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Stochastic Behavioral Asset Pricing Models and the Stylized Facts

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Stochastic Behavioral Asset Pricing Models and the Stylized Facts[1]

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Abstract: High-frequency financial data are characterized by a set of ubiquitous statistical properties that prevail with surprising uniformity, both in their qualitative appearance as well as their numerical statistical identification. While these 'stylized facts' have been well-known for decades and have spurred a plethora of descriptive modeling approaches in financial econometrics, few approaches towards a behavioral explanation of these most pervasive time series characteristics exist in the literature. A systematic explanation or replication of these salient features has been attempted only recently with simple stochastic models of interacting traders. This chapter first outlines the empirical 'laws' that seem to govern financial prices independently of time and space, and highlights the challenge that these laws constitute for financial economics. In particular, the heaviness of the tail of financial returns and the persistence of volatility autocorrelations indicate a kind of universal preasymptotic behavior of high-frequency data with convergence to the Normal distribution occurring only at relatively low frequencies. After the introduction to the stylized facts, we turn to models that provide candidate explanations for these particular empirical findings. While these models draw some inspiration and justification from the broader current of behavioral finance, their design is closer in spirit to models of multi-particle interaction in physics than traditional asset-pricing models in finance. This reflects a basic insight in the natural sciences which contains that similar regularities like those observed in financial markets (denoted as 'scaling laws' in physics) are often signatures of the complex interaction within multi-unit systems and can only be explained via the macroscopic interactions of their constituent parts. Since these emergent properties should be independent of the microscopic details of the system, this viewpoint advocates negligence of the details of the determination of individuals' market behavior and focuses on the study of a few plausible rules of behavior and the emergence of macroscopic statistical regularities in a market with a large ensemble of traders. This chapter will review the philosophy of this new approach, its various implementations, and its contribution to an explanation of stylized facts in finance. The chapter concludes

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with a discussion of the relationship of this approach to more traditional asset pricing models and open questions to be pursued in future research.

Contents

1 Introduction

The finance literature hast a somewhat ambiguous approach towards the salient empirical features that characterize financial markets. While they are identified as 'stylized facts' in recent surveys (de Vries, 1994; Pagan, 1996), they have more often been christened as 'anomalies' in the past (cf. Frankfurter and McGoun, 2001, who argue that the increasing (mis)use of the term 'anomaly' in the finance literature is evidence of a propagandistic "effort to imply that ... the reigning paradigm is irreplaceable..." (from their abstract)). The difference in language is perplexing: while the former notion implies an identification of robust features of the data that call for a scientific explanation, the later rather appears to denounce the same features as a minor nuisance for the established theoretical framework. One certainly does not do injustice to a large body of theoretical research in finance by stating that it had almost entirely ignored some of the most pervasive characteristics of financial markets for quite some time. While this does not hold for all of the stylized facts, it is certainly undisputable for two important regularities that have motivated a large part of the empirical finance literature: the fat tails of asset returns and the characteristic time-variation of their fluctuations. To be honest, a few attempts at explaining these features on the base of standard modeling frameworks do exist in recent literature (cf. Vanden, 2005)[2], but at least there has been no systematic theoretical approach towards their explanation within 'mainstream' models.

However, it also needs to be emphasized that mainstream finance had not been careless about empirical results altogether: on the contrary, one of the most important empirical findings, the martingale character of prices, is at the heart of its main paradigmatic approach, the efficient market hypothesis. It appears, however, that focusing on the explanation of this single feature, other equally universal findings have been deliberately neglected and marginalized as anomalies. The point that will be made in this chapter is that, from a different perspective, what has been found to be strange and unexpected behavior of markets, might appear as revealing charac-

[2]However, his results are rather supportive of an alternative approach. Studying the capacity of representative agent equilibrium models to account for volatility clustering, he concludes that "it it doubtful that there exist any representative equilibrium model... that is consistent with the data" (p. 374).

teristics that could guide the scientist towards a candidate explanation of price dynamics in financial markets. The surprising insight here is that - when presented in an appropriate format - the stylized facts so well known to econometricians and market practitioners would immediately be identified as *scaling laws* by natural scientists. Viewed from this perspective, a picture emerges that differs enormously from that of traditional finance: scaling laws in natural science are viewed as imprints of "complex" systems composed of many interacting subunits that have to be explained as a result of their microscopic interaction. This motivates an approach towards modeling of financial markets that focusing on the interaction of many actors rather than intertemporal optimization of representative investors. Models with such an emphasis have been proposed from the early nineties both by economists dissatisfied with the representative agent methodology as well as by physicists in the evolving 'econophysics' movement. To some extent, the promise of the scaling approach seems to have materialized: models with interacting agents of a certain type appear to be quite robust generators of the formerly mysterious anomalies of fat tails and clustered volatility. This explanatory power for some of the previously unexplained characteristics of financial markets might lend some credibility to this new approach.

The remainder of this chapter starts with an outline of the empirical stylized facts that have been of such utmost importance for the development of stochastic agent-based models. In section 2 we discuss in turn: the martingale property, fat tails and clustering of volatility and have a cursory look at other reported regularities. Section 3 highlights the interpretation of these stylized facts as 'scaling laws' and the connotations of this view for theoretical modeling. Section 4 goes into details about some representative models in the area: we start with a short exposition of sources of inspiration for these models in 4.1 in the older literature on interaction of different groups of speculators (e.g. fundamentalists vs. chartists) and then move on to models that are very explicitly based on microscopic interactions: Kirman's (1993) 'ant' model and its financial interpretations is dealt with in 4.2 and the models of interacting speculators proposed by Lux and Marchesi are featured in sec 4.3. More complicated models with a lattice-based topological structure are considered in sec. 4.4. Section 5 concludes and tries to provide an assessment of the state of this new approach vis-à-vis

other approaches in the field of behavioral finance.

2 The Stylized Facts of Financial Data

2.1 Martingales, Lack of Predictability and Informational Efficiency

The one empirical feature that has become a core ingredient of theoretical models and which a broad literature attempts to explain is the martingale property of financial prices. It can simply be stated as:

$$E[P_{t+1}|I_t] = P_t \tag{1}$$

where P_t denotes the prize of the asset at time t and I_t is the available information set at date t. As a consequence, ownership of the asset can be viewed as a *fair game* with expected pay-off equal to zero:

$$E[P_{t+1} - P_t|I_t] = 0 \tag{2}$$

and the realized price change is a random variable driven by the news arrival process that leads to a price at time $t+1$ after new information arrivals that differs from its date t conditional expectation:

$$P_{t+1} - P_t = P_{t+1} - E[P_{t+1}|I_t] = \varepsilon_t \tag{3}$$

with $E[\varepsilon_t] = 0$ due to the stochasticity of new information arrivals. With returns being defined as $r_{t+1} = \frac{P_{t+1}-P_t}{P_t}$ and

$$E[r_{t+1}|I_t] = \frac{E[P_{t+1}|I_t] - P_t}{P_t}, \tag{4}$$

the randomness of price changes carries over to this quantity as well.

A glance at any financial returns series reveals that the lack of predictability of price changes, $E[\varepsilon_t] = 0$, is at least a very reasonable characterization of the data: at first view, the increments of high-frequency returns appear like random fluctuations about a mean value close to zero with no apparent asymmetry between positive and negative realizations (cf. the well-known

examples exhibited in Fig.1). The random nature of price changes is explained by the *Efficient Market Hypothesis* (EMH) as the imprint of informational efficiency, i.e. all currently available information of importance in evaluating the asset in question is already incorporated in the market price. Therefore, only new information could lead to price changes which then would be the immediate and unbiased reaction of the market on any new information item. It is worthwhile to note that the EMH is a theory about *market outcomes* and originally had only suggested a relatively vague concept of how this macroscopic outcome might emerge from the microscopic interaction of a diversity of agents in the market place. This missing behavioral underpinning has been added by the literature on market microstructure and asymmetric information (cf. Glosten and Milgrom, 1985; Kyle, 1985; O'Hara, 1995) who show how the private interaction of some agents will be revealed via their trading activity and how the market over time approaches a state of complete revelation of any formerly private information. Since what is revealed of the private information of some better informed agents becomes public information, these models support the so called *semi-strong* version of the EMH that specifies I_t as the information available to all market participants. The stronger version with I_t including even all private information is only valid asymptotically, i.e. after an infinite number of trading rounds involving the better informed agents. In these seminal contributions, the price process in the repeated trading scenario can be shown to follow a martingale.

Traditional finance, thus, provides a well-established body of literature offering a plausible generic explanation of the martingale property, that can be supported by microeconomic models of price formation under various institutional settings.

Of course, there are many qualifications to be made from different angles: first, the lack of predictability of price changes has been questioned in tons of papers: variance-ratio tests try to recover long swings in stock prices, trading rules have been tested in-sample and out-of-sample for their ability to track hidden patterns in price records and artificial intelligence and data mining techniques have been used for the same purpose (cf. Taylor, 2005 for a comprehensive review). On the theoretical front it is well-known that allowing for risk aversion instead of the assumed risk neutrality of early

microstructure models leads to efficient markets without the martingale property (cf. Leroy, 1989).

We do not attempt to go into detail on any of these points in this chapter, but simply note that markets might only be close to martingale behavior and that there might be good reasons for why we should expect them to deviate from perfect efficiency and complete randomness of price movements.

The point we wish to emphasize is rather that the traditional framework while providing a generic explanation for one of the striking features of Fig. 1, leaves unexplained the remaining set of similarly ubiquitous findings.

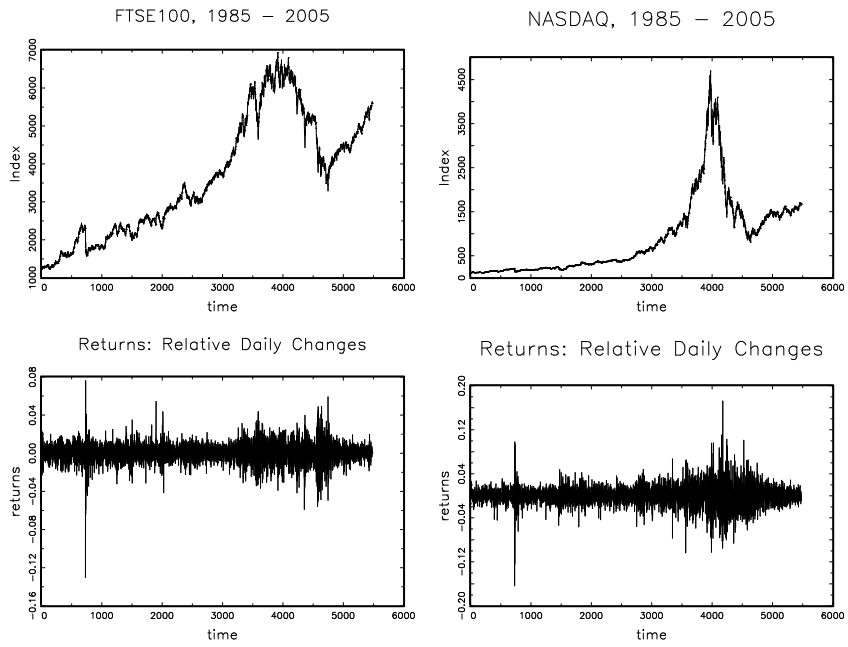

Figure 1: Two typical financial time series: the index evolution and daily increments of the UK FTSE 100 index and the U.S. NASDAQ.

2.2 Fat Tails of Asset Returns

Fig. 2 highlights the distributional properties of the returns series exhibited in Fig. 1. A very natural benchmark for characterizing the unconditional distribution is the Normal distribution. As is well-known at least since the early sixties (Fama, 1963; Mandelbrot, 1963), however, the Normal distribution provides a very poor fit to financial returns. As can be seen

from Fig. 2, empirical distributions are, in fact, quite nicely bell-shaped and symmetric, but typically have more probability mass in their center and tails than the Normal distribution. While the predominance of small fluctuations (smaller than expected under the Normal with the same standard deviation) is apparent from the histogram, the importance of fat tails can be better grasped from a comparison of empirical returns with simulated Gaussians (cf. Fig. 3). As can be seen, positive and negative events exceeding, for example, 5 times the sample standard deviation occur quite regularly in empirical data while they would have negligible probability in a Gaussian market. Table 1 provides some evidence that this behavior is truly universal: for a number of assets it lists the kurtosis statistics and the tail index (see below for details) for various definitions of the tail region of the data. Kurtosis is defined as the standardized fourth moment:

$$\kappa = \frac{1}{N}\sum_{t=1}^{N}(\frac{r_t - \overline{r}}{\sigma})^4 - 3 \tag{5}$$

with $\overline{r}$ the mean value, and σ the standard deviation of the sample. The benchmark of $\kappa = 0$ characterizes the Normal distribution and separates platykurtic ($\kappa < 0$) from leptokurtic ($\kappa > 0$) distributions. Leptokurtosis (at least for uni-modal distributions) has the visual appearance of higher peaks around the mean and heavier tails than the Normal which is the kind of shape that we *always* encounter in returns.

The finding of non-Normality and leptokurtosis as universal properties of financial returns has spurred a long-lasting debate on the appropriate stochastic model for the innovations in eq. (3). Stochasticity of returns quite naturally leads to the hypothesis that aggregate returns should obey the Central Limit Law and, hence, would have to approach the Normal distribution. However, despite their aggregation over large numbers of high-frequency price changes, daily returns are apparently non-Normal. Mandelbrot (1963) and Fama (1963) provided a solution for this conundrum evoking the *Generalized Central Limit Law*. The basis tenet of this more general convergence theorem is that the distribution of sums of random variables converges to an appropriate member of the family of Levy stable distributions. If the second moment exists, the pertinent member is the Normal distribution (as a special parametric case of the Levy stable distributions). If the second moment does not exist, other members of this family are the limiting distri-

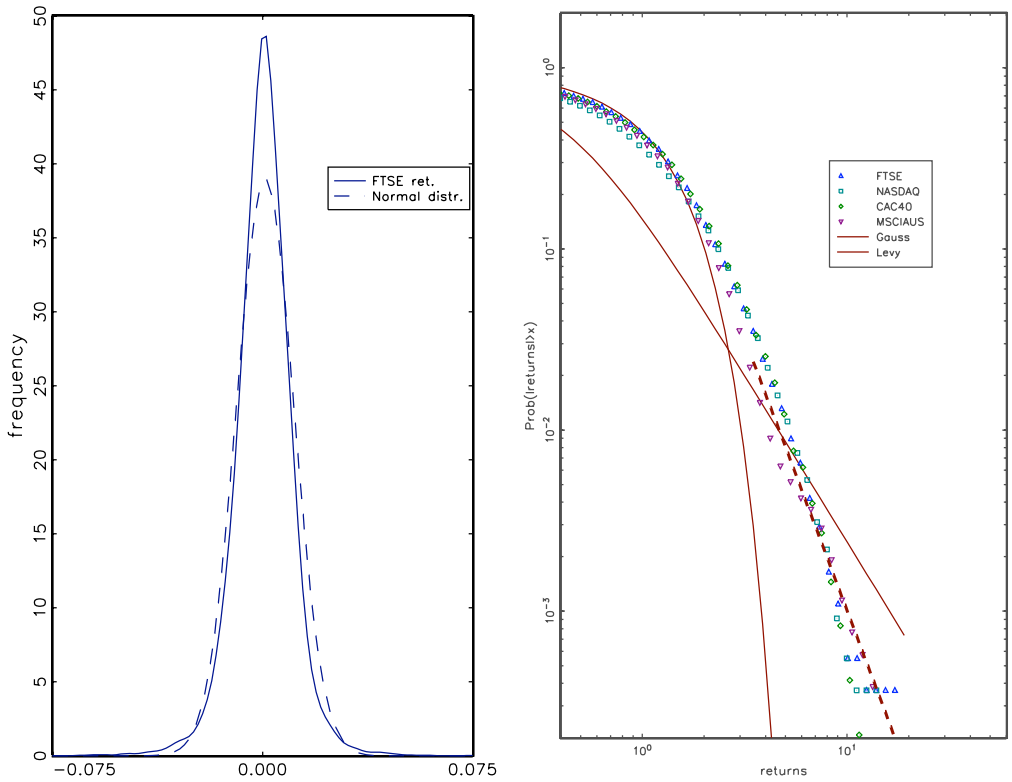

Figure 2: Distributional properties of returns: The left panel exhibits the distribution of returns of the FTSE (smoothed via a Gaussian Kernel estimator) in comparison to the Normal distribution with the same mean and standard deviation. The right panel shows the empirical complement of the cumulative distribution of absolute returns for four financial indices (the FTSE 100, NASDAQ, CAC 40 and the MSCI Australia). Note that under the first case of hyperbolic tail behavior in eq. 7, this amounts to $\text{Prob}(|\text{returns}| > x) \sim x^{-\alpha}$. In all cases we observe the typical *preasymptotic* distribution: the tails are more elongated than under a Normal distribution, but thinner than under a Levy stable distribution. The broken line illustrates the decay factor of -3 of the 'universal cubic law' claimed by Gopikrishnan et al. (1998). Through not an equally good fit for all indices, the inverse cubic decay is close to the empirical behavior of all financial assets.

butions of sums. In particular, these alternative limiting distributions are all leptokurtic and share the typical deviation of the empirical histogram from the Normal distribution. While the Mandelbrot/Fama hypothesis has motivated a large literature on parameter estimation and practical application of Levy distributions, it eventually turned out that these models would largely overstate the frequency of large returns (cf. Figs. 2 and 3 for illustrations). Much of this evidence is owed to the introduction of concepts from statistical extreme value theory in empirical finance. The key concept

of extreme value theory is the so-called *tail index* that allows a classification of the extremal behavior of empirical data and distribution functions, (cf. Beirlant, Teugels and Vynckier, 1996). The basic result is the classification of extreme values (maxima and minima) from i.i.d. random variables with continuous distributions. Denoting by $M = \max(x_1, \ldots, x_n)$ the maximum of a sample of observations $\{x_i\}$, it can be shown that after appropriate change of location and scale the limiting distribution of M belongs to one of only three classes of distribution functions. More formally, the distribution of the normalized maximum, $\text{Prob}[a_n M + b_n \leq x]$ converges to one of the following *extreme value distributions* (GEVs)

$$G_{1,\alpha}(x) = \begin{cases} 0 & x \leq 0 \\ \exp(-x^{-\alpha}) & x > 0, \end{cases}$$

$$G_{2,\alpha}(x) = \begin{cases} \exp(-(-x)^{\alpha}) & x \leq 0 \\ 1 & x > 0, \end{cases} \tag{6}$$

$$G_3(x) = \exp(-e^{-x}) \quad x \in \Re.$$

From this typology of extremal behavior, a similar classification of the underlying distribution's asymptotic behavior in its outer parts, i.e. tails, can be inferred. Namely, denoting probabilities ($\text{Prob}(x_i \leq x) \equiv W$ it follows directly from the classification of extremes in (6) that if the maximum of a distribution follows a GEV of type $j (j = 1, 2, 3)$, then its upper tail asymptotically converges to the pertinent distribution from the following list:

$$\begin{aligned} W_{1,\alpha} &= 1 - x^{-\alpha}, x \geq 1, \\ W_{2,\alpha} &= 1 - (-x)^{\alpha}, -1 \leq x \leq 0, \\ W_3 &= 1 - \exp(-x), x \geq 0. \end{aligned} \tag{7}$$

These three types of tail behavior can be described as hyperbolic decline ($W_{1,\alpha}$), distributions with finite endpoints ($W_{2,\alpha}$), and exponential decline (W_3). In order to nest all three alternatives, one can integrate the three limit laws into a unified representation:

$$W_\gamma = 1 - (1 + \frac{\gamma x}{\sigma})^{-1/\gamma} \tag{8}$$

with $\gamma = 1/\alpha$ ($\gamma = -1/\alpha$) in the cases $W_{1,\alpha}$ and $W_{2,\alpha}$ and W_3 being covered as the limit $\gamma \to 0$ (σ is a parameter for scale adjustment). Estimation of the *tail index* α allows to determine whether a particular distribution falls into classes 1,2, or 3 and in the case of the one-parameter families$W_{1,\alpha}$ and $W_{2,\alpha}$ simultaneously enables one to estimate the *decay parameter* of the tail. These estimates would allow to assess whether certain distributional hypotheses are in conformity or not with the empirical behavior. For example, an empirical α ($= 1/\gamma$) significantly above 0 would allow rejection of the Normal distribution as well as any other distribution with exponentially declining tails. The indication of hyperbolic decline would also exclude a finite endpoint as implied by $W_{2,\alpha}$ type distributions. Needless to say, the estimated α would be an extremely valuable tool in financial engineering as it could be easily used to compute the probability of large losses and gains (cf. Lux, 2001).

Table 1 shows that - with some variation depending on the selection of the tail size - empirical estimates hover within the interval of about 2 to 4. 95 percent intervals from the asymptotic distribution of the pertinent maximum likelihood estimates allow to demarcate even more sharply the set of distribution functions that would or would not be in harmony with such extremal behavior. As an important consequence, the family of Levy-stable distributions proposed by Mandelbrot (1963) and Fama (1963) would have heavier tails than the empirical records with their α being restricted to the interval $]0, 2]$. Any empirical α significantly above 2 (as we mostly find it) would, therefore, speak against the Levy stable model (which, as a consequence, would hugely *overstate* the risks of large returns). On the positive side, an admissible candidate for the unconditional distribution would be the Student t whose degrees of freedom are equal to its tail index so that it could be tuned in a way to conform to empirical shapes of return distributions. Fergussen and Platen (2006) show that for a variety of stock indices the parameter estimates of a very general family of distributions (the generalized hyperbolic distributions) cluster in the neighborhood of a Student t with 3 d.f.

What implications does this phenomenological characterization have for theoretical models? First, from the viewpoint of the efficient market theory, price increments only have to be random. The innovations in eq. 3 could,

	$\alpha_{2.5\%}$	$\alpha_{5\%}$	$\alpha_{10\%}$	κ
FTSE 100	3.21 (2.68, 3.75)	3.06 (2.70, 2.56)	2.80 (2.56, 3.03)	11.10
NASDAQ	3.31 (2.76, 3.86)	3.23 (2.85, 3.62)	2.69 (2.46, 2.91)	7.36
CAC 40	3.64 (2.99, 4.29)	3.17 (2.77, 3.57)	2.87 (2.62,3.13)	4.72
MSCI Aus	3.16 (2.62, 3.71)	3.61 (3.17, 4.05)	3.17 (2.90, 3.44)	46.21

Table 1: Kurtosis statistics and maximum likelihood estimates for the Pareto tail index characterizing the extremal law $G_{1,\alpha}$ and tail distribution $W_{1,\alpha}$. Data are the same as in Fig. 2 with daily sampling frequency and time horizon 1985 to 2005. The estimates are given for three different sizes of the tail region (2.5%, 5%, and 10%) with asymptotic 95% confidence intervals shown in brackets. All results are in good overall agreement with a 'universal cubic law' as postulated in the pertinent literature. The tendency for a decrease of the estimated coefficient with increasing tail size is usually seen as reflecting contamination of tail data with observations from the center of the distribution. With estimated tail indices significantly below 4, the fourth moment would not exist. The expected divergence of the kurtosis statistics would lead to unstable estimates in finite samples that increase with sample size.

therefore, be drawn from a Student t as well as any other distribution function that meets the minimum requirement $E[\varepsilon_t] = 0$. Since ε_t reflects the news arrival process its realm is outside economics and the EMH is agnostic as to what the the joint distribution of all relevant news items might look like. However, there is a more subtle issue here: returns over longer time intervals are aggregates of high-frequency returns (at least under continuous compounding, i.e. for $r_t = \ln(p_t) - \ln(p_{t-1})$ and approximately so for $r_t = \frac{p_t - p_{t-1}}{p_{t-1}}$). If all high-frequency returns are reflections of i.i.d. news arrival processes, their aggregates should converge towards the Normal distribution irrespective of the underlying distribution of single high-frequency returns. This is simply a consequence of the central limit law. One might argue that at the level of daily data (the time horizon we have investigated above), returns within liquid markets are already sums of thousands of intra-daily price changes so that we should have gone through the better part of the convergence towards the Gaussian shape at this level of time

aggregation. Nevertheless, as we have seen above, daily returns are quite different still from Normally distributed random variates[3]. With higher levels of time aggregation (e.g. monthly returns), we indeed get closer to the Gaussian as would be expected from the central limit law. The intriguing aspect about this phenomenology is that if we look at financial data of a particular time horizon (e.g. daily) we find a kind of *universal preasymptotic behavior* which seems to be independent of location, time and details of the market structure. To appreciate this universality of the approximately *cubic law of asset returns* (Gopikrishnan *et al.*, 1998) note that it appears to apply to practically all types of financial markets, e.g. various developed stock markets, foreign exchange markets, precious metals and emerging markets (Jansen and de Vries, 1991; Longin 1996; Koedijk *et al.*, 1990; Lux, 1996, Rockinger and Jondeau, 2003). The relevant news arrival processes might be quite different for all these markets. At least there is no *a priori* reason to assume that they should all obey a roughly identical distribution of news. Furthermore, one might argue that the velocity of efficient information processing in the trading process might have increased over time due to technical advances like electronic trading. Nevertheless, we have no indication that the shape of the return distribution has undergone any remarkable changes over the past decades reflecting an increase in information transmission. It seems that the set of potential events (summarized in the distribution of returns), during a day in a financial market is always pretty much the same - irrespective of whether trading is organized via order-driven and quote-driven systems, whether shares are traded on the floor or electronically and broadly independent of the size of the market. Despite the agnostic view of distributional properties by the EMH, we might feel somewhat uncomfortable about the apparent *universality* of the type of randomness of price changes. If the return distribution is that robust, additional factors besides new information in the trading process might be held responsible for this particular outcome of the market process. However, if this were the case, the EMH would not offer a full explanation of financial

[3]This finding had actually motivated the proposal by Mandelbrot and Fama of the Levy stable distributions. According to a *Generalized Central Limit Law*, these are the limiting distributions of sums of random variables with *infinite* second moment (whereas with a finite second moment, we are back at the classical central limit law). Unfortunately, the data also speak against the Levy stable hypothesis.

price movements and prices would *not* solely reflect new information.

One might, then, argue that the universality of distributional properties of asset returns should have its behavioral roots within the trading process and needs to the explained from the way human subjects interact in financial exchanges. Sec. 3 will further pursue this avenue by considering it from the perspective of 'scaling' theory developed in the natural sciences. Before we turn to this unfamiliar approach, we expand on other ubiquitous regularities in the following subsections.

2.3 Volatility Clustering and Dependency in Higher Moments

The martingale property of financial prices implies that price differences define a martingale difference process and are, thus, uncorrelated. In empirical time series, one typically finds marginally significant positive or negative autocorrelations at the first few lags for stock and currency returns, respectively. These are, however, believed to reflect particular structural characteristics of particular markets and the way in which prices are recorded: in stock markets, small positive autocorrelations are probably due to infrequent trading for single stocks, and certain common news factors of importance for the individual components of stock indices. In foreign exchange markets bounces between the bid and ask price for currency quotes lead to negative correlation of recorded transaction prices. Since these autocorrelations, though statistically significant could mostly not be exploited via pertinent trading strategies they are usually not classified as strong evidence against the EMH.

However, while almost uncorrelated, asset returns are not i.i.d. stochastic processes. Another glance at Fig. 1 reveals that while the ensemble of returns over a longer horizon leads to fairly similar distributions across different sets of data, on shorter time scales we encounter less homogeneous behavior. The comparison of empirical returns and simulated Gaussian and Levy stable data in Fig. 3 makes the difference particularly transparent: while the latter have a very uniform degree of fluctuations, the former switch between periods of tranquility and more turbulent episodes. The

returns generating process is, thus, characterized by non-homogeneity of its higher moments. This variability in the extent of fluctuations is actually the reason for the introduction of the concept of a martingale process in financial economics as it makes no requirement on the noise term except for $E[\varepsilon_t] = 0$.[4]

The lack of i.i.d. properties is also reflected in autocorrelations of simple transformations of returns. Considering various powers of absolute returns $\{|r_t|^\lambda\}$, one typically observes much higher and longer lasting autocorrelations than for the raw series. Fig. 3 illustrates this finding for the most frequent choices $\lambda = 1, 2$. As can be seen there is strong dependence in these higher moments. Since powers of absolute returns can be interpreted as measures of volatility (as they all drop the sign and only preserve the extent of fluctuations), these results indicate a high degree of predictability of volatility (in the absence of significant predictability of the *direction* of price movements).

While the volatility clustering phenomenon has been known for a long time, models covering this feature have appeared first with Engle's (1983) seminal proposal of the ARCH framework that has spurred a plethora of models with nonlinear dependency in second movements (cf. Taylor, 2005, for an overview). While most early literature had considered only the second moment ($\lambda = 2$), Taylor (1986) pointed out that the first absolute moment has even more pronounced dependence than the second. Ding, Engle and Granger (1993) discuss a whole range of positive λ's and find that the highest degree of autocorrelation is typically found for $\lambda \approx 1$. This hierarchy of strength of dependency meanwhile also counts as an established stylized fact (Lobato and Savin, 1998).

An important facet of the empirical findings on higher-order dependencies is the distinction between short-memory and long-memory in autocorrelation structures. While short-memory processes are characterized by exponentially decaying ACF functions (ARMA models as well as GARCH models are standard samples), a long-memory process has hyperbolically decaying autocorrelations which implies a much slower decay with long lasting after

[4] In contrast to the more restricted concept of a random walk which would require a constant variance $\sigma_\varepsilon^2 = Var[\varepsilon_t]$ of the fluctuations.

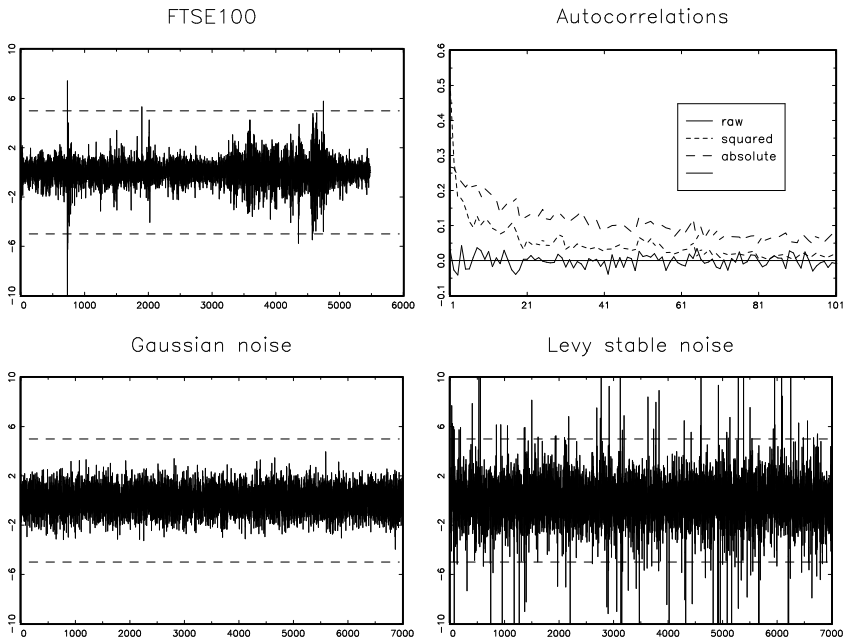

Figure 3: Returns of FTSE 100 compared to simulated Gaussian noise and Levy stable noise without temporal correlation. For better comparability, all three series have been rescaled so that the sample variance is equal to unity. The tail parameter of the Levy distribution has been set equal to 1.7, a typical estimate for stock returns. The upper right-hand shows the pronounced, hyperbolically decaying autocorrelations of squared and absolute returns which indicate volatility clustering and time-variation of the degree of fluctuations.

effects of innovations. Inspection of Fig. 3 suggests that the autocorrelations of absolute and squared returns are examples of hyperbolic rather than exponential decline. Indeed, if one considers very long series, the autocorrelations stay significant over perplexingly long horizons: for daily S&P 500 data over the period 1928-1990 Granger and Ding (1996) report significant autocorrelations over 2500 lags, i.e. more than 10 years! The decay of autocorrelations of squared and absolute returns is, in fact, indicative of a hyperbolic decline. This implies that, for example, covariances of absolute returns, would decay according to:

$$E[|r_t r_{t+\triangle t}|] \sim \triangle t^{-\lambda} \tag{9}$$

Processes with long-memory or long-term dependence have properties very different from those that only display short memory (cf. Beran, 1994). In particular, the variance of the sample mean decays to zero at a rate slower than n^{-1}, and the spectral density diverges at the origin. Findings of

long-memory in the mean of certain series have motivated the development of fractional Brownian motion and autoregressive fractionally integrated processes while the finding of long-term dependence in the second moment of financial data has inspired the development of pertinent extensions of GARCH type and stochastic volatility models (cf. Baillie, Bollerslev and Mikkelsen, 1996).

2.4 Other Stylized Facts

One important recent addition to the set of time-series characteristics of financial data is long-memory of trading volume. It has been known for quite some time that volume is highly comtemporaneously correlated with volatility. This pronounced comovement might suggest that both series have more common characteristics. Convincing evidence for long-term dependence in volume has been presented in Lobato and Velasco (1998) although the authors also point out that volatility and volume do not share exactly the same degree of long-term dependence (i.e. have different decay parameters λ).

Recent work on U.S. high-frequency stock market data has come up with the additional finding of fat tails in the unconditional distribution of transaction volume (Gopikrishnan *et al.*, 2001) and the number of trades within a time interval. Gabaix *et al.* (2003) provide a theoretical framework in which they combine these findings, the power law of returns and a Zipf's law for the size distribution of mutual funds within a choice-theoretic setting for the trading activity of large investors. However, empirical evidence for the new regularities is so far restricted to the U.S. data sets investigated by these authors.

3 The Stylized Facts as 'Scaling Laws'

The neglect of almost all the prominent features of asset prices except for their martingale property by the efficient markets paradigm is not too hard to explain. If one shares the view of informational efficiency being reflected in the unpredictability of price changes, the price increments are simply

one-to-one mappings of the news arrival process. As noted above, neither fat tails nor long-term dependence as properties of price changes are, therefore, in contradiction to the EMH. One might, however, be aware that as a consequence from accepting the empirical facts and the validity of the traditional EMH view, one would have to concede that 'news' in all times and all places seem to come with the very same underlying distribution.

Interestingly, scientists with a different background who have stumbled over one of the huge data sets from financial markets have typically arrived at very different conclusions after detecting the above 'stylized facts'. Since financial data represent the largest available records of human activity, they have indeed attracted curiosity from various other disciplines. A strong recent current is that of physicists engaging in empirical analysis and theoretical modeling of financial markets. The reaction of these researchers to the well-known stylized facts of empirical finance was entirely homogeneous and totally different from the received viewpoint recalled above: natural scientists saw these as imprints of a complex system with a large number of interacting microscopic entities. As Stanley *et al.* wrote in an influential early contribution pointing out this viewpoint:

"Statistical physics has determined that physical systems which consist of a large number of interacting particles obey universal laws that are independent of the microscopic details. This progress was mainly due to the development of scaling theory. Since economic systems also consist of a large number of interacting units, it is plausible that scaling theory can be applied to economics" (Stanley *et al.*, 1996)

This statement basically argues that since one finds in financial markets statistical properties similar to those of certain physical (or biological) systems, the explanation of these characteristics could also be broadly similar. This assertion has (at least) three components which we may consider in turn for their empirical validity or plausibility:

(i) Financial stylized facts are analogous to the scaling laws that play a prominent role in statistical physics. We have spent some effort above on outlining how empirical finance had arrived at a very parsimonious characterizations of the fat tails and clustered volatility of returns. Both the unconditional distribution of large returns (eq. (7)) and the

conditional dependence structures of their fluctuations (eq. (9)) can be expressed by hyperbolic decay rates. Such hyperbolic distributional characteristics are, however, exactly what is denoted as a power or scaling law in statistical physics. As far as the existence of these 'laws' counts as well-established in empirical finance, financial data in this descriptive sense share the scaling laws of various natural records.

(ii) Scaling laws (stylized facts) should typically be robust (universal) and should, therefore, hold for similar phenomena independent of the microscopic details. In finance, one could interpret these apparently unimportant microscopic details as the particular institutional details of the market microstructure (floor trading vs. electronic trading, quote versus order driven markets etc.). In fact, while many other empirical findings do somehow depend on the microstructure, the hyperbolic scaling laws are those features that can be found everywhere, e.g. the cubic law of large returns seems to govern both stock markets as well as foreign exchange markets with their very different organization of the trading process.

(iii) Scaling laws are the signature of systems with a large ensemble of interacting units which emerge from the interaction of these subunits (particles, molecules) and are only dependent on a few basic principles of interaction.

While this is simply an observation across many different categories of dynamic processes in physics and other areas of natural science (examples are turbulent flows or evolutionary processes in biology), it seems harder to accept this viewpoint for man-made systems. It appears to imply that we have to disregard the importance of individual rational choice which is one of the basic tenets of economic theory. While we would certainly not deny the rationality (or from a behavioral perspective rather: attempt towards rational behavior) of economic agents, we could argue that the diversity of micro-motives, preferences, endowments, access to information, and degrees of rationality and deliberation of these agents could be better captured by a statistical approach than by the optimization of one representative agent. It might well be that in the presence of this large ensemble of heterogenous agents, a few basic principles of interaction can be

found that exert a dominating influence on the macroscopic market behavior and that prevail in more or less the same way in different institutional settings (microscopic details). It would, then, be the task of a theory motivated by the analogy between scaling in physics and finance to show that this possibility can be substantiated by sensible stochastic models of asset price dynamics. The relevant literature will be reviewed in the next section.

Some words of caution on the 'scaling approach' have to be added: there might well be an exaggeration of both the statistical basis and the potential implications of scaling laws as signatures of complex dynamics in the pertinent literature. As concerns the statistical validity, detection of scaling in the natural sciences is typically based on apparent linearity in some kind of log-log plot such as Fig. 2: With the necessity of 'binning' the data (i.e., grouping it into intervals) and the violation of the independence assumption in the linear regression this approach appears questionable from a methodological viewpoint and has often been criticized by economists (Brock, 1999; Durlauf, 2005, Gallegatti *et al.*, 2006): the ubiquitous declaration of statistical objects as fractal, self-similar or scaling has also been attacked in a recent paper in *Science*. The authors (Avnir *et al.*, 1998) had surveyed all 96 articles in the *Physical Review* journals over the period 1990 to 1996 that contained some empirical scaling analysis of natural or experimental time series. They conclude that the "... scaling range of experimentally declared fractality is extremely limited, centered around, 1.3 order of magnitude... " while a true self-similar or fractal object in the mathematical sense would require infinitely many orders of power-law scaling. They find it doubtful to accept the claims of most of the pertinent studies that with their often very small range of magnitude they would indeed have detected scaling behavior.

A number of papers also point out that one could obtain 'spurious' or 'apparent' scaling behavior for processes without a 'true' asymptotic power law. Gielens *et al.* (1996) show that one can always find local alternatives to fat-tailed distribution that possess thin tails (i.e., decline exponentially) while tail index estimates would indicate a power-law behavior. The temporal scaling characteristics (long-term dependence of absolute moments) could be obtained as a spurious outcome of certain specifications of GARCH models (Crato and de Lima, 1996), stochastic volatility models (LeBaron,

2001), regime-switching processes or even uncorrelated stochastic processes with heavy tails (Barndorff-Nielsen and Prause, 2001). However, mostly this alternatives would require a certain degree of fine-tuning of parameters in order to 'fool' the pertinent statistical tests. Given enough flexibility of parameter selection, it would always be possible to design a local alternative to a process with power-law characteristics that has no 'true' scaling behavior, but comes arbitrarily close to scaling and could, then, not be distinguished from a generic power-law mechanism with finite samples. An example is the Markov-switching multifractal model introduced by Calvet and Fisher (2001) which has 'long memory over a finite interval' that could be made arbitrarily long by appropriate choice of the specification. This model had indeed been designed as a well-behaved stochastic process that provides a close resemblance to the statistically more cumbersome first vintage of multifractal models of asset returns with true scaling behavior (cf. Mandelbrot *et al.*, 1997). The ubiquity of fat tails and long memory for financial data might, however, be viewed as support for models that have these feature generically rather than apparent scaling for particular sets of parameter values. As concerns the rigor of statistical analysis and the sample sizes of empirical data for which 'scaling' has been declared, pertinent studies in finance are in a *better* position than most of the studies in natural sciences criticized by Avnir *et al.* (1998). First, financial econometricians routinely apply more rigorous methods than log-log plots. Most of the research on fat tails in finance is based on the theoretical concepts of extreme value theory and has adopted state-of-the-art estimators of the tail index (mostly without reference to the concept of scaling). Similarly, research on temporal dependence has also used more refined methods from stochastics (cf. Lux and Ausloos, 2002, for a comparison of the tools used by physicists and financial econometricians). Second, as for the sample sizes, the literature had started out typically with daily recorded series, but has moved on to the immense data-bases of intra-daily high-frequency returns. The pertinent literature indicates that both the findings on fat tails and long-term dependence of volatility in daily data are confirmed for intra-daily records (cf. Abhyankar *et al.*, 1995, Dacorogna *et al.*, 2001, Bollerslev and Wright, 2000).

Another concern might be the alleged relationship between power-laws and

'complex' interactions of heterogenous subunits. The evidence for such a relation is mainly illustrative in nature: physics and biology offer a variety of examples where the non-linear interactions of elementary units results in overall system characteristics that can be described by power laws. A famous case is the leading example of self-organized criticality: dropping grains on piles of sand (or other materials like rice) always leads to a power-law distribution of avalanches that can be explained by a stochastic process of the change of local gradients (Jensen, 1998). However, it has also been critically discussed recently how useful the diffuse labels of complexity theory are (Horgan, 1995). It might also be noted that there exist some simple explanations for power laws: power-law distributions could be generated via a combination of exponentials, by taking the inverse of quantities that themselves obey harmless distributions, or by splitting processes, among others (cf. Newman, 2004). As has been demonstrated by Granger (1980), long memory in aggregate data could result from the aggregation process of heterogenous individual behaviour (a principle that has recently inspired a new branch of empirical literature in political science, cf. Box-Steffensmeier and Smith, 1996). However, none of the simple generating mechanisms has ever been proposed as a source of power laws in financial data and aggregation of individual behaviour might not be inconsistent with the view of financial markets as a system of interacting agents. It, therefore, seems worthwhile to explore the 'complexity' approach that views scaling as the consequence of phase transitions and critical phenomena.

4 Behavioral Asset Pricing Models with Interacting Agents

4.1 Interaction of Chartists and Fundamentalists in Complex Nonlinear Dynamics

From about the late eighties and early nineties, behavioral approaches to financial markets gained in momentum. The literature on excess volatility and overreaction of asset prices to news suggested that psychological mechanisms and boundedly rational behavior might provide explanations

for these and other mysterious 'anomalies'. At the same time, surveys of trading strategies and expectation formation mechanisms of real-life traders pointed to the importance of technical trading and adaptive expectations (Allen and Taylor,1990, Taylor and Allen, 1992). The dollar bubble of the early eighties was believed to have been at least partially due to positive feedback trading (Frankel and Froot, 1986) and this perception gave rise to new interest in models of interacting groups of chartist and fundamentalist speculators (Beja and Goldman, 1980; Day and Huang, 1990). These models were framed as systems of difference or differential equations that contained the asset price as well as some characteristics of investors as state variables. Some of the pertinent models assumed permanent market clearing, while others used a sluggish price adjustment rule as a proxy for market making activities in the presence of excess demand (ED). While excess demand functions of the different groups of traders could either be formulated in an *ad-hoc* fashion or were derived from particular utility functions, traders were typically not rational as neither of both groups properly takes into account the effect of its own trading activity on subsequent price movements. In a sense (to be detailed below), these contributions were already motivated by the idea to explain market-wide phenomena as emergent characteristics from complex interactions, but restricted the level of disaggregation to a small number of behavioral types with complete homogeneity within groups.

The seminal paper by Beja and Goldman (1980) provides a simple example of the legacy of models of chartist-fundamentalist interaction. Beja and Goldman assume simple *ad-hoc* functional forms for excess demand of fundamentalists and chartists. Fundamentalists' excess demand depends on the difference between the fundamental value P_f (assumed to be known to them) and the current market price P_t:

$$ED_f = a(P_f - P_t) \tag{10}$$

where a is a coefficient for the sensitivity of fundamentalists' excess demand to deviations of the price from the underlying fundamental value. Assuming an expected reversal of the market price towards P_t together with constant risk aversion and constant expected volatility such a function could also be derived from myopic utility maximization using a mean-variance framework or a negative exponential CARA utility function together with Normally distributed expected price changes. While this format of fundamentalists'

excess demand is pretty standard in the literature and can already be found in early contributions like Baumol (1957), there is more variation in this literature in the formulation of chartists' excess demand. The particular hypothesis employed by Beja and Goldman (1980) is that their excess demand depends on the expected price change π (i.e. expected capital gains or losses)

$$ED_c = b\pi \tag{11}$$

where b again captures the sensitivity of the order flow of this group to expected gains or losses. In a continuous-time framework π is the *subjective* expectation of the infinitesimal price change $\frac{dp}{dt}$. Beja and Goldman (1980) invoke a market maker mechanism in order to justify sluggish Walrasian price adjustment:

$$\frac{dP}{dt} = P'(t) = \lambda(ED_f + ED_c) = \lambda(a(P_f - P_t) + b\pi) \tag{12}$$

with λ the price adjustment speed. While this is a phenomenological characterization without microeconomic motivation from the optimization problem of a market maker, one may note that it closely resembles the microfounded price adjustment rules of the literature on price formation under asymmetric information (e.g. Kyle, 1985).

Given the trading strategies of the two groups of investors, price changes result endogenously from the total imbalance between demand and supply so that the chartists' expectations might be confirmed or not. In the presence of a deviation of expected from realized price movements, chartists are assumed to adaptively adjust their expectations:

$$\frac{d\pi}{dt} = \pi'(t) = c(P'(t) - \pi) \tag{13}$$

where c is a parameter for the speed of adaptation of expectations.

Neither of both groups is characterized by rational expectation formation: chartists react adaptively by assumption so that they will hardly ever correctly predict price changes. Fundamentalists neglect the existence of chartists and their influence on price changes so that even if the price reverts towards its fundamental value (which might not be guaranteed), the speed of its reversal towards P_f might be different from the hypothesized adjustment coefficient of eq. (10).

The model of speculative activity by Beja and Goldman (1980) boils down to a system of two differential equations (eqs. (13) and (14)). It is typical for a large body of literature in that it consists of a dynamic system covering the market price plus some group characteristics that undergo changes over time in response to the asset price dynamics. It is also quite characteristic of the broader literature in its main results. The interest of the authors of this and many subsequent contributions in this vein is mainly in the *existence* and *stability* of a fundamental equilibrium in the presence of non-rational speculative activity. It is easy to see that the conditions for existence of a stationary state of the joint dynamics of P and π, $P'(t) = \pi'(t) = 0$ lead to a dynamic equilibrium $P^* = P_f, \pi^* = 0$. The only possible steady state is, therefore, obtained if both the price equals its fundamental value and chartists expect no further price changes. In this case, excess demand of both groups of traders equals zero and the price remains unchanged. It is slightly more demanding to arrive at results on the stability or instability of this steady state. Applying the standard stability criteria for systems of autonomous differential equations, we find that the system converges asymptotically towards its steady state if the following necessary and sufficient condition is met[5]:

$$a\lambda + c(1 - b\lambda) > 0 \tag{14}$$

This condition yields the following plausible insights:

(i) high sensitivity of fundamentalists' (chartists') excess demand is stabilizing (destabilizing),

(ii) whether increased price adjustment speed is destabilizing or not, depends on the relative sensitivity of the chartists' and fundamentalists' demand schedules. Higher price adjustment speed has a more stabilizing (destabilizing) tendency , if $a > (<) c \cdot b$,

(iii) the influence of the speed of expectation adjustment of chartists is ambiguous: if $1 - b\lambda > 0$ the systems is always stable independent of the value of c (since with $b < \frac{1}{\lambda}$ the market maker's price adjustment succeeds in reducing chartists' excess demand over time). Conversely: if $b > \frac{1}{\lambda}$, price adjustment triggers even higher order volumes by chartists due to pronounced bandwagon effects. In this case,

[5]See Beja and Goldman (1980) for details.

increasing adjustment speed c in their adaptive expectation formation would have a destabilizing tendency.

The second and third item above illustrate that stabilizing and destabilizing features of particular chartists strategies could be subtle and could easily change under different specifications of their strategies. It is worthwhile to note that this model provides a potential explanation of certain empirical regularities. If stability condition (14) is satisfied and the eigenvalues of the dynamic system are complex conjugate numbers (which happens in an open set of parameter values), the model exhibits overshooting and subsequent mean reversal in the presence of new information. To see this assume that the fundamental value of the asset increases from $P_{f,1}$ to $P_{f,2}$ (cf. Fig. 4). Fundamentalists knowing of the increase of the intrinsic value will start buying shares. This leads to excess demand and exerts upward pressure on market prices. The increase of the asset price is interpreted as a positive trend by chartists who subsequently also start buying shares. Due to this non-informed source of additional demand, the price will overshoot its new fundamental value and fundamentalists will switch from the demand to the supply side leading to mean reversion towards $P_{f,2}$. In the following, the price will converge to its new fundamental value with damped oscillations.

If the stability criterion (14) is not satisfied, these oscillations would - because of strong feedback effects from chartists - display an increase rather than decrease in amplitude. Since the model by Beja and Goldman is framed as a *linear* system of differential equations, there would be no limit to the divergence of the price from the underlying fundamental value. Of course, such a scenario is unrealistic which essentially means that this baseline model is silent on the dynamics one would expect under local instability of the fundamental equilibrium.

Similar dynamic models like the one proposed by Beja and Goldman with added nonlinear ingredients have been studied by a number of authors: even prior to Beja and Goldman, Zeeman (1974) had published a very similar model that assumed a non-linear reaction function of chartists on observed price changes which flattens out further away from the fundamental equilibrium. While Zeeman's interest is in the application of concepts from catastrophe theory (demonstrating the possibility of sudden stock market

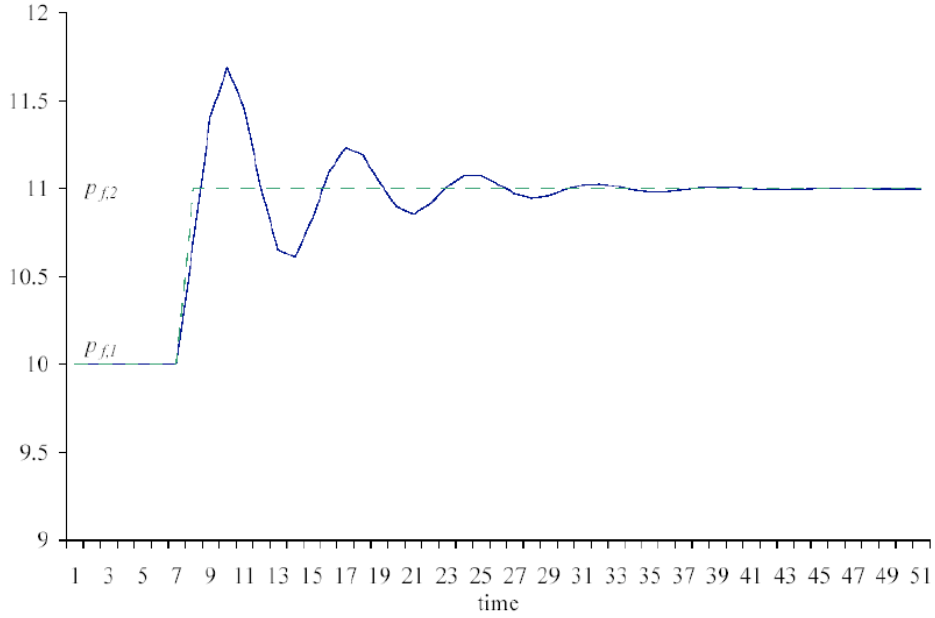

Figure 4: Overshooting and mean reversion of market prices after arrival of new information. Simulation results with $a = 0.7$, $b = 0.8$, $c = 0.9$, $\lambda = 1$, $P_{f,1} = 10$ and $P_{f,2} = 11$.

crashes), Chiarella (1992) showed that the Beja/Goldman model would generate periodic oscillations around the fundamental equilibrium in the unstable case if chartists' excess demand function gets sufficiently flat far from the equilibrium price. Day and Huang (1990) consider another similar model formulated in discrete time whose 'information traders' (equivalent to the above fundamentalists) trade the more aggressively the farther the market price is from the fundamental value. With a strong reaction of chartists destabilizing the fundamental equilibrium, the assumed nonlinear reactions result in the same combination of centripetal and centrifugal forces like in Zeeman (1974) and Chiarella (1992): strong reaction of chartists prevents convergence to the fundamental equilibrium and generates bubble episodes of overvaluation or undervaluation of the asset. However, once the deviation of the market price from P_f becomes too large, either the chartists become more cautious or the fundamentalists step in more aggressively so that the price process does not diverge endlessly but rather reaches a turning point at which the attraction toward the fundamental value dominates over the positive feedback effect. The global dynamics is, therefore, *bounded* but not

asymptotically stable. It does not converge to its (unique) equilibrium, but also does not exhibit unbounded deviations from the equilibrium.

While Chiarella (1992) in a nonlinear version of the above setting in continuous time ends up with a closed orbit with constant amplitude, Day and Huang get an even more exciting outcome: depending on parameter values, the market could exhibit *chaotic* fluctuations.[6] Despite the deterministic excess demand functions and price formation rule the price trajectories then appear like the realization of a stochastic process with random switches between bear and bull markets. The difference in outcomes is mainly due to the mathematical formalization of the speculative dynamics: while systems of differential equations are capable of generating chaos only if they consist of at least three first-order equations (Beja/Goldman and Chiarella only have two equations), even difference equations of first order can generate chaotic attractors. The erratic appearance of price paths from a deterministic system and the lack of predictability of chaotic systems provided a new avenue towards an explanation of the stylized facts: despite deterministic behavioral sources, the systematic forces of the market interactions could become 'invisible' due to the apparent randomness of the chaotic dynamics. A similar avenue is pursued within a model of the foreign exchange market by DeGrauwe *et al.* (1993). Assuming simple versions of moving average rules applied by chartists, they end up with a higher-order system of difference equations that yields chaotic attractors for a broad range of parameter values. Interestingly, their model also allows to explain stylized facts of foreign exchange markets other than merely the deviation between market exchange rates and their fundamental value. In particular, they demonstrate that their chaotic process is hard to distinguish from a unit root process (martingale) by standard statistical tests and that the overall dynamics could explain the forward premium puzzle (the finding that forward rates are poor and biased predictors of subsequent exchange rate movements). Experiments with a macroeconomic news arrival process in-

[6] Gu (1995) analyzed market mediating behavior of an active market maker in the framework of Day and Huang demonstrating that it would be in the interest of this agent to churn the market rather than calming it down, i.e. choose a price adjustment speed in the chaotic zone of parameter values.

dicate that while this incoming information is incorporated into exchange rates over longer horizons, there is no one-to-one mapping between exchange rate changes and macroeconomic news in the short-run. The connection between the currency movements and macroeconomic factors might, then, at times appear quite loose, explaining the so-called 'disconnect' puzzle and the failure of macroeconomic models to predict exchange rates.

A closely related recent branch of literature is that on "adaptive belief systems". In contrast to the contributions reviewed above, this class of models allows agents to switch between different prediction functions (mostly chosen from the typical chartist and fundamentalist varieties). Thus, the fractions of agents using particular predictors become additional state variables in addition to the market price. Early work in this vein was mainly concerned with the possible bifurcation routes towards chaotic attractors in these systems (Brock and Hommes, 1997, 1998). Various extensions have considered a broad variety of prediction functions, have allowed for transaction costs, and have studied the endogenous development of wealth of agent groups as an alternative to switches due to the success or failure of their predictions (Gaunersdorfer, 2000; Chiarella, Dieci and Gardini, 2002, Chiarella and He, 2002, Brock, Hommes and Wagener, 2005, DeGrauwe and Grimaldi, 2006). A recent paper by Gaunersdorfer and Hommes, 2007, is concerned with a possible mechanism for volatility clustering within this framework. They demonstrate that in a scenario with coexistence of a locally stable fixed point and an additional cycle or chaotic attractor superimposed stochastic disturbances would lead to recurrent switches between both attractors. In the vicinity of the fixed point, fluctuations will be confined to the stochastic disturbance, while the endogenous dynamics of the cycle or chaotic attractors will magnify the stochastic fluctuations. As a consequence, switching between both attractors will come along with the impression of volatility clustering and, therefore, could provide a possible explanation of this stylized fact. Gaunersdorfer and Hommes show that estimation of GARCH parameters produces numbers close to those of empirical data for some parameterizations of the model. In a related framework, He and Li (2007) point out that an appropriate combination of noise factors in an otherwise deterministic chartist-fundamentalist model (both a stochastic fundamental value and an additional noise component in aggregate excess demand are

assumed) could lead to absence of autocorrelation in raw returns together with apparent hyperbolic decay of autocorrelations in squared and absolute returns.

The adaptive belief models have a close resemblance to a class of models using machine-learning tools for agents' expectation formation. The prototype of this strand of literature on artificial financial markets is the Santa Fe artificial stock market (Arthur *et al.*, 1997, LeBaron et al, 1999) that had already been launched in the early nineties. In this model, traders are equipped with a set of classifiers of chartist and fundamentalist type to categorize the configuration of the market and formulate expectations of future returns based on this classification. Both classifiers and forecast parameters evolve via genetic operations. The main finding of this project is that dominance of either chartist of fundamentalist components depend on the frequency of activation of the genetic operations. Under frequent activation, chartist behavior was found to dominate while with a lower frequency of activation, fundamentalist classifiers gained in importance. Imposing 'short-termism' on the artificial agents, they are apparently forced to focus on trends rather than on, for example, price-to-dividend ratios. It has also been shown that the chartist regime had higher volatility than the fundamentalist regime, and the later exhibited excess kurtosis as well as positive correlation between volume and volatility.

Other recent artificial markets include Chen and Yeh (2002) whose traders are equipped with genetic programming tools rather than classifiers systems. Simpler models with artificial agents have used genetic algorithms for parameter selection of trading strategies (Arifovic, 1996, Dawid, 1999, Szpiro, 1997, Lux and Schornstein, 2005 and Georges, 2006). Due to the inherent stochasticity of the evolutionary learning mechanism, some of these models are closer in spirit to the stochastic models discussed below than to the deterministic approaches of the early 'chaos' literature.

One concern on the body of literature on chartist-fundamentalist models with a deterministic structure is the lack of convincing evidence of chaotic dynamics in financial markets. While there had been some hope of detecting low-dimensional deterministic chaos in financial returns in the early literature on this subject (Eckmann *et al.* 1988, Scheinkmann and LeBaron, 1988) it soon turned out that the daily data sets used in these studies were

too small for reliable estimation of, for example, the correlation dimension of a chaotic attractor (Ruelle, 1990). The consensus that emerged from this body of literature was that the empirical evidence for low-dimensional chaos is weak. One should also note that despite their sensitivity with respect to initial conditions, low-dimensional attractors are characterized by recurrent patterns that could probably be exploited too easily by advanced methods from the toolbox of nonlinear dynamics. Nevertheless, the literature agrees that there is strong evidence for nonlinearity in that all standard tests for IID-ness would typically reject their null hypotheses when applied to financial returns. However, this nonlinear dependency is mostly confined to higher moments (GARCH effects) and it might not be uniformly present in the data. As an interesting exercise by de Lima (1998) demonstrates, rejection of IID-ness by the popular BDS test in S&P 500 returns over the eighties happens only if one uses data including the crash of 87. If the series stops before this event, the data, in fact, look like white noise and would not reject the null hypothesis. These findings indicate that financial data have a structure that is even more complex than that of chaotic processes. It might, therefore, be important to allow for both deterministic and stochastic factors whose interaction could give rise to different behavior in different time windows.

4.2 Kirman's Model of Opinion Formation and Speculation

While a few attempts at modeling stochastic economies of interacting agents have been published decades ago (most notably Föllmer's seminal 1974 paper), a more systematic analysis of stochastic interactions only started in the nineties and was largely confined to models of trading in financial markets. The first study that gained wider prominence within the economics literature is Kirmans's (1991, 1993) model of herding through pair-wise contacts. Its mechanism of contagion of opinions - which in principle could be applied to a variety of problems in economics and beyond - has also been used as an ingredient in models of interacting chartists and fundamentalists and serves to highlight the differences in results brought about by an intrinsically stochastic rather than deterministic framework.

We start with the basic stochastic interactions considered in Kirman's approach. The motivation for Kirmans's model stems from experiments on information transmission among ants. If a group of foraging ants is offered two identical sources of food in the vicinity of their nest, a majority of the population will be found to exploit one of both resources at any point in time. This concentration comes about by chemical information transmission via pheromones by which successful pioneer ants recruit followers and guide them to the same manger. The higher concentration of pheromone on one of the two paths to both food sources stimulates more and more ants to exploit the same source. However, if experiments last long enough, random switches are observed in the preferred source and, averaging over time, a bimodal distribution is found for the number of ants collecting food from one source. The switch is believed to be caused by evaporation of pheromones together with random search of ants not yet recruited for the exploitation of one resource. This combination of concentrated exploitation and random search is often viewed as an evolved optimal foraging strategy that achieves a balance between the costs and benefits of undirected search and exploitation of known resources (cf. Deneubourg *et al.* 1990). It also counts as one of the leading examples of natural optimization and has motivated together with similar findings of seemingly purposeful self-organized behavior in insect societies the new brand of "ant algorithms" in the artificial intelligence literature (Bonabeau, Dorigo and Theraulaz, 1999).

Kirman (1993) has come up with a stochastic model of this recruitment process of foraging ants. Following the experimental setup, he assumes ants (agents) have two alternatives at their disposal (which might be food sources, opinions, or strategies like chartism and fundamentalism). Each individual is assumed to adhere to one of both alternatives at any point in time. There exists a fixed number of N agents, k of whom are currently subscribing to alternative 1. Hence, the probability that a randomly chosen agent belongs to group 1 (2) is $\frac{k}{N}$ and $\frac{N-k}{N}$, respectively.

The state of the system, then changes over time by a combination of recruitment and random changes (random search):

1. individuals meet pairwise and exchange information on their respective strategies or opinions. From these meetings, any agent might come out convinced or persuaded that the choice of the other is more

preferable. This happens with a constant probability $1-\delta$ (δ denoting the probability of holding on to one's own former strategy or opinion).

2. individuals can also change their opinion or strategy without meeting others in an autonomous fashion, say due to idiosyncratic factors. This random change happens with a probability ϵ.

Within a small time interval (small enough to allow for at most one pairwise encounter) the number k of individuals of type 1 can, consequently, undergo the following changes:

$$k \rightarrow \begin{cases} k+1 & \text{with probability} \quad p_1, \\ k & \text{with probability} \quad 1-p_1-p_2, \\ k-1 & \text{with probability} \quad p_2. \end{cases} \tag{15}$$

Both probabilities in eq. (15) are determined by simple combinatorial considerations:

$$p_1 = \text{Prob}((k \rightarrow k+1) \quad = \quad \frac{N-k}{N}(\epsilon + (1-\delta)\frac{k}{N-1}) \tag{16}$$

$$p_2 = \text{Prob}(k \rightarrow k-1) \quad = \quad \frac{k}{N}(\epsilon + (1-\delta)\frac{N-k}{N-1}) \tag{17}$$

The resulting stochastic process converges to a limiting distribution which for large N and small ϵ can be approximated by the symmetric Beta distribution:

$$f(x) \quad = \quad \text{const.} x^{\alpha-1}(1-x)^{\alpha-1} \tag{18}$$

where $x \equiv \frac{k}{N}$ and the shape parameter $\alpha \equiv \frac{\epsilon(N-1)}{1-\delta}$ depends on the relative strength of the autonomous component ϵ and the recruitment probability $1-\delta$. The equilibrium distribution may have a unimodal or bimodal shape: for $\epsilon > \frac{1-\delta}{N-1}$ the distribution will be unimodal with a concentration of probability mass around the mean value $\frac{k}{2}$. If we increase the herding propensity, however, the population dynamics will undergo what is denoted a "phase transition" at $\epsilon = \frac{1-\delta}{N-1}$ with the equilibrium distribution changing from unimodal to bimodal. Fig. 5 illustrates the different possibilities for the distribution of opinions together with examples of stochastic simulations for the uni-modal and bi-modal case. Note that in the bi-modal case, the mean value of the equilibrium distribution is still $\frac{k}{2}$ but it is the *least probable* realization to be observed in a simulated time series. Probability

mass is rather concentrated at the extreme ends which means that a majority of agents will follow most of the time one of both alternatives. This also means that although all agents are governed by the same conditional probabilistic laws, the emergent global configuration may be inhomogeneous with alternating phases of dominance of one or the other strategy. If we assume that the two alternatives in question are chartist and fundamentalist strategies, we would observe waves of popularity of one or the other alternative among traders due to non-economic forces (Frankel and Froot, 1986, and Liu, 1996, offer some evidence for changes in population of both types of strategies in foreign exchange markets). From a certain perspective, such an added non-economic explanation for the popularity of chartist and fundamentalist trading strategies could have some appeal: in an efficient market, both alternatives would be inferior to a single buy-and-hold strategy so that some reasons outside the realm of economics would be needed to explain their perseverance and popularity among traders.[7] However, these non-economic forces would lead to dependence between agents. The lack of independence of individuals' deviations from rational behavior would prevent applicability of a suitable law of large numbers. As a consequence, irrationality would not be washed out in the aggregate but would exert a non-negligible influence on equilibrium prices.

Kirman (1991) had already incorporated his recruitment mechanism into a chartist-fundamentalist framework. His model is formulated as a monetary model of the foreign exchange market. This implies that the equilibrium exchange rate is determined by the uncovered interest parity condition (UIP). Summarizing the fundamental macroeconomic factors influencing the domestic and foreign interest rates via a compound contemporaneous macro variable x_t, the equilibrium log exchange rate is obtained as:

$$S_t = x_t + \delta E_{m,t}[S_{t+1}] \tag{19}$$

[7] Kaldor (1939) already argued that there might be a representation bias that might lead to a steady inflow of new speculators even in the presence of the net losses of the population of existing speculators as a whole: "... even if speculation as a whole is attended by a net loss, rather than a net gain, this will not prove, even in the long run, self-corrective. For the losses of a floating population of unsuccessful speculators will be sufficient to entertain permanently a small body of successful speculators; and the existence of this body of successful speculators will be a sufficient attraction to secure a permanent supply of this floating population." (p.2).

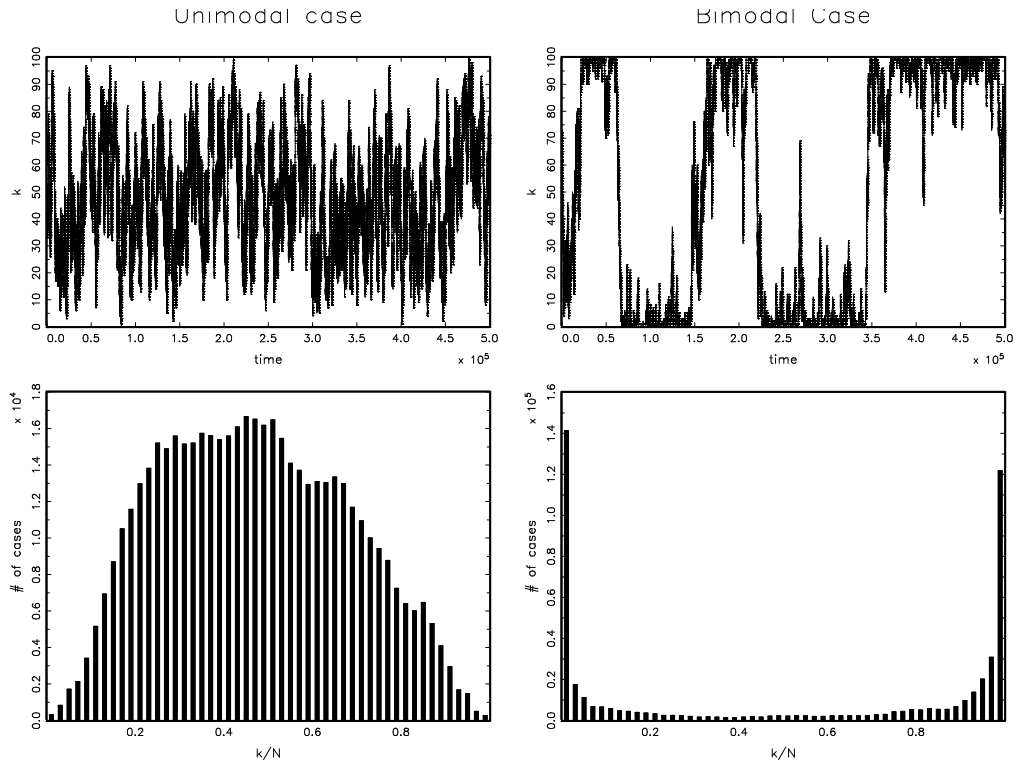

Figure 5: The two possible scenarios of Kirman's ant model: The population might fluctuate around the mean value $k/2$ with an equal number of agents in both groups (left-hand side) or it might tend towards a uniform state $k = 0$ or $k = N$ with intermittent switches between both polar cases due to the randomness of the recruitment process (right-hand side). Parameters are $\delta = 0.15, N = 100$ and $\epsilon = 0.02$ and 0.002, respectively. The upper panels show simulations of both cases, the lower panels exhibit the frequency of observations $\frac{k}{N}$ in these simulations.

where δ is the discount factor and $E_{m,t}[S_{t+1}]$ is the market-wide expectation of the future exchange rate entering via UIP.

The traditional monetary approach would, of course, assume rational expectation formation. Any current information on future changes of macroeconomic fundamentals (components of $x_\tau, \tau = t+1, t+2, ...$) would, then, be incorporated into current spot rates via their correctly predicted influence on future equilibrium exchange rates. Upon iterative solution of (19), the fundamental value $S_{f,t}$ would be obtained. Analogously to Beja and Goldman and the related literature on speculative models of asset price formation, rational expectations are replaced in Kirman's model by the non-rational expectations of chartists and fundamentalists. Kirman assumes the standard format of fundamentalists' expectations:

$$E_{f,t}[\Delta S_{t+1}] = a(S_{f,t} - S_{t-1}) \tag{20}$$

with $\Delta S_{t+1} = S_{t+1} - S_t$ and a simple trend following rule for chartists. In our representation of chartists' expectation function we slightly modify Kirman's original set-up:

$$E_{c,t}[\Delta S_{t+1}] = b(S_{t-1} - S_{t-2}). \tag{21}$$

The number of agents formulating their expectations in one or the other way is assumed to change under the influence of the stochastic recruitment process so that the aggregate forecast of the exchange rate, $E_m[S_{t+1}]$, is given as a weighted average whose weights change stochastically with the group occupation numbers:

$$\begin{aligned} E_{m,t}[S_{t+1}] &= E_{m,t}[\Delta S_{t+1}] + S_t \\ &= S_t + w_t E_{f,t}[\Delta S_{t+1}] + (1 - w_t) E_{c,t}[\Delta S_{t+1}]. \end{aligned} \tag{22}$$

Setting $w_t = \frac{k_t}{N}$, we obtain:

$$E_{m,t}[S_{t+1}] = S_t + \frac{k}{N} E_{f,t}[\Delta S_{t+1}] + \frac{N-k}{N} E_{c,t}[\Delta S_{t+1}]. \tag{23}$$

In Kirman's simulations of this model, weights are not exactly identical to group occupation numbers, but are given by agents' assessment of what the

majority opinion might be. For this purpose, every agent is assumed to receive a noisy signal of the majority opinion. Assuming that agents follow this perceived majority, the aggregate of these signals is, then, used instead of the raw outcome from the population model, k_t. In addition, the social dynamics occurs at a faster time scale than price formation in the foreign exchange market. In particular, in the results reported in various papers (Kirman, 1991, 1992; Kirman and Teyssière, 2002), the group occupation numbers are sampled after 10,000 pairwise encounters in order to implement the weights in the unit time steps of eq. (23). However, all these refinements are of minor importance. The more important insight is that we end up with a complex dynamic system in which the process of social interactions between agents exerts a crucial influence on the relatively conventional (in the light of the previous sec. 4.1) speculative process component. To see this, plug (21) and (20) into (19):

$$S_t = x_t + w_t a(S_{f,t} - S_{t-1}) + (1 - w_t)b(S_{t-1} - S_{t-2}). \tag{24}$$

For constant fractions of chartists and fundamentalists, this is only slightly different from the model of Beja and Goldman. Despite the formulation in the tradition of monetary models of the exchange rate, the different formalization of chartists' expectations and the discrete rather than continuous-time framework, we easily recover the stabilizing and destabilizing tendencies of both groups. In particular, for the extreme cases $w_t = 1$ and $w_t = 0$, we immediately see that the system would be unconditionally unstable if all traders adopted the chartist forecast rule ($w_t = 0$). In the case of complete dominance of fundamentalists, we would find stability of the fundamental equilibrium in the case $a < 1$ (as with $a > 1$ fundamentalists would overreact to a discrepancy between the current price and the fundamental value). Furthermore, for the system of two interacting groups, stability conditions can be expressed in terms of group occupation numbers. For a constant $w_t = \bar{w}$, this second-order difference equation would have an asymptotically stable equilibrium if the following conditions were satisfied:

(i) $\bar{w} > 1 - \frac{1}{b}$

(ii) $\bar{w} < \frac{2b+1}{2b+a}$

Note that the second condition is always met if fundamentalists' reaction is not excessive ($a < 1$). The autonomous recruitment dynamics can be

seen as a driving factor that sweeps the speculative dynamics back and forth between stable and unstable configurations. As can be inferred from a typical simulation, the stochastic fluctuations of the prevailing majorities lead to different characteristic phases in the market's dynamics. During phases dominated by fundamentalists, the exchange rate stays close to its fundamental value while speculative bubbles emerge if the majority turns to the chartist forecast rule. Bubbles collapse together with the stochastic switches from the chartist majority back to the fundamental majority. Kirman (1992) shows that standard tests would mostly not reject the unit root hypothesis for simulated time series while Kirman and Teyssière (2002) testing for long-term dependence in absolute and squared returns find robust indication of long-term dependence with decay parameters in the range of those obtained with empirical data. Although we do not get the full set of stylized facts reviewed in sec. 2, the sweeping through a bifurcation value (threshold for a qualitative change of the dynamics) due to superimposed stochastic forces is a more general phenomenon that also occurs in other models of interacting agents. As we will see below, in a slightly different framework, it appears to be a potential key mechanism generating fat tails and clustered volatility.

Alfarano, Lux and Wagner (2007) study a continuous-time version of the 'ant process'. In their model, the speculative dynamics is closer to the Beja and Goldman legacy. They assume constant fractions of fundamentalists and chartists, but have the number of buyers and sellers among chartists being determined by the social interaction dynamics above. Using tools from statistical physics, they derive approximate closed-form solutions for conditional and unconditional moments of returns of their asset price process. They show that leptokurtosis and volatility persistence are generic features of this model, although neither the unconditional distribution nor the autocorrelations exhibit 'true' power-law decay. However, Alfarano and Lux (2007) show in a closely related model that up to a characteristic time scale the temporal characteristics would closely resemble that of a process with 'true' long memory and the deviation from 'true' asymptotic scaling behavior could only be detected for very long (simulated) time series. Gilli and Winker (2003) have estimated via a heuristic grid search method Kirman's (1991) model for the U.S.$-DEM exchange rate and have obtained parame-

ter estimates within the bi-modal regime. Alfarano, Lux and Wagner (2005) developed an approximate ML approach for a similar model that generalizes the previous framework by allowing for asymmetric autonomous transition rates between groups. This added feature leads to arbitrary asymmetries in the limiting distribution of the population configuration depending on parameter values which translates into subtle asymmetries in the distribution of returns. Since a higher autonomous tendency towards one group leads to a certain dominance of one strategy, the empirical parameter estimates provide some evidence on the average population composition in the market under investigation. As it turns out, parameter estimates of this asymmetric ant model indicate that stock markets mostly have a higher fraction of chartists than foreign exchange markets.

4.3 Beyond Local Interactions: Socio-Economic Group Dynamics in Financial Markets

4.3.1 Social Interactions: A General Framework

Pairwise interactions as they appear in the seminal ant model are just one way to formalize the interpersonal influences between economic agents. The first systematic investigation into the effects of mutual non-economic interaction between agents in an economic setting is due to Föllmer (1974). He studied the existence and uniqueness of the equilibria in a system of markets and pointed to the existence of a *phase transition* from a unique prize vector to a "polarized" multi-modal state with increasing strength of interpersonal spillovers. This phenomenon is similar to the bifurcation from uni-modality to bi-modality in the ant model and can be found in various broadly similar approaches.

In the following section, we outline another model in the chartist-fundamentalist tradition that uses a formalization of interactions that can be viewed as the opposite extreme to pairwise influences: traders will be assumed to be influenced by the overall mood of the market, i.e. an average of the influence from all their fellow traders. Such an overall influence allows to study the macroscopic dynamics more easily via so-called mean-field approximations. As it will be seen most qualitative results of the simple

model that follows are in harmony with the findings reported above. The added advantage (besides the generalization of previous results) will be that this framework allows to illustrate a general avenue towards the analysis of macroscopic quantities in stochastic systems of interacting agents that can be compared to typical stochastic models applied in empirical finance. Our particular framework is adopted from Lux (1995). As in Kirman's model, a population is divided into two camps, say, optimistic and pessimistic (or: bullish and bearish) individuals, whose average mood can be captured by the opinion index x:

$$x = \frac{n_+ - n_-}{2N} \tag{25}$$

with $n_+(n_-)$ the current number of optimists (pessimists) and $2N$ the overall number of agents. Individuals are assumed to revisit their choice of opinion from time to time and to have a tendency to switch to the majority opinion. With an overall 'field' effect (i.e., all other individuals exerting the same influence on any one), the group pressure can be modeled via some feedback effect from the macroscopic configuration x on individual decisions. This feedback leads to migration of individuals between both groups under the influence of the overall 'field' of the average opinion. Formally, these transitions might be specified by Poisson processes in continuous time with rates p_{+-} and p_{-+} for an individual from the "-" group to switch to the "+" group and *vice versa*. The canonical function used for transitions in particle physics is the exponential which motivates an *ansatz* of the following type:

$$p_{+-} = v \cdot \exp(\alpha x); p_{-+} = v \cdot \exp(-\alpha x). \tag{26}$$

Obviously, eq. (26) supposes positive probabilities for agents to migrate between groups, but hypothesizes a stronger tendency for migration following the dominating opinion: if $x > 0$ ($x < 0$), the majority of the population can be found in the "+" ("-") group and the probability for other agents to join this group is larger than that of members of the majority to switch to the minority view. We, thus, have a very direct formalization of social interaction or herding among the members of our population. Note that eq. (26) includes two parameters: v which captures the general frequency of revision of opinion within our population and α which parameterizes the strength of the herding effect. Because of the assumed Poisson nature of

the switches of opinion, we can easily come up with probabilities for movements of agents from one group to another during a certain time intervalΔt. For small time increments Δt the simultaneous movements of two or more individuals during an interval Δt become increasingly unlikely and can be completely neglected in the limit $\Delta t \to 0$. In addition, the probability for an individual to switch from one group to the other converges to$p_{+-}\Delta t$ and $p_{-+}\Delta t$. Since these Poisson processes are assumed to be the same for all members of the optimistic and pessimistic group, respectively, we can infer the transition rates for group occupation numbers as the limiting cases of conditional probabilities $w(n_i + 1, \Delta t|n_i, t)$ for $i \in \{\text{“}+\text{”}, \text{“}-\text{”}\}$:

$$\lim_{\Delta t \to 0} \frac{w(n_+ + 1, t + \Delta t|n_+, t)}{\Delta t} \equiv w(n_+ + 1|n_+, t) = n_- p_{+-}, \tag{27}$$

$$\lim_{\Delta t \to 0} \frac{w(n_- + 1, t + \Delta t|n_-, t)}{\Delta t} \equiv w(n_- + 1|n_-, t) = n_+ p_{-+}. \tag{28}$$

Denoting by $n = \frac{1}{2}(n_+ - n_-) = xN$ the *socio-economic* configuration with $n \in \{-N, -N+1, \ldots, N-1, N\}$, and $x \in \{-1, -1+\frac{1}{N}, \ldots, 0, \ldots, \frac{N-1}{N}, 1\}$, the opinion dynamics leads to a sequence of switches from n to one of the neighboring values $n \pm 1$ (or from x to $x \pm \frac{1}{N}$) in irregularly spaced time intervals. A complete description of the dynamic process is obtained via the so-called Master equation which captures the change in time of the probabilities $\tilde{Q}(n, t)$ or $Q(x, t)$ over all candidate states n or x, respectively. This amounts to a system of differential equations for the probability flux which in our case can be written as:

$$\frac{dQ(x;t)}{dt} = \underbrace{w_\downarrow(x + \frac{1}{N})P(x + \frac{1}{N};t) + w_\uparrow(x - \frac{1}{N})P(x - \frac{1}{N};t)}_{\textit{inflow of prob. to state } x} \tag{29}$$

$$- \underbrace{(w_\downarrow(x) + w_\uparrow(x))P(x;t)}_{\textit{outflow of prob. from } x}.$$

The transition rates for the changes of x by one unit $\pm\frac{1}{N}$ are identical to the rates introduced in eq. (27) and (28) translated into the pertinent transition rates of the intensity x:

$$w_\uparrow(x) = n_- p_{+-} = (1 - x)Np_{+-}, w_\downarrow(x) = n_+ p_{-+} = (1 + x)Np_{-+}. \tag{30}$$

Note that the rates $w_{\uparrow}(x)$ and $w_{\downarrow}(x)$ are both state dependent and non-linear due to our formalization of the individual transition rates p_{+-} and p_{-+}. While one could in principle use the Master equation in order to simulate the time development of our dynamic process, it is certainly too complicated to allow an analytical solution. The major advantage of this formalization consists, however, in its use as a starting point to derive more manageable approximations. One potential avenue consists in performing a Taylor series approximation to the Master equation itself leading to the so-called *Fokker-Planck equation* whose use is illustrated in Fig. 7. A second complementary approach is to investigate macroscopic characteristics of the dynamic process, e.g. first, second or higher moments, whose implementation also requires the Master equation formalism. The details of both approaches have been nicely laid out in the monographs Weidlich and Haag (1983), Aoki (1996) and Weidlich (2002). We will not go into too much detail here but simply illustrate some of the main results that can be obtained for our opinion dynamics.

Let us start with the first moment of the opinion index, i.e. $\overline{x_t}$, whose time change characterizes the most probable development of the system conditional on the initial condition x_0 and time $t = 0$. Since the mean is defined by:

$$\overline{x_t} = \sum_{x=-1}^{1} xQ(x;t), \tag{31}$$

its change in time can be computed exactly only under complete knowledge of the dynamics of the probability distribution over all states x:

$$\frac{d\overline{x_t}}{dt} = \sum_{x=-1}^{1} x\frac{dQ(x;t)}{dt}. \tag{32}$$

The exact time evolution covered in eq. (32) can be approximated in a Taylor series expansion around the current mean $\overline{x_t}$ to various degrees of accuracy. To first order, we obtain a self-consistent differential equation[8]:

$$\frac{d\overline{x_t}}{dt} = a_{x,1}(\overline{x_t}), \tag{33}$$

[8]Note that the first derivative vanishes because of $E[(x - \bar{x})a'_{x,1}(\overline{x})] = 0$

while to second-order accuracy, a correction term involving the varianceσ_x^2 enters the equation:

$$\frac{d\overline{x_t}}{dt} = a_{x,1}(\overline{x}) + \frac{1}{2}\sigma_x^2 a''_{x,1}(\overline{x}). \tag{34}$$

The function $a_{x,1}$ in eqs. (33) and (34) is denoted the *first jump moment*. It gives the expected change of the system conditional on the previous realization. Evaluated at the current expectation, $\overline{x}_t$ (conditional on some critical condition) it allows to track the mean-value dynamics of the system. In the case of an infinite population, (33) would be exact. For finite populations, however, the influence of higher moments has to be taken into account. Eq. (34) includes the next higher term in the Taylor series expression of (32) involving the second moment.

Higher-order expansions would involve higher-order moments in the additional entries on the right-hand side of the equation. An example for the determination of the jump moment $a_{x,1}$ can be found below.

The dynamics of higher moments are obtained analogously. For example, the second moment $x_t^2 = \sum_x x^2 Q(x;t)$ changes over time according to:

$$\frac{d}{dt}\overline{x_t^2} = \sum_x x^2 \frac{dQ(x;t)}{dt}, \tag{35}$$

while the time change of the variance σ_x^2 is given by:

$$\frac{d}{dt}\sigma_x^2 = \frac{d}{dt}(\overline{x^2} - \overline{x}^2) = \frac{d}{dt}\overline{x^2} - 2\overline{x}\frac{d}{dt}\overline{x} \tag{36}$$

and can be solved using eqs. (34) and (31). Taylor series expansions of these exact equations again lead to approximations of various orders of accuracy involving non-linear functions of various moments on the right-hand side. One immediate consequence is that any model involving a group dynamics like the one under investigation (or a broad range of alternative population processes) entails *autoregressive dependence in higher moments*, as well as *cross-dependencies between moments*. It has already been pointed out by Braglia (1990) and Ramsey (1996) that stochastic systems based on microscopic interactions (like our illustrative example) provide a generic avenue towards interesting non-trivial dynamics in higher moments and, therefore,

a potential behavioral explanation for the ubiquitous ARCH effects in financial data. Of course, it would have to be seen whether the direction and scope of autoregressive dependency in any hypothesized micro model is in qualitative and quantitative agreement with the empirical stylized facts. Interestingly, Braglia (1990) already argued that it would be natural to interpret a cross-dependence between the mean and second moment as a fads effect. Lux (1998) provides a fully worked out analysis of the interactions between first and second moments in an asset pricing model with non-rational speculators along the lines of our present framework.

Let us return to our particular model of social imitation. Implementing eq. (33), the exact mean-value dynamics turns out to be determined by the average change of the configuration, $a_{x,1}$:

$$\frac{d\overline{x_t}}{dt} = \sum_x \sum_{x'} (x' - x) w_{x'x} Q(x,t) = \sum_x a_{x,1} Q(x,t). \tag{37}$$

Since possible movements within an infinitesimal time step are restricted to neighboring states, $x' - x$ can only assume values $\frac{1}{N}$ and $-\frac{1}{N}$, so that:

$$a_{x,1} = \frac{1}{N} w_\uparrow(x) + (-\frac{1}{N}) w_\downarrow(x). \tag{38}$$

Since $w_\uparrow$ and $w_\downarrow(x)$ are identical to $w_\uparrow(n)$ and $w_\downarrow(n)$ in eq.(30) we end up with

$$\begin{aligned} a_{x,1} &= \frac{1}{N} n_- p_{+-} - \frac{1}{N} n_+ p_{-+} \\ &= (1-x) v e^{\alpha x} - (1-x) v e^{-\alpha x} \end{aligned} \tag{39}$$

Using the hyperbolic trigonometric functions this can be rewritten as:

$$a_{x,1} = 2v\{\tanh(\alpha x) - x\} \cosh(\alpha x). \tag{40}$$

The exact mean value equation is thus:

$$\frac{d\bar{x}_t}{dt} = \sum_x 2v\{\tanh(\alpha x) - x\} \cosh(\alpha x) P(x;t). \tag{41}$$

Its first-order Taylor series approximation around $\bar{x}_t$ leads to the self-consistent differential equation:

$$\frac{d\bar{x}_t}{dt} = 2v\{\tanh(\alpha\bar{x}) - \bar{x}\}\cosh(\alpha\bar{x}), \tag{42}$$

while the second-order approximation involves a correction factor due to fluctuations around the mean:

$$\frac{d\bar{x}_t}{dt} = 2v\{\tanh(\alpha\bar{x}) - \bar{x}\}\cosh(\alpha\bar{x}) +$$

$$v\{(\alpha^2 - 2\alpha)\sinh(\alpha\bar{x}) - \bar{x}\alpha\cosh(\alpha\bar{x}))\sigma_x^2 \tag{43}$$

which already reveals a rich nonlinear structure of interactions between first and second moments. Note that σ_x^2 is time-changing as well. Approximating the dynamic law (36) for σ_x^2 to first-order one would arrive at an equation that also depends on both the first and second moments,$\bar{x}_t$ and σ_x^2. Combining the second-order approximations of the first moment and the first-order approximation of the second moment would, thus, lead to a self-consistent system of two (highly nonlinear) first-order differential equations.

We proceed by investigating the properties of the first-order approximation, eq. (42). Since $\cosh(.) > 0$ for all x, the condition for a steady state of the mean value dynamics is:

$$\frac{d\bar{x}_t}{dt} = 0 \Leftrightarrow \bar{x^*} = \tanh(\alpha\bar{x^*}). \tag{44}$$

Since $\tanh(.)$ is bounded between -1 and 1 and its local slope at 0 is equal to 1, we arrive at the following insights concerning the equilibria of the system:

- $\alpha \leq 1$ implies existence of a stable, unique equilibrium $\overline{x_0^*} = 0$,
- $\alpha > 1$ gives rise to multiple equilibria, $\bar{x_-^*}, \bar{x_0^*}, \bar{x_t^*}$ with $\bar{x_t^*} = -\tilde{x_-^*} > 0$, of which the outer ones are stable and the middle one, $\bar{x_0^*}$, is unstable, cf. Fig. 6.

The bifurcation from a unique steady state to multiple steady states shows that the interaction intensity needs to surpass a certain critical value for a 'polarized' state to emerge. If interaction is weak ($\alpha \leq 1$) the system would fluctuate around a balanced state with, on average and in expectation, as

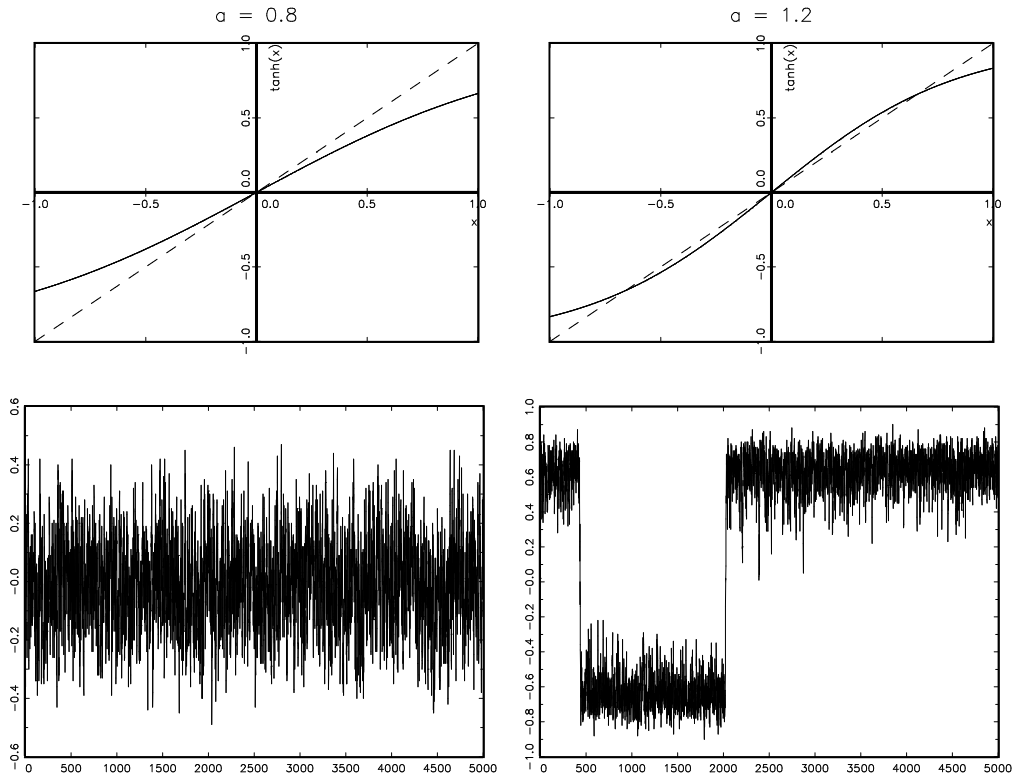

Figure 6: Two cases of the social dynamics with mean-field effect. If the interaction intensity is weak ($\alpha = 0.8$), minor fluctuations around a balanced disposition among the population occur, while a majority of "+" or "-" agents emerges with strong interactions (e.g. for $\alpha = 1.2$). The bifurcation is similar to the one in Kirman's model, but there is no tendency to totally uniform behavior.

many optimistic as pessimistic agents. Beyond the critical value ($\alpha > 1$), however, a snow-ball like process of infection would result in the emergence of either a majority of "+" or "-" agents. Note that the level of the majority $\overline{x^*_\pm}$ depends on the intensity α (cf. Fig. 6). Multiplicity of equilibria of the mean-value dynamics corresponds to bi-modality of the stationary distribution. The steady states $\overline{x^*_\pm}$ correspond to the two modes of the distribution while the unstable steady state $\overline{x^*_0}$ is identical to the anti-mode, i.e. the local minimum of the stationary distribution. In the bi-modal case, the dynamics of the probability distribution would switch from a concentration around the initial state to a bimodal shape according to the two equally likely paths the system could take in the medium and long run. Fig. 7 shows an example of such a transient density simulated via numerical integration of an approximation to the Master equation (the so-called Fokker-Planck equation) equation. In contrast to the complete characterization of the stochastic process via its transient density, the mean-value dynamics would 'only' indicate the most likely path leading to the nearby mode $\bar{x^*_+}$ or $\bar{x^*_-}$. This quasi-deterministic approximation would, therefore neglect possible recurrent switches between $\bar{x^*_+}$ and $\bar{x^*_-}$ due to stochastic fluctuations.

If we interpret our social dynamics as a formalization of a 'fads' process in our asset market, the agents could be viewed as noise traders switching between bullish and bearish disposition. How much this fads component influences the asset price, would, then, depend on the intensity of interaction: with α small, no dominating majority opinion would emerge among the noise traders and their influence would be minor because the irrational influences would cancel out each other in the aggregate (in fact, if both the optimistic and pessimistic noise traders would have the same order volume, average excess demand of the irrational agents would be equal to zero, see below). If, on the contrary, interaction is strong, one would observe more coherence among the noise traders' activities leading to a dominance of either optimists or pessimists at any point in time. The resulting dominance of either buyers of sellers would presumably lead to price changes away from some rationally determined fundamental value. The next section concretizes these thoughts by embedding our model of social imitation into a simple asset pricing model along the lines of Beja and Goldman (1980).

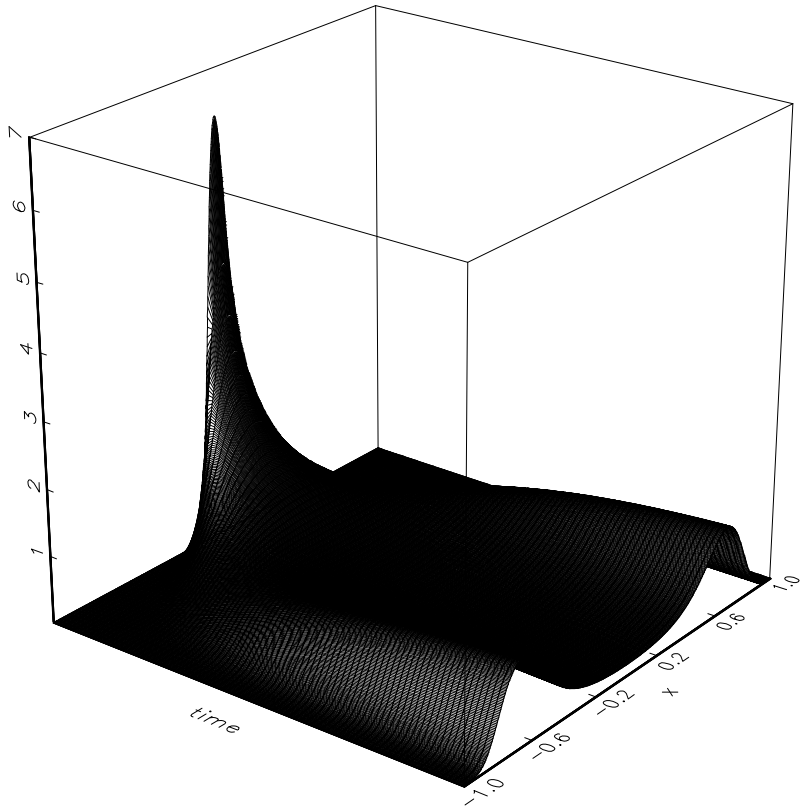

Figure 7: Example of a transient density computed via numerical integration of the Fokker-Planck equation. Parameters are $\alpha = 1.2, v = 4$ and $N = 50$. From a deterministic initial condition, $x_0 = 0$, the system is seen to converge towards the known bi-modal stationary distribution over time (the time horizon shown is$T = 2$ unit time intervals).

4.3.2 An Asset Pricing Model with Social Interactions

In this application, we interpret the former "+" and "-" groups as bullish and bearish speculators, who are influenced by herd effects together with observed price changes. Their transition probabilities, therefore, include two terms:

$$p_{+-} = \nu \exp(\alpha_1 x + \frac{\alpha_2}{\nu} p^{'}(t)), \tag{45}$$

$$p_{-+} = \nu \exp(-\alpha_1 x - \frac{\alpha_2}{\nu} p'(t)), \tag{46}$$

where the price change $p^{'}(t)$ reinforces or weakens the herding tendency depending on whether its sign is in harmony or not with a bullish (bearish) attitude.[9] Following the lines of our previous derivations, we can establish the mean value dynamics for the opinion index for the average bullish or bearish market sentiment (which is pretty close in its structure to some published indices of investor sentiment):

$$\frac{d\bar{x}_t}{dt} = 2\nu\{\tanh(\alpha_1 \bar{x}_t + \frac{\alpha_2}{\nu} p'(t)) - \bar{x}_t \cosh(\alpha_1 \bar{x}_t + \frac{\alpha_2}{\nu} p'(t)) \tag{47}$$

In order to close the model, we have to add a hypothesis for price adjustment. A simple possibility is Walrasian price adjustment in reaction to excess demand (ED) with a certain adjustment speed β:

$$P'(t) = \frac{dP}{dt} = \beta ED. \tag{48}$$

Following Beja and Goldman (1980), excess demand in our financial market could be decomposed into two components: excess demand by chartists (ED_c) and excess demand by fundamentalist traders (ED_f).
The chartists might be just those whom we have classified as bullish or bearish in the agent-based component of the model. If chartists have a trading volume t_c this amounts to:

$$ED_c = (n_+ - n_-)t_c = 2Nxt_c = xT_c \quad \text{with} \quad T_c = 2Nt_c \tag{49}$$

following the definition of the opinion index $x = \frac{n_+ - n_-}{2N}$. Fundamentalists, in contrast will have their excess demand depending on the difference between

[9] Division by ν of the second term is for technical reasons: An agent considers the price change during the mean time interval between switches between groups (which is ν^{-1}).

the perceived fundamental value p_f and the current market price:

$$ED_f = T_f(P_f - P), \tag{50}$$

with T_f the proportional trading volume of fundamentalists. Putting both components together, we arrive at the price adjustment equation:[10]

$$\frac{dP}{dt} = \beta(\overline{x}T_c + T_f(P_f - P)). \tag{51}$$

Eqs. (47) and (51) formalize our interdependent dynamic system in which the group dynamics influences the price dynamics and the price development feeds back on investor sentiment.
In studying the resulting system, we might first explore the question of existence and uniqueness or multiplicity of equilibria. Steady states of the joint opinion and price dynamics require $\frac{d\bar{x}_t}{dt} = \frac{dP}{dt} = 0$. Since this implies that the new second component of the herding probabilities is zero in any steady state, we arrive at the joint condition:

$$\frac{d\bar{x}_t}{dt} = \frac{dP}{dt} = 0 \Longrightarrow \tanh(\alpha_1 \overline{x_t}) = \overline{x_t} \quad \text{and} \quad P_t^* = \frac{T_c}{T_f}\overline{x} + P_f. \tag{52}$$

Inspection reveals the following:

(i) for $\alpha_1 \leq 1$ we have a unique equilibrium $\overline{x_0}$ together with $P^* = P_f$.

(ii) for $\alpha > 1$ we encounter the two majority equilibria $\overline{x_+^*}$ and $\overline{x_-^*}$ (now bullish and bearish majorities) with pertinent prices $P_\pm^* = \frac{T_c}{T_f}\overline{x_\pm^*} + P_f$.

Hence, if herding is weak (case (i)) the price converges to the fundamental value (on average); if herding is strong (case(ii)), the equilibrium price comes along with an overvaluation or undervaluation of the asset compared to its fundamentals.

However, there are additional possibilities in this more complex system: both $\overline{x_0^*}$ and the majority states $\overline{x_\pm^*}$ could be unstable (stability conditions are more involved than in the one-dimensional case). In such a scenario, the market performs almost regular cycles between overvaluation and undervaluation accompanied by investor sentiment oscillating between bullish

[10] The price equation could in principle, also be formalized as a Poisson process with transition probabilities for price changes in upward and downward direction, cf. Lux (1997).

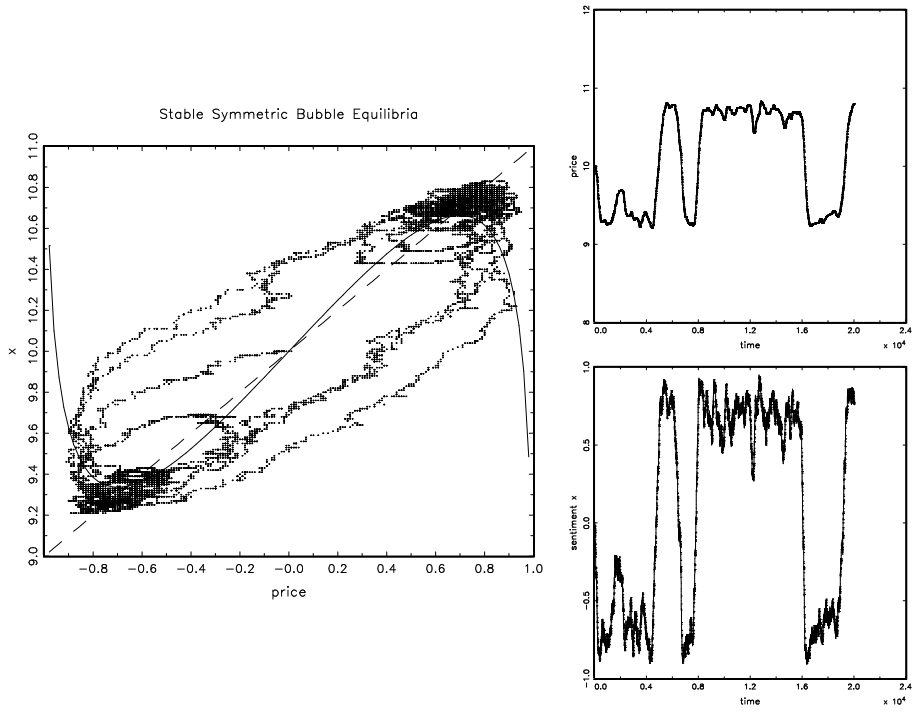

Figure 8a: The case of symmetric "bubble" equilibria: the system tends towards$(\overline{x_+^*}, P_+^*)$ or $(\overline{x_-^*}, P_-^*)$ but also switches occasionally between phases with overvaluation and undervaluation. Parameters are $\nu = 0.5, \beta = 1, T_c = T_f = 0.5, P_f = 10, \alpha_1 = 1.2, \alpha_2 = 0.75$, and $N = 100$.

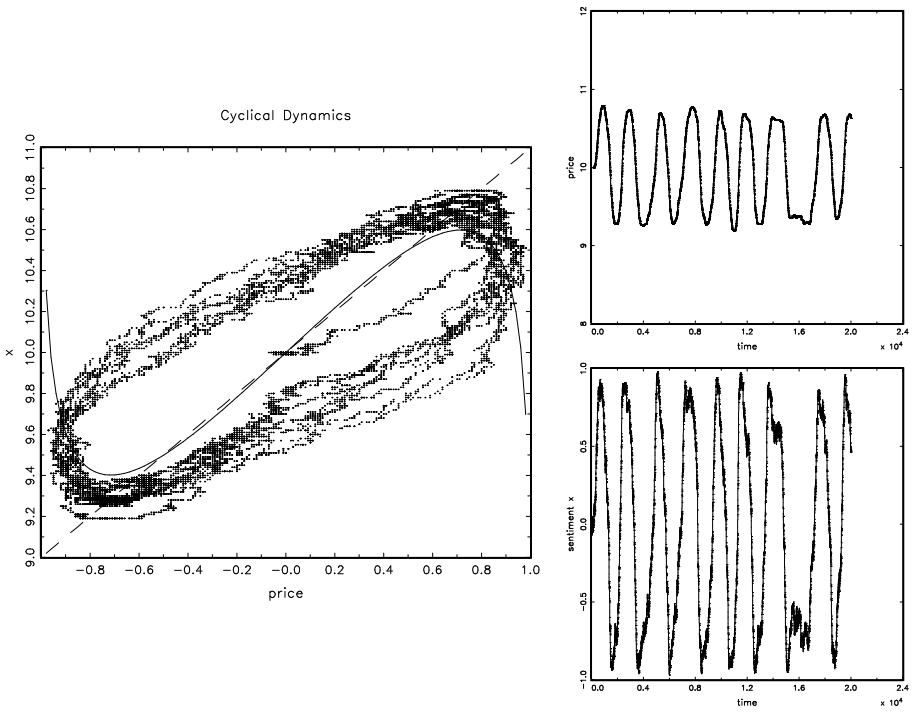

Figure 8b: The case of cyclical variation between bullish and bearish phases. Parameters as before except for $\alpha_1 = 1.1, \alpha_2 = 0.95$.

and bearish majorities, cf. Fig. 8. Expanding our methodology above to the 2D case, we could also characterize the fluctuations in different market phases via the variance dynamics and the time development of the covariance between P and $\overline{x}$, cf. Lux (1997).

4.3.3 Realistic Dynamics and the 'Stylized Facts'

Of course, neither stationary bubbles nor persistent cycles are realistic scenarios for financial markets.[11] One obvious criticism is that agents maintain

[11]The second part of this statement needs some modification: note that a cycle in mean values will appear more or less blurred in single realizations of the stochastic process. This distortion might go as far as to leave no apparent trace of cyclical dynamics. Lux and Schornstein (2005) investigate a more complicated model of a foreign exchange market with agents using genetic algorithms to evolve their strategies. Simulations of this model look extremely realistic in terms of returns and their statistical properties. Nevertheless, an analysis of the mean-value dynamics reveals a clear cyclical underlying dynamics which materializes itself only with a very large population of

their potentially unprofitable strategies forever without learning from past experience. This criticism could be faced by allowing agents to adopt to their environment using some learning or artificial intelligence algorithm for the choice and adaptation of their strategies.

We have reviewed some of the contributions in this vein above in sec. 4.1. Here we adopt a very simple mechanism to slightly increase the degree of smartness of our agents. Following Lux and Marchesi (1999, 2000) we allow agents to switch between the fundamentalist and chartist (or noise trader) strategy on the base of a rough measure of their supposed profitability. As it will be seen, this slight extension suffices to remove predictability of market movements to a large degree and also leads to simulated asset prices that share the ubiquitous stylized facts or scaling laws of empirical data. We will argue later that this example might also serve to reveal a general mechanism for generating realistic behavior that could also be identified in some alternative models.

The new ingredient of switches between noise traders and fundamentalists is introduced via exponential transition probabilities along the lines of eqs. (45) and (46). Formally, four new Poisson transition rates have to be introduced for the propensity of fundamentalists to switch to the optimistic (pessimistic) noise trader camp and *vice versa*:

$$\begin{aligned} p_{+f} &= v_2 \frac{n_+}{2N} \exp(U_{2,1}), \\ p_{f+} &= v_2 \frac{n_f}{2N} \exp(-U_{2,1}), \\ p_{-f} &= v_2 \frac{n_-}{2N} \exp(U_{2,2}), \\ p_{f-} &= v_2 \frac{n_f}{2N} \exp(-U_{2,2}). \end{aligned}$$

The forcing functions $U_{2,1}$ and $U_{2,2}$ depend on the difference between the momentary profits earned by noise traders and fundamentalists, respectively. We specify these functions as:

$$U_{2,1} = \alpha_3 \{ \frac{r + \frac{1}{v_2}\frac{dP}{dt}}{P} - R - s|\frac{P_f - P}{P}|\} \tag{53}$$

$$U_{2,2} = \alpha_3 \{ R - \frac{r + \frac{1}{v_2}\frac{dP}{dt}}{P} - s|\frac{P_f - P}{P}|\} \tag{54}$$

traders.

The first term of both functions represents the current profit of noise traders from the optimistic and pessimistic camp, respectively. The second term is the expected profit of fundamentalists after reversal to the fundamental value. Excess profits of the optimistic chartists consist of nominal dividends (r) and capital gains (dP/dt). Division by the actual market price (P) yields the revenue per unit of the asset. Subtracting the average real returns of alternative investments (or safe interest rate R) gives excess returns. Pessimistic noise traders, in contrast, leave the market for the asset under consideration so that their excess profits consist of the alternative return R minus the sum of forgiven dividends plus capital gains of the pertinent stock. It is somewhat harder to come up with a formalization of fundamentalists' profits. Fundamental activity is based on a perceived discrepancy between the market price and the fundamental value $P \neq P_f$. Profits from the pertinent traders are, however, *expected* profits only and will be materialized only if the stock price will have reverted towards its fundamental value. Because of the time needed for a reversal towards fundamental valuation and the potential uncertainty of this reversal, expected profits by fundamentalists have to be discounted by a factor $s < 1$. Otherwise, we tread fundamentalist speculation in periods of overvaluation and undervaluation symmetrically by computing the expected gain per unit of the asset as $|\frac{P_f - P}{P}|$. Note that the fundamentalists' profits did not contain dividends: This negligence is due to the assumption that they use the long-run expected asset price P_f for computing real dividends and that $r/P_f = R$, i.e. (risk-adjusted) dividends are the same for alternative investments if the price is equal to its fundamental value.

This new component endogenizes the fraction of chartists and fundamentalists which necessitates some adjustment in the $x - P$ dynamics as well. In particular, the opinion index x now refers to the numbers of optimists and pessimists within the noise trader group whose overall population is also changing over time, $x = \frac{n_+ - n_-}{n_c}$. Furthermore, the formalization of excess demand has to take into account the changing numbers of noise traders and fundamentalists as well. Denoting by z the fraction of noise traders, $z = \frac{n_c}{2N}$, we modify eq (51) accordingly:

$$\frac{dP}{dt} = \beta(ED_f + ED_c) = \beta(xzT_f + (1-z)T_f(P_f - P)). \tag{55}$$

Investigating the overall mean-value dynamics, the system evolution can

be characterized by the time change of the expectations of x, z and p. We restrict ourself here to reporting the main results for the pertinent system of three differential equations. As detailed in Lux and Marchesi (2000), for this quasi-deterministic system the following characterization of its steady states can be obtained:

- there are three types of steady states:
  (i) $x^* = 0, P^* = P_f$ with arbitrary z,
  (ii) $x^* = 0, z^* = 1$ with arbitrary P,
  (iii) $z^* = 0$, $P^* = P_f$ with arbitrary x,
- no steady states exist with both $x^* \neq 0$ and $P^* \neq P_f$.

The second result indicates that the additional assumption of switching between strategies due to profit differentials prevents emergence of stationary bubbles. Quite obviously, such lasting situations of overvaluation or undervaluation would give rise to differences in profits between groups so that they could not persist any more. The first part of the results indicates the types of equilibria that would be admitted under flexible strategy choice: there could be either a price equal to its fundamental value (on average) together with a balanced disposition of noise traders and an *arbitrary* composition of the overall population with respect to noise traders and fundamentalist strategy (i), there could be a dominance of noise traders ($z^* = 1$) with an arbitrary price development (ii), or a dominance of fundamentalists with P again equal to P_f on average (iii). All three categories are continua of equilibria rather that isolated fixed points as there is one 'free' variable. The more interesting of these possibilities is (i), while (ii) and (iii) are relatively uninteresting so-called absorbing states whose existence is hard to avoid in a population dynamics (if one group dies out by chance, it has no way to get into existence again). Inspecting type (i) equilibria, their most interesting feature is the indeterminateness of the population (i.e. of z). After some reflection, this outcome seems quite natural: If agents are allowed to switch between strategies, then, in an equilibrium, none of the surviving strategies should have a higher pay-off than others. This is the case in our model almost by definition of a steady state: if there are no price changes any more and the price is equal to its fundamental value, both the noise traders and fundamentalists would report excess profits equal to zero. Switching between subgroups would then occur unsystematically leading to

permanent changes of z along the continuum of steady states due to the stochastic elements of our process.

The set of results obtained for the mean-value dynamics of the extended model appears to indicate that the slight steps towards more rationality of agents represented in eqs. (53) and (54) weeds out the weird cyclical and bubble processes of the simpler model presented in sec. 4.3.2. In any case, the interesting equilibria are characterized by a price fluctuating around its fundamental value and a balanced disposition of noise traders, i.e. no more built-up of coherent optimistic or pessimistic majorities. A glance at the stability properties of these equilibria adds some additional insights. As shown in Lux and Marchesi (2000), an equilibrium along the line $(x^* = 0, P^* = P_f, z^*)$ is unstable [12] if

$$2z^* v_1(\alpha_1 + \alpha_2 \frac{\beta}{v_1} z^* T_c - 1) + 2(1 - z^*)\alpha_3 \beta z^* T_c / P_f - \beta(1 - z^*)T_f > 0 \qquad (56)$$

or

$$\alpha_1 > 1 + \alpha_3 \frac{v_2 T_c R}{v_1 T_f P_f} \qquad (57)$$

holds. The most interesting aspect of these results is that (56) defines a region $z^* \in [0, \overline{z}]$ in which the dynamics reverts to the continuum after disturbances, while for z beyond the threshold value $\overline{z}$, the dynamics becomes unstable. Since the stochastic components lead to ongoing changes of z *along* the continuum of equilibria, the system might wander from time to time from the subset of stable equilibria ($z < \overline{z}$) to that with repelling dynamics. The instability in the region $z > \overline{z}$ is due to the strong reaction on price changes in a population dominated by noise traders. Fig. 9 shows that stronger fluctuations set in if the system gets close to or surpasses the threshold $\overline{z}$. Apparently, these fluctuations have the appearance of volatility clusters. They hold on for some time but die out due to inherent stabilizing tendencies that become effective out-of-equilibrium. Namely, strong fluctuations lead to relatively large deviations from the fundamental value and former noise traders are induced to switch to fundamentalist behavior in large numbers.

[12] Since we have a continuum rather than isolated fixed points it is more convenient to express the stability properties in terms of conditions for instability rather than for stability.

The combination of deterministic and stochastic forces (incorporated in the stochastic formalization of agents' behavior) leads to repeated switches between turbulent and tranquil episodes. It is worthwhile to emphasize that despite a certain number of free behavioral parameters the qualitative outcome of this process is entirely *generic: all* combinations of parameters lead to a continuum of equilibria with stochastic switching between attractive and repulsive phases.[13] As demonstrated in Lux and Marchesi (1999) and Chen, Lux and Marchesi (2001) the apparent proximity of simulated returns to empirical records is reflected in agreement of many important statistics with empirical stylized facts. In particular, both the scaling laws of large returns and the hyperbolic decay of autocorrelations of squared and absolute returns are reproduced by the data from this artificial market and the pertinent estimates of, for example, tail indices and decay exponents of autocorrelations of squared and absolute returns, are numerically close to their typical values for empirical data. Switching between strategies also eliminates autocorrelations in raw returns to a large extent so that the apparent predictability of cyclical ups and downs of the simpler model of sec. 4.3.2 does not carry over to the extended framework. The lack of predictability seems plausible since the outbreak of fluctuations is triggered by the stochastic part of unsystematic population movements in the vicinity of the fundamental equilibrium. As a result, the market appears to be characterized by *speculative efficiency*. Allowing for an additional news arrival process, the market price is found to closely track the fundamental value albeit with temporary deviations that manifest themselves in a broader leptokurtotic distribution of returns compared to changes of the fundamental value.

Lux and Marchesi (2000) argue that the underlying mechanism of periodic switching between stable and unstable states due to stochastic forces constitutes a relatively general scenario to generate realistic ARCH type dynamics. In behavioral models, it seems natural that an equilibrium will be characterized by an arbitrary mixture of equally successful strategies and that the stability of such steady states will depend on the current distribution of strategies among the population. Models with similar features

[13] The dynamics is also qualitatively similar in the extreme cases where $\bar{z} = 0$ as still stabilizing forces out-of-equilibrium prevail.

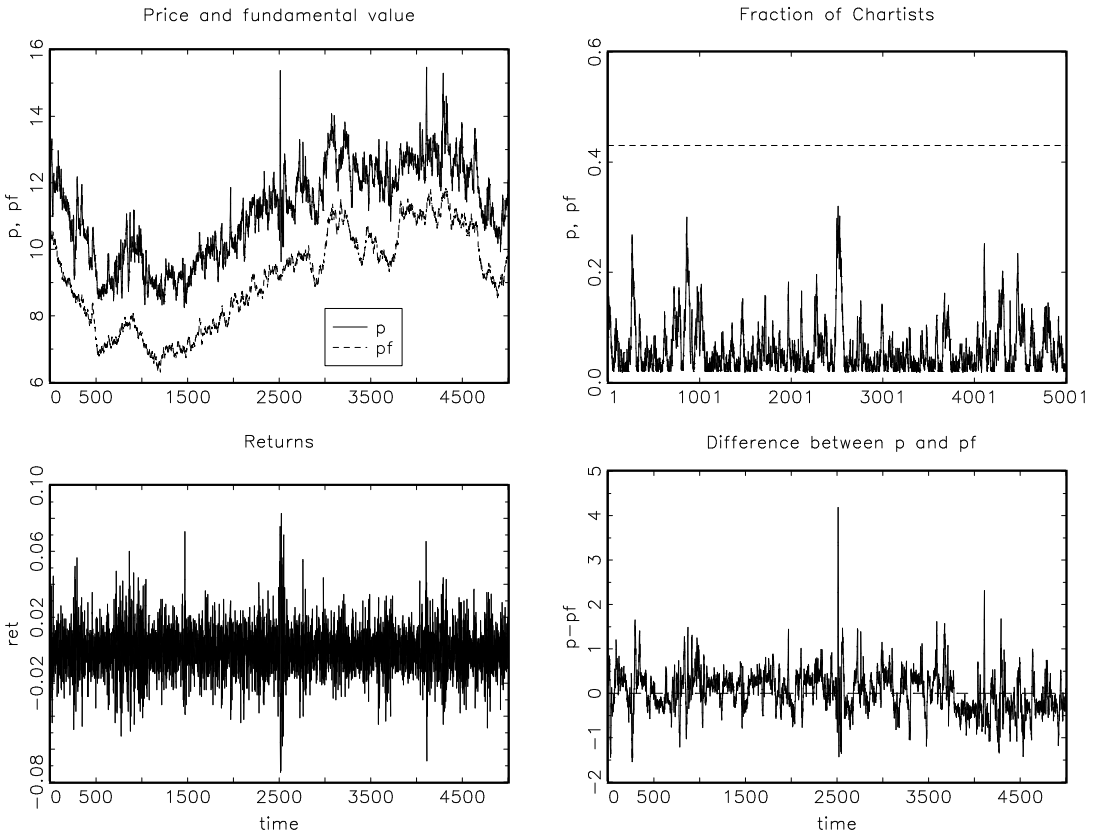

Figure 9: Example of a simulation of the model of Lux and Marchesi (1999). Parameters are $\alpha_1 = 0.8, \alpha_2 = 1, \alpha_3 = 0.5, v_1 = 1, v_2 = 0.6, T_c = T_f = 2.5, \delta = 0.75$ and $R = 0.0004$. The fundamental value has been shifted downward by two units to provide better visibility. As can be observed, the price mostly tracks closely the fundamental value but shows occasional large deviations from this benchmark. The development of the fraction of noise traders or chartists in the upper right-hand panel indicates that large fluctuations of returns and large degrees of mispricing occur if many traders follow the chartist strategy. The broken line in the upper right-hand panel demarcates the theoretical bifurcation value $\overline{z} = 0.46$ in this case.

have been proposed by Giardina and Bouchaud (2003) with a larger set of strategies, and Arifovic and Gencay (2000), and Lux and Schornstein (2005). The latter have a totally different set-up, a two-country general equilibrium model of the foreign exchange market with agents choosing consumption and interest strategies via genetic algorithms. Despite this very different framework, the dynamics of returns seems to be governed by a similar mechanism like that of the above stock market dynamics: there is a continuum of steady states with indeterminateness of investment decisions (in steady state, the revenue from domestic and foreign assets is the same), but random deviations from the steady state (brought about by the inherent randomness of genetic algorithms) destabilize this steady state and lead to an onset of fluctuations. The original framework by Lux and Marchesi has recently been extended by Pape (2007a, b) who reformulates traders' behavior as position-based trading (in this way keeping track of their inventories) and adds both a second risky asset and a risk-free bond. As it turned out, the main mechanisms of the original model still are found to be at work in this richer set-up leading to similarly realistic simulations. It is worthwhile to point out that the combination of stochastic and deterministic forces in these models is also similar to that of Kirman's population dynamics reviewed in sec 4.2. in that it leads to movements back and forth across a stability threshold of the underlying deterministic benchmark system.

4.4 Lattice Topologies of Agents' Connections

The models reviewed in secs. 4.2 and 4.3 are among the first contributions to allow for social interactions among agents in an economic context. However, they adopt very different assumptions as the design of their social interactions: while Kirman (1993) allows for pair-wise interactions only (after random encounters of agents), Lux (1995) and Lux and Marchesi (1999) use a mean-field approach. The later implies that all agents influence all other agents with the same intensity or - in the language of network theory - that the social interactions are embedded in a fully connected network with equal weights of its modes. Although we do not have reliable information on market participants' social networks, both of the above alternatives might not be very realistic. Even if a certain simple topology of interactions might

be acceptable for a first approach towards social influences, one could be concerned about the intensity of social interactions in relation to the size of the market (the number of agents). As Egenter *et al.* (1999) show, stylized facts do vanish in the model of Lux and Marchesi (1999), if one increases the number of agents while keeping the parameters of social interactions constant. The reason is that due to the law of large numbers, fluctuations of noise traders' mood become more and more moderate with increasing N. With the opinion index x staying close to its steady state value $x^* = 0$ most of the time, the emergence of an optimistic or pessimistic majority occurs less often so that the frequency of small price bubbles declines. The intrinsic chartist 'information' component, then, loses its importance against the fundamental component in eqs. (53) and (54) so that the profit differential works in favor of the fundamentalist strategy. As a consequence, the average fraction of noise traders gets smaller and smaller with increasing N, and the distribution of returns gets closer and closer to the assumed Gaussian distribution of the news arrival process. The 'interesting dynamics' with their fat tails and clustered volatility are, therefore, a *finite size* effect and do not survive in the limit $N \to \infty$. Essentially, this is a consequence of the law of large numbers as the market excess demand is an aggregate over N Poisson processes for individual traders. Obviously, the correlation between agents brought about by their social interactions is not strong enough to undo the effect of aggregation. With more than about 5000 socially interacting agents the model converges to returns following a pure white noise. A similar result is obtained for the very different artificial foreign exchange market with genetically generated strategies in Lux and Schornstein (2005). As it seems, the genetic operations of selection, recombination and mutation also lead to a reduced intensity of interpersonal coupling with an increasing number of agents, so that the dynamics loses its stochastic appearance with increasing numbers of market participants. Again, interesting and realistic dynamics are only obtained for markets with up to a few thousand traders. These findings are disturbing in so far, as empirical stylized facts are observed in quite the same way with practically the same estimated scaling exponents for markets of all sizes. In this sense, the *universality* of the empirical records is not reproduced by the above stochastic models. On the other hand, the universality of non-Gaussian behavior of all known financial markets implies that there probably is strong coupling between traders

in real life. With the largest markets having populations of the order of 10^6 or more market participants, the law of large numbers would imply Gaussian behavior if all these agents would act independently (or with sufficiently weak correlation). The universal non-Gaussianity, then, appears to indicate that financial markets have a typical number of *effectively independent* agents which is much smaller than their nominal number of market participants.

The challenge for models of social interactions would, therefore, be to come up with an explanation of this insensitivity with respect to system size. Alfarano, Lux and Wagner (2007) discuss this problem for a variant of Kirman's ant model. They show that if the frequency of pairwise encounters increases linearly with the number of agents, the resulting dynamics remains qualitatively the same for any number N of agents. In contrast, if the frequency of encounters is kept constant, the system converges to a Gaussian limit with increasing N. Quite similar to the experiments of Egenter *et al.* (1999), the relative importance of the herding component against the autonomous switching propensity declines if one does not adjust the former to the system size. The *intensity of interpersonal coupling* is only preserved in this model, if the frequency of pair-wise exchange increases with the number of potential partners for exchange.[14] Certainly, an ever increasing probability of pair-wise exchange is somewhat hard to digest in its literal interpretation. Departing from the extremes of either pair-wise interactions or fully connected social systems, network topologies of agents' social interactions might be a promising avenue to explore how the intensity of social coupling might plausibly change with system size. Alfarano and Milakovíc (2007) modify the ant model by replacing pair interactions by neighborhood effects within various network topologies. Increasing the number of agents but keeping the parameters of the network generating mechanism fixed, they note that most popular network designs (regular, scale-free and 'small-world' networks) cannot overcome the N-dependency within their generating mechanism, i.e. without adapting crucial parameters. The only case in which the generating mechanism keeps the intensity of communication constant for varying numbers of agents by the very na-

[14] Finite-size effects in alternative models of opinion formation are investigated in Toral and Tessone (2007).

ture of its construction is the random network. While these results sound somewhat disappointing, they have only focused so far on the mechanical structure of various topologies. Incentives of agents to form links could lead to changes of the connectivity with system size which remains to be investigated. Interestingly, Alfarano and Milakovíc (2007) also show that allowing for a small number of independent agents who only influence others without being prone to social influences themselves (unilateral links) changes the outcome and allows for prevalence of interesting dynamics whatever the number of herding agents (cf. also Schmalz, 2007).

A different type of network structure has been used in a related paper by Cont and Bouchaud (2000). Essentially, their contribution is an adaptation of the seminal percolation model from statistical physics. In this framework, agents are situated on a lattice with periodic boundary conditions. Each site of this lattice might initially be 'occupied' with a certain probability p or empty with probability $1 - p$. Groups of occupied neighboring states form clusters. In Cont and Bouchaud (2000) occupied sites are traders and clusters are subsets of synchronized trading behavior (i.e. all members of a cluster are buyers or sellers, or remain inactive). The type of activity of a cluster is determined via random draws. The market price is again driven by an auctioneer equation depending on excess demand over all clusters. Since the underlying formal structure has been extensively studied in physics, certain known results for the cluster size distribution can be evoked and due to the simple link between cluster distribution and price changes carry over to returns. In particular, one knows that both distributions will follow a power law if the probability for the connection of lattice sites is close to a critical value, the so-called percolation threshold. However, the power-law is characterized by an exponent 1.5, in contrast to the empirical law with decay rate ~ 3. In the baseline version of the model, higher moments are uncorrelated so that the percolation model could not explain volatility clustering either. Despite (or because of) these deficits, the framework of Cont and Bouchaud has spawned a sizeable literature (mostly published in physics periodicals) that tries to get its time series characteristics closer to empirical scaling laws. Interesting extensions of the original model include Stauffer *et al.* (1999) and Eguiluz and Zimmerman (2000) who generate autocorrelations in higher moments via sluggish changes of cluster configu-

rations. In view of the above discussion, it is worthwhile to note that the critical connection probability at the percolation threshold is N-dependent and, therefore, has to be adjusted with system size in order to guarantee a power-law distribution of the clusters.

More realistic time series are obtained in some alternative lattice models: Iori (2002) considers an Ising type model with interactions restricted to nearest neighbors, while Bartolozzi and Thomas (2004) propose a cellular automaton structure with similar neighborhood influences between traders. In both models, realistic time series seem to be a robust outcome without the need of fine-tuning certain parameter values. However, due to the complexity of these structures, it is hard to single out what key features of these models are responsible for the interesting dynamics. It is unknown so far whether the realistic features of those models persist for large populations of traders or not.

5 Conclusions

The present chapter has reviewed recent models that try to explain the characteristics of financial markets as emergent properties of interactions and dispersed activities of a large ensemble of agents populating the market place. This view has a certain tradition starting in the early nineties (or even earlier if one includes contributions of the 70s like Zeeman's, 1974) when chaotic processes based on simple behavioral assumptions have been proposed as an explanation of the apparent randomness of financial data. As it turned out in subsequent research, market statistics are in all likelihood more 'complex' than data from low-dimensional chaotic attractors and seem to be characterized by an intricate mixture of randomness and nonlinear structure in higher moments. The most pervasive characteristics of the particular stochastic nature of financial markets are the power laws for large returns and autocorrelations of volatility. Similar system-wide features are the typical imprints of large systems of interacting subunits in the natural sciences.

Inspired by these analogies, some recent models have proposed simple structures that could reproduce the empirical findings to a high degree with

statistics that are even quantitatively close. This appears the more remarkable since 'mainstream' theory has offered hardly any hint at the generating forces behind the stylized facts, let alone models with precise numerical predictions. Offering explanations for hitherto unexplained observations is typically what characterizes a new, superior paradigm. This new view also opens the stage for entirely new avenues of research and questions that could not even have been formulated before. Among these questions, the most important task for future research might be the explanation of the *universal preasymptotic* behavior of financial markets, i.e. the answer to the question why they are not subject to the law of large numbers (as they should if they were populated by independent agents).

From the viewpoint of mainstream finance, it might be a threatening experience to see some basic stylized facts explained by models that have hardly anything in common with a traditional representative-agent approach. However, what the above models offer are just those ingredients that critics of the mainstream have been emphasizing for a long time. As a prominent example, Kindleberger (1989) has stressed the importance of psychological factors and irrational behavior in explaining historical financial crises. In fact, recent micro structure literature has allowed for irrational components like overconfidence or framing (e.g. Daniel *et al.*, 1998, Barberis and Huang, 2001), with highly interesting results. While the analysis of certain types of non-rational behavior and its consequences might explain important facets of reality, an explanation of the overall characteristics of the market might require a different approach. Proponents of mainstream finance have, in fact, criticized a lack of unifying framework in the behavioral finance literature. Most notably, Fama (1998) noted that a variety of psychological biases could be used to explain various anomalies, but that behavioral finance models were unable to explain the 'big picture' and to capture the 'menu of anomalies better than market efficiency' (Fama, 1998, p. 241). While stochastic models of interacting agents have so far not focused on overreaction and other return anomalies they appear to be able to provide generic explanations for the 'deeper' anomalies of fat tail and volatility clustering. Although they are mostly not micro-based in the sense of featuring utility maximization or alternative psychological decision mechanisms, they might provide a broader macroscopic picture of emergent properties of microeco-

nomic interaction embedding the diverse deviations from perfect rationality at the micro level. Since we probably encounter a wide variety of trading motives, strategies, and degrees of (non-)rationality and (lack of) foresight among agents, a stochastic approach might be required to compensate for our ignorance of the microscopic details. This is the starting point of the above models. The present stochastic approach could, therefore, be seen as complementary to the focus of the previous strands of the behavioral finance literature on particular behavioral observations in that it tries to infer macroscopic regularities via a simple representation of the diverse collection of the boundedly-rational behaviorial types in real markets.

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