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BOUNDEDNESS THEOREMS FOR FLOWERS AND SHARPS

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Abstract. We show that the Σ^1_1- and Σ^1_2-boundedness theorems extend to the category of continuous dilators. We then apply these results to conclude the corresponding theorems for the category of sharps of real numbers, thus establishing another connection between Proof Theory and Set Theory, and extending work of Girard-Normann and Kechris-Woodin.

1. Introduction

The Σ^1_1-boundedness theorem is a fundamental result in descriptive set theory with applications throughout many branches of logic. It states that if $A \subset \mathsf{Ord}$ is Σ^1_1, then there is $\alpha < \omega_1^{ck}$ which bounds every element of A. Here, it is assumed that (some) ordinals are represented by real numbers according to some fixed coding mechanism, such as Kleene's $\mathcal{O}$. Throughout the article, we will abuse notation by identifying codes of ordinals with their order types. The theorem can be generalized in many ways. For instance, it can be generalized to larger complexity classes using more elaborate coding devices for larger ordinals (see Koellner-Woodin [15] for some examples). Alternatively, it can be generalized to categories with more structure than the ordinals, and it is this line of study that we pursue here.

We think of the class of ordinals Ord as a category where morphisms are strictly increasing functions. A functor d in this category is said to be a *dilator* if it commutes with direct limits and pullbacks. The class of dilators can be regarded as a category of functors where natural transformations serve as morphisms. A dilator is *countable* if it maps countable ordinals to countable ordinals. As was the case with countable ordinals, countable dilators can be coded by real numbers. The Σ^1_1-boundedness theorem for dilators states that if A is a Σ^1_1 set of dilators, then there is a recursive dilator which bounds every element of A. It is due independently to Kechris-Woodin [14] and to Girard-Normann [8].

In this article, we consider extensions of this result where the dilators are supposed to possess some additional property and conclude that the bounding dilator can also be assumed to satisfy that property. Specifically, we consider the categories of flowers and sharps. According to a characterization proved below, a *flower* is a dilator whose restriction to Ord is continuous; and the *sharp* of a real x is the theory $x^\sharp$ of $L[x]$ in the language of set theory with constants for infinitely many order-indiscernibles. Interestingly, the statement of this result for the lightface class Σ^1_1 is vacuous, but this is not true for the boldface class $\boldsymbol{\Sigma}^1_1$, so it is an example of a theorem in effective descriptive set theory the simplest proof of which does not

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relativize. After proving the boundedness theorems for these categories, we derive some additional consequences thereof, including forms of the boundedness theorems for Σ^1_2 sets.

An interesting aspect of the work presented here is that it involves some results which are purely set-theoretic, but whose proofs involve the theory of dilators, which by some people is regarded as pertaining to proof theory. This type of interaction, although unusual, is not without precedent (cf. e.g., the work of Arai [3, 4]).

2. Dilators

Let Ord denote the category of ordinal numbers with strictly increasing injections as morphisms. A *dilator* is a functor d on Ord which commutes with pullbacks and direct limits. Dilators themselves can be viewed as a functor category DIL with natural transformations as morphisms. We refer the reader to Girard [7] for background on dilators and Π^1_2-logic.

A dilator is *countable* if it maps countable (equivalently, finite) ordinals to countable ordinals. A dilator d is completely determined by its action on finite ordinals and on morphisms between two finite ordinals. Thus, countable dilators can be coded by real numbers in many ways; we shall adopt the formalization of Girard-Ressayre [9]. It follows from commutation with direct limits that countable dilators map arbitrary countable ordinals to countable ordinals.

2.1. **Ordinal denotations.** A dilator d can be identified with a system of ordinal denotations in which an ordinal $<d(\alpha)$ is represented by an expression of the form

$$(C; x_0, ..., x_n; \alpha),$$

where $x_0 < ... < x_n$, are ordinal parameters, α is given in advance, $x_n < \alpha$, and C is the *configuration.* Here, we include the degenerate case $n = -1$ in which the list of parameters is empty. In order to obtain the equivalence between ordinal denotations and dilators, some constraints are imposed, namely:

(1) For each α and each $z < d(\alpha)$, the representation of z as an expression of the form $(C; x_0, ..., x_n; \alpha)$ is unique;
(2) If $(C; x_0, ..., x_n; \alpha)$ is a denotation and if $x'_0 < ... < x'_n < \alpha'$, then so too is $(C; x'_0, ..., x'_n; \alpha')$.
(3) If $(C; x_0, ..., x_n; \alpha) < (D; y_0, ..., y_m; \alpha)$ and if $x'_0 < ... < x'_n < \alpha'$ and $y'_0 < ... < y'_m < \alpha'$ are such that $x_i < y_j$ iff $x'_i < y'_j$ for each $i = 0, 1, ..., n$ and $j = 0, 1, ..., m$, then $(C; x'_0, ..., x'_n; \alpha') < (D; y'_0, ..., y'_m; \alpha')$.

2.2. **Flowers.** Given ordinals x, y with $x \leq y$, we denote by E_{xy} the order-preserving function with domain x and co-domain y given by

$$E_{xy}(z) = z$$

for $z < x$. A dilator F with the property

$$F(E_{nm}) = E_{F(n)F(m)} \tag{1}$$

for all $n, m \in \mathbb{N}$ with $n \leq m$ is called a *flower.* Note that it follows from F being a dilator that condition (1) holds for all natural numbers n if and only if it holds for all ordinals α.

Girard [7, Proposition 2.4.7] has shown that a dilator is a flower precisely when the value of denotations $(C; x_0, ..., x_n; \alpha)$ is independent of α. In these cases, we will simply write $(C; x_0, \dots, x_n)$.

We prove another characterization of flowers:

Theorem 1. *Let D be a dilator. Then, the following are equivalent:*

(1) *D is a flower;*
(2) *The function on ordinals given by $\alpha \mapsto D(\alpha)$ is continuous.*

Proof. The fact that (1) implies (2) is well-known and essentially proved by Aczel [1] before the notion of a dilator was isolated. We refer the reader to Girard [7] or to Freund-Rathjen [6] for a proof.

Assume that α and β are ordinals and that $D(E_{\alpha\beta})$ is not the inclusion map $E_{D(\alpha)D(\beta)}$; equivalently, the range of $D(E_{\alpha\beta})$ is not an initial segment of Ord. Viewing D as a denotation system, this means that we can find ordinals $\alpha_0, \ldots, \alpha_m; \beta_0, \ldots, \beta_n; \alpha'_0, \ldots, \alpha'_k$ and configurations σ, τ with $n \neq -1$ and

$$\alpha_0 < \cdots < \alpha_m < \alpha \leq \beta_0 < \cdots < \beta_n < \beta$$

and $\alpha'_0 < \cdots < \alpha'_k < \alpha$ such that

$$(\sigma; \alpha_0, \ldots, \alpha_m, \beta_0, \ldots, \beta_n; \beta) < (\tau; \alpha'_0, \ldots, \alpha'_k; \beta). \tag{2}$$

Let λ be a limit ordinal large enough so that $\lambda = \beta + \lambda$ and $D(\gamma) < \lambda$ for all $\gamma < \lambda$. Then,

$$\sup\{D(\gamma) | \gamma < \lambda\} \leq \lambda.$$

To complete the proof, it suffices to establish $D(\lambda) > \lambda$. For each $\gamma < \lambda$ let $\sigma(\gamma)$ be the ordinal represented by the term

$$(\sigma; \alpha_0, \ldots, \alpha_m, \beta_0, \ldots, \beta_{n-1}, \beta + \gamma; \lambda) \in D(\lambda).$$

By the usual properties of denotations, the function with domain λ given by $\gamma \mapsto \sigma(\gamma)$ is strictly monotone, so $\gamma < \gamma' < \lambda$ implies $\sigma(\gamma) < \sigma(\gamma') < D(\lambda)$. By (2),

$$(\sigma; \alpha_0, \ldots, \alpha_m, \beta_0, \ldots, \beta_n; \lambda) < (\tau; \alpha'_0, \ldots, \alpha'_k; \lambda).$$

Since $\alpha'_k < \alpha \leq \beta_n$, the displayed inequality remains true if we replace β_n by any ordinal between β and λ, and in particular by $\beta + \gamma$, so we also have

$$\sigma(\gamma) < (\tau, \alpha'_0, \ldots, \alpha'_k; \lambda).$$

We have shown that $D(\lambda)$ is at least $\lambda + 1$, which contradicts the continuity of D. □

3. Boundedness by continuous dilators

The goal of this section is to state and prove the boundedness theorem for flowers. The proof will rely on some known constructions on dilators and facts about them. The first such construction is a simple addition operator; the second one is an integral operator.

If ξ is an ordinal and d is a dilator, we denote by $\xi + d$ the function defined by

$$(\xi + d)(\gamma) = \xi + d(\gamma)$$

and if $h : \alpha \to \beta$ is a strictly increasing function,

$$(\xi + d)(h)(\gamma) = \begin{cases} \gamma, & \text{if } \gamma < \xi, \\ \xi + d(h)(-\xi + \gamma), & \text{if } \xi \leq \gamma. \end{cases}$$

We state some facts about $\alpha + d$ which are easy to verify. Here $-\xi + \gamma$ is the unique ordinal δ such that $\xi + \delta = \gamma$.

Lemma 2. *Let ξ be an ordinal and d, f, and g be dilators.*

(1) *$\xi + d$ is a dilator.*
(2) *Suppose there is a natural transformation $T : f \to g$ and ξ is an ordinal. Then, there is a natural transformation $T' : \xi + f \to \xi + g$.*
(3) *Suppose that there is a natural transformation $T : f \to g$ and ξ is an ordinal. Then, there is a natural transformation $T' : f \to \xi + g$.*

Proof. The first claim is easy to verify. For the second and third claims, we simply state what the components of the transformation are and leave the verification that these are as desired to the reader. For the second claim, let α be an ordinal and η_α be the αth component of T. Then, the αth component of T' is μ_α given by $\mu_\alpha(\gamma) = \gamma$ if $\gamma < \xi$ and $\mu_\alpha(\xi+\gamma) = \xi+\eta_\alpha(\gamma)$. For the third claim, if η_α is as above, then μ_α is defined by $\mu_\alpha(\gamma) = \xi + \eta_\alpha(\gamma)$. □

Given a dilator d, the dilator $\int d$ is defined by

$$\left(\int d\right)(\alpha) = \sum_{\beta<\alpha} d(\beta)$$

and, if $f : \alpha \to \alpha'$ is strictly increasing, $\gamma < d(\beta')$, $\beta' < \alpha$, then

$$\left(\int d\right)(f)\Big(\sum_{\beta<\beta'} d(\beta) + \gamma\Big) = \sum_{\beta<f(\beta')} d(\beta) + d(g)(\gamma),$$

where $g : \beta' \to f(\beta')$ is the function given by $g(x) = f(x)$. Girard [7, Example 2.4.9(i)] has shown that a dilator d is a flower if and only if d is of the form

$$d = \alpha + \int d'$$

for some ordinal α and some dilator d'. The construction shows that if d is a countable flower, then both α and d' can be obtained recursively from d.

Lemma 3. *Let f and g be dilators and suppose that $T : f \to g$ is a natural transformation. Then, there is a natural transformation*

$$\int T : \int f \to \int g.$$

Proof. Given an ordinal α, let η_α be the corresponding component of T. We define

$$\mu_\alpha : \left(\int f\right)(\alpha) \to \left(\int g\right)(\alpha)$$

as follows: for $\gamma < \left(\int f\right)(\alpha)$, find $\beta < \alpha$ and $\gamma' < f(\beta)$ such that

$$\gamma = \left(\int f\right)(\beta) + \gamma'.$$

Such β and γ' must exist, by choice of γ. Then, we set

$$\mu_\alpha(\gamma) = \left(\int g\right)(\beta) + \eta_\beta(\gamma').$$

We claim that these functions are the components of a natural transformation. To see this, first observe that each η_β is strictly increasing (since it is the component of a natural transformation) and hence so is μ_α. Now, fix a strictly increasing function

$$h : \alpha \to \bar{\alpha};$$

we show that
$$\left(\int g\right)(h)\circ\mu_\alpha=\mu_{\bar\alpha}\circ\left(\int f\right)(h).$$
Let $\gamma<\left(\int f\right)(\alpha)$ and find β and γ' as above. We have
$$\begin{aligned}\left(\int g\right)(h)\circ\mu_\alpha(\gamma)&=\left(\int g\right)(h)\left(\left(\int g\right)(\beta)+\eta_\beta(\gamma')\right)\\&=\left(\int g\right)(h)\left(\sum_{\beta'<\beta}g(\beta')+\eta_\beta(\gamma')\right).\end{aligned}$$
Since $\eta_\beta: f(\beta)\to g(\beta)$, $\eta_\beta(\gamma')<g(\beta)$ so by the definition of the integral, letting
$$\bar h:\beta\to h(\beta)$$
be the restriction of h to β, we have
$$\begin{aligned}\left(\int g\right)(h)\left(\sum_{\beta'<\beta}g(\beta')+\eta_\beta(\gamma')\right)&=\sum_{\beta'<h(\beta)}g(\beta')+g(\bar h)(\eta_\beta(\gamma'))\\&=\left(\int g\right)(h(\beta))+g(\bar h)(\eta_\beta(\gamma')).\end{aligned}$$
By hypothesis, η_β is a component of the natural transformation T, so
$$g(\bar h)(\eta_\beta(\gamma'))=\eta_{h(\beta)}(f(\bar h)(\gamma')).$$
Since $f(\bar h)$ is a function from $f(\beta)$ to $f(h(\beta))$, we have $f(\bar h)(\gamma')<f(h(\beta))$, so we can now argue as follows:
$$\begin{aligned}\left(\int g\right)(h(\beta))+g(\bar h)(\eta_\beta(\gamma'))&=\left(\int g\right)(h(\beta))+\eta_{h(\beta)}(f(\bar h)(\gamma'))\\&=\mu_{\bar\alpha}\left(\left(\int f\right)(h(\beta))+f(\bar h)(\gamma')\right)\\&=\mu_{\bar\alpha}\left(\sum_{\beta'<h(\beta)}f(\beta')+f(\bar h)(\gamma')\right)\\&=\mu_{\bar\alpha}\circ\left(\int f\right)(h)\left(\sum_{\beta'<\beta}f(\beta')+\gamma'\right)\\&=\mu_{\bar\alpha}\circ\left(\int f\right)(h)(\gamma).\end{aligned}$$
This proves that the μ_α's are as desired. □

We can now prove the boundedness theorem for flowers.

Theorem 4. *Suppose that A is a Σ^1_1 set of flowers. Then, there is a recursive flower F_0, such that every $F\in A$ naturally transforms into F_0.*

Proof. Let A be a Σ^1_1 set of flowers. Define
$$B=\left\{d\;\middle|\;\left(\alpha+\int d\right)\in A\text{ for some ordinal }\alpha\right\}$$

and

$$C = \{f(0) | f \in A\}.$$

By Girard's characterization, every flower f is of the form $\alpha + \int d$, and necessarily we must have $\alpha = f(0)$, so to each $f \in A$ corresponds some $d \in B$ and some $\alpha \in C$. Moreover, d and $d(0)$ are recursive in f, so it follows that both B and C are Σ^1_1 sets.

Since B is a Σ^1_1 set of dilators, the boundedness theorem for dilators yields a recursive dilator d_0 such that every $b \in B$ transforms naturally into d_0. Since C is a Σ^1_1 set of ordinals, the boundedness theorem for ordinals yields a recursive ordinal α_0 which bounds all $\alpha \in C$. We let

$$F_0 = \alpha_0 + \int d_0.$$

If $f \in A$, then f is of the form $\alpha + \int d$ for some $\alpha \in C$ and some $d \in B$. By choice of d_0, there is a natural transformation from d into d_0. By Lemma 3, there is a natural transformation from $\int d$ into $\int d_0$. By Lemma 2(2), there is a natural transformation from $\alpha + \int d$ into $\alpha + \int d_0$. By choice of α_0, we have $\alpha < \alpha_0$. By Lemma 2(3) applied to f and $-\alpha + \alpha_0$, there is a natural transformation from $\alpha + \int d$ into $\alpha_0 + \int d_0$, as desired. □

Other boundedness theorems, such as the ones of Kechris [13] and Girard-Normann [8] generalize similarly. We state and prove one such generalization below. We will use the following straightforward lemma.

Lemma 5. *Let f be a dilator and ξ be an ordinal and define g by*

$$g(\alpha) = f(\xi + \alpha)$$

and if $h : \alpha \to \beta$ is strictly increasing, then

$$g(h) : f(\xi + \alpha) \to f(\xi + \beta)$$

is given as follows: for $\xi_0 < \xi_1 < \cdots < \xi_k < \xi$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha$, we set

$$g(h)(C; \xi_0, \ldots, \xi_k, \xi+\alpha_0, \ldots, \xi+\alpha_n; \xi+\alpha) = (C; \xi_0, \ldots, \xi_k, \xi+h(\alpha_0), \ldots, \xi+h(\alpha_n); \xi+\beta).$$

Then, g is a dilator and there is a natural transformation from f to g.

Proof. This follows directly from the fact that the composition of two dilators is a dilator. □

Recall that a binary dilator $d(x, y)$ is said to be a *bilator* if for every x, the partial functor $d(x, \cdot)$ is a flower and d is not constant in y, i.e., $d(x, y)$ is not of the form $\hat{d}(x)$ for some $\hat{d}$.

Theorem 6. *Suppose that A is a Σ^1_2 set of flowers. Then, there is a recursive bilator $F_0(\cdot, \cdot)$ such that every $F \in A$ naturally transforms into $F_0(\alpha, \cdot)$ for some α.*

Proof. Let A be a Σ^1_2 set of flowers. If A consists entirely of constant flowers, then F_0 is easy to find, so we suppose otherwise. As before, let

$$B = \left\{ d \,\middle|\, \left(\alpha + \int d\right) \in A \text{ for some ordinal } \alpha \right\}.$$

Then, B is a Σ^1_2 set of dilators. We appeal to the boundedness theorem for Σ^1_2 sets of dilators of Girard-Normann [8, Theorem 4.5], whereby there is a recursive dilator

$d_0(x, y)$, in two variables, such that every $d \in B$ can be naturally transformed into $d_0(\alpha, \cdot)$ for some countable ordinal α. We define

$$F_0(\alpha, \beta) = \alpha + \int_\beta d_0(\alpha, \beta),$$

where $\int_\beta$ denotes integration over β (with α constant). We claim that F_0 is as desired. First, observe that F_0 is a dilator. Moreover, for each α, $F_0(\alpha, \cdot)$ is a flower. Finally, F_0 cannot be constant in β, for otherwise d_0 would be constant in β. Since every non-constant countable dilator maps countable ordinals to arbitrarily large ordinals below ω_1, this contradicts the assumption that A had a non-constant element. Hence, F_0 is a bilator.

Now, let $F \in A$ and write $F = \xi + \int d$, with ξ a countable ordinal and $d \in B$. By choice of d_0, there is an ordinal α and a natural transformation from d into $d_0(\alpha, \cdot)$. By Lemma 5, there is a natural transformation from $d_0(x, y)$ into $d_0(\xi + x, y)$ (as two-variable dilators), and thus there is a natural transformation from $d_0(\alpha, \cdot)$ into $d_0(\xi + \alpha, \cdot)$. Hence, there is one from d into $d_0(\xi + \alpha, \cdot)$. By Lemma 3 there is a natural transformation from $\int d$ into $\int_\beta d_0(\xi + \alpha, \beta)$. By Lemma 2(3), there is a natural transformation from $\int_\beta d_0(\xi + \alpha, \beta)$ into $\alpha + \int_\beta d_0(\xi + \alpha, \beta)$, and thus one from $\int d$ into $\alpha + \int_\beta d_0(\xi + \alpha, \beta)$. Finally, by Lemma 2(2), there is a natural transformation from $\xi + \int d$ (i.e., from F) into $\xi + \alpha + \int_\beta d_0(\xi + \alpha, \beta)$ (i.e., into $F_0(\xi + \alpha, \cdot)$), as desired. $\square$

4. Boundedness for sharps

4.1. **Preliminaries.** Let $x \in \mathbb{R}$ and let $L[x]$ denote the class of sets constructible from x. The statement that "$x^\sharp$ exists" means that there is a closed, cofinal class I_x of ordinals that are order-indiscernible in $L[x]$; we call these *x-indiscernibles*. If it exists, I_x is uniquely determined. We enumerate these ordinals by $\{c^x_\iota : \iota \in \mathsf{Ord}\}$ if they exist, in which case $x^\sharp$ denotes the theory of $L[x]$ in the language $\mathcal{L}^\sharp$ of set theory with additional constants $\dot{x}$ for x and $\dot{c}_i$ for c^x_i ($i < \omega$). If $x^\sharp$ exists, then $L[x]$ is the Skolem hull of its class of indiscernibles. Since the fundamental work of Silver [17] and Solovay [18], sharps have been thoroughly studied, together with their foundational and descriptive-set-theoretic properties. We refer the reader to Devlin [5], Jech [11], Kanamori [12], and Moschovakis [16] for background. Below, we write

$$\mathcal{S} = \{x^\sharp | x \in \mathbb{R}\}.$$

Throughout this section, we assume that $x^\sharp$ exists for all $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$. For our purposes, a *term* is an expression

$$t^{L[x]}(\dot{c}_{k(0)}, \ldots, \dot{c}_{k(n)}),$$

where $k(0) < k(1) < \ldots < k(n)$ are non-negative integers and t is a formula with one free variable y in the language $\mathcal{L}^\sharp$. Given such a term, it can be evaluated as follows: first, interpret the constant $\dot{x}$ as x; then, interpret the constants $\dot{c}_{k(0)}, \ldots, \dot{c}_{k(n)}$ as ordinals; then, the value of the term is set to be α if α is the unique ordinal such that the formula becomes true in $L[x]$ when y is replaced by α. If no such α exists, or α is not unique, then the value of the term is set to be 0. Most commonly, we will evaluate terms by interpreting the constants $\dot{c}_{k(i)}$ as indiscernibles c^x_ι. We may indicate the intended interpretation by writing down the ordinals themselves instead of the constants, e.g.:

$$t^{L[x]}(c^x_1, c^x_\omega, \omega_4).$$

By a theorem of Solovay (see e.g., Kanamori [12]), if $x^\sharp$ exists, then every $\alpha < c^x_\iota$ is definable in $L[x]$ from indiscernibles smaller than c^x_ι. Let $x, y \in \mathbb{R}$. We say that $x^\sharp$ *embeds into* $y^\sharp$ if there is an assignment f of terms $f(t)$ to terms t such that

$$t^{L[x]}(c_0, \ldots, c_n) \leq s^{L[x]}(c_0, \ldots, c_m)$$

if and only if

$$f(t)^{L[y]}(c_0, \ldots, c_n) \leq f(s)^{L[y]}(c_0, \ldots, c_m)$$

for all terms t and s, whenever $c_0, c_1, \ldots, c_{\max\{m,n\}}$ are uniform indiscernibles (i.e., indiscernibles for all real numbers simultaneously). By indiscernibility, t and $f(t)$ will be terms of the same arity, save possibly due to the appearance of dummy variables. This notion was implicit in Magidor's proof for the Σ^1_3-correctness of the core model K; see also Theorem 2.1 of Hjorth [10].

As observed by Girard [7], $x^\sharp$ can be regarded as a countable dilator as follows:

Definition 7. Given $x \in \mathbb{R}$, we define a dilator F_x by

$$F_x(\alpha) = c^x_\alpha$$

and, for strictly increasing $f : \alpha \to \beta$,

$$\begin{gathered} F_x(f) : c^x_\alpha \to c^x_\beta \\ t^{L[x]}(c^x_{\alpha_1}, \ldots, c^x_{\alpha_n}) \mapsto t^{L[x]}(c^x_{f(\alpha_1)}, \ldots, c^x_{f(\alpha_n)}). \end{gathered}$$

Here, we remark that $F_x(f)$ is well defined (i.e., it does not depend on the choice of t) and strictly increasing; both these facts follow from indiscernibility. Since our goal is to study Σ^1_1-boundedness theorems, it is worth checking that the question of whether a set is Σ^1_1-definable from $x^\sharp$ does not depend on whether $x^\sharp$ is presented as a theory or as a dilator.

Lemma 8. $(F_x, x) \equiv_{hyp} x^\sharp$.

Proof. In order to compute F_x, we need to determine its action on finite ordinals and on morphisms between finite ordinals. These objects can be computed easily from the theory of $L[x]$ with indiscernibles, so we get the stronger result that $F_x \leq_T x^\sharp$. For the other direction, observe that the collection of F_x-recursive ordinals is closed under F_x, and in particular c^x_ω is an F_x-recursive ordinal. The sequence

$$s := \{c^x_n : n \in \mathbb{N}\}$$

is the image of $\mathbb{N}$ under F_x and hence

$$c^x_\omega < \omega_1^{F_x}$$

by the admissibility of $L_{\omega_1^{F_x}}[F_x]$. Since $x^\sharp$ is computable from the theory of $L_{c^x_\omega}[s, x]$, it is (F_x, x)-hyperarithmetical. □

Lemma 9. *For all $x \in \mathbb{R}$, F_x is a flower.*

Proof. This is immediate from Theorem 1, since the class of indiscernibles is closed. □

We have two notions of embeddability for sharps: the one defined above directly, and the one inherited from the category of dilators. However, these notions coincide:

Lemma 10. *The following are equivalent:*

(1) $x^\sharp$ *embeds into* $y^\sharp$*,*
(2) *there is a natural transformation* $T : F_x \to F_y$*.*

Proof. The proof consists mainly of untangling each definition and tangling it back into the other; we sketch it. If $x^\sharp$ embeds into $y^\sharp$, as witnessed by an assignment of terms f, one can define a natural transformation where the component $\eta_\alpha : c^x_\alpha \to c^y_\alpha$ is given by

$$\eta_\alpha(t^{L[x]}(c^x_{\alpha_1}, \ldots, c^x_{\alpha_n})) = f(t)^{L[y]}(c^y_{\alpha_1}, \ldots, c^y_{\alpha_n}).$$

Conversely, suppose there is a natural transformation T between F_x and F_y and denote by η_α its αth component. Given a term t of arity n, choose $f(t)$ so that

$$f(t)^{L[y]}(c^y_1, \ldots, c^y_n) = \eta_{n+1}\left(t^{L[x]}(c^x_1, \ldots, c^x_n)\right) \tag{3}$$

Since η_{n+1} is a function from c^x_{n+1} to c^y_{n+1}, such an $f(t)$ will exist. It follows from naturalness that (3) remains true after shifting the indiscernibles, i.e., that

$$f(t)^{L[y]}(c^y_{\alpha_1}, \ldots, c^y_{\alpha_n}) = \eta_\alpha\left(t^{L[x]}(c^x_{\alpha_1}, \ldots, c^x_{\alpha_n})\right).$$

holds whenever, $\alpha_1 < \cdots < \alpha_n < \alpha$ are ordinals. In particular, this holds for the uniform indiscernibles, as desired. □

4.2. **Boundedness theorems for sharps.** Below, recall that we identify countable dilators with the real numbers which code them.

Proposition 11. *Suppose F is a countable flower and $F \in L[x]$. Then, F naturally transforms into F_x.*

Proof. Since $F \subset \mathbb{N}$ is countable in $L[x]$, we have $F \in L_{c^x_0}[x]$ (this is because every subset of $\mathbb{N}$ in $L[x]$ belongs to $L_{\omega_1^{L[x]}}[x]$ and $\omega_1^{L[x]} < c^x_0$), so F is definable in $L[x]$ by a formula in the language of set theory with no parameters other than x. Choose a system S of ordinal denotations for F such that $S \in L[x]$. Such a system exists, since F is a dilator in $L[x]$. Since $S \in L[x]$, we can find a formula in the language of set theory φ such that for every configuration C in S and every increasing sequence of ordinals $\alpha_1, \ldots, \alpha_n$ of the arity of C, we have

$$L[x] \models \varphi(x, C, \alpha_1, \ldots, \alpha_n, \beta) \text{ if and only if } \beta = (C; \alpha_1, \ldots, \alpha_n).$$

Thus, we may define terms t_C uniformly in C such that

$$t_C^{L[x]}(\alpha_1, \ldots, \alpha_n) = (C; \alpha_1, \ldots, \alpha_n)$$

for all C and all $\alpha_1 < \cdots < \alpha_n$.

In order to define the natural transformation, to each $\alpha \in \mathsf{Ord}$ we associate the component

$$\eta_\alpha : F(\alpha) \to c^x_\alpha$$

given by

$$\eta_\alpha(C; \alpha_1, \ldots, \alpha_n) = t_C^{L[x]}(c^x_{\alpha_1}, \ldots, c^x_{\alpha_n})$$

for each configuration C in S of arity n and each sequence of ordnals $\alpha_1 < \ldots < \alpha_n < \alpha$.

We have to verify that this is indeed a natural transformation. First, we note that each component η_α is strictly increasing, since they were defined using the ordinal denotations $(C; \alpha_1, \ldots, \alpha_n)$. Let $\alpha < \beta$ be ordinals, $f : \alpha \to \beta$ be an increasing function, and $\gamma < F(\alpha)$. Choose a configuration C and parameters $\alpha_1, \ldots, \alpha_n$ such that

$$\gamma = (C; \alpha_1, \ldots, \alpha_n).$$

Then,

$$\begin{aligned}
\eta_\beta \circ F(f)(\gamma) &= \eta_\beta \circ F(f)(C; \alpha_1, \ldots, \alpha_n) \\
&= \eta_\beta(C; f(\alpha_1), \ldots, f(\alpha_n)) \\
&= t_C^{L[x]}(c^x_{f(\alpha_1)}, \ldots, c^x_{f(\alpha_n)}) \\
&= F_x(f)\Big(t_C^{L[x]}(c^x_{\alpha_1}, \ldots, c^x_{\alpha_n})\Big) \\
&= F_x(f) \circ \eta_\alpha(C; \alpha_1, \ldots, \alpha_n) \\
&= F_x(f) \circ \eta_\alpha(\gamma),
\end{aligned}$$

as desired. □

Remark 12. The assumption that F is a flower cannot be removed from Proposition 11. For instance, the natural dilator given by $x \mapsto x + 1$ cannot be embedded into any flower and in particular it cannot be embedded into any sharp.

The following result is the analogue of the Σ^1_1-boundedness theorem for sharps.

Theorem 13. *Suppose that $A \subset \mathcal{S}$ is $\Sigma^1_1(x)$. Then, $a^\sharp$ embeds into $x^\sharp$ for every $a^\sharp \in A$.*

Proof. Let x and A be as in the statement of the theorem. By Lemma 8,

$$F_A := \{F_a | a^\sharp \in A\}$$

is a $\Sigma^1_1(x)$ set of flowers. Theorem 4 then yields a flower F recursive in x such that every F_a with $a^\sharp \in A$ naturally transforms into F. By Proposition 11, F naturally transforms into F_x, so the result follows from Lemma 10. □

W. H. Woodin had previously shown (unpublished) that Theorem 13 follows from the Axiom of Determinacy.

Remark 14. The variants of the boundedness theorems in which $\boldsymbol{\Sigma}^1_1$ is replaced by its lightface form Σ^1_1 are trivial: every Σ^1_1 subset of $\mathcal{S}$ is empty, by absoluteness. However, once one allows for parameters in definitions, one can define nonempty $\boldsymbol{\Sigma}^1_1$ sets of sharps. For instance, $\{x^\sharp\}$ is definable from $x^\sharp$. In a way, the content of Theorem 13 is that $\boldsymbol{\Sigma}^1_1$ sets of sharps are not much more complicated than this.

Theorem 15. *Suppose that $A \subset \mathcal{S}$ is $\boldsymbol{\Sigma}^1_1$ and nonempty. Then, there is $y \in \mathbb{R}$ such that every y-indiscernible is simultaneously an a-indiscernible for all $a^\sharp \in A$.*

Proof. Suppose that $A \subset \mathcal{S}$ is $\boldsymbol{\Sigma}^1_1$ and nonempty. By Theorem 13, there is $x \in \mathbb{R}$ such that $a^\sharp$ embeds into $x^\sharp$ for every $a^\sharp \in A$. Let

$$C = \{\alpha | \alpha = c^x_\alpha\}.$$

Since I_x is closed and cofinal below ω_1, so too is C. Moreover, each element of C is simultaneously an a-indiscernible for every $a^\sharp \in A$. To see this, suppose otherwise, and let $\alpha \in C$ and $a^\sharp \in A$ be such that α is not an a-indiscernible. Hence, there are ordinals $\alpha_1 < \alpha_2 < \ldots < \alpha_n < \alpha$ such that each α_i is an a-indiscernible and α is definable in $L[a]$ from $\alpha_1, \ldots, \alpha_n$. It follows that the set of ordinals definable from the first $(\alpha_n + 1)$-many indiscernibles in $L[a]$ has order-type $\geq\alpha$. Since $a^\sharp$ embeds into $x^\sharp$, the set of ordinals definable from the first $(\alpha_n + 1)$-many indiscernibles in $L[x]$ has order-type $\geq\alpha$. Thus, α is definable in $L[x]$ from finitely many indiscernibles below $c^x_{\alpha_n+1}$. However, since $\alpha \in C$ and $\alpha_n < \alpha$, $c^x_{\alpha_n+1} < \alpha$, so α is definable in $L[x]$ from finitely many ordinals below α, contradicting the fact that α is an x-indiscernible.

Hence, whenever $a^\sharp \in A$, we have $C \subset I_a$. Let $y = x^\sharp$. It follows from Lemma 8 that if $L_\gamma[y]$ is admissible, then it is closed under the function

$$\alpha \mapsto c^x_\alpha.$$

If γ is a y-indiscernible, then $L_\gamma[y]$ is clearly admissible. Thus, we have

$$I_y \subset C \subset I_a,$$

so y is as desired. □

Remark 16. It is shown in [2] that, assuming the existence of large cardinals, there are projective sets of sharps A such that

$$\bigcap_{x^\sharp \in A} I_x$$

contains a finite, nonzero amount of countable ordinals. By Theorem 15, "projective" cannot be weakened to "analytic." Conversely, in the statement of Theorem 15 one cannot replace "analytic" by "projective." We will see below that Theorem 15 fails already at the level of $\mathbf{\Pi}^1_1$.

The following lemmata concern the relation between $x^\sharp$ and sharps of generic extensions of $L[x]$ by collapse algebras of the form $\mathrm{Coll}(\omega, \kappa)$. Here, we identify a generic $g \subset \mathrm{Coll}(\omega, \kappa)$ with $\bigcup g : \omega \to \kappa$ and in turn $\bigcup g$ with the relation on $\mathbb{N}$ given by

$$n \preceq m \leftrightarrow \left(\bigcup g\right)(n) \leq \left(\bigcup g\right)(m). \tag{4}$$

Thus, we think of g as a real number.

Lemma 17. *Let $x \in \mathbb{R}$ and let ξ be an ordinal, or $\xi = -1$, in which case we write $c^x_\xi := 0$. Suppose that $\alpha \in [c^x_\xi, c^x_{\xi+1})$ and $g \subset \mathrm{Coll}(\omega, \alpha)$ is $L[x]$-generic. Then, $c^{(x,g)}_\iota = c^x_{\xi+1+\iota}$ for all ι.*

Proof. Suppose towards a contradiction that $c^{(x,g)}_\iota \neq c^x_{\xi+1+\iota}$ for some least ι. Since $L[x] \subset L[x, g]$, the only possibility is that $c^x_{\xi+1+\iota}$ is not an (x, g)-indiscernible. Hence, it is definable by some term t in $L[x, g]$ from x, g and indiscernibles $c^x_{\iota_1}, \ldots, c^x_{\iota_n}$. Let $p \in \mathrm{Coll}(\omega, \kappa)$ be such that

$$L[x] \models \text{“}p \Vdash c^x_{\xi+1+\iota} = t^{L[x,g]}(c^x_{\iota_1}, \ldots, c^x_{\iota_n}).\text{”}$$

Since the forcing relation is definable from the partial order $\mathrm{Coll}(\omega, \alpha)$, it follows that $c^x_{\xi+1+\iota}$ is definable in $L[x]$ from x together with p, the indiscernibles $c^x_{\iota_1}, \ldots, c^x_{\iota_n}$, and α. Since $\alpha < c^x_{\xi+1}$ and $p \in \mathrm{Coll}(\omega, \alpha)$, $p \in L_{c^x_{\xi+1}}[x]$ and both p and α are definable in $L[x]$ from x and some sequence of indiscernibles $c^x_{\xi_1}, \ldots, c^x_{\xi_m}$, with $\xi_m < \xi$. It follows that $c^x_{\xi+1+\iota}$ is definable in $L[x]$ from x and $c^x_{\iota_1}, \ldots, c^x_{\iota_n}$, which is impossible. □

Lemma 18. *Let $x \in \mathbb{R}$ and let ξ be an ordinal. Suppose that $g \subset \mathrm{Coll}(\omega, \gamma)$ is $L[x]$-generic, for $\gamma \in [c^x_\xi, c^x_{\xi+1})$. Then there is a natural isomorphism between $F_{(x,g)}$ and $F_x^{+(\xi+1)}$*

Proof. We first observe that

$$F_{(x,g)}(\iota) = c_\iota^{(x,g)} = c^x_{\xi+1+\iota} = F_x^{+\xi+1}(\iota)$$

holds for any ordinal ι by Lemma 17. To define the natural isomorphism, we declare that each component

$$T_\iota : c_\iota^{(x,g)} \to c^x_{\xi+1+\iota}$$

is the identity. It remains to show that these identity maps are natural with respect to the functors $F_{(x,g)}$ and $F_x^{+\xi+1}$. This means that we must establish

$$F_{(x,g)}(f)(\alpha) = F_x^{+\xi+1}(f)(\alpha)$$

for any morphism $f : \iota \to \iota'$ and any $\alpha < c_\iota^{(x,g)} = c^x_{\xi+1+\iota}$. Pick a representation

$$\alpha = t^{L[x]}(c^x_{\xi_1}, \dots, c^x_{\xi_m}, c^x_{\xi+1+\alpha_1}, \dots, c^x_{\xi+1+\alpha_n})$$

with $\xi_1 < \ldots < \xi_m \leq \xi$ and $\alpha_1 < \ldots < \alpha_n < \iota$. Since $L[x]$ can be defined in $L[x,g]$, we can find a term $\hat{t}$ with

$$\hat{t}^{L[x,g]}(c^x_{\xi_1}, \dots, c^x_{\xi_m}, c^{(x,g)}_{\gamma_1}, \dots, c^{(x,g)}_{\gamma_n}) = t^{L[x]}(c^x_{\xi_1}, \dots, c^x_{\xi_m}, c^x_{\xi+1+\gamma_1}, \dots, c^x_{\xi+1+\gamma_n})$$

for all ordinals $\gamma_1 < \ldots < \gamma_n$. Now $\xi_i \leq \xi$ entails that $c^x_{\xi_i}$ is countable in $L[x,g]$, hence smaller than $c_0^{(x,g)}$ and thus definable in $L[x,g]$ without indiscernibles. By incorporating the definitions of the $c^x_{\xi_i}$, we can transform $\hat{t}$ into a term $\bar{t}$ with

$$\bar{t}^{L[x,g]}(c^{(x,g)}_{\gamma_1}, \dots, c^{(x,g)}_{\gamma_n}) = \hat{t}^{L[x,g]}(c^x_{\xi_1}, \dots, c^x_{\xi_m}, c^{(x,g)}_{\gamma_1}, \dots, c^{(x,g)}_{\gamma_n})$$

for all $\gamma_1 < \ldots < \gamma_n$. Note that $\bar{t}$ depends not only on t but also on the ordinals ξ_i. We now get

$$\begin{aligned}
F_{(x,g)}(f)(\alpha) &= F_{(x,g)}(f)(\bar{t}^{L[x,g]}(c^{(x,g)}_{\alpha_1}, \dots, c^{(x,g)}_{\alpha_n})) = \\
&= \bar{t}^{L[x,g]}(c^{(x,g)}_{f(\alpha_1)}, \dots, c^{(x,g)}_{f(\alpha_n)}) \\
&= \hat{t}^{L[x,g]}(c^x_{\xi_1}, \dots, c^x_{\xi_m}, c^{(x,g)}_{f(\alpha_1)}, \dots, c^{(x,g)}_{f(\alpha_n)}) \\
&= t^{L[x]}(c^x_{\xi_1}, \dots, c^x_{\xi_m}, c^x_{\xi+1+f(\alpha_1)}, \dots, c^x_{\xi+1+f(\alpha_n)}) \\
&= F_x^{+\xi+1}(f)(\alpha),
\end{aligned}$$

as required. □

Theorem 19. *Suppose that $A \subset \mathcal{S}$ is $\Sigma^1_2(x)$. Then, every F_a with $a^\sharp \in A$ naturally transforms into $F_x^{+\alpha}$ for some ordinal α.*

Proof. Let

$$F_A := \{F_a | a^\sharp \in A\}.$$

Then F_A is a $\Sigma^1_2(x)$ set of flowers, so Theorem 6 yields a bilator F recursive in x such that every $f \in F_A$ naturally transforms into $F(\alpha, \cdot)$ for some countable ordinal α. Fix such an f and α. Let $g \subset \mathrm{Coll}(\omega, \alpha)$ be $L[x]$-generic. Such a g exists because $\alpha < \omega_1$ and, since $x^\sharp$ exists, it follows that ω_1 is inaccessible in $L[x]$, so there are only countably many dense subsets of the partial order $\mathrm{Coll}(\omega, \alpha)$ in $L[x]$. Then, $F(\alpha, \cdot)$ is a countable flower in $L[x, g]$, so there is a natural transformation from

$F(\alpha,\cdot)$ (and thus from f) into $F_{(x,g)}$, by Proposition 11. The result now follows from the previous two lemmata. $\square$

The statement of Theorem 15 is false if one replaces $\Sigma^1_1(x)$ by $\Sigma^1_2(x)$ (this follows from the argument of Theorem 23 below); however, we do have the following analogue:

Theorem 20. *Suppose that $A \subset \mathcal{S}$ is $\boldsymbol{\Sigma}^1_2$. Then, there is a $y \in \mathbb{R}$ such that for every $a^\sharp \in A$, I_a contains $I_y \setminus \alpha$ for some α.*

Proof. Find $x \in \mathbb{R}$ such that A is $\Sigma^1_2(x)$. By Theorem 19, every F_a with $a^\sharp \in A$ naturally transforms into $F_x^{+\alpha}$ for some ordinal α. Arguing as in the proof of Theorem 15, we see that for every F_a with $a^\sharp \in A$,

$$\{\gamma < \omega_1 | \gamma = c^x_\gamma\} \setminus \alpha + 1 \subset I_a.$$

Let $y = x^\sharp$. Then, every y-indiscernible ξ is closed under the function $\beta \mapsto c^x_\beta$ (e.g., because $L_\xi[y]$ is admissible), so we have

$$I_y \setminus \alpha + 1 \subset \{\gamma < \omega_1 | \gamma = c^x_\gamma\} \setminus \alpha + 1 \subset I_a,$$

as desired. $\square$

5. Concluding remarks

The existence of a natural transformation from a functor f to a functor g can be regarded as an assertion that f is "simpler" than g. In the particular case that f and g are countable and presented as a real number, there are other notions of relative complexity to consider, such as Turing reducibility. In general, the two notions are not comparable. Clearly, $f \leq_T g$ does not imply that there is a natural transformation from f to g. The converse can be negated in a very strong way:

Proposition 21. *There exist pairs of reals (x, y) such that*

(1) *$y \in L[x]$;*
(2) *$x \notin L[y]$;*
(3) *$x^\sharp$ embeds into $y^\sharp$.*

Proof. Letting $g \subset \mathrm{Coll}(\omega, \omega)$ be $L[x]$-generic, the three properties hold of x and (x, g) by Lemma 18. $\square$

5.1. Π^1_1 sets.

We finish this article by clarifying whether the Σ^1_1-boundedness theorems extend to Π^1_1 sets. In the case of flowers and dilators, the answer is easily seen to be negative.

Proposition 22. *There is a Π^1_1 set of dilators (in fact, flowers) which cannot be bounded by any recursive dilator.*

Proof. Let A be the set of all constant dilators with recursive value, i.e., the set of all dilators with constant value α for some $\alpha < \omega_1^{ck}$. This can be coded by a Π^1_1 set e.g., by choosing representations of these ordinals in terms of Kleene's $\mathcal{O}$. Any recursive dilator d will satisfy $d(1) < \omega_1^{ck}$ and thus cannot bound this set. $\square$

A similar result holds for $\boldsymbol{\Pi}^1_1$ sets of sharps. It is worth noting that every Π^1_1 set of sharps can be bounded by a sharp, simply because any such set is empty (see Remark 14). Hence, the following result can be regarded as a curious example of a fact about Π^1_1 sets which does not relativize.

Theorem 23. *There is a $\Pi^1_1(0^\sharp)$ set of sharps of real numbers which cannot be bounded by a sharp of a real number.*

Proof. Let $\hat{A} = \{F_0^{+\alpha} | \alpha < \omega_1\}$, where F_0 is the dilator given by $0^\sharp$ and $F_0^{+\alpha}$ is defined as in Lemma 18. Clearly $\hat{A}$ cannot be bounded by a sharp of a real number. The idea is to express $\hat{A}$ as a set of sharps of real numbers. Recall our convention whereby we identify generic subsets of collapse partial orders $\mathrm{Coll}(\omega, \alpha)$ with the real number which codes the relation given by equation (4) on p. 11. Let B be the set of all triples (x, y, z) such that

(1) x and y are codes for countable ordinals $|x|$ and $|y|$;
(2) $z \subset \mathrm{Coll}(\omega, |x|)$ is $L_{|y|}$-generic;
(3) $|x|^{+L} < |y|$;
(4) x is recursive in z and y is recursive in z and $0^\sharp$.

The fourth condition essentially imposes the condition that x and y do not code very complicated information aside from their order-type. The third condition states that $|y|$ is greater than the cardinal successor of $|x|$ in L. This implies that

$$\mathcal{P}(\mathrm{Coll}(\omega, |x|)) \subset L_{|y|},$$

so every dense subset of $\mathrm{Coll}(\omega, |x|)$ belongs to $L_{|y|}$. By the second condition, z is L-generic.

Claim 24. *Suppose α is a countable ordinal. Then, there is a wellordering x of $\mathbb{N}$ of length α such that $(x, y, z) \in B$ for some y and z.*

Proof. Let $z \subset \mathrm{Coll}(\omega, \alpha)$ be L-generic. Then, $\alpha < \omega_1^z$, so there is some z-recursive wellordering x of $\mathbb{N}$ of length α. Let $x + 1$ be the extension of x by a new element and let $f : x + 1 \to \alpha + 1$ be the isomorphism. Note that α and α^{+L} are definable in L from indiscernibles strictly below $c^0_{\alpha+1}$. In view of this, we can pick a term t and elements $x_1, \ldots, x_n \in x + 1$ with

$$t^L(c^0_{f(x_1)}, \ldots, c^0_{f(x_n)}) = \alpha^{+L}.$$

We now define $y \subset \mathbb{N}$ as the set of all (codes of) expressions of the form $s^L(c^0_{y_1}, \ldots, c^0_{y_m})$, where s is a term and $y_1, \ldots, y_m$ are such that

(1) $s^L(c^0_{f(y_1)}, \ldots, c^0_{f(y_m)}) < t^L(c^0_{f(x_1)}, \ldots, c^0_{f(x_n)})$,
(2) no expression with code smaller than $s^L(c^0_{y_1}, \ldots, c^0_{y_m})$ yields the same value (under the function g defined below).

Note that y is recursive in x and $0^\sharp$: after comparing the y_i and x_j in $x+1$, we can use $0^\sharp$ to decide (1). Thus, we obtain a bijection $g : y \to \alpha^{+L}$ by setting

$$g(s^L(c^0_{y_1}, \ldots, c^0_{y_m})) = s^L(c^0_{f(y_1)}, \ldots, c^0_{f(y_m)}).$$

The induced order on y has order type α^{+L} and is also recursive in x and $0^\sharp$ (by the same algorithm that decides (1)). It follows that $(x, y + 1, z) \in B$. □

Suppose $(x, y, z) \in B$. By Lemma 18, F_z is naturally isomorphic to $F_0^{+\alpha}$ for some α. (Here, α depends only on how many indiscernibles were collapsed by z.) We consider the set

$$A := \{z^\sharp | \exists x \, \exists y \, (x, y, z) \in B\}.$$

Since one can find elements of A with z collapsing arbitrarily many countable indiscernibles of L, this set cannot be bounded by a sharp of a real number. It remains to prove that

$$A \in \Pi^1_1(0^\sharp).$$

The condition $\exists x\, \exists y\, (x, y, z) \in B$ is equivalent to

$$\exists x \leq_T z\, \exists y \leq_T (z, 0^\sharp)\, (x, y, z) \in B$$

and thus $\Pi^1_1(z, 0^\sharp)$ so it suffices to show that $z^\sharp$ can be uniformly defined in a Π^1_1 way with $0^\sharp$ and z as parameters under the assumption that $(x, y, z) \in B$. For such a z, the z-indiscernibles are all sufficiently large 0-indiscernibles, by Lemma 18, so, using the Forcing Theorem, we can find ordinals α and β such that

$$\begin{aligned} L[z] \models \phi(c^z_{n_0}, \ldots, c^z_{n_k}) &\leftrightarrow L[z] \models \phi(c^0_{\beta+n_0}, \ldots, c^0_{\beta+n_k}) \\ &\leftrightarrow \exists p \in z \left(p \Vdash^L_{\mathrm{Coll}(\omega,\alpha)} \phi(c^0_{\beta+n_0}, \ldots, c^0_{\beta+n_k})\right) \\ &\leftrightarrow \exists p \in z \left(p \Vdash^L_{\mathrm{Coll}(\omega,\alpha)} \phi(c^0_{n_0}, \ldots, c^0_{n_k})\right) \end{aligned}$$

The truth-value of the formula on the right-hand side can be extracted from $0^\sharp$, z, and α; and α can be computed from z. Hence, the truth-value of the formula can be computed (uniformly) from z and $0^\sharp$. □

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