THE ACTIVITY AND APPLICATION OF MATHEMATICS

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There are four so-called philosophies of mathematics that I regard as totally discredited, and I think that there is a consensus that three of them, formalism, intuitionism, and logicism, can no longer be taken seriously as definitive. They were attempts to guarantee the certainty of mathematics, three vehicles in the mathematical lane of the dead-end street of cartesian philosophy. Just as I know of no one that has been made a Christian by seeing a proof of the existence and attributes of God, I know of no one that has been convinced of the adequacy of an ism as an account even of written-down mathematics. In mathematics as in life we proceed on faith, whether in God or in mathematics itself. Bourbaki is often quoted as noting that "for twenty-five centuries mathematicians have been correcting their errors, and seeing their science enriched and not impoverished in consequence; and this gives them the right to contemplate the future with equanimity. [1]" I think that Christians are on safer ground than Bourbaki.

Unlike the other three discredited philosophies, the fourth, Platonism, is alarmingly popular. By Platonism I mean to refer to a variety of convictions that attribute truth to mathematical statements because they capture the facts the way "It is raining" is a true statement in the rain.[2] This view assumes, usually implicitly an ontological commitment to mathematical objects, a quasiphysical existence that requires being somewhere. Plato frankly gave them a special place, Aristotle less physically found them in the things around us, and a frequent Christian device is to put them in the mind of God. No one I know takes Plato's heaven seriously, and it stretches the imagination enough to find differentiability in nature without trying to cope with non-standard models and pseudo-differential operators. While I see nothing irreverent in placing ideas of mathematical objects in the mind of God, this measure is as useless as Plato's because of its irrelevance. It does mathematics no more good to imagine its objects in the mind of the supremely real God than in a purely imaginary Platonic heaven. We have no access to the objects wherever we may imagine them, and even if we had, such access gives rise to empirical knowledge not mathematical.[3] It is therefore entirely beside the point to choose axioms in pure mathematics because they are true in the everyday sense of true.[4]

It is a common feature of the ism philosophies of mathematics that they do not take the application of mathematics very seriously. All but logicism make it very puzzling that mathematics is applicable. Stefan Körner in his sympathetic little book The Philosophy of Mathematics[5] indicates with a historical survey that the slighting of application is typical of the philosophical consideration of mathematics ever since mathematics welcomed philosophy to the company of intellectual pursuits twenty-five hundred years ago. One sometimes forgets that mathematics has been growing for five thousand years and that--however beautiful a tree it has grown to be--applications are its roots and its fruit. Before spending a few minutes explaining a view of mathematics that takes applications seriously, I want to say something about applying mathematics.

At Christmas a combination of ailments kept me in bed for ten days, during which I had ample time to read the proceedings of previous conferences. I think that I detected a consensus that there is no Christian mathematics[6], only Christian mathematicians. I also saw less on applications than I might have. Now it seems to me that there are two Christian outcomes that knowing a bit of mathematics might have. In keeping with the tradition of servanthood we can help those that want or need to know more about mathematics by teaching it to them, and we can help those that want or need to do mathematics by helping them to do it. The former may be pure mathematics, but the latter will almost always be applied. The person helped may be rather ignorant or as knowledgeable as the mathematician but involved chiefly in some other activity. Applied mathematics, like teaching, may therefore be viewed as a ministry. I view it so. As well it sheds light for me by practical experience as the only philosophical issue about mathematics that really interests me, namely, how it is applied. I did pure mathematics as a student only because it came more easily than statistics or theoretical mechanics. Even before trying to do what is normally regarded as applied mathematics, I had worked at applying topological groups to differential topology and finite groups, graphs, and computing to combinatorial topology.[7] Everyone is familiar with that sort of thing; how have real applications worked?

My first attempt was operations research, but I found that, because I live well away from such centres of commerce and industry as Canada has, I would have great trouble to find projects. The only one I did manage to find in Winnipeg indicated what I can suggest to others that do not live in large urban areas. It was a project within the university itself showing how to reroute the student bus service to be more useful[8] and how to do it cheaply.[9] Any educational institution is full of non-mathematicians needing to do or use mathematics that they are not prepared for, whether by training, time, or inclination. Their use of computers can be a stimulus to this. In the two Canadian universities I am most familiar with, the statisticians run consulting services, the computer centres do not, and the mathematicians think about it. I urge it upon you. R. V. Benson at the conference two years ago quoted Branko Grünbaum as writing

> Mathematicians frequently regard it as demeaning to work on problems related to "elementary geometry" in euclidean space of two or three dimensions. In fact, we believe that many are unable, both by inclination and training, to make meaningful contributions to the more "concrete" type of mathematics, yet it is precisely these and similar considerations that include the results and techniques needed by workers in other disciplines.[10]

One of my recent projects was on thread patterns in weaving[11] and arose from a weaver's reading a paper by Grünbaum in Mathematics Magazine.[12] What is the good of this activity, one may ask.

The benefits of being helpful are for reaching. Society benefits because the application is achieved--I am assuming a beneficial application. A large institution benefits because bridges are built between persons, disciplines, faculties. The mathematical community benefits both by the not uncommon generation of new mathematics and by the breaking down of its austere and forbidding image. The mathematician benefits from the stimulus he receives and the learning he must do. Lastly students benefit. The students of the non-mathematician benefit from any improvement in his attitude toward mathematics. And the mathematician's students benefit from his first-hand experience with applications; certainly it is becoming a common complaint that they suffer from the lack of such experience.[13] The Hilton Committee recommended government-funded summer institutes "to retrain members of the faculty in applications of mathematics.[14]" Helpful as such instruction would be in broadening horizons, no practical art is learned exclusively from instruction. Who will take seriously a non-swimmer as a swimming instructor?[15] I have mentioned benefits to counter a possible scepticism, but my basic argument is one of principle not politics.

There is another scepticism about which something must be said both here and in mathematics courses emphasizing applications. It is important both here and there not to be an apostle of what has been called "instrumental reason"; it is also important not to seem to be one. By "instrumental reason"[16] I mean the use of rationality as a means to any end whatever, without regard for the suitability of that end. With a few exceptions, all tools are what their users make of them, good or bad. The application of mathematics is what we make of it; it is certainly not a value-free activity--not that I can think of a value-free activity. It seems to me to be primarily in commenting on what can be done with mathematics that we have opportunity to let our students know where we stand on the use of mathematics, not to mention other knowledge. Briefly, there are things one would not do; it is worth noting and mentioning this.

I should now like to turn to a brief presentation of my view of mathematics. It is of course a human activity with everything that entails, history, sociology, psychology, motives, matters of taste. What I am going to limit myself to is what is publicly available in published form, most of which is far from formal--not because I am a behaviourist. The rest is important, but what is in print is both central and objective; it is also what can be applied. There is nothing preculiar to mathematics among human activities in the centrality of its objective residue. We can consider the inner springs of the actions of politicians and artists, but we can study their laws and their art. From the leavings of Babylonian and Egyptian mathematics we know that they had numbers, rules, algorithms and answers because they wrote them down. We conclude that they did not have axioms or theorems because they did not write them down. While it is controversial, I think that a culture's mathematics can be reconstructed from its accumulated written expressions.[17]

When we turn to written mathematics, what do we find? We find questions, answers, axioms, theorems, proofs, conjectures, disproofs, problems, structures. I submit that these are what mathematics at one level is about. It is often said that mathematics is quantitative and spatial, meaning about numbers and figures or even just numbers.[18] Even without matroids and probability distributions to cast doubt on the accuracy of this view, I think that the mistake being made is like saying that English literature is about the characters in plays, the dialogue in novels, the locations of the poems. The persons, speeches, and places do appear in literature but they are devised by the authors with greater or lesser similarity to persons living or dead in order to appear in plays, novels, and poems. They are the imaginary building blocks of human creations of a totally different kind, their correspondence to reality being of little relevance to the literary effect. I hope that the analogy is sufficiently transparent. What are usually called mathematical objects are the imaginary building blocks that we use to construct and communicate our questions, answers, axioms, and so forth. While some of them correspond to things outside mathematics, and those correspondences and various personal intuitions are very important psychologically to mathematicians,[19] their correspondence is of little relevance to the mathematical effect. How well they are devised and then wielded makes all the difference.

I do not mean to suggest that the placing of primary emphasis is of tremendous importance, but in asking yourself whether it is correct consider whether you admire functions or theorems, polygons or proofs. And if, like me, you do admire hexagons as well as the higher-level artifacts, then ask yourself whether that admiration is specifically mathematical or aesthetic and visual like any non-mathematician's. I want now to move down a level to the content of conjectures, disproofs, problems and so forth. What do they say? They differ in form, being statements that we guess, disprove, or wonder about, but what is conjectured, disproved, or wondered about seems to me always to be relations. Take a simple theorem like the equality of the base angles of an isosceles triangle. Equality is a relation; the base angles are those opposite the equal sides, and opposite is a relation; an angle is a relation between intersecting line segments, and intersection is a relation; finally a triangle is a triple of points not collinear and collinearity and cardinal equivalences are relations. Or look at the axioms of arithmetic; everything is in terms of the equality and successor relations. This is a sort of thesis that is hardly susceptible of proof, since it is partly a point of view, but if you look at a piece of mathematics you <u>can</u> see a network of relations of various kinds.

Allow me to pursue the explanation of this view for a few moments. An axiom becomes a relation statement that is postulated. An axiom system relates several such statements by conjunction. A theorem becomes a statement that certain hypotheses, usually the conjunction of an axiom system and some special relations, imply certain special relations as conclusions according to some programme of inference. A programme of inference is, among other things,[20] a repertoire of ways of transmitting the relation of implication. A proof is a setting out, formally or informally, of a chain of implications linking hypotheses to conclusions. A problem is frequently a statement of ignorance of the precise relation among some relations with the suggestion that some interesting relation is implicit but unknown. A solution is then finding an interesting relation, often not unique. A problem is more interesting if it has a number of solutions unlike the typically single-answer or even single-solution problems in schoolbooks. Proof according to this view is relative to choice of programme of influence. In pure mathematics one must choose whether to operate within intuitionism or constructivism[21] or the usual technique of "no holds barred". Not to care about one's logic, like not to care about one's topology or one's religion, is to make a choice even if it is unconsidered and indefensible. Finally, to return to the axioms, it is clear that they attribute no character to the entitites they relate in certain ways. It seems reasonable to me to recognize this explicitly and admit that they need have no characters but may be regarded just a grammatical necessities for the symbolic expression of relations. I think that the formalists were right to recognize that mathematics makes no ontological commitment[22] to its objects, no more than a writer of fiction makes an ontological commitment to fictional characters even if they are modelled on historical characters. One can of course make whatever ontological commitments one likes. There need be no such commitment to the relation either; they need only be intelligibly related by axioms.[23] I am not saying that onotology must not deal with mathematical objects. There is an obvious sense in which we have to speak of mathematical existence. Popper invents a world to put theorems and lower-level objects into. In Objective Knowledge he wrote

> The main point here I owe to Lakotos's philosophy of mathematics. It is that mathematics (and not only the natural sciences) grows through the criticism of guesses, and bold informal proofs. This presupposes the linguistic formulation of these guesses and proofs, and thus their status in the third world. Language, at first merely a means of communicating descriptions of prelinguistic objects, becomes thereby an essential part of the scientific enterprise, even in mathematics, which in its turn becomes part of the third world.[24]

While the formalists seem to have been mistaken in their search for infallibility, I think that they were right in not attributing meaning, in anything like an ordinary sense of the word, to axioms and theorems.[25] In themselves axioms have intelligibility, which is rare enough, and theorems can be correct, which is rare in meaningful discourse, and they have applicability. Application is the process of giving meaning[26] to a statement that lacks meaning in itself. $1 + 1 = 2$, and $\$1 + \$1 = \$2$ when arithmetic is applied to money. Vector fields become gravitational forces when they are applied to physics. The variable t becomes a time co-ordinate in many sciences. If these things had meanings of their own, their application would be seriously confusing. Indeed, when everything geometrical was given a mechanical interpretation in the nineteenth century, the vector fields of electromagnetism were a tremendous puzzle to the British like Lord Kelvin but did not bother Europeans like Pierre Duhem who broke out of the mechanical trap.[27] There is nothing either reprehensible or frivolous about a statement without intrinsic meaning. "He served it to her" depends for its meaning on whether he is playing tennis, being a waiter, or working for the sheriff. By giving referents to the pronouns in the sentence we make real ones of the potential meanings that it had out of context. "They did it to him" depends for its meaning on whether he ends the process elected or lynched. I think that mathematics is put to work in much the same way when it is applied to a subject matter.

What is the point, one may say, of taking up a view of mathematics that is so idiosyncractic, even if it does tie in a bit with some well known views? Part of the point is just to present an illuminating point of view, even if it is idiosyncractic; not surprisingly, no view of mathematics that I have yet seen has seemed altogether right or even if right as far as it goes to be any help in explaining what is sometimes called the unreasonable effectiveness[28] of applied mathematics. The reason that I have spent the time to reach my meagre conclusion is that I think it important to be able to explain to non-mathematicians what mathematicians, pure and applied, do and do not do.[29]

If logicism had lasted more than thirty years, it would have been excellent for explanatory purposes since everyone has some idea of what logic is.[30] I want to pause and look at logic for a moment. It is no accident that logicism happened when it did because, as Morris Kline points out throughout his book Mathematical thought from ancient to modern times,[31] it is the latest thing that its proponents say is everything, whether the numbers of the Pythagoreans or projective geometry in the nineteenth century. What happened to logic, which is the empirical study of inference, was that it became an application of mathematics. And a not very satisfactory application it is too, as anyone that has tried to sell material implication to unsuspecting sophmores might well agree. Even if Principia mathematica had succeeded, it would have shown only that all of mathematics could be based on one of its own applications. I mention this here in order to claim symbolic logic as a branch of applied mathematics[32] in which, for example, propositional calculus is given meaning by being made to refer to propositions. The various formal logics are available to anyone prepared to be rigorous in his choice of a programme of inference when doing mathematics.

There are a couple of consequences of my point of view that I find attractive and worth mentioning. One is that it encourages a mathematical, or strictly speaking metamathematical, use of the word "true" that corresponds to its use outside. The whole statement "The axioms $A_1, \ldots, A_a$ and the special hypotheses $H_1, \ldots, H_h$ imply conclusion C according to program of inference Π" is true if in fact that is the case. One might write

$$A_1 \,\&\, \ldots \,\&\, A_a \,\&\, H_1 \,\&\, \ldots \,\&\, H_h \Rightarrow C \pmod{\Pi}$$

Such a statement is independent of the suitability of programme Π for other purposes. It is independent of the consistency of the axioms and hypotheses, since inconsistency renders material implication trivial not false. It is independent of ontology. It is independent of our knowledge of a proof. If we have a proof in accordance with Π, then we know the statement is true.[33] If we have no such proof, then we do not know that the statement is true. It may even be false. Certainly truth and falsity should not be confused with our knowledge of them.

The other important cosequence that I want to mention, again for talking to non-mathematicians, is that the potential-knowledge view of mathematics makes its applicability easy to appreciate. For if mathematics draws relational consequences of structures of relations in accordance with a programme of inference Π, then, whenever relations in some intellectual field have that structure, the relational consequences can be depended upon modulo Π and whenever the structure is approximated the relational consequences are at least heuristically suggested. In some applications of finite mathematics like Boolean algebra and graph theory, the postulated relations may hold precisely. In many of the classical applications like Newtonian mechanics the relations are very good approximations. We can accordingly deduce exact relations or very good approximations modulo Π. In order to apply mathematics with confidence, we must be prepared to live with our choice of Π, not to mention the consistency of our axioms and hypotheses. Within classical applied mathematics we are perfectly confident of Π. The American and Soviet space programmes are compelling verifications of celestial mechanics; one thing no astronaut has lost sleep worrying about is a faulty Π.

My final topic is the importance of choosing a programme of inference. If mathematics were the meaningless game that some persons thought the formalists reduced it to, then that choice would be of no importance. But mathematics can easily be given meaning of the most concrete possible kind. And it is important that when it is applied it is not wrong. Our programmes of inference must be adequate. Publicity like Morris Kline's _Mathematics: the loss of certainty_[34] we can live without, but if we are irresponsible we shall have worse. Our logics must be intelligible, and preferably intelligible to non-specialists. They will probably be formal logics and therefore themselves applied mathematics. Since logic is an empirical science, one can not unreasonably regard mathematics as the laboratory of logic. There is no reason to expect consensus on a single programme of inference in the near future, if only because so many persons delight in working with an arm tied up behind their backs, e.g. intuitionists. There is certainly no consensus at present. In fact the present state of mathematicians' logic is worse than that, if only because of our typical ignorance. Most mathematicians, including me, get along with a simple-minded two-valued logic, perhaps unaware of arguments for its inadequacy, and freely apply the law of the excluded middle though we may know that it is dangerous. On the other hand, the law of the excluded middle is apparently often appropriate, but we do not know when. The only advice actually offered is "never".

This situation brings me back to where I began with Christian mathematians and an emphasis on what we do besides teach. The state of affairs that I have just sketched seems to me one that Christian mathematicians are particularly suited to address because their faith, unlike Bourbaki's, is placed in God rather than in their mathematics. We can afford to explore alternative logics without having the feeling that we are undermining our personal foundations. To judge from the tenor of Kline's uncertainty book, whose third and fourth chapters could be mistaken for an atheist tract, this sort of matter makes some persons very uncomfortable indeed. My contention is that Christians have no need to be uncomfortable in our examination of logical matters. While I am constitutionally very traditional and do not expect any radical improvements in the logic that I use at present, I am certain that it can be improved and I commend study and experiment to any mathematician that does not want to involve himself in applications _outside_ mathematics. Here _inside_ is an important application. Please note that I am not suggesting here, as Harold Heie[35] did at the conference of two years ago, taking up mathematical logic as a theoretical persuit; that was one small part of my earlier suggestion of applied mathematics. What I am suggesting here is an alternative, that even within pure mathematics knowledge of the results of logical research and experiment based on that knowledge are needed. And so I end where began with the suggestions that next to teaching mathematics, appropriate activities for a Christiam mathematician are either applications outside of mathematics, like teaching, practical expressions of Christian love, or failing that adventurous applications within mathematics itself, theoretical explorations of Christian freedom.

NOTES

1. The conclusion to the introduction of Bourbaki's treatise Eléments de mathématique translated on p. 330 of G. T. Kneebone, Mathematrical logic and the foundations of mathematics, Van Nostrand, 1963.

2. I have in mind Alfred Tarski's theory of truth, particularly as defended by K. R. Popper in Chapter 9 of his Objective knowledge, Clarendon Press, 1972.

3. This is a standard objection to the usefulness, not the truth, of the Platonic view. I have seen it most recently mentioned on pp. 394f. of P. J. Davis and R. Hersh, The mathematical experience, Boston: Houghton Mifflin, 1982.

4. As for instance on p. 2 of Schoenfield's text on mathematical logic.

5. Hutchinson University Library, London, 1960.

6. One hardly expects unanimity, but I find no content to correspond to the notion of Christian mathematics as far as the outcome is concerned; the difference is in the worker.

7. 'The equality of a manifold's rank and dimension', Canad. Math. Bull. 12 (1969), 183-184; 'Isotopic closed non-conjugate braids' (with K. Murasugi), Proc. Amer. Math. Soc. 33 (1972), 137-139; 'The structure of the fundamental braids', Quart. J. of Math (Oxford 2) 26 (1975), 283-288; 'Garside's braid-conjugacy solution implemented' (with B. T. Paley), Utilitas Math. 6 (1974), 321-335.

8. 'Postal codes locate special transit demand' (with J. M. Wells), The Logistics and Transportation Review 14 (1978), 90; Canadian postal codes are much more precise than U.S. codes.

9. 'Multiple-origin single-destination transit routing' (with J. M. Wells), INTERFACES 10 (1980), 41-43.

10. P. 117.

11. 'Conditions for isonemal arrays on a Cartesian grid' (with W. D. Hoskins), Linear Algebra and its Applications, to appear. The weaver was J. A. Hoskins.

12. 'Satins and twills: An introduction to the geometry of fabrics' (with G. C. Shephard), Vol. 53(1980), pp. 139-161.

13. Some comments of Chandler Davis were quoted by Paul J. Zwier on p. 131 of the proceedings of the Third Conference.

14. <u>Ad hoc</u> committee on applied mathematics training, The role of applications in the undergraduate mathematics curriculum, Washington, D.C.: National Academy of Sciences, 1979, p. 16. Cf. Recommendation 4, p. 20. See also Committee on the undergraduate program in mathematics, Applied Mathematics in the undergraduate curriculum and Recommendations for a general mathematical sciences program, Washington, D.C.: Mathematical Association of America, 1972 and 1981.

15. E.g., Davis and Hersh, op. cit., p. 85, remark

> In a typical book on applied mathematics, one finds, for example, a discussion of the Laplace problem for a two-dimensional region. This has important applications, says the author, in electrodynamics and in hydrodynamics. So it may be, but one should like to see the application pinpointed at the level of common utility rather than of pious potentiality.

If those authors are sceptical, what of the undergraduate?

16. By "instrumental reason", Charles Davis, the Montreal sociologist of religion means the use of reason primarily for control where success is the only goal and means are not questioned. He explains the alternative as the subordination of rationality as technique to reason as the intelligent participation in the order of reality as an order of truth and values, as a responsive pursuit of the true and the good. Only in the pursuit of a transcendent end can we reasonably and responsibly evaluate particular ends. I quote from my notes on a lecture given at Wycliffe College, Toronto, February 21, 1983, and based on a forthcoming book.

17. This is controversial; for example Davis and Hersh disagree, op. cit., p. 33.

18. E.g., Pierre Duhem, The aim and structure of physical theory, New York: Atheneum, 1974, p. 113.

19. Davis and Hersh in their fine book, op. cit., do seem to blur the two distinctions between philosophy and psychology and between existence and objectivity. They sort out the latter somewhat, p. 409, but not the former. I do not doubt, for instance, that the objects of really informal mathematics, p. 351, are merely mental and personal. Making more precise to produce communicable and provable mathematics is not the process of dispensing with them but of removing logical dependence on them. In Popper's terms, op. cit., Chapter 4, this is the shift from World 2 to World 3; it allows communication, criticism, and objectivity.

20. E.g., restrictions on postulates.

21. E. Bishop makes a remarkably good case for constructivism as a method of doing mathematics at the beginning of his Foundations of constructive analysis, New York: McGraw-Hill, 1967. I think that no mathematician should fail to read that case.

22. I mean in the ordinary not the Quinean sense.

23. The _nature_ of the relations matters as little as the nature of the objects. (They have great psychological importance to the mathematician; no doubt the author of a fiction knows more of his characters than he sets down.) What matters to the matehamatics is the postulated relations among the relations. My favorite example of this is the theory of planar graphs, which can be "about" _regions separated_ by _boundaries_ or _vertices joined_ by _edges_. The objects are very different and the relations almost opposite, but the way they are related has the same structure, is mathematically indistinguishable; hence it is a single theory.

24. P. 136.

25. I hasten to note that this does not make mathematics a meaningless game. In much mathematics neither the psychology nor the result is at all gamelike, and there are never competitors in the game sense. If it is playing, it is more like playing music than a game. Secondly, mathematics is no more meaningless than grammar and logic, where statements are also studied separately from what they could be made to refer to. The possibility of translation proves that there is meaning in _some_ sense to be translated.

26. I found most interesting the link that Davis and Hersh, _op. cit._, pp. 342f., make between formalism and logical positivism. Certainly both must be rejected for the same reason, that they attempt to reduce complex human activities to simple subsets. The link is a historical question, and it would be worth something to see a substantial answer. Formalism lacked the apparently atheistic motivation of logical positivism, which was definitely in the philosophical lane of the cartesian dead-end street (see my first paragraph).

27. And described the matter for us, _op. cit_.

28. E. P. Wigner, 'The unreasonable effectiveness of mathematics in the natural sciences', _Comm. Pure Appl. Math._ _13_ (1960), 1-14; R. W. Hamming, 'The unreasonable effectiveness of mathematics', _Amer. Math. Monthly_ _87_ (1980), 91-90; as I view it, it is not at all unreasonable.

29. And do not do. A question that has interested me is why mathematics is so little use in theology. The 1981 bibliography indicates two authors.

30. If I were asked how mathematics differs from logic, I would elaborate my contention that the one is a specific application of a small part of the other by pointing out that mathematics supplies a framework for reasoning that is not just about atomic propositions and secondly that it is interesting on its own account.

31. Oxford University Press, New York, 1972.

32. There is, first, the appearance of a work like _Principia mathematica_ and second Russell's "regret that more attention was not paid to the mathematical techniques evolved in the work" (F. Coplesten, _History of philosophy_, Vol. 8, Part II, p. 198). Russell felt that it was mathematically interesting.

33. We may be mistaken, being fallible.

34. Oxford University Press, New York, 1980.

35. P. 59 of _A third conference on mathematics from a Christian perspective_, ed. R. L. Brabenec, 1981.