehat{\mathcal{P}^*}) = (H\overline{P},\overline{\alpha})$ by the minimality of $\Phi(\widehat{\mathcal{P}^*})$. This means $\widehat{\mathcal{P}^*} = (HP,\alpha)/\mathbf{P}(T)$ by the virtue of Φ, i.e., $\mathcal{P}^* = (HP,\alpha)$.
$(4) \Rightarrow (2)$. Let $\mathcal{P}^*$ be a minimal prime ideal of T. Then by part (4) there exists a minimal prime ideal P of R such that P is completely prime and $\mathcal{P}^* = (HP,\alpha)$. From Proposition 3.1 we infer that P is an α-ideal. Then the same method as in the proof of Proposition 3.1 shows that $T/\mathcal{P}^* \cong (H((R/P)),\overline{\alpha})$ is a domain,

where $\overline{\alpha}$ is the induced endomorphism on R/P. Thus $\mathcal{P}^*$ is a completely prime ideal in T. By [32, Proposition 1.11], it implies that T is a 2-primal ring, and the result follows. □

Corollary 3.6. *Let R be a ring such that $char(R/\boldsymbol{P}(R)) = 0$ and $\boldsymbol{P}(R)$ is a nilpotent ideal of R. Then the Hurwitz series ring HR is 2-primal if and only if R is 2-primal if and only if R is 2-primal and $nil(HR) = (H(nil(R))$ if and only if for every minimal prime ideal $\mathcal{P}^*$ of HR there exists a minimal prime ideal P of R such that P is completely prime and $\mathcal{P}^* = HP$.*

Proposition 3.7. *Let R be a 2-primal ring such that $char(R/\boldsymbol{P}(R)) = 0$ and α an endomorphism of R. Assume that R is α-compatible and $\boldsymbol{P}(R)$ is a nilpotent ideal of R. Then $\mathcal{P}^*$ is a minimal prime ideal of (HR,α) if and only if there exists a minimal prime ideal P of R such that $\mathcal{P}^* = (HP,\alpha)$.*

Proof. Suppose that P is a minimal prime of R. Then (HP,α) is a completely prime ideal of (HR,α) by Proposition 3.1. We claim that (HP,α) is a minimal prime ideal of (HR,α). Assume on the contrary that there exists a minimal prime ideal $\mathcal{P}^*$ in (HR,α) such that $\mathcal{P}^* \subset (HP,\alpha)$. Then by the same argument as in the proof of Theorem 3.5, there exists a minimal prime ideal Q of R such that Q is completely prime and $\mathcal{P}^* = (HQ,\alpha)$. This means $(HQ,\alpha) \subset (HP,\alpha)$. Thus we have $Q \subset P$, contradicting the minimality of P. Therefore (HP,α) is a minimal prime ideal of (HR,α). The converse follows directly from Theorem 3.5, and the proof is complete. □

By combining Proposition 3.7 and Lemma 2.3, we obtain the following:

Corollary 3.8. *Let R be a ring that is torsion free as a $\mathbb{Z}$-module and α an endomorphism of R. If R is α-rigid, then $\mathcal{P}^*$ is a minimal prime ideal of (HR,α) if and only if there exists a minimal prime ideal P of R such that $\mathcal{P}^* = (HP,\alpha)$.*

Corollary 3.9. *Let R be a 2-primal ring such that $char(R/\boldsymbol{P}(R)) = 0$ and α an endomorphism of R. If R is α-compatible and $\boldsymbol{P}(R)$ is a nilpotent ideal and P_i ($i \in I$) are all minimal prime ideals of R, then:*

$$\boldsymbol{P}((HR,\alpha)) = \bigcap_{i\in I}(HP_i,\alpha).$$

Proof. It is a direct consequence of Proposition 3.7. □

The following corollary offers a criterion for a skew Hurwitz series ring to have only finitely many minimal prime ideals.

Corollary 3.10. *Let R be a 2-primal ring such that $char(R/\boldsymbol{P}(R)) = 0$ and α an endomorphism of R. Assume that R is α-compatible and $\boldsymbol{P}(R)$ is nilpotent. If the number of minimal prime ideals of R is finite, then the number of minimal prime ideals of (HR, α) is finite.*

Let R be a ring, $End(R,+)$ the ring of additive endomorphisms of R and Φ a subset of $End(R,+)$. Recall from [14], that a sequence $(a_0, a_1, \ldots, a_n, \ldots)$ of elements of R is called an *Φ-m-sequence* if for any $i \in \mathbb{N}$ there exist $\varphi_i, \varphi_i^{'} \in \Phi$ and $r_i \in R$ such that $a_{i+1} = \varphi_i(a_i) r_i \varphi_i^{'}(a_i)$. An element $a \in R$ is called *strongly Φ-nilpotent* if every Φ-m-sequence starting with a eventually vanishes. Now suppose that α is an endomorphism of R and let Φ be the subsemigroup of $End(R,+)$ generated by α. From [14, Proposition 1.11] it follows that $\mathcal{P}_\alpha(R)$ is precisely the set of all strongly Φ-nilpotent elements of R. If $\Phi = \{id_R\}$ we recover the corresponding classical notions (for more details see [14]).

The following result generalizes [32, Proposition 1.11].

Proposition 3.11. *Let R be an α-compatible ring, where α is an automorphism of R. Then $\mathcal{P}_\alpha(R) = nil(R)$ if and only if every minimal α-prime ideal of R is completely prime.*

Proof. Suppose that $\mathcal{P}_\alpha(R) = nil(R)$. Clearly, the ring $\overline{R} := R/\mathcal{P}_\alpha(R)$ is reduced and $\mathcal{P}_\alpha(R)$ is an α-ideal of R. By the same method as in the proof of Lemma 2.5 we can show that $\overline{R}$ is $\overline{\alpha}$-rigid, where $\overline{\alpha} : \overline{R} \to \overline{R}$ is the induced endomorphism. Now, let P be a minimal α-prime ideal of R, then $\overline{P}$ is a minimal $\overline{\alpha}$-prime ideal of $\overline{R}$. Since $R/P \cong \overline{R}/\overline{P}$, then by the part (i) of Lemma 2.4 implies that P is completely prime. Conversely, assume that every minimal α-prime ideal of R is completely prime. Let $\{P_i\}_{i\in I}$ be a family of all minimal α-prime ideals of R. Then $\mathcal{P}_\alpha(R) = \bigcap\limits_{i\in I} P_i$ and so $R/\mathcal{P}_\alpha(R)$ embeds in $\prod\limits_{i\in I} R/P_i$. Since R/P_i is a domain for all $i \in I$, it follows that $R/\mathcal{P}_\alpha(R)$ is reduced and so $nil(R) \subseteq \mathcal{P}_\alpha(R)$. It is clear $\mathcal{P}_\alpha(R) \subseteq nil(R)$, and the result follows. $\square$

A proper α-ideal I of R is said to be *a α-semiprime ideal* if whenever J is an ideal of R and m is an integer such that $J\alpha^t(J) \subseteq I$ for all $m \leq t$, then $J \subseteq I$. The ring R is said to be *α-semiprime* if the zero ideal is an α-semiprime ideal of R. We next compute the prime radical of a skew Hurwitz series ring in terms of the α- prime radical of R.

Theorem 3.12. *Let R be a ring such that $char(R/\boldsymbol{P}(R)) = 0$ and α an automorphism of R. Assume that $\mathcal{P}_\alpha(R)$ is a nilpotent ideal. Then*

$$\boldsymbol{P}((HR, \alpha)) = (H(\mathcal{P}_\alpha(R)), \alpha).$$

Proof. Suppose that $\pi : R \to R/\mathcal{P}_\alpha(R) = \overline{R}$ be the canonical map. Clearly, $\mathcal{P}_\alpha(R)$ is an α-ideal. Therefore $J = (H(\mathcal{P}_\alpha(R)), \alpha)$ is an ideal of $T = (HR, \alpha)$, by Proposition 2.1. Now, we have an automorphism $\overline{\alpha} : \overline{R} \to \overline{R}$ given by $\overline{\alpha}(\overline{x}) = \overline{\alpha(x)}$ for each $\overline{x} \in \overline{R}$. The surjective ring homomorphism $T \to (H\overline{R}, \overline{\alpha})$ given by $f \to \pi o f$ induces an isomorphism $T/J \cong (H\overline{R}, \overline{\alpha})$. Since $\overline{R}$ is $\overline{\alpha}$-semiprime, and so, by [26, Theorem 7], $(H\overline{R}, \overline{\alpha})$ is semiprime. This means that J is semiprime ideal in T, so we have $\mathbf{P}((HR, \alpha)) \subseteq (H(\mathcal{P}_\alpha(R)), \alpha)$. Conversely, since $\mathbf{P}(R)$ is a nilpotent α-ideal, $(H(\mathcal{P}_\alpha(R)), \alpha)$ is also nilpotent and so $(H(\mathcal{P}_\alpha(R)), \alpha) \subseteq \mathbf{P}((HR, \alpha))$. □

Below we provide a characterization of 2-primal skew Hurwitz series rings (HR, α).

Theorem 3.13. *Let R be a ring such that $char(R/\mathbf{P}(R)) = 0$ and α an automorphism of R. Suppose that R is α-compatible and $\mathbf{P}(R)$ is nilpotent. Then the following are equivalent:*

(1) *R is 2-primal;*

(2) *$\mathcal{P}_\alpha(R) = nil(R)$;*

(3) *(HR, α) is 2-primal;*

(4) *$\mathbf{P}((HR, \alpha)) = (H(nil(R)), \alpha)$;*

(5) *R is 2-primal and $nil((HR, \alpha)) = (H(nil(R)), \alpha)$;*

(6) *every minimal α-prime ideal of R is completely prime;*

(7) *for every minimal prime ideal $\mathcal{P}^*$ in (HR, α) there exists a minimal prime ideal P of R such that P is completely prime and $\mathcal{P}^* = (HP, \alpha)$.*

Proof. By Theorem 3.5, the parts (1), (3), (5) and (7) are equivalent. It follows from Proposition 3.11 that the parts (2) and (6) also are equivalent so it suffices to prove that (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (2).
(1) $\Rightarrow$ (4). Assume that R is 2-primal. Since $nil(R)$ is a nilpotent ideal of R, $nil((HR, \alpha)) = (H(nil(R)), \alpha)$, by Theorem 3.2. Also, Theorem 3.3 and the part (ii) of Lemma 2.3 implies that $\mathbf{P}((HR, \alpha)) = (H(nil(R)), \alpha)$. Therefore, we have $\mathbf{P}((HR, \alpha)) = (H(nil(R)), \alpha)$.
(4) $\Rightarrow$ (2). Suppose that $\mathbf{P}((HR, \alpha)) = (H(nil(R)), \alpha)$. Since $\mathbf{P}(R)$ is a nilpotent ideal of R, it follows that $\mathcal{P}_\alpha(R)$ is nilpotent. Hence by Theorem 3.12, $\mathbf{P}((HR, \alpha)) = (H(\mathcal{P}_\alpha(R)), \alpha)$. Thus $\mathcal{P}_\alpha(R) = nil(R)$.
(2) $\Rightarrow$ (1). Since $\mathcal{P}_\alpha(R) \subseteq \mathbf{P}(R) \subseteq nil(R)$, it follows that R is 2-primal. □

We close this paper by extending to skew Hurwitz series rings a characterization of 2-primal Öre extensions due to Nasr-Isfahani (see [22, Corollary 2.11]).

Corollary 3.14. *Suppose that R is either a ring with ascending chain condition (ACC) on both right and left annihilators, or is left or right Goldie, or has the ACC on ideals, or has right Krull dimension. Let R be an α-compatible ring such that $char(R/\boldsymbol{P}(R)) = 0$, where α is an automorphism of R. Then the following are equivalent:*

(1) *R is 2-primal;*

(2) *$\mathcal{P}_\alpha(R) = nil(R)$;*

(3) *(HR, α) is 2-primal;*

(4) *$\boldsymbol{P}((HR, \alpha)) = (H(nil(R)), \alpha)$;*

(5) *R is 2-primal and $nil((HR, \alpha)) = (H(nil(R)), \alpha)$;*

(6) *every minimal α-prime ideal of R is completely prime;*

(7) *for every minimal prime ideal $\mathcal{P}^*$ in (HR, α) there exists a minimal prime ideal P of R such that P is completely prime and $\mathcal{P}^* = (HP, \alpha)$.*

Proof. If R has the ACC on ideals or R is right Goldie or satisfies the ascending chain condition on both right and left annihilators, then by [31, Lemma 2.6.22], [15, Theorem 1] and [6, Theorem 1], $\mathbf{P}(R)$ is nilpotent, respectively. If R has right Krull dimension, then by [17], $\mathbf{P}(R)$ is nilpotent. Also, if R is a ring with ACC on both right and left annihilators, then by [2, Theorem 1.34], $\mathbf{P}(R)$ is nilpotent. Now, the result follows by Theorem 3.13. □

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