01 Jun 2020

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Recommended Citation

U. D. Jentschura, "Fifth Force and Hyperfine Splitting in Bound Systems," *Physical Review A*, vol. 101, no. 6, American Physical Society, Jun 2020.
The definitive version is available at https://doi.org/10.1103/PhysRevA.101.062503

Fifth force and hyperfine splitting in bound systems

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(Received 10 March 2020; accepted 5 May 2020; published 1 June 2020)

Two recent experimental observations at the ATOMKI Institute of the Hungarian Academy of Sciences (regarding the angular emission pattern of electron-positron pairs from nuclear transitions from excited states in ^{8}Be and ^{4}He) indicate the possible existence of a particle of a rest mass energy of roughly 17 MeV. The so-called X17 particle constitutes a virtual state in the process, preceding the emission of the electron-positron pair. Based on the symmetry of the nuclear transitions ($1^+ \to 0^+$ and $0^- \to 0^+$), the X17 could either be a vector, or a pseudoscalar particle. Here, we calculate the effective potentials generated by the X17, for hyperfine interactions in simple atomic systems, for both the pseudoscalar as well as the vector X17 hypotheses. The effective Hamiltonians are obtained in a general form which is applicable to both electronic as well as muonic bound systems. The effect of virtual annihilation and its contribution to the hyperfine splitting also is considered. Because of the short range of the X17-generated potentials, the most promising pathway for the observation of the X17-mediated effects in bound systems concerns hyperfine interactions, which, for S states, are given by modifications of short-range (Dirac-δ) potentials in coordinate space. For the pseudoscalar hypothesis, the exchange of one virtual X17 quantum between the bound lepton and the nucleus leads to hyperfine effects, but does not affect the Lamb shift. Effects due to the X17 are shown to be drastically enhanced for muonic bound systems. Prospects for the detection of hyperfine effects mediated by X17 exchange are analyzed for muonic deuterium, muonic hydrogen, muonium, true muonium ($\mu^+\mu^-$ bound system), and positronium.

DOI: 10.1103/PhysRevA.101.062503

I. INTRODUCTION

For decades, atomic physicists have tried to push the accuracy of experiments and theoretical predictions of transitions in simple atomic systems higher [1]. The accurate measurements have led to stringent limits on the time variation of fundamental constants [2–4], and enabled us to determine a number of important fundamental physical constants [5] with unprecedented accuracy. Yet, a third motivation (see, e.g., Refs. [6,7]), hitherto not crowned with success, has been the quest to find signs of a possible low-energy extension of the Standard Model, based on a deviation of experimental results and theoretical predictions.

Recently, the possible existence of a fifth-force particle, commonly referred to as the "X17" particle because of the observed rest mass of 16.7 MeV, has been investigated in Refs. [8–10], based on a peak in the emission spectrum of electron-positron pairs in nuclear transitions of excited helium and beryllium nuclei. Two conceivable theoretical explanations have been put forward, both being based on low-energy additions to the Standard Model. The first of these involves a vector particle (a "massive, dark photon"; see Refs. [11,12]), and the second offers a pseudoscalar particle (see Ref. [13]), which couples to light fermions as well as hadrons.

The findings of Refs. [8–10] have not yet been confirmed by any other experiment and remain to be independently verified (for an overview of other experimental searches and conceivable alternative interpretations of the ATOMKI results, see Refs. [14,15]). However, we believe that, with the advent of consistent observations in two nuclear transitions in ^{8}Be and ^{4}He, it is justified to carry out a calculation of the effects induced by the X17 boson in atomic systems. In more general terms, we ask the question of which effects could be expected from a potential "light pseudoscalar Higgs"-type particle in atomic spectra, as envisioned in Ref. [13].

Somewhat unfortunately, the rest mass of 16.7 MeV makes the X17 particle hard to detect in atomic physics experiments. The observed X17 rest mass energy is larger than the binding energy scale for both electronic as well as muonic bound systems [16]. Even more importantly, the Compton wavelength of the X17 particle (about 11.8 fm) is smaller than the effective Bohr radius for both electronic as well as muonic bound systems. Because the Compton wavelength of the X17 particle determines the range of the Yukawa potential, the effects of the X17 are hard to distinguish from nuclear-size effects in atomic spectroscopy experiments [16].

We recall that the Bohr radius amounts to $a_0 \sim 5 \times 10^4$ fm, while the effective Bohr radius of a muonic hydrogen atom is $a_0 \sim \hbar/(\alpha m_\mu c) \sim 256$ fm. It is thus hard to find an atomic system, even a muonic one, where one could hope to distinguish the effect of the X17 particle on the Lamb shift from the nuclear-finite-size correction to the energy. A possible circumvention has been discussed in Ref. [16], based on a muonic carbon ion, where the effective Bohr radius approaches the range of the Yukawa potential induced by the X17, in view of the larger nuclear charge number. However, it was concluded in Ref. [16] that considerable additional effort would be required in terms of an accurate understanding of nuclear-size effects, before the X17 signal could be extracted reliably.

The definition of the Lamb shift $\mathcal{L}$, as envisaged in Ref. [17] and used in many other places, e.g., in Eq. (67) of Ref. [18], explicitly excludes hyperfine effects. Conversely, hyperfine effects, at least for S states, are induced, in leading order, by the Dirac-δ peak of the magnetic dipole field of the atomic nucleus at the origin (see Eq. (9) of Ref. [19]). The Fermi contact interaction, which gives rise to the leading-order contribution to the hyperfine splitting for S states, is proportional to a Dirac δ in coordinate space, commensurate with the fact that the atomic nucleus has a radius not exceeding the femtometer scale. The effect of short-range potentials is thus less suppressed when we consider the hyperfine splitting, as compared to the Lamb shift. The *aficionados* of bound states thus realize that, if we consider hyperfine effects, we have a much better chance of extracting the effect induced by the X17, which, on the ranking scale of the contributions, occupies a much higher place than for the Lamb shift alone. Despite the large mass m_X of the X17 particle, which leads to a short-range potential proportional to $\exp(-m_X\, r)$ (in natural units with $\hbar = c = \epsilon_0 = 1$, which are used throughout the current paper), the effect of the X17 could thus be visible in the hyperfine splitting in muonic atoms.

Here, we elaborate on this idea, and derive the leading corrections to the hyperfine splitting of nS, $nP_{1/2}$, and $nP_{3/2}$ states in ordinary as well as muonic hydrogenlike systems, due to the X17 particle, by matching the nuclear-spin-dependent terms in the scattering amplitude with the effective Hamiltonian. Anticipating some results, we can say that the relative correction (expressed in terms of the leading Fermi term) is proportional to m_r/m_X, where m_r is the reduced mass of the two-body bound system, while m_X is the X17 boson mass. The effect is thus enhanced for muonic in comparison to electronic bound systems.

This paper is organized as follows. In Sec. II, we summarize the interaction Lagrangians for both a hypothetical X17 vector exchange [11,12], as well as a pseudoscalar exchange [13], with corresponding conventions for the coupling parameters. In Sec. III, we derive the effective hyperfine Hamiltonians for both vector and pseudoscalar exchanges. In Sec. IV, we evaluate general expressions for the corrections to hyperfine energies induced by the X17 particle, for S and P states. In Sec. V, we derive bounds on the coupling parameters for both models in the muon sector, based on the muon g factor. Finally, in Sec. VI, we apply the obtained results to muonic hydrogen, muonic deuterium, muonium, true muonium (bound $\mu^+\mu^-$ system), and positronium. We also discuss the measurability of the X17 effects in the hyperfine structure of the mentioned atomic systems. Conclusions are reserved for Sec. VII.

II. INTERACTION LAGRANGIANS

In the following, we intend to study both the interaction of X17 vector and pseudoscalar particles with bound leptons (electrons and muons) and nucleons (protons and deuterons). Vector interactions are denoted by the subscript V, while pseudoscalar interactions carry the subscript A, as is customary in the particle physics literature. We write the interaction Lagrangian $\mathcal{L}_{X,V}$ for the interaction of an X17 vector boson with the fermion fields $f = e, \mu$ (electron and muon) and the nucleons $N = p, n$ (proton and neutron) as follows:

$$\mathcal{L}_{X,V} = -\sum_f \varepsilon_f e\, \bar\psi_f\, \gamma^\mu\, \psi_f X_\mu - \sum_N \varepsilon_N e\, \bar\psi_N\, \gamma^\mu\, \psi_N X_\mu, \tag{1a}$$

where we follow the conventions delineated in the remarks following Eq. (1) of Ref. [11] and Eq. (10) of Ref. [12]. Here, ε_f and ε_N are the flavor-dependent coupling parameters for the fermions and nucleons, while $e = -\sqrt{4\pi\alpha} = -0.091$ is the electron charge. The fermion and nucleon field operators (the latter, interpreted as field operators for the composite particles) and denoted as ψ_f and ψ_N, while the X_μ is the X17 field operator. For reasons which will become obvious later, we use, in Eq. (1), the alternative conventions,

$$h'_f = \varepsilon_f\, e, \quad h'_N = \varepsilon_N\, e, \tag{1b}$$

for the coupling parameters to the hypothetical X17 vector boson. Our conventions imply that for $\varepsilon_N > 0$, the coupling parameter h'_N parametrizes a "negatively charged" nucleon under the additional $U(1)$ gauge group of the vector X particle.

According to a remark following the text after Eq. (9) of Ref. [12], conservation of X charge implies that the couplings to the proton and neutron currents fulfill the relationships

$$\varepsilon_p = 2\varepsilon_u + \varepsilon_d, \quad \varepsilon_n = \varepsilon_u + 2\varepsilon_d, \tag{2}$$

where the up and down quark couplings are denoted by the subscripts u and d. Numerically, one finds (see the detailed discussion around Eqs. (38) and (39) of Ref. [12]) that the electron-positron field coupling ε_e needs to fulfill the relationship

$$2 \times 10^{-4} < \varepsilon_e < 1.4 \times 10^{-3}. \tag{3}$$

Furthermore, in order to explain the experimental observations [8,9], one needs the neutron coupling to fulfill (see Eq. (10) of Ref. [11])

$$|\varepsilon_n| = |\varepsilon_u + 2\varepsilon_d| \approx \left|\tfrac{3}{2}\, \varepsilon_d\right| \approx \tfrac{1}{100}. \tag{4}$$

Because the hypothetical X vector particle acts like a "dark photon" which is hardly distinguishable from the ordinary photon in the high-energy domain, the proton coupling ε_p is highly constrained. According to Eqs. (8) and (9) of Ref. [12], and Eq. (35) of Ref. [12], one needs to have

$$|\varepsilon_p| = |2\varepsilon_u + \varepsilon_d| \lesssim 8 \times 10^{-4}. \tag{5}$$

This is why the conjectured X17 vector boson is referred to as "protophobic" in Refs. [11,12].

Following Ref. [13], we write the interaction Lagrangian for the fermions interacting with the pseudoscalar candidate of the X17 particle as follows:

$$\mathcal{L}_{X,A} = -\sum_f h_f\, \bar\psi_f\, \mathrm{i}\,\gamma^5\, \psi_f\, A - \sum_N h_N\, \bar\psi_N\, \mathrm{i}\,\gamma^5\, \psi_N\, A, \tag{6}$$

where A is the field operator of the pseudoscalar field. Inspired by an analogy with putative pseudoscalar Higgs couplings [20], the pseudoscalar couplings have been estimated in Refs. [13,20] to be of the functional form

$$h_f = \xi_f\, \frac{m_f}{v}, \quad h_N = \xi_N\, \frac{m_N}{v}, \tag{7}$$

where $v = 246$ GeV is the vacuum expectation value of the Higgs (or Englert-Brout-Higgs [21,22]) field, m_f is the fermion mass, and m_N is the nucleon's mass. Furthermore, the parameters ξ_f and ξ_N could in principle be assumed to be of order unity. Note that the spin parity of the Standard Model

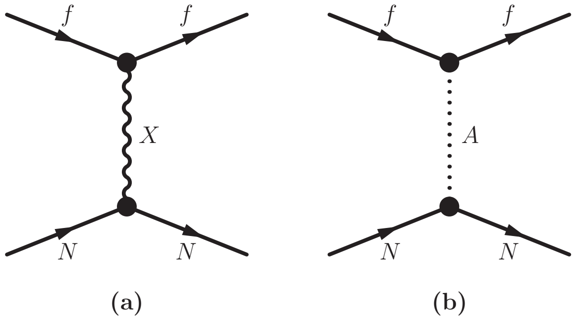

FIG. 1. The one-quantum exchange scattering amplitude for the X17 particle is matched against the effective Hamiltonian, for (a) the vector hypothesis and (b) the pseudoscalar hypothesis. The arrow of time is from left to right.

Higgs boson has recently been determined to be consistent with a scalar, not pseudoscalar, particle [23], but it is still intuitively suggested to parametrize the couplings to the novel putative pseudoscalar X17 in the same way as one would otherwise parametrize the couplings to the Higgs particle.

According to Eq. (2.7) and the remark following Eq. (3.12) of Ref. [13], the nucleon couplings can roughly be estimated as

$$\hbar_p = \frac{m_p}{v}(-0.40\,\xi_u - 1.71\,\xi_d) \approx -2.4\times 10^{-3}, \tag{8a}$$

$$\hbar_n = \frac{m_n}{v}(-0.40\,\xi_u + 0.85\,\xi_d) \approx 5.1\times 10^{-4}, \tag{8b}$$

where we have assumed $\xi_u \approx \xi_d \approx 0.3$. For the electron-positron field, based on other constraints detailed in Ref. [13], one has to require that (see Eq. (4.2) of Ref. [13])

$$\xi_e \overset{!}{>} 4, \quad \hbar_e \overset{!}{>} \frac{4\,m_e}{v} = 8.13\times 10^{-6}. \tag{9}$$

Based on a combination of experimental data [24] and theoretical considerations [25–27], one can also derive an upper bound,

$$\xi_e \overset{!}{<} 500, \quad \hbar_e \overset{!}{<} \frac{500\,m_e}{v} = 10^{-3}, \tag{10}$$

which is used in the following.

III. MATCHING OF THE SCATTERING AMPLITUDE

In order to match the scattering amplitude (see Fig. 1) with the effective Hamiltonian, we use the approach outlined in Chap. 83 of Ref. [28], but with a slightly altered normalization for the propagators, better adapted to a natural unit system ($\hbar = c = \epsilon_0 = 1$). Specifically, we use the bispinors in the representation (cf. Eq. (83.7) of Ref. [28])

$$u_{f,N} = \begin{pmatrix} \left(1 - \frac{\vec{p}_{f,N}^{\,2}}{8m^2}\right) w_{f,N} \\ \frac{\vec{\sigma}\cdot\vec{p}}{2m}\, w_{f,N} \end{pmatrix}, \tag{11}$$

where f, N stands for the bound fermion, or the nucleus, and $w_{f,N}$ are the nonrelativistic spinors. Of course, two two-component spinors constitute the four-component bispinor $u_{f,N}$ of the same field. The massive photon propagator (for the X17 vector hypothesis) is used in the following normalization (we may ignore the frequency of the photon in the order of approximation relevant for the current article):

$$D_{00}(\vec{q}) = -\frac{1}{\vec{q}^{\,2} + m_X^2}, \tag{12a}$$

$$D_{ij}(\vec{q}) = -\frac{1}{\vec{q}^{\,2} + m_X^2}\left[\delta^{ij} - \frac{q^i\,q^j}{\vec{q}^{\,2} + m_X^2}\right]. \tag{12b}$$

The derivation of the massive vector boson propagator in the Coulomb gauge, which is best adapted to bound-state calculations and involves a certain subtlety is discussed in the Appendix. The pseudoscalar propagator is used in the normalization

$$D_A(\vec{q}) = -\frac{1}{\vec{q}^{\,2} + m_X^2}, \tag{13}$$

where we also ignore the frequency. The scattering amplitude for the X17 vector particle reads as

$$\begin{aligned} M_{fi,V} = {} & \hbar_f\,\hbar_N\,\{(\bar{u}'_f\,\gamma^0\,\bar{u}_f)\,(\bar{u}'_N\,\gamma^0\,\bar{u}_N)\,D_{00} \\ & + (\bar{u}'_f\,\gamma^i\,\bar{u}_f)\,(\bar{u}'_N\,\gamma^j\,\bar{u}_N)\,D_{ij}\}, \end{aligned} \tag{14}$$

and

$$M_{fi,A} = \hbar_f\,\hbar_N\,(\bar{u}'_f\,\mathrm{i}\,\gamma^5\,\bar{u}_f)\,(\bar{u}'_N\,\mathrm{i}\,\gamma^5\,\bar{u}_N)\,D_A, \tag{15}$$

for the pseudoscalar case. Here, we denote the final states of the scattering process by a prime, $u'_f = u_f(\vec{p}'_f)$, $u'_N = u_N(\vec{p}'_N)$, while the initial states are $u_f = u_f(\vec{p}_f)$ and $u_N = u_N(\vec{p}_N)$, and the bar denotes the Dirac adjoint. Analogous definitions are used for the $w'_{f,N}$ and $w_{f,N}$ in Eq. (11). Furthermore, we have $\vec{p}_f + \vec{p}_N = \vec{p}\,'_f + \vec{p}\,'_N$. The momentum transfer is $\vec{q} = \vec{p}\,'_f - \vec{p}_f = \vec{p}_N - \vec{p}\,'_N$.

The form (11) is valid for the bispinors if the Dirac equation is solved in the Dirac representation of the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}. \tag{16}$$

The scattering amplitudes are matched against the effective Hamiltonian by the relation

$$M_{fi} = -(w_f'^{+}\,w_N'^{+})\,U(\vec{p}_f, \vec{p}_N, \vec{q})\,(w_f\,w_N), \tag{17}$$

where $U(\vec{p}_f, \vec{p}_N, \vec{q})$ is the effective Hamiltonian. The scattering amplitude M_{fi} is a matrix element involving four spinors, two of which represent the final and initial states of the two-particle system.

The scattering amplitude, evaluated between four spinors (cf. Eq. (83.8) of Ref. [28]), must now be matched against a Hamiltonian which acts on only one wave function in the end. We need to remember that the scattering amplitude corresponds to a matrix element of the Hamiltonian. Any matrix element of the Hamiltonian, even in the one-particle setting, is sandwiched between two wave functions, not one. Then, going into the center-of-mass frame $\vec{q} = \vec{p}\,'_f - \vec{p}_f = \vec{p}_N - \vec{p}\,'_N$ means that the wave function is written in terms of a center-of-mass coordinate $\vec{R}$, and a relative coordinate $\vec{r}$. In the center-of-mass frame, one eliminates the dependence on the center-of-mass coordinate $\vec{R}$ and the total momentum $\vec{P} = \vec{p}_f + \vec{p}_N$. Fourier transformation under the condition

$\vec{p}\,'_f + \vec{p}\,'_N = \vec{p}_f + \vec{p}_N$ leads to the effective Hamiltonian (cf. Eq. (83.15) of Ref. [28]).

For the record, we note that in the X17 vector case, the 00 component of the photon propagator gives the leading, spin-independent term in the effective Hamiltonian,

$$H_0 = \frac{h'_f\, h'_N}{4\pi r} \exp(-m_X\, r). \tag{18}$$

Under the replacements $h'_f \to e$ and $h'_N \to -e$, in the massless limit $m_X \to 0$, one recovers the Coulomb potential, $H_0 \to -\frac{e^2}{4\pi r} = -\frac{\alpha}{r}$. One finally extracts the terms responsible for the hyperfine structure, i.e., those involving the nuclear spin operator $\vec{\sigma}_N$, and obtains the following hyperfine Hamiltonian for a vector X17 particle:

$$\begin{aligned} H_{\mathrm{HFS},V} = \frac{h'_f\, h'_N}{16\,\pi\, m_f\, m_N} \Bigg[& -\frac{8\pi}{3} \delta^{(3)}(\vec{r})\, \vec{\sigma}_f \cdot \vec{\sigma}_N \\ & - \frac{m_X^2\, (\vec{\sigma}_f \cdot \vec{r}\, \vec{\sigma}_N \cdot \vec{r} - r^2\, \vec{\sigma}_f \cdot \vec{\sigma}_N)}{r^3}\, \mathrm{e}^{-m_X\, r} \\ & - (1 + m_X\, r) \frac{3\, \vec{\sigma}_f \cdot \vec{r}\, \vec{\sigma}_N \cdot \vec{r} - r^2\, \vec{\sigma}_f \cdot \vec{\sigma}_N}{r^5}\, \mathrm{e}^{-m_X\, r} \\ & - \left(2 + \frac{m_f}{m_N}\right) (1 + m_X\, r) \frac{\vec{\sigma}_N \cdot \vec{L}}{r^3} \mathrm{e}^{-m_X\, r} \Bigg]. \end{aligned} \tag{19}$$

Taking the limit $m_X \to 0$, and replacing

$$h'_f \to e, \quad h'_N \to \frac{\mathrm{g}_N\, (-e)}{2} = \frac{\mathrm{g}_N\, |e|}{2}, \quad e^2 = 4\pi\alpha, \tag{20}$$

one recovers the Fermi Hamiltonian H_F (see Eq. (10) of Ref. [19]),

$$\begin{aligned} H_F = \frac{\mathrm{g}_N\, \alpha}{m_f\, m_N} \Bigg[& \frac{\pi}{3}\, \vec{\sigma}_f \cdot \vec{\sigma}_N\, \delta^{(3)}(\vec{r}) \\ & + \frac{3\, \vec{\sigma}_f \cdot \vec{r}\, \vec{\sigma}_N \cdot \vec{r} - r^2\, \vec{\sigma}_f \cdot \vec{\sigma}_N}{8\, r^5} + \frac{\vec{\sigma}_N \cdot \vec{L}}{4\, r^3} \Bigg], \end{aligned} \tag{21}$$

where we have ignored the reduced-mass correction proportional to m_f/m_N in the $\vec{\sigma}_N \cdot \vec{L}$ term in Eq. (19). For a pseudoscalar exchange, one has

$$\begin{aligned} H_{\mathrm{HFS},A} = \frac{h_f\, h_N}{16\,\pi\, m_f\, m_N} \Bigg[& \frac{4\pi}{3} \delta^{(3)}(\vec{r})\, \vec{\sigma}_f \cdot \vec{\sigma}_N \\ & - \frac{m_X^2\, \vec{\sigma}_f \cdot \vec{r}\, \vec{\sigma}_N \cdot \vec{r}}{r^3}\, \mathrm{e}^{-m_X\, r} + (1 + m_X\, r) \\ & \times \frac{3\, \vec{\sigma}_f \cdot \vec{r}\, \vec{\sigma}_N \cdot \vec{r} - \vec{\sigma}_f \cdot \vec{\sigma}_N\, r^2}{r^5}\, \mathrm{e}^{-m_X\, r} \Bigg]. \end{aligned} \tag{22}$$

Note that the Hamiltonian given in Eq. (22) constitutes the complete Hamiltonian derived from pseudoscalar exchange, which, in view of the γ^5 matrix in the Lagrangian given in Eq. (6), contributes only to the hyperfine splitting, but not to the Lamb shift, in leading order [i.e., via the exchange of one virtual particle, as given in Fig. 1(a)]. For a deuteron nucleus, the spin matrix $\vec{\sigma}_N$ has to be replaced by $2\,\vec{I}_N$, where $\vec{I}_N$ is the spin operator of the deuteron, corresponding to the spin-1 particle. Important bounds on the coupling parameters h'_μ and h_μ can be derived from the muon anomalous magnetic moment (see Fig. 2).

IV. HYPERFINE STRUCTURE CORRECTIONS

A. X17 boson exchange

In order to analyze the S-state hyperfine splitting, we extract from Eqs. (19) and (22) the terms which are nonzero when evaluated on a spherically symmetric wave function. This entails the replacements

$$\vec{\sigma}_f \cdot \vec{r}\, \vec{\sigma}_N \cdot \vec{r} \to \frac{1}{3}\, r^2\, \vec{\sigma}_f \cdot \vec{\sigma}_N, \quad \vec{\sigma}_N \cdot \vec{L} \to 0, \tag{23a}$$

$$H_{\mathrm{HFS},V} \to -\frac{h'_f\, h'_N\, \vec{\sigma}_f \cdot \vec{\sigma}_N}{24\,\pi\, m_f\, m_N} \left[4\pi\, \delta^{(3)}(\vec{r}) - \frac{m_X^2}{r}\, \mathrm{e}^{-m_X\, r} \right], \tag{23b}$$

$$H_{\mathrm{HFS},A} \to \frac{h_f\, h_N\, \vec{\sigma}_f \cdot \vec{\sigma}_N}{48\,\pi\, m_f\, m_N} \left[4\pi\, \delta^{(3)}(\vec{r}) - \frac{m_X^2}{r}\, \mathrm{e}^{-m_X\, r} \right]. \tag{23c}$$

The expectation value of the Fermi Hamiltonian is

$$E_F(nS) = \langle nS_{1/2} | H_F | nS_{1/2} \rangle = \mathrm{g}_N\, \frac{\alpha\, (Z\alpha)^3\, m_r^3}{3\, n^3\, m_f\, m_N}\, \langle \vec{\sigma}_f \cdot \vec{\sigma}_N \rangle. \tag{24}$$

Here, Z is the nuclear charge number, and $m_r = m_f m_N/(m_f + m_N)$ is the reduced mass of the system. We use the nuclear g factor in the normalization

$$\vec{\mu}_N = \mathrm{g}_N\, \frac{|e|}{2m_N}\, \frac{\vec{\sigma}_N}{2}, \tag{25}$$

which can more easily be extended to more general two-body systems than a definition in terms of the nuclear magneton. For the proton, one has $\mathrm{g}_p = 5.5856\ldots$ as the proton's g factor [29,30], while definition (25) implies that $\mathrm{g}_d = 1.713\ldots$ for the deuteron [5]. For true muonium ($\mu^+\mu^-$ bound system) and positronium, one has $\mathrm{g}_N = 2$ according to the definition (25).

By contrast, in the limit $m_X \to \infty$, one verifies that

$$\lim_{m_X \to \infty} \left\{ \frac{m_X^2}{r}\, \mathrm{e}^{-m_X\, r} \right\} = 4\pi\, \delta^{(3)}(\vec{r}), \tag{26}$$

and the two Hamiltonians given in Eqs. (23b) and (23c) vanish in the limit of an infinitely heavy X17 particle. This implies that the expectation values of S states of the Hamiltonians in Eqs. (23b) and (23c) have to carry at least one power of m_X in the denominator, and in particular, that the correction to the hyperfine energy will be of order $\alpha(Z\alpha)^4$, not $\alpha(Z\alpha)^3$, as one would otherwise expect from the two individual terms in Eqs. (23b) and (23c). For the vector hypothesis, one finds that $E_{X,V}(nS_{1/2}) = \langle nS_{1/2} | H_{\mathrm{HFS},V} | nS_{1/2} \rangle$, for the leading and subleading terms in the expansion in inverse powers of m_X, can be expressed as

$$\begin{aligned} E_{X,V}(nS_{1/2}) = h'_f\, h'_N \Bigg(& -\frac{2(Z\alpha)^4}{3\pi n^3}\, \frac{m_r^4}{m_f\, m_N\, m_X} + \frac{5(Z\alpha)^5}{3\pi n^3} \\ & \times \left(1 + \frac{1}{5n^2}\right) \frac{m_r^5}{m_f\, m_N\, m_X^2} \Bigg) \langle \vec{\sigma}_f \cdot \vec{\sigma}_N \rangle_{S_{1/2},F}. \end{aligned} \tag{27}$$

We have neglected relative corrections of higher than first order in $\alpha\, m_f/m_X$ and m_f/m_N. For the pseudovector hypothesis, one finds that $E_{X,A}(nS_{1/2}) = \langle nS_{1/2}|H_{\mathrm{HFS},A}|nS_{1/2}\rangle$ is given as follows:

$$E_{X,A}(nS_{1/2}) = \hbar_f\,\hbar_N \left(\frac{(Z\alpha)^4}{3\pi n^3}\,\frac{m_r^4}{m_f\,m_N\,m_X} - \frac{5(Z\alpha)^5}{6\pi n^3} \times \left(1 + \frac{1}{5n^2}\right)\frac{m_r^5}{m_f\,m_N\,m_X^2}\right)\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{S_{1/2},F}. \tag{28}$$

The S-state splitting is obtained from the following expectation values:

$$\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{S_{1/2},F=1} = 1, \quad \langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{S_{1/2},F=0} = -3. \tag{29}$$

Expressed as a relative correction to the leading term, given in Eq. (21), one has the following corrections due to the X17 particle,

$$\frac{E_{X,V}(nS_{1/2})}{E_F(nS_{1/2})} \approx -\frac{2\hbar'_f\hbar'_N}{g_N\pi}\,\frac{Z\,m_r}{m_X}, \tag{30a}$$

$$\frac{E_{X,A}(nS_{1/2})}{E_F(nS_{1/2})} \approx \frac{\hbar_f\hbar_N}{g_N\pi}\,\frac{Z\,m_r}{m_X}, \tag{30b}$$

depending on the vector (V) or pseudoscalar (A) hypothesis. One notices the different sign of the correction, depending on the symmetry group of the new particle. We observe that the *relative* correction to the Fermi splitting is enhanced for muonic bound systems, by a factor $m_r/m_X \sim m_\mu/m_X$ as compared to electronic bound systems, because the corresponding factor m_e/m_X is two orders of magnitude smaller.

For $nP_{1/2}$ states, whose wave function vanishes at the nucleus in the nonrelativistic approximation, one finds for the first-order corrections $E_{X,V}(nP_{1/2}) = \langle nP_{1/2}|H_{\mathrm{HFS},V}|nP_{1/2}\rangle$ and $E_{X,A}(nP_{1/2}) = \langle nP_{1/2}|H_{\mathrm{HFS},A}|nP_{1/2}\rangle$,

$$E_{X,V}(nP_{1/2}) = \hbar'_f\,\hbar'_N\,\frac{(Z\alpha)^5}{\pi n^3}\left(1 - \frac{1}{n^2}\right) \times \frac{m_r^5}{m_f\,m_N\,m_X^2}\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{nP_{1/2},F}, \tag{31a}$$

$$E_{X,A}(nP_{1/2}) = \hbar_f\,\hbar_N\,\frac{(Z\alpha)^5}{2\pi n^3}\left(1 - \frac{1}{n^2}\right) \times \frac{m_r^5}{m_f\,m_N\,m_X^2}\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{nP_{1/2},F}. \tag{31b}$$

In these results, matrix elements of tensor structures proportional to $\langle\vec{\sigma}_f\cdot\vec{r}\,\vec{\sigma}_N\cdot\vec{r}\rangle$ in Eqs. (19) and (22) have been reduced to simpler structures $\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle$ by angular algebra reduction formulas, which are familiar in atomic physics [31]. Under the replacement $\hbar'_f \to \hbar_f$ and $\hbar'_N \to \hbar_N$, the correction, for a vector X17 particle, assumes the same form as for the pseudoscalar hypothesis, up to an additional overall factor 1/2. For $nP_{1/2}$ states, the expectation values are

$$\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{P_{1/2},F=1} = -\tfrac{1}{3}, \quad \langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{P_{1/2},F=0} = 1. \tag{32}$$

The leading term in the hyperfine splitting for $nP_{1/2}$ states is well known to be equal to

$$E_F(nP_{1/2}) = \langle nP_{1/2}|H_F|nP_{1/2}\rangle = -g_N\,\frac{\alpha\,(Z\alpha)^3\,m_r^3}{3\,n^3\,m_f\,m_N}\,\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{nP_{1/2},F}. \tag{33}$$

Expressed in terms of the leading term, one obtains the following corrections due to the X17 particle for $nP_{3/2}$ states:

$$\frac{E_{X,V}(nP_{1/2})}{E_F(nP_{1/2})} \approx -\frac{3\hbar'_f\hbar'_N}{g_N\pi}\,\frac{Z\,m_r}{m_X}\left(1 - \frac{1}{n^2}\right)\left(\frac{Z\alpha m_r}{m_X}\right), \tag{34a}$$

$$\frac{E_{X,A}(nP_{1/2})}{E_F(nP_{1/2})} \approx -\frac{3\hbar_f\hbar_N}{2g_N\pi}\,\frac{Z\,m_r}{m_X}\left(1 - \frac{1}{n^2}\right)\left(\frac{Z\alpha m_r}{m_X}\right). \tag{34b}$$

Parametrically, these are suppressed with respect to the results for S states, by an additional factor $Z\alpha m_r/m_X$. For the $nP_{3/2}$ states, one considers the corrections $E_{X,V}(nP_{3/2}) = \langle nP_{3/2}|H_{\mathrm{HFS},V}|nP_{3/2}\rangle$ and $E_{X,A}(nP_{3/2}) = \langle nP_{3/2}|H_{\mathrm{HFS},A}|nP_{3/2}\rangle$, with the results

$$E_{X,V}(nP_{3/2}) = -\frac{(Z\alpha)^5}{12\pi n^3}\left(1 - \frac{1}{n^2}\right)\frac{m_r^5}{m_N^2\,m_X^2} \times \hbar'_f\,\hbar'_N\,\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{nP_{3/2},V}, \tag{35a}$$

$$E_{X,A}(nP_{3/2}) = \frac{2(Z\alpha)^6}{45\pi n^3}\left(1 - \frac{1}{n^2}\right)\frac{m_r^6}{m_f\,m_N\,m_X^3} \times \hbar_f\,\hbar_N\,\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{nP_{3/2},A}. \tag{35b}$$

Here, the expectation values are

$$\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{P_{3/2},F=2} = 1, \quad \langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{P_{3/2},F=1} = -\tfrac{5}{3}. \tag{36}$$

The leading term in the hyperfine splitting for $nP_{3/2}$ states is well known to be equal to

$$E_F(nP_{3/2}) = \langle nP_{3/2}|H_F|nP_{3/2}\rangle = g_N\,\frac{\alpha\,(Z\alpha)^3\,m_r^3}{15\,n^3\,m_f\,m_N}\,\langle\vec{\sigma}_f\cdot\vec{\sigma}_N\rangle_{nP_{3/2}}. \tag{37}$$

Expressed in terms of the leading term, one obtains the following corrections due to the X17 particle for $nP_{1/2}$ states:

$$\frac{E_{X,V}(nP_{3/2})}{E_F(nP_{3/2})} \approx -\frac{5\hbar'_f\hbar'_N}{4g_N\pi}\,\frac{Z\,m_r}{m_X}\,\frac{m_f}{m_N}\left(1 - \frac{1}{n^2}\right)\left(\frac{Z\alpha m_r}{m_X}\right), \tag{38a}$$

$$\frac{E_{X,A}(nP_{3/2})}{E_F(nP_{3/2})} \approx \frac{2\hbar_f\hbar_N}{3g_N\pi}\,\frac{Z\,m_r}{m_X}\left(1 - \frac{1}{n^2}\right)\left(\frac{Z\alpha m_r}{m_X}\right)^2. \tag{38b}$$

Parametrically, in comparison to $nP_{1/2}$ states, the correction for $nP_{3/2}$ states is suppressed for a vector X17 by an additional factor m_f/m_N, while for a pseudoscalar X17, the suppression factor is $Z\alpha m_r/m_X$. For electrons bound to protons and other nuclei, both suppression factors are approximately of the same order of magnitude, while for muonic hydrogen and deuterium, the vector contribution dominates over the pseudoscalar one.

B. Virtual annihilation

For bound systems consisting of a particle and antiparticle, virtual annihilation processes also need to be considered (see Fig. 3). The resulting effective potentials are local (proportional to a Dirac δ) and affect S states. We here consider positronium and true muonium ($f = e, \mu$), for which one has $Z = 1$, $N = \overline{f}$ (antifermion), $m_r = m_f/2$, $m_N = m_f$. In this

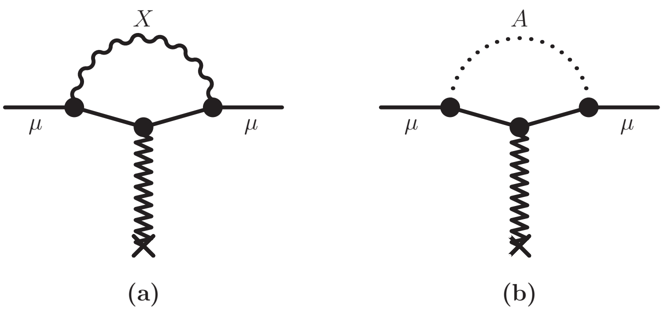

FIG. 2. The X17 particle induces vertex corrections to the anomalous magnetic moment of the muon: (a) X17 vector hypothesis and (b) pseudoscalar hypothesis. The interaction with the external magnetic field is denoted by a zigzag line.

case, the Fermi energy, as *defined* in Eq. (24), assumes the form (it is no longer equal to the full leading-order result for the hyperfine splitting, as we will see)

$$E_{F,f\bar{f}}(nS) = \frac{\alpha^4\, m_f}{12\, n^3} \langle \vec{\sigma}_f \cdot \vec{\sigma}_{\bar{f}} \rangle. \tag{39}$$

When one replaces the vector X17 boson in Fig. 3(a) by a photon, one obtains the annihilation potential (see Eq. (83.24) of Ref. [28]),

$$H_{\mathrm{ANN},\gamma} = \frac{\pi\alpha}{2m_f^2} (\vec{\sigma}_f \cdot \vec{\sigma}_{\bar{f}} + 3)\, \delta^{(3)}(\vec{r}). \tag{40}$$

Based on the identity

$$\langle \vec{\sigma}_f \cdot \vec{\sigma}_{\bar{f}} \rangle = 2S(S+1) - 3, \tag{41}$$

where $S = 1$ for an ortho state, and $S = 0$ for a para state, one can see that the annihilation process into a virtual vector particle is relevant only for ortho states, consistent with the conservation of total angular momentum in the virtual transition to the photon, which has an intrinsic spin of unity. The virtual annihilation contribution to the hyperfine splitting is

$$E_{\mathrm{ANN},\gamma}(nS) = \frac{\alpha^4\, m_f}{16\, n^3} \langle \vec{\sigma}_f \cdot \vec{\sigma}_{\bar{f}} \rangle. \tag{42}$$

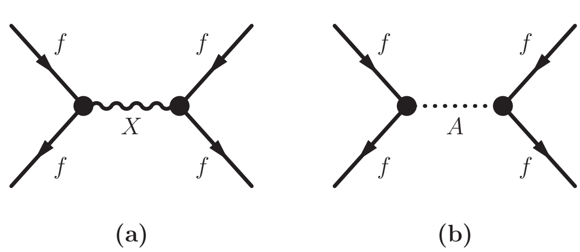

FIG. 3. For bound systems consisting of a lepton and antilepton there is an additional correction to the energy levels induced by virtual annihilation (the arrow of time is from left to right). We here consider $f = e$ (positronium) and $f = \mu$ (true muonium). The resulting effective potential is proportional to a Dirac δ and affects S states. (a) Diagram is relevant for orthopositronium and ortho true muonium ($S = 1$, annihilation into a vector X17 boson) and (b) diagram is relevant for para states (with total spin $S = 0$, which allows for an annihilation into a hypothetical pseudoscalar X17 boson). Both virtual processes contribute to the hyperfine splitting.

The difference between the expectation values for ortho and para states is the well-known result $[\langle\langle \vec{\sigma}_f \cdot \vec{\sigma}_{\bar{f}} \rangle\rangle = 1 - (-3) = 4]$

$$\Delta E_{\mathrm{HFS}}(nS) = \langle\langle E_F(nS) + E_{\mathrm{ANN},\gamma}(nS) \rangle\rangle = \frac{7}{12} \frac{\alpha^4\, m_f}{n^3}. \tag{43}$$

The exchange of a virtual photon contributes a fraction of 4/7 to this result, while the virtual annihilation yields the remaining fraction of 3/7.

The generalization of Eq. (40) to a vector X17 exchange is immediate,

$$H_{\mathrm{ANN},V} = \frac{(h'_f)^2}{8\left(m_f^2 - \frac{1}{4} m_X^2\right)} (\vec{\sigma}_f \cdot \vec{\sigma}_{\bar{f}} + 3) \delta^{(3)}(\vec{r}), \tag{44}$$

with the expectation value (we select the term relevant to the hyperfine splitting)

$$E_{\mathrm{ANN},V}(nS) = \frac{(h'_f)^2\, (\alpha m_f)^3}{64\left(m_f^2 - \frac{1}{4} m_X^2\right) n^3} \langle \vec{\sigma}_f \cdot \vec{\sigma}_{\bar{f}} \rangle. \tag{45}$$

In the calculation, one precisely follows Eqs. (83.18)–(83.24) of Ref. [28] and adjusts for the mass of the X17 in the propagator denominator. The relative correction to the hyperfine splitting due to annihilation channel, for a virtual vector X17 particle (for S states), is

$$\frac{E_{\mathrm{ANN},V}(nS)}{E_{\mathrm{ANN},\gamma}(nS)} = \frac{(h'_f)^2}{4\pi\alpha} \frac{m_f^2}{m_f^2 - \frac{1}{4} m_X^2}. \tag{46}$$

For the virtual annihilation into a pseudoscalar particle, one can also follow Eqs. (83.18)–(83.24) of Ref. [28], but one has to adjust for the different interaction Lagrangian, which now involves the fifth current, and one also needs to adjust for the mass of the X17 in the pseudoscalar propagator denominator. The result in Eq. (83.22) of Ref. [28] for the Fierz transformation of the currents has to be adapted to the pseudoscalar current, i.e., to the last entry in Eq. (28.17) of Ref. [28]. The result is

$$H_{\mathrm{ANN},A} = \frac{3h_f^2}{8\left(m_f^2 - \frac{1}{4} m_X^2\right)} (\vec{\sigma}_f \cdot \vec{\sigma}_{\bar{f}} - 1) \delta^{(3)}(\vec{r}). \tag{47}$$

The expectation value of this effective Hamiltonian is nonvanishing only for para states ($S = 0$), as had to be expected in view of the pseudoscalar nature of the virtual particle (the intrinsic parity of para states of positronium and true muonium is negative, allowing for the virtual transition to the pseudoscalar X17). In contrast to Eq. (22), we observe a small shift of the hyperfine centroid for the fermion-antifermion system, due to the term that is added to the scalar product of the spin operators. The general expression for the expectation value in an nS state is (we select the term relevant to the hyperfine splitting)

$$E_{\mathrm{ANN},A}(nS) = \frac{3h_f^2 (\alpha m_f)^3}{64\left(m_f^2 - \frac{1}{4} m_X^2\right) n^3} \langle \vec{\sigma}_f \cdot \vec{\sigma}_{\bar{f}} \rangle. \tag{48}$$

The relative correction to the hyperfine splitting due to the pseudoscalar annihilation channel, for S states, is

$$\frac{E_{\mathrm{ANN},A}(nS)}{E_{\mathrm{ANN},\gamma}(nS)} = \frac{3(h_f)^2}{4\pi\alpha} \frac{m_f^2}{m_f^2 - m_X^2}. \tag{49}$$

For the fermion-antifermion bound system, the corrections given in Eq. (30) specialize as follows:

$$\frac{E_{X,V}(nS)}{E_F(nS)} \approx -\frac{(h'_f)^2}{2\pi}\frac{m_f}{m_X}, \tag{50a}$$

$$\frac{E_{X,A}(nS)}{E_F(nS)} \approx \frac{(h_f)^2}{4\pi}\frac{m_f}{m_X}. \tag{50b}$$

The relative correction to the total hyperfine splitting, due to the X17 exchange and annihilation channels, is

$$\chi_V(nS) = \frac{4}{7}\frac{E_{X,V}(nS)}{E_F(nS)} + \frac{3}{7}\frac{E_{\mathrm{ANN},V}(nS)}{E_{\mathrm{ANN},\gamma}(nS)}, \tag{51a}$$

$$\chi_A(nS) = \frac{4}{7}\frac{E_{X,A}(nS)}{E_F(nS)} + \frac{3}{7}\frac{E_{\mathrm{ANN},A}(nS)}{E_{\mathrm{ANN},\gamma}(nS)}, \tag{51b}$$

with the individual contributions given in Eqs. (30), (46), (49), and (50).

V. X17 PARTICLE AND MUON ANOMALOUS MAGNETIC MOMENT

One aim of our investigations is to explore the possibility of a detection of the X17 particle in the hyperfine splitting of muonic bound systems. To this end, it is instructive to derive upper bounds on the coupling parameters h'_μ and h_μ, for the muon. The contribution of a massive pseudoscalar loop to the muon anomaly [see Fig. 2(b)] has been studied for a long time [32–36], and recent updates of theoretical contributions [37–39] has confirmed the existence of a (roughly) 3.5σ discrepancy of theory and experiment. The contribution of a massive vector exchange [see Fig. 2(a)] has recently been revisited in Ref. [36]. Specifically, the experimental results for the muon anomaly a_μ (see Eqs. (1.1) and (3.36) of Ref. [38]) are as follows:

$$a_\mu^{(\mathrm{expt})} = 0.00116592091(54)(33), \tag{52}$$

$$a_\mu^{(\mathrm{theor})} = 0.001165918204(356). \tag{53}$$

The 3.7σ discrepancy $a_\mu^{(\mathrm{expt})}$–$a_\mu^{(\mathrm{theor})} \approx 2.7 \times 10^{-9}$ needs to be explained.

According to Eq. (41) of Ref. [36], we have the following correction to the muon anomaly due to the vector X vertex correction in Fig. 2(a),

$$\begin{aligned}\Delta a_\mu &= \frac{(h'_\mu)^2}{8\pi^2}\frac{m_\mu^2}{m_X^2}\int_0^1 \frac{dx\, x^2\,(2-x)}{(1-x)\left[1-\frac{m_\mu^2}{m_X^2}\right]+\frac{m_\mu^2}{m_X^2}x} \\ &= 8.64\times 10^{-3}\,(h'_\mu)^2,\end{aligned} \tag{54}$$

where we have used the numerical value $m_X = 16.7$ MeV. The following numerical value

$$h'_\mu = (h'_\mu)_{\mathrm{opt}} = 5.6\times 10^{-4} \tag{55}$$

is "optimum" in the sense that it precisely remedies the discrepancy described by Eq. (52) and will be taken as the input datum for all subsequent evaluations of corrections to the hyperfine splitting in muonic bound systems. Note that, even if the vector X17 particle does not provide for an explanation of the muon anomaly discrepancy, the order of magnitude of the coupling parameter h'_μ could not be larger than the value indicated in Eq. (55), because otherwise, the theoretical value of a_μ would increase too much beyond the experimental result.

According to Eq. (20) of Ref. [36], the vertex correction due to a virtual pseudoscalar X17 leads to the following correction:

$$\begin{aligned}\Delta a_\mu &= -\frac{(h_\mu)^2}{4\pi^2}\frac{m_\mu^2}{m_X^2}\int_0^1 \frac{dx\, x^3}{(1-x)\left[1-\frac{m_\mu^2}{m_X^2}\right]+\frac{m_\mu}{m_X}^2 x} \\ &= -1.19\times 10^{-3}\,(h_\mu)^2.\end{aligned} \tag{56}$$

Here, because the correction is negative and decreases the value of a_μ, the experimental-theoretical discrepancy given in Eq. (52) can only be enhanced by the pseudoscalar X17 particle. If we demand that the discrepancy not be increased beyond 6σ, then we obtain the condition that $|h_\mu|$ could not exceed a numerical value of 3.8×10^{-4}. In the following, we take the maximum permissible value of

$$h_\mu = (h_\mu)_{\max} = 3.8\times 10^{-4} \tag{57}$$

in order to estimate the magnitude of corrections to the hyperfine splitting in muonic bound systems, induced by a hypothetical pseudoscalar X17 particle.

VI. NUMERICAL ESTIMATES AND EXPERIMENTAL VERIFICATION

A. Overview

The relative corrections to the hyperfine splitting due to the X17 particle, expressed in terms of the leading Fermi interaction, for S and P states, are given in Eqs. (30), (34a), (34b), (38a), and (38b). All of the formulas involve at least one factor of m_r/m_X, and so experiments appear to be more attractive for muonic rather than electronic bound systems. Furthermore, parametrically, the corrections are largest for S states, which is understandable because the range of the X17 potential is limited to its Compton wavelength of about 11.8 fm, and so its effects should be more pronounced for states whose probability density does not vanish at the origin, i.e., for S states. This intuitive understanding is confirmed by our calculations. Note that the formulas for the corrections to the hyperfine splitting, given in Eqs. (27), (28), (31a), (31b), (35a), and (35b), are generally applicable to bound systems with a heavy nucleus, upon a suitable reinterpretation of the nuclear spin matrix $\vec{\sigma}_N$ in terms of a nuclear spin operator.

B. Muonic deuterium

In view of a recent theoretical work [40] which describes a 5σ discrepancy of theory and experiment for muonic deuterium, it appears indicated to analyze this system first. Indeed, the theory of nuclear-structure effects in muonic deuterium has been updated a number of times in recent years [40–42]. Expressed in terms of the Fermi term, the discrepancy $\delta E_{\mathrm{HFS}}(2S)$ observed in Ref. [40] can be written as

$$\frac{\Delta E_{\mathrm{HFS}}(2S_{1/2})}{E_F(2S_{1/2})} = 0.0094(18). \tag{58}$$

In order to evaluate an estimate for the correction due to the X17 vector particle, we observe that the interaction is protophobic [11,12]. Hence, we can assume that the sign of the coupling parameter of the deuteron is approximately equal to that of the neutron. We will assume the opposite sign for the coupling parameter of the deuteron (neutron), as compared to the sign of the coupling parameter in Eq. (55). This choice is inspired by the opposite charge of the muon and nucleus with respect to the $U(1)$ gauge group of quantum electrodynamics. In view of Eq. (10) of Ref. [11] and Eq. (4) here, we thus have the estimate

$$h'_d \approx h'_n = -\frac{1}{100}\sqrt{4\pi\alpha} = -3.02 \times 10^{-3}. \qquad (59)$$

Because of the numerical dominance of the proton coupling to the pseudoscalar particle over that of the neutron [see Eq. (8)], we estimate the pseudoscalar coupling of the deuteron to be of the order of

$$h_d \approx h_p = -2.4 \times 10^{-3}. \qquad (60)$$

In view of Eq. (30), we obtain the estimates

$$\frac{E^{(\mu d)}_{X,V}(nS_{1/2})}{E_F(nS_{1/2})} \mathrel{\dot\approx} 3.8 \times 10^{-6}, \qquad (61a)$$

$$\frac{E^{(\mu d)}_{X,A}(nS_{1/2})}{E_F(nS_{1/2})} \mathrel{\dot\approx} -1.0 \times 10^{-6}, \qquad (61b)$$

where the symbol $\dot\approx$ is used to denote an estimate for the quantity specified on the left, including its sign, based on the estimates of the coupling parameters of the hypothetical vector and pseudoscalar X17 particle, as described in the current work. As already explained, the modulus of our estimates for the coupling parameters is close to the upper end of the allowed range; the same thus applies to the absolute magnitude of our estimates for the X17-mediated corrections to hyperfine energies. Note that the vector X17 contribution slightly decreases the discrepancy noted in Ref. [40], while the hypothetical pseudoscalar effect slightly increases the discrepancy, yet on a numerically almost negligible level.

Similar considerations, based on Eqs. (34a) and (34a), lead to the following results for P states,

$$\frac{E^{(\mu d)}_{X,V}(nP_{1/2})}{E_F(nP_{1/2})} \mathrel{\dot\approx} 2.5 \times 10^{-7}\left(1 - \frac{1}{n^2}\right), \qquad (62a)$$

$$\frac{E^{(\mu d)}_{X,A}(nP_{1/2})}{E_F(nP_{1/2})} \mathrel{\dot\approx} 6.6 \times 10^{-8}\left(1 - \frac{1}{n^2}\right), \qquad (62b)$$

which might be measurable in future experiments. Specifically, there is a nuclear-structure correction to the $P_{1/2}$-state hyperfine splitting due to the lower component of the Dirac $nP_{1/2}$ wave function, which has S-state symmetry. However, the lower component of the wave function is suppressed by a factor α, which implies that the $P_{1/2}$ state nuclear-structure correction is suppressed in relation to $E_F(nP_{1/2})$ by a factor α^2. An order of magnitude of the achievable theoretical uncertainty for the $P_{1/2}$-state result can be given by an appropriate scaling of the current theoretical uncertainty for S states, given in Eq. (58). The result is that the achievable theoretical uncertainty should be better than 10^{-7}, which would make the effect given in Eq. (62) measurable. Also, according to Ref. [43], the experimental accuracy should be improved into the range 10^{-6}–10^{-7} in the next round of planned experiments. Results for the Sternheim [44] weighted differences $[n^3 E_{X,V}(nS_{1/2}) - E_{X,V}(1S_{1/2})]/E_F(1S_{1/2})$ and correspondingly $[n^3 E_{X,A}(nS_{1/2}) - E_{X,A}(1S_{1/2})]/E_F(1S_{1/2})$ are of the same order of magnitude as for the individual $P_{1/2}$ states.

C. Muonic hydrogen

The considerations are analogous to those for muonic deuterium. However, the coupling parameter for the nucleus, for the protophobic vector model, is constrained by Eq. (5),

$$h'_p \approx -8 \times 10^{-4}\sqrt{4\pi\alpha} = -2.42 \times 10^{-5}, \qquad (63)$$

which is much smaller than for the deuteron nucleus. The coupling parameter of the proton, for the pseudoscalar model, can be estimated according to Eq. (8a). One obtains

$$\frac{E^{(\mu p)}_{X,V}(nS_{1/2})}{E_F(nS_{1/2})} \mathrel{\dot\approx} 8.8 \times 10^{-9}, \qquad (64a)$$

$$\frac{E^{(\mu p)}_{X,A}(nS_{1/2})}{E_F(nS_{1/2})} \mathrel{\dot\approx} -2.9 \times 10^{-7}, \qquad (64b)$$

for the S-state effects, and

$$\frac{E^{(\mu p)}_{X,V}(nP_{1/2})}{E_F(nP_{1/2})} \mathrel{\dot\approx} 5.8 \times 10^{-10}\left(1 - \frac{1}{n^2}\right), \qquad (65a)$$

$$\frac{E^{(\mu p)}_{X,A}(nP_{1/2})}{E_F(nP_{1/2})} \mathrel{\dot\approx} 1.8 \times 10^{-8}\left(1 - \frac{1}{n^2}\right), \qquad (65b)$$

for $P_{1/2}$ states. Results of the same order of magnitude as for individual $P_{1/2}$ states are obtained for the Sternheim difference of S states. The effects, in muonic hydrogen, for the vector model are seen to be numerically suppressed. The contributions of the X17 particle need to be compared to the proton structure effects, which have recently been analyzed in Refs. [45–49]. According to Ref. [48], the numerical accuracy of the theoretical prediction for the $2S$ hyperfine splitting in muonic hydrogen is currently about 72 ppm [$E_{\rm HFS}(2S) = 22.8108(16)$ meV].

D. Muonium

Muonium is the bound system consisting of a positively charged antimuon (μ^+) and an electron (e^-). Its ground-state hyperfine splitting has been studied in Ref. [50] with a result of $\Delta\nu^{\rm (expt)}_{\rm HFS} = 4\,463\,302\,765(53)$ Hz, i.e., with an accuracy of 1.2×10^{-8}. The theoretical uncertainty is about one order of magnitude worse and amounts to 1.2×10^{-7} (see Ref. [51]), with the current status being summarized in the theoretical prediction $\Delta\nu^{\rm (theor)}_{\rm HFS} = 4\,463\,302\,872(515)$ Hz.

Coupling parameters for the muon have been estimated in Eqs. (55) and (57) for the vector and pseudoscalar models, respectively, and we take the antimuon coupling estimate as the negative value of the coupling parameters for the muon. For the coupling parameters, we use the maximum allowed value for the electron in the vector model [see Eq. (3)],

$$h'_e \mathrel{\dot\approx} 1.4 \times 10^{-3}\sqrt{4\pi\alpha} = 4.2 \times 10^{-4}, \qquad (66)$$

and for the pseudoscalar model [see Eq. (10)],

$$h_e \mathrel{\dot\approx} 500\,\frac{m_e}{v} = 1.0 \times 10^{-3}. \qquad (67)$$

One obtains the estimates

$$\frac{E_{X,V}^{(\mu\mu)}(nS_{1/2})}{E_F(nS_{1/2})} \approx 2.3 \times 10^{-9}, \tag{68a}$$

$$\frac{E_{X,A}^{(\mu\mu)}(nS_{1/2})}{E_F(nS_{1/2})} \approx -1.8 \times 10^{-9}. \tag{68b}$$

Because the reduced mass of muonium is close to the electron mass, the effect of the X17 boson is parametrically suppressed. It will take considerable effort to increase experimental precision beyond the level attained in Ref. [50]. On the other hand, hadronic vacuum polarization effects are suppressed in muonium, and their uncertainty [52] is less than the X17-induced effect in the hyperfine splitting of muonium. It is thus not completely hopeless to detect X17-induced effects in muonium in the future.

E. True muonium ($\mu^+\mu^-$) system

Taking into account the exchange and annihilation channels, and using the same coupling parameter estimates as for muonium, one obtains the estimates [see Eq. (51)]

$$\chi_V(nS) \doteq 1.3 \times 10^{-6}, \quad \chi_A(nS) \approx 2.1 \times 10^{-6}. \tag{69}$$

The annihilation channel contribution numerically dominates over the exchange channel. For S states, the contribution of hadronic vacuum polarization in the annihilation channel has been improved to the level of 2 ppm, as a result of gradual progress achieved over the last decades (see Eq. (41) of Ref. [53,54] as well as Refs. [53,55,56], and the recent work [57]). A very modest progress in the theoretical determination of the hadronic vacuum-polarization contribution would make the effect of the X17 visible.

F. Positronium

Quite considerable efforts have recently been invested in the calculation of the $m\alpha^7$ corrections to the positronium hyperfine splitting, and related effects [58–71]. In positronium, effects of the X17 particle are suppressed in view of the smaller reduced mass of the bound system. Under these assumptions, the estimates for S states are as follows [see Eq. (51)]:

$$\chi_V(nS) \doteq -3.6 \times 10^{-9}, \quad \chi_A(nS) \doteq -5.1 \times 10^{-9}. \tag{70}$$

The effects are thus numerically smaller than the $m\alpha^7$ effects currently under study [59–71].

VII. CONCLUSIONS

The conceivable existence of the X17 particle [8–10] provides atomic physicists with a long-awaited opportunity to detect a very serious candidate for a low-energy (fifth force) addition to the Standard Model. The energy range of about 17 MeV provides for a certain challenge from the viewpoint of atomic physics; the range of the X17-induced interaction potentials is smaller than the effective Bohr radii in electronic and muonic bound systems. Rather than looking at the Lamb shift [16], we here advocate a closer look at the effects induced by the X17 on the hyperfine splitting, for both S and P states, in simple electronic and muonic bound systems. This notion is based on two observations:

(i) Hyperfine effects for S states are induced by a contact interaction (the Fermi contact term) and, thus, naturally confined to a distance range very close to the atomic nucleus. As far as hyperfine effects are concerned, the virtual exchange of an X17 is thus not masked by its small range.

(ii) In the pseudoscalar model [13], the one-quantum exchange of an X17 exclusively leads to hyperfine effects, but leaves the Lamb shift invariant. For a fermion-antifermion system, this statement should be taken *with a small grain of salt*, namely, it holds up to the numerically tiny shift of the hyperfine centroid, induced by the virtual annihilation channel [see Eq. (47)]. This finding could be of interest irrespective of whether the results of the experimental observations reported in Refs. [8–10] can be independently confirmed by other groups.

We have derived limits on the coupling parameters of the X17 particle in the muonic sector in Sec. V and compiled estimates for the X17-induced effects in Sec. VI, for a number of simple atomic systems. The results can be summarized as follows.

(a) We show in Sec. V that the pseudoscalar model [13] enhances the experimental-theoretical discrepancy in the muon anomaly, while the vector model [11,12] could eliminate it. Stringent bounds on the magnitude of the muon coupling parameters to the X17 particle can be derived based on the muon anomaly. Note that the order of magnitude of the maximum permissible coupling to the pseudoscalar, given in Eq. (57), also leads to a tension with the parametrization $h_f = \xi_f(m_f/v)$ given in Ref. [13] (applied to the case $f = \mu$, i.e., to the muon). Namely, the parametrization could be read as suggesting a likely increase of the pseudoscalar coupling parameter with the mass of the particle. While ξ_e is bound from below by the condition $\xi_e > 4$ [see Eq. (9)] the corresponding parameter in the muon sector must fulfill $\xi_\mu < 0.9$ [see Eq. (57)].

(b) The relative correction to the hyperfine splitting for both S and P states is enhanced in muonic as compared to electronic bound systems by two orders of magnitude, in view of the scaling of the relative corrections with m_r/m_X, where m_r is the reduced mass of the two-body bound system.

(c) In muonic deuterium, the correction, for S states, is of order 3.8×10^{-6} (vector X17) and -1.0×10^{-6} (pseudoscalar X17) in units of the Fermi energy, while the experimental accuracy for the S-state hyperfine splitting is of order 10^{-3}, and there is a 5σ discrepancy of theory and experiment, in view of a recent calculation of the nuclear polarizability effects [40]. One concludes that the experimental accuracy would have to be improved by three orders of magnitude before the effects of the X17 become visible, and the understanding of the nuclear effects would likewise have to be improved by a similar factor.

(d) In muonic deuterium, for the hyperfine splitting of $P_{1/2}$ states, the X17-mediated correction to the hyperfine splitting is of order 2.5×10^{-7} for the vector model, and of order 6.6×10^{-8} for the pseudoscalar model. These effects are not suppressed by challenging nuclear structure effects and could be measurable in the next round of experiments [43]. The same applies to the Sternheim weighted difference [44] of the hyperfine splitting of S states, where the effect induced by the X17 particle is of the same order of magnitude as for the $P_{1/2}$ splitting.

(e) In muonic hydrogen, because of the protophobic character of the vector model, effects of the vector X17 are suppressed (order 10^{-9} for the S-state splitting and order 10^{-10} for the P-state splitting and the Sternheim weighted difference). For the pseudoscalar model, the S-state splitting is affected at relative order 10^{-7}, and the P-state splitting as well as the Sternheim difference are affected at order 10^{-8}. These effects could be measurable in the future.

(f) For muonium, the situation is not hopeless: While the effects induced by the X17 shift the hyperfine splitting only on the level of 10^{-9}, which is two orders of magnitude lower than current theoretical predictions [51], we can say that, at least, the uncertainty in the theoretical treatment of the hadronic vacuum polarization [52] would not impede an experimental detection of the X17. Still, it would seem that more attractive possibilities exist in muonic systems.

(g) For true muonium, one expects effects (for the hyperfine splitting of S states) on the order of 10^{-6} for the vector X17 model as well as the pseudoscalar model. These have to be compared to the uncertainty from the hadronic vacuum polarization, which has recently been improved to the level of 2 ppm [57]. This implies that a modest progress in the determination of the R ratio, namely, the ratio of the cross section of an electron-positron pair going into hadrons versus an electron-positron pair going into muons (see Eq. (9) of Ref. [57]), could render the effect visible in true muonium.

(h) For positronium, the effects of the X17 are suppressed by the small reduced mass of the system. They are bound not to exceed the level of 10^{-9} for the vector model and pseudoscalar models. Thus, even taking into account all the $m\alpha^7$ corrections currently under study [59–71], the detection of an X17-induced signal in the hyperfine splitting appears to be extremely challenging in positronium.

One concludes that the most promising approach toward a conceivable detection of the X17 in high-precision atomic physics experiments would probably concern the hyperfine splitting of $P_{1/2}$ states in muonic deuterium, and the related Sternheim difference, where the effects are enhanced because of the large mass ratio of the reduced mass of the atomic system to the mass of the X17, and nuclear structure effects are suppressed because the P-state wave function (as well as the weighted difference of the S states) vanishes at the origin. Furthermore, very attractive prospects in true muonium [53–57] should not be overlooked.

ACKNOWLEDGMENTS

The author acknowledges support from the National Science Foundation (Grant No. PHY-1710856). Also, the author acknowledges utmost insightful conversations with K. Pachucki. Helpful conversations with A. Krasznahorkay, U. Ellwanger, S. Moretti, and W. Shepherd are also gratefully acknowledged. The author also thanks the late Professor G. Soff for insightful discussions on general aspects of the true muonium bound system.

APPENDIX: COULOMB-GAUGE PROPAGATOR FOR MASSIVE VECTOR BOSONS

Even if the Coulomb-gauge propagator for massive vector bosons, given in Eqs. (12a) and (12b), has been used in the literature before (see Eq. (6) of Ref. [72] and Eqs. (16) and (17) of Ref. [73]), separate notes on its derivation can clarify the role of the "Coulomb gauge" for massive vector bosons.

Generally, in order to calculate the vector boson propagator in a specific gauge, one can write the relation of the vector potential A^μ to the currents J^ν (in the gauge under investigation), and observe that the operator mediating the relation is the propagator. However, one can also ask the question which terms could possibly be added to the propagator, initially obtained in a specific gauge, without changing the fields.

Let us start with the massless case, i.e., the photon. One should remember [74] that the most general form of a gauge transformation of the photon propagator is [in relativistic notation with $k^\mu = (\omega, \vec{k})$]

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2} + \frac{1}{2k^2}(f^\mu\, k^\nu + f^\nu\, k^\mu). \qquad \text{(A1)}$$

The first term is the Feynman gauge result. The added terms do not change the fields, because the term proportional to $f^\mu k^\nu$ vanishes in view of current conservation, while the term proportional to $f^\nu k^\mu$ amounts to a gauge transformation of the four-vector potential A^μ, as a careful inspection shows. The choice

$$f^0 = \frac{k^0}{\vec{k}^2}, \quad f^i = -\frac{k^i}{\vec{k}^2}, \qquad \text{(A2)}$$

leads to the Coulomb gauge. The fact that the fields remain unaffected holds even for a "noncovariant" form of the f^μ, i.e., for a form where the f^μ do not transform as components of a four-vector under Lorentz transformations.

In order to generalize the result to a massive vector particle, it is necessary to observe that, in a more general sense, the "Coulomb gauge" for the calculation of bound states is defined to be the gauge in which the photon propagator component D_{00} is exactly static, i.e., has no dependence on the photon frequency. Otherwise, one would incur additional corrections in the Breit Hamiltonian, which is generated by the spatial components of the photon propagator.

The generalization of Eq. (A1) to the massive vector exchange reads as [$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$]

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2 - m^2} + \frac{1}{2(k^2 - m^2)}(f^\mu\, k^\nu + f^\nu\, k^\mu). \qquad \text{(A3)}$$

Here, $m = m_X$ denotes the vector boson mass, and we start from the Feynman gauge result $g_{\mu\nu}/(k^2 - m^2)$ (see Eqs. (3.147) and (3.149) in Ref. [75], with $\lambda = 1$, in the notation of Ref. [75]). Now, choosing

$$f^0 = \frac{k^0}{\vec{k}^2 + m^2}, \quad f^i = -\frac{k^i}{\vec{k}^2 + m^2}, \qquad \text{(A4)}$$

one (almost) derives the result given in Eqs. (12a) and (12b) (and previously used in Eq. (6) of Ref. [72] and in Eqs. (16) and (17) of Ref. [73]), with one caveat. Namely, in the spatial components of the propagator (denoted by the

Latin indices ij), one replaces $k^2 = \omega^2 - \vec{k}^2 \to -\vec{k}^2$ in the order of approximation of interest here. This is because the frequency dependence of the propagator denominator leads to higher-order corrections, which, for the photon exchange, are summarized in the Salpeter recoil correction [76].

The transition to the massive Coulomb gauge is well known to be useful in bound-state theory but is perhaps less familiar in the particle physics community; pertinent remarks are thus in order when it comes to the possible detection of a new particle in low-energy experiments.

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