MATHEMATICAL REFLECTIONS: STANDING ON THE SHOULDERS OF GIANTS

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Once upon a time, in the days of the Roman Empire, a mob was gathered in the Coliseum to watch as a Christian was thrown to a hungry lion. The spectators cheered as the wild beast went after its prey. But the Christian quickly whispered something in the lion's ear and the beast backed away with obvious terror on his face. No amount of calling and foot stomping by the audience could get the lion to approach the Christian again. Fearlessly, the Christian walked from the arena.

The Emperor was so amazed at what had happened that he sent for the Christian and offered him his freedom if he would say what he had done to make the ferocious beast cower in fear. The Christian bowed before the Emperor and said, "I merely whispered in the lion's ear: After dinner, you'll be required to say a few words."

I read recently of a poll concerning the fears of people in the United States, and found that the number one fear is being asked to give a speech -- even greater than that of dying. I won't tell you that I've been dying to give a talk to your group, but I will tell you I'm really honored to be the first invited speaker on your program this year. You have already made me feel welcome, even though I've never attended one of your conventions before. I am a practicing mathematician and I have turned my life over to Jesus, so now that I've found this group I know I'm hooked.

I'm a little wary, however, because of a warning given by St. Augustine about the danger of mixing mathematicians and Christians. And I quote:

The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of hell.

A rephrasing of this admonition for the present day could well be:

The good citizen (which, we hope includes all good Christians) should be AWARE of mathematicians and all those who make practical use of mathematics, particularly BECAUSE they speak the truth. The fact is, that these people, in league with other scholars, will enlighten the souls and minds of a nation, and set them free from the sources of ignorance.

Even though my students would probably agree with this quotation, we all know better, don't we?

Being the first speaker on your program is a little intimidating. I'm reminded of a young clergyman who was asked to deliver a sermon at the campus church at Yale University. He announced that his sermon would be divided into four parts -- each initialed, so to speak, by the four letters of the university's name. Starting with Y, he talked about the God of the Jews, using His Hebrew name, Yahweh. Moving on to A, he talked about Amos, or maybe it was Aaron. In like manner, he found inspiration in the leters L and E. After the service, two students who had heard the visiting clergyman talk were discussing what they heard. "Quite ingenious," said one, "building his sermon aound Y-A-L-E. "It sure was," said the other, "but I'm glad I didn't hear him preach at the California Institute of Technology." You don't have to worry tonight; I shall try to be brief. I remember a prof I had at UCLA who told us to be kind to our audiences. He used to say, "If you haven't struck oil in 20 minutes, stop boring."

When I first started college I was told that if you steal from one person, its plagiarism; if you steal from 10 people its research; and if you steal from hundreds, you're a scholar. In that respect, I do qualify as a scholar since most of what I'm going to do today is taken from many resources. In fact, in mathematics we rarely have the opportunity to do original work, but instead stand on the shoulders of the giants: Einstein, Gauss, Abel, Polya and Euler.

Hanging on my office wall is a treasured Charles Schulz original. It states a mathematical problem and ends with the desperate cry of Peppermint Patty, "ABANDON SHIP." I keep this comic strip at eye level and mind level to warn me of the limitations of abstract thought in mathematics.

Yet I cannot help wondering why the layperson's reaction to mathematics or a mathematical problem is often that of Peppermint Patty. Why, when we tell people our occupation, do we more often than not hear about their unpleasant experiences with mathematics. They often tell us, "I studied algebra in high school and haven't used it since!" Indeed, that may be true. However, we can reply, "How many times has algebra been used on you?" We are all restricted, and protected, by the formulas of mathematics. In the mysterious metaphors we have agreed to call mathematics, all creation is involved, from the symbol-happy logician down to those cunning geometers, the bees. When I trust myself to a ladder, I lean upon mathematics. Every baby is a formula-baby, for when we say that its growth is a function of its nourishment, what are we citing but a case of the calculus?

There's a story about how Albert Einstein was traveling to universities in a chauffeur driven car, delivering lectures on his theory of relativity. One day while in transit, the chauffeur remarked: "Dr. Einstein, I've heard you deliver that lecture about 30 times. I know it by heart and bet I could give it myself."

"Well, I'll give you the chance," said Einstein, "They don't know me at the next school, so when we get there I'll put on your cap, and you introduce yourself as me and give the lecture." The chauffeur delivered Einstein's lecture flawlessly. When he finished, he started to leave, but one of the professors stopped him and asked a complex question filled with mathematical equations and formulas. The chauffeur thought fast. "The solution to that problem is so simple," he said, "I'm going to ask my chauffeur to come up here and answer your question."

There are no nonmathematical minds, but only nonmathematical teachers. The masses who draw back in fear at the sound of the word "mathematics" are often merely suffering from a bad persistent case of early pedagogy. I believe one of the first steps in motivating the unmotivated is in getting them to realize that WE, the teachers, are NOT the enemy.

Enjoyment of mathematics is NOT dependent upon "book learning." It is NOT dependent on arithmetical skills. We have progressed from the days where an engineering student could set himself apart from the rest of the student population with a "slip-stick" hanging from his belt, to the day when EVERY student (not just mathmaticians and engineers) can have more calculating power at their fingertips than we would have dreamed possible just 10 years ago. We not only have calculators under $10, but calculators to figure our monthly payments and interest rates, tell us the temperature, do our banking, find a stock's annual yield, monitor our heartbeat, remember our telephone numbers, or even print out a biorhythm chart all with the push of a single button. The big discussion at the MAA meeting this year in San Antonio was about the new calculators that do symbolic manipulation, calculus, and function graphing. Nearly every student in every class I teach now has his or her own calculator, and how we respond to these new technologies is extremely important. Just as mathematics has freed itself from its practical origins, calculation has become less and less important.

When Niels Abel, often regarded as Norway's greatest mathematician, was once asked his formula for so rapidly forging ahead to become a first rate mathematician, he replied, "By studying the masters and not their pupils." I'd like to digress for a moment to tell you a story about Abel, as reported by Howard Eves. Few things can be more exasperating to an author than to have his manuscript lost or misplaced after it has been submitted for publication. Niels Abel once left one of his greatest masterpieces on transcendental functions in Cauchy's care to be presented to the French Academy for judgment and possible publication. Cauchy, being very busy with research of his own, passed the paper over to Hachette, who made the presentation to the Academy. The outcome was that Legendre and Cauchy were appointed to referee the paper -- Legendre was at the time seventy-four and Cauchy was thirty-nine. A couple of years then passed, during which Abel heard nothing about his paper. He finally wrote to Jacobi, telling of the paper and what was in it. So three years after Abel submitted his paper, Jacobi wrote to Legendre in an effort to find out the fate of the paper. Legendre replied that the manuscript was barely legible, the ink hardly visible, and the letters badly formed, and that it was agreed the author should be asked to submit a more readable copy. Cauchy took the paper home, presumably to communicate with Abel on its rewriting, misplaced the paper and soon forgot all about it. In time the author's unheeded inquiries about his paper became known in Norway, and the Norwegian counsul at Paris, in an effort to stir up the Academy and secure some sort of report on the disposition of the paper, raised a minor diplomatic fuss about the missing manuscript, and finally five years after it was submitted, the French Academy made amends for all it blundering and awarded Abel, jointly with Jacobi, the Grand Prize in Mathematics. But, alas, this story has a sad ending. It was too late, Abel was then dead. There is a story that Norm Shaumbauger tells about a prof teaching an open enrollment college class in New York City. The first day of class after all the students had entered the room the prof would look up and say, "Someday everyone enrolled in this class will be dead." There was some snickering at the back of the room. The prof repeated, "Someday everyone enrolled in this class will be dead." More snickering. The prof asked the disruptive student what was so funny, and the student replied, "I'm not enrolled in this class."

Let's take Abel's advice and look at a master, Karl Gauss, the so-called Prince of Mathematics. There is so much we can say about Gauss. In all the history of mathematics there is nothing approaching the precocity of Gauss as a child. Gauss showed his caliber before he was three years old. In later life Gauss used to joke that he knew how to reckon before he could talk. In seems that when he was three he corrected his father's computations on a payroll report. Shortly after his seventh birthday Gauss entered his first school, a hell-hole which was a squalid relic of the Middle Ages run by a virile brute, Herr Buttner whose idea of teaching the hundred or so boys in his charge was to thrash them into such a state of terrified stupidity that they forgot their own names. Upon Gauss' admission to an arithmetic class Buttner gave a long problem in addition:

81,297 + 81,495 + 81,693 + ... + 101,097.

Gauss recongnized it as an arithmetic progresion of 100 terms with a common difference of 198, so he wrote the answer on his slate and put it on the teacher's desk; he said "Ligget se" (there it lies, in his peasant dialect). Then, for the ensuing hour, while the other boys toiled, he sat with his hands folded, favored now and then by a sarcastic glance from Buttner, who imagined the youngest pupil in the class was just another blockhead. At the end of the period Buttner looked over the slates. On Gauss' slate there appeared but a single number. To the end of his days Gauss loved to tell how the number he had written was the correct answer and how all the others were wrong. Today, happily, we are no longer compelled to waste valuable time in the swamp of mere reckoning. Instead, we can take Gauss' lead and look for patterns, relationships, and interesting results. Consider the following problems:

Suppose you were offered a job with a starting salary of $21,000 and were offered the following options.

A. $1200 annual increase

B. $300 semiannual increase

C. $75 quarterly increase

Which option would you choose?

The best option is C; to see why, look at the cumulative earnings.

YEAR	A	B	C
1	$21,000	$21,300	$21,450
2	$43,200	$43,800	$44,100
3	$66,600	$67,500	$67,950

Find the general formula.

OPTION A

First year: 21,000

Second year: 21,000 + (21,000 + 1200) = 43,200

Third year: 21,000 + (21,000 + 1200) + (21,000 + 1200 + 1200) = 66,600

...

nth year: $21,000 + (21,000 + 1200) + (21,000 + 2\cdot 1200) + \ldots + [21,000 + (n - 1)1200]$

$= 21,000n + 1200[1 + 2 + 3 + \ldots + (n - 1)]$

$= 21,000n + 1200[(n - 1)n/2]$

$= 21,000n + 600n^2 - 600n$

OPTION B

First year: $10{,}500 + (10{,}500 + 300) = 21{,}300$

Second year: $10{,}500 + (10{,}500 + 300) + (10{,}500 + 300 + 300)$

$+ (10{,}500 + 300 + 300 + 300) = 43{,}800$

...

nth year: $10{,}500 + (10{,}500 + 300) + (10{,}500 + 2\cdot 300) + (10{,}500 + 3\cdot 300)$

$+ \ldots + [10{,}500 + (2n - 1)300]$

$= 10{,}500(2n) + 300[1 + 2 + 3 + \ldots + (2n - 1)]$

$= 21{,}000n + 300[(2n - 1)(2n)/2]$

$= 21{,}000n + 600n^2 - 300n$

OPTION C

First year: $5250 + (5250 + 75) + (5250 + 75 + 75) + (5250 + 75 + 75 + 75) = 21{,}450$

Second year: $5250 + (5250 + 75) + \ldots + (5250 + 7\cdot 75) = 44{,}100$

...

nth year: $4n(5250) + 75[(4n - 1)(4n)/2] = 21{,}000n + 600n^2 - 150n$

If you are dealing with students who are unmotivated in mathematics, you must grab them where they are; you cannot begin at some preconceived teacher position. St. Thomas Aquinas, who knew more about education and persuasion that almost anybody who ever lived, once said that when you want to convert someone to your view, you go over to where he is standing, take him by the hand (mentally speaking), and guide him. You don't stand across the room and shout at him; you don't call him a dummy; you don't order him to come over to where you are. You start where he is, and work from that position. That's the only way to get him to budge. How can we be sure of beginning where the student is? There are so many students, so much to do, and it seems as if each student is at a different location.

When students are not successful, or are unmotivated, it is often because they are not properly placed in the correct class, or because they have unrealistic expectations about their future. Educators like to blame the students, the administrators, the buildings, or trick us into thinking that some people "have an aptitude for mathematics" and others do not. But such is not the case. As J. F. Herbart put it:

The idea that aptitude for mathematics is rarer than aptitude for other subjects is merely an illusion which is caused by belated or neglected beginners.

I would add that it is also caused by incompetent teachers (but maybe they are just neglected beginners anyway). To be sure, everyone has different talents, but to develop any talent takes practice, practice, and more practice. What do we practice in our everyday lives? That which we like to do. What do we like to do? That at which we are successful. At what things are we successful? Those things which we practice. You see the vicious cycle; and dislike or ineptitude in mathematics probably traces its origins back to bad teaching somewhere in our past, which begins the cycle. About 10 years ago I decided I could do something about the situation to convey my attitude about mathematics and mathematics teaching by writing a series of mathematics textbooks all tied together by common denominators of good teaching. People ask me how can I keep writing new mathematics textbooks, "Isn't algebra just algebra?" they ask. What can be new in algebra? The answer rests in pedagogy -- new ways of making mathematics interesting and relevant to the student in an everchanging world. To begin at the student's position and lead him or her to a new position -- not demand or expect that the student begin at some preconceived teacher or book location. Historically, mathematics has been presented by reasoning from the abstract to the concrete; that is giving the general statement, then perhaps a proof or a derivation, some examples, and finally some applied problems. But I believe this is contrary to the way people learn! Students learn by reasoning from the concrete to the abstract.

One of the first to advocate this reversal of learning procedure was George Polya in his book HOW TO SOLVE IT. I have been privileged to personally hear Professor Poyla. He is fond of telling fables or stories. Here is one which he says he first heard 40 years ago from Professor Mandelbrodt at the Ecole Normal Polytechnique.

A mathematician is given a stove and a pan filled with water on an adjacent table. He is asked to boil the water. He complies by placing the pan on the stove, and boiling the water in the usual way. The next day, he is given the same problem with a single exception -- the pan filled with water is on a chair next to the table. The mathematician removed the pan from the chair, places it on the table and then quits, announcing that the problem is now solved, having been reduced to a previous problem.

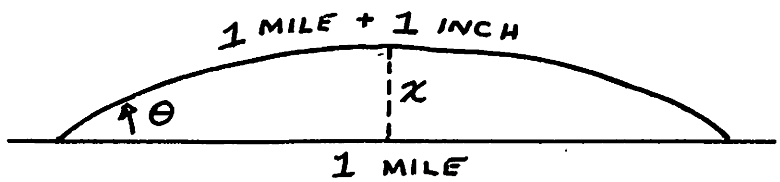
Inspired by this fable, a mathematics student approached me and made the following request: "You are given a stove and a pan filled with water on an adjacent table. Your task is to boil the water. Explain how you would solve the problem." I thought for a moment, and then proceeded: "First look in the pan; if the water is boiling we are done. Otherwise ..."

My son is a tennis player, and I was watching a match where a fellow who was playing kept hitting the ball into the net. Time after time he hit the ball into the net. Finally I went over to him and said, "You know what your problem is?"

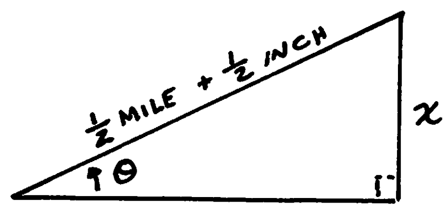
"No," he responded, "what is my problem?"

"Well," I said, "you keep hitting the ball into the net." The problem with most mathematicians is that they make statements which are absolutely correct, but also absolutely useless. Here is an absolutely useless problem, but one which I often give to some of my classes simply because it has such a surprising result.

Suppose a single railroad track is laid one mile over level ground. It is firmly secured at the ends so that they cannot move. If, in the heat of the day, the track expands one inch over its length and arcs up above the ground, then how high is the arc at its center?



Consider a triangle which may be used to approximate x, since θ is so small (approximately 0.03°)



$$(1/2 \text{ mile} + 1/2 \text{ inch})^2 = x^2 + (1/2 \text{ mile})^2$$

$$(31{,}680.5)^2 = x^2 + (31{,}680)^2 \quad \text{in inches}$$

$$x \approx 177.989466$$

$$\approx 14.8 \text{ feet}$$

Casey Jones can go right under the track!

In the 1960s we heard a lot about the "new math." What happened to the new math? Some are calling it a failure, and the buzzword for the 1980s is "back to basics." But this implies that there are only two ways to teach mathematics: "old math" and "new math." This is false. Mathematical pedagogy should always be evolving and changing and the only requirement should be: Does it teach mathematics because it MAKES THE STUDENT WANT TO LEARN, or does it try to teach mathematics by force. Commercial advertising takes the position that the consumer should remember a product in spite of himself, and as educators we should do the same.

Another important aspect in the motivation of students is using the element of surprise. I don't want my students to be able to predict exactly what I'm going to do on any particular day. Now, I'm not saying that you shouldn't have a classroom routine, but I am saying that there are variations on the usual mathematics classroom routine of working problems, presenting new material, and then assigning homework. If every day is the same as the previous, the students tend to become uninterested and bored. Look to current events or to the great mathematicians for famous problems and ideas. One of the greatest mathematicians of all times was Leonhard Euler. A few years ago I purchased a book simply called ELEMENTS OF ALGEBRA by Euler (reprinted by Springer Verlag from Euler's fifth edition last published in 1840) and was amazed at how clearly it was written. You know, I often use Euler as a one word test of mathematics competency. I write E-U-L-E-R on a piece of paper and ask the student to pronounce this mathematician's name. If they can pronounce his name, they pass the test. Anyway, you know Euler was the most prolific mathematican of all time -- he had 13 children! He was also blind for the last 17 years of his life. When Euler lost the sight of his right eye, it is reported that he commented, "Now I will have less distraction."

I think it is worthwhile to tell our students not only of the things that Euler could do, but also those things that he couldn't do. Take for example, Fermat's four square problem. You might recall that this conjecture states that any positive integer can be expressed as the sum of four squares. Fermat sent this problem to Euler, who in turn could not prove the conjecture, so he did what every good mathematician does with a good problem he can't solve -- send it to another mathematician.

You know this procedure of sending challenging problems goes all the way back to biblical times. Solomon, when he was King of Jerusalem, sent problems to Hirom (king of Tyre) to be solved, and desired he would send others back for him to solve, and that he who could not solve the problems proposed to him, should pay money to him that solved them. When Hiram had agreed to the proposals, but was not able to solve the problems, he was obliged to pay a great deal of money to Solomon, as penalty for the same. Later another person named Abdemon, a man of Tyre, came along and did solve the problems, and proposed others which Solomon could not solve and thereby was obliged to repay a great deal of money back to Hirom.

Anyway, back to Euler. He sent Fermat's four square problem to Goldbach and

Goldbach sent back to Euler HIS famous conjecture. Now Euler had two problems that he could not solve. Goldbach's conjecture was presented in a letter he sent to Euler in 1742; in short, Goldbach observed that every even integer except 2 seemed representable as the sum of two primes. I remember I presented this one evening to a Math 10 class I was teaching. It was offered on Tuesday and Thursday evenings. I made the mistake of presenting it on a Thursday evening. You know, Tuesday to Thursday is not too bad, but Thursday to Tuesday -- and it is gone. On the following Tuesday I asked if they remembered what we had talked about in the previous class. Silence. Finally with some prodding, one student said he remembered -- it was chuck numbers. "Chuck numbers!" I cried. We talked about prime numbers! "Well, said he, I knew it had something to do with meat." I'm really a pretty easy grader in that class. A means alright, B means bad, C means catastrophic, and D means dead.

When you present famous problems such as Goldbach's conjecture to the class, you need to do it in such a way that the students become involved with the problem. I begin by having the students do some work researching the history of this conjecture. In that search they will find that the conjecture is presently not proved. Then I give them an article from THE JOURNAL OF IRREPRODUCABLE RESULTS which purports to prove Goldbach's conjecture. I ask them to reconcile the article (which, of course is false) with what they have found in their search of the literature.

Another example of a famous problem I use to motivate my students is the four-color conjecture. Recall, the four-color conjecture says, roughly, that no more than four colors are required to color a map in such a way that four contiguous regions are assigned different colors. The proof of this theorem was reported in 1976. At the same time I show them an article from the April 1975 issue of SCIENTIFIC AMERICAN which reports a five color map; that is a map which requires five colors. Just between you and me, notice that it is the April issue and this was Martin Gardner's April fool's joke. I don't tell my students that, but simply ask them to reconcile the two.

It is interesting to observe how the number five occurs in different parts of mathematics:

1. Five points uniquely determine a conic.
2. There are exactly five regular polyhedra.

3. The alternating groups of order five or less are simple.
4. All groups of order five or less are commutative.
5. The general algebraic equation of degree five or higher cannot be solved in terms of radicals.
6. The number of divisions required to find the greatest common divisor of two numbers is never greater than five times the number of digits in the smaller number.
7. Every positive integer can be expressed as the sum of at least five distinct positive or negative integral cubes.

I'm sure you can add many others. The point I'm making is that we need to convey to the student that mathematics does not rule us like a tyrant. It seems rather to be part of the structure of our minds, more akin to memory than to a learnable discipline. The toddler, gratified to discover a correspondence between the toes on her right foot and those of her left, has taken her first steps in number theory. The amateur paper hanger who has forgotten all his high school geometry but somehow manages the job of papering his wall with a minimum of waste, is a whistle-stop Euclid.

I must hasten to add that proper assessment, beginning at the student's location, looking at the mathematical giants, selection of the correct textural materials, is not enough. These do not replace good teaching. Enjoyment of mathematics is not dependent upon "book learning."

The most important ingredient in motivating your students is to let your enthusiasm shine. Mathematics is many things to many people. Like music, it resists definition. Bertrand Russell had this to say about mathematics: "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." Einstein, with his customary mildness tells us that, "so far as the theorems of mathematics are about reality, they are not certain; so far as they are certain, they are not about reality." Aristotle, who was as sure of everything as anyone can be of anything, thought mathematics the study of quantity; whereas Russell, in a less playful mood, thinks of it as the "Class of all propositions of the type p implies q" which seems to have little to do with quantity. Willard Gibbs thought of mathematics as a language. Hilbert thought of it as a game. For Benjamin Pierce it was "the science that draws necessary conclusions." Hardy joyfully stressed its uselessness; Hogben stressed its practicality. Mill thought it an empirical science, whereas to Sullivan it was an art, and to the wonderful J.J. Sylvester, it was "the music of reason."

I find this ambiguity consoling. It suggests that mathematics has so many mansions that there is room for all of us; it does not appeal merely to one type of mind. If mathematics can be so many things to so many great thinkers of the ages, then as teachers we can assume no less diversity among our students. We need to go beyond the mediocre and the drudgery of day-to-day lesson planning, and concerns of our own research, to challenge, to stimulate, and to interest even the most disinterested student. Yes, we need good textbooks, good teacher's guides, and motivated materials; but we also need to convey that special spark of enthusiasm to our classes. We need to tell each student that he or she is special to us, and that we CARE about their progress. We need to remember what Alfred North Whitehead has said, "It is the function of a teacher not to cover the subject, but to uncover it for the students."