Three Essays on the Effects of Product Recall on Decision Making

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Abstract

Three Essays on the Effects of Product Recall on Decision Making

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The effects of product recalls can be utterly disastrous for the firm responsible for the recall. A recalling firm may bear substantial external failure costs along with sales loss due to a tarnished goodwill. Deciding the pre-recall and post-recall advertising, price and quality are crucial for both the recalling firm and its rivals. Using a differential game theoretic modelling approach, we develop and analyze theoretical models associated with a product recall and study the effect of the same on supply chain decision making. In our first essay, we analyze a scenario with two manufacturers under goodwill-advertising competition. Either one firm or both firms can be susceptible to recall a product, and both are aware of the recall likelihood ex-ante. We examine the pre-crisis and post-crisis equilibrium advertising efforts and effect of recall on firms' profits when firms are farsighted or "hazard myopic. We show that the variance of advertising in the two periods and the profits of the firms depend on crisis likelihood and impact. The trade-off between the likelihood of crisis and its impact explains the previous conflicting findings of the literature. In our second essay, we investigate joint decisions of pricing and advertising for mitigating the harmful effects of a product recall on a firm and the negative spillover effect on a rival firm. We recommend the equilibrium pricing and advertising policies of the two competing firms under product recalls of different impact and likelihood. We compare the case when the focal firm is a market leader to the case when both firms are similar. In the third essay, we define two contracts for managing "collateral damage" during a product recall. We show that adopting a cost-sharing contract can be beneficial for suppliers and manufacturers. We found that cost-sharing decisions, advertising efforts and quality efforts vary under different scenarios. There may be individual motivation for the firms to move against a contract under different crisis likelihood and impact. Nevertheless, the supply chain profit is always higher under a contract.

Dedication

To my parents for their inspiration,
To my wife for her constant and amazing support,
To my daughter for making my every day memorable.

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Chapter 1

Introduction

A product recall is an undesirable event that may disrupt a supply chain severely. The recalled items could be dangerous for life and may damage the environment. Consequently, recalls are detrimental to the image of a firm and could threaten the existence of the associated firm. For example, the recall of airbags in 2016 by the Japanese manufacturer, "Takata", has been described by NHTSA (The National Highway Traffic Safety Administration) as "the largest and most complex safety recall in U.S. history". The impact was 11 deaths and approximately 180 injuries. Consequently recalls of about 42 million vehicles from 19 manufacturers were issued. The damage enforced Takata, an 80-year-old company, to file for bankruptcy in 2017.

Despite the numerous product recalls in industries such as food, pharmaceutical, automobile, and consumer goods. not much research is available to understand the recall events (Wowak and Boone 2015). Nevertheless, few empirical studies have covered various recall events and, some papers used mathematical modelling methodology to investigate the effect of product recalls on supply chains. In the proposed research, our goal is to understand the recall phenomena and its repercussion on decision making in a supply chain.

The existing literature on product recalls cover various recall contexts and touch various related issues such as supply chain traceability (Dai, Tseng, and Zipkin 2015), brand image (Dawar and Pillutla 2000), spillover effects (Borah and Tellis 2016), quality (Chao, Iravani, and Savaskan 2009) and timing of a product recall (Yao and Parlar 2019). The extant literature which investigates decision making during product recall encompasses different areas like marketing, operations management and financial implications. A comprehensive literature review on product recall (Wowak and Boone 2015) compiled the opinion of industry experts and researchers to provide the four key aspects of product recalls that needs to be researched/studied in detail. These key aspects of product-recall are *product recall precursors, product recall processes, product recall impacts* and *mitigation approaches*. The authors highlighted that the literature is lacking analytical research with regards to the above four phases of product recalls. Motivated by the paucity of research in this important area, we investigate the impacts of a product recall on decisions like advertising, pricing and quality. Our work also validates if the decisions mentioned above can mitigate the negative effects of a product recall. We address three key themes in this dissertation:

- *Impact of a product recall on firm goodwill and the mitigating effects of advertising decisions*
- *Spillover effect of product recall and its effect on pricing and advertising in a supply chain*
- *Quality cost-sharing contracts in a supply chain to mitigate the effects of product recalls*.

Research Design and Methodology: To analyze our research themes proposed above section, we choose mathematical modelling, especially differential game theory as our research methodology. The temporal dimension of decision making in dynamic games makes our models realistic. Assuming random recall time, we derive the equilibrium decisions and policies to answer our various research questions. We develop and analyze three models to deal with the different scenarios of product recalls, separately. The modelling involves the following steps –

1. Review literature or industry reports to generate supply chain involvement scenarios in a product recall.
2. Develop a model for each scenario. Once we select the appropriate modelling technique, we abide by certain assumptions
3. We develop analytical solutions for the model problems and conduct numerical experiments to gain further insights.

In the following section, we highlight the positioning of our research with respect to the present literature. We briefly discuss the contribution of our research.

1.1. Product Recall and Advertising

When a manufacturer issues a recall, there is a damaging effect on the manufacturer's brand. The damaging effect, however, depends on the firm's existing brand image and the magnitude of the recall (H. Van Heerde, Helsen, and Dekimpe 2007; Cleeren, Dekimpe, and Helsen 2008). Brand advertising can potentially help in overcoming this harmful impact of a product recall. Using the extension of Vidale-Wolfe model by Sethi 1983, Rubel, Naik, and Srinivasan 2011 finds the optimal advertising decisions while product harm crisis may occur. The analytical model, under the framework of dynamic game, shows that managers discount the present more if they anticipate the recall compared to when they do not expect it. The paper also throws light on the effect of damage and crisis likelihood on the optimal decisions. The authors finally test their model by the sales from the automobile industry.

Our first essay (Chapter 2) is closely related to the study of Rubel, Naik, and Srinivasan 2011. In Chapter 2, we construct a model where demand functions of two manufacturers are affected by goodwill and show how the advertising decisions of the two firms in a duopoly market are affected by product recalls of variable impacts. We primarily re-examine the validity of some of the results presented in (Rubel, Naik, and Srinivasan 2011) under a different model. Besides, we introduce the concept of "hazard myopia" and also extend the literature to investigate the advertising decisions when two firms, with individual risks of recall, compete for goodwill. (Rubel 2018) studied such a model with pricing competition, but the model assumes a monopoly market after a recall. We relax this assumption to extend the finding in case of partial recalls.

1.2. Spillover Effect of Product Recall

Empirical research shows that when a recall happens for one product, the ramification of the occurrence felt across categories or even by competitors (Freedman, Kearney, and Lederman 2012; Borah and Tellis 2016). As an aftermath of a recall, consumer reaction can affect a brand which has not recalled the product but sells similar products. This reaction of consumers diffusing to rival brands is the so-called spillover effect of a product recall. For instance, frozen food recall of one brand may negatively affect the demands of other brands producing similar items as well as other frozen food items. The impact is, in general, more severe in high valued complex items such as car where recall is due to one critical component and the demand for competing brands reduced just because competition shared some similarities such as common supplier, existing brand image and country of origin. On the other hand, competitors based in other countries can invest more in advertising or sales to boost their profit. A most cited example is very recent, Toyota recalls. When Toyota recalled few models, demand for Honda cars also declined, but GM was able to make more profit by demand shift.

In the second essay (Chapter 3), we model the spillover effect of a product recall when two manufacturers compete for goodwill and price. To the best of our knowledge, the extant empirical literature has studied the spillover effect, but there is a paucity of research in the modelling literature. While the focus of most of the empirical studied is the event of spillover, we contribute by investigating the advertising and pricing decisions when spillover occurs. Moreover, our model incorporates both positive and negative spillover under the scope of a single framework.

1.3. Collateral Damage of Product Recall in a Supply Chain

A product recall can have a rippling effect on a supply chain. When a recall is issued, not only the firm at fault but also the other supply chain members can suffer alike. For example, when a supplier's faulty parts lead to a recall, the manufacturer, retailer and possibly other involved members of the supply chain may face drastic consequences. Samsung issued a recall for the Galaxy Note 7 due to the fault of the battery suppliers resulting in considerable financial damage to Samsung. In such cases, contractual agreements between a manufacturer and a supplier may help to overcome the deterring effect of recall.

In the third essay (Chapter 4), we study cooperation instead of competition. We propose two cost-sharing contracts offered by a manufacturer to her suppliers. We examine how the cost-sharing mechanisms vary before and after a recall. The extant literature has scarcely focused on the viability of such contracts during product recalls. We fulfil this gap by analyzing the proposed contracts and show that partial cost-sharing might be a better option when crisis likelihood is low. In terms of the research focus, our paper is close to that of (Chao, Iravani, and Savaskan 2009), who also discuss two different cost-sharing contracts in the wake of product harm crisis.

The last piece of this work, Chapter 5 provides some concluding remarks and directions for future research.

Chapter 2

The Impact of Product Recall on Advertising Decisions While Envisioning Crisis or Being Hazard Myopic

2.1. Introduction

Product recalls are common occurrences in industries like consumer products, vehicles, food products and health products. The Transport Canada database shows that there have been 10402 recalls issued only in the automobile industry starting from 2010 till January 2018. In a contemporary competitive market, rival firms strive for quality excellence. However, the data from Transport Canada shows that despite firms' best efforts to ensure quality, product-recalls occur often and therefore, should be anticipated during a management's decision making. The impact of product-recalls depends on the harm caused by the products under recall. While small product-recalls, causing little media attention and customer awareness may go unnoticed, the major recalls often result in huge losses for the firms at fault as consumers lose confidence in the brand, supplier relationships get affected negatively, or competitors take advantage of the situation (Craig and Thomas 1996; A. M. Eilert 2013). The anticipation of product harm crisis is important from a managerial point of view because it can affect decisions like advertising, pricing or quality investments and largely impact the firm's reputation and financial performance (Rubel 2018; Rubel, Naik, and Srinivasan 2011; Chao, Iravani, and Savaskan 2009; Cleeren, Dekimpe, and Helsen 2008; Gao et al. 2015; Y. Chen, Ganesan, and Yong Liu 2009; M. Eilert et al. 2017). Examples of strategic decision making to alleviate recall impacts are abundant. Salmonella contamination led to the recall of some Cadbury chocolate flavours from the UK market. After the recall, Cadbury's marketing efforts increased substantially resulting in an increase in advertising and the launch of new products (Walsh 2006). The increased efforts of marketing or advertising were in the expectation of regaining lost market. In a more recent incident, Samsung globally recalled Note 7. The impact was huge, and it wiped off $16 billion of the company's market value initially (Reuters 2016). Following the recall, Samsung made extensive advertising campaigns to regain customers' trust. The advertising went to the extent of showing the new eight-point battery check that exceeded the industry quality standards (Fenech 2017) demonstrating how Samsung signalled quality improvement via advertising.

Most of the marketing literature uses the words product-recall and product harm crisis synonymously. The strict meaning of the two words is not the same(Cleeren, Dekimpe, and Heerde 2017). While a product harm crisis is an antecedent, a product recall is a consequence. However, for the sake of this paper, we use the words "product recall", "crisis" or product harm crisis" to mean the same thing - product recall.

The motivation of our paper stems out from the dilemma that the previous literature and the various industry reports yield about a firm's advertising efforts after a recall. Furthermore, there is a relative lack of research in the area of decision making in marketing channels to mitigate recall impacts (Wowak and Boone 2015). We refer to a firm issuing a recall as a focal firm. We investigate the equilibrium advertising policies of two competing firms when one firm anticipates a recall (one focal firm) or both firms (two focal firms) predict recalls. Additionally, we analyze the situation where a firm ignores the crisis likelihood. Moreover, we examine the effects of crisis likelihood and damage on the firm profit for both the cases: competition with one focal firm and competition with two focal firms.

High pre-crisis brand equity can protect a firm from the recall crisis and therefore, a firm, in anticipation of a recall, can advertise more during pre-crisis regime to build its brand image (Cleeren, Dekimpe, and Helsen 2008). Alternatively, management might have an urge to make more advertising efforts in the post-crisis regime in order to gain back its image and lost market share. An increase in the advertising can be a lost effort if the consumers are unforgiving towards the firm at fault. Consequently, the competitor might also exhibit opportunistic behaviour, thereby increasing its advertising (Cleeren, H. J. Van Heerde, and Dekimpe 2013; Craig and Thomas 1996). Both the arguments, favouring increased advertisement spending before or after a recall are valid. Every firm has a certain likelihood and damage rate for a particular product recall. Managers of a firm should envision the risk of recall in their advertising decisions throughout the planning horizon because crisis anticipation increases managers' time preference (Rubel, Naik, and Srinivasan 2011).

Uncertain occurrence, substantial damage which incurs additional costs like costs for managing reverse logistics, customer compensations, litigation costs and above all an erosion of brand value or goodwill are the characteristics of a product recall. Our paper focuses mainly on analyzing optimal advertising decision when firms compete with respect to brand image or goodwill.

Product recall affects the baseline sales of the recalling firm (H. Van Heerde, Helsen, and Dekimpe 2007). Very severe damage due to quality failure and subsequent recall can jeopardize the image of the firm at fault. The car airbags recall by Takata Corporation of Japan cost the company in billions. Consequently, Takata filed bankruptcy in 2016 (Tajitsu 2017). In another incident, the Westland/Hallmark Meatpacking company, accused of improper cattle handling, recalled more than 143 million pounds of beef in 2008 due to an intervention by the FDA. The company, under a $ 500 million settlement with various plaintiffs, had to file bankruptcy in 2012. A highly publicized and costly recall caused swift goodwill erosion thereby accelerating the firm's downfall.

High goodwill results in better market share and hence, better performance. A firm's goodwill is often linked with advertising. Advertising is a major factor in maximizing brand equity (Meenaghan 1995; Achenbaum 1989; Lindsay 1990). Meenaghan 1995 observes - "at all levels of marketing imagery advertising is identified as one of the principal components of image creation". For example, Nike's famous "Just do it" campaign not only boosted the company's brand image but also increased the sales from $877 million to $9.2 billion within a decade. A high brand image has a positive impact on the market share or sales of a firm. Numerous goodwill based demand models assume the positive effect of goodwill on demand (Dockner et al. 2000; Karray and Zaccour 2005). If consumers' perceived brand image is congruent with a consumer's social, actual, ideal images, the brand is purchased, and the sales of the brand are impacted positively (Ataman and Ülengin 2003).

We wish to validate the findings related to optimal advertising during recall (Rubel, Naik, and Srinivasan 2011; Cleeren, H. J. Van Heerde, and Dekimpe 2013; Cleeren, Dekimpe, and Helsen 2008). Firms fear the goodwill loss as a result of a recall. To the best of our knowledge, the extant literature, related to the effect of product recalls, did not shed light on the optimal advertising policies when goodwill of the competing firms affect their demands. We develop a dynamic game-theoretic model where demand is a function of the firms' goodwills. In our model, a product recall affects goodwill, and consequently demand. Thus our model investigates optimal advertising decisions while envisioning three significant aspects of a product recall – uncertain recall time, the effect of advertising on goodwill and impact of the recall risk and damage on the optimal policies.

The previous literature, based on mathematical modelling framework, has stressed the importance of envisioning the impending product recall while making decisions (Rubel 2018; Rubel, Naik, and Srinivasan 2011). We believe that the assumption that firms have perfect information about the hazard rate might not always be applicable. For example, a firm can be a new entrant in the market. In such cases, the entrant or the incumbent might not predict or consider the crisis possibility of the entrant. If a firm has no recall history, and it maintains a high quality or brand value, the firm itself or the competitor can also possibly ignore the possibility of a recall hazard. In this context, we propose the term "hazard myopic" for a firm if it ignores the chances of a recall. In the marketing literature, "managerial myopia" is used in a different sense with respect to advertising. Marketing myopia with respect to advertising refers to the strategy where the focus is promoting a product rather than building the brand or paying attention towards the customers needs (H. H. Friedman and L. Friedman 1976; Sharma 2015; Levitt 1960). The models under the dynamic game framework in the marketing literature have used the term myopia to signify that a player ignores the state evolution of the system, e.g. (Taboubi and Zaccour 2002; Benchekroun, Martín-Herrán, and Taboubi 2009; Zu and L. Chen 2017). We consider another form of myopia - overlooking hazard possibility while determining the long-term profit (hazard myopic). In a situation where we have this type of myopia, a viable case of interest is to examine the firms' equilibrium advertising policies and profits. When firms are not "hazard myopic" we call them farsighted in this paper. Thus, farsighted firms envision their respective hazard rates.

By analyzing our model, we answer the following research questions:

1. What are the equilibrium advertising policies of the individual players before product recall and after product recall?

2. How are the equilibrium advertisings of the competing firms affected by the intensity of the recall and the hazard rate of the recall?

3. Under the model assumptions, is post recall advertising always higher than the pre-recall advertising?

4. What is the impact of the impending recall on the performance of the competing firms?

5. What is the impact of hazard myopia on advertising and performance of the competing firms?

In this paper, we have developed a brand image based demand function and a dynamic model which captures advertising competition and uncertain recall time. The recall is also "precise," i.e. the firms know the exact items to be recalled and can observe the "drop" in goodwill as a result of the damage caused by the recall (Ketchen Jr, Wowak, and Craighead 2014). Besides, the recall is partial, and non-defective products remain in the market after the recall. We have analyzed the advertising decisions and the impact of a recall in two scenarios -
(i) A focal firm and a non-focal firm compete
(ii) Two focal firms compete.
Our results augment or support those of the previous literature (Rubel, Naik, and Srinivasan 2011; Rubel 2018; Cleeren, H. J. Van Heerde, and Dekimpe 2013; Gao et al. 2015). We find that a firm's profit margin and sensitivity to competition affect its equilibrium advertising policies. Whether a firm should increase or decrease advertising in the post-recall regime depends on what profit margin the firm wants to attain, how sensitive the market is towards the firms' goodwills and, most

importantly on the (χ, η) pair, where χ in the recalling firm's hazard rate and $\eta \in (0,1)$ is the damage to goodwill caused by the recall. Crisis intensity and hazard rate are major determining factors for firms' profits. A low impact low recall hazard rate can positively affect the performance of a firm. High competition can increase profit for a firm. High recall probability or intensity negatively affects a firm's profit. We have not considered price as a decision variable. However, some of our results point towards the importance of pricing before and after a product recall. The findings from the model also highlight that a brand having high initial goodwill suffers from less profit loss by a recall. However, we found that a high-intensity recall can substantially erode firm profit irrespective of its initial goodwill. We summarize our contribution to the existing literature in the following paragraph.

First, we have investigated the equilibrium advertising policies, and long-term expected profits of the competing firms under a dynamic goodwill based game theoretic duopoly model when product recall can strike at an uncertain time. While the previous studies (Rubel, Naik, and Srinivasan 2011; Cleeren, Dekimpe, and Helsen 2008; Gao et al. 2015) considered advertising decisions and the impact of the impending recall on the same, our study articulates the effect of a recall on a focal as well a non-focal firms' advertising strategies and the resulting long-term expected profits.

Second, we have considered another form of myopia, different from the concept of the existing literature - overlooking hazard possibility while determining the long-term profit (hazard myopic). In a situation where we have this type of myopia, a viable case of interest is to examine the firms' equilibrium advertising policies and profits. When firms are not "hazard myopic" we call them farsighted in this paper. Thus, farsighted firms envision their respective hazard rates.

Third, in contrast with the previous studies (Rubel 2018), we believe that this paper is the first to have developed the feedback (closed loop) strategies in the case when two focal firms with different crisis likelihoods exist and the duopoly competition continues even after both the crises occur. The analysis of the above model provides us with some useful managerial implications. Table 0 presents the details of our contribution and the comparison of the same with the related literature. The table is partly adapted from a literature review in the area of marketing and product harm crisis (Cleeren, Dekimpe, and Heerde 2017), but is modified to highlight our contribution.

The rest of the paper is arranged in the following manner. In section 2, we present a detailed description of the model. We show how uncertain recall time, the damaging effect of recall can be incorporated into the model and propose the solution procedure. Section 3 gives a comprehensive analysis of the model and illustrates how the advertising policies differ for a single focal firm, myopic firms or two focal firms. Section 4 supports the analysis done in Section 3 by numerical experiments and elucidate the significance of our analytical findings. Finally, in section 5, we conclude with managerial implications of our research and future research directions.

Authors	Regime of Analysis	Focus	Theory/Approach	Dependent variables	Main effects	Interaction Effects
Cleeren et al. (2008)	Post-crisis	Consumers	Hazard model	Timing of first purchase after the crisis	Advertising, Loyalty, Familiarity, Usage	Loyalty*time
Rubel et al. (2011)	Pre-crisis, Post-crisis	Firms, Consumers	Stochastic control problem	Sales of affected brands, competitors and number of recalls	Advertising	Crisis likelihood*Damage Crisis likelihood*Advertising
Zhao et al. (2011)	Pre-crisis, Crisis, Post-cisis	Consumers	Consumer learning model	Brand choice of affected brands and non-affected brands	Advertising, Price, Risk Aversion	
Liu and Shankar (2015)	Pre-crisis, Crisis, Post-cisis	Firms, Consumers	State-space model with random coefficients	Market share of affected and non-affected brands		Advertising*publicity Advertising*Severity of Crisis
Gao et al. (2015)	Event window of [0, +1] days	Investors	Event study	Future product reliability, injuries and recall frequency	Newness of recalled product, severity of hazard	Pre-recall advertising adjustment*newness of recalled product; Pre-recall advertising adjustment*severity of hazard
Our paper	Pre-crisis, Post-crisis	Firms	Differential game	Demand/Sales	Advertising, Profit	**For one focal firm**: a) Crisis likelihood*Damage to goodwill*Advertising; b) Crisis likelihood*Damage to Goodwill* Firm profit; c) Hazard myopia*Damage to Goodwill *Advertising; d) Hazard myopia*Damage*Firm Profit; **For Two Focal Firms**: a) Crises likelihoods*Damage to Goodwills*Advertisings; b) Crises likelihoods*Damage to Goodwills* Firm profits;

Table 0: Contribution of our Paper

2.2. Model Formulation

In this section, we define the model comprising of the demand functions, the process of recall occurrence, the effect of recall and the manufacturers' profit-maximizing problems in the different scenarios. We consider a market structure involving two manufacturers, competing for goodwill. We will refer the manufacturers as M_1 and M_2. Consumers are sensitive towards the difference in the goodwill of the manufacturers. While a manufacturer's goodwill positively affects its demand, the competitor's goodwill negatively affects its demand. The manufacturers try to maintain high brand goodwill by continuous advertising efforts. Product-recall can occur at an uncertain time within the planning horizon, $[0,\infty)$. A recall dampens the goodwill of the focal firm and consequently decreases the firm's demand, thereby affecting the firm's profit. Therefore, advertising, a commonly used weapon for maintaining or increasing goodwill, can be a viable strategy to counter the adverse effects of the product recall. The main objective of our model is to find the equilibrium advertising in pre-crisis and post-crisis regimes and analyze the effect of recall probability and intensity on the equilibrium strategies and firms' profits. We assume that the players take the advertising decisions simultaneously and the equilibrium strategies are, therefore, solutions of the Nash game described later.

Table 2.1 shows the equilibrium advertising decisions A^*_{ij} in different cases. We consider two different scenarios depending on the number of focal firms. First, we investigate the case when

Case	Players	Behaviour	Regime 1	Regime 2	Regime 3
One Focal firm M_1	M_1	Farsighted	A^*_{11}	A^*_{12}	NA
	M_2		A^*_{21}	A^*_{22}	NA
	M_1	Myopic	A^*_{11}	A^*_{12}	A^*_{13}
	M_2		A^*_{21}	A^*_{22}	A^*_{23}
Two Focal firms	M_1	Farsighted	A^*_{11}	A^*_{12}	A^*_{13}
	M_2		A^*_{21}	A^*_{22}	A^*_{13}

Table 2.1: Firm decisions and regimes

there is one focal firm, i.e., one of the manufacturers is prone to a product recall. Second, we extend our model to the situation where there are two focal firms, i.e., both the manufacturers are prone to recall. We analyze these two cases separately. For each of the above

Model Parameter	**Description**
$\theta_i(t)$	Goodwill of manufacturer i at time t. $\theta_i(t) \in [0, \infty)$
$A_{ij}(t)$	Advertising effort by the manufacturer i in regime j (Measured in monetary values)
$A^*_{ij}(t)$	Equilibrium advertising effort of the manufacturer i in regime j
α_i	Market demand (for firm i) in the absence of goodwill. $\alpha_i \in [0, \infty)$
β_i	Consumer sensitivity of brand image difference. $\beta_i \in [0, \infty)$
k_{ij}	Advertising effectiveness of firm i in regime j. $k_{ij} \in [0, \infty)$
δ_{ij}	Absorption of goodwill for manufacturer i in regime j. $\delta_{ij} \in [0, 1]$
r	Discounting factor. $r \in [0, 1]$
η_i	Damage to goodwill caused by the recall $\eta_i \in [0, 1]$
χ	Hazard rate
$D_i(\theta_1, \theta_2)$	Demand for manufacturer i in regime j
m_{ij}	Unit profit margin for manufacturer i in regime j
V_{ij}	Value function for manufacturer i in regime j
μ_i	Proportionality constant for advertising cost of Manufacturer i

Table 2.2: Model Parameters

two scenarios, we consider that the firms are farsighted, i.e., they perfectly predict the hazard rate at the beginning of the planning horizon. We further extend the one focal firm model to the case when the firms are "hazard myopic". We define hazard myopia for a firm as the behaviour of not foreseeing the chance of a recall. A firm, when myopic, will become aware of a recall when it is issued because of the widespread media attention. It is plausible that the firms reconsider their advertising decisions at the time when the firm itself or the rival issues a recall. Table 2.1 summarizes the different scenarios and decisions, while Table 2.2 summarizes the model notations.

2.2.1. The Recall Occurrence:

The planning horizon in our study is $[0,\infty)$. We also assume that during this infinite planning horizon, recall occurs only once by any manufacturer. In our subsequent description, we call the manufacturer subjected to product recall as a focal firm and the other without the occurrence of product recall as a non-focal firm. When we consider only one focal firm, let t_r be the random time of the recall. Consider χ to be the hazard rate. While χ is not necessarily between $(0,1)$, for our numerical computations we consider variations of χ in the range of (0,1). We define the probabilistic switching of the pre-crisis and the post-crisis regime by means of the stochastic process $[R(t):t\geq 0]$ defined below:

$$\begin{aligned}
\lim_{dt\to 0}\frac{P[R(t+dt)=2|R(t)=1]}{dt} &= \chi, \\
\lim_{dt\to 0}\frac{P[R(t+dt)=1|R(t)=2]}{dt} &= 0. \qquad (2.1)
\end{aligned}$$

$R(t)=1$ signifies the pre-crisis regime and $R(t)=2$ signifies the post-crisis regime. When both the firms are prone to recall, t_{r1} and t_{r2} are the random times of recall. The firms do not know ex ante if $t_{r1}>t_{r2}$ or $t_{r1}<t_{r2}$. We exclude the case of $t_{r1}=t_{r2}$. Each firm i has its own hazard rate χ_i following a stochastic process of occurrence, $[R_i(t):t\geq 0]$. The regime switches are given by the following equations.

$$\begin{aligned}
\lim_{dt\to 0}\frac{P[R_i(t+dt)=2|R_i(t)=1]}{dt} &= \chi_i, \\
\lim_{dt\to 0}\frac{P[R_i(t+dt)=1|R_i(t)=2]}{dt} &= 0. \qquad (2.2)
\end{aligned}$$

Applications of such regime switching and piecewise deterministic games can be found in the closely related studies (Boukas, Haurie, and Michel 1990; Haurie and Moresino 2006; Rubel, Naik, and Srinivasan 2011). The above definition assumes a constant hazard rate. However, this does not mean that the probability of the crisis is fixed. In many major recall events, the likelihood of crisis is predetermined by a number of factors related to product failure (Thirumalai and Sinha 2011). Hence use of a fixed likelihood is justified. For a more generic analysis, one can use a time dependent hazard rate. But the analysis of the problem becomes more complex.

The random timings of a recall essentially split the planning horizon into different decision epochs or regimes. When there is one focal firm, there are two possible decision epochs. When there are two focal firms, there are three possible decision epochs. Figure (2.1) and Figure (2.2) depicts the recall occurrence and the regimes for decision making.

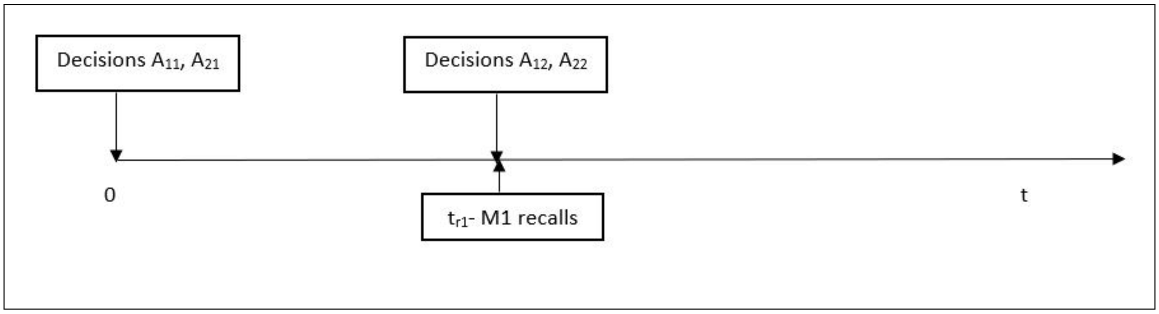


Figure 2.1: Advertising decisions - one focal firm

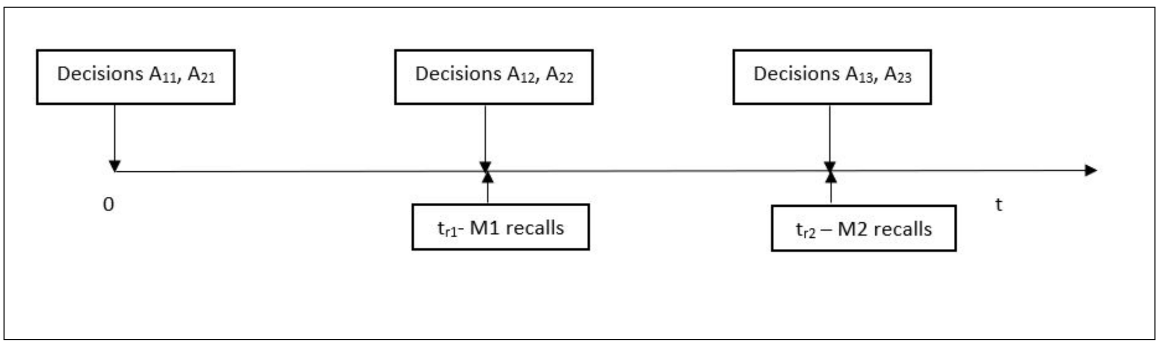


Figure 2.2: Advertising decisions -two focal firms

2.2.2. Demand Functions

We consider dynamic linear demand functions for the manufacturers M_1 and M_2. One underlying assumption is that the manufacturers have no restrictions or capacity issues and are able to meet demand if it is high. Since decisions are taken in pre-crisis and post-crisis regimes, and decisions affect the state trajectory, ideally we need a time index j for the demand functions, state variables and the decision variables. However, to avoid notational complexity and confusion *we drop the time index from the notation of demand and state variables*. We use the indices i, j for the decision variables (A_{ij}), profit margins (m_{ij}) and value functions (V_{ij}). $D_i(\theta_1(t), \theta_2(t)(t))$ is the demand for manufacturer M_i where $i \in \{1,2\}$. Thus, the demands for the two manufacturers M_1 and M_2 are respectively:

$$\begin{aligned} D_1(t) &= \alpha_1 + \beta_1(\theta_1(t) - \theta_2(t)), \\ D_2(t) &= \alpha_2 + \beta_2(\theta_2(t) - \theta_1(t)). \end{aligned} \tag{2.3}$$

The relatively simple nature of the demand function assures tractability of our model. Morover, some studies in the previous literature adopt a structurally similar price-dependent demand model (Balasubramanian and Bhardwaj 2004). In the above demand functions, $\theta_i(t)$ is the goodwill of the manufacturer i, α is the initial market size , β_i is the consumer sensitivity towards difference of the goodwills of M_i and M_j. While the positivity of the demand functions might not be guaranteed if we consider $\theta_i(t) \in [0,\infty)$, we assume that α_i is large enough and demand will remain positive for the solutions of our games. The states evolve according to the following differential equations:

$$\begin{aligned} \dot{\theta}_1(t) &= k_{1j}A_{1j}(t) - \delta_{1j}\theta_1(t), \quad \forall\, t \in [0,\infty), \theta_1(0) = \tilde{\theta}_1, \\ \dot{\theta}_2(t) &= k_{2j}A_{2j}(t) - \delta_{2j}\theta_2(t), \quad \forall\, t \in [0,\infty), \theta_2(0) = \tilde{\theta}_2. \end{aligned} \tag{2.4}$$

A_{ij} is the advertising effort of M_i in regime j and δ_i is the decay parameter (Nerlove and Arrow 1962). $\tilde{\theta}_1 \geq 0, \tilde{\theta}_2 \geq 0$ are the initial values of the brand images at the beginning of the planning horizon. k_{ij} are the advertising effectiveness.

The demand functions (2.3) and the state variables (2.4) of our model are based on the following assumptions:

1. The demand for a firm is positively affected by its own goodwill and negatively by the goodwill of its rival. This represents the competition in the duopoly.

2. We do not incorporate any negative effect of a firm's advertising directly in the state evolution of its rival. This is consistent with some of the literature (Erickson 1995).

3. A firm's marginal goodwill is positively affected by its own advertising efforts.

While our assumption 1 is captured by the demand functions, the assumptions 2 is reflected in the evolution of the states of our dynamic system, the states being - the brand images $\theta_1(t)$, $\theta_2(t)$. Assumptions 1 and 2 comport with some of the models discussed in the previous literature (Dockner et al. 2000; Karray and Zaccour 2005; Nair and Narasimhan 2006) . While Dockner et al. 2000 proposed a multiplicative dynamic demand model with linear state equations, Karray and Zaccour 2005 considers a linear demand function which is affected positively by the brand image and the efforts (investments) a firm makes to maintain its brand image. C. K. Kim and Chung 1997 showed that brand image is positively related to market share. Using the Lanchester model, one study found that the closed-loop strategies of firms depend on the realized market share (Chintagunta and Vilcassim 1992). We also investigate the closed loop strategies. However, in our case, the advertising strategies will be degenerate stationary Markov strategies.

2.2.3. Capturing the effect of recall as a jump state:

In the case of a product recall, there is a damaging effect of the recall on the recalling firm's baseline sales (H. Van Heerde, Helsen, and Dekimpe 2007; Rubel, Naik, and Srinivasan 2011). The damaging effect, however, depends on the firm's existing goodwill and the magnitude of the recall. Product recall causes erosion of brand goodwill (A. M. Eilert 2013; Fenech 2017). In our model the damage due to product recall is captured by the diminishing goodwill of the focal firm after the recall. If, in general, the manufacturer M_i recalls, We express the jump state by $\theta_{i2}(t_r^+) = (1-\eta_i)\theta_{i1}(t_r^-))$, where t_r is the time of the recall for M_i. Similar applications of jump states have been considered in the related literature (Kamien and Schwartz 2012). The focal firm's demand function increases with its own goodwill. So a loss in goodwill lowers its demand. The solution procedure of our model is given in section 2.4. In the following section, we illustrate how to incorporate the jump state using state equations (2.8).

After a product recall, advertising effectiveness can reduce (Yan Liu and Shankar 2015; Yi Zhao, Ying Zhao, and Helsen 2011) and this reduction of effectiveness again depends on the brand equity (H. Van Heerde, Helsen, and Dekimpe 2007). We capture this by the parameter k_{ij}. We assume that after a recall which causes a significant damage to the goodwill of a firm i, $k_{i1} > k_{i2}$, loss of advertising effectiveness occurs. The absorption parameter of the Nerlove-Arrow model can be used to capture the ineffectiveness of advertisement or brand forgetting. The marginal brand image is negatively affected by the absorption factor, δ_{ij}. Thus, after a recall of substantial intensity by a firm i, $\delta_{i2} > \delta_{i1}$. Thus the damage to the recalling firm can occur by damage to goodwill, loss of advertising effectiveness and increase in brand forgetting.

2.2.4. The Manufacturers' Decision Problems in presence of one focal firm

2.2.4.1. Farsighted Firms

Without the loss of generality, we assume that M_1 is the focal firm. Both M_1 and M_2 anticipate the hazard rate, χ at the beginning of the planning horizon. The manufacturers face their individual profit maximizing problems. We have considered the quadratic costs of advertising efforts. Such a cost function, given by $C(A_{ij}) = \frac{\mu_i}{2}A_{ij}^2$ has been widely used in the literature (e.g. (Karray and Zaccour 2005)). μ_i is the proportionality constant of the quadratic cost. Since, in case of one focal firm, a product-recall splits the decision horizon into two regimes - pre-crisis and post-crisis, in general the decisions made by the firms will differ in those two regimes. The time of recall is stochastic. Therefore, the long-term profit is an expected sum of the profits in the post-crisis and pre-crisis regimes. Let A_{ij} be the advertising effort and π_{ij} be the profit for manufacturer M_i in regime j.

For Manufacturer i, the profits in each regime are given by:

$$\begin{aligned}
\pi_{i1} = \pi_{i1}(A_{i1}) &= \int_0^{t_r} e^{-rt}[m_{i1}D_1(\theta_1,\theta_2) - \frac{\mu_1}{2}A_{i1}^2]dt \\
&= \int_0^{t_r} e^{-rt} J_{i1}(A_{i1})dt, \\
\pi_{i2} = \pi_{i2}(A_{i2}) &= \int_{t_r}^{\infty} e^{-rt}[m_{i2}D_2(\theta_1,\theta_2) - \frac{\mu_1}{2}A_{i2}^2]dt \\
&= \int_{t_r}^{\infty} e^{-rt} J_{i2}(A_{i2})dt.
\end{aligned} \tag{2.5}$$

Now, t_r, the time of recall, is a random time which is not known in advance. This implies that the profits in the two periods are random variables. So the long-term expected profit is given by, $\Pi_i(A_{i1},A_{i2}) = E[\pi_{i1} + e^{-rt_r}\pi_{i2}]$ where the expectation E[.] is taken with respect to the crisis occurrence process. The discount factor e^{-rt_r} appears because π_{i1} accrues at $t = 0$ and π_{i2} at $t = t_r$. Thus, it is required to discount π_{12} back to $t = 0$ to add the two long-term profits. The value of the long-term profit will therefore depends on the strategies (A_{i1},A_{i2}) chosen in the two regimes for $i \in \{1,2\}$. The problem above is a random stopping problem. If $f(t)$ and $F(t)$ are the probability density and cumulative density functions of the stochastic occurrence process, then the hazard rate is $h(t) = \chi, F(t) = 1 - e^{-\int_0^t h(s)ds}$. Therefore, $f(t) = \chi e^{-\int_0^t h(s)ds}$. The long-term expected profit for the manufacturer i can be written as:

$$\Pi_i(A_{i1},A_{i2}) = E[\int_0^{t_r} e^{-rs} J_{i1} ds + e^{-rt_r}\pi_{i2}] \text{ where } i \in \{1,2\}, \tag{2.6}$$

where the first term under the expectation gives the profit of the pre-crisis period and the second term gives the profit of the post-crisis regime. Therefore, the sum of the two profits gives the long-term profit over the planning horizon. Integrating by parts and making algebraic manipulations (Haurie and Moresino 2006; Rubel, Naik, and Srinivasan 2011) the above expression can be transformed into the following equation (see appendix A):

$$\Pi_i(A_{i1},A_{i2},\chi) = \int_0^{\infty} e^{-(r+\chi)t}\{J_{i1} + \chi\pi_{i2}\}dt. \tag{2.7}$$

The management of both the firms would like to optimize their long-term profit. Thus, given the demand dynamics and the state evolution, they would like to choose the advertising efforts which would maximize the expected profit over the planning horizon. In order to solve the problems for the manufacturers, we have to start by solving for the value functions of the post-crisis regime first (Rubel, Naik, and Srinivasan 2011; Haurie and Moresino 2006) . The value functions for the M_1

and M_2 in the second regime is given by:

$$\begin{aligned} V_{12}(\theta_1,\theta_2) &= \underset{A_{12}}{\text{Max}} \int_{t_r}^{\infty} e^{-rt}[m_{12}D_1(t) - \frac{\mu_1}{2}A_{12}^2]dt, \\ V_{22}(\theta_1,\theta_2) &= \underset{A_{22}}{\text{Max}} \int_{t_r}^{\infty} e^{-rt}[m_{22}D_2(t) - \frac{\mu_2}{2}A_{22}^2]dt, \\ &\text{Subject to} \\ \dot{\theta}_{12}(t) &= k_{12}A_{12}(t) - \delta_{12}\theta_1(t), \quad \theta_1(t_r+) = (1-\eta)\theta_1(t_r-), \\ \dot{\theta}_{22}(t) &= k_{22}A_{22}(t) - \delta_{22}\theta_2(t), \quad \theta_2(t_r) = \theta_{2t_r}, \end{aligned} \tag{2.8}$$

where θ_{2t_r} is the goodwill of firm 2 at time t_r . We want to find the feedback strategies for the firms. Therefore, as a standard solution procedure we start by writing the Hamilton-Jacobi-Bellman (HJB) equations (Dockner et al. 2000). Subsequently, the first order conditions on the decision variables will help us in finding the equilibrium strategies from the HJB equations. We note that the value functions $V_{ij}(\theta_1,\theta_2)$ are concave in A_{ij} since $\frac{\partial^2 V_{ij}}{\partial A_{ij}^2} = -\mu_i < 0$. The HJB equations for the manufacturers 1 and 2 in the second regime are respectively given by:

$$\begin{aligned} rV_{12}(\theta_1,\theta_2) &= \underset{A_{12}}{\text{Max}}[(\alpha_1 + \beta_1(\theta_1 - \theta_2)m_{12} - \frac{\mu_1}{2}A_{12}^2 + \frac{\partial V_{12}}{\partial \theta_1}\dot{\theta}_1(t) + \\ &\quad \frac{\partial V_{12}}{\partial \theta_2}\dot{\theta}_2(t)], \\ rV_{22}(\theta_1,\theta_2) &= \underset{A_{22}}{\text{Max}}[(\alpha_2 + \beta_2(\theta_2 - \theta_1))m_{22} - \frac{\mu_2}{2}A_{12}^2 + \frac{\partial V_{22}}{\partial \theta_1}\dot{\theta}_1(t) + \\ &\quad \frac{\partial V_{22}}{\partial \theta_2}\dot{\theta}_2(t)]. \end{aligned} \tag{2.9}$$

The problems for the firms in the first regime, in accordance with equation (2.7) are thus given below:

$$\begin{aligned} V_{11}(\theta_1,\theta_2) &= \underset{A_{11}}{\text{Max}} \int_0^{\infty} e^{-(r+\chi)t}[m_{11}D_1(t) - \frac{\mu_1}{2}A_{11}^2 + \chi V_{12}((1-\eta)\theta_1,\theta_2)]dt, \\ V_{21}(\theta_1,\theta_2) &= \underset{A_{21}}{\text{Max}} \int_0^{\infty} e^{-(r+\chi)t}[m_{21}D_2(t) - \frac{\mu_2}{2}A_{21}^2 + \chi V_{22}((1-\eta)\theta_1,\theta_2)]dt, \\ &\text{Subject to} \\ \dot{\theta}_1(t) &= k_{11}A_{11}(t) - \delta_{11}\theta_1(t), \quad \forall t \in [0,\infty), \theta_1(0) = \theta_{10}, \\ \dot{\theta}_2(t) &= k_{21}A_{21}(t)) - \delta_{21}\theta_2(t), \quad \forall t \in [0,\infty), \theta_2(0) = \theta_{20}, \end{aligned} \tag{2.10}$$

where θ_{10}, θ_{20} are the initial goodwills of firm 1 and firm2.

From equation (2.10), the pre-crisis HJB equations for the manufacturers 1 and 2 are given by

:

$$(r+\chi)V_{11}(\theta_1,\theta_2) = \underset{A_{11}}{\text{Max}}[(\alpha_1+\beta_1(\theta_1-\theta_2))m_{11}-\frac{\mu_1}{2}A_{11}^2+\frac{\partial V_{11}}{\partial \theta_1}\dot{\theta}_1(t)+ \frac{\partial V_{11}}{\partial \theta_2}\dot{\theta}_2(t)+\chi V_{12}((1-\eta)\theta_1,\theta_2)], \tag{2.11}$$

$$(r+\chi)V_{21}(\theta_1,\theta_2) = \underset{A_{21}}{\text{Max}}[(\alpha_2+\beta_2(\theta_2-\theta_1))m_{21}-\frac{\mu_2}{2}A_{21}^2+\frac{\partial V_{21}}{\partial \theta_1}\dot{\theta}_1(t)+ \frac{\partial V_{21}}{\partial \theta_2}\dot{\theta}_2(t)+\chi V_{22}((1-\eta)\theta_1,\theta_2)].$$

2.2.4.2. Hazard Myopic Firms

The marketing literature has stressed the importance of envisioning the impending product recall while making decisions. However, the assumption that the rival firms would always know and consider each others hazard rate can be questionable. Then some viable questions can be -

- What is the effect of ignoring crisis probability on the decision and performance of the firms?
- Is there a condition under which a myopic firm has a better performance?

Though the firms overlook the hazard probability at the beginning of the planning horizon, it is understandable that they will become aware of a recall when it is announced.i.e. they will notice the change in goodwill as the damage η occurs. Hence, the firms can potentially change the equilibrium advertising policies to ensure better profits. We assume that both M_1 and M_2 do not foresee χ, the hazard rate.

When both M_1 and M_2 are myopic, the solution procedure starts with solving for the equilibrium decisions and value functions for the first regime and then doing the same for the second regime. With crisis not in sight, the decision problems for the manufacturers in the first regime are:

$$V_{11}(\theta_1,\theta_2) = \underset{A_{11}}{\text{Max}}\int_0^\infty e^{-rt}[m_{11}D_1(t)-\frac{\mu_1}{2}A_{11}^2]dt, \tag{2.12}$$

$$V_{21}(\theta_1,\theta_2) = \underset{A_{21}}{\text{Max}}\int_0^\infty e^{-rt}[m_{21}D_2(t)-\frac{\mu_2}{2}A_{21}^2]dt,$$

Subject to

$$\dot{\theta}_1(t) = k_{11}A_{11}(t)-\delta_{11}\theta_1(t), \quad \forall t\in[0,\infty),\theta_1(0)=\theta_{10},$$

$$\dot{\theta}_2(t) = k_{21}A_{21}(t))-\delta_{21}\theta_2(t), \quad \forall t\in[0,\infty),\theta_2(0)=\theta_{20}.$$

The second regime decision problems are same as (2.12) with the exception that the margins change to m_{12} and m_{22} respectively for the focal and the non focal firms, respectively. The state equations also change as the damage occurs. Thus, the second regime problems are given by:

$$
\begin{aligned}
V_{12}(\theta_1,\theta_2) &= \underset{A_{12}}{\text{Max}} \int_{t_r}^{\infty} e^{-rt}[m_{12}D_1(t) - \frac{\mu_1}{2}A_{12}^2]dt, \\
V_{22}(\theta_1,\theta_2) &= \underset{A_{22}}{\text{Max}} \int_{t_r}^{\infty} e^{-rt}[m_{22}D_2(t) - \frac{\mu_2}{2}A_{22}^2]dt, \\
\text{Subject to} & \\
\dot{\theta}_1(t) &= k_{12}A_{12}(t) - \delta_{12}\theta_1(t), \quad \forall\, t \in (t_r,\infty),\ \theta_1(t_r^+) = (1-\eta)\theta_1(t_r^-), \\
\dot{\theta}_2(t) &= k_{22}A_{22}(t)) - \delta_{22}\theta_2(t), \quad \forall t \in (t_r,\infty), \theta_2(t_r) = \theta_{t_r}.
\end{aligned}
\tag{2.13}
$$

Proceeding similarly as in the far sighted firms' cases, we can similarly derive the HJB equations for the myopic firms and solve for the equilibrium feedback strategies.

2.2.5. The Manufacturers' Decision Problems in presence of two focal firms

In this case, the market consists of two focal manufacturers M_i. Both manufacturers are prone to recall and each has a specific hazard rate of χ_i where $i \in \{1,2\}$. We are assuming that after the recall both the manufacturers still sell their products in the market or in other words the recalls are partial. For example, in Feb 2018, Fujitsu Canada, recalled only certain computers with defective batteries. Other units or models of the same (E-series) were still available in the market after the recall.

When the firms are farsighted, both the hazard rates are common knowledge. However, at the beginning of the horizon neither firm knows who will recall first. Consequently, the value function of firm i at the beginning of the horizon is given by:

$$
\begin{aligned}
V_{i1}(\theta_1,\theta_2) = \underset{A_{i1}}{\text{Max}} \underset{R_i,R_j}{\mathbb{E}} \Big(\int_0^{\min\{t_i,t_j\}} e^{-rt}\Pi_{i1}(t)dt + e^{-rt_i}V_{i2}(\theta_i,\theta_j).\boldsymbol{\Phi}[\boldsymbol{t_i < t_j}] \\
+ e^{-rt_j}\hat{V}_{i2}(\theta_i,\theta_j).\boldsymbol{\Phi}[\boldsymbol{t_j < t_i}]\Big),
\end{aligned}
\tag{2.14}
$$

where the operators $\boldsymbol{\Phi}[\boldsymbol{t_i < t_j}]$ and $\boldsymbol{\Phi}[\boldsymbol{t_j < t_i}]$ are defined in the following manner:

$$
\begin{aligned}
\boldsymbol{\Phi}[\boldsymbol{t_i < t_j}] &= 1, \quad \text{if} \quad t_i < t_j \\
&= 0, \quad \text{otherwise}, \\
\boldsymbol{\Phi}[\boldsymbol{t_j < t_i}] &= 1, \quad \text{if} \quad t_j < t_i \\
&= 0 \quad \text{otherwise}.
\end{aligned}
\tag{2.15}
$$

and the second regime value functions V_{i2} are given by

$$\begin{aligned} V_{i2}(\theta_i,\theta_j) &= \underset{A_{i2}}{\text{Max}} \int_0^{\infty} e^{-(r+\chi_j)t}\{\Pi_{i2}(t)dt + \chi_j V_{i3}(\theta_i,(1-\eta_j)\theta_j)\}, \\ \hat{V}_{i2}(\theta_i,\theta_j) &= \underset{A_{i2}}{\text{Max}} \int_0^{\infty} e^{-(r+\chi_i)t}\{\Pi_{i2}(t)dt + \chi_i V_{i3}((1-\eta_i)\theta_i,\theta_j)\}. \end{aligned} \tag{2.16}$$

Here $\Pi_{i1}(t)$ is the profit of the manufacturer i in the pre-recall regime. Similar formulation of value function can be found in (Rubel 2018) where one of the firms becomes a monopolist after the recall as products are taken off the market. We have relaxed this assumption in our model because recalls can be partial as well. Hence both the firms continue to compete after recall. In other words, the market remains a duopoly market in all the three decision regimes - the pre-recall regime, the regime between the first recall and second recall and the post-recall regime for both the firms. Hence, the horizon now has three decision epochs - *the pre-recall epoch*, *the epoch between the recalls made by M_1 and M_2* and *the post-recall epoch*.

The derivation of the value function for the firm i in regime 1 are based on the premises that the occurrences of the two recalls are independent of one another. The probability of the random variable $\min\{t_i,t_j\}$ can be defined as - $\Pr(\min\{t_i,t_j\} > t)$=$\Pr\{(t_i > t) \cap (t_j > t)\}$=$\Pr(t_i > t)$.$\Pr(t_j > t)$=$e^{-t(\chi_i+\chi_j)}$. Thus proceeding in a similar manner as in equations (2.10) and making some algebraic manipulations, we get:

$$\begin{aligned} V_{i1}(\theta_i,\theta_j) = \underset{A_{i1}}{\text{Max}} \int_0^{\infty} e^{-(r+\chi_i+\chi_j)t}\{&\Pi_{i1}(t)dt + \chi_i V_{i2}(\theta_i,\theta_j) \\ &+\chi_j \hat{V}_{i2}(\theta_i,\theta_j)\}. \end{aligned} \tag{2.17}$$

Similarly, for the j^{th} firm, the value function in the first regime is given by:

$$\begin{aligned} V_{j1}(\theta_i,\theta_j) = \underset{A_{j1}}{\text{Max}} \int_0^{\infty} e^{-(r+\chi_i+\chi_j)t}\{&\Pi_{j1}(t)dt + \chi_i V_{j2}(\theta_i,\theta_j) \\ &+\chi_j \hat{V}_{j2}(\theta_i,\theta_j)\}. \end{aligned} \tag{2.18}$$

The details of the derivation of the first regime value function is given in the appendix A.

Remark: The recall damages are still captured by the jump states in this case when two focal firms compete. However, it is only after the first recall , i.e., at t_{r1} that we know who has recalled first. If M_i recalls first at t_{r1} and M_j recalls second at t_{r2}, then

$$\begin{aligned}\theta_i(t_{r1}^{+}) &= (1-\eta_i)\theta_i(t_{r1}^{-}),\\ \theta_j(t_{r2}^{+}) &= (1-\eta_j)\theta_j(t_{r2}^{-}),\end{aligned}$$

capture the jump states and the impacts of the recalls.

2.3. Analytical Results and Discussion

First, we want to reiterate that what our goals are and how the model elements are articulated to help us reach our goals. Our goals are to find the equilibrium advertising of the manufacturers in the pre-crisis and post-crisis regimes in the two cases - one focal firm issuing a recall, two focal firms susceptible to recall. First, we solve the manufacturers' problems to find equilibrium advertising strategies. We start with finding the value function for the post-crisis regime using HJB equations (2.9). Then using equations (2.11) we find the equilibrium advertising for the first regime. Proofs of all the propositions, unless trivial, are given in the appendix A.

2.3.1. Equilibrium Advertising and Profits - One Focal Firm

We investigate the scenario when one of the competing firms has a probability of recall and both the firms make their decisions taking into account the hazard rate of the focal firm. In other words, the hazard rate here is common knowledge for both the players. A_{ij}^* is the equilibrium advertising effort of manufacturer i in regime j, where $i,j \in \{1,2\}$.

2.3.1.1. Decisions of the Farsighted Firms

Proposition 1. ***(a)*** *The post-crisis equilibrium advertising efforts of the manufacturers M_1 and M_2 are respectively given by:*

$$\begin{aligned}A_{12}^* &= \frac{k_{12}m_{12}\beta_1}{\mu_1(r+\delta_{12})},\\ A_{22}^* &= \frac{k_{22}m_{22}\beta_2}{\mu_2(r+\delta_{22})}.\end{aligned} \tag{2.19}$$

(b) *For the pre-crisis advertising efforts of the manufacturers M_1 and M_2 are respectively given by:*

$$\begin{aligned}A_{11}^* &= \frac{k_{11}\beta_1\big(m_{11}(r+\delta_{12})+m_{12}(1-\eta)\chi\big)}{\mu_1(r+\chi+\delta_{11})(r+\delta_{12})},\\ A_{21}^* &= \frac{k_{21}\beta_2\big(m_{21}(r+\delta_{22})+\chi m_{22}\big)}{\mu_2(r+\chi+\delta_{21})(r+\delta_{22})}.\end{aligned} \tag{2.20}$$

The above are the feedback advertising strategies of the two manufacturers in the two regimes. The strategies, in this case, are stationary and hence Markov perfect. In the absence of a crisis, i.e when the hazard rate, $\chi = 0$ and hence the damage, $\eta = 0$, the strategies of both the regimes are identical. Once a recall has occurred, the firm M_1 does not anticipate any more recall during the planning horizon. This is a primary assumption of our model. Hence, the equilibrium advertisements are free from the hazard rate in the second regime. The equilibrium advertising efforts in the pre-crisis regime incorporate the hazard rate χ . This emphasizes the foresightedness of the management of both the focal and non-focal firms as they take the probability of recall into account when they make their advertising decisions even before the recall occurs. This is consistent with the previous findings that risk creates impatience. (Rubel, Naik, and Srinivasan 2011).

In the following lemmas, we present some properties of the equilibrium advertising policies with respect to the parameters. Then we perform a comparative analysis of the equilibrium advertising expressions and infer on its variance in the pre and post-crisis regimes.

Lemma 1. *The equilibrium advertisements efforts of both the firms in the second regime are:*

(a) increasing with respective margins of the firms m_{12} or m_{22},

(b) increasing with the sensitivity of demand towards brand difference β_i,

(c) decreasing with the absorption δ_{ij} and discounting factor r.

Lemma 1 has some significant implications. Higher margins result in higher equilibrium advertising in the second regime for both the manufacturers. If a manufacturer wants a higher margin, it can either reduce manufacturing costs or increase the prices. We argue that the M_1 produces more thereby benefiting from the economies of scale and lowering the cost. In the post-crisis regime, this production cost savings can help the manufacturer to nullify the effects of the profit loss by the recall. Moreover, the resulting excess advertising will help it in gaining more market and positively impact its own revenue. The argument that the M_1 will increase the price after the recall is quite flimsy on the grounds of its recent goodwill loss. M_2 can take advantage of the situation by increasing its price (thereby increasing margin, advertisement and hence demand) or produce more at a lower cost as it expects to sale more in the wake of its rival's crisis.

For a firm with higher goodwill, in general, it can be assumed that brand forgetting δ_{ij} will be lesser than its rival with lower brand equity. We found that the advertising efforts of the firms are decreasing with δ_{ij}. The above finding is counter intuitive as one might expect that a brand will invest more in advertising if the forgetting is high. However, δ_{ij} is only one factor that affects goodwill. The firm might cut advertising when absorption factor is high and put efforts in other areas like increasing advertising effectiveness or consumers' perception of the brand difference. The effort of the firm ensures that it finds a trade-off amongst different parameters affecting advertising to obtain the equilibrium profit.

For two firms which are symmetric in all parameters except δ_{1j} and δ_{2j}, either $\delta_{1j} > \delta_{2j}$ or $\delta_{1j} < \delta_{2j}$. If $\delta_{1j} > \delta_{2j}$, the non-focal firm knows that the rival's marginal brand image is subject to heavier erosion than its own. Therefore, it behaves opportunistically by advertising significantly more than the recalling firm to capitalize on the situation.

Lemma 2. *(a) In the pre-crisis regime, for the focal firm M_1, the equilibrium advertising is decreasing with the hazard rate if:*

$$\eta > 1 - \frac{m_{11}(r+\delta_{12})}{m_{12}(r+\delta_{11})}. \tag{2.21}$$

(b) *In the pre-crisis regime, for the non-focal firm M_2, the equilibrium advertising is always decreasing with the hazard rate if,*

$$m_{22}(r+\delta_{21}) < m_{21}(r+\delta_{22}) \tag{2.22}$$

.

From the above lemma, for a focal firm, the pre-crisis advertising decreases with the likelihood of crisis if the crisis impact is larger than a certain threshold given by (2.21). If impact is very low and there is no change in the margins or for absorption δ_{1j}, then condition (2.21) reduces to $\eta > 0$. Thus for any positive low impact recall the advertising will always decrease with the likelihood. On the other hand, for a high impact recall for the focal, the variation of advertising efforts with crisis likelihood will depend on the margins and the absorption δ_{1j} in the two periods. This explains that if a high recall probability is anticipated by the recalling firm, it may not put a lot of efforts in advertising as high advertising may result in high sales and consequently high recall costs.

If we assume that the absorption δ_{2j} remains unchanged in the two periods for the rival firm M_2, the advertising efforts of M_2 will decrease in the hazard rate only if its second regime unit profit margin is lower than the first regimes profit margin. This reflects that if M_2 reduces profit margin by strategies like price reduction, it saves on advertising cost for a high hazard rate. A high likelihood of crisis gives M_2 complacency and it expects to gain advantage from the crisis opportunity.

Lemma 3. *The equilibrium advertising effort for the focal firm in pre-crisis regime is decreasing with the damaging effect, η.*

Lemma 3 is consistent with the previous literature, mainly the model of (Rubel, Naik, and

Srinivasan 2011) based on (Sethi 1983). The damage effect η cannot be estimated ex-ante unless the recall root causes and affected units are known. A high damage in goodwill means lesser profit in the post-crisis period. Thus the significance of Lemma 3 is that, if a high damage rate is anticipated, the management might want to cut off advertisement costs in regime 1 in order to advertise more in the post-crisis regime thereby maximizing goodwill and sales. In our model, the damaging effect does not directly impact demand. But the damage rate accounts for the jump in the state variable θ_1 which is the brand image of manufacturer 1. This, in turn, affects the demand negatively. Therefore, if sales or demand is affected by the damage, either directly or indirectly, and the management anticipates the recall, it is likely that a high damage expectation will reduce advertisement spending in the first regime.

2.3.1.2. Is the post-crisis advertising higher than the pre-crisis advertising?

The previous literature found empirical evidence that ad spending increases after the recall. In our study, we find the conditions under which the post-crisis advertising efforts are more than the pre-crisis advertising. The mathematical structures of the advertising efforts that we derived, equations (2.19) and (2.20), reflect that there is no direct answer to the question - should advertising before recall be more than that after the recall i.e. is

$A^*_{11} < (>) A^*_{12}$ or $A^*_{21} < (>) A^*_{22}$? The obvious answer is that it depends on the parameter values of our model, most important of which are the (χ, η) pair.

Proposition 2. ***(a)*** *For the focal firm(M_1), the post-crisis advertising effort is more than the pre-crisis advertising effort if,*

$$\chi > \frac{k_{11}m_{11}(r+\delta_{12}) - k_{12}m_{12}(r+\delta_{11})}{m_{12}\big(k_{12} - k_{11}(1-\eta)\big)}. \tag{2.23}$$

(b) *For the non-focal firm (M_2), the post-crisis advertising effort is more than the pre-crisis advertising effort if,*

$$\chi > \frac{k_{21}m_{21}(r+\delta_{22}) - k_{22}m_{22}(r+\delta_{21})}{m_{22}(k_{21} - k_{22})}. \tag{2.24}$$

.

Proposition 2 shows that the focal firm's equilibrium advertising will be more or less in the post-crisis period depending on the crisis likelihood χ. However, the threshold for this χ also depends on the margins. Assuming that the focal firm's second regime's margin is affected by the recall damage η, the advertising efforts in the post and pre-crisis period depends on χ (directly) and η (indirectly). More about this is discussed under numerical analysis.

The previous literature (Rubel, Naik, and Srinivasan 2011) showed that in a competitive environment non-focal firms should increase post-crisis ad spending in order to exploit the situation. However, our analysis suggests that equilibrium advertising for the non-focal firm should be definitely more in the second regime, if the firm's second regime's margin is higher or equal to the margin in the first regime. Otherwise, the crisis likelihood will determine the level of advertising efforts.

From a strategic point of view, the profit margins can be influenced by pricing decisions. However, pricing strategies are not in the scope of our paper, albeit our analysis accentuates the importance of dual decision making with respect to pricing and advertising during a recall as suggested by some of the existing literature (Cleeren 2015).

2.3.1.3. The effect of recall on the expected profits of M_1 and M_2

The value functions were explicitly derived by the method of comparison of coefficients. These are the instantaneous profits of the firms under consideration. It is of interest to have a look into the structure of the value functions as we can possibly find out the influence of the different parameters on the value functions. This in turn will give us an estimate of how the long-term profit can be influenced by the parameters.

Proposition 3. ***(a)*** *The value function for manufacturer M_1 in regime 2 is given by:* $V_{12} = a_{12}\theta_1(t) + b_{12}\theta_2(t) + c_{12}$ *where a_{12}, b_{12}, c_{12} are given by:*

$$\begin{aligned} a_{12} &= \frac{m_{12}\beta_1}{(r+\delta_{12})}, \\ b_{12} &= \frac{-m_{12}\beta_1}{(r+\delta_{22})}, \\ c_{12} &= \frac{m_{12}\left(2\alpha_1 + \beta_1\left(\frac{k_{12}^2 m_{12}\beta_1}{(r+\delta_{12})^2\mu_1} - \frac{2k_{22}^2 m_{22}\beta_2}{(r+\delta_{22})^2\mu_2}\right)\right)}{2r}. \end{aligned} \tag{2.25}$$

(b)) *The value function for manufacturer M_1 in regime 1 is given by $V_{11} = a_{11}\theta_1(t) + b_{11}\theta_2(t) + c_{11}$ where a_{11}, b_{11}, c_{11} are defined as:*

$$\begin{aligned} a_{11} &= \frac{\beta_1\left(m_{11}(r+\delta_{12}) + m_{12}(1-\eta)\chi\right)}{(r+\delta_{12})(r+\delta_{11}+\chi)}, \\ b_{11} &= -\frac{\beta_1\left(m_{11}(r+\delta_{22}) + m_{12}\chi\right)}{(r+\delta_{12})(r+\delta_{11}+\chi)}, \\ c_{11} &= \frac{1}{(r+\chi)}\left[m_{11}\alpha_1 + \frac{(k_{11}a_{11})^2}{2\mu_1} + \frac{k_{21}^2 b_{21} b_{11}}{\mu_2} + \chi c_{12}\right]. \end{aligned} \tag{2.26}$$

From proposition (3 b) we can evaluate the long term profit of the firm M1 by using the initial values of the state variables $\theta_1(0)$ and $\theta_2(0)$ at $t = 0$. Noting that $a_{11} > 0$ and $b_{11} < 0$, we infer that the focal firm's profit increases with its own goodwill and decreases with the goodwill of the rival. The expression of c_{11}, the constant term is very complicated and its sign depends on a complex relationship amongst the model parameters. However, for our set of parameters, we verified numerically that the constant term is positive.

Proposition 4. ***(a)*** *The value function for manufacturer M_2 in regime 2 is given by:* $V_{22} = a_{22}\theta_1(t) + b_{22}\theta_2(t) + c_{22}$ *where a_{22}, b_{22}, c_{22} are given by:*

$$\begin{aligned} a_{22} &= \frac{-m_{22}\beta_2}{(r+\delta_1)}, \\ b_{22} &= \frac{m_{22}\beta_2}{(r+\delta_2)}, \\ c_{22} &= \frac{m_{22}\left(2\alpha_2 + \beta_2\left(\frac{k_{22}^2 m_{22}\beta_2}{(r+\delta_{22})^2\mu_2} - \frac{2k_{12}^2 m_{12}\beta_1}{(r+\delta_{12})^2\mu_1}\right)\right)}{2r}. \end{aligned} \tag{2.27}$$

(b) *The value function for manufacturer 2 in regime 1 is given by $V_{21} = a_{21}\theta_1(t) + b_{21}\theta_2(t) + c_{21}$ where a_{21}, b_{21}, c_{21} are given by:*

$$\begin{aligned} a_{21} &= -\frac{\beta_2\left(m_{21}(r+\delta_{12}) - m_{22}(1-\eta)\chi)\right)}{(r+\delta_{12})(r+\delta_{11}+\chi)}, \\ b_{21} &= \frac{\beta_2\left(m_{21}(r+\delta_{22}) + m_{22}\chi\right)}{(r+\delta_{22})(r+\delta_{21}+\chi)}, \\ c_{21} &= \frac{1}{(r+\chi)}\left[m_{21}\alpha_1 + \frac{(k_{21}b_{21})^2}{2\mu_2} + \frac{k_{11}^2 a_{21}a_{11}}{\mu_1} + \chi c_{22}\right]. \end{aligned} \tag{2.28}$$

From proposition (4 b) we can evaluate the long term profit of the firm M2, the non-focal firm, by using the initial values of the state variables $\theta_1(0)$ and $\theta_2(0)$ at $t = 0$. In this case, $a_{21} < 0$ and $b_{21} > 0$. Thus, the non-focal firm's profit increases with its own goodwill and decreases with the goodwill of the focal firm. The expression of c_{21}, the constant term is as complex as c_{11}. However, for the non-focal firm also, with our set of parameters values, we have verified numerically that the constant term is positive. Table 2.3 summarizes the relationship amongst the state variables and the Value functions of the two firms in the two regimes.

	θ_1	θ_2
V_{11}	$\uparrow$	$\downarrow$
V_{12}	$\uparrow$	$\downarrow$
V_{21}	$\downarrow$	$\uparrow$
V_{22}	$\downarrow$	$\uparrow$

Table 2.3: Value function vs state variables

2.3.1.4. Decisions of Hazard Myopic Firms

The hazard myopia leads to policies where the firms do not anticipate the hazard rate, χ or damage, η. As a result, unlike the far-sighted case, these factors do not appear in the mathematical expressions of the pre-crisis policies. However, the damage η appears in the expression of the value functions of the firms. This is because the firms observe the damage caused after the recall. Thus, myopia might lead to identical policies in the two regimes but affects the firms' performances as the firms observe the goodwill erosion.

Proposition 5. *In the presence of one focal firm(M_1), when both the firms are hazard myopic, the equilibrium advertising policies are given by:*

$$\begin{aligned} A_{1j}^* &= \frac{k_{1j}m_{1j}\beta_1}{\mu_1(r+\delta_{1j})}, \\ A_{2j}^* &= \frac{k_{2j}m_{2j}\beta_2}{\mu_2(r+\delta_{2j})}, \textit{ for } j \in \{1,2\}. \end{aligned}$$

Since we considered the firms to be myopic, the hazard rate χ, does not affect the value function of the firms (Proposition 5). However, to understand the effect of η, on firm's profits, we need to elaborate the procedure of the calculation of value functions for the firms under this scenario.

The procedure to find the value function (expected long-term profit) of the myopic firm is different from case when the firms are far sighted. The procedure is given below:

1. Using appropriate Hamilton-Jacobi-Bellman equations find the equilibrium policies of the focal firm.

2. Using the equilibrium policy, evaluate
$V_{11}(\theta_1,\theta_2) = \int_0^{t_r} e^{-rt}[m_{12}D_1(t) - \frac{\mu_1}{2}{A_{11}^*}^2]dt + V_{12}(\theta_1,\theta_2)$ where
$V_{12}(\theta_1,\theta_2) = a_{12}(1-\eta)\theta_1(t_r) + b_{12}\theta_2(t_r) + c_{12}$.

Similarly, the non focal firm's value function can also be evaluated. A myopic firm does not anticipate the recall occurrence or the damage intensity. Thus its problem is deterministic and the profit will change for different values of t_r. In the numerical experiments section we compare the myopic and non myopic value functions and discuss more about this.

2.3.2. Equilibrium advertising in presence of two focal firms

When two firms anticipate recall, there are three decision epochs - pre-crisis epoch (1), the epoch between two recalls (2), post-crisis epoch (3). It is not known in advance who will recall first. The two recalls are characterized by two different stochastic processes with hazard rates χ_1 and χ_2 for M_1 and M_2 respectively. We first find the equilibrium advertising decisions in the three regimes. While the decisions at the beginning of epoch 1 and epoch 3 are unambiguously determined, the decision at the beginning of the second epoch depends on who recalls first.

Proposition 6. *The equilibrium advertising of the i^{th} firm in the three epochs are given by:*

$$\begin{aligned}
A_{i1}^* &= \frac{k_{i1}[(\beta_i m_{i1} + \chi_j \hat{a}_{i2} + \chi_i a_{i2}]}{\mu_i(r+\chi_i+\chi_j+\delta_{i1})}, \\
A_{i2}^* &= \frac{k_{i2}}{\mu_i(r+\chi_j+\delta_{i2})}[\beta_i m_{i2} + \chi_j a_{i3}] \text{ if manufacturer i recalls first }, \\
\hat{A}_{i2}^* &= \frac{\hat{k}_{i2}}{\mu_i(r+\chi_i+\hat{\delta}_{i2})}[\beta_i \hat{m}_{i2} + \chi_i(1-\eta_i)a_{i3}] \text{ if manufacturer i recalls second }, \\
A_{i3}^* &= \frac{k_{i3}m_{i3}\beta_i}{\mu_i(r+\delta_{i3})},
\end{aligned} \tag{2.29}$$

where $a_{i3} = \frac{m_{i3}\beta_i}{r+\delta_{i3}}$, $a_{i2} = \frac{\beta_i m_{i2} + \chi_j a_{i3}}{r+\delta_{i2}+\chi_j}$, $\hat{a}_{i2} = \frac{\beta_i \hat{m}_{i2} + \chi_i(1-\eta_i)a_{i3}}{r+\hat{\delta}_{i2}+\chi_i}$.

Here a_{ij}, b_{ij}, c_{ij}, $\hat{a}_{ij}, \hat{b}_{ij}, \hat{c}_{ij}$ are the coefficients of the state variables of the value functions in regimes j. If M_i recalls first, the second regime value function is given by $V(\theta_i,\theta_j) = a_{i2}\theta_i + b_{i2}\theta_j + c_{i2}$. If M_i recalls second, the second regime value function is given by $V(\theta_i,\theta_j) = \hat{a}_{i2}\theta_i + \hat{b}_{i2}\theta_j + \hat{c}_{i2}$. The expressions of these coefficients are given in the appendix A. We also denote the second regime margins differently depending on who recalls first. M_i recalls first, the margin is denoted by m_{i2} , otherwise it is denoted by $\hat{m}_{i2}$. The logic behind using different notations for margin is that, m_{i2} and $\hat{m}_{i2}$ are not necessarily same. For example, the profit margin may drop after a recall, and remain same when the rival recalls. In that case, $m_{i2} = m_{i1} > \hat{m}_{i2}$.

The equilibrium decisions are complex expressions of the model parameters. It is clear that for the i^{th} firm the first regime advertising is decreasing with η_i. The second regime advertising is decreasing with η_i if the i^{th} firm recalls second. The variance of the advertising with respect to χ_i and χ_j depends on certain conditions of our parameters. We present one important proposition.

Proposition 7. ***(a)**:The first regime equilibrium advertising efforts A_{i1}^* of M_i is decreasing with the damage η_i.*

(b) *When M_i recalls first, the second regime advertising A^*_{i2} of M_i is increasing in χ_j (the competitor's crisis likelihood) if $m_{i3} > m_{i2}$, i.e. the third period unit profit margin of M_i is greater than its second period unit profit margin. Otherwise, A^*_{i2} is decreasing in χ_j.*
(c) *Is M_i recalls second, it's second regime equilibrium advertising efforts is decreasing with the damage η_i.*

The finding that the first period advertising is decreasing with η_i is again consistent with case of one focal firm in our paper and the previous literature. The above result highlights whether pre-crisis advertising should increase with a firm's hazard rate depends on whether the competitor's hazard rate is sufficiently high or not. Knowing the competitor's chances of recall therefore gives a strategic benefit to the rival. By increasing the pre-recall advertising (here recall refers to the recall by the competitor) the firm accumulates high goodwill which ensures high pre-crisis and post-crisis profits. If the first recall happens for the rival, the i^{th} firm anticipates only its own possible future recall. Thus it reduces its advertising if the impact it foresees is high. This is analogous to the one-focal firm case. The firm can believed to cut advertising costs to optimize profits.

Proposition 8. *The long term profit of the firm M_i can be determined from the value function given by $V_{i1}(\theta_i(t),\theta_j(t)) = a_{i1}\theta_i(t) + b_{i1}\theta_j(t) + c_{i1}$ at $t = 0$, where*

$$
\begin{aligned}
a_{i1} &= \frac{(\beta_i m_{i1} + \chi_i a_{i2} + \chi_j \hat{a}_{i2})}{(r + \chi_i + \chi_j + \delta_i)}, \\
b_{i1} &= \frac{(-\beta_i m_{i1} + \chi_i b_{i2} + \chi_j \hat{b}_{i2})}{(r + \chi_i + \chi_j + \delta_j)}, \\
c_{i1} &= \frac{(m_{i1}\alpha_i - \mu_i {A^*_{i1}}^2/2 + k a_{i1} A^*_{i1} + k b_{i1} A^*_{j1} + \chi_i c_{i2} + \chi_j \widehat{c_{i2}})}{(r + \chi_i + \chi_j)},
\end{aligned}
$$

and the initial values $\theta_i(0)$ and $\theta_j(0)$ are known.

2.4. Numerical Experiments

We have obtained the equilibrium strategies and value functions in closed forms for all scenarios under consideration. We found some straight forward relationships among the equilibrium advertisement efforts, the value functions and the system parameters. However, our system has many parameters and we present some numerical experiments to emphasize the efficiencies of our analysis. The numerical results help us draw some insights about the influence of our model parameters on firm performance. The model parameters are -

$$\alpha_i, \beta_i, k_{ij}, \chi, \mu_i, \eta, \delta_{ij}, r, m_{ij}$$

We study the impact of hazard probability and damage intensity on

- The equilibrium advertising $A^*_{ij}(t)$
- State Variables $\theta_i(t)$
- Firms' profits $V_{ij}(0)$

Since our model is goodwill based, we investigate the effect of recall on a firm with high initial goodwill as compared to a firm with lower initial goodwill. We have chosen the parameter values in such a manner that equilibrium advertising efforts, profits, the goodwill, the margins, all remain within a small range of values and this helps us in scaling our figures properly. The parameter values also ensure the constraint that $\theta_1(t) \geq 0$ and $\theta_2(t) \geq 0$. Some parameters are fixed for all the models. These are

$$\begin{aligned} \alpha_i = 1, k_{i1} = 1, \\ \mu_1 = \mu_2 = 200, \\ r = .06, \delta_{i1} = .03, \\ \beta_i = .5 \text{ where } (i \in \{1,2\}) \end{aligned} \tag{2.30}$$

In our analyses, we found that the damage intensity η and hazard rate χ have significant impacts on the firms' decisions, goodwills and profits. We, therefore, numerically study the effect of these two parameters on advertising, state variables and the profits. We classify the recalls based on the (χ, η) pair as -

1. (Low probability, Low impact)=(.05,.05)
2. (High probability, High impact)=(.7,.7)
3. (Low probability, High impact)=(.05,.7)
4. (High probability, Low impact)=(.7,.05)
5. (Benchmark, Benchmark)=(.3,.3)

Previous empirical analysis found that in the automobile industry, both the benchmark recall probability and damage intensity are approximately 30% (Rubel, Naik, and Srinivasan 2011). Therefore, our benchmark case considers$(\eta, \chi) = (.3, .3)$. We have not considered any recall costs in our models. Therefore, we believe our results will provide more accurate insights if we consider the decrease in margin of the focal firm in the second period as an implicit indicator of potential recall costs. We have assumed that the first period margin is always 1, i.e. $m_{i1} = 1,\ i = 1,2$. When recall impact or damage is low ,i.e. $\eta = .05$, second period margin $m_{i2} = 1$ implying there is no loss in margin for the focal firm i. If $\eta = .3, m_{i2} = .8$ signifying a 20% loss in margin for the focal firm i. If the recall intensity or damage is high, .7, then $m_{i2} = .5$ for the focal firm i.

2.4.1. Equilibrium Advertising, Firm Performance, State Trajectories: One Focal Firm

2.4.1.1. Effect of Product Recall on the Pre-crisis and Post-crisis Advertising

We compare the equilibrium advertising for both the firms in the two regimes using the parameter values as given in the previous section. We precisely depict the curve determined by the (χ, eta) pairs which dictates the level of advertising in the two regimes. For a given level of impact, the curve changes as demonstrated in figures (2.3) and (2.4).

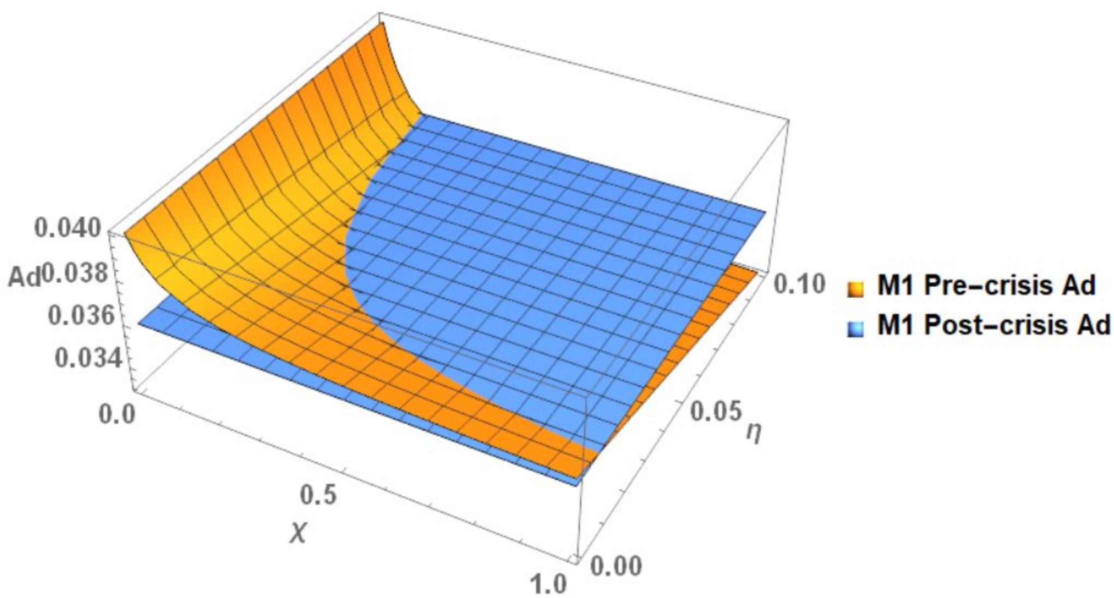


Figure 2.3: Low Impact Recall - Pre-crisis and post-crisis Advertising for M1

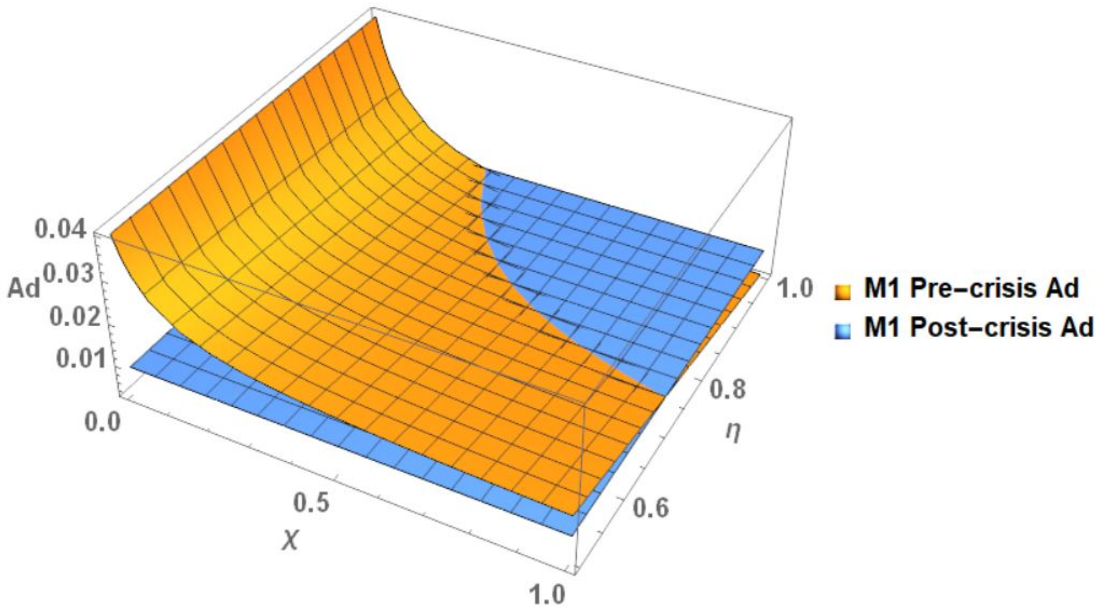


Figure 2.4: High Impact Recall - Pre-crisis and post-crisis Advertising for M1

Observation 1: *The post-crisis advertising is higher than the pre-crisis advertising when there is a high probability of a recall capable of causing substantial damage*.

This finding comports with the previous literature and explains the fact that the firms would engage in aggressive advertising after a recall of moderate intensity has been issued. We believe that the classification of the impact of recall as "low" or "high" will depend on the firm size or the recall costs incurred by the firm. However, in our theoretical results, we consider $\eta \leq .05$ to be fairly low impact. Even if we consider .05 to be low intensity if there is a high probability of such recall, firms post-crisis advertising increases in order to protect its image. Bad news spread fast and firm's image may get tarnished even due to a small recall because of brand switching or possible scepticism about its products that may arise. Sufficient advertising after the recall can prevent the public perception of potential recall damage to be more than the actual severity of the recall.

We also find that even a very low probability of high-intensity recall results in higher pre-crisis advertising. The focal firm will fear a slight risk of high-intensity recall because if the recall occurs it can be devastating for the firm. Thus, there might be a motivation for the firm to have high pre-crisis advertising in order to make the most profit from the pre-crisis market. Our finding also supports the empirical analysis of (Gao et al. 2015) who conclude that for a firm, if recall involves minor hazard, the pre-crisis advertising should be more. This is because more advertising sends a positive signal to the market about the firm's confidence in its product.

As figures 2.5 and 2.6 show, for the non-focal firm, pre-crisis and post-crisis advertising efforts coincide when recall impact is low. This is a direct consequence of the Proposition 1 and the fact that the parameter values do not change in the two periods for the firms. When recall impact is high, we can assume at least the advertising effectiveness of the non-focal firm would increase if not the margin also. Thus the advertising efforts will increase.

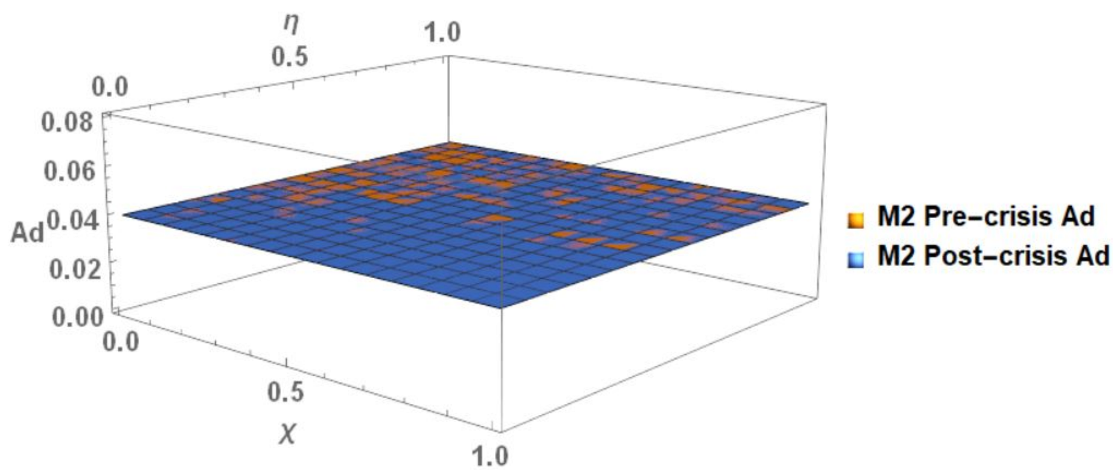


Figure 2.5: Low impact recall- pre-crisis and post-crisis ad variation for M2

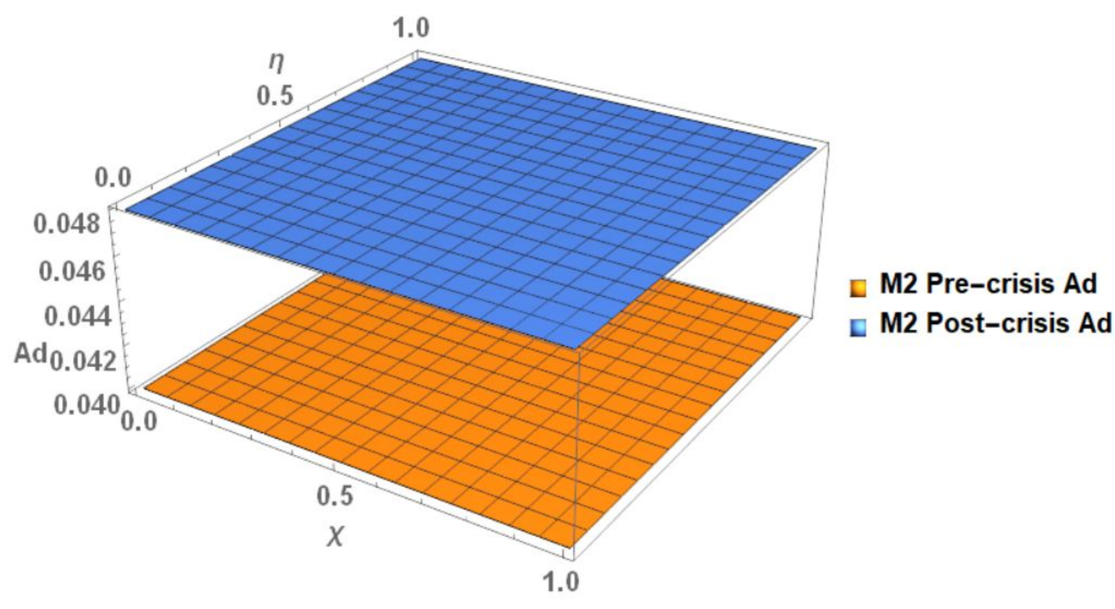


Figure 2.6: High impact recall- pre-crisis and post-crisis ad variation for M2

The Role of Profit Margin:

We found that margin plays an important role in determining whether the pre-crisis advertising is higher. Our numerical experiments show that if the margin in the post-crisis regime is at least the same as that of the pre-crisis regime, then the post-crisis advertising is higher for almost all practical cases for the focal firm. Numerically, there are instances when the post-crisis advertising may be lower at $m_{11} = m_{12}$. However, the value of η is practically negligible for such cases and hence such cases are excluded from our analysis. From a managerial point of view, our findings highlight the importance of estimating three things- *the recall probability, the recall impact* and *the possible loss of margin after the recall.*

For the non-focal firm M_2, we find that as long as $m_{22} \geq m_{21}$, for all $\chi > 0$, the post-crisis advertising is always equal to or higher than the pre-crisis advertising. Under the model assumptions $m_{22} \geq m_{21}$. Thus, we can posit that. in the second regime the non-focal firm will always advertise at least as much as the first regime. This can be interpreted as the reaction of the non-focal firm as it adjusts equilibrium advertising to increase its demand when crisis damages its competitor's goodwill. It is worth noting that in the previous empirical estimations of recall probability (Rubel, Naik, and Srinivasan 2011), in the automobile industry, the expected crisis time is 2 to 3 years and the baseline sales are affected by 35%. This is in agreement with the maximum value of η (around .32) that we derived using numerical analysis and the analytical condition in proposition 2-(a) (Figure (2.6)). Thus, we can confidently assert that our parameter values are not contrived.

2.4.1.2. Effect of Product Recall on Goodwill

We examine the behavior of the state variables, θ_1, θ_2 for different values of χ (crisis probability) and η (damage intensity). We find that the interpretations of the variance of state variables largely depend on the two regime margins of the firms. We obtained the solutions of the state variables in the two regimes.

Pre-crisis solutions:

$$\theta_1(t) = \frac{e^{-t\delta_1}\left(-A_{11}^*k + e^{t\delta 1}A_{11}^*k + \theta_{10}\delta 1\right)}{\delta 1} \qquad \text{with initial condition } \theta_1(0) = \theta_{10},$$

$$\theta_2(t) = \frac{e^{-t\delta_2}\left(-A_{21}^*k + e^{t\delta 2}A_{21}^*k + \theta_{20}\delta 2\right)}{\delta 2} \qquad \text{with initial condition } \theta_2(0) = \theta_{20},$$

A_{i1}^* are the equilibrium advertising of the manufacturers in the pre-crisis regimes.

The post-crisis solutions are:

$$\theta_1(t) = \frac{e^{-t\delta_1}(e^{t\delta_1}A_{12}^*k - A_{12}^*ke^{t_r\delta_1} + ae^{tr\delta_1}\delta_1)}{\delta_1}, \theta_1(t_r+) = a = (1-\eta)\theta_1(t_r-),$$

$$\theta_2(t) = \frac{e^{-t\delta_2}(e^{t\delta_2}A_{22}^*k - A_{22}^*ke^{t_r\delta_2} + be^{tr\delta_2}\delta_1)}{\delta_2}, \theta_2(t_r+) = b = \theta_2(t_r-).$$

The different scenarios of recall impact and probabilities are considered. The scenarios are given in table below.

	Low impact, Low likelihood	Low impact, High likelihood	High impact, Low likelihood	High impact, High likelihood
χ	0.05	0.7	0.05	0.7
η	0.05	0.05	0.7	0.7

Table 2.4: Recall Likelihood and Impact

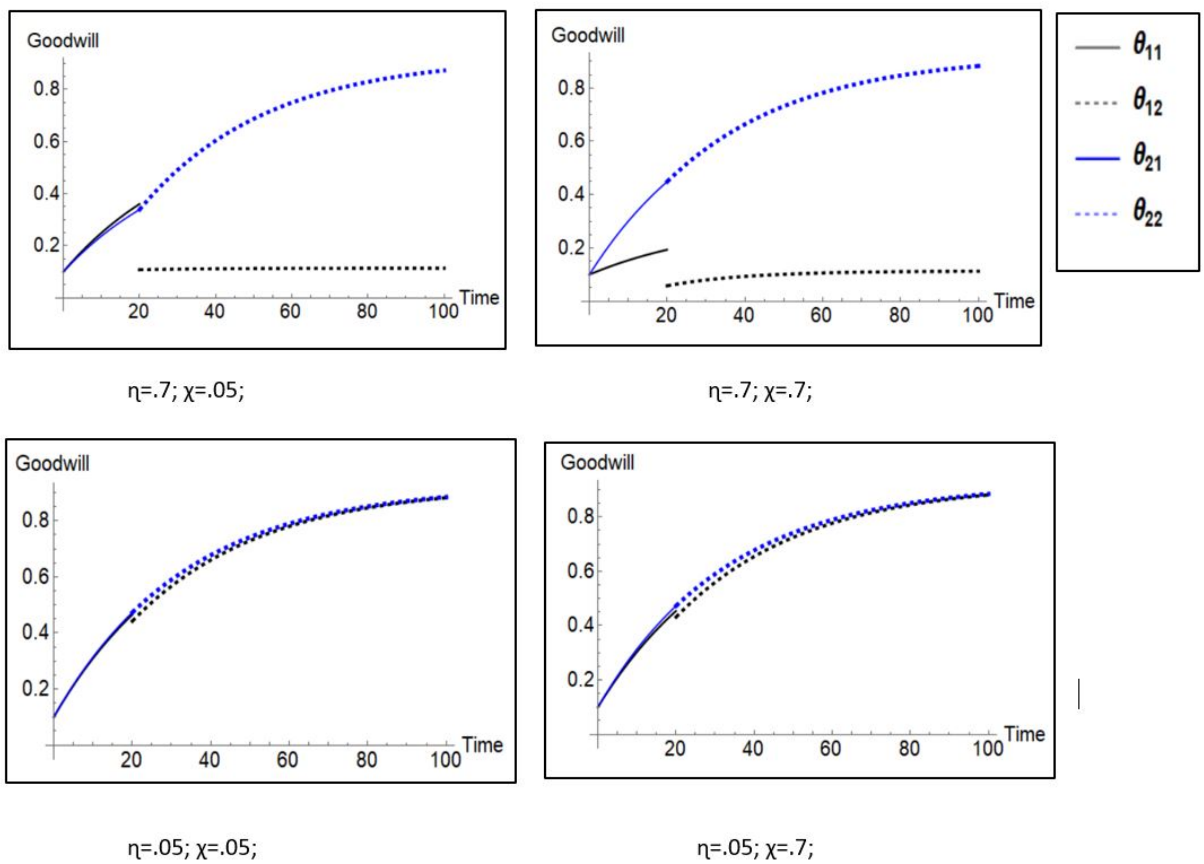


Figure 2.7: State Trajectories - Similar Firms

Figure 2.7 shows how the firms' goodwill vary before and after recall for similar firms (same initial goodwill; same sensitivity to brand difference and same absorption capacity). We believe that our numbers are sufficiently indicative to illustrate the true effect of recalls on the state variables.Clearly, a recall of substantial impact has a disastrous effect on the goodwill of the focal firm and such an effect may go beyond recovery. This is agreement with our model assumptions. Since advertising effectiveness decreases largely after a crisis, the goodwill suffers. For low impact however, there is chance of recovery for the focal firm as its post-crisis optimal advertising is higher.

A product recall is often followed by market overreaction (Govindaraj, Jaggi, and Lin 2004). Therefore, in case of small impact recalls, the focal firm advertises aggressively so that the stock of goodwill is high enough to nullify the effects of overreaction. Moreover, from a consumer's perspective, high goodwill and high quality are often associated. Consequently, if a brand's goodwill is high, then despite a recall, customers can be still loyal to the brand diminishing the potential damage (Kalaignanam, Kushwaha, and M. Eilert 2013). Whenever there is a recall, invariably in each case, the goodwill of M_2 rises. This is intuitive because M_2 advertises more in the second period as long as $m_{22} \geq m_{21}$.

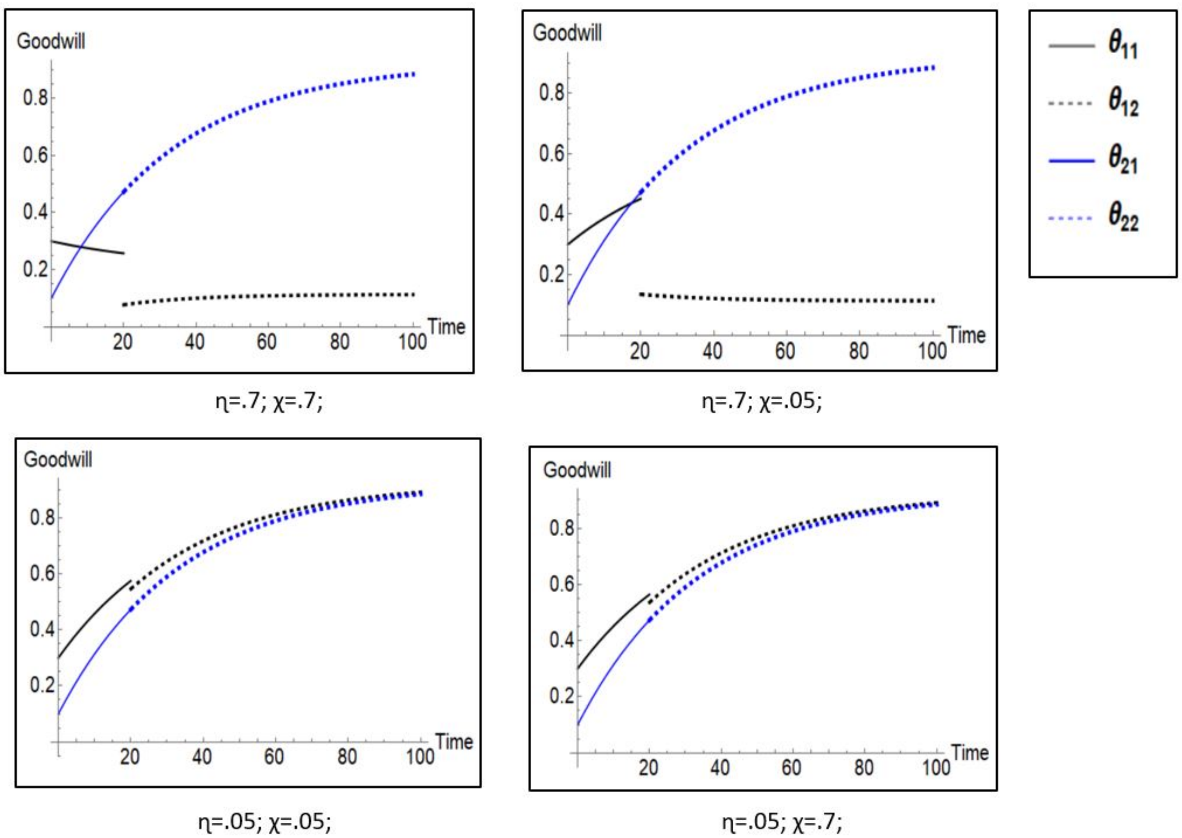


Figure 2.8: State Trajectories - Focal Firm Dominant

From Figure 2.8 when the focal firm has sufficiently higher initial goodwill, the initial goodwill acts as a buffer and the firm continues to dominate the non-focal firm in terms of goodwill even after a small impact recall. However, in case of a high impact recall, if the crisis likelihood is low, the firm still attains a high pre-crisis goodwill by elevated levels of advertising. However, post-crisis goodwill drops and does not recover naturally. On the contrary, for a high likelihood recall the pre-crisis decreases with time, has a further post-crisis jump and then recovers marginally. When a firm advertises more for goodwill before a recall, a positive signal or confidence may result in increased sales and profit but when the recall actually occurs, it might not be good for the firm in terms of goodwill. When the recall likelihood is high, the firm takes a cautious approach to advertising resulting in goodwill drop as compared to the case when the likelihood was low.

2.4.1.3. Effect of Product Recall on Firm Performance

Since, the expressions of the value function are complex, we numerically examine the effect of the recall likelihood and the intensity of the damage on the profit. The analytic investigation leaves us with a complex parametric space and it is hard to infer anything insightful from the analysis. We evaluate the value functions i.e the long-term expected profits, $V_{11}(t)$ and $V_{21}(t)$ at time t=0. This gives us the long-term expected profit of the manufacturers. An existing high brand equity can insure a firm against potential damage during crisis (Dawar and Pillutla 2000; Cleeren, Dekimpe, and Helsen 2008). Hence, we consider following three scenarios:
i) M_1, the recalling firm and M_2 are symmetric implying that they have a similar brand image initially at t=0
ii) M_1 is a bigger brand than M_2 before the recall implying $\theta_1(0) > \theta_2(0)$
iii) M_2 is bigger brand than M_1 before the recall implying $\theta_2(0) > \theta_1(0)$

Using Mathematica software we depict the long-term profits corresponding to the above scenarios using the values of Table 2.6 : We compare the long-term profits of the firms for the two

	$\theta_1(0)$	$\theta_2(0)$	η	
Similar Firms	0.1	0.1	0.05(m2=1)	0.7(m2=.5)
Focal firm is bigger	0.2	0.1	0.05(m2=1)	0.7(m2=.5)
Non focal firm is bigger	0.1	0.2	0.05(m2=1)	0.7(m2=.5)

Table 2.5: Scenarios - similar firms and dissimilar firms

extreme cases:
1. Damage intensity η is low at 5% and the effect on profit margin is negligible $(m_{11} = m_{12} = 1)$,
2. Damage intensity η is high at 70% and there is a high margin erosion $(m_{11} = 1, m_{12} = .5)$.

The long-term expected profits vary with the hazard rate (others parameters remaining fixed). Therefore we express the value function as a function of χ and find the local maximum by using the first order condition on the value function. The first order conditions yield a polynomial of degree 5 in χ. We use mathematica software to solve the first order condition and find the real positive roots of the polynomial. This gives us the values of χ for which $V_{11}(t = 0)$ or $V_{21}(t = 0)$

are maximum.

Observation 2 :

(a) *When recall impact is low, the focal firm's profit decreases very little with the the hazard rate. However, a high impact recall is always severely detrimental to the firm's profit.*

(b) *A "no-recall" scenario is more profitable for the focal firm.*

(c) *The non-focal firm has a significantly higher profit when the recall impact is high and its profit increases with the hazard rate.*

(d) *For a low impact recall, the non-focal firm's profit increases with the hazard rate initially followed by a descending trend.*

Figure 2.9 shows that for a high impact recall the expected profit of the focal firm is sharply decreasing with the crisis likelihood. On the other hand, the non-focal firm's profit is gradually increasing with the crisis likelihood. While this variation of the profits are expected, one important inference (which is not obvious from analytical results) is that the crisis likelihood χ has a lesser impact on the profit of the non-focal firm. For a low impact recall, the expected profit for the focal firm initially drops and then increases with the crisis likelihood. Numerically we found that the cusp beyond which the profit starts increasing for M1 is $\chi = 0.130841$. Consequently, the non-focal firm's profit initially increases for a low impact recall and starts declining after the above threshold value of χ.

A product recall can be categorized as an unsystematic risk and low likelihood of such risks can positively affect firm performance (Aaker and Jacobson 1987). The focal firm's profit is monotonically decreasing with the hazard rate for a high impact crisis. This is an expected result because if the impact is high the advertising will be low, post-recall goodwill erosion will be high and consequently the profit is low. The non focal firm's profits are complementary to that of the focal firm. This follows from our model's mathematical structure. The managerial implications of the above observations are significant. When two similar firms compete, a low impact recall can be beneficial for the firm at fault.

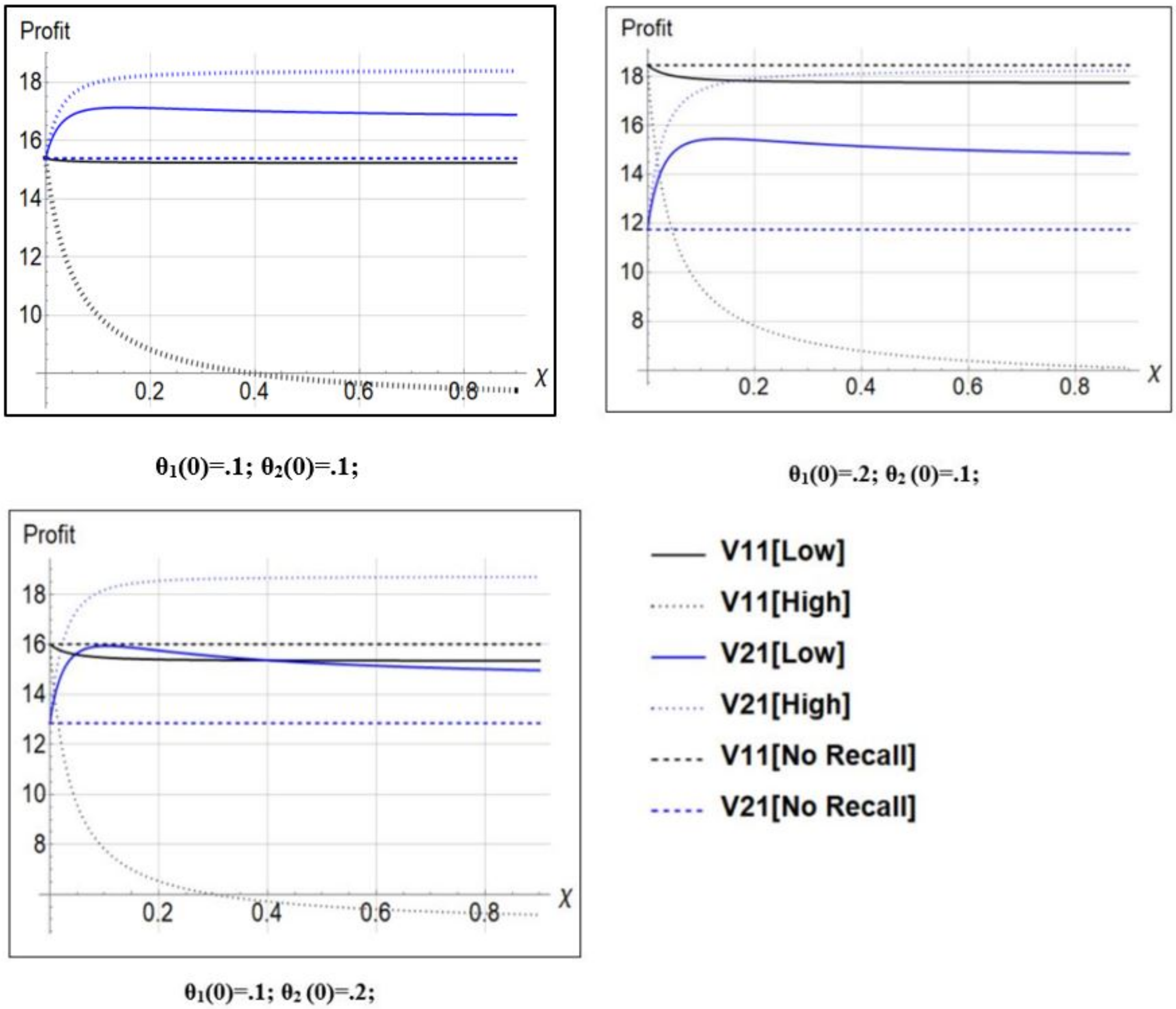


Figure 2.9: Long-term expected profit for the two firms

2.4.2. Can Hazard Myopia be Profitable?

We ask whether a myopic focal firm or non-focal firm can be better off in terms of profit under any circumstances during a recall. To compute the profit of the myopic firms we need to assume some recall time t_r as the profit depends on the time of recall. An early recall, $t_r = 5$ or a recall later in the planning horizon $t_r = 10$ are considered. The farsighted focal firms' profit will also depend on χ which the myopic firm never foresees. Moreover, the farsighted firm will be aware of the impending recall. As a result, the firm will make efforts to minimize post-crisis brand forgetting i.e. reduction in (δ_{12}) through announcements or campaigns and also strategically manage operational costs to reduce the loss in unit profit margin. Our numerical analysis incorporates these assumptions. The profits we consider are given in table 2.6.

η	Profit for farsighted firm ($= .05$)	Profit for farsighted firm ($= .7$)	Profit for myopic firm ($t_r = 3$)	Profit for myopic firm ($t_r = 10$)
.05	15.17	15.06	17.53	21.822
.2	13.85	12.48	12.140	16.07
.6	12.35	9.51	9.156	13.01
.7	11.41	7.54	5.132	9.08

Table 2.6: Myopic vs Farsighted Focal Firm Profit

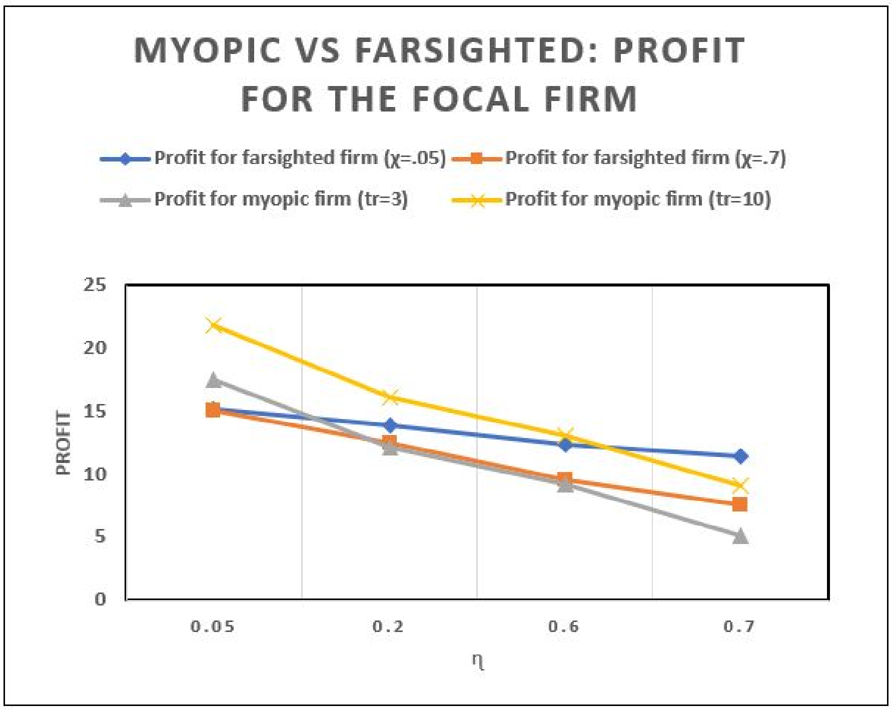


Figure 2.10: Expected profits -hazard myopic vs farsighted firms

Observation 3:

(a) *A farsighted focal firm is more profitable than a myopic focal firm if recall is early in the planning horizon. For a late recall, a myopic focal firm always makes more profit than a farsighted focal firm except when crisis impact if very high.*

(b) *The myopic non-focal firm is always better off than the farsighted non-focal firm. The myopic non-focal firm's profit decreases with an early recall.*

As mentioned above, the myopic focal firm is unprepared for a recall and therefore does not make any efforts for managing a recall. Therefore, potentially more goodwill absorption in the post-crisis period and a bigger loss in unit profit margin caused by potential operational issues lead to lower profits specially when the recall is early. The myopic firm cannot capitalize on the pre-crisis market and makes most of the profit from the post-crisis market when its image is already tarnished. Therefore, it has lower profit than a farsighted firm.

However, for a recall occurring later in the horizon, a myopic firm may be more profitable by ignoring a potential risk. This might look surprising. In the above case, the firm makes most of its profit from the favourable market when its image is better. Moreover, as the firm does not see the recall coming, its time preference remains unchanged. This factor also has an impact on the profit. However, for a recall of very high impact, the myopic firm can still be less profitable than a farsighted firm as shown in the Table 2.6 and figure 2.10 above.

From a managerial perspective, the results show that a firm must remain vigilant and estimate the likelihood of a crisis, possible timing of a crisis and also the potential impact of the crisis. While taking advertising decisions, a focal firm may ignore low impact future crisis which is possibly far away in time (as it will make more profit being myopic). However, if a high impact or highly likely recall is anticipated, it is better to incorporate the hazard rate while making the equilibrium advertising policy. The non-focal firm's management, on the other hand, would be better off ignoring the rival's hazard rate in their decision. Non-focal firms, mainly belonging to the same country of origin of the focal firm or selling similar products, are often negatively affected by a spillover effect of the recall. Consumers get sceptical about the non-focal firm's products, consequently decreasing its profit. In such cases, it may be advantageous to consider the crisis likelihood. Furthermore, if the focal firm's advertising drops due to high margin loss, it may decrease advertising, and the non-focal firm's advertising efforts will be way above those of the focal firm. Consumers may also perceive the higher advertising by the non-focal firm as opportunistic behaviour and have formed a negative mindset and might not buy the competitor's products (Gao et al. 2015). The examination of the effects of consumer's negative perception or spillover is beyond the scope of this study.

2.4.3. Equilibrium advertising and firm performance for two focal firms

The experiments conducted for two focal firms are categorized by two situations - i^{th} *firm recalls first* or i^{th} *firm recalls second*. The most important parameters that we are varying for these experiments are :

$$\chi_i, \chi_j, \eta_i, \eta_j, m_{i1}, m_{i2}, m_{i3}$$

Advertising for all three regimes are not affected by all the parameters that we are varying for our experiments. To easily visualize the effect of the above mentioned parameters on the the equilibrium advertising, we present the following Table 2.7, where "Yes" corresponding to a parameter means that the corresponding advertising is a function of the parameter. For example, A_{i1} is a function of $\chi_i, \chi_j, \eta_i, m_{i1}, m_{i2}, \hat{m}_{i2}, m_{i3}$.

Description	Equilibrium Advertising	Parameters							
		χ_i	χ_j	η_i	η_j	m_{i1}	m_{i2}	$\hat{m}_{i2}$	m_{i3}
1st Period Ad	A_{i1}	Yes	Yes	Yes		Yes	Yes	Yes	Yes
2nd Period Ad - *i* recalls first	A_{i2}		Yes				Yes		Yes
2ndd Period Ad - *i* recalls second	$\hat{A}_{i2}$	Yes		Yes				Yes	Yes
3rd Period Ad	A_{i3}								Yes

Table 2.7: Parameters Affecting Equilibrium Advertising

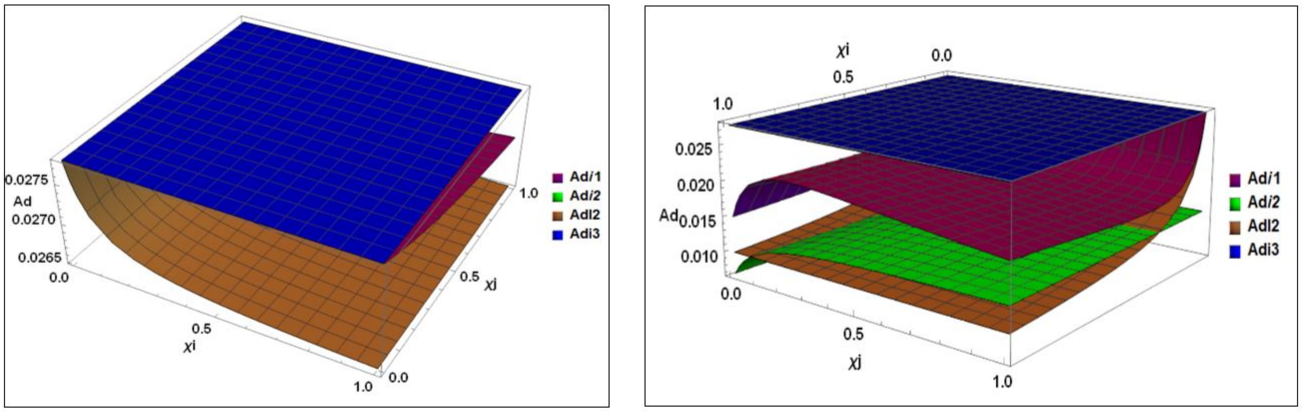
We follow some assumptions about the margins in the different periods.

Assumptions about margins

1. If the i^{th} firm recalls first, we consider $m_{i1} > m_{i2} = m_{i3}$
2. If the i^{th} firm recalls second then $m_{i1} = m_{i2} > m_{i3}$

The complexity of the analysis of the equilibrium advertising decisions and the performance of the two focal firms case is multiplied due to the dimension of the parametric space. We now have the extra parameters, for hazard rate, damage effect and margins. Our goal is to reduce the analytical complexity without reducing the dimension of the problem. While many possible cases are imperative, we have focused on the cases which we believe to be most feasible. The assumptions about the margins is an example of such feasibility. Similar to the one focal firm case, we consider the combinations when damage and crisis are high or low and the benchmark case when damage or crisis equals 0.3. Thus there are four parameters $\chi_i, \chi_j, \eta_i, \eta_j$ and these take values high(0.7) or low (.05). We have thus 2^4+1(benchmark case) = 17 cases to consider. However, from Table 2.9, we notice that the goodwill damage of the j^{th} firm, η_j, does not affect the advertising decision of the i^{th} firm. Hence, the number of cases reduce to 2^3+1=9. We present the most important cases diagrammatically -

High Impact Recall ($\eta_i = \eta_j = .7$);
Low Impact Recall ($\eta_i = \eta_j = .05$).
We found that the behaviour of equilibrium advertising for benchmark case and high impact case are similar. We perform 3D plot in mathematica and vary the equilibrium advertising over all values of χ_i and χ_j. Therefore, these figures not only cover all the cases in the above mentioned 9 scenarios, but also spans over all likelihood scenarios of both the rivals. This makes our findings more inclusive and robust.



Low Impact Recall **High Impact Recall**

Figure 2.11: Equilibrium Advertising in Three Regimes - Two Focal Firms

Noting that the rival's damage, η_j, does not affect the advertising policies of firm M_i, the following cases are equivalent with respect to advertising:
i) $\eta_i = .05; \eta_j = .7$ & $\eta_i = .05; \eta_j = .05$
ii) $\eta_i = .7; \eta_j = .7$ & $\eta_i = .7; \eta_j = .05$

Observation 4:

(a) *M_i recalls first*:

- *During a low impact recall, irrespective of the likelihoods of recall of each firm, M_i's equilibrium advertising is always higher in the second and third regimes than in the first regime.*

- *If recall impact is the benchmark or high, the first period advertising of M_i is higher than the second or third period advertising for all values of χ_i if the rival's crisis likelihood χ_j is low (Figure 2.11).*

(b) *M_i recalls second*:

- *The first regime advertising of M_i is always higher than the second regime advertising irrespective of recall impact or likelihood of each firm.*

- *For moderate and high likelihood of a crisis, the post-crisis (third regime) advertising of M_i is higher than it's pre-crisis advertising.*

When M_i recalls first, in the second regime M_i already knows the impact of the recall and that only its rival has a chance of recall in future. Also, a low impact recall means no or very little margin erosion. Therefore, the firm M_i uses higher advertising as a double-edged sword to overcome any negative effect of its own recall and also to react to the possible higher advertising by the rival. On the other hand, a moderate or high crisis impact makes M_i more cautious and its first-period advertising is higher if the rival's crisis likelihood is low. This is consistent with Proposition 7. M_i tries to make the most profit out of the market in the first regime when there is a high impact high likelihood recall possibility. However, if the rival also has a high possibility of high impact recall, M_i's advertising level is low in the first period as it can potentially benefit from the competitor's misfortune.

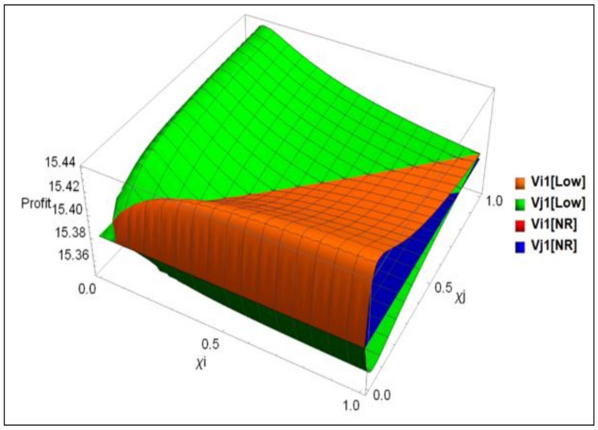
If M_i recalls second, it's second regime advertising level drops. In the second period, M_i's rival has already issued a recall and is presumably coping with the negative effects. Moreover, M_i knows that there is a possibility of a recall that it itself might issue in the future. Therefore, M_i's advertising drops in the second period as low advertising can well give M_i the optimal profit while its rival suffers from a recall and at the same time M_i higher post-crisis advertising (third regime) helps it to overcome the effects of its own recall.

Observation 5:

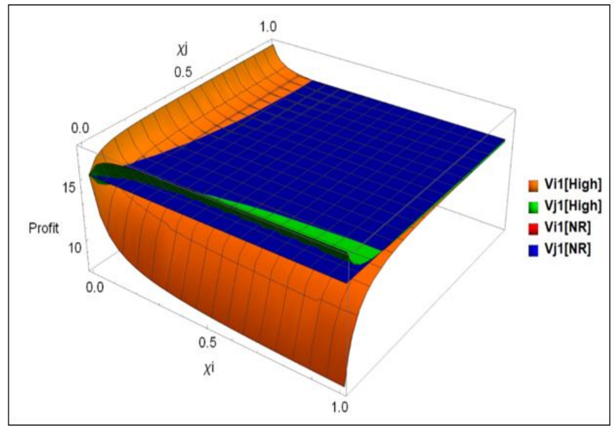
(a) *Irrespective of all crises likelihoods, the no recall scenario is strictly more profitable for* M_i *only when* M_i *has a very high impact recall and* M_j *has a low impact recall. On the other hand, irrespective of all crises likelihoods, the no recall scenario is strictly less profitable for* M_i *when* M_i*'s recall impact is low and* M_j*'s recall impact is high (Figures 12 (d) & (e).*

(b) *When both firms have the same impact of recall*:

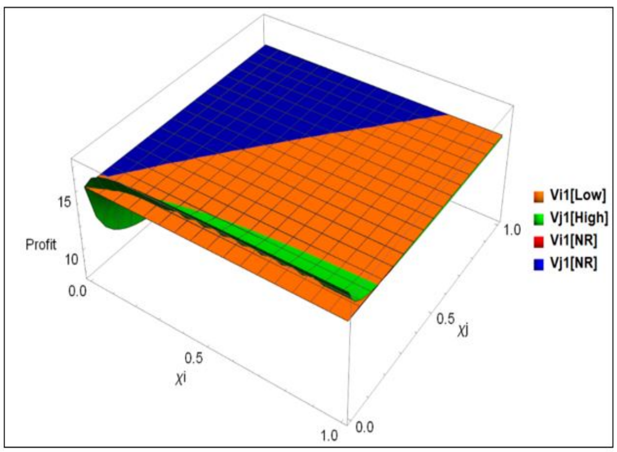
- *In a high impact recall scenario,* M_i*'s profit is higher than the no recall scenario when* M_i*'s crisis likelihood is low*.
- *Surprisingly, when there is a low impact recall, the firm profit is increasing with its own hazard rate*.

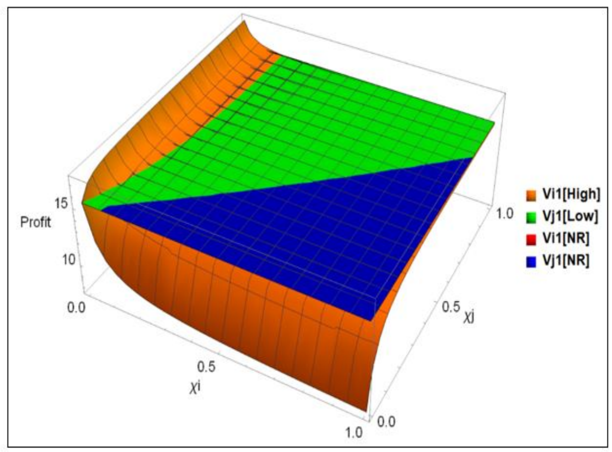


Profit -Low Impact and No Recall



Profit -High Impact and No Recall





Profit -Low Impact, High Impact and No Recall

Figure 2.12: Firm profits in Presence of Two Focal Firms

The results show that during a competition between two focal firms, the no recall scenario may not always be the maximum profitable. If the expected damage caused by own recall is small, a firm will have no incentive to avoid a recall as long as the competitor is more risky than the firm itself. When a firm is at a risk of causing a high damage, it is best to make efforts to avoid the recall because a no recall scenario is more profitable for it. Consequently, the firm may seek to invest in quality or strategy in order to avoid a recall. When the expected recall damage is equal for both the firms, a firm should aim to keep the recall probability low when the damage is high. On the other hand if expected damage is low for both the firms, a firm has no incentive to reduce the crisis likelihood as a high likelihood of crisis increases it's profit. The main takeaway from our findings is that in the presence of a risky competitor a firm should try to avoid a recall if the expected damage for itself or both the firms is high and consequently invest in other areas of quality or possibly supplier management. If the firm's own risk is low, the firm can invest more in advertising without making any effort to avoid the recall.

2.5. Manegerial Implications and Conclusion

The marketing literature has investigated whether advertising should increase after a product recall. The findings have revealed conflicting results. While some studies emphasize the need for increased pre-crisis advertising, others found support for the increased post-crisis advertising. One common conclusion in the product recall literature highlights the firms loss of goodwill as a result of the recall (Craig and Thomas 1996; Cleeren, H. J. Van Heerde, and Dekimpe 2013). Hence, in this study, we built a theoretical model with goodwill based demand function and analyzed advertising decision models under the lens of differential game theory.

In this study, we make some significant contributions to the existing literature related to product recall. First, we re-investigated not only the advertising decisions of firms in a competitive market but also the effect of a product recall on firm profit. We found that whether a firms' advertising level will increase after the product recall, depends on the the severity and likelihood of the recall. This claim is consistent with the empirical findings of (Gao et al. 2015). We also found that when similar firms compete or the focal firm is bigger, a small crisis likelihood increases the focal firm's profit. This is consistent with another study which presents price competition in presence of product recall (Rubel 2018). Second, we have introduced the concept of "hazard myopia", a firm's inability to foresee recall occurrence. This concept differs from the conventional myopia (ignoring state evolution) in the context of differential games. We showed how to solve a hazard myopic firm's decision problem and found an interesting case when hazard myopia can be more profitable than envisioning a recall. Our study enhances the literature by studying the equilibrium advertising policies and profits of two farsighted focal firms.

Although price is not a decision variable in our model, the profit margin has a very important role to play when it comes to advertising decision making. Our analytical and numerical findings give a number of insights which are important from a managerial point of view. We summarize our key findings. The investigation results answer our research questions and provide new managerial insights in addition to verifying previous research results.

Managerial Implications: In a duopoly market with a product recall risk, it is important for the management of both firms to understand if the rival is risky or not because a firm's advertising strategy depends on the rival's product recall likelihood. By risk we mean the combination of recall likelihood and intensity of the damage. Thus, a firm should estimate its own and the rival's recall likelihood and possible damage caused by any expected recall. The damage should not only be measured in terms of the goodwill-loss, but also in monetary value. Since we showed that advertising levels depend on profit margins, the monetary value measure of a recall damage will enable the firm to evaluate the profit margin and advertise accordingly. It is also important to know how much a damage can potentially erode a profit margin. Advertising decisions are seldom the only decisions made. Product pricing is another important decision. The prevalence of profit margins of the different regimes in the expressions of our equilibrium advertising decisions shows that price might also be used as a potent tool for mitigating the effects of a product recall. When a recall is expected to cause low damage, managers may well ignore the chances of recall and advertise myopically. Managers know that a low likelihood of recall can increase profit. Therefore, at the beginning of the planning horizon, if a focal firm's management knows that the firm's crisis likelihood is high, they can put efforts like quality maintenance or inspection to minimize the chances of a recall and signal these efforts to the market. Consequently, its rival also estimates a low crisis likelihood for the focal firm. This may soften competition and increase profit. Hazard myopia might look attractive as it generates high long-term profit, but, it is important for the firm's management to estimate the hazard rate rather than ignoring it because a high hazard rate might be disastrous. When both firms have different likelihoods of product recall, it is important for the managements of the firms to estimate the likelihoods and the expected damage severity. The effectiveness of the resulting advertising policies in the three regimes depend on this knowledge likelihood and expected damage.

Our work can be extended in a number of directions. First, we have not considered the recall costs as a separate cost in our problem. Instead, we assumed that the margin will drop in the post-crisis period due to possible recall costs. Modelling recall costs as a function of the recall damage and the recall time can result in valuable insights. Second, the damage caused by the recall is assumed to be a constant fraction by which the brand image is eroded. This ensures the mathematical tractability of our model. However, this damage might not be a constant factor and can be a decreasing function of time. This is because as the product overcomes the pernicious consequences of the recall, the damaging effect might fade after some time i.e., there can be recovery. Empirical research shows that as products move further into the life cycle, the chances of recall decreases (Hora, Bapuji, and Roth 2011). Other distributions with a decreasing hazard rate can be considered. Third, we have not considered channel leadership in our model. In fact, due to the structure of our game, the Stackelberg game and the Nash game coincide. The firms' advertising strategies and performance can be analyzed for a model where there is a market leader. This will help in finding if market leadership gives any advantage during a recall. Last, it is challenging but possible to investigate multiple decision like price and advertising or quality in the same recall scenario.

Chapter 3

Pricing and Advertising to Mitigate the Negative Spillover Effect of a Product Recall

3.1. Introduction

Over the past few decades, with the globalization of modern business and complexity of supply chains, product harm crisis incidents have become very common. When such a crisis, especially of high impact occurs, the firm at fault is subject to substantial loss of reputation. The news of a product harm crisis and consequent product recall spreads rapidly as word of mouth or through online chatter and can cause substantial damage to consumer perception of product quality or brand (Tirunillai and Tellis 2012; Borah and Tellis 2016). Consequently, the sales of a firm can dampen considerably (H. Van Heerde, Helsen, and Dekimpe 2007; Asur and Huberman 2010; Dellarocas, X. M. Zhang, and Awad 2007). During such a crisis, when a firm at fault suffers from the reputation damage and loss of sales, rival firms can behave opportunistically to increase their sales. For example, rivals can increase advertising or decrease price in order to capture a large share of the market. However, empirical research has shown that such reputation damage of the focal firm can extend to the rivals of the firm when the rivals are from the same country or sell the products of the same category (Borah and Tellis 2016; Roehm and Tybout 2006). In contrast, rival brands with a different country of origin may benefit from the crisis of the competitor through brand switching and increase of sales. When a rival firm is adversely affected by its competitor's product recall, we call the event a negative spillover. On the other hand, when the rival firm is benefited from the recalling firm's crisis, we refer to the event as a positive spillover. The spillover effect of product recalls can be very complex and intricate depending on how the consumers perceive about the brand at fault and how well the crisis is managed by the brand using different strategies like pricing, advertising or quality assurance (Mackalski and Belisle 2015; Borah and Tellis 2016; Dawar and Pillutla 2000; Dahlen and Lange 2006).

Pricing and advertising are two potent marketing tools to fight the negative effects of a product recall. When a product-recall affects the focal firm and its rival, it is interesting to investigate how the two firms decide on advertising and pricing of the product under recall (for the focal firm) or a similar product of the competitor. The existence of the focal firm as a market leader can also significantly affect the firms' strategies. The existing literature on the spillover effect of product recall study different aspects of the phenomenon like measuring the extent of spillover, type of spillover based on size of the firm, location of the firms, effect of apology advertising and impact of product harm on the effectiveness of advertising and price (Borah and Tellis 2016; Arnade, Calvin, and Kuchler 2009; Mackalski and Belisle 2015; Roehm and Tybout 2006; H. Van Heerde, Helsen, and Dekimpe 2007; GAOH, ZHANG, et al. 2012; Seo et al. 2014). Our study integrates two steams of research - *spillover effect of a product recall* and *pricing and advertising strategies during product-recall*. We investigate how farsighted firms use equilibrium advertising and pricing in the pre-crisis and post-crisis regimes to fight the effects of product-recall under a duopoly competition.

(Dahlen and Lange 2006) analyze how a brand crisis can affect the product category and also rival brands. Based on two studies, the research shows that consumer perception may change for the entire brand category and depending on the similarity of the brands, the spillover effect on the rival firms vary. The authors found that consumers may reconstruct the brand schema after a crisis and under such circumstances, decreased post-crisis advertising might be a better strategy for the brand at fault. The authors conclude that as the crisis event reconstructs brand schemas, brand managers must keep a close eye on the competition and react on their behalf as well. The authors do not discuss pricing strategies. In our study, goodwill is synonymous to brand value. We re-validate the authors' claim about the advertising policies of the firms using equilibrium advertising policies under competition. Our jump state model captures one aspect of the reconstruction of the consumer's brand schema.

(Arnade, Calvin, and Kuchler 2009) studies the impact of the FDA announcement of an E. coli outbreak linked to spinach using a two-stage AIDS (Almost Ideal Demand System) model which separated the effect of prices and trends from the impact of the FDA announcement shock. The article determines the influence of the spinach shock on demand for several related leafy green products (the spillover effect on similar products). The E-coli outbreak resulted in 204 illness, 104 hospitalizations and three deaths and some other complications in the USA in 2006. The spillover was positive for different non-spinach salads(rivals or different category) and negative for similar category or brand. All brands from the USA suffered from negative spillover. Retailers increased the price of spinach. Rival products showed more price responsiveness. Prices might have induced brand switching. The article does not discuss advertising strategies.

(Mackalski and Belisle 2015) found that during a butter recall by Land O'Lakes, the entire

Land O'Lakes umbrella family is adversely affected. So negative spillover is prominent across categories. Moreover, spillover is extremely quick, and promotion and advertising positively affect the sales of the "brand ecosystem" (the brand and its competitor selling a similar product or category). However, this study did not find significant negative spillover for the competing brands. The study did not elucidate on pricing strategies.

In a more recent study of recalls in the automobile industry, (Borah and Tellis 2016) find the evidence of extensive "perverse halo spillover" via negative chatter. Moreover, the spillover is more prominent from a dominant brand to a less dominant brand. Indeed the same country of origin intensifies the perverse halo. The authors also confirm substantial sales loss due to the spillover effect. While the article did not discuss pricing strategies, the authors found that apology advertising can be detrimental to the recalled brand and its rivals because by raising consumer scepticism about the brands. We extend the literature by introducing the effect of goodwill advertising and pricing decisions. Our assumptions of the direction of negative and strength of spillover comport with the above study.

A stream of literature has analyzed the pricing and advertising decisions during the pre-recall period or the post-recall period. However, the findings of such literature are contradictory. (H. Van Heerde, Helsen, and Dekimpe 2007) found that decreasing price, especially for bigger brands, can be an effective way of gaining back the lost market after a recall. However, for smaller brands, price sensitivity of products can remain stable in the pre-crisis period but can increase substantially after a recall. (Yi Zhao, Ying Zhao, and Helsen 2011) found conflicting results and found that consumers may become less price-sensitive during a crisis and may hail the quality of a product as a primary determinant for purchase. (Cleeren, H. J. Van Heerde, and Dekimpe 2013), based on a sample of 60 product harm cases, investigated these conflicts and found that while price changes may not serve as a viable weapon to mitigate loss for affected brands, non-affected brands can reduce prices to gain market share. The authors also found that acknowledging blame can affect pricing and advertising decisions. (Rubel 2018) analyzed pricing decisions of two firms based on a Hotelling model of price competition and two regimes - competitive and monopoly. The author found that market leadership can induce a higher price, but a market leader need not always be more profitable. Moreover, anticipating a crisis can lead to higher profit for risky products due to softening of price competition.

The above discussion reveals that scholars hold different opinions about product harm crisis and pricing decisions. We adopt the assumptions of some of the modelling literature (Rubel, Naik, and Srinivasan 2011; Rubel 2018) that envisioning product harm can affect the pricing and advertising decisions and following them model product recall as a random event during a planning horizon. Based on the assumption, we investigate the pricing decisions during a crisis for the focal firm and the non-focal firm suffering from spillover. We show how a trade-off between crisis likelihood and impact can help us explain some of the conflicting findings.

Several studies analyze the pre and post-crisis advertising decisions (Gao et al. 2015; Cleeren, H. J. Van Heerde, and Dekimpe 2013; Cleeren, Dekimpe, and Helsen 2008; Rubel, Naik, and Srinivasan 2011; Kalaignanam, Kushwaha, and M. Eilert 2013). While some academics argue that aggressive advertising efforts in the pre-crisis regime can buffer a brand from potential image loss during the recall, others conclude that post-crisis advertising should raise in order to protect the brand. Due to the regular occurrence of product recalls in the different industries like automobile, food, consumer products and healthcare, it is reasonable to believe that the management of firms in the industries as mentioned earlier should expect recalls to occur. However, depending on the brand, industry, product, the crisis likelihood and expected impacts may vary. Alike (Rubel, Naik, and Srinivasan 2011), we examine how forward-looking managers should take advertising and pricing decisions. However, our model is distinct, and the strategies we obtain gives us feedback pricing and advertising policies which depend on the firms' goodwill, the time of the decision, the impact of the crisis and the likelihood of a crisis. Moreover, we introduce a spillover function which realistically captures the effect of the negative reputation of a recalling brand to its rival.

Our synthesis of the above literature in the two streams (spillover effect of a product recall and pricing and advertising strategies during product-recall) shows that there is little consensus amongst researchers about the true effect of product recall on spillover and how pricing and advertising can be as instruments of damage control for two rival firms during a product recall. We, therefore, investigate the following research questions hoping to seek a mechanism of reconciliation of the above conflicts:

1. When firms are similar in terms of market presence, what are the pre-crisis and post-crisis equilibrium advertising and pricing policies of the manufacturers?

2. When the focal firm is a leader, what are the pre-crisis and post-crisis equilibrium advertising and pricing policies of the manufacturers?

3. How do intensity and likelihood of a product recall affect the equilibrium advertising and pricing decisions of the two firms?

4. What is the impact of the recall on the performance of the competing firms?

5. Is market leadership helpful in overcoming the effects of the recall?

To answer the above questions, we formulate a dynamic model in which during the planning horizon, a non zero crisis likelihood exists. The occurrence of a crisis event that hurts a brand's goodwill, advertising effectiveness and consequently erodes the firms' sales. The occurrence of a product recall is at a random time. We consider price and advertising competition in our dynamic model. We assume that the forward-looking management of both the firms estimates the crisis likelihood and impact ex-ante to decide product price and advertising efforts. We also define a spillover function which mimics the real-world situation - spillover effect is stronger from a dominant brand to a less dominant brand than vice versa.

The rest of the paper is arranged in the following manner. Section 3.2 describes the model in details. In section 3.3, we discuss the equilibrium results of the differential game under consideration and provide some insights about the results complementing with numerical analysis. In section 3.4, we present our concluding remarks along with managerial implications and directions for future research.

3.2. Model Setup

Two manufacturers selling differentiated products are involved in a duopoly competition concerning advertising and pricing. Our motive is to study the effects of recall on the focal firm and the recall's negative spillover effect on the non-focal firm. Hence we consider only one focal firm for whom product recall can occur with a certain probability. The other firm, possibly having the same country of origin, suffers from a negative spillover effect of the recall. The firms do not recall all products and continue to compete after the recall. We analyze the possibilities of both firms having similar market potential or the focal firm being a market leader. Pricing and advertising decisions are taken simultaneously in the first case. The focal firm, being a leader announces its decisions earlier in the second case.

Consumers are sensitive towards the price difference and difference in the advertising efforts of the firms. Own goodwill positively affects a firm's demand. On the other hand, price increase negatively affects the firm's demand. We adopt dynamic linear demand functions for the two manufacturers M_1 (focal firm responsible for the recall) and M_2 (non-focal firm suffering from the spillover).

$$
\begin{aligned}
Q_{1j} &= \alpha_1 - p_{1j} + \beta_1 p_{2j} + \theta_{1j} - \gamma_1 \theta_{2j} \\
Q_{2j} &= \alpha_2 - p_{2j} + \beta_2 p_{1j} + \theta_{1j} - \gamma_2 \theta_{1j}
\end{aligned}
\tag{3.1}
$$

where α_i is the market potential for firm i, Q_{ij} is the demand for firm i in regime j (j is the index for time) and $i, j \in \{1,2\}$. Price for firm i in period j is p_{ij}. θ_{ij} is the goodwill of manufacturer i in regime j. Own price sensitivity is normalized to 1. The cross-price sensitivity parameters β_i are assumed to be in the range $0 < \beta_i < 1$ where $i \in \{1,2\}$. Therefore, the sensitivity of a firm's demand towards the firm's price (1 in this case) is greater than that of the competitor. Similarly, the self goodwill sensitivity is normalized to 1 and cross goodwill sensitivities follow $0 < \gamma_i < 1$ for $i \in \{1,2\}$. The solution to our model is complex and to make insightful analysis of our model we need to have a minimal number of parameters. Therefore, without loss of generality, we normalize the price and goodwill sensitivity of the firms to 1 in the demand function. This helps us in reducing the number of parameters. The manufacturers try to maintain a high brand goodwill by continuous advertising efforts. Brand advertising can have an adverse or complementary effect on a rival brand's sales and profits (Dubé and Manchanda 2005; Karray and Martın-Herrán 2009; Erdem and Sun 2002). Consequently, some scholars have adopted differential game models with negative effect of competitor's advertisement while others have also incorporated a positive effect of advertising on the rival firms sales or profits ((Viscolani and Zaccour 2009; Nair and Narasimhan 2006; Cellini and Lambertini 2003)). We consider a duopoly market where firms compete on price and goodwill. Following (Erickson 1995), we adopt an approach where the evolution of the stock of goodwill for a firm is affected by the firm's own advertising only. However, in our model, the positive parameter γ_i in a firm's demand function guarantees that a rival's rising goodwill negatively affects the firm's demand.

$\theta_{ij}(t)$ is the goodwill of the manufacturer i, in regime j, where $\{i, j\} \in \{1,2\}$. The states evolve according to the following differential equations:

$$\dot{\theta}_{ij}(t) \quad = \quad k_{ij}a_{ij}(t) - \delta_{ij}\theta_{ij}(t), \qquad \forall t \in 0 \leq t \leq \infty, \theta_{ij}(0) = \tilde{\theta_{ij}} \tag{3.2}$$

where $i, j \in \{1,2\}$. a_{ij} is the advertising effort of M_i in period j and δ_{ij} is the decay parameter (Nerlove & Arrow 1962). Marketing literature has evidence that the advertising effectiveness decreases after a recall crisis depending on the initial brand goodwill (H. Van Heerde, Helsen, and Dekimpe 2007; Yi Zhao, Ying Zhao, and Helsen 2011). Therefore, we assume that in general, k_{ij}, the advertising effectiveness, of the firms (specially the focal firm) may be different in the two regimes. Specifically, $k_{i1} \geq k_{i2}$. There is also a possibility that brand forgetting will increase in the post-crisis period. This can be simulated by choosing $\delta_{i1} \geq \delta_{i2}$. Following the previous literature (Karray and Zaccour 2005; Dockner et al. 2000; etc.), we consider the advertising costs to be quadratic in the efforts. The manufacturer's advertising costs in the regime j is given by $\dfrac{\mu_i a_{ij}^2(t)}{2}$ where μ_i is the proportionality constant.

We consider two cases - (i) *There is no market leadership* and (ii) *The focal firm is the market leader*. One important point to note here is that market leadership will affect the mathematical expressions of only pricing decisions. Due to the structure of the games, the Nash game and the Stackelberg game with respect to advertising will be structurally similar. But the value of the advertising effort will be different in the two games. In the Nash game the manufacturers decide the price and advertising simultaneously. In the Stackelberg game, the focal firm announces its price and advertising first, and the competitor who suffers from spillover effect moves next. We analyze these two cases separately.

3.2.1. The Occurrence and Effect of Recall

The planning horizon in our study is $[0,\infty)$. We also assume that during the planning horizon, recall occurs only once. Let t_r be the random time of the recall. Consider $\chi \in (0,1)$ to be the hazard rate. We define the probabilistic switching of the pre-crisis and the post-crisis regime by means of the stochastic process $[R(t) : t \geq 0]$ defined below:

$$\begin{aligned}\lim_{dt \to 0} \frac{P[R(t+dt)=2|R(t)=1]}{dt} &= \chi, \\ \lim_{dt \to 0} \frac{P[R(t+dt)=1|R(t)=2]}{dt} &= 0. \end{aligned} \tag{3.3}$$

$R(t) = 1$ signifies the pre-crisis regime and $R(t) = 2$ signifies the post-crisis regime. Many of the studies (Boukas, Haurie, and Michel 1990; Haurie and Moresino 2006; Rubel, Naik, and Srinivasan 2011) explore decision making under such regime-switching giving rise to piecewise deterministic games. The random timings of a recall essentially split the planning horizon into different decision epochs or regimes - the pre-crisis regime and the post-crisis regime when spillover effect is prominent.

The goodwill of the focal firm (the firm issuing the recall) declines immediately after the recall. Therefore, the impact of recall is captured by the jump in the state variable, goodwill at the instance t_r. For the focal firm, the following equation represents the jump in goodwill:

$$\theta_{12}(t_r^+) = (1-\eta)\theta_{11}(t_r^-)$$

For the rival, the spillover effect may cause the post-crisis goodwill to rise or fall depending on the category of the rival. As mentioned earlier, a rival from same geography and selling similar products may suffer from negative spillover while competitors with foundation at different countries may benefit from an increase in goodwill or positive spillover. However, the focus of our study is only negative spillovers.

Spillover Function:

In our model the spillover function is the jump in the state variable $\theta_{21}(t)$ at time t_r for the non focal firm. We introduce a spillover function in the following manner:

$$\phi = \nu\eta \frac{\theta_{11}(0)}{(1+\theta_{11}(0))(1+\theta_{21}(0))} \tag{3.4}$$

where ν is interpreted as the sensitivity of the non-focal firm's customers towards the goodwill damage η of the focal firm. The spillover function satisfies the following desirable properties:

1. The damage to the goodwill of the non-focal firm, caused by negative spillover is less than the goodwill damage caused to the focal firm. This is captured by the term $\nu\eta$ where η is the intensity of jump of the focal firm's goodwill at t_r and $0 \leq \eta \leq 1$, $0 \leq \nu \leq 1$.

2. The spillover is increasing in the focal brand's initial goodwill $\theta_{11}(0)$ and decreasing in the non focal firm's goodwill $\theta_{21}(0)$.

3. The functional form of spillover ensures that the value of the spillover, $\phi \in (0,1)$.

4. When recall damage η is high, ν will be high and vice versa.

Thus, for the non focal firm, the following equation represents the jump in goodwill:

$$\theta_{22}(t_r^+) = (1-\phi)\theta_{21}(t_r^-)$$

where ϕ is given in equation (3.4). A spillover effect can be short term, around days to weeks, however, in many instances, like in the automobile industry or pharmaceutical, concerns arise consistently over a long period and if not addressed by the manufacturer, can lead to long term lingering effect of spillover (Borah and Tellis 2016; Mackalski and Belisle 2015). Our study focuses on this type of spillovers which may be long-lasting.

3.2.2. Long term profit under the impact of product recall

In this section, we show how the expected profits of the two regimes for both the manufacturers can be transformed into long term expected profit as functions of recall likelihood parameter χ and the state variables- the goodwill, θ_1 and θ_2, of the two firms. We follow a similar approach as described in the existing literature (Haurie and Moresino 2006; Rubel, Naik, and Srinivasan 2011). In a continuous time framework, if t_r be the random time of product recall, then it divides the infinite planning horizon into two decision periods: pre-crisis period $[0,t_r]$ and post-crisis period (t_r,∞). Let $J_{ij} = J_{ij}(\theta_1,\theta_2)$ denote the instantaneous profit of the manufacturer i in the j^{th} regime where $j \in \{1,2\}$). $j = 1$ implies pre-recall period and $j = 2$ implies post recall period. The instantaneous profits of the firms are given by $J_{ij}(t) = Q_{ij}(t)p_{ij}(t) - \frac{\mu_i a_{ij}(t)^2}{2}$ where Q_{ij} is the demand of the manufacturer i at j^{th} period, $\mu_i \frac{a_{ij}^2}{2}$ denotes the quadratic advertising costs of the manufacturer i in period j. If r be the discount rate over the planing horizon, then the net present value of the profit of manufacturer i channel member in pre-recall and post-recall period are given respectively as

$$\begin{aligned}\pi_{i1} &= \int_0^{t_r} e^{-rt} J_{i1}(\theta_{i1}, \theta_{(3-i)1}, p_{i1}, p_{(3-i)1}, a_{i1})dt, \\ \pi_{i2} &= \int_{t_r}^{\infty} e^{-rt} J_{i2}(\theta_{i2}, \theta_{(3-i)2}, p_{i2}, p_{(3-i)2}, a_{i2})dt \end{aligned} \tag{3.5}$$

where π_{i1} represents the total discounted profit of i^{th} manufacturer during the pre-crisis regime $[0, t_r]$ and π_{12} is the total post-crisis discounted profit of the i^{th} manufacturer at time $t = t_r$. Now the time to product recall, t_r is random and can take any value on the planing horizon $[0, \infty)$. t_r being a random variable, the profits in both the period, as functions of t_r are random variables. Hence the long term expected profit of the i^{th} firm is:

$$\Pi_i = E\left[\pi_{i_1} + e^{-rt_r}\pi_{i2}\right], \tag{3.6}$$

where the expectation is taken with respect to the stochastic process as described in (3). The problem above is a random stopping problem. If $f(t)$ and $F(t)$ are the probability density and cumulative density functions of the stochastic occurrence process, then the hazard rate is $h(t) = \chi, F(t) = 1 - e^{-\int_0^t h(s)ds}$. Therefore, $f(t) = \chi e^{-\int_0^t h(s)ds}$. The long-term expected profit for the manufacturer i can be written as:

$$\begin{aligned}\Pi_i &= E[\int_0^{t_r} e^{-rs} J_{i1} ds + e^{-rt_r}\pi_{i2}], \\ &= \int_0^{\infty} \left[\int_0^{t_r} e^{-rs} J_{i_1} ds + e^{-rt}\pi_{i_2}\right]\chi e^{-\chi t} dt. \end{aligned} \tag{3.7}$$

where the profit of the pre-crisis period is the first term of the expression and the second term gives the profit of the post-crisis regime. It can be shown that the the sum of the two expected profits reduces to the following from when we integrate by parts:

$$\Pi_i = \int_0^{\infty} e^{-(r+\chi)t}\left\{J_{i_1} + \chi\pi_{i_2}\right\} dt. \tag{3.8}$$

Thus, expression (3.8) give the long term expected profit of any firm. We use this expression to find the optimal decisions of each of the two manufacturers.

3.2.3. Decision Problems of the Manufacturers

According to the above discussions, we define the decision problems of the competing manufacturers in the two regimes, pre-crisis and post-crisis. Thereafter, we can solve the second regime's problem first, followed by the first regime's problem. We denote the second regime's value functions of manufacturer 1 and manufacturer 2 by $V_{12}(\theta_{1j}(t), \theta_{2j}(t))$ and $V_{22}(\theta_{1j}(t), \theta_{2j}(t))$ respectively. The first regime's value functions are denoted by $V_{11}(\theta_{1j}(t), \theta_{2j}(t))$ and $V_{21}(\theta_{1j}(t), \theta_{2j}(t))$.

Both the manufacturers optimize their individual long term profits. The second regime's decision problems of the the manufactures are given by:

$$
\begin{aligned}
V_{12}(\theta_{12}(t), \theta_{22}(t)) &= \underset{p_{12}(t), a_{12}(t)}{\text{Max}} \int_{t_r}^{\infty} e^{-rt}[Q_{12}p_{12}(t) - \frac{\mu_1 a_{12}^2(t)}{2}]dt \\
V_{22}(\theta_{12}(t), \theta_{22}(t)) &= \underset{p_{22}(t), a_{22}(t)}{\text{Max}} \int_{t_r}^{\infty} e^{-rt}[Q_{22}p_{22}(t) - \frac{\mu_2 a_{22}^2(t)}{2}]dt \\
\text{Subject to} & \\
\dot{\theta}_{12}(t) &= k_{12}a_{12}(t) - \delta_{12}\theta_{12} \\
\dot{\theta}_{22}(t) &= k_{22}a_{22}(t) - \delta_{22}\theta_{22} \\
\text{and with the following initial conditions} & \\
\theta_{12}(t_r^+) &= (1-\eta)\theta_{11}(t_r^-) \\
\theta_{22}(t_r^+) &= (1-\phi)\theta_{22}(t_r^-) \qquad (3.9)
\end{aligned}
$$

Similarly, the pre-crisis decision problems , in accordance with equation (3.7) are given by:

$$
\begin{aligned}
V_{11}(\theta_{11}(t), \theta_{21}(t)) &= \underset{p_{11}(t), a_{11}(t)}{\text{Max}} \int_{0}^{\infty} e^{-rt}[Q_{11}p_{11}(t) - \frac{\mu_1 a_{11}^2(t)}{2} \\
&\quad + \chi V_{12}\big((1-\eta)\theta_{11}(t), (1-\phi)\theta_{21}(t)\big)]dt \\
V_{21}(\theta_{11}(t), \theta_{21}(t)) &= \underset{p_{21}(t), a_{21}(t)}{\text{Max}} \int_{0}^{\infty} e^{-rt}[Q_{21}p_{21}(t) - \frac{\mu_2 a_{21}^2(t)}{2} \\
&\quad + \chi V_{22}((1-\eta)\theta_{11}(t), (1-\phi)\theta_{21}(t)]dt \\
\text{Subject to} & \\
\dot{\theta}_{11}(t) &= k_{11}a_{11}(t) - \delta_{11}\theta_{11} \qquad (3.10) \\
\dot{\theta}_{21}(t) &= k_{21}a_{21}(t) - \delta_{21}\theta_{21} \\
\text{and with the following initial conditions} & \\
\theta_{11}(0) &= \theta_{10} \geq 0 \\
\theta_{22}(0) &= \theta_{20} \geq 0 \qquad (3.11)
\end{aligned}
$$

We investigate two different games - The Nash Game and the Stackelberg game. Under the Nash game, the manufacturers are similar in terms of market potential, brand image and market power. They take simultaneous pricing and advertising decisions and at the times $t = 0$ and $t = t_r$. Under the Stackelberg game, the focal firm is the market leader and announces its pricing and advertising decisions first. The non-focal firm is the follower and moves second.

3.3. Results and Discussions

The analytical solutions of the model enables us to find the feedback pricing and advertising policies for the firms. As a standard solution procedure of solving the differential game, we start by stating the Hamilton-Jacobi-Bellman (HJB) equations (Dockner et al. 2000) for each firm. Subsequently, the first order conditions on the decision variables will help us in finding the equilibrium strategies from the HJB equations. We note that the value functions V_{ij} are concave in a_{ij} and p_{ij}. From equations (3.9), the HJB equations for the second regime are given by:

$$\begin{aligned} rV_{12}(\theta_{12},\theta_{22}) &= \underset{a_{12},p_{12}}{\text{Max}}\,[(Q_{12})p_{12} - \frac{\mu_1}{2}a_{12}^2(t) + \frac{\partial V_{12}}{\partial \theta_{12}}\dot{\theta}_{12}(t) + \frac{\partial V_{12}}{\partial \theta_{22}}\dot{\theta}_{22}(t)] \\ rV_{22}(\theta_{12},\theta_{22}) &= \underset{a_{22},p_{22}}{\text{Max}}\,[(Q_{22})p_{22} - \frac{\mu_2}{2}a_{22}^2(t) + \frac{\partial V_{22}}{\partial \theta_{12}}\dot{\theta}_{12}(t) + \frac{\partial V_{22}}{\partial \theta_{22}}\dot{\theta}_{22}(t)] \end{aligned} \tag{3.12}$$

where Q_{ij}s are the demand functions in equation (3.1). Similarly, from equations (3.10) and (3.11) the first period HJB equations of the players are given by:

$$\begin{aligned} (r+\chi)V_{11}(\theta_{11},\theta_{21}) &= \underset{a_{11},p_{11}}{\text{Max}}\,[(Q_{11})p_{11} - \frac{\mu_1}{2}a_{11}^2(t) + \frac{\partial V_{11}}{\partial \theta_{11}}\dot{\theta}_{11}(t) + \frac{\partial V_{11}}{\partial \theta_{21}}\dot{\theta}_{21}(t) \\ &\quad + \chi V_{12}((1-\eta)\theta_{11}(t),(1-\phi)\theta_{21}(t)] \\ (r+\chi)V_{22}(\theta_{11},\theta_{21}) &= \underset{a_{21},p_{21}}{\text{Max}}\,[(Q_{21})p_{21} - \frac{\mu_2}{2}a_{21}^2(t) + \frac{\partial V_{21}}{\partial \theta_{11}}\dot{\theta}_{11}(t) + \frac{\partial V_{21}}{\partial \theta_{21}}\dot{\theta}_{21}(t)] \\ &\quad + \chi V_{22}((1-\eta)\theta_{11}(t),(1-\phi)\theta_{21}(t)] \end{aligned} \tag{3.13}$$

To understand the impact of the recall, we use numerical analysis along with our mathematical analysis. Three parameters of our model, crisis likelihood χ, damage to goodwill η and the spillover effect ϕ illustrate the impact of a recall. We consider a crisis of different likelihoods and damages and show how the pricing strategies and advertising strategies vary for the two firms in the post and pre-crisis periods. For the computational analysis, we use the following parameter values-

(i) $\alpha_1 = 1; \alpha_2 = 1;$ (market size)
$\alpha_1 = 1.5$ when focal firm is Stackelberg leader

(ii) $\mu_1 = 200; \mu_2 = 200;$ (proportionality constant for advertising cost)

(iii) $\delta_1 = .06; \delta_2 = .06;$ (goodwill forgetting)

(iv) $r = .06$ discounting

(v) $\beta_1 = .5; \beta_2 = .5$; (cross price sensitivity)

(vi) $\gamma_1 = .5; \gamma_2 = .5$; (cross goodwill sensitivity)

(vii) $k_{11}, k_{12}, k_{21}, k_{22}$ (advertising effectiveness)

(viii) χ (crisis likelihood)

(ix) η (crisis impact or jump in goodwill after recall)

(x) ν (Spillover sensitivity to recall impact)

The value of the parameters (vii) to (x) depend on the recall impact. For example, for a high severity recall, advertising effectiveness will reduce considerably while for a low severity recall the reduction is small. For all cases, where we compare the pre-crisis and post-crisis decisions, we assume that the time to recall $t_r = 20$. The pre-crisis and post-crisis values of the parameters (Vi) to (x) are given in the following table.

	Likelihood(χ)	Impact(η)	Spillover Sensitivity (ν)	k_{11}	k_{12}	k_{21}	k_{22}
Benchmark Case	0.3	0.3	0.7	1	0.75	1	0.9
Low Likelihood Low Impact	0.05	0.05	0.5	1	1	1	1
Low Likelihood High Impact	0.05	0.6	0.9	1	0.5	1	0.7
High Likelihood Low Impact	0.6	0.05	0.5	1	1	1	1
High Likelihood High Impact	0.6	0.6	0.9	1	0.5	1	0.7

Table 3.1: Cases for Numerical Analysis

We use the Mathematica 12 software to perform our Numerical Analysis. Proof of all the Propositions are given in the Appendix.

The value functions have the following form:
$V_{ij}(\theta_{1j}, \theta_{2j}) = A_{ij}\theta_{1j}(t)^2 + B_{ij}\theta_{2j}(t)^2 + C_{ij}\theta_{1j}(t)\theta_{2j}(t) + D_{ij}\theta_{1j}(t) + E_{ij}\theta_{2j}(t) + F_{ij}$. We obtain the coefficients $A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}$ and F_{ij} of the value functions by solving a set of 12 nonlinear equations. These are given in the appendix.

3.3.1. Nash Game

When two similar manufacturers compete, we assume that there is no market leader and the rivals take their advertising and pricing decision simultaneously. The first order conditions of the HJB equations wih respect to the decision variables, allows us to obtain the following proposition.

Proposition 1:*Under a Nash game, the equilibrium pre-crisis and post-crisis advertising and pricing decision of the two firms are:*

$$p^*_{ij}(t) = \frac{(2\alpha_i + \alpha_{3-i}\beta_i + (2-\beta_i\gamma_{3-i})\theta_{ij}(t) + (\beta_i - 2\gamma_i)\theta_{(3-i)j}(t))}{(4-\beta_1\beta_2)} \tag{3.14}$$

$$a^*_{1j}(t) = \frac{k_{1j}(2A_{1j}\theta_{1j}(t) + C_{1j}\theta_{2j}(t) + D_{1j})}{\mu_1} \tag{3.15}$$

$$a^*_{2j}(t) = \frac{k_{2j}(2B_{2j}\theta_{2j}(t) + C_{2j}\theta_{1j}(t) + E_{2j})}{\mu_2}.$$

The value functions of the firms are of the form:

$$\begin{aligned} V_{1j}(\theta_{1j},\theta_{2j}) &= A_{1j}\theta_{1j}(t)^2 + B_{1j}\theta_{2j}(t)^2 + C_{1j}\theta_{1j}(t)\theta_{2j}(t) + D_{1j}\theta_{1j}(t) + E_{1j}\theta_{2j}(t) + F_{1j} \\ V_{2j}(\theta_{1j},\theta_{2j}) &= A_{2j}\theta_{1j}(t)^2 + B_{2j}\theta_{2j}(t)^2 + C_{2j}\theta_{1j}(t)\theta_{2j}(t) + D_{2j}\theta_{1j}(t) + E_{2j}\theta_{2j}(t) + F_{2j} \end{aligned} \tag{3.16}$$

where the equations for deriving the coefficients $A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}$ and F_{ij} are listed in the Appendix 2.

Proposition 1 reveals that the feedback pricing strategies are free from the the coefficients of the state variables in the value function. The advertising strategies depend on the coefficients of the state variables in the value function. We will show in the Appendix that the solutions to the Hamilton-Jacobi-Bellman equation is not unique in our model. In fact, the equations give four solutions to each coefficient of the value function. Hence, we have to select the right combination to form an admissible solution. We have assumed that the advertising and pricing decisions hold positive values only. This enables us to find a unique solution which results in positive price and advertising.

Nash Pricing Decisions: We analyze the effect of a firm's own goodwill and its competitor's goodwill on the pricing decisions. The above proposition shows that the pricing decisions of a firm is a linear function of the firm's own and its rival's goodwill. The following lemma shows the variation of price of each firm with the rival's and its own goodwill.

Lemma 1. *In the Nash game, for a firm $i \in \{1,2\}$, price p_{ij} is always increasing in its goodwill θ_{ij}*

and increases with the rival's goodwill, $\theta_{(3-i)j}$ *if* $\beta_i > 2\gamma_i$ *and decreases with the rival's goodwill if* $\beta_i < 2\gamma_i$ *.*

Proof: The proof follows from the first order condition of the firm price with respect to the goodwill. From equation (3.14), the coefficient of θ_i in a firm's price, $\frac{\partial p_{ij}}{\partial \theta_i} = \frac{2-\beta_i\gamma_{(3-i)}}{4-\beta_1\beta_2}$. By our assumptions, $0 < \beta_i, \gamma_i < 1$. It follows that, $(2-\beta_i\gamma_{(3-i)}) > 0$ and $(4-\beta_1\beta_2) > 0$ for $i \in \{1,2\}$. Therefore, $\frac{\partial p_{ij}}{\partial \theta_i} = \frac{2-\beta_i\gamma_{(3-i)}}{4-\beta_1\beta_2} > 0$.

The first order condition of p_{ij} with respect to $\theta_{(3-i)}$ gives $\frac{\partial p_{ij}}{\partial \theta_{(3-i)}} = \frac{\beta_i - 2\gamma_i}{4-\beta_1\beta_2}$. Therefore, the price decided by manufacturer i is increasing in the rival's goodwill if $\frac{\beta_i - 2\gamma_i}{4-\beta_1\beta_2} > 0$ i.e. $\beta_i > 2\gamma_i$ and decreasing in the rival's goodwill if $\beta_i < 2\gamma_i$.

The above findings have some significant insights. This first result highlights that a firm's own goodwill has a positive effect on pricing. Thus, a firm with a high goodwill can charge a higher price as compared to a firm with lower goodwill. However, for goodwill to be high, higher advertising efforts are needed. Hence, the manufacturer needs to invest more in advertising in order to charge a higher price. This is consistent with the finding of the previous literature (Lu, J. Zhang, and Tang 2019; De Giovanni 2019).

The second result about the inter-relationship between the cross price sensitivity and the cross goodwill(brand) sensitivity shows that a firm's price can increase with the rival's goodwill if cross price sensitivity of the firm if greater than twice the cross goodwill sensitivity. In other words, cross-price sensitivity has a more pronounced effect on the a firm's demand and pricing policies than the cross-brand sensitivity. Previous empirical research has shows that an increase in non-price advertising leads to lower price sensitivity among consumers (Kaul and Wittink 1995) and brand credibility decreases price sensitivity (Erdem, Swait, and Louviere 2002). We may hypothesize that goodwill is a measure of brand credibility and hence price sensitivity is reduced by a higher goodwill. Our findings in this paper comport with the above results.

In addition to the above findings, Figure 3.1 shows how the Nash prices of the two firms vary with the goodwill of both the firms.

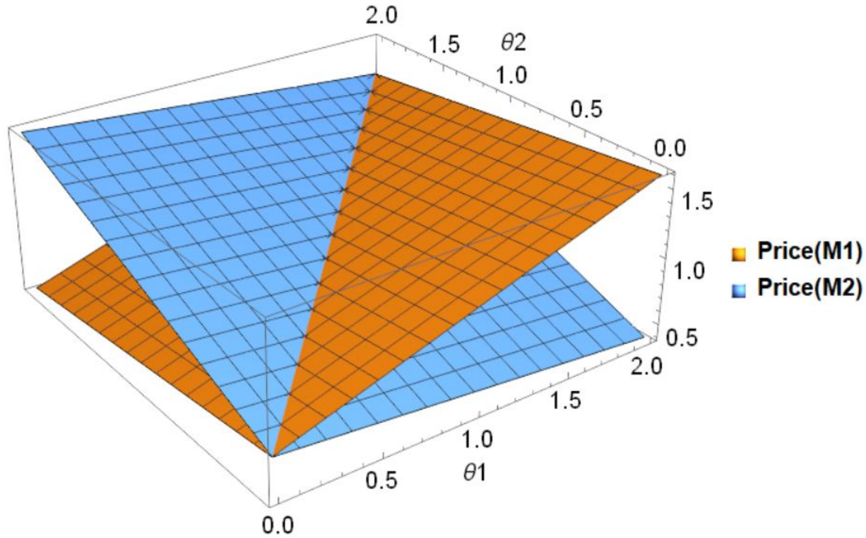


Figure 3.1: Nash Pricing and Goodwill

Nash Advertising Decisions: A complete analysis of the advertising decisions can be made if the signs of the coefficients of the state variables are known. Advertising effort, a_{ij} is increasing in the advertising effectiveness k_{ij}, which shows that firms should invest more in advertising if effectiveness is high. However, following the empirical evidence, we have assumed that advertising effectiveness decreases in the post-crisis period. Hence, we may say that the loss of this advertising effectiveness has the potential to diminish post-crisis advertising. However, the presence of the coefficients of the value function also influence the advertising efforts. For example, $A_{1j} > 0$ (or $C_{1j} > 0$) would indicate that M1 should increase advertising with its goodwill θ_{1j} (or its rival's goodwill θ_{2j}) and $B_{2j} > 0$ (or C_{2j}) implies that M2 should increase its advertising efforts with its own goodwill, θ_{2j} (or its rival's goodwill θ_{1j}). Our computational analysis shows that the coefficients of value functions $A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}$ and F_{ij} have four possible solutions. In the feasible solution with positive values of price and advertising, $A_{1j} > 0, B_{1j} < 0, D_{1j} > 0$ and $B_{2j} > 0, C_{2j} < 0, E_{2j} > 0$. The above findings highlight that a firm's advertising increases with its brand image and plummets when the rival's brand image rises.

Apparently the expressions of the advertising decisions in the two periods seem similar. However, this is because we are presenting the advertising efforts in terms of the coefficients of the value functions. From equations (B.18) and (B.26) in the Appendix B, it is clear that the first period coefficients of the value functions depend on the second period coefficients as well as χ (crisis likelihood), η (damage to goodwill due to recall) and spillover ϕ. Due to the non-linear and complex nature of the equations in many variables, we solve the equations numerically and obtain the insights about the influence of model parameters.

3.3.1.1. Impact of Crisis on Nash Pricing and Advertising Decisions

The impact of crisis is characterized by the parameters χ (Crisis likelihood) and η (damage to goodwill). We examine five cases - Benchmark likelihood and impact, Low likelihood and low impact, low likelihood and high impact, high likelihood and low impact, high likelihood and high impact and find the corresponding pricing policies. We plot the pricing policies of the two similar firms and show the variation of prices with time. We assume that recall occurs at $t = t_r$. The regime when $t > t_r$ is the spillover (post-crisis) regime. Previous literature has estimated that in the auto industry the likelihood of crisis is around 30% and the impact is also around 30% (Rubel, Naik, and Srinivasan 2011). While this may change from industry to industry, there is not enough empirical research confirming the crisis likelihood across industries. We therefore assume that $(\chi = .3, \eta = .3)$ is the benchmark case. Such a likelihood and impact might be considered to be moderate. As mentioned in table 3.1, low impact and high impact correspond to $\eta = .05, \eta = .6$ respectively and low likelihood and high likelihood correspond to $\chi = .05, \chi = .6$ respectively.

Immediate Impact of Crisis

Immediately after the crisis, the goodwills of both the firms take a jump. For the manufacturer 1, a loss of goodwill results in a decline in goodwill and we have

$$\theta_{12}(t_r^+) = (1-\eta)\theta_{11}(t_r^-). \tag{3.17}$$

For the manufacturer 2, a spillover occurs and we have the following equation

$$\theta_{22}(t_r^+) = \left(1 - \nu\eta \frac{\theta_{11}(0)}{(1+\theta_{11}(0))(1+\theta_{21}(0))}\right)\theta_{21}(t_r^-) \tag{3.18}$$

From equations (3.16) and (3.20), we compare the immediate effect of recall on price. The result is the following Lemma.

Lemma 2. *For the focal firm, manufacturer 1, the pre-crisis price at time t_r^- is higher than the the post-crisis price at t_r^+ if,*

$$\theta_{11}(t_r^-) > \frac{\phi(2\gamma_1 - \beta_1)}{\eta(2-\beta_1\gamma_2)}\theta_{21}(t_r^-). \tag{3.19}$$

We have considered that the goodwill θ_{ij} for any firm in our model is positive. Therefore, condition (3.19) is trivial when $\beta_1 > 2\gamma_1$ which also implies that the non-focal firm's goodwill has a positive impact on the focal firm's price (follows from Lemma 1). Thus the focal firm will always raise post-crisis prices if the non-focal firm's goodwill positively affects the focal firm's price. On the other hand, if the non-focal firm's goodwill has a negative impact on the focal firm's price, then the focal firm decreases price immediately after the recall if its pre-crisis goodwill is sufficiently higher than a threshold based on the non-focal firm's pre-crisis goodwill.

Lemma 3. *For the non-focal firm, manufacturer 2, the pre-crisis price at time t_r^- is higher than the the post-crisis price at t_r^+ if,*

$$\theta_{21}(t_r^-) > \frac{\eta(2\gamma_2 - \beta_2)}{\phi(2 - \beta_2\gamma_1)}\theta_{11}(t_r^-). \tag{3.20}$$

How Do Crisis Likelihood and Intensity Affect Nash Pricing and Advertising Decisions?

We compare the pricing and advertising decisions of both the firms numerically for the different cases given in Table 3.1. Our numerical analysis provide several novel insights about the effect of crisis likelihood and intensity on the decision of the firms. As depicted in Figures 3.2 and 3.3, when the recall impact is low, the decisions of the firms do not change much with the crisis likelihood. Since a low impact recall causes minimal damage to a firm's reputation, the likelihood of crisis does not alarm the firms ex-ante. The focal firm maintains a higher pre-crisis price and price drops marginally after the recall. Moreover, during the pre-crisis period, the focal firm's price has a decreasing trend signifying that the firm is trying to capture more market. A steady drop in the post-crisis price can send a negative signal to consumers about the value of the focal firm's brand. Therefore, the focal firm's post-crisis price, though drops marginally after the recall, remains relatively constant.

Reacting to the competition, the non-focal firm decreases prices in the pre-crisis period. The non-focal firm's price marginally increases immediately after the crisis followed by a decrease during the spillover period. However, we note that the increased price is still below the price of the focal firm. The price increase is thus merely an opportunistic behaviour wherein the non-focal firm believes that due to the recall, consumer perception of the focal brand might have changed. Consequently, the non-focal firm's demand can increase even if its price is a bit higher than before. We also believe that the price increase can potentially convey a positive signal about the goodwill of the non-focal firm.

The advertising efforts of both the firm increase after the recall. However, the focal firm's advertising remains higher than the non-focal firm in both regimes. Few empirical studies (Rubel, Naik, and Srinivasan 2011; Cleeren, H. J. Van Heerde, and Dekimpe 2013) support the finding that advertising should increase after a recall.

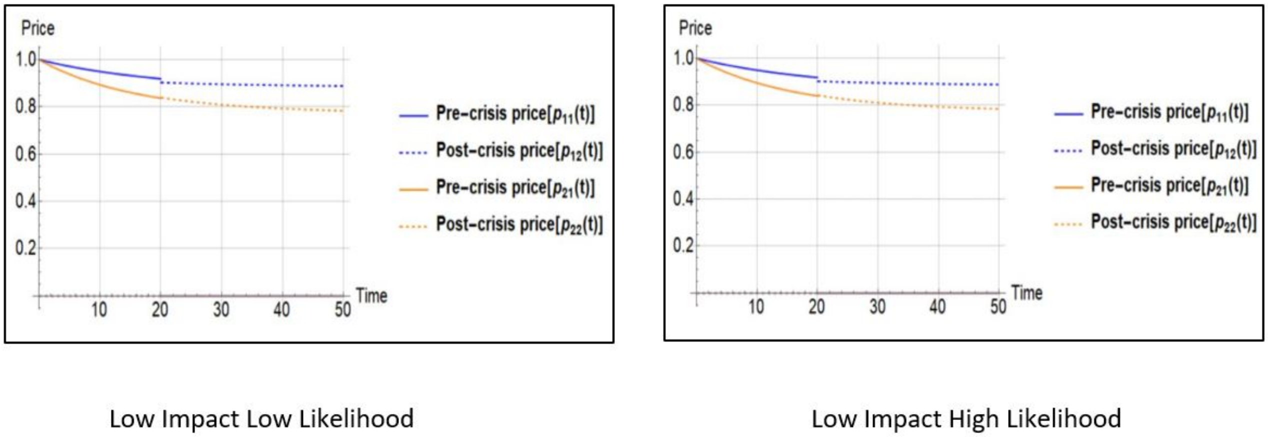


Figure 3.2: Nash Pricing Policies for Low Impact ($\chi = .05, \eta = .05$) Recall

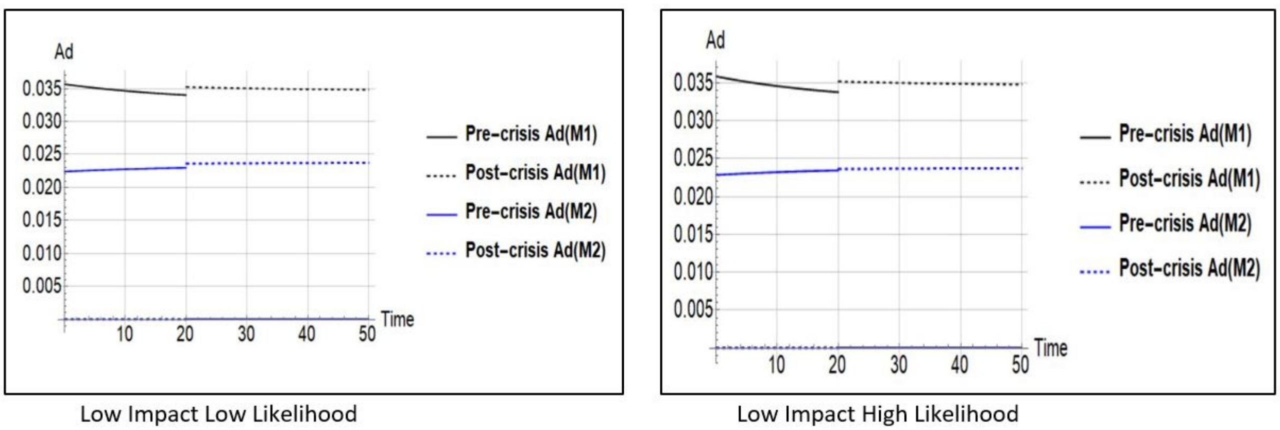


Figure 3.3: Nash Advertising Policies for Low Impact ($\chi = .05, \eta = .05$) Recall

The result of our analysis in figures 3.4 and 3.5 shows that as crisis impact increases, the non-focal firm charges prices higher than the focal firm not only in the post-crisis period but also in the pre-crisis period for a high impact high likelihood recall. Such pricing decisions are again a manifestation of the non-focal firm's dual motivation - signal positive and higher product quality or goodwill to the consumers and make extra profit by increasing price. The focal firm, decreases price in order to make a profit with an impending decline in post-crisis reputation. Moreover, the focal firm cuts pre-crisis advertising costs by reducing efforts in the pre-crisis regime.

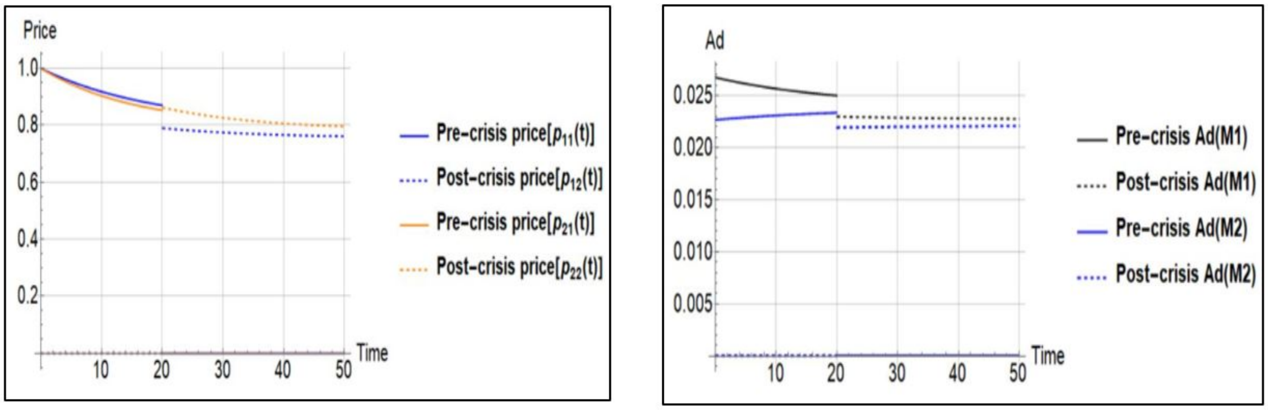
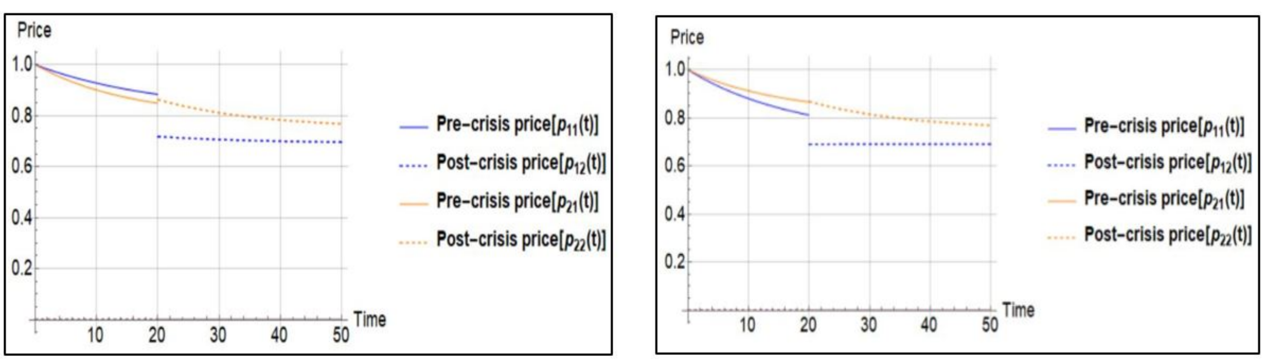


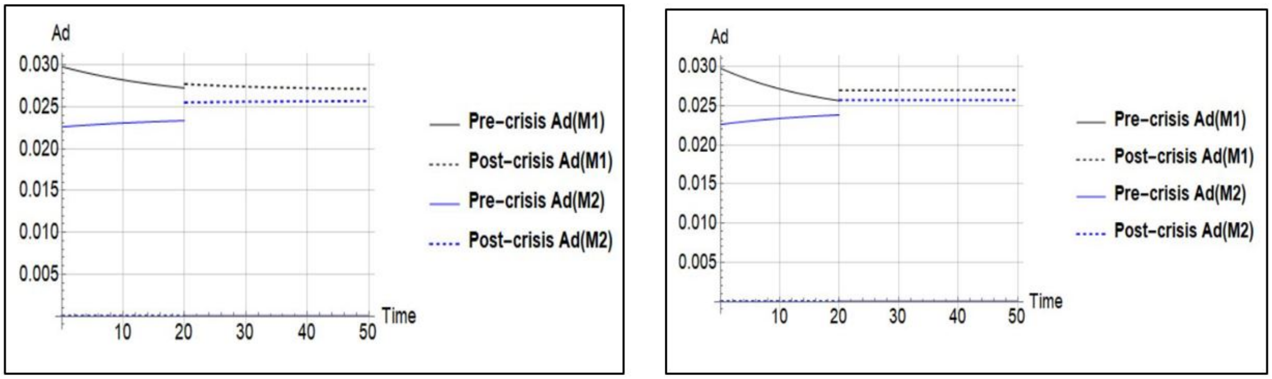
Figure 3.4: Nash Pricing Policies- Benchmark Case ($\chi = .3, \eta = .3$)

On the contrary, the non-focal firm raises pre-crisis advertising efforts. While this enhanced advertising increases the non-focal firm's advertising costs, it potentially increases goodwill enough to buffer against the spillover of a negative reputation in the post-crisis regime. Surprisingly, we find that the benchmark case necessitates equilibrium advertising of both firms to be lower in the post-crisis regime. We believe that the non-linearity of the advertising functions of both firms causes the above variation of advertising. The finding, however, is consistent with some findings which state that whether pre-crisis advertising should be higher depends on the (χ, η) pair (Mukherjee and Chauhan 2019). Some empirical evidence also suggests that "laying low" might be a feasible advertising strategy during brand crisis (Dahlen and Lange 2006). While the study does not emphasize the importance of envisioning the recall, we posit that the prior estimation or knowledge of (η, χ) pair dictates the firm's equilibrium strategies and hence post-crisis advertising level drops in some cases.



High Impact Low Likelihood High Impact High Likelihood

Figure 3.5: Nash Pricing Policies for High Impact ($\chi = .6, \eta = .6$) Recall



High Impact Low Likelihood High Impact High Likelihood

Figure 3.6: Nash Advertising Policies for High Impact ($\chi = .6, \eta = .6$) Recall

As an outcome of the numerical analysis and our discussions above, we can propose the equilibrium advertising policies of either of the firms for a given impact and a likelihood of a product recall. For ready reference, we summarize the resulting pricing and advertising policies of the focal firm in the Table 3.2. The strategies correspond to product recalls of different impacts and likelihoods. Table 3 summarizes the resulting pricing and advertising policies for the non-focal firm suffering from a negative spillover effect. The strategies are classified based on the combination of a likelihood and an impact of a product recall.

Crisis Likelihood (η)	**Crisis Impact (χ)**	**Focal Firm**	
		Pricing Strtegy	**Advertising Strategy**
Low	Low	Maintain a higher pre-crisis and post-crisis price during spillover than the non-focal firm. Reduce price marginally after crisis. Hold price fairly constant after crisis.	Start with a high pre-crisis ad. Reduce ad gradually during pre-crisis regime. Increase and maintain post-crisis ad.
High	Low	Same as Low likelihood low impact case.	Same as Low likelihood low impact case.
Low	High	Maintain a higher pre-crisis price than the non-focal firm. Decrease price immediately after crisis and maintain a steady low price during spillover.	Start with lower pre-crisis ad than the low impact case. Decrease pre-crisis more than low impact case. Increase and maintain post-crisis ad.
High	High	Maintain a lower pre-crisis and post-crisis price during spillover than the non-focal firm. Reduce price after crisis. Hold price fairly constant after crisis during spillover.	Same as high impact low likelihood case.

Table 3.2: Nash Game- Focal firm Pricing and Advertising

Crisis Likelihood (η)	Crisis Impact (χ)	Non-Focal Firm	
		Pricing Strategy	Advertising Strategy
Low	Low	Gradually decrease price in both the regimes. Keep price lower than the focal firm's price.	Gradually increase pre-crisis ad. Increase ad even more after the crisis during spillover and hold steady.
High	Low	Same as the low impact low likelihood case.	Same as low impact low likelihood case.
Low	High	Keep a lower pre-crisis pricing than the focal firm. Increase price immediately after crisis and gradually decrease price during the spillover.	Gradually increase pre-crisis ad. Increase ad substantially after the crisis and keep high post-crisis advertising.
High	High	Maintain a higher pre-crisis price than the focal firm. Gradually decrease price during spillover but keep the price higher than the focal firm.	Same as the low likelihood high impact case.

Table 3.3: Nash Game - Non- Focal firm's Pricing and Advertising Policies

3.3.2. Stackelberg Game

A legitimate question in a duopoly competition is whether a market leadership can be advantageous for a focal firm. We presume that the pricing and advertising decisions will change when the focal firm is a leader and makes its decisions based on the follower's reaction functions. Our investigation of the above question leads to the following proposition.

Proposition 2:*Under a Stackelberg game, with the focal firm as leader, the equilibrium pre-crisis and post-crisis advertising and pricing decision of the two firms are:*

$$p_{1j}^*(t) = \frac{(2\alpha_1 + \alpha_2\beta_1 + (2-\beta_1\gamma_2)\theta_{1j}(t) + (\beta_1 - 2\gamma_1)\theta_{2j}(t))}{2(2-\beta_1\beta_2)} \tag{3.21}$$

$$p_{2j}^*(t) = \frac{(4\alpha_2 + 2\alpha_1\beta_2 - \alpha_2\beta_1\beta_2 + (2\beta_2 + \beta_1\beta_2\gamma_2 - 4\gamma_2)\theta_{1j}(t) + (4 - \beta_1\beta_2 - 2\beta_2\gamma_1)\theta_{2j}(t))}{4(2-\beta_1\beta_2)}$$

$$a_{1j}^*(t) = \frac{k_{1j}(2A_{1j}\theta_{1j}(t) + C_{1j}\theta_{2j}(t) + D_{1j})}{\mu_1} \tag{3.22}$$

$$a_{2j}^*(t) = \frac{k_{2j}(2B_{2j}\theta_{2j}(t) + C_{2j}\theta_{1j}(t) + E_{2j})}{\mu_2}.$$

The value functions of the firms are of the form:

$$\begin{aligned} V_{1j}(\theta_{1j},\theta_{2j}) &= A_{1j}\theta_{1j}(t)^2 + B_{ij}\theta_{2j}(t)^2 + C_{1j}\theta_{1j}(t)\theta_{2j}(t) + D_{1j}\theta_{1j}(t) + E_{1j}\theta_{2j}(t) + F_{1j} \\ V_{2j}(\theta_{1j},\theta_{2j}) &= A_{2j}\theta_{1j}(t)^2 + B_{2j}\theta_{2j}(t)^2 + C_{2j}\theta_{1j}(t)\theta_{2j}(t) + D_{2j}\theta_{1j}(t) + E_{2j}\theta_{2j}(t) + F_{2j} \end{aligned} \tag{3.23}$$

where the equations for deriving the coefficients $A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}$ and F_{ij} are listed in the Appendix 2.

The pricing strategies for a Stackelberg game are significantly different from the Nash pricing strategies. However, the advertising strategies are structurally identical in both the games. However, the solution of the coefficient of value functions will differ in the two games as the price is different and the parameter values different. For example, in the Stackelberg game, initial goodwill $\theta)1(0)$ can be greater than $\theta_2(0)$ or the market potential of the leader is greater than the follower $\alpha_1 > \alpha_2$. Therefore, though the advertising policies in the Nash and the Stackleberg game looks similar, they are indeed different when evaluated. Thus, the value function and hence the long term expected profits of the firms will also be different.

Stackelberg Pricing Decisions: We analyze the effect of a firm's own goodwill and its competitor's goodwill on the pricing decisions. Proposition 2 shows that the Stackelberg pricing decision of a firm is a linear function of the firm's and its rival's goodwill. Unlike the Nash game, in the Stackelberg game, the mathematical structures of the two firms' prices are different. Therefore, we analyze the effect of the model parameters on the prices of the two firms separately. We have the following lemma for the leader.

Lemma 4. *In the Stackelberg game, the Leader's price p_{1j} is always increasing in its goodwill θ_{1j} and increases with the follower's goodwill, θ_{2j} if $\beta_1 > 2\gamma_1$ and decreases with the follower's goodwill if $\beta_1 < 2\gamma_1$.*

Proof: From equation (3.21), the first order condition of p_{1j} with respect to θ_{1j} gives, $\frac{\partial p_{1j}}{\partial \theta_{1j}} = \frac{2-\beta_1\gamma_2}{2(2-\beta_1\beta_2)} > 0$ since $0 < \beta_1, \beta_2, \gamma_2 < 1$. Therefore the leader's price is increasing in its goodwill.

Furthermore, $\frac{\partial p_{1j}}{\partial \theta_{2j}} = \frac{\beta_1 - 2\gamma_1}{2(2-\beta_1\beta_2)} > 0$ if $\beta_1 > 2\gamma_1$. Thus the leader's price increases with the rival's goodwill if the cross-price sensitivity of the leader is greater than twice the cross goodwill sensitivity.

The conditions obtained for the leader are similar to the conditions in the Nash game. Hence, the insights obtained are same as in the Nash game. We have the following lemma for the non-focal firm.

Lemma 5. *In the Stackelberg game, the follower's price p_{2j} is increasing in its goodwill, θ_{2J} and decreasing with the leader's goodwill, θ_{1j} if $\beta_2 < \frac{4\gamma_2}{(2+\beta_1\gamma_2)}$ and increasing with the leader's goodwill if $\beta_2 > \frac{4\gamma_2}{(2+\beta_1\gamma_2)}$.*

Proof: From equations (3.21), for the non-focal firm, the first order condition gives $\frac{\partial p_{2j}}{\partial \theta_{2j}} =$

$\frac{4-\beta_1\beta_2-2\beta_2\gamma_1}{4(2-\beta_1\beta_2)} > 0$ since both the numerator and denominators are greater than zero from the model assumptions $0 < \beta_i < 1$ and $0 < \gamma_i < 1$. Thus the follower's price increases in its own goodwill.

In addition, $\frac{\partial p_{2j}}{\partial \theta_{1j}} = \frac{2\beta_2+\beta_1\beta_2\gamma_2-4\gamma_2}{4(2-\beta_1\beta_2)}$. Clearly, the denominator is positive. The sign of the expression therefore depends on the sign of the numerator. The numerator's negativity (or positivity) gives the following conditions - $(2\beta_2+\beta_1\beta_2\gamma_2-4\gamma_2) < (\text{ or } >) 0 \implies$, $\beta_2 < (\text{ or } >) \frac{4\gamma_2}{(2+\beta_1\gamma_2)}$.

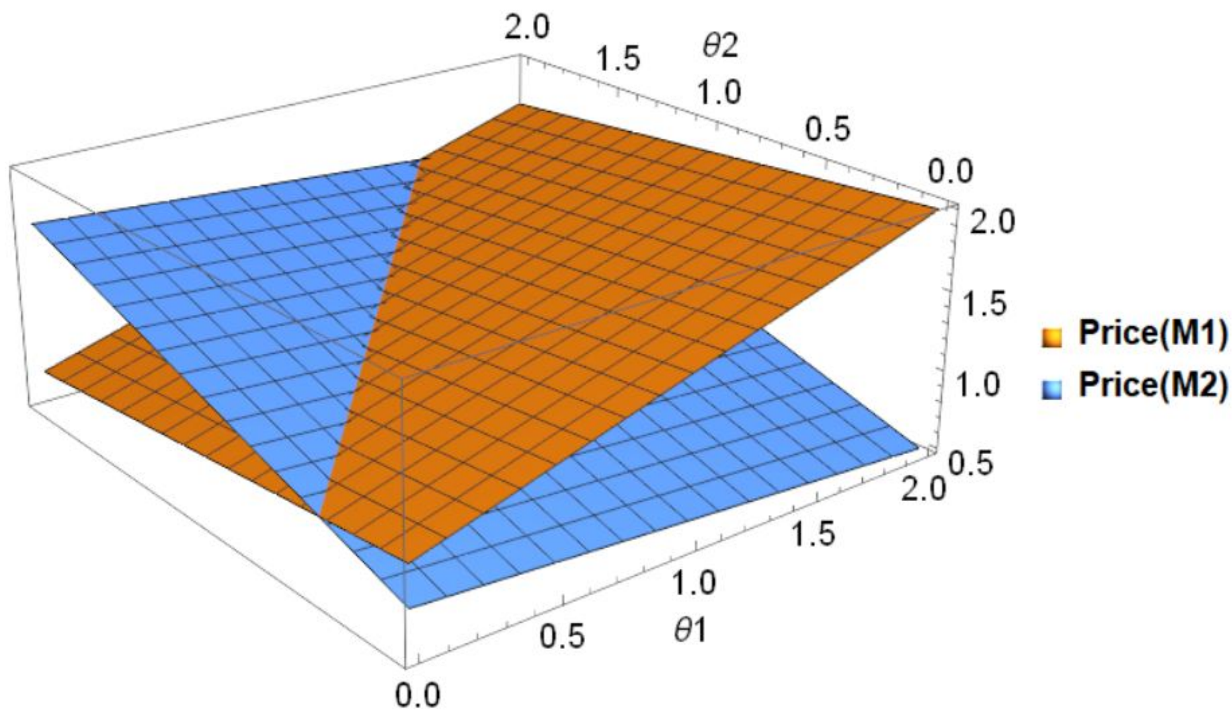


Figure 3.7: Stackelberg Price and Goodwill

Stackelberg Advertising: The focal firm as a market leader announces its pricing decisions and advertising decisions first. We have seen that contrary to the pricing decisions, the structural form of the advertising decision do not change in the two games - Nash and Stackelberg. However, as discussed earlier the solutions of value function coefficients are different in the Stackleberg game. Similar to the Nash game, in the feasible solution which gives positive values of price and advertising, $A_{1j} > 0, B_{1j} < 0, D_{1j} > 0$ and $B_{2j} > 0, C_{2j} < 0, E_{2j} > 0$. Therefore, like the Nash game a firm's advertising increases with its brand image and plummets when the rival's brand image rises.

Lemma 6. *In both the pre-crisis and post-crisis regime, the focal firm as a market leader would charge higher prices than in the Nash game if,*

(i) $\beta_1 > 2\gamma_1$ *and*

(ii) $\theta_{ij}^S > \dfrac{2(2-\beta_1\beta_2)}{4-\beta_1\beta_2}\theta_{ij}^N$

where the Nash and Stackelberg goodwills are $\theta_{ij}^N(t)$ *and* $\theta_{ij}^S(t)$ *respectively.*

Proof: For the proof of this lemma, in order to avoid confusion with notations, we denote the Nash prices by $p_{ij}^N(t)$ and the Stackelberg prices by $p_{ij}^S(t)$. The Nash and Stackelberg brand images are $\theta_{ij}^N(t)$ and $\theta_{ij}^S(t)$ respectively. For brevity we drop the time variable t from the functions.

Comparing the two prices from equations (3.21) and (3.14), the Stackelberg price is higher if,

$$\begin{aligned} p_{ij}^S - p_{ij}^N &= \frac{(2\alpha_1+\alpha_2\beta_1+(2-\beta_1\gamma_2)\theta_{1j}^S+(\beta_1-2\gamma_1)\theta_{2j}^S}{2(2-\beta_1\beta_2)} - \\ &\quad \frac{(2\alpha_i+\alpha_{3-i}\beta_i+(2-\beta_i\gamma_{3-i})\theta_{ij}^N(t)+(\beta_i-2\gamma_i)\theta_{(3-i)j}^N(t))}{(4-\beta_1\beta_2)} \\ &> 0. \end{aligned}$$

Algebraic manipulations give,

$$\begin{aligned} p_{ij}^S - p_{ij}^N &= \frac{\beta_1\beta_2(2\alpha_1+\alpha_2\beta_1)}{2(2-\beta_1\beta_2)(4-\beta_1\beta_2)} + \\ &\quad (2-\beta_1\gamma_2)\left(\frac{\theta_{1j}^S}{2(2-\beta_1\beta_2)} - \frac{\theta_{1j}^N}{4-\beta_1\beta_2}\right) + \\ &\quad (\beta_1-2\gamma_1)\left(\frac{\theta_{2j}^S}{2(2-\beta_1\beta_2)} - \frac{\theta_{2j}^N}{4-\beta_1\beta_2}\right). \end{aligned}$$

The three terms in the above expression are positive if $(\beta_1-2\gamma_1)>0$ and $\theta_{ij}^S > \dfrac{2(2-\beta_1\beta_2)}{4-\beta_1\beta_2}\theta_{ij}^N$.

The above lemma signifies that the Stackelberg pricing is guaranteed to be higher than the Nash pricing for the focal firm if its cross-price sensitivity is considerably bigger than the cross goodwill sensitivity and the Stackelberg goodwill of each firm is greater than the Nash goodwill.

As a special case, if we consider the brand goodwills to be identical in the two games, then the Stackelberg price is greater than the Nash price if, $\dfrac{\Gamma}{2(2-\beta_1\beta_2)} > \dfrac{\Gamma}{4-\beta_1\beta_2}$ where $\Gamma = (2\alpha_1 + \alpha_2\beta_1 + (2-\beta_1\gamma_2)\theta_{1j}(t) + (\beta_1-2\gamma_1)\theta_{2j}(t))$. Simplification gives the condition, $\beta_1\beta_2 > 0$ which is always true since $0<\beta_i<1$ by our model assumptions.

Lemma 7. *In both the pre-crisis and post-crisis regime, the non-focal firm M2 would charge higher prices than in the Nash game if,*

(i) $\beta_2 > 2\gamma_2$

(ii) $\theta_{1j}^S > \dfrac{\Delta(2-\beta_2\gamma_1)}{(2\beta_2+\beta_1\beta_2-4\gamma_2)}\theta_{1j}^N$

(iii) $\theta_{2j}^S > \dfrac{\Delta(\beta_2-2\gamma_2)}{(4-\beta_1\beta_2-2\beta_2\gamma_1)}\theta_{2j}^N$

. where $\Delta = \dfrac{4(2-\beta_1\beta_2)}{4-\beta_1\beta_2}$.

Proof: The prrof of the above lemma is similar to the proof of lemma 6.

3.3.2.1. The Impact of Crisis on Price and Advertising in the Stackelberg Game

We analyze the impact of recall on pricing and advertising policies in the Stackelberg leadership game under different levels of impact and likelihood - benchmark(medium), low and high. The value function coefficients play a critical role in the following analysis. Therefore, we present a table which compares the value of coefficients in the Nash and Stackelberg games.

Coefficients	**Stackelberg >Nash**	**Stackelberg <Nash**
A_{11}	✓	
B_{11}	✓	
C_{11}		✓
D_{11}	✓	
E_{11}		✓
A_{21}	✓	
B_{21}	✓	
C_{21}		✓
D_{21}	✓	
E_{21}		✓

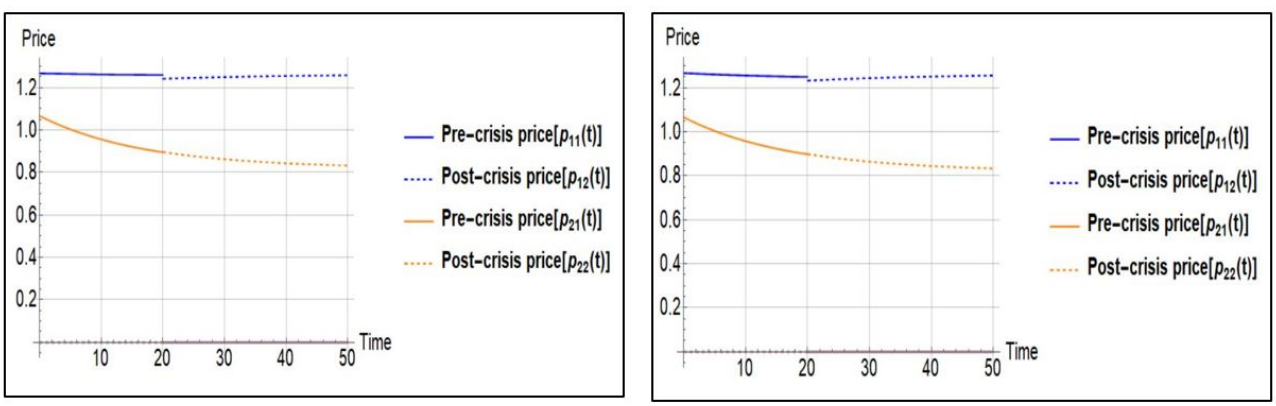
Table 3.4

How Do Crisis Likelihood and Intensity Affect Stackelberg Pricing and Advertising Decisions?

Previous research shows that a Stackelberg leader can charge higher prices during a product recall (Rubel 2018). We compare the pricing and advertising decisions of both the firms numerically for the different cases given in Table 3.1. Figures 3.8-3.12 depict the numerical findings and Table

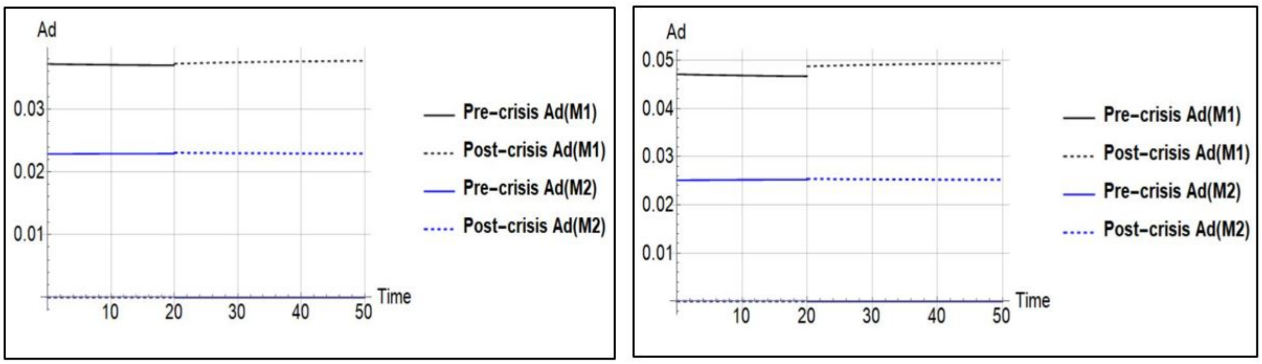
3.5 (for the focal firm) and Table 3.6 (for the non-focal firm) prescribe the equilibrium advertising and pricing policies for the Stackelberg game.

Our findings reveal that some of the Stackelberg equilibrium policies are significantly different from the Nash policies. Firstly, we note that the prices of products for both the firms are always higher in the Stackelberg game than in the Nash game. In the Stackelberg game, for a low damage crisis, the decisions of the firms do not change much with the likelihood. The focal firm, by the privilege of leadership, can also increase pricing slowly during the post-crisis period though the price drops marginally after the recall. The non-focal firm decreases prices in the pre-crisis period, presumably trying to increase the demand for its product. The immediate post-crisis price of the non-focal firm raises slightly and then decreases with time. Advertising for both the firms remains steady in both the periods with marginal increase after the recall. The elevated advertising is again to protect the goodwill of the firms. For a low impact recall, the damage is not severe enough to tarnish the image of the leader. Therefore, the advertising efforts of the firms remain low. The leader is not seriously concerned with the minor goodwill loss, and the follower is neither alarmed by the insignificant effect of the spillover sensitivity, v. Moreover, the follower knows that the low damage is not severe enough to perhaps change consumer perceptions of the leading brand. Hence increased advertising will enhance the follower's costs without having much positive influence on its demand.



Low Impact Low Likelihood Low Impact High Likelihood

Figure 3.8: Stackelberg Pricing Policies- Low Impact



Low Impact Low Likelihood

Low Impact High Likelihood

Figure 3.9: Stackelberg Advertising Policies- Low Impact

Starting with a high price is a strategic decision of the leader. When the impact of the recall is severe, the leader drops the price considerably, even though starting with a higher price. Such a decision insulates against the losses caused by the loss in demand. Since, at the beginning of the planning horizon, the firm envisions a possibly severe recall, the prices are set high so that the post-crisis price drop minimizes the loss of profit. Moreover, the leader cuts down on the advertising costs by reducing its advertising efforts. During a high impact recall, excessive advertising can backfire, especially when a market leader does so. Consumers might perceive the excess advertising as a signal of covering up the damage and hence respond negatively. (Gao et al. 2015) found that for existing products a firm might reduce marketing expenditures by decreasing advertising after a recall without significantly affecting the stock value. We augment the above finding by illustrating that the firm can also decide to decrease post-crisis advertising when recall impact is high. We argue that the follower here resorts to opportunistic behaviour and tries to capture as much market as she can by indulging in higher pre-crisis and post-crisis advertising than the leader. Consequently, the leader responds to the competition by adjusting its advertising. Moreover, under the Stackelberg game, the leader's higher price may hurt its sales. Therefore, the leader reduces advertising costs to maintain profits.

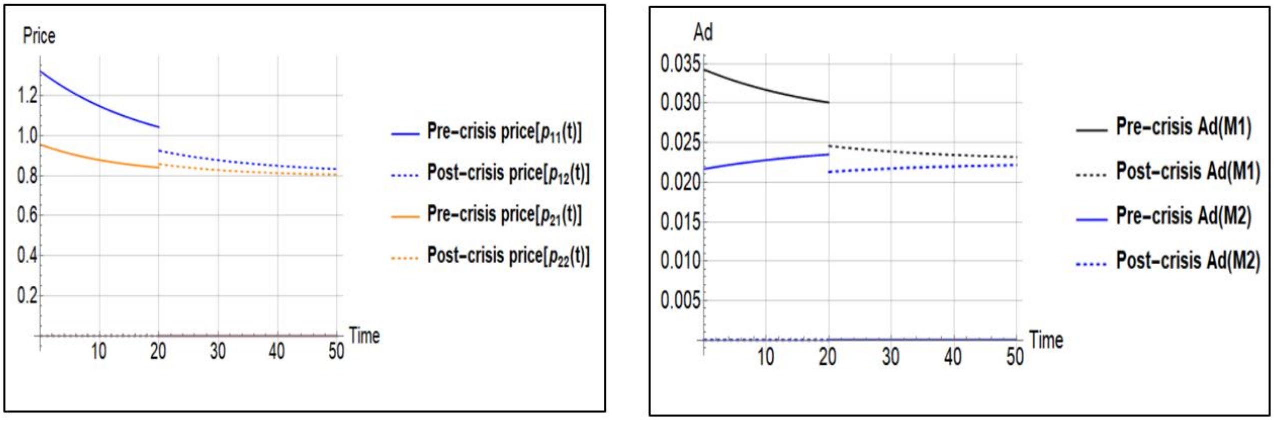
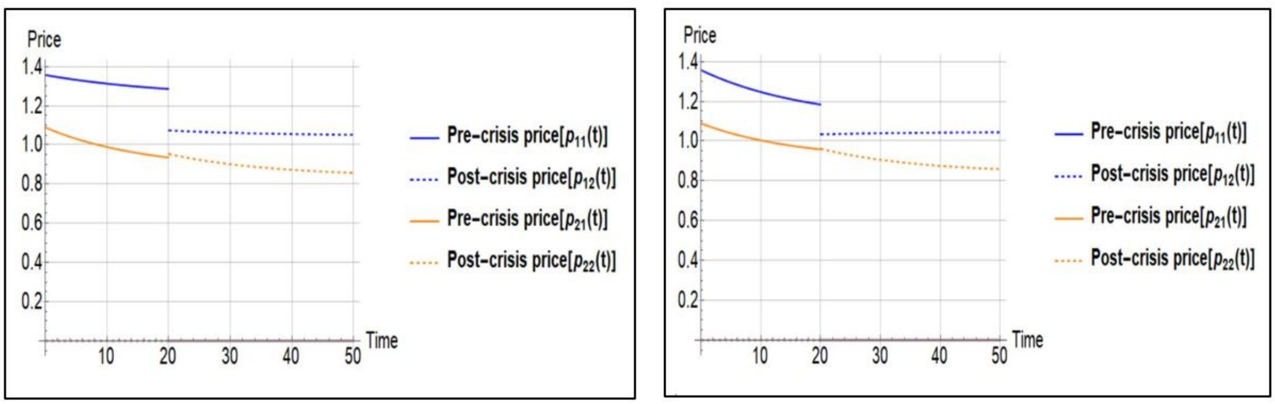


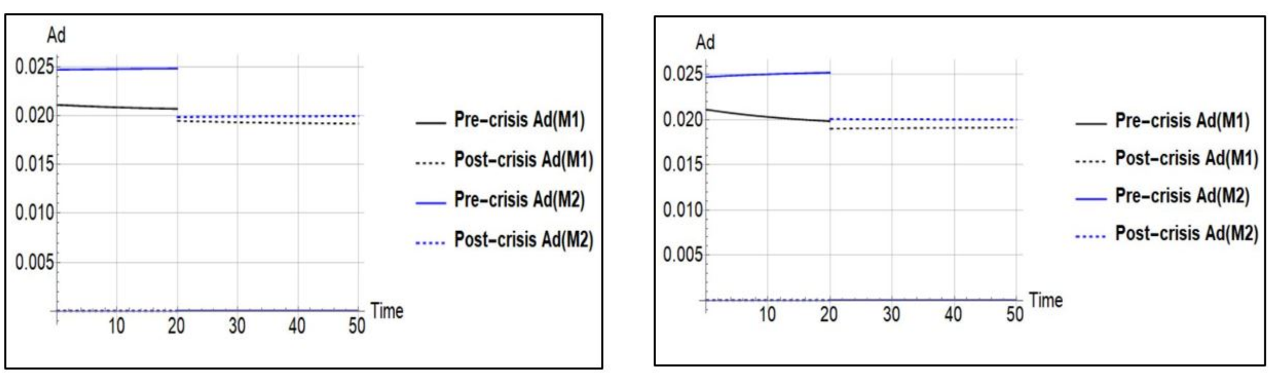
Figure 3.10: Stackelberg Pricing and Advertising Policies- Benchmark Case



High Impact Low Likelihood

High Impact High likelihood

Figure 3.11: Stackelberg Pricing Policies- High Impact



High Impact Low Likelihood

High Impact High Likelihood

Figure 3.12: Stackelberg Pricing Policies- High Impact

In the Stackelberg game, the above discussions of the numerical analysis of firm strategies lead us to several policies for both the firm given an impact and a likelihood of product recall. For the Stackelberg game, we summarize the resulting pricing and advertising policies of the leader (focal firm) in Table 3.4. The strategies correspond to product recalls of different impacts and likelihoods. Table 5 summarizes the resulting pricing and advertising policies of the follower (non-focal firm) suffering from a negative spillover effect. The strategies are classified based on likelihood and impact of a product recall.

<table>
<tr><th rowspan="2">Crisis Likelihood (η)</th><th rowspan="2">Crisis Impact (χ)</th><th colspan="2">Focal Firm</th></tr>
<tr><th>Pricing Strtegy</th><th>Advertising Strategy</th></tr>
<tr><td>Low</td><td>Low</td><td rowspan="2">Maintain a steady pre-crisis price with very little or no decrease. Marginally drop post-crisis price. Gradually increase price during post-crisis regime.</td><td>Maintain steady pre-crisis advertising efforts. Boost ad after the crisis and maintain the higher advertising efforts during post-crisis regime.</td></tr>
<tr><td>High</td><td>Low</td><td>The strategy is similar to Low likelihood low impact case. However, overall advertising efforts are higher when likelihood is high.</td></tr>
<tr><td>Low</td><td>High</td><td>Start with a higher price than low impact recall. Decrease price during pre-crisis period. Drop price after recall. Keep post-crisis price steady.</td><td>Decrease pre-crisis ad gradually. Drop post-crisis ad.</td></tr>
<tr><td>High</td><td>High</td><td>Start with a high price. Substantially decrease pre-criss price. Drop post-crisis price and maintain.</td><td>Decrease pre-crisis advertising gradually. Drop post-crisis advertising Further lower than the low likelihood high impact case.</td></tr>
</table>

Table 3.5: Stackelberg Game- Focal firm's Pricing and Advertising Policies

Crisis Likelihood (η)	Crisis Impact (χ)	**Non- Focal Firm**	
		Pricing Strategy	**Advertising Strategy**
Low	Low	Gradually decrease price in both the regimes. Slow down rate of post-crisis price decrease. Keep price lower than the focal firm's price.	Maintain steady pre-crisis ad. Increase ad and maintain the same after the crisis during spillover.
High	Low	Same as the low impact low likelihood case.	Same as low impact low likelihod case. Level of advertising can be slightly higher than low likelihood case.
Low	High	Keep a lower pre-crisis pricing than the focal firm. Increase price immediately after crisis and gradually decrease price during the spillover.	Maintain higher pre-crisis advertising than the rival leader. Gradually increase pre-crisis ad. Decrease post-crisis ad.
High	High	Same as high impact low likelihood case.	Same as the low likelihood high impact case.

Table 3.6: Stackelberg Game - Non- Focal firm's Pricing and Advertising Policies

3.3.3. The Impact of Recall on Long Term Expected Profit

Recall that the value function, $V_{ij}(\theta_{1j},\theta_{2j})$ of the firm i in regime j is given by the following equation:

$$V_{ij}(\theta_{1j},\theta_{2j}) \quad = \quad A_{ij}\theta_{1j}(t)^2+B_{ij}\theta_{2j}(t)^2+C_{ij}\theta_{1j}(t)\theta_{2j}(t)+D_{ij}\theta_{1j}(t)+E_{ij}\theta_{2j}(t)+F_{ij}.$$

The value functions of the firms at time $t=0($ and $j=1)$ give the long term expected profits of the firms. However, it is interesting to analyze the effects of the state variables on the instantaneous profits. First, we note that the value functions have quadratic terms of the goodwills of both the firms, a multiplicative association of the two goodwills and linear goodwill terms apart from the constant terms. As discussed earlier, the HJB equation in our case has only one admissible solution irrespective of the likelihood and impact of the recall. For that solution, the signs of the coefficients of the value functions are:

$\mathbf{A_{ij}>0, B_{ij}>0}$: This implies that the quadratic presence of the state variable benefits each firm,

$\mathbf{C_{ij}<0}$**:** The multiplicative association of the goodwills has a negative effect on firm profit. The significance of this term is that, if a firms goodwill is high, it would want the other firm's goodwill to below in order to have a minimum value of $C_{ij}\theta_{1j}(t)\theta_{2j}(t)$.

$\mathbf{D_{1j}>0, D_{2j}<0, E_{1j}<0, E2j>0}$**:** This relation signifies that the rivals firm's high goodwill can dampen a firm's profit and vice versa.

$\mathbf{F_{ij}>0}$**:** The constant terms always has a positive effect on the firms' profits.

From the above discussion on the variation of signs of the value function coefficients for the equilibrium solutions, we see that the presence of the rivals firm's goodwill in a firm's value function has a dual effect of increasing firm profit (by quadratic presence) and also decreasing it (multiplicative association and linear presence).

The two most important parameters of our study are the crisis likelihood and the impact. The spillover effect, as described in table (3.1) will depend on the impact of the recall. In the following section, we present how firm profit if affected by crisis likelihood χ and impact η.

Profit under Nash Game - Similar firms compete: For a low impact recall, the profit of the focal firm increases with crisis likelihood. The above finding might be surprising. The presence of χ in a nonlinear form in the value functions of the two firms brings in two opposing tendencies of lowering the function or increasing it. When the crisis-impact η is small, we find that the "beneficial force" of χ is more prominent for the focal firm. On the other hand, the non-focal firm's profit is always increasing in the crisis likelihood. Note that both price and advertising of the focal firm remains much higher than the non-focal firm for a low impact crisis. Thus, the focal firm is more profitable than the rival suffering from a spillover effect. The trade-off of advertising costs (quadratic in advertising efforts), cost of lost sales due to loss in demand by recall impact and the increment of the price works in favour of the focal firm resulting in better profits than the rival.

As the recall impact increases, the focal firm's profit decreases with an increase in recall likelihood. Thus for high impact recall, the detrimental effect of χ is more prominent. Therefore, we find that as impact increases, the profit for the focal firm drops. For similar firms, with identical parameter values, the focal firm's profit drops below the non-focal firm's profit for a high likelihood of recall. Figure 3.11 shows the profit variations with different levels of impact. Figure 3.12 shows the threshold curve of the (χ, η) pairs beyond which the profits for the two firms increase (or decrease).

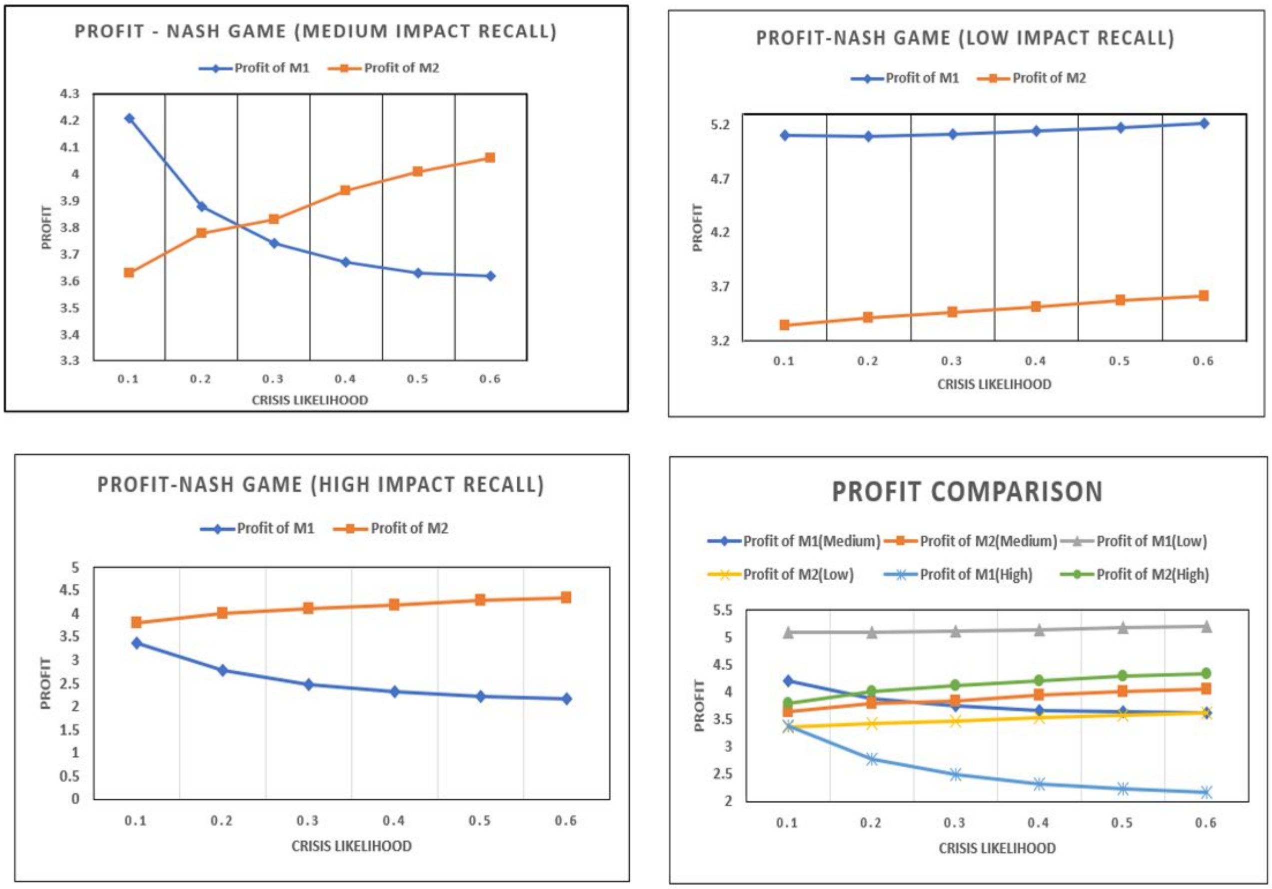


Figure 3.13: Nash - Profit vs Crisis Likelihood

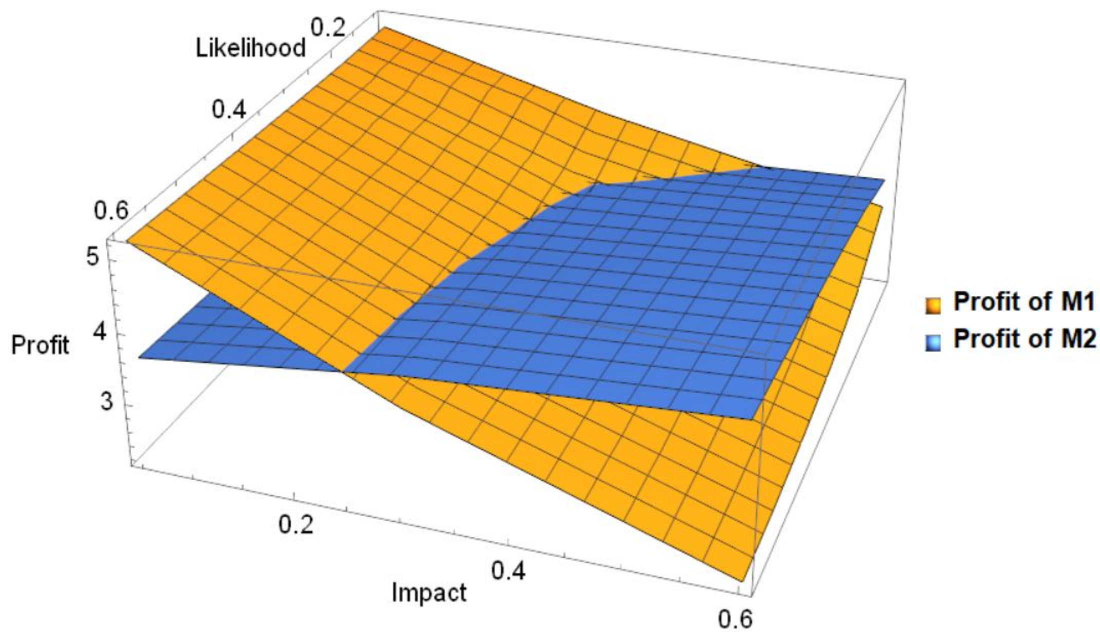


Figure 3.14: Nash Profit Variation with Likelihood and Impact

Profit Under Stackelberg Game- Focal Firm is the Market Leader: In general, for a low impact recall or a high impact recall, the Stackelberg game generates more profit for both the focal and the non-focal firms than their profits in the Nash game. The overall behaviour of profit with respect to crisis likelihood and impact remains more or less the same in the Stackelberg game. However, there are some exceptions. For example, for a high impact recall, the focal firm can be more profitable than the non-focal firm until a higher crisis likelihood as compared to the Nash game. The above behaviour of profit is under the focal firm's power of leadership. Under a medium impact recall, the focal firm may not always be better off in the Stackelberg game than the Nash game. We found that for a higher likelihood of a crisis, ($\chi \geq .4$), the focal firm's profit in the Nash game can be higher than its profit in the Stackelberg game.

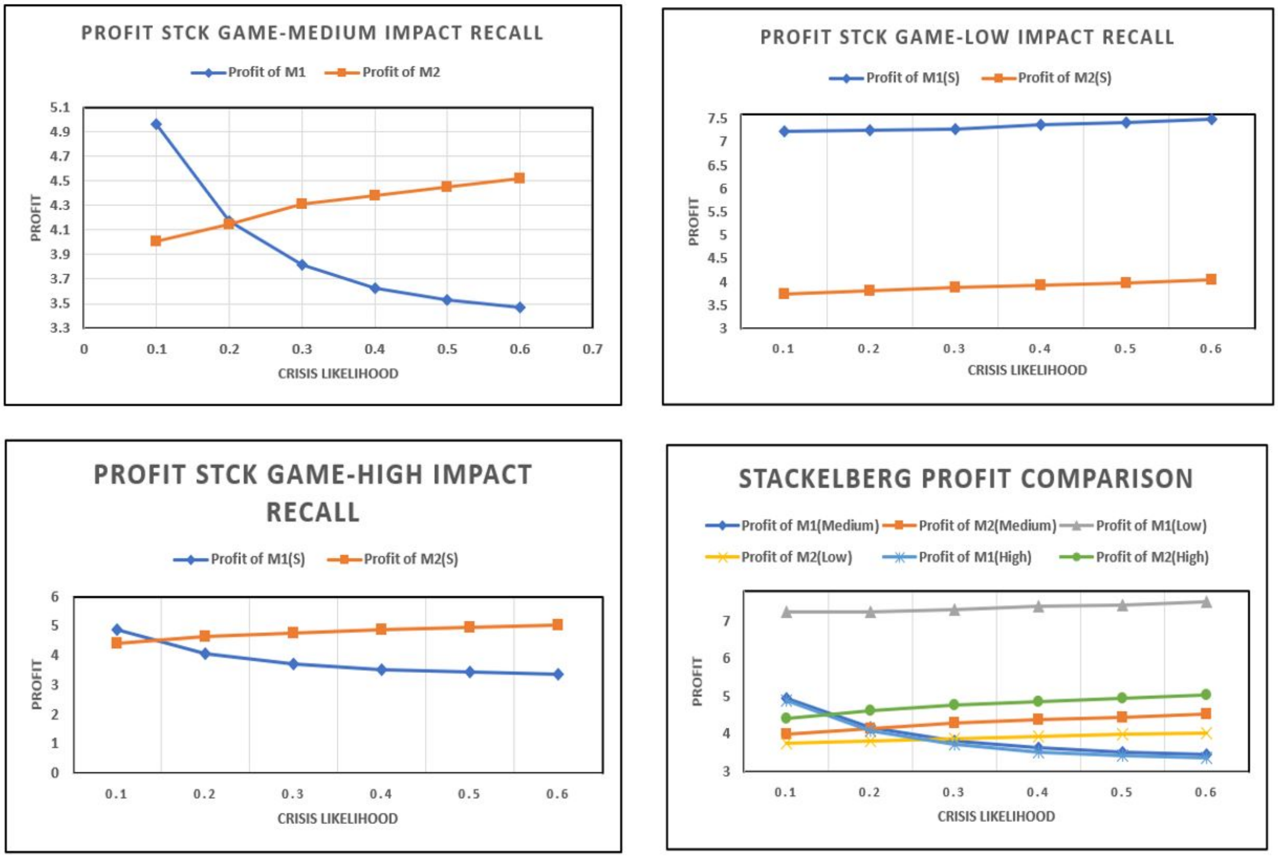


Figure 3.15: Stackelberg Game - Profit vs Crisis Likelihood

Is Leadership Always Profitable?

In the Stackelberg game, the non-focal firm yields better profits than in the Nash game under any impact of the recall, low, medium or high. On the other hand, the focal firm has higher profit than the Nash game when the impact is low or high. For a medium impact ($\eta = .3$), the focal firm's profit is lower in the Stackelberg game when the likelihood of crisis is high ($\chi > 0.3$). Thus for the focal firm, we cannot always say that market leadership can be beneficial, and the benefits depend on the crisis impact and likelihood. Figure 3.13 illustrates the profits in the Stackelberg game for both the firms and the variations of profits with the crisis likelihood and impact.

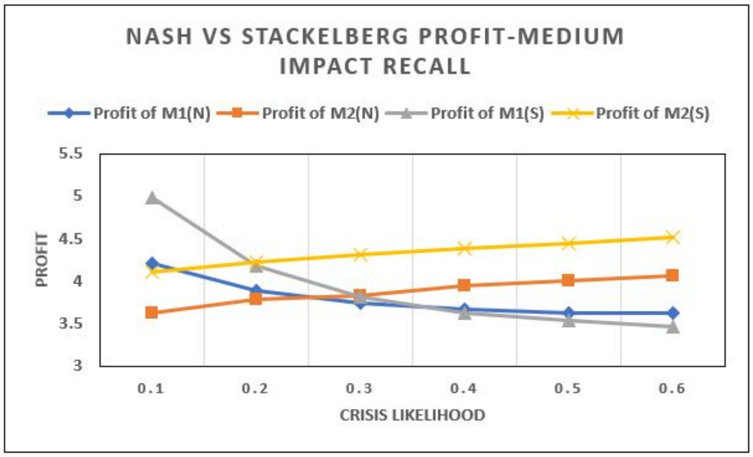


Figure 3.16: Nash Pricing Policies- Benchmark Case

3.4. Conclusions

Product recalls of different magnitudes impact firms and their rivals in various ways. In this paper, we studied the equilibrium pricing and advertising decisions of two competing firms under the negative impact of product recalls when both the firms envision the recall. In particular, we illustrate the different pricing and advertising policies of the firms - a focal firm and its rival suffering from spillover effect under crises of different levels (low, medium and high). We found that the policies differ with the impact of the recall, likelihood of the recall and the market leadership of the focal firm.

Managerial Implications: We obtained several insights. Firms undergoing a crisis need to consider goodwill advertising and pricing strategies depending on the estimated likelihood and impact of a recall. In general, our study underscores the importance of envisioning the crisis impact as the likelihood and impact of crisis change the firms' pricing and advertising policies and such decisions can affect firms profits. The above implications comport with the previous literature (Rubel 2018; Rubel, Naik, and Srinivasan 2011; Mukherjee and Chauhan 2019).

Our study focuses on negative spillover. Empirical literature shows that negative spillover happens to similar firms from the same country (Borah and Tellis 2016). We examined the cases when the focal firm is similar to the rival, and the focal firm is a market leader and found that the pricing and advertising policies for both the firms differ in the two cases. Therefore, rival firms should be aware of recall events of competitors from the same country and also the size of the competitor.

We have considered a dynamic and evolving brand image for both the firms over the planning horizon. The feedback pricing and advertising strategies and their variation with the brand goodwill or parameters necessitate that the management of both the firms constantly monitors the factors that may affect brand goodwill and model parameters like cross-price sensitivity and cross goodwill sensitivity. A thorough introspection and elimination of the above determinants ensure that the results of our analysis do not change by the mediation of other factors.

Future Research: Our study has some limitations, and future research can extend the same. Spillover effect, in some cases, can be short-term. Our model does not cover this possibility because of the with the present model it might have been very challenging or impossible to analyze the feedback strategies. Consequently we might not have any analytical results if such a finite horizon problem was considered. However, a modified model may be used for future research to incorporate short-term spillover effects.

The empirical literature suggests that rival firms from different countries of origin can under positive spillover effect during product recalls. We have not examined this case in our model. The above is a viable case of interest, and our model can be easily extended in that direction.

Price sensitivity and goodwill sensitivity of firms may change in the post-crisis period. Our model can be extended by assuming different values of β_1 and γ_i in the two periods. These variables can be considered to be state variables. However, in such a case makes the solution of the game more difficult due to the dimensional issue.

Chapter 4

Quality Cost-sharing Contracts to Manage Collateral Damage by Product Recalls

4.1. Introduction

In recent times product recalls have become increasingly common. Frequently enough, the cause for a product-harm crisis and a consequent recall issued by a manufacturer is the faulty or inadequate quality parts provided by the suppliers. For example, GM recalled more than 2.7 million vehicles in 2014. The reason for the recall was faulty ignition switches supplied by Delphi (Lawrence 2017). In another incident, U.S. firm CTS Corp. supplied poor quality accelerator pedals to Toyota. As a result, Toyota had to recall 2.3 million vehicles (Kimiko de Freytas-Tamura 2010). The Ford SUV recalls due to faulty tire and treads separation problems of Bridgestone tires brought an end to a very long-lasting buyer-supplier partnership (Gifford 2015). Similarly, Samsung recalled the Samsung-Note mobile phones wiping off much of Samsung's market share. In this case, the battery suppliers of Samsung were at fault (Tilley 2017). The above recalls were due to supplier's quality failures, and the product harms have even resulted in multiple deaths. The instances of such product harm spread across industries. Companies often outsource raw materials or parts of products and manufacturing activities to other members of the supply chain. When a manufacturer has less visibility or control over the supply chain or when there are multiple members of the upstream supply chain (for example many suppliers), there is an element of increased risk as far as the quality of the final product is concerned (Foster Jr 2008; Robinson and Malhotra 2005). If the quality of a supplier's product is compromised, the quality of the finished good may suffer. Such quality failures can lead to product recalls causing severe damage to a firm's reputation along with other indirect external failure costs which include costs of warranty, return, rework or lawsuits by a customer and regulatory bodies (**ref1**).

High product quality can boost a firm's goodwill and therefore, sales. Therefore, in order to ensure the quality of their products, manufacturers often want to have better visibility of supplier risks and closely track the supplier's quality performance. Consequently, manufacturers form contractual relationships with suppliers to have better visibility and control over the supplier's product quality. The supply chain literature has examined cost-sharing contracts formed to incentivize the suppliers by sharing their quality costs. However, the viability of such contracts is questionable when suppliers are prone to quality risks, and such risks can potentially result in product recalls. A manufacturer's intention behind a cost-sharing contract is motivating the supplier to render high-quality products which in turn increases the manufacturer's reputation and sales. Product recall due to a supplier's fault may signify an apparent failure of conformance to minimal quality standards on the supplier's side. External failure costs and cost of lost goodwill are added adverse effects following a recall (Gao et al. 2015; Cleeren, Dekimpe, and Helsen 2008; Cleeren, H. J. Van Heerde, and Dekimpe 2013; Rubel, Naik, and Srinivasan 2011; Lawrence 2017; Chao, Iravani, and Savaskan 2009). However, even if a manufacturer knows that a supplier is risky, it may share quality costs with the supplier because supplier switching might be very costly and cost-sharing might mean that the risky supplier will adhere to high standards of quality improvement and conformance, thereby avoid a major costly recall. Based on a single manufacturer, single supplier scenario, a study analyzes recall cost-sharing to induce supplier's quality improvement (Chao, Iravani, and Savaskan 2009) when there is a product recall.

The main motivation of our paper comes from the different types of cost-sharing contracts that are offered to suppliers by manufacturers, who are the Stackelberg leaders. In the Supply Chain literature, many papers consider cost-sharing mechanisms where the manufacturer typically offers a fixed cost-sharing (Reyniers and Tapiero 1995b; Reyniers and Tapiero 1995a; Baiman, Netessine, and Kunreuther 2004; Baiman, Fischer, and Rajan 2000; El Ouardighi and B. Kim 2010). When a manufacturer offers such a contract two significant questions are imminent - (i) Which costs should be shared? (ii) What proportion of costs should be shared? Cost-sharing mechanisms could be partial, which focuses on sharing only quality improvement costs; or full, which are designed to share the full costs of quality improvement and quality failure.

A product recall is often anticipated. For example, in the case of Delphi's ignition switches, the GM officials were aware of the lacking quality (Harrison 2014) In other words, the officials were aware of a possible crisis due to the supplier's fault. Such quality flaws can remain latent, resulting in a recall after customers suffer product harm and can result in huge external failure costs. In the case of GM, the recall costs were close to $ 1.3 Billion. While there is no guarantee that sharing quality costs (like quality improvement, conformance) can completely avoid a product recall, nonetheless this can reduce the chance or intensity of a recall. Moreover, the suppliers and manufacturers can adjust their quality decisions or cost-sharing contracts to maximize the long term profit if a recall occurs. Such managerial decisions can be executed efficiently if the recall likelihood, associated lack of quality and loss of brand goodwill can be estimated ex-ante at the beginning of the planning horizon. Envisioning a recall can affect decisions like advertising, quality and pricing (Rubel, Naik, and Srinivasan 2011; Rubel 2018; Mackalski and Belisle 2015). In this article, we investigate a scenario where a manufacturer and its two suppliers are aware of a potential recall due to the failure of one of the suppliers quality. At the beginning of the planning horizon, the unreliable supplier also knows that its increasing quality efforts can increase the quality level of the product and possibly lower the crisis impact. Due to the unreliable supplier's quality failure, the manufacturer and the second supplier suffers from "collateral damage" through the loss of after-recall sales.

Goodwill advertising is another potential weapon to mitigate the negative effects of a product recall by raising goodwill. Knowing that a recall can occur, a manufacturer thus often faces multiple decision-making problems -(i) How much effort in advertising should be made before and after a possible recall? (ii) What proportion of quality cost should be shared with a supplier before and after a possible recall? The available literature finds that advertising efforts to protect a brand image may increase after a recall (Cleeren, H. J. Van Heerde, and Dekimpe 2013; Rubel, Naik, and Srinivasan 2011; Gao et al. 2015). However, to the best of our knowledge, the joint effect of advertising and quality cost-sharing decisions have not been researched in the context of product recalls. Our study addresses this gap with a simple supply chain structure and linear-quadratic differential game which enables tractability and analysis of the equilibrium decisions.

We consider two types of quality cost-sharing contracts offered by the manufacturer to both the suppliers who provide different parts necessary to assemble a finished product. The contracts are (i) Quality Improvement Cost-sharing Contract (QICS) and (ii) Comprehensive Quality Cost-sharing Contract (CQCS). On the one hand, the unreliable supplier incurs three types of quality costs - quality improvement costs, conformation costs and external failure costs. On the other hand, the reliable supplier fully conforms and is not accountable for any product recall. Thus it has to contend with only quality improvement costs. Consequently, the two contracts are similar for the reliable supplier, but the presence of the unreliable supplier may affect the cost-sharing decision of the manufacturer. Given a crisis likelihood and recall impact, our main research questions are:

1. What are the equilibrium quality efforts of the two suppliers before and after the recall? How are the equilibrium quality decisions affected by crisis likelihood and intensity?

2. What are the equilibrium advertising policies and cost-sharing decisions of the manufacturer before product recall and after product recall? What is the effect of crisis likelihood and intensity on the manufacturer's equilibrium decisions?

3. What is the impact of the recall on the long term expected profit of the firms?

4. Which cost-sharing contract is suitable or sustainable for the firms under different crisis likelihood and intensity?

Our research contributes incrementally to the literature mentioned above. We introduce two contracts which are motivated from partial and full cost-sharing agreements between manufacturers and suppliers. In contrast to the closely related papers like (Chao, Iravani, and Savaskan 2009) which discuss static models, we have developed a dynamic model which is close to reality, tractable and touches a few streams of literature related to cost-sharing contracts, quality efforts and advertising. The empirical research literature considers many aftermaths of a product recall which we have captured in our model. These attributes are the reduced advertising effectiveness and sales after recall (H. Van Heerde, Helsen, and Dekimpe 2007; Rubel, Naik, and Srinivasan 2011; Yi Zhao, Ying Zhao, and Helsen 2011), quality efforts depending on recall impact and crisis likelihood, learning after a recall (Thirumalai and Sinha 2011) and the dependency of the negative effect of recall on the initial goodwill of the firm (Cleeren, H. J. Van Heerde, and Dekimpe 2013; H. Van Heerde, Helsen, and Dekimpe 2007). We articulate the critical empirical findings in one model and believe that the insights and results of our research are novel.

Secondly, we introduce two contracts - QICS (quality improvement Cost-sharing) and CQCS (Comprehensive Quality Cost-sharing) contracts and analyze which contract is suitable for the players who envision an impending product recall. While doing so, unlike most of the literature, we allow the cost-sharing proportion to be a decision variable of the manufacturer. Considering share of costs as a decision variable enables the manufacturer to react to a product recall and modify its cost-sharing decision at a random time when the recall occurs.

Third, while most of the previous modelling literature analyzing cost-sharing or quality decisions consider two-player buyer-supplier games (Reyniers and Tapiero 1995b; Reyniers and Tapiero 1995a; Zhu, R. Q. Zhang, and Tsung 2007; Chao, Iravani, and Savaskan 2009; Baiman, Fischer, and Rajan 2000), we consider three players, two of whom are victims of "collateral damage". Contractual agreements in the presence of collateral damage is an essential but overlooked topic in the literature that integrates product recall and operations management. In our study, we examine the reaction of the firms who are not at fault but have to suffer due to the quality failure of the other member of the supply chain.

Fourth, our model captures the effect of quality efforts on a firm's goodwill and also incorporates the relationship between the firm's goodwill erosion and drop in quality level after a recall. Empirical work (Cleeren, Dekimpe, and Helsen 2008; Cleeren, H. J. Van Heerde, and Dekimpe 2013; Yi Zhao, Ying Zhao, and Helsen 2011) has verified that consumers may react differently towards a recall depending on the goodwill of the firm. We believe our model is one of the first to capture this in a simple form.

We also introduce the marginal quality level as a function of crisis likelihood and capture the unreliable supplier's negligence of likelihood in the model. Most significantly, the feedback strategies that we derive provide the opportunity for adjustment of quality decisions of the suppliers and advertising and cost-sharing decisions of the manufacturers before and after the recall.

The rest of the paper is arranged in the following manner. Section 4.2 describes the model in details. In section 4.3, we discuss the equilibria of the differential game under consideration and provide some insights about the results. In section 4.4, computational analysis is presented to strengthen our insights. We conclude the paper in section 4.5 with some managerial implications and directions for future research.

4.2. The Model

We consider a supply chain consisting of three decision-makers, two suppliers S1, S2 and a manufacturer, M. The manufacturer assembles a final product using the distinctive components from these suppliers. The consumer demand for the final product depends on the manufacturer's goodwill and product quality. We adopt a linear dynamic demand function. Similar demand functions have been frequently adopted in the literature (G. Liu, J. Zhang, and Tang 2015; De Giovanni 2011).

$$D(G(t),Q(t)) \quad = \quad \alpha+\beta G(t)+\gamma Q(t) \tag{4.1}$$

$\beta \geq 0$ and $\gamma \geq 0$ are the consumers' sensitivity towards the manufacturer's goodwill $G(t)$ and the Quality level $Q(t)$ at any time t in the infinite planning horizon $[0,\infty)$. Goodwill $G(t)$ and quality level $Q(t)$ are the state variables of our model. In the context of our paper, quality level Q(t) refers to design quality, fitness for use or utility for the consumers (El Ouardighi and B. Kim 2010; G. Liu, J. Zhang, and Tang 2015), and in our study, both suppliers are responsible for implementing quality efforts under the basic scenario. The manufacturer shares these quality efforts with the suppliers under cost-sharing contract scenarios. A product recall would mean that the goodwill and the quality levels have undergone some deterioration. We will illustrate in a subsequent paragraph that there if a company anticipates a product recall, there will be two decision regimes (pre-recall and post-recall). Therefore, we use two indices- i as the index for the supplier and j as the regime index. Goodwill is influenced by the manufacturer's advertising decision $A_j(t)$ as well as the suppliers' quality efforts $q_{ij}(t)$. Besides, the manufacturer also decides on what fraction of quality costs it should share with the suppliers under cost-sharing contract scenarios. The cost-sharing variables are $\phi_j(t)$ (proportion of quality-related costs shared with S1) and $\sigma_j(t)$ (proportion of quality costs shared with S2). The three players in the market make the decisions to maximize their long term expected profits. The degree of risk or reliability of the suppliers and manufacturer are common knowledge. The manufacturer shares the quality cost with the suppliers from the beginning of the planning horizon. Therefore, the manufacturer expects a commitment to quality efforts by the suppliers. Knowing that the manufacturer will share the costs, the suppliers announce their quality efforts. Thus, in our game model, the manufacturer is the Stackelberg leader and offers a cost-sharing contract to the suppliers. The market leadership assumption of the manufacturer is consistent with the previous literature (Chao, Iravani, and Savaskan 2009).

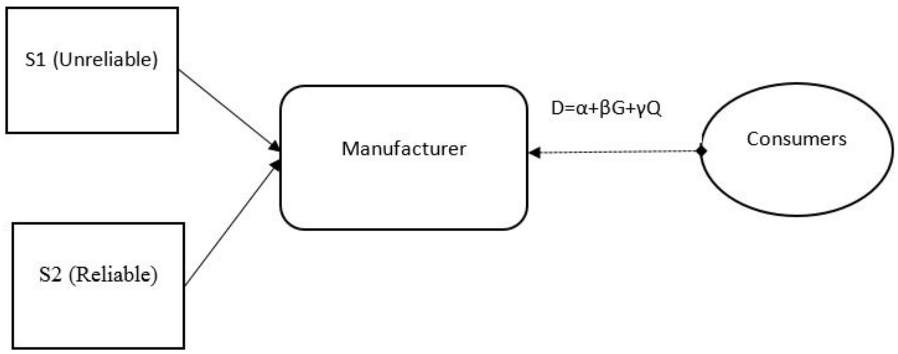


Figure 4.1: The Supply chain

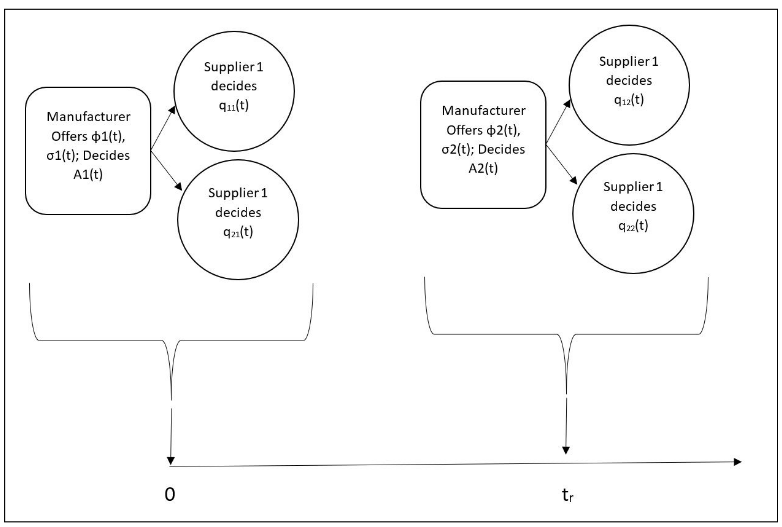


Figure 4.2: Decision Sequence

A product recall affects goodwill $G(t)$ and quality level $Q(t)$. Therefore, before defining the evolution of the states, we describe the occurrence process of the recall. Between the two suppliers, supplier 1 is a risky supplier with a likelihood (χ) of product failure and subsequent product recall. A recall can occur at a random time t_r in the planning horizon $[0,\infty)$. Let χ be the hazard rate. We define the probabilistic switching of the pre-crisis and the post-crisis period using the stochastic process $[R(t) : t \geq 0]$. The occurrence of a recall, therefore, means that there are two regimes - pre-crisis regime and the post-crisis regime where the players make the quality, advertising and cost-sharing decisions. The pre-crisis regime is defined by $[R(t) = 1]$ and $[R(t) = 2]$ represents the post-crisis regime. Similar definitions are found in (Boukas, Haurie, and Michel 1990; Haurie and Moresino 2006; Rubel, Naik, and Srinivasan 2011). The occurrence of recall is assumed to follow an exponential distribution.

$$\begin{aligned}\lim_{dt\to 0}\frac{P[R(t+dt)=2|R(t)=1]}{dt} &= \chi \\ \lim_{dt\to 0}\frac{P[R(t+dt)=1|R(t)=2]}{dt} &= 0\end{aligned} \tag{4.2}$$

The supplier 2 is perfectly reliable and its products are not prone to recall. If the manufacturer issues a recall due to the compromised quality of supplier 1, the demand for the manufacturer's product is affected by a loss of goodwill and quality. Also, the manufacturer has to share recall costs. Supplier1 and the manufacturer bear the costs of the recall.Supplier 2 does not incur any loss/cost for the items sold in the first period. However, Supplier 2 suffers due to post-crisis demand loss. The situation is that of collateral damage caused by the fault of Supplier 1.
Table 4.1 below enlists the model parameters.

Model Parameters	Description
$G(t)$	Goodwill of the manufacturer. $G(t) \in [0,\infty)$
$Q(t)$	Quality level of the product. $Q(t) \in [0,\infty)$
$A_j(t)$	Advertising effort (monetary value, say in dollars) of the manufacturer in period j
$q_{ij}(t)$	Quality effort (monetary value) of supplier i in regime j
α	Demand in absence of goodwill and quality. $\alpha \in [0,\infty)$
β	Consumer sensitivity towards goodwill. $\beta \in [0,\infty)$
γ	Consumer sensitivity towards Quality. $\gamma \in [0,\infty)$
δ_j	Goodwill decay. $\delta_j \in [0,1]$
r	Discounting factor. $r \in [0,1]$
k_{gj}	Advertising effectiveness
k_{ij}	Consumer's confidence or trust on the supplier i's quality effort in period j
χ	Hazard rate
ω	Proportionality constant for quality conformance cost
θ	Proportionality constant for external quality failure cost
μ_{s1}	Proportionality constant for quality improvement cost of Supplier S1
μ_{s2}	Proportionality constant for quality conformance cost improvement cost of Supplier S2
μ_m	Proportionality constant for advertising cost of manufacturer M
$V_i(G,Q)$	Value Function of player i at $t=0$
$\hat{M}_i,\ i \in \{m,s1,s2\}$	Unit profit margins for manufacturer($i=m$) or the supplier($i=s1,s2$) in the first regime
$M_i,\ i \in \{m,s1,s2\}$	Unit profit margins for manufacturer($i=m$) or the supplier($i=s1,s2$) in the second regime
$D(G(t),Q(t))$	Product demand at the manufacturer's end

Table 4.1: Model Parameters

Manufacturer's Goodwill

At any time t during the planning horizon, goodwill is influenced by the manufacturer's advertising efforts $A_j(t)$ and the suppliers' quality efforts $q_{ij}(t)$ where $i,j \in \{1,2\}$. We assume that goodwill evolves according to the modified Nerlove-Arrow dynamics (equation 4.2). (Nerlove and Arrow 1962; Dockner et al. 2000). Ideally, the state variables $G(t)$ and $Q(t)$ should have the regime index j. However, for brevity, we drop the index.

$$\dot{G}(t) \quad = \quad k_{gj}A_j(t) + k_{1j}q_{1j}(t) + k_{2j}q_{2j}(t) - \delta_i G(t) \tag{4.3}$$

Extant literature shows that the advertising effectiveness decreases after a recall crisis depending on the initial brand goodwill (H. Van Heerde, Helsen, and Dekimpe 2007; Yi Zhao, Ying Zhao, and Helsen 2011). Therefore, we assume that in general, k_{gj}, the advertising effectiveness may be different in the two regimes. Specifically, $k_{g1} \geq k_{g2}$. Moreover, the consumers' trust in quality can reduce after the crisis (Yi Zhao, Ying Zhao, and Helsen 2011). We interpret the parameters k_{ij} for $\{i,j\} \in \{1,2\}$ as the consumers trust or confidence in the product quality efforts of the suppliers. Therefore, a lack of confidence in the post-crisis period can be captured by the relationship $k_{i1} \geq k_{i2}$. There is also a possibility that brand forgetting will increase in the post-crisis period. This can be simulated by choosing $\delta_1 \geq \delta_2$. Following the previous literature (Karray and Zaccour 2005; Dockner et al. 2000), we consider the advertising costs to be quadratic in the efforts. The manufacturer's advertising costs in the regime j is given by $\frac{\mu_m A_j^2(t)}{2}$ where μ_m is the proportionality constant.

Quality of Suppliers

Quality is a multidimensional concept. In the context of our paper, we assume that the quality efforts of each supplier are channelized to multiple directions like quality improvement, appraisal costs, enhancing fitness for use, improving consumer utility, improving perceived quality, conforming to manufacturer's quality standards and investing in external quality failure mitigation procedures. A supplier's continuous quality improvement efforts may prevent product failures and thereby minimize the chances of a product recall. The quality of the manufacturer's finished product depends on the quality of the supplier's parts, and a product recall can occur due to the faulty parts provided by the supplier. The supplier i decides on the quality efforts $q_{ij}(t)$ at time t. Thus, in general, as the supplier increases the efforts $q_{ij}(t)$, a quality level $Q(t)$ improves. However, if there is a high hazard rate, the quality efforts of the suppliers need to be higher to maintain the same level of quality than when the hazard rate is low. Accordingly, we have the following equation, which gives the evolution of quality level $Q(t)$.

$$\begin{aligned} \dot{Q}(t) &= l_{1j}q_{1j}(t)e^{-\lambda\chi} + l_{2j}q_{2j}(t) \\ \text{and } \chi &> 0 \text{ for } 0 \leq t \leq t_r \\ \chi &= 0 \text{ for } t_r < t < \infty. \end{aligned} \tag{4.4}$$

In equation (4.4), l_{ij} are the effectiveness of the quality efforts of the suppliers. Higher the effectiveness, higher will the product quality level be. Moreover, effectiveness of such quality eff forts can vary before and after recall. Recall experience can lead to improved learning and a consequent enhancement in product quality (Thirumalai and Sinha 2011). If there is learning as a consequence of a recall, $l_{i2} > l_{i1}$.

$\lambda \geq 0$ represents the unreliable supplier's negligence of the crisis likelihood. A firm may avoid or overlook the importance of crisis likelihood and avoid the immediate costs of quality (A. X. Liu, Yong Liu, and Luo 2016). Alternatively, a firm can be proactive and try to enhance the quality level by increasing efforts. The marginal quality level becomes lower if λ is higher (since $\chi \geq 0$). $\lambda = 0$ signifies that the supplier 1 is most proactive and sensitive towards the hazard rate. The multiplicative association, $\lambda\chi$ signifies that higher likelihood coupled with higher negligence (or less proactivity) results in a slower evolution of quality.

Our focus is to study the loss in quality due to the unreliable supplier 1 and the subsequent collateral damage. Therefore, like some existing works (El Ouardighi and B. Kim 2010; G. Liu, J. Zhang, and Tang 2015) we do not consider any natural loss of quality under the purview of this paper. Since the quality level $Q(t)$ of the supplier positively influences the consumer demand, the manufacturer will have incentives to improve quality at the supplier's end. Therefore, the manufacturer shares a part of the supplier's quality cost after the supplier commits quality efforts. The unreliable supplier, S1 and the manufacturer M may jointly bear the burden of two quality costs in the pre-crisis period. These are *(i) Quality improvement costs given by* $\frac{\mu_{s1}q_{11}^2}{2}$ and *(ii) Quality conformance costs given by* $\frac{(1-\chi)\omega q_{11}^2}{2}$ where μ_{s1} and ω are the proportionality constants of the cost parameters. The cost of conformance is decreasing with the hazard rate. Therefore, the supplier's conformance cost is maximum when $\chi = 0$ and minimum when he does not spend on quality conformance ($\chi = 1$).

In the post-crisis regime, there is an additional cost of a recall or the external failure cost for S1, which may or may not be shared by the manufacturer. We hypothesize that this cost is proportional to the drop in of the quality level, ξ. Thus this cost is given by $\frac{\xi\theta q_{12}^2(t)}{2}$. Both goodwill loss and quality loss capture the damage caused by a recall. Since we have assumed that η, the loss in goodwill, is a function of ξ, considering the external failure cost to be a function of ξ is sufficient.

We assume only one recall over the planning horizon. This means that the post-recall conformance cost of the supplier is at the maximum i.e. $\frac{\omega q_{12}^2(t)}{2}$ since $\chi = 0$.

The reliable supplier S2 and the manufacturer M bear the cost $\frac{\mu_{s2}q_{2j}^2(t)}{2}$ in the two periods and mu_{s2} is the proportionality constant.

The Impact of the Recall

The jumps in the state variables, goodwill and quality level, capture the impact of the recall. Our model also considers external failure costs proportional to the drop in quality level. These costs include costs of repair, warranty,re-manufacturing, and improving perceived quality, liability claims and lawsuits. We capture this by considering the post-crisis profit margin of the firms to be less than the pre-crisis profit margin. A substantial economic impact on the firm due to the recall can raise the operational cost to the extent where profit margin will suffer. Higher the loss of goodwill and quality level, higher the loss in profit margin. However, we have assumed that the firms do not get bankrupt or affected to the extent that they completely withdraw all units of the products from the market. In other words, the recall is partial. Very severe impact recalls can cause a firm to become bankrupt. For example, Takata corporation files bankruptcy in 2017 after the airbags recall (Reuters 2016). Such cases are beyond the scope of our model.

When a recall occurs at a random time t_r, due to quality failure, the goodwill of the manufacturer suffers. The jump states capture quality deficiency and goodwill loss:

$$G(t_r^+) = (1-\eta)G(t_r^-)$$
$$Q(t_r^+) = (1-\xi)Q(t_r^-)$$

t_r^+ is the time just after the recall, and t_r^- is the time just before the recall. For tractability of our analysis, we assume that a firm issues recall as soon as it becomes aware of quality failure and the goodwill drop results simultaneously. These assumptions are not far from reality. As the previous literature highlights, the negative effects of a recall can spread fast (Tirunillai and Tellis 2014; Borah and Tellis 2016), and as such any time lag between issuing a recall and drop in goodwill or quality level may be ignored. Although the quality level might have been dropping from some time before the recall, the issue becomes noticeable to the public only after the recall. Therefore, the costs related to product recall occur only in the post-crisis period.

Relationship Between Goodwill Damage and Drop in Quality Level

An important question to ask is whether there is a relationship between the damage to goodwill and the drop in in quality that has necessitated a product recall. The goodwill damage is a result of the product recall, and the quality deterioration has lead to a product recall. When an initial brand image is high, a drop in quality will have a smaller impact on the manufacturer's goodwill than in the case where initial goodwill is low (Yi Zhao, Ying Zhao, and Helsen 2011). For our study, especially the numerical analysis, we have considered G(0)=.15 as high goodwill and G(0)=.1 as low goodwill. Hence, we propose that if $G(0) = .15$, i.e, the initial goodwill is high, then $\eta = a\xi^2$, $a \leq 1$ signifying a slower loss in goodwill damage for a specific drop in in quality level. When $G(0) = .1$, i.e, the initial brand image is low $\eta = b\xi$, $b \leq 1$ signifying that the drop in goodwill is higher. The conditions $a \leq 1$ and $b \leq 1$ are necessary to ensure that $\eta \leq 1$. Moreover, $a \leq b$ so that the linear relation always means higher loss in goodwill for a given loss in quality. In both cases, $0 \leq \{\eta, \xi\} \leq 1$ is satisfied and importantly, $\xi = 0 \implies \eta = 0$ and $\xi = 1 \implies \eta = 1$.

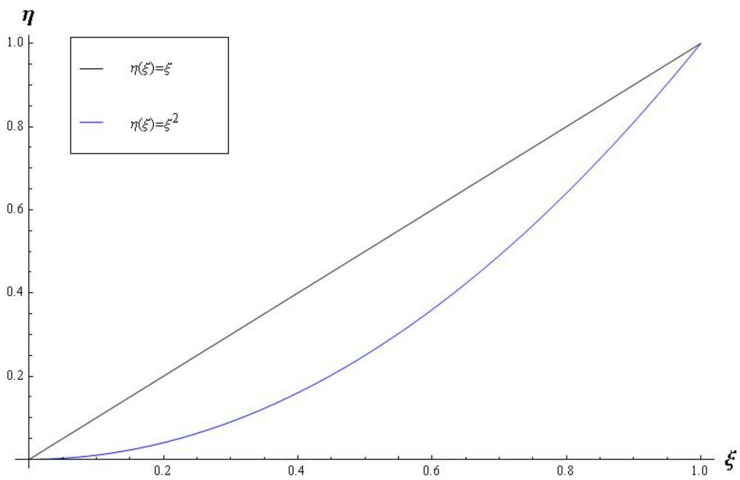


Figure 4.3: Relationship between η and ξ

4.2.1. Cost-sharing Contracts

From the above discussion, we see that a variety of costs are associated with quality. In the context of our paper, the costs borne by the different players differ in the two regimes. Moreover, the costs depend on the type of contract. Cost-sharing in a supply chain can undergo different contractual agreements. We propose two different cost-sharing contracts, a) *Quality Improvement Cost-sharing Contract(QICS)* and b) *Comprehensive Quality Cost-sharing Contract (CQCS)*.

4.2.1.1. Quality Improvement Cost-sharing (QICS) Contract

In this section, we discuss the different costs borne by the different players and the mechanism of Cost-sharing. Under the QICS contract, we assume that the manufacturer shares only the cost of quality improvement with both the suppliers. Details of the costs for the different players are discussed below.

i) The Manufacturer M: The manufacturer bears the advertising costs and the costs of quality improvement that it shares with the suppliers. Thus, the manufacturer's costs in the regime j:

$$\begin{aligned}\text{Advertising Cost} &= \frac{\mu_m A_j^2(t)}{2}\\ \text{Cost shared with S1} &= \frac{\phi_j(t)\mu_{s1} q_{1j}^2(t)}{2}\\ \text{Cost shared with S2} &= \frac{\sigma_j(t)\mu_{s2} q_{2j}^2(t)}{2}\end{aligned}$$

ii) The Supplier S1: The unreliable supplier bears its share of the quality improvement costs and the full cost of its unreliability/non- conformance in the pre-crisis regime. There is an added cost of quality loss in the post-crisis period. This cost, interpreted as a cost of external quality failure, is assumed to be linearly increasing with the drop in quality ξ. Besides, we assumed that a crisis occurs only once. Therefore $\chi = 0$ in the second regime and hence the quality conformance cost is maximum. Thus the costs of S1 in regime j are:

$$\begin{aligned}
\text{Cost shared with M} &= \frac{(1-\phi_j)\mu_{s1}q_{1j}^2(t)}{2} \\
\text{Cost of conformance} &= \frac{(1-\chi)\omega q_{1j}^2(t)}{2} \\
\text{Cost of external failure} &= \frac{\tau_j \xi \theta q_{1j}^2(t)}{2} \\
\text{where } \tau_j &= 0 \text{ for } j=1 \\
\tau_j &= 1 \text{ for } j=2 \\
\chi &= 0 \text{ for } j=2.
\end{aligned}$$

iii) The Supplier S2: The only cost of the reliable supplier is its share of quality improvement cost in regime j.

$$\text{Cost shared with M} = \frac{(1-\sigma_j)\mu_{s2}q_{2j}^2(t)}{2}$$

4.2.1.2. Comprehensive Quality Cost-sharing (CQCS) Contract

Under the CQCS contract, the manufacturer shares all the quality improvement, quality conformance or quality failure costs of the unreliable supplier S1 and the quality improvement costs of the reliable supplier S2. Hence the name - Comprehensive Quality Cost-sharing. Thus the members of the supply chain bear the different costs in the following manner.

i) The Manufacturer M: The manufacturer bears the advertising costs and the costs of quality improvement that it shares with the suppliers. Thus the manufacturer's costs in the regime j are:

$$\begin{aligned}
\text{Advertising Cost} &= \frac{\mu_m A_j^2(t)}{2} \\
\text{Cost shared with S1} &= \frac{\phi_j(t)\Big(\mu_{s1} + (1-\chi)\omega + \tau_j \xi \theta\Big) q_{1j}^2(t)}{2} \\
\text{where } \tau_j &= 0 \text{ for } j = 1 \\
\tau_j &= 1 \text{ for } j = 2 \\
\chi &= 0 \text{ for } j = 2 \\
\text{Cost shared with S2} &= \frac{\sigma_j(t)\mu_{s2} q_{2j}^2(t)}{2}
\end{aligned}$$

ii) The Supplier S1: The unreliable supplier bears its share $(1-\phi)$ of all the costs of quality - improvement, conformance and failure costs. Thus the costs of S1 in regime j are:

$$\begin{aligned}
\text{Cost shared with M} &= \frac{(1-\phi_j(t))\Big(\mu_{s1} + (1-\chi)\omega + \tau_j \xi \theta\Big) q_{1j}^2(t)}{2} \\
\text{where } \tau_j &= 0 \text{ for } j = 1 \\
\tau_j &= 1 \text{ for } j = 2 \\
\chi &= 0 \text{ for } j = 2
\end{aligned}$$

iii) The Supplier S2: The only cost of the reliable supplier is its own share of quality improvement cost in regime j.

$$\text{Cost shared with M} = (1-\sigma_j)\mu_{s2} q_{2j}^2(t)/2$$

Apparently in both the contracts the reliable supplier S2's costs remain the same. However, the costs of quality are differently shared for S1 and M in the two contracts.

4.2.2. The Manufacturer's and supplier's decision problems

The manufacturer and the suppliers individually try to maximize their individual profits. We first show how, in general, the long term expected profit can be calculated for any firm. The same method can be used to calculate the profit of the manufacturer and the two suppliers.

If the product recall occurs at a random time t_r there will be two decision regimes - the pre-crisis regime, $[0, t_r]$ and the post-crisis regime, (t_r, ∞). The profits for any firm $i \in M, S1, S2$, in the two regimes are given by:

$$\begin{aligned}
\pi_{i1} &= \int_0^{t_r} e^{-rt}[m_{i1}D_{i1} - C_{i1}]dt \qquad (4.5)\\
&= \int_0^{t_r} e^{-rt} J_{i1} dt\\
\pi_{i2} &= \int_{t_r}^{\infty} e^{-rt}[m_{i2}D_{i2} - C_{i2}]dt\\
&= \int_{t_r}^{\infty} e^{-rt} J_{i2} dt
\end{aligned}$$

where D_{ij} is the demand of firm i in period j, m_{ij} is the profit margin of firm i in period j and C_{ij} is the cost incurred by firm i in period j. Since, t_r, the time of recall, is a random time which is not known in advance, the profits in the two periods are random variables. So the long-term expected profit is given by, $\Pi_i = E[\pi_{i1} + e^{-rt_r}\pi_{i2}]$ where the expectation E[.] is taken with respect to the crisis occurrence process. The discount factor e^{-rt_r} appears because π_{i1} accrues at $t = 0$ and π_{i2} at $t = t_r$. Thus, it is required to discount π_{i2} back to $t = 0$ to add the two long-term profits.

Let $(\tau_{1j}, \tau_{2j}, ..., \tau_{kj})$ be the generic set of k strategies for the firm i in the regime $j \in \{1,2\}$. Then the value of the long-term profit will depend on the strategies $(\tau_{11}, \tau_{21}, ..., \tau_{k1})$ and $(\tau_{12}, \tau_{22}, ..., \tau_{k2})$. The problem above is a random stopping problem. If $f(t)$ and $F(t)$ are the probability density and cumulative density functions of the stochastic occurrence process, then the hazard rate is $h(t) = \chi, F(t) = 1 - e^{-\int_0^t h(s)ds}$. Therefore, $f(t) = \chi e^{-\int_0^t h(s)ds}$. The long-term expected profit for the manufacturer i can be written as:

$$\Pi_i(\tau_{11}, \tau_{21}, ..., \tau_{k1}, \tau_{12}, \tau_{22}, ..., \tau_{k2}) = E[\int_0^{t_r} e^{-rs} J_{i1} ds + e^{-rt_r}\pi_{i2}], \qquad (4.6)$$

where the first term under the expectation gives the profit of the pre-crisis period and the second term gives the profit of the post-crisis regime. Therefore, the sum of the two profits gives the long-term profit over the planning horizon. Integrating by parts and making algebraic manipulations (Haurie and Moresino 2006; Rubel, Naik, and Srinivasan 2011) the above expression can be transformed into the following equation (see appendix C):

$$\Pi_i(S1_1, ...Sk_1, S1_2, ...Sk_2, \chi) = \int_0^{\infty} e^{-(r+\chi)t}\{\pi_{i1} + \chi\pi_{i2}\}dt. \qquad (4.7)$$

In our case the strategies Si_j refer to the quality efforts in case of the suppliers and to the advertising effort and quality cost-sharing proportion for the manufacturer.

The decision problems of the players will vary depending on which contract we are considering. We present the decision problems under the two contracts.

4.2.2.1. Decision Problems Under Quality Improvement Cost-sharing (QICS) Contract

The discussion in the above section implies that we need to define the decision problems of the manufacturer and the suppliers in the two regimes, pre-crisis and post-crisis. Thereafter, we can solve the second regime's problem first, followed the first regime's problem. Accordingly, we denote the second regime's value functions of the supplier 1, supplier 2 and the manufacturer by $V_{s1}(G(t),Q(t))$, $V_{s2}(G(t),Q(t))$ and $V_m(G(t),Q(t))$ respectively. The first regime's value functions are denoted by $\hat{V}_{s1}(G(t),Q(t))$, $\hat{V}_{s2}(G(t),Q(t))$ and $\hat{V}_m(G(t),Q(t))$.

The second regime's decision problems of the suppliers and the manufacturer under this QICS contract are given by(the Model parameters are given in Table 4.1):

$$\begin{aligned}
V_{s1}(G(t),Q(t)) &= \underset{q_{12}(t)}{\text{Max}} \int_{t_r}^{\infty} e^{-rt}[D(G(t),Q(t))M_{s1} - \frac{(1-\phi_2)\mu_{s1}q_{12}^2(t)}{2} \\
&\quad - \frac{\omega q_{12}^2(t)}{2} - \frac{\xi\theta q_{12}^2(t)}{2}]dt \\
V_{s2}(G(t),Q(t)) &= \underset{q_{22}(t)}{\text{Max}} \int_{t_r}^{\infty} e^{-rt}[D(G(t),Q(t))M_{s2} - \frac{(1-\sigma_2)\mu_{s2}q_{22}^2(t)}{2}]dt \\
V_m(G(t),Q(t)) &= \underset{A_2(t),\phi_2(t),\sigma_2(t)}{\text{Max}} \int_{t_r}^{\infty} e^{-rt}[D(G(t),Q(t))M_m - \frac{\mu_m A_2^2(t)}{2} - \\
&\quad \frac{\phi_2(t)\mu_{s1}q_{12}^2(t)}{2} - \frac{\sigma_2(t)\mu_{s2}q_{22}^2(t)}{2}]dt
\end{aligned} \tag{4.8}$$

Subject to

$$\begin{aligned}
\dot{G}(t) &= k_{g2}A_2(t) + k_{12}q_{12}(t) + k_{22}q_{22}(t) - \delta_2 G(t) \\
\dot{Q}(t) &= l_{12}q_{12}(t) + l_{22}q_{22}(t)
\end{aligned} \tag{4.9}$$

and with initial conditions

$$\begin{aligned}
G(t_r^+) &= (1-\eta)G(t_r^-) \\
Q(t_r^+) &= (1-\xi)Q(t_r^-)
\end{aligned} \tag{4.10}$$

The pre-crisis decision problems , in accordance with equation (4.7) are given by:

$$\begin{aligned}
\hat{V}_{s1}(G(t),Q(t)) &= \underset{q_{11}(t)}{\text{Max}} \int_0^\infty e^{-(r+\chi)t}[D(G(t),Q(t))\hat{M}_{s1} - \frac{(1-\phi_1)\mu_{s1}q_{11}^2(t)}{2} \\
&\quad - \frac{(1-\chi)\omega q_{11}^2(t)}{2} + \chi V_{s1}((1-\eta)G(t),(1-\xi)Q(t))]dt \\
\hat{V}_{s2}(G(t),Q(t)) &= \underset{q_{21}(t)}{\text{Max}} \int_0^\infty e^{-(r+\chi)t}[D(G(t),Q(t))\hat{M}_{s2} - \frac{(1-\sigma_1)\mu_{s2}q_{21}^2(t)}{2} \\
&\quad + \chi V_{s2}((1-\eta)G(t),(1-\xi)Q(t))]dt \\
\hat{V}_m(G(t),Q(t)) &= \underset{A_1(t),\phi_1(t),\sigma_1(t)}{\text{Max}} \int_0^\infty e^{-(r+\chi)t}[D(G(t),Q(t))\hat{M}_m - \frac{\mu_m A_2^2(t)}{2} \\
&\quad - \frac{\phi_1(t)\mu_{s1}q_{12}^2(t)}{2} - \frac{\sigma_1(t)\mu_{s2}q_{21}^2(t)}{2} \\
&\quad + \chi V_m((1-\eta)G(t),(1-\xi)Q(t))]dt \qquad (4.11)
\end{aligned}$$

Subject to

$$\begin{aligned}
\dot{G}(t) &= k_{g1}A_1(t) + k_{11}q_{11}(t) + k_{21}q_{21}(t) - \delta_1 G(t) \\
\dot{Q}(t) &= l_{11}q_{11}(t)e^{-\lambda\chi} + l_{21}q_{21}(t)
\end{aligned}$$

and with initial conditions

$$\begin{aligned}
G(0) &= G_0 \geq 0 \\
Q(0) &= Q_0 \geq 0 \qquad (4.12)
\end{aligned}$$

4.2.2.2. Decision Problems Under Comprehensive Quality Cost-sharing (CQCS) Contract

The second regime's decision problems of the suppliers and the manufacturer under this CQCS contract are given by:

$$\begin{aligned}
V_{s1}(G(t),Q(t)) &= \underset{q_{12}(t)}{\mathrm{Max}} \int_{t_r}^{\infty} e^{-rt}[D(G(t),Q(t))M_{s1} \\
&\quad - \frac{(1-\phi_2(t))\left(\mu_{s1}+\omega+\xi\theta\right)q_{12}^2(t)}{2}]dt \\
V_{s2}(G(t),Q(t)) &= \underset{q_{22}(t)}{\mathrm{Max}} \int_{t_r}^{\infty} e^{-rt}[D(G(t),Q(t))M_{s2} - \frac{(1-\sigma_2)\mu_{s2}q_{22}^2(t)}{2}]dt \\
V_m(G(t),Q(t)) &= \underset{A_2(t),\phi_2(t),\sigma_2(t)}{\mathrm{Max}} \int_{t_r}^{\infty} e^{-rt}[D(G(t),Q(t))M_m - \frac{\mu_m A_2^2(t)}{2} - \\
&\quad - \frac{\phi_2(t)\left(\mu_{s1}+\omega+\xi\theta\right)q_{12}^2(t)}{2} - \frac{\sigma_2(t)\mu_{s2}q_{22}^2(t)}{2}]dt \quad (4.13)
\end{aligned}$$

Subject to

$$\begin{aligned}
\dot{G}(t) &= k_{g2}A_2(t)+k_{12}q_{12}(t)+k_{22}q_{22}(t)-\delta_2 G(t) \\
\dot{Q}(t) &= l_{12}q_{12}(t)+l_{22}q_{22}(t) \quad (4.14)
\end{aligned}$$

and with initial conditions

$$\begin{aligned}
G(t_r^+) &= (1-\eta)G(t_r^-) \\
Q(t_r^+) &= (1-\xi)Q(t_r^-) \quad (4.15)
\end{aligned}$$

The pre-crisis decision problems , in accordance with equation (4.7) are given by:

$$\begin{aligned}
\hat{V}_{s1}(G(t),Q(t)) &= \underset{q_{11}(t)}{\text{Max}} \int_0^{\infty} e^{-(r+\chi)t}[D(G(t),Q(t))\hat{M}_{s1} - \\
&\quad \frac{(1-\phi_1(t))\left(\mu_{s1}+(1-\chi)\omega\right)q_{11}^2(t)}{2} \\
&\quad +\chi V_{s1}((1-\eta)G(t),(1-\xi)Q(t))]dt \\
\hat{V}_{s2}(G(t),Q(t)) &= \underset{q_{21}(t)}{\text{Max}} \int_0^{\infty} e^{-(r+\chi)t}[D(G(t),Q(t))\hat{M}_{s2} - \frac{(1-\sigma_1)\mu_{s2}q_{21}^2(t)}{2} \\
&\quad +\chi V_{s2}((1-\eta)G(t),(1-\xi)Q(t))]dt \\
\hat{V}_{m}(G(t),Q(t)) &= \underset{A_1(t),\phi_1(t),\sigma_1(t)}{\text{Max}} \int_0^{\infty} e^{-(r+\chi)t}[D(G(t),Q(t))\hat{M}_{m} - \frac{\mu_m A_2^2(t)}{2} - \\
&\quad \frac{\phi_1(t)\left(\mu_{s1}+(1-\chi)\omega\right)q_{11}^2(t)}{2} - \frac{\sigma_1(t)\mu_{s2}q_{21}^2(t)}{2} \\
&\quad +\chi V_{m}((1-\eta)G(t),(1-\xi)Q(t))]dt
\end{aligned} \tag{4.16}$$

Subject to

$$\dot{G}(t) = k_{g1}A_1(t)+k_{11}q_{11}(t)+k_{21}q_{21}(t)-\delta_1 G(t) \tag{4.17}$$

$$\begin{aligned}
\dot{Q}(t) &= l_{11}q_{11}(t)e^{-\lambda\chi}+l_{21}q_{21}(t) \\
G(0) &= G_0 \geq 0 \\
Q(0) &= Q_0 \geq 0
\end{aligned} \tag{4.18}$$

Our model includes a number of parameters. For the convenience of the reader, a summary of all the model assumptions are given below.

Assumption 1: *The three players, two suppliers and a manufacturer know the consumer demand at the manufacturer's end. The demand is sensitive to goodwill and quality level of the product.*

Assumption 2: *S1 is unreliable, i.e prone to recall with a hazard rate of* χ *whereas the S2 and M are reliable and free from recall risk. The hazard rate is common knowledge.*

Assumption 3: *The players continue to exist in the market after the recall, implying that the recall is partial.*

Assumption 4: *The evolution of quality level is decreasing with the hazard rate of S1 in the first period. Crisis can occur only once in the planning horizon because S1(or S1 and M) ensure hazard free quality at an extra cost in the post crisis period.*

Assumption 5: *The profit margins of S1 and M are lower in the second period when a recall of medium or high impact occurs. This assumption is supported by the fact that S1 and M may have to involve in recall management efforts and costs like reverse logistics, insurance claims, compen-*

sating the supplier S2 suffering from collateral damage etc..However, when the recall impact is low (about 5% loss of goodwill), we assume that no change in post-crisis profit margin occurs).

Assumptions About the Variations in Model Parameters:

- $k_{g1} \geq k_{g2}$ because advertising effectiveness may reduce after a recall (H. Van Heerde, Helsen, and Dekimpe 2007).
- $k_{11} \geq k_{12}$ as the faulty supplier's quality efforts may lose credibility after the recall and the sensitivity of the goodwill evolution towards the efforts may reduce.
- $\delta_1 \geq \delta_2$ as natural brand absorption may increase after the recall.
- $l_{11} \leq l_{12}$ because previous recall experience can lead to improved learning and a consequent enhancement in product quality (Thirumalai and Sinha 2011). This parameter indicates that due to a potential learning improvement, the evolution of product quality can raise with the quality efforts of supplier 1.
- For supplier 2, $k_{21} = k_{22}$ and $l_{12} = l_{22}$ and $\hat{M}_{s2} = M_{s2}$.
- As recall may reduce profit margin for supplier 1 and the manufacturer, $\hat{M}_{s1} \geq M_{s1}$ and $\hat{M}_m \geq M_m$.

4.3. Equilibrium Analysis

In this section, we provide an analysis of the equilibrium decisions of the different players and the impact of the important system parameters on the decisions. We want to investigate the feedback strategies for the firms. Therefore, as a standard solution procedure we start by writing the Hamilton Jacobi Bellman (HJB) equations (Dockner et al. 2000). Subsequently, the first order conditions on the decision variables will help us in finding the equilibrium strategies from the HJB equations. We note that the value functions $\hat{V}_{s1}, \hat{V}_{s2}, \hat{V}_m, V_{s1}, V_{s2}$ and V_m are individually concave in A_j, q_{ij}, ϕ_j and σ_j. The HJB equations for the manufacturer and the suppliers will vary depending on the contract as the cost-sharing mechanisms in the two different contracts vary. To save space and avoid repetition we illustrate the generalized HJB by denoting the costs of the Manufacturer and the suppliers by C_M, C_{s1} and C_{s2}. It is understood that when QICS contract is being considered, the costs will assume values given in the section 4.2.1.1 and when CQCS contract is considered, the costs will assume values from discussions in section 4.2.1.2. Thus, from equation (4.8) (for

QICS) or equation (4.13) (for CQCS), the HJB equations for the second regime are given by:

$$
\begin{aligned}
rV_M(G,Q) &= \underset{A_2,\phi_2,\sigma_2}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_m - C_M + \frac{\partial V_m}{\partial G}\dot{G}(t) + \frac{\partial V_m}{\partial Q}\dot{Q}(t)] \\
rV_{s1}(G,Q) &= \underset{q_{12}}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_{s1} - C_{s1} + \frac{\partial V_{s1}}{\partial G}\dot{G}(t) + \frac{\partial V_{s1}}{\partial Q}\dot{Q}(t)] \\
rV_{s2}(G,Q) &= \underset{q_{22}}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_{s2} - C_{s2} + \frac{\partial V_{s2}}{\partial G}\dot{G}(t) + \frac{\partial V_{s2}}{\partial Q}\dot{Q}(t)]
\end{aligned} \tag{4.19}
$$

Similarly, from equations (4.11) (for QICS) and (4.16) (for CQCS) the first period HJB equations of the players are given by:

$$
\begin{aligned}
(r+\chi)\hat{V}_M(G,Q) &= \underset{A_1,\phi_1,\sigma_1}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_m - C_M + \frac{\partial \hat{V}_m}{\partial G}\dot{G}(t) + \frac{\partial \hat{V}_m}{\partial Q}\dot{Q}(t) \\
&\quad + \chi V_m((1-\eta)G(t),(1-\xi)Q(t)] \\
(r+\chi)\hat{V}_{s1}(G,Q) &= \underset{q_{11}}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_{s1} - C_{s1} + \frac{\partial \hat{V}_{s1}}{\partial G}\dot{G}(t) + \frac{\partial \hat{V}_{s1}}{\partial Q}\dot{Q}(t) \\
&\quad + \chi V_{s1}((1-\eta)G(t),(1-\xi)Q(t)] \\
(r+\chi)\hat{V}_{s2}(G,Q) &= \underset{q_{21}}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_{s2} - C_{s2} + \frac{\partial \hat{V}_{s2}}{\partial G}\dot{G}(t) + \frac{\partial \hat{V}_{s2}}{\partial Q}\dot{Q}(t) \\
&\quad + \chi V_{s2}((1-\eta)G(t),(1-\xi)Q(t)]
\end{aligned} \tag{4.20}
$$

In this section we present the results for the two cost-sharing contracts and the benchmark case of no cost-sharing amongst the manufacturer and the suppliers. The value functions are linear in the state variables (see Proposition 6). The value functions are of the form $\hat{V}_i(G(t),Q(t)) = \hat{X}_iG(t) + \hat{Y}_iQ(t) + \hat{Z}_i$ where $i \in \{M, s1, s2\}$. For clarity and ready reference we present the expressions of the coefficients in Table 4.2. The value functions are given in propositions 9.

	Coefficient of $G(t)$	**Coefficient of $Q(t)$**
Supplier 1	$\hat{X}_{s1} = \frac{\beta\left(\hat{M}_{s1}(r+\delta_2)+(1-\eta)\chi M_{s1}\right)}{(r+\delta_2)(r+\delta_1+\chi)}$	$\hat{Y}_{s1} = \frac{\gamma\left(\hat{M}_{s1}r+(1-\xi)\chi M_{s1}\right)}{r(r+\chi)}$
Supplier 2	$\hat{X}_{s2} = \frac{\beta\left(\hat{M}_{s2}(r+\delta_2)+(1-\eta)\chi M_{s2}\right)}{(r+\delta_2)(r+\delta_1+\chi)}$	$\hat{Y}_{s2} = \frac{\gamma\left(\hat{M}_{s2}r+(1-\xi)\chi M_{s2}\right)}{r(r+\chi)}$
Manufacturer	$\hat{X}_m = \frac{\beta\left(\hat{M}_m(r+\delta_2)+(1-\eta)\chi M_m\right)}{(r+\delta_2)(r+\delta_1+\chi)}$	$\hat{Y}_m = \frac{\gamma\left(\hat{M}_m r+(1-\xi)\chi M_{s1}\right)}{r(r+\chi)}$

Table 4.2: Value Function Coefficients for Pre-crisis Regime

We solve the HJB equations for the manufacturer and the suppliers 1 and 2 and get some insightful analytical results which underscore the relationships amongst the model parameters and the equilibrium decisions of the different players.

4.3.1. Quality decisions of the suppliers

We investigate the suppliers' quality decisions under the two cost-sharing contracts and also in the no cost-sharing scenario. The benchmark case is that of the no cost-sharing (NCS).

Proposition 1. *Under both the QICS and CQCS contracts, the equilibrium quality effort levels for the unreliable supplier (S1) in the pre-crisis and post crisis regimes are given by*

$$\begin{aligned} q_{11}^{*}(t) &= \frac{1}{2(\mu_{s1}+(1-\chi)\omega)}\Bigg(\frac{\gamma l_{11}e^{-\lambda\chi}(\chi(2M_m+(1-\xi)M_{s1})+r(2\hat{M}_m+\hat{M}_{s1})}{r(r+\chi)} \\ &\quad +\frac{\beta k_{11}((2\hat{M}_m+\hat{M}_{s1})(r+\delta_2)+(1-\eta)\chi(2M_m+M_{s1}))}{(r+\delta_2)(r+\delta_1+\chi)}\Bigg), \\ q_{12}^{*}(t) &= \frac{(2M_m+M_{s1})(k_{12}r\beta+l_{12}\gamma(r+\delta_2))}{2r(r+\delta_2)(\mu_{s1}+\theta\xi+\omega)}. \end{aligned} \tag{4.21}$$

Proposition 1 has an interesting revelation. The equilibrium quality efforts of the supplier 1 remain same irrespective of the contract chosen. However, the supplier's quality efforts differ in terms of the structure if it is expressed as a function of $\phi_2*(t)$. For example, under the QICS contract,

$$q_{11}^{QICS*}(\phi_1^{QICS*}(t)) = \frac{k_{11}\hat{X}_{s1}+l_{11}\hat{Y}_{s1}e^{-\lambda\chi}}{\mu_{s1}(1-\phi_1^{QICS*})+\omega(1-\chi)},$$

whereas under the CQCS contract

$$q_{11}^{CQCS*}(\phi_1^{CQCS*}(t)) = \frac{k_{11}\hat{X}_{s1}+l_{11}\hat{Y}_{s1}e^{-\lambda\chi}}{(1-\phi_1^{CQCS*})(\mu_{s1}+(1-\chi)\omega)}.$$

The above finding underscores the first mover or leadership advantage of the manufacturer. The manufacturer adjusts its cost-sharing decision based on the committed quality efforts (reaction function) of the suppliers. Hence the cost-sharing decisions vary depending on the contract. Intuitively. the manufacturer shares a lesser proportion of cost in the comprehensive quality cost sharing contract.

Lemma 1. *(a) In both the pre-crisis and post-crisis regimes, the equilibrium quality efforts of supplier 1 increase with β, the consumer's sensitivity towards goodwill and γ, the consumer's sensitivity towards quality level.*

(b) The quality effort of supplier 1 in the pre-crisis regime is decreasing with the goodwill damage η and quality loss ξ.

(c) The quality effort of supplier 1 in both the regimes are increasing with the unit profit margins of both the supplier and the manufacturer.

Proof: (a) The proof of the above lemma follows from the sign of the first derivative of $q_{12}^*(t)$ with respect to η, β and γ. From the positivity of all the parameters, it is clear that,

$$\begin{aligned}
\frac{\partial q_{11}^*}{\partial \beta} &= \frac{k_{11}\Big((2\hat{M}_m+\hat{M}_{s1})(r+\delta_2)+(1-\eta)\chi(2M_m+M_{s1})\Big)}{2(\mu_{s1}+(1-\chi)\omega)(r+\delta_2)(r+\delta_1+\chi)} > 0,\\
\frac{\partial q_{11}^*}{\partial \gamma} &= \frac{l_{11}e^{-\lambda\chi}(\chi(2M_m+(1-\xi)M_{s1}))}{2r(r+\chi)(\mu_{s1}+(1-\chi)\omega)} > 0,\\
\frac{\partial q_{12}^*}{\partial \beta} &= \frac{(2M_m+M_{s1})k_{12}}{2(r+\delta_2)(\mu_{s1}+\theta\xi+\omega)} > 0,\\
\frac{\partial q_{12}^*}{\partial \gamma} &= \frac{(2M_m+M_{s1})l_{12}}{2r(\mu_{s1}+\theta\xi+\omega)} > 0.
\end{aligned}$$

The proofs of parts (b) and (c) are trivial and similarly follow from the sign of the first derivative of the quality decisions with respect to the parameters in question.

The profits of the suppliers as well as the manufacturer increase with the demand. In both the regimes, pre-crisis and post-crisis, the consumer demand is increasing linearly with goodwill and quality level as well the consumer sensitivity towards these factors. Therefore, the best interest of the supplier 1 is to put more efforts on the quality level if consumer sensitivity towards Goodwill and quality level are high.

***Quality Fade Effect*:** Quality fade refers to the action of reducing product quality to reduce cost and make more profit (Midler 2007). In the pre-crisis regime, the decrease in quality effort with increasing anticipated loss of Goodwill or quality can be attributed towards the "quality fade" effect. This act of cost reduction need not always be intentional but can also have unintended sources like complacency, lack of proper standardization of quality etc. (Whipple and Roh 2010). Apart from the quality fade possibility, we believe that when the supplier 1 anticipates a recall with a certain impact,it might reduce quality cost so that the post-crisis costs related to quality or other external failure costs can be paid off. A higher profit margin will allow the supplier 1 to have more budget to invest in quality and thereby improve profitability, competitive advantage etc. Moreover, the manufacturer's high margin would imply that the manufacturer can share a higher proportion of the quality costs. Therefore, the quality efforts of the supplier 1 are increasing with the manufacturer's profit margin as well.

***Quality Decision and Crisis Likelihood*:** Apart from the above implications of the system parameters and quality decisions of supplier 1, another important topic for discussion is the influence of crisis likelihood on the suppliers' quality decisions. In the absence of crisis (i.e. $\chi = 0, \xi = 0, l_{11} = l_{12}, k_{11} = k_{12}$), we have $q_{11}^*(t) = q_{12}^*(t)$ as it should be. When there is a positive likelihood of a crisis, we search for the condition when the pre-crisis quality decision off the supplier 1 increase with the likelihood of a crisis. Analytically, the derivative of $q_{11}(t)$ with respect to χ is complex polynomial in χ and the first-order condition doesn't provide us with any fruitful insights. Therefore, we investigate this effect of χ on the quality level using numerical experiments in section 4.4.

Proposition 2. *Under both QICS and CQCS contracts, the equilibrium quality effort levels for the reliable supplier (S2) in the pre-crisis and post-crisis period are given by*

$$\begin{aligned} q_{21}^*(t) &= \frac{k_{21}\hat{X}_{s2} + l_{21}\hat{Y}_{s2}}{\mu_{s2}(1-\sigma_1^*)}, \\ &= \frac{1}{\mu_{s2}}\left(\chi\left(\frac{k_{21}\beta(1-\eta)}{(r+\delta_2)(r+\delta_1+\chi)} + \frac{l_{21}\gamma(2M_m+(1-\xi)M_{s2})}{r(r+\chi)}\right) + \right. \\ &\quad \left. (2\hat{M}_m+\hat{M}_{s2})\left(\frac{k_{21}\beta}{r+\delta_1+\chi} + \frac{l_{21}\gamma}{r+\chi}\right)\right) \\ q_{22}^*(t) &= \frac{(2M_m+M_{s2})(k_{22}r\beta + l_{22}\gamma(r+\delta_2))}{2r(r+\delta_2)\mu_{s2}}. \end{aligned} \tag{4.22}$$

Lemma 2. *(a) In both the pre-crisis and post-crisis regimes, the equilibrium quality efforts of the reliable supplier increase with β, the consumer's sensitivity towards goodwill and γ, the consumer's sensitivity towards quality level.*
(b) The quality effort of supplier 2 in the pre-crisis regime is decreasing in the goodwill damage η and quality loss ξ.
(c) The quality effort of supplier 2 in both the regimes are increasing in the unit profit margins of both supplier 2 and the manufacturer.

The proof of Lemma 2 is similar to the proof of Lemma 1. The equilibrium quality efforts of the reliable and the unreliable supplier are structurally similar with respect to the model parameters but are not identical. However, we find that the model parameters have same effect effect on the two suppliers quality efforts in the pre-crisis and post-crisis periods. However, in general we have assumed that the post-crisis margin of supplier 2 will be higher than the post-crisis margin of the supplier 1 even if the suppliers are similar. This is because the supplier 2 will not bear any recall costs and is only affected by the loss of sales due to recall. Therefore, the post-crisis quality levels of the two suppliers are definitely not equal. From equations (4.21) and (4.22), the pre-crisis quality expressions for both the suppliers are different and hence the efforts are different.

We consider the case of identical suppliers for whom $M_{s2} = M_{s1}, \mu_{s1} = \mu_{s2}$ and $k_1 = k_2$. The following Lemma holds for identical suppliers. For similar suppliers we have the following lemma.

Lemma 3. *For two identical suppliers, quality effort of the reliable supplier, is always higher than that of the unreliable supplier in the second regime.*

This might be surprising at a first observation. However, the supplier 1 has to account for the external failure costs of quality as well as full conformance costs. This can make the quality effort level of supplier 1 less than that of the supplier 2.

***No Cost-Sharing*:** A no coast sharing scenario is a benchmark scenario against which any contract's usefulness can be evaluated. In other words, if a contracting agreement results in more profit than a no cost-sharing scenario, then contracting is indeed desirable. Therefore, we present the equilibrium quality decisions of the suppliers under no cost sharing. The decision of the manufacturer are the advertising efforts. The advertising decisions are not affected by the quality cost-sharing. Hence, the manufacturer's equilibrium advertising efforts are same when there is no cost-sharing or when a contract is adopted.

Proposition 3. *Under no cost-sharing, the equilibrium quality effort levels for the unreliable supplier S1 in the pre-crisis and post crisis regimes are given by*

$$\begin{aligned} q_{11}^*(t) &= \frac{1}{(\mu_{s1}+(1-\chi)\omega)}\Bigg(\frac{\gamma l_{11} e^{-\lambda\chi}\big(\chi(1-\xi)M_{s1}+r\hat{M}_{s1}\big)}{r(r+\chi)} \\ &\quad + \frac{\beta k_{11}\big((r+\delta_2)\hat{M}_{s1}+(1-\eta)\chi M_{s1}\big)}{(r+\delta_2)(r+\delta_1+\chi)}\Bigg), \\ q_{12}^*(t) &= \frac{M_{s1}(k_{12}r\beta+l_{12}\gamma(r+\delta_2))}{r(r+\delta_2)(\mu_{s1}+\theta\xi+\omega)}. \end{aligned} \tag{4.23}$$

As compared to the QICS and CQCS cost-sharing quality efforts, the quality efforts of the suppliers under no cost-sharing is not affected by the manufacturers unit profit margin. In section (4.3.2) we show that for cost-sharing to be possible in our model, a condition is $2M_m > M_{s1}$. An important question is where cost-sharing is an option, can no-cost sharing ever lead to higher supplier product quality? We answer this question in section (4.3.2).

4.3.2. Manufacturer's cost-sharing and advertising decisions

We consider a dynamic game, but the linear quadratic structure of the game enables us to find decisions which are stationary in the pre-crisis and post-crisis regimes. This means the contract need not be changed very often. Frequent negotiations of a contract is indeed undesirable. In our case, the contracts are revised only at the beginning of the planning horizon and when a product is recalled. This assures the sub-game perfectness of the contracts in all sub-games in regime 1 and regime 2. We have incorporated the effect of quality efforts in the firm's goodwill.

Proposition 4. *Under the Quality Improvement Cost-sharing (QICS) contract, the manufacturer's equilibrium pre-crisis share $\phi_1(t)$ and post-crisis share $\phi_2(t)$ of the quality improvement cost of the* *unreliable supplier (S1) are given by*

$$\phi_1^{QICS*}(t) = \frac{\Big(e^{\lambda\chi}k_{11}\big(2\hat{X}_m-\hat{X}_{s1}\big)+l_{11}\big(2\hat{Y}_m-\hat{Y}_{s1}\big)\Big)(\mu_{s1}+\omega(1-\chi))}{\Big(e^{\lambda\chi}k_{11}\big(2\hat{X}_m+\hat{X}_{s1}\big)+l_{11}\big(2\hat{Y}_m+\hat{Y}_{s1}\big)\Big)\mu_{s1}}, \quad (4.24)$$

$$\phi_2^{QICS*}(t) = \frac{(2M_m-M_{s1})(\mu_{s1}+\theta\xi+\omega)}{(2M_m+M_{s1})\mu_{s1}}.$$

For equations (4.24) to be meaningful, we must have $0 \le \phi_i^*(t) \le 1$. Using this constraint on $\phi_2^*(t)$, we get the upper and lower bounds of M_m given by $\frac{M_{s1}}{2} \le M_m \le (\frac{1}{2}+\frac{\mu_{s1}}{\theta\xi})M_{s1}$. Thus the equilibrium decision of the manufacturer is to share no cost of quality if $\frac{M_{s1}}{2} = M_m$. On the other hand, the manufacturer shares the full cost of quality in the second period if $M_m = (\frac{1}{2}+\frac{\mu_{s1}}{\theta\xi})M_{s1}$. Similarly, considering $0 \le \phi_1^*(t) \le 1$, we get the lower bounds $\frac{M_{s1}}{2} \le M_m$ and $\frac{\hat{M}_{s1}}{2} \le \hat{M}_m$.

It is easy to verify that in equation (4.24) the expression $\frac{\Big(e^{\lambda\chi}k_{11}\big(2\hat{X}_m-\hat{X}_{s1}\big)+l_{11}\big(2\hat{Y}_m-\hat{Y}_{s1}\big)\Big)}{\Big(e^{\lambda\chi}k_{11}\big(2\hat{X}_m+\hat{X}_{s1}\big)+l_{11}\big(2\hat{Y}_m+\hat{Y}_{s1}\big)\Big)} = \zeta < 1$. Thus we note that when there is a recall certainty, i.e. $\chi = 1$, the upper bound of 1 for $\phi_1(t)$ is always satisfied. Otherwise, the upper bound, 1 of $\phi_1^*(t)$ is satisfied if

$$\frac{\zeta(\mu_{s1}+\omega(1-\chi))}{\mu_{s1}} < 1$$
$$\implies \omega < \frac{\mu_{s1}(1-\zeta)}{(1-\chi)\zeta}.$$

We recall that ω and μ_{s1} are the proportionality constants for quality conformance cost and quality improvement cost. The above condition gives us a necessary relationship between conformance costs and quality improvement costs for the Quality improvement cost-sharing contract to be meaningful.

The above analysis shows that the unreliable supplier has a motivation to reduce its unit profit margins in both the regimes in order to force the manufacturer to share the cost of quality. By reducing profit margin the supplier can ensure that $\frac{M_{s1}}{2} \le M_m$. In the pre-crisis regime, the supplier cannot make the manufacturer share the full cost of quality until and unless $M_{s1} = \hat{M}_{s1} = 0$ and $\chi = 1$. Noting the expressions of $\hat{X}_{s1}$ and $\hat{Y}_{s1}$ from Table 4.2, we can conclude that the upper bound 1 of cost-sharing is attained only if $M_{s1} = \hat{M}_{s1} = 0$. In the context of our study maintaining no profit margin might be irrational for the supplier. However, we believe that this scenario is explainable from a strategic point of view. The supplier knows that it is a risky supplier and might want to maintain future contracts with the manufacturer in the case a recall occurs. Therefore, at the beginning of the planning horizon S1 is motivated to reduce its margin to 0 (offer a minimal price) and thereby allow the manufacturer enough affordability to sponsor the full cost of quality and repeat the cost-sharing contractual agreement in the second period provided the supplier bears the external failure costs. Pricing decisions are out of the scope of this study but the relationships between the supplier's and manufacturer's profit margins emphasizes the importance of pricing decisions.

Proposition 5. *Under the Comprehensive Quality Cost-sharing (CQCS) contract, the manufacturer's equilibrium pre-crisis share $\phi_1(t)$ and post-crisis share $\phi_2(t)$ of the quality improvement cost of the unreliable supplier (S1) are given by*

$$\begin{aligned} \phi_1^{CQCS*}(t) &= \frac{\left(e^{\lambda\chi}k_{11}\left(2\hat{X}_m-\hat{X}_{s1}\right)+l_{11}\left(2\hat{Y}_m-\hat{Y}_{s1}\right)\right)}{\left(e^{\lambda\chi}k_{11}\left(2\hat{X}_m+\hat{X}_{s1}\right)+l_{11}\left(2\hat{Y}_m+\hat{Y}_{s1}\right)\right)}, \\ \phi_2^{CQCS*}(t) &= \frac{(2M_m-M_{s1})}{(2M_m+M_{s1})}. \end{aligned} \tag{4.25}$$

Since all the model parameters are positive and from the Table 4.2, $\hat{X_m}, \hat{X_{s1}}, \hat{Y_m}$ and $\hat{Y_{s1}}$ are positive, the inequality $\phi_1^*(t) \leq 1$ always holds true.

Lemma 4. *(i) The manufacturer's share of the quality cost to be shared with the unreliable supplier, is increasing with the quality drop ξ in the second period.*
(ii)In the pre-crisis regime, the manufacturer prefers to a share a higher proportion of quality improvement costs than the comprehensive quality costs with the unreliable supplier.

Proof:(i) The first order condition of $\phi_2^*(t)$, with respect to ξ gives, $\frac{\partial \phi_2^*}{\partial \xi} = \frac{\theta(2M_m-M_{s1})}{(2M_m+M_{s1})\mu_{s1}}$. Clearly, the manufacturer's equilibrium quality cost shared with the unreliable supplier is increasing with the drop in quality, ξ if $(2M_m - M_{s1}) > 0$. But for the share $\phi_2(t)$ to be positive we have to impose the condition $(2M_m - M_{s1}) > 0$.

(ii) Comparing the expressions of the cost-sharing decisions from equations (4.24) and (4.25), we note that:

$$\begin{aligned} \phi_1^{QICS*}(t) &= \phi_1^{CQCS*}(t)\frac{(\mu_{s1}+\omega(1-\chi))}{\mu_{s1}} \\ \phi_2^{QICS*}(t) &= \phi_2^{CQCS*}(t)\frac{(\mu_{s1}+\omega+\theta\xi)}{\mu_{s1}}. \end{aligned} \tag{4.26}$$

From the positivity of all parameters, $(\mu_{s1}+\omega(1-\chi)) \geq \mu_{s1}$. It follows that $\phi_1^{QICS*}(t) \geq \phi_1^{CQCS*}(t)$.

Similarly, $(\mu_{s1}+\omega+\theta\xi) \geq \mu_{s1}$, ensures that $\phi_2^{QICS*}(t) \geq \phi_2^{CQCS*}(t)$.

Under the QICS contract, the manufacturer has to share only the quality improvement costs and hence can afford to share a higher proportion of the quality improvement costs.

Lemma 5. *Whenever cost-sharing is possible, no cost-sharing leads to lower post-crisis supplier quality.*

The comparison of q_{i2} from equations (4.21), (4.22) and (4.23) shows that quality effort of any supplier is more during no cost-sharing only if $2M_m < M_{s1}$ or $2M_m < M_{s1}$, i.e. when cost-sharing is not possible. Thus, whenever cost-sharing is possible, a supplier's post-crisis quality is always better under quality cost-sharing contract.

***Supplier's negligence**:* An unreliable supplier may neglect crisis likelihood for several reasons - ignorance, incompetency or avoiding immediate possible costs. Most powerful manufacturers scrutinize their suppliers' performance. Therefore, it is reasonable to believe that the manufacturer will become aware of such negligence. Consequently, the manufacturer may reduce the share of quality costs. This is evident from the following proposition.

Proposition 6. *Supplier's negligence of crisis likelihood lowers the manufacturer's pre-crisis cost-sharing proportions.*

Proof: Recall that the cost-sharing proportions of the manufacturer under the two contracts are given by :

$$\phi_1^{QICS*}(t) = \frac{\left(e^{\lambda\chi}k_{11}\left(2\hat{X}_m-\hat{X}_{s1}\right)+l_{11}\left(2\hat{Y}_m-\hat{Y}_{s1}\right)\right)(\mu_{s1}+\omega(1-\chi))}{\left(e^{\lambda\chi}k_{11}\left(2\hat{X}_m+\hat{X}_{s1}\right)+l_{11}\left(2\hat{Y}_m+\hat{Y}_{s1}\right)\right)\mu_{s1}}, \tag{4.27}$$

$$\phi_1^{CQCS*}(t) = \frac{\left(e^{\lambda\chi}k_{11}\left(2\hat{X}_m-\hat{X}_{s1}\right)+l_{11}\left(2\hat{Y}_m-\hat{Y}_{s1}\right)\right)}{\left(e^{\lambda\chi}k_{11}\left(2\hat{X}_m+\hat{X}_{s1}\right)+l_{11}\left(2\hat{Y}_m+\hat{Y}_{s1}\right)\right)}.$$

and that the parameter λ represents the supplier1's negligence of crisis likelihood. We want to prove that: $\frac{\partial\phi_1^{QICS*}(t)}{\partial\lambda} < 0$ and $\frac{\partial\phi_1^{CQCS*}(t)}{\partial\lambda} < 0$. The coefficients of the value functions, i.e $\hat{X}_i$ and $\hat{Y}_i$ for $i \in \{s1, m\}$ are free of λ. Therefore, we do not reduce to the above expressions of ϕ_i any further. We show in the Appendix that the condition $(2M_m - M_{s1}) > 0$ ensures $\frac{\partial\phi_1^{QICS*}(t)}{\partial\lambda} < 0$ and $\frac{\partial\phi_1^{CQCS*}(t)}{\partial\lambda} < 0$. Hence, unreliable supplier's negligence of crisis likelihood lowers the manufacturer's pre-crisis cost-sharing proportions.

The above result is intuitive. Since the supplier's negligence negatively affects the quality level and in turn the manufacturer's demand, a manufacturer who expects quality level commitments from a supplier would negotiate a contract where it would share lesser proportion of the quality costs.

Proposition 7. *Under both the QICS and CQCS contracts, the manufacturer's pre-crisis and post-crisis equilibrium share of the supplier 2(S2)'s quality improvement costs are given by*

$$\begin{aligned} \sigma_1^*(t) &= \frac{k_{21}\left(2\hat{X}_m - \hat{X}_{s2}\right) + l_{21}\left(2\hat{Y}_m - \hat{Y}_{s2}\right)}{k_{21}\left(2\hat{X}_m + \hat{X}_{s2}\right) + l_{21}\left(2\hat{Y}_m + \hat{Y}_{s2}\right)} \\ \sigma_2^*(t) &= \frac{(2M_m - M_{s2})}{(2M_m + M_{s2})}. \end{aligned} \tag{4.28}$$

The type of the contract does not change the manufacturer's share of the reliable supplier's costs. This is obvious as the cost-sharing mechanism varies depending on how the manufacturer shares the cost with the reliable supplier. But, the cost-sharing decisions of the manufacturer can change the profit of the reliable supplier depending upon the contract.

4.3.3. Manufacturer's Advertising Decisions

Proposition 8. *The pre-crisis and post-crisis equilibrium advertising of M are given by*

$$\begin{aligned} A_1^*(t) &= \frac{k_{g1}\beta(\hat{M}_m(r+\delta_2) + M_m\chi(1-\eta))}{\mu_m(r+\delta_2)(r+\delta_1+\chi)} \\ A_2^*(t) &= \frac{k_{g2}M_m\beta}{(r+\delta_2)\mu_m}. \end{aligned} \tag{4.29}$$

Like the quality and cost-sharing decisions, the above equilibrium advertising decisions are influenced by the parameters χ and η. Clearly, if $\chi = 0$, the advertising efforts of the two periods coincide. We discuss some important properties of the advertising in the following lemma.

Lemma 6. *(i) The pre-crisis advertising is always decreasing with the damage rate η.*
(ii) The pre-crisis advertising is decreasing with the crisis likelihood, χ if, $\eta > 1 - \dfrac{\hat{M}_m(r+\delta_2)}{M_m(r+\delta_1)}$.

Proof: For part (i), the first order condition of pre-crisis advertising yields:

$\dfrac{\partial A_1^*(t)}{\partial \eta} = \dfrac{-k_{g1}\beta M_m\chi}{\mu_m(r+\delta_2)(r+\delta_1+\chi)} < 0$ since all the parameters in the fraction are positive.

For part (ii), proceeding similarly and simplifying the first order conditions of $A_1^*(t)$ with respect to χ, we get:

$$\frac{\partial A_1^*(t)}{\partial \chi} = \frac{-k_{g1}\beta\left(\hat{M}_m(r+\delta_2) - M_m(r+\delta_1)(1-\eta)\right)}{\mu_m(r+\delta_2)(r+\delta_1+\chi)^2}.$$

The above expression is negative if $\hat{M}_m(r+\delta_2) - M_m(r+\delta_1)(1-\eta) > 0$. Simplification gives, $\eta > 1 - \frac{\hat{M}_m(r+\delta_2)}{M_m(r+\delta_1)}$.

The finding, advertising of the manufacturer is decreasing in the damage rate η, is consistent with the previous literature (Rubel, Naik, and Srinivasan 2011; Mukherjee and Chauhan 2019). In our model, demand is directly affected by goodwill and goodwill is positively affected by advertising. Therefore, by a transitive relation, a higher advertising investment results in high demand. By reducing the pre-crisis advertising, the management of the recalling firm can serve a dual purpose - plan to invest more in advertising in the post-crisis period thereby reducing the effect on goodwill damage and invest more in supplier quality which is also an integral part of our investigation.

Our model considers goodwill advertising. The manufacturer's advertising is decreasing with the crisis likelihood if the ratio of the margin of the first period to that of the second period is higher than $1-\eta$. If the two-period margins are equal or $\hat{M}_m > M_m$, the advertising is always decreasing in the crisis likelihood because $1-\eta \leq 0$. Thus the only instance when the advertising can be increasing in the likelihood of crisis is when the post-crisis regime's unit profit margin is higher than the first regime's unit profit margin.

4.3.4. Equilibrium Profits of the Suppliers and the Manufacturer

In this section, we analyze the long term expected profit of the suppliers and the manufacturer. The value function at time t gives the expected profit at time t. Therefore, we need to find the expression of the value function and evaluate it at $t = 0$ to find the long term expected profit.

Under both the QICS and CQCS contracts, the value functions of the firms have the same coefficients for the state variables goodwill G(t) and quality level Q(t). However, depending on the contract we choose, the constant term in the value functions will vary. We summarize the findings in the following Proposition.

Proposition 9. *The long term expected profit of the player i is given by the value function:*

$V_i(G(t),Q(t)) = \hat{X}_i G(t) + \hat{Y}_i Q(t) + \hat{Z}_i$ at $t=0$ where :

$$\begin{aligned}
\hat{X}_i &= \frac{\beta\big(\hat{M}_i(r+\delta_2)+(1-\eta)\chi M_i\big)}{(r+\delta_2)(r+\delta_1+\chi)}, \qquad (4.30)\\
\hat{Y}_i &= \frac{\gamma\big(\hat{M}_i r+(1-\xi)\chi M_i\big)}{r(r+\chi)},\\
\hat{Z}_i &= \Delta_i^{QICS}\text{(for the QICS contract)} = \Delta_i^{CQCS}\text{(for the CQCS contract)},
\end{aligned}$$

where $\hat{M}_i$ is the player, i's pre-crisis unit profit margin and M_i is the player i's post-crisis unit profit margin. Δ_i is the constant term of the value function, and its expression is given in the appendix. $i \in \{M, S1, S2\}$, where $S1$ is supplier 1, $S2$ is supplier 2 and M is manufacturer.

The coefficients of goodwill and quality level are positive. Since the goodwill and quality levels are also positive, the coefficients have a positive impact on the firms' profit. Clearly, the unit profit margins of the firms also have positive impact on the profit. However, the recall impact dampens profits. The presence of the crisis likelihood in each of the coefficients has two opposing effects. It appears in the numerator,thereby increasing the profit and again appears in the denominator, thereby decreasing the profit. In fact, χ also appears in Δ_i. Thus it is hard to analytically infer the effect of χ. The constant term Δ_i has a very complex structure and is evaluated numerically later on.

4.4. Numerical Experiments

In this section, we complement our analytical results by numerical experiments. Considering the large parametric space that we have, the numerical analysis allows us to investigate the effect of a product recall on the firm decisions and profits and find out which contract is more efficient and when. Due to the complex mathematical interrelationship of the decision variables and the profit functions, especially with the crisis likelihood variable χ, it is infeasible to analytically get some insights about the effect of χ on the decision variables and the firm profit. We investigate this numerically.

A firm's initial goodwill and the intensity or likelihood of a recall can affect its decisions, and its long term expected profit. The previous literature supports this fact (Rubel, Naik, and Srinivasan 2011; Cleeren, Dekimpe, and Helsen 2008; Gao et al. 2015). Figure 4.4, underpinned to the above findings in the literature, shows a hierarchical structure of the different scenarios that we have considered for analysis under the two cost-sharing contracts in questions. Besides, we also considered the benchmark case of "no cost-sharingg" and compare the profits with the cost-sharing cases.

A high initial goodwill scenario is represented by $G(0) = .15$ and $Q(0) = .1$ is characterized as high goodwill and a low initial goodwill scenario is characterized by $G(0) = .1$ and $Q(0) = .1$.

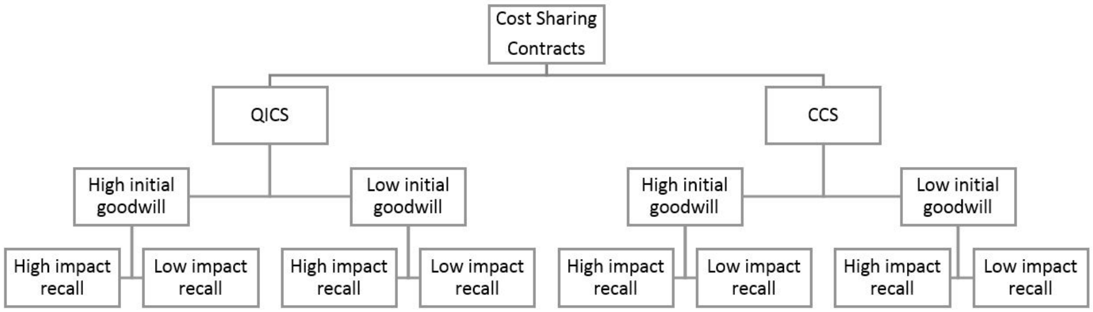


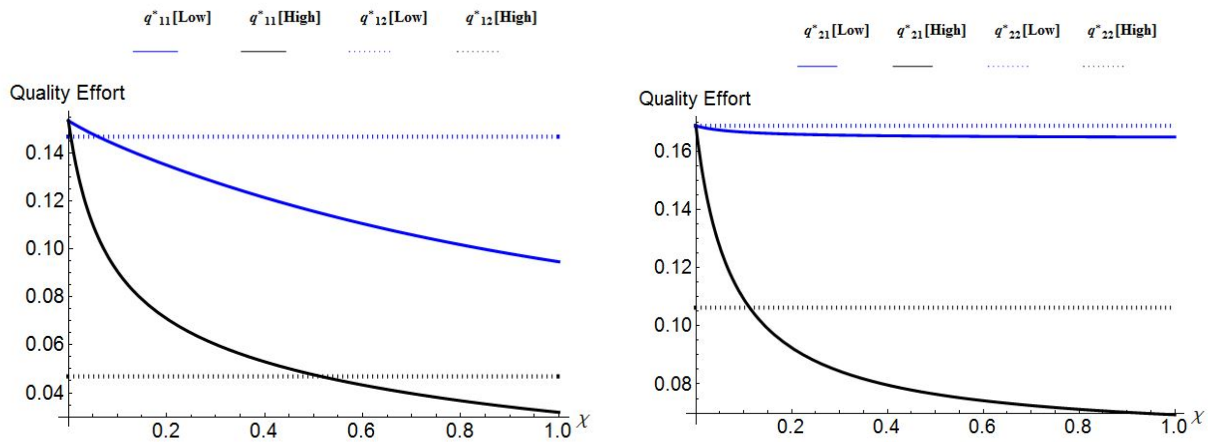
Figure 4.4: Possible Cases

The parameters of the numerical analysis which remain unchanged in the two regimes are:
$\alpha = 1; \beta = .5; \gamma = .03; \delta = .06; r = .06; l = 1; \lambda = 1;$
$\hat{M}_m = 1; \hat{M}_{s1} = .7; \hat{M}_{s2} = .7;$
$\mu_{s1} = 100; \mu_m = 100; \theta = 150; \mu_{s2} = 100; \omega = 10$
$k_{g1} = k_{11} = k_{21} = l_{11} = l_{21} = k_{22} = 1;$

The literature shows that typically external failure costs are much higher than the conformance costs and quality improvement costs. To mimic this we assume $\omega < \mu_{s1} = \mu_{s2} = \mu_m < \theta$. The parameters which may change depending on the impact of the recall are:
$k_{g2}, k_{12}, l_{12}, M_m, M_{s1}, M_{s2}$.
For a low impact recall, when $\eta = .05$,
$k_{g2} = k_{12} = k_{22} = l_{12} = l_{22} = 1; M_m = 1; M_{s1} = M_{s2} = .7.$
For a high impact recall, when $\eta = .7$,
$k_{g2} = k_{12} = k_{12} = .5; l_{12} = 1.2; M_m = .5; M_{s1} = .35; M_{s2} = .7.$

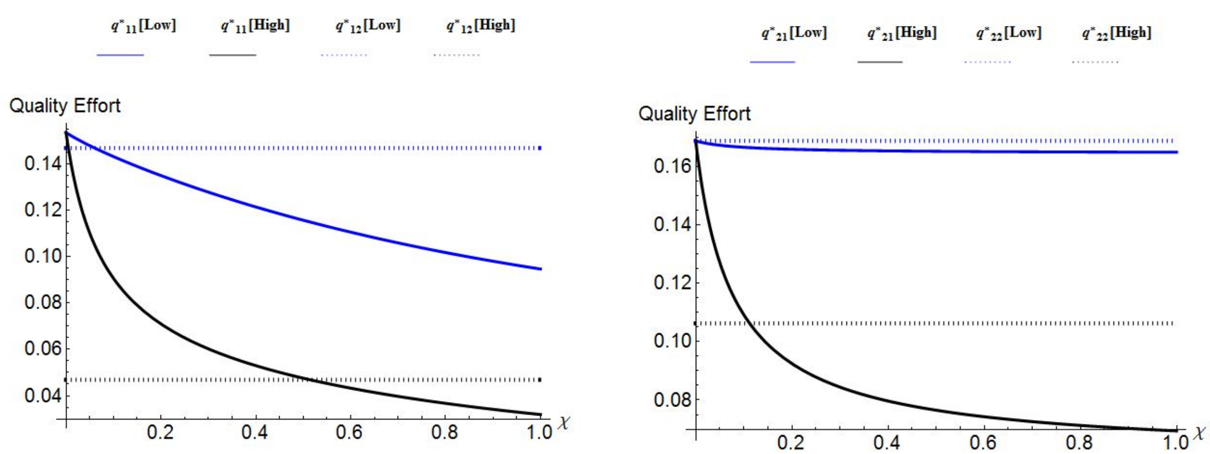
4.4.1. Quality Decisions

The following figures 5, 6 depict the variation of quality efforts of the two suppliers with the crisis likelihood χ. Here $q_{ij}[Low]$ means the quality effort of supplier i in regime j when impact of the recall is low i.e $\xi = .05$ and $q_{ij}[High]$ denotes quality efforts when the recall impact is high $\xi = .7$.



(a) Supplier 1's quality decisions (b) Supplier 2's quality decisions

Figure 4.5: Quality Decisions in pre-crisis and post-crisis regimes under QICS Contract



(a) Supplier 1's quality decisions (b) Supplier 2's quality decisions

Figure 4.6: Quality decisions in pre-crisis and post-crisis regimes under CQCS Contract

Claim 1:*Under both QICS and CQCS contracts,*
a)*In the pre-crisis regime, quality efforts of both the suppliers decrease as crisis likelihood increases,*
b)*the post-crisis quality efforts of the suppliers are higher than pre-crisis quality efforts for high crisis likelihood.*

At first glance, the result may seem surprising. Ideally, suppliers should invest more in quality if there is a considerable risk of crisis. However, a high likelihood of crisis means that there is a high chance of incurring external failure costs and high conformance costs in the post-crisis period. Therefore, we believe that the suppliers reduce pre-crisis quality efforts with a high likelihood to be able to bear the additional post-crisis costs. The post-crisis quality efforts do not depend on the likelihood of recall (Figures 4.5, a, b; Figures 4.6, a, b).

For the computational analysis, we have considered the suppliers to be symmetric, i.e. they have identical parameter values). We note that the supplier 2 typically puts in more quality efforts than the unreliable supplier. This finding is consistent with our assumption that the supplier 1 is risky. In other words, the unreliable supplier, by putting in little efforts, pose a risk of recall to the entire supply chain, thereby causing collateral damage.

The collateral damage (profit loss) for the reliable supplier during the low impact recall is negligible compared to high impact recall. Consequently, a low impact recall does not motivate the reliable supplier to invest significantly higher in the post-crisis quality efforts. The post-crisis quality effort of the reliable supplier is always higher than pre-crisis efforts for a high-impact high-likelihood recall. The higher effort helps in mitigating the losses related to both goodwill and quality.

Claim 2: *Supplier's pre-crisis quality efforts increase with the manufacturer's share of costs.*

From section 4.3.1, we recall that the quality effort of unreliable supplier can be presented as a function of the manufacturer's share of cost share $\phi_1(t)$. This functional form varies from the QICS to the CQCS contract.

$$q_{11}^{QICS*}(\phi_1^*(t)) = \frac{k_{11}\hat{X}_{s1} + l_{11}\hat{Y}_{s1}e^{-\lambda\chi}}{\mu_{s1}(1-\phi_1^*) + \omega(1-\chi)},$$
$$q_{11}^{CQCS*}(\phi_1^*(t)) = \frac{k_{11}\hat{X}_{s1} + l_{11}\hat{Y}_{s1}e^{-\lambda\chi}}{(1-\phi_1^*)(\mu_{s1} + (1-\chi)\omega)}.$$

We examine the variations of the supplier's quality efforts when crisis likelihood χ and the manufacturer's share of cost, $\phi_1(t)$ jointly vary in pre-crisis period. We ask if higher cost-sharing can lead to higher quality efforts. Assuming a recall of benchmark impact, $\xi = .3$, we find that the pre-crisis quality efforts of Supplier 1 indeed increase with the manufacturer's cost-sharing proportion (Figure 4.7). Thus the manufacturer's cost-sharing intentions even with a risky supplier is justifiable.

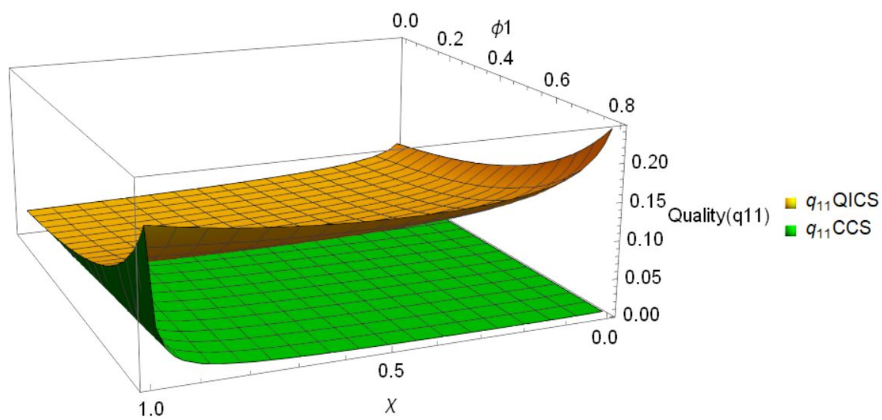


Figure 4.7: Supplier 1- Quality Effort variance with χ and $\phi_1(t)$

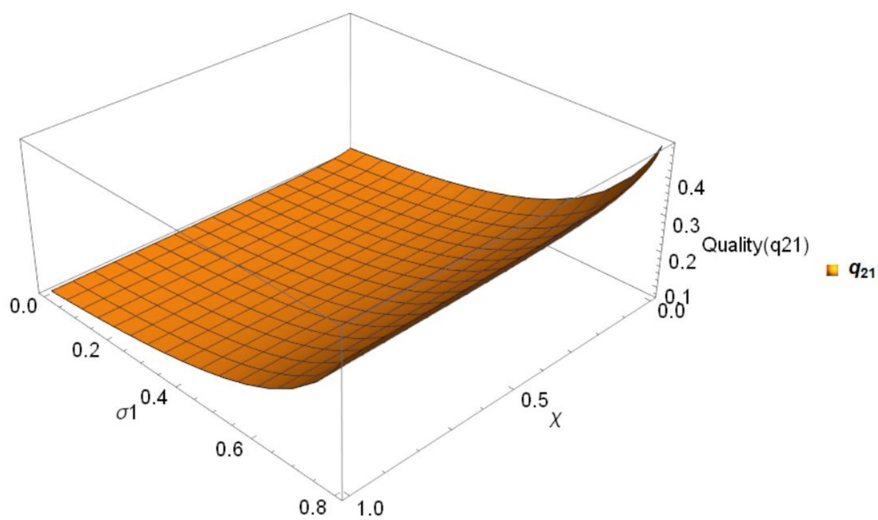


Figure 4.8: Supplier 2- Quality Effort variance with χ and $\sigma_1(t)$

The reliable supplier also increases the optimal pre-crisis quality efforts when share of the manufacturer's cost is high. Since the reliable supplier fully conforms and does not have any other quality costs, its efforts for quality are typically higher than the unreliable supplier for a given share of the manufacturer's.

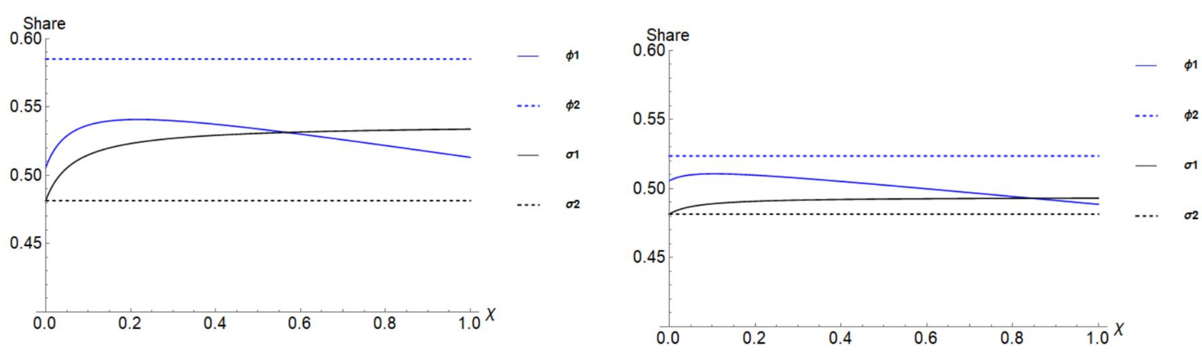
4.4.2. Cost-sharing Decisions

In this section, we present an analysis of the equilibrium cost-sharing decisions of the manufacturer. Cost-sharing decisions depend on multiple parameters. We examine how the decision changes with crisis likelihood χ, the damage intensities η and ξ and the level of initial goodwill $G(0)$. In our study, a high initial goodwill ($G(0) = 0.15$) would mean $\eta = \xi^2$ and a low initial goodwill ($G(0) = 0.10$) would mean $\eta = \xi$.

Figures 5,6,7 and 8 show the variation of the equilibrium cost-sharing decisions of the manufacturer with the crisis likelihood χ under the two contracts. Since our model assumes only one recall during the planning horizon, the second-period cost-sharing decisions of the manufacturer do not depend on the crisis likelihood but change with the crisis intensity and initial goodwill of the firms. The pre-crisis decisions, however, vary with χ. For a convenient interpretation of the figures below, we reiterate the following:

ϕ_1 = Manufacturer's pre-crisis cost-sharing proportion with S1
ϕ_2 = Manufacturer's post-crisis cost-sharing proportion with S1
σ_1 = Manufacturer's pre-crisis cost-sharing with S2
σ_2 = Manufacturer's post-crisis cost-sharing with S2
In the figures below ϕ and σ are the equilibrium proportions.

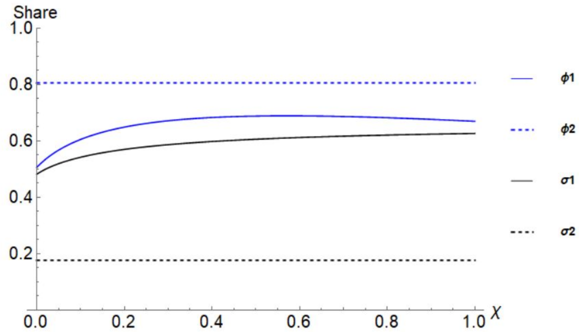
QICS Cost Sharing



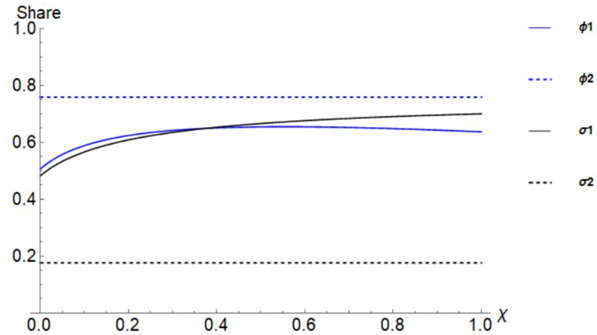
(a) High Initial Goodwill

(b) Low Initial Goodwill

Figure 4.9: Variation of cost-sharing proportions with χ under QICS Contract -low impact recall



(a) High Initial Goodwill



(b) Low Initial Goodwill

Figure 4.10: Variation of cost-sharing proportions with χ under QICS Contract - high impact recall

Claim 3:*Under QICS contract for both low impact and high impact recall scenarios,*
i) the manufacturer's post-crisis cost-sharing proportion with the risky supplier 1 is higher than the pre-crisis proportion,
ii) the manufacturer's post-crisis cost-sharing proportion with the reliable supplier 2 is lower than the pre-crisis proportion.

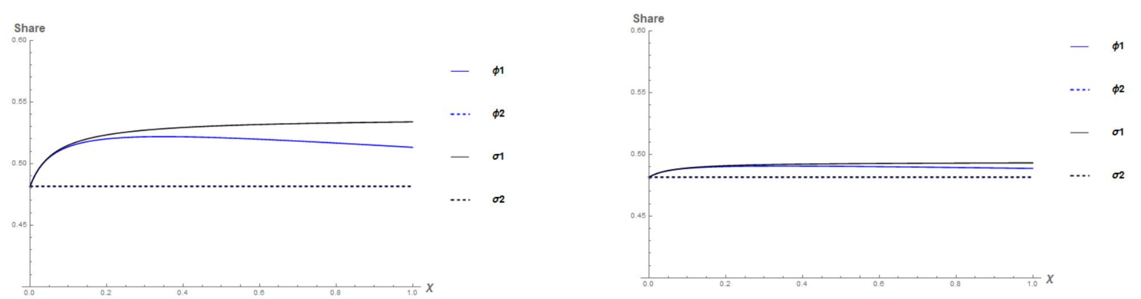
The above claim comports with the inference that the manufacturer balances its optimal cost-sharing efforts with the two suppliers. To nullify the adverse effects of recall and increased cost-sharing with supplier 1, the manufacturer reduces the optimal cost-sharing proportion of supplier 2, who is already a reliable supplier. Moreover, since supplier quality will affect the goodwill (recall that $\dot{G}(t) = k_{gj}a(t) + k_{1j}q_{1j}(t) + k_2q_{2j}(t) - \delta_jG(t)$), the manufacturer, to whom brand image is a primary source of competitive advantage, engages in aggressive post-recall cost-sharing. Moreover, the intensity of the impact of recall plays a vital role in determining the equilibrium cost-sharing decision. Clearly, during a high impact recall, a manufacturer with high initial goodwill shares more than 80 % of the post-crisis quality improvement costs and during the low impact recall the proportion is around 58%. When the level of initial goodwill is low, the high impact recall can induce more efforts in post-crisis cost-sharing from the manufacturer.

Claim 4:*The manufacturer takes judicial quality cost-sharing decisions to control collateral damage caused by the supplier 1.*

A noticeable feature of the manufacturer behaviour is its intention of quality improvement when crisis impact and likelihood are low. On such occasions, the manufacturer decides to increase its share of quality costs. However, after a certain threshold of χ, the share drops signifying the manufacturer is now expecting an impending recall. Consequently, the manufacture is provisioning financial resources to manage the post-crisis costs of potential future contingencies (Figures 4.9, a, b). When the impact of the recall is high, the manufacturer shows a more lenient behaviour towards the risky supplier. The change in the manufacturer's share of cost is not very prominent as the crisis likelihood increases. This is consistent with the explanation that a high impact recall might generate considerable post-crisis cost. So the manufacturer makes an effort to avoid the recall at a higher cost and uplift the supplier's quality by increasing its share of cost in the pre-crisis period. The above behavioural decisions of the manufacturer are poised to manage the collateral damage caused by the risky supplier's products in case a recall is issued (Figures 4.10, a, b).

Irrespective of the impact, low or high, the manufacturer's share of cost with the reliable supplier 2 increases with the crisis likelihood. This is again a strategic decision because the manufacturer realizes that the post-crisis optimal decision will reduce the share of the reliable supplier's cost. The pre-crisis share σ_1, therefore, increases with χ to make sure that the pre-crisis quality efforts of supplier 2 are high enough to increase both quality level and goodwill. High pre-crisis goodwill and quality act as a buffer to nullify the effects of the product recall and higher pre-crisis quality efforts might guarantee post-crisis reliability and consumer confidence (Cleeren, H. J. Van Heerde, and Dekimpe 2013; Kalaignanam, Kushwaha, and M. Eilert 2013; H. Van Heerde, Helsen, and Dekimpe 2007; Yi Zhao, Ying Zhao, and Helsen 2011).

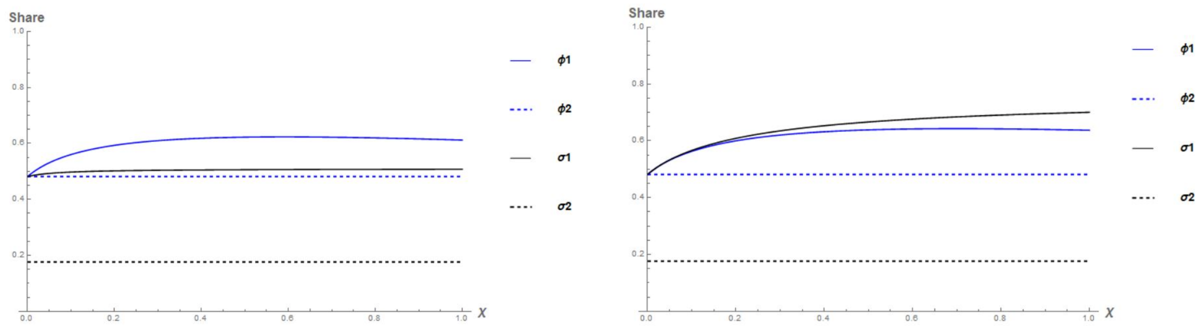
CQCS Cost Sharing



(a) High Initial Goodwill

(b) Low Initial Goodwill

Figure 4.11: Variation of the cost-sharing proportions with χ under CQCS Contract with low impact recall



(a) High Initial Goodwill (b) Low Initial Goodwill

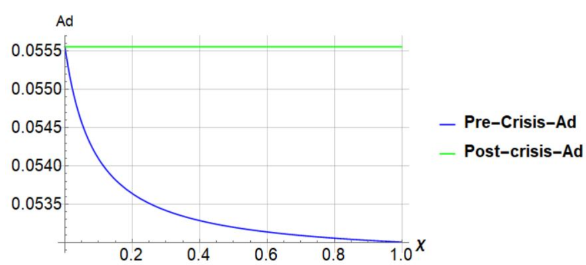
Figure 4.12: Variation of the cost-sharing proportions with χ under CQCS Contract with high impact recall

Claim 5:*Under CCS contract,*
i) the manufacturer's post-crisis cost-sharing proportions with both the suppliers are always lesser than the pre-crisis proportions, i.e., $\phi_2(\cdot) < \phi_1(\cdot)$ *and* $\sigma_2(\cdot) < \sigma_1(\cdot)$*;*
ii) for a low impact recall with no unit margin loss for the manufacturer and supplier 1, post-crisis shares of the manufacturer with the both suppliers are the same, i.e., $\sigma_2(t) = \phi_2(t)$ *(since we assumed symmetric suppliers with* $M_{s1} = M_{s2}$*);*
iii) for a high impact recall, the manufacturer's post-crisis cost-sharing proportion, $\sigma_2(t)$ *with the reliable supplier 2, is lower than the post-crisis cost-sharing* $\phi_2(t)$ *with supplier 1, i.e.,* $\sigma_2(t) < \phi_2(t)$*.*

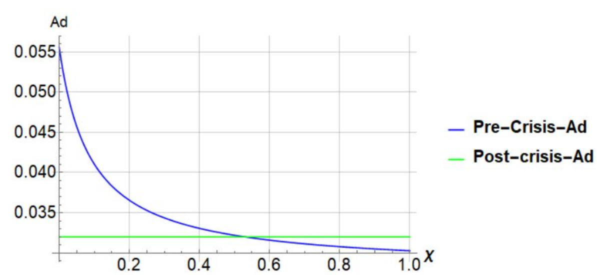
Under CQCS contract, the manufacturer bears more costs in both the regimes. We notice that the post-crisis shares of the manufacturer are given by,
$\sigma_2^{CQCS*}(t) = \dfrac{(2M_m - M_{s2})}{(2M_m + M_{s2})}$ and $\phi_2^{CQCS*}(t) = \frac{(2M_m - M_{s1})}{(2M_m + M_{s1})}$.
If $M_{s1} = M_{s2}$, clearly proportions are equal. Hence under the low impact recall with no margin loss, the two shares coincide in Figure 4.10. Again, both the proportions are decreasing with $Ms1$ or M_{s2}. Since a high impact recalls implies, $M_{s1} \leq M_{s2}$, hence $\phi_2(t) \geq \sigma_2(t)$. The manufacturer's pre-crisis cost-sharing behaviour remains similar to the case of QICS contract, but the proportions of the cost-shared with the suppliers are slightly lower than the QICS contract. The manufacturer has a higher burden of costs of share. Hence the proportion decreases slightly in the pre-crisis period. Moreover, the manufacturer bears a proportion of the recall costs in the post-crisis regime. Hence, the manufacturer adopts a more cautious approach of sharing a lesser proportion of the costs in the pre-crisis regime.

4.4.3. Advertising

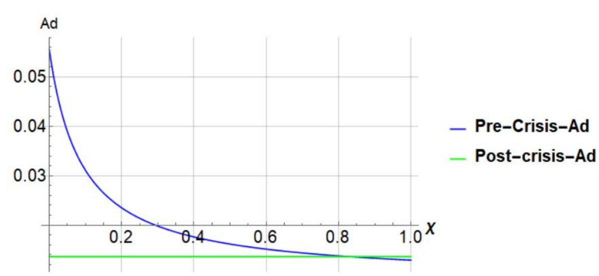
As the previous literature suggests, the pre-crisis and post-crisis advertising decisions vary, and typically post-crisis ad spending increases (Gao et al. 2015; Cleeren, Dekimpe, and Helsen 2008; Rubel, Naik, and Srinivasan 2011). We find that advertising decisions depend on crisis likelihood and impact. The following Figure 4.11 demonstrates the variations in advertisement efforts in the two regimes when recall likelihood and impact vary. We find that for a low impact recall, post-crisis advertising is higher than the pre-crisis advertising efforts (Figure 4.12, a). On the other hand, as crisis impact increases, the choice of advertising efforts depends on the crisis likelihood. For example, when $\xi = .3$, the likelihood of $\chi = .3$ would mean pre-crisis advertising is higher than the post-crisis advertising (Figure 4.12, b). Whereas, when recall impact is very high, $\xi = .7$, a likelihood of $\chi = .6$ means a higher pre-crisis ad will yield better profits. Whereas, if $\chi = .9$, then post-crisis advertising is higher than pre-crisis advertising (Figure 4.12, c). The decision dilemma of choosing an appropriate advertising level in the two regimes is thus determined by the crisis likelihood, given a specific anticipated impact. As the impact increases, the manufacturer takes a cautious approach. Therefore, when the likelihood of crisis is low, the manufacturer invests in pre-crisis advertising to raise the brand image. Secondly, the manufacturer might invest more in quality costs



(a) Low Impact Recall



(b) Medium Impact Recall



(c) High Impact Recall

Figure 4.13: Variation of Advertising Efforts with Crisis Impact and Likelihood

4.4.4. Goodwill and Quality Level

The solutions of the state equations give us the goodwill and the quality trajectories. The generic solutions are:

$$G(t) = \frac{e^{-t\delta_j}\left(\tilde{G}\delta_j + \left(e^{t\delta_j}-1\right)(A_j k_{\text{gj}} + k_{1j}q_{1j} + k_{2j}q_{2j})\right)}{\delta_j} \tag{4.31}$$

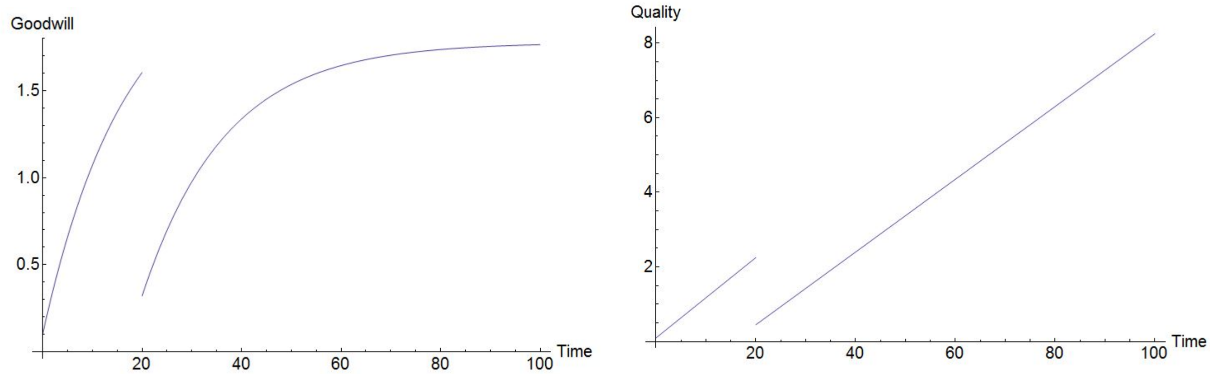
$$Q(t) = e^{-\lambda\chi}\left(Q_0 e^{\lambda\chi} + l_{21}q_{21}te^{\lambda\chi} + l_{11}q_{11}t\right) \text{ for } 0 \le t \le t_r \tag{4.32}$$

$$Q(t) = Q_{t_r} + l_{12}q_{12}t + l_{22}q_{22}t \text{ for } t_r < t < \infty \tag{4.33}$$

The time index is $j \in \{1,2\}$. For the pre-crisis period $\tilde{G} = G_0 = G(0)$ and for the post-crisis period, $\tilde{G} = G_{t_r} = G(t_r^+) = (1-\eta)G(t_r^-)$ are the required initial values of the goodwill required to obtain the solution of the above equations. Similarly, $Q(0) = Q_0$ and $Q_{tr} = Q(t_r^+) = (1-\xi)Q(t_r^-)$. For the numerical analysis we solve the state differential equations to find the state trajectories for four cases - *High impact High Likelihood Recall (*$\eta = \xi = .8, \chi = .8$*), High Impact Low Likelihood Recall (*$\eta = \xi = .8, \chi = .05$*), Low Impact High Likelihood Recall (*$\eta = \xi = .05, \chi = .8$*)* and *Low Impact Low Likelihood Recall (*$\eta = \xi = .05, \chi = .05$*)*. We have assumed the initial values $G(0) = .Q(0) = .1$ to numerically solve the differential state equations.

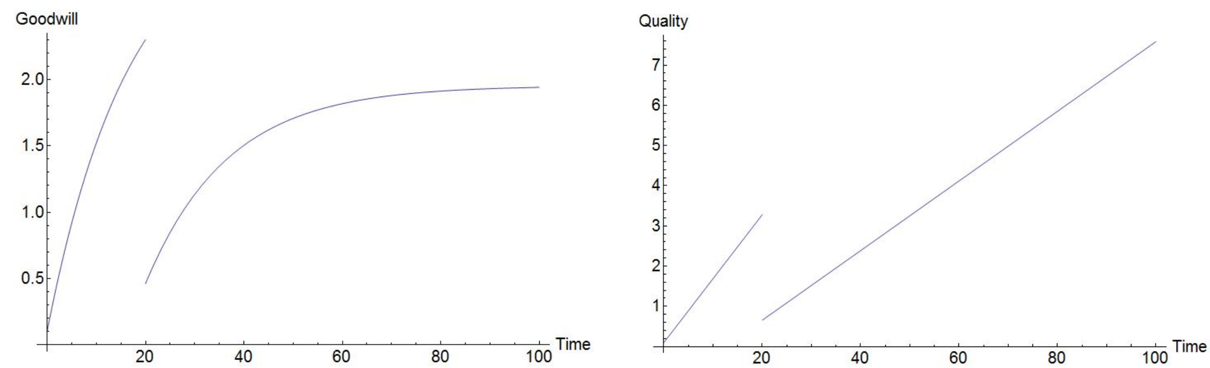
According to Figures 14 and 15 (where we compare the goodwill and quality levels of the players under a high impact recall) a low likelihood of a high impact recall can enhance pre-crisis goodwill and quality when compared to the high likelihood high impact crisis. A low likelihood crisis can possibly bring complacency and doubts about whether a recall will occur at all. In such cases, advertising and quality efforts of the firms are high, and consequently the state trajectories reach a higher level (recall that goodwill is affected by quality decisions as well).

On the other hand, Figures 16 and 17 shows that when recall probability is high, the suppliers and the manufacturer exhibit a cautious approach, possibly anticipating the post-recall quality improvement costs, and hence put lesser efforts during the pre-crisis period. Consequently the state trajectories and goodwill reach a lower level. The cautious approach however acts as a double edged sword - it takes the quality and goodwill to a lower level and as a consequence, the state jump due to recall brings the quality level and goodwill further down if the recall happens. Thus it will take longer for the firms to recover from crisis when the recall likelihood is higher and the recall actually occurs.



(a) High Impact High Probability Recall- Goodwill (b) High Impact High Probability Recall - Quality

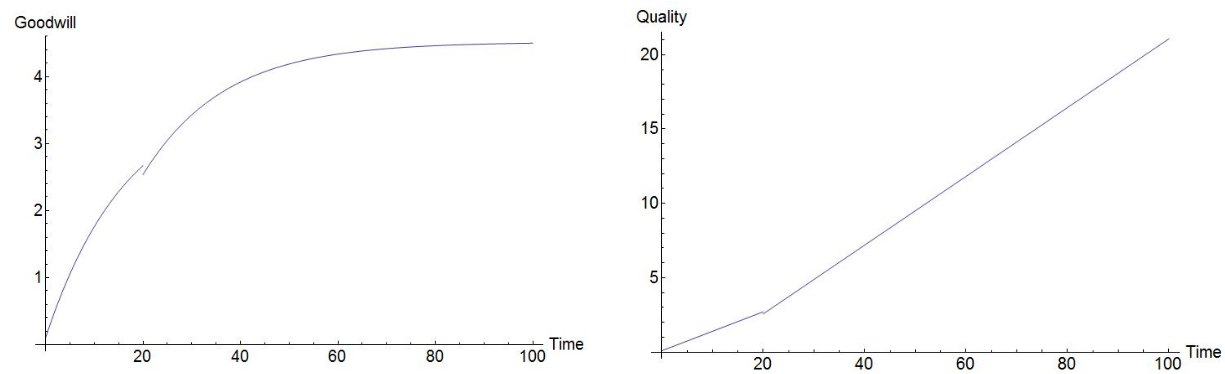
Figure 4.14: Goodwill and Quality for High Impact High Likelihood Recall



(a) High Impact Low Probability Recall- Goodwill (b) High Impact Low Probability Recall - Quality

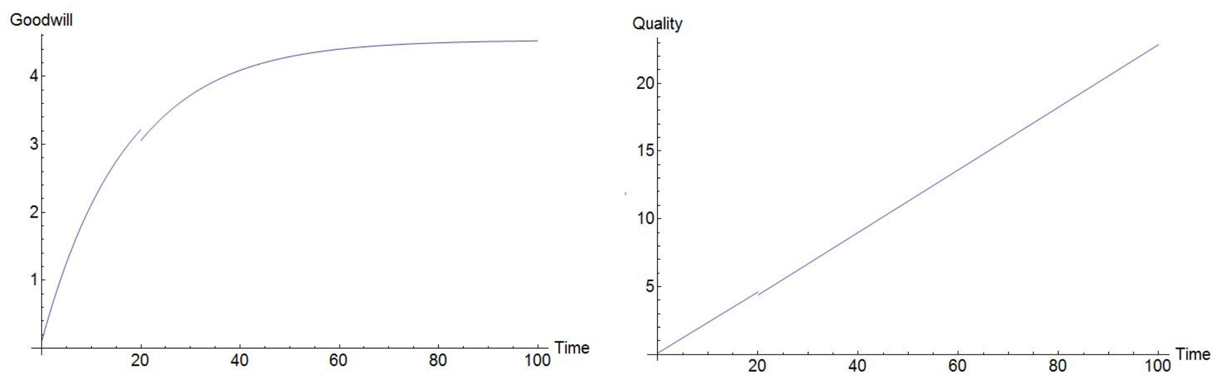
Figure 4.15: Goodwill and Quality for High Impact Low Likelihood Recall

When impact of the crisis is low, intuitively, the goodwill and quality levels attain a high value. The contrasts of quality levels are quite staggeringly different than the high impact case. In fact for our parameter values, the quality level for the low impact case, in the post-crisis period is almost three times the quality level in the high impact case. This underscores the importance of a recall impact. Even when a firm recovers from a high impact recall, recovery can be delayed and attaining the desired quality level may be difficult due to the losses already borne and the lack of sufficient budget to invest in quality. The above is also supported by the fact that the suppliers typically invest more in post-crisis quality for a low impact recall (Figure 4.5 and 4.6).



(a) Low Impact High Probability Recall- Goodwill (b) Low Impact High Probability Recall - Quality

Figure 4.16: Goodwill and Quality for Low Impact High Likelihood Recall



(a) Low Impact Low Probability Recall- Goodwill (b) Low Impact Low Probability Recall - Quality

Figure 4.17: Goodwill and Quality for Low Impact Low Likelihood Recall

4.4.5. Supplier's Negligence and Collateral Damage

Claim 5:*Higher negligence of crisis likelihood by the risky supplier results in greater collateral damage.*

In our study, we represent the risky supplier's degree of negligence of crisis by the parameter λ. We have shown that the risky supplier's degree of negligence can compel the manufacturer to change the cost-sharing decision. Another important question is to ask how negligence impacts the firm profits. We find that if supplier 1, the risky supplier, avoids quality investments or efforts and is not sensitive towards the crisis likelihood, the long term expected profit could indeed be impacted negatively.

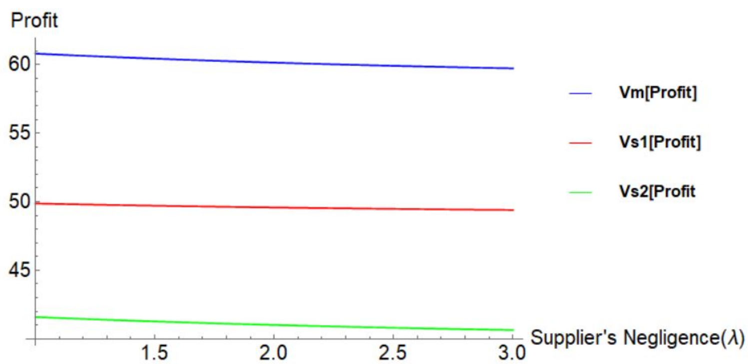


Figure 4.18: Negligence of unreliable supplier vs profit

Figure 4.17 shows the variation of the long term expected profits, Vm, Vs1 and Vs2 of the manufacturer, supplier 1 and the supplier 2, respectively with respect to the negligence of the supplier 1. The presence of collateral damage is very prominent. As negligence increases, the profit of each firm decreases. The experiment was conducted using a benchmark crisis likelihood of 30% ($\chi = .3$), margin loss of about 20% for supplier 1 and the manufacturer ($\hat{M}_{s1} = .7, M_{s1} = .56, \hat{M}_m = 1, M_m = .8$) , 20% loss of advertising effectiveness ($k_{g1} = 1, k_{g2} = .8$), no change in learning for supplier 1 ($l_{11} = l_{12} = 1$) and 20% decline is post-crisis consumer confidence in supplier 1's quality efforts ($k_{11} = 1, k_{12} = .8$).

4.4.6. Effect of Product Recall on Individual Firm's Profits

The value functions evaluated at time $t = 0$ gives the long term expected profit of the firms. However, due to the complexity of our parameter space, it is hard to find substantial analytical insights from the closed-form solutions of the value functions. To support our analysis, we further evaluate the value functions at $t = 0$ and assume different values of the initial goodwill $G(0)$ and initial quality level $Q(0)$. We find the profits for low impact and high impact recalls with different crisis likelihoods. Under three scenarios, QICS contract, CQCS contract and no cost-sharing (NCS), we consider the following cases under both high initial goodwill and low initial goodwill :

- $\chi = .05, \eta = .05, \xi = .05$ (low likelihood low impact recall)
- $\chi = .3, \eta = .05, \xi = .05$ (benchmark likelihood low impact recall)
- $\chi = .7, \eta = .05, \xi = .05$ (high likelihood low impact recall)
- $\chi = .05, \eta = .7, \xi = .7$ (low likelihood high impact recall)
- $\chi = .3, \eta = .7, \xi = .7$ (benchmark likelihood high impact recall)

- $\chi = .7, \eta = .7, \xi = .7$ (high likelihood high impact recall)

Figure 4.19: Profit Comparison

We observe that under any contract or even NCS scenario, higher initial goodwill can produce more long term expected profit for the manufacturer. The above finding is consistent with the previous research, which predicts the reduced effect of product harm crisis on brand goodwill loss for firms which have a high brand image. (Yi Zhao, Ying Zhao, and Helsen 2011; H. Van Heerde, Helsen, and Dekimpe 2007; Cleeren, H. J. Van Heerde, and Dekimpe 2013). However, the suppliers may not always observe a higher profit when the initial goodwill is high because the manufacturer is the market leader and adjusts the share of quality costs based on the contract. The suppliers' profits are dependent on this cost-sharing proportion and hence may not always be high with high initial goodwill.

When the crisis likelihood is low or benchmark, the QICS contract is the best option for all the three players and therefore coordinates the chain as it produces better results for all the channel members as compared to no cost-sharing contract (NCS) scenario, provided crisis impact is low. The supplier 2 is indifferent between the QICS and CQCS contracts under low impact crisis, but will not prefer NCS contract. Thus, the Cost-sharing mechanism with the faulty supplier controls the "collateral damage" caused to the manufacturer as well as the reliable supplier.

High crisis likelihood may fail to coordinate the supply chain as the manufacturer's best interest is to opt for the no Cost-sharing option as opposed to the best interests of supplier 1 and supplier 2, who would prefer a CQCS contract. However, our model assumes that the manufacturer needs the two suppliers, raw materials or products. Hence, if supplier 1 is reluctant to accept the NCS scenario either because he produces specialized parts or has some market power, supplier switching is an option. However, supplier switching might be costly and in some cases, beget more risks resulting in the buyer becoming inert to such switching decisions (Wagner and Friedl 2007). In such cases, depending on the power of the suppliers, the manufacturer may still offer a CQCS or QICS contract. The details of such discussion are out of the purview of our model. Alternatively, at the beginning of the planning horizon, the manufacturer may not even choose the supplier who is very risky and poses a threat of very high chance of recall.

4.4.7. Effect of Product Recall on Supply Chain Profit:

In the above discussions, we found that the manufacturer may be reluctant to offer a quality cost-sharing contract under a product recall risk, especially if the risk is high. We ask what happens to the supply chain profit under the different cost-sharing contracts and the NCS scenario. Tables 5 and 6 demonstrate the chain profits. We find that compared to an NCS scenario, the chain profit is always higher when the players adopt a cost-sharing contract. The enhanced profit with cost-sharing is another motivation for adopting a cost-sharing contract. Irrespective of the impact of the recall, the QICS contract seems better suited for the chain as it brings in more profit. Of course, a high impact recall results in much lesser profit. Like in our other cases of numerical analysis, we investigate three instances of recall likelihood,low (.05), benchmark (.3) and high (.7) and recall impact either low or high. Tables 4.3 and 4.4 show that no cost-sharing is detrimental to chain profit when QICS contract is available.

Low impact recall- chain profit			
Crisis Likelihood (χ)	**QICS**	**CQCS**	**NCS**
0.05	190.0954	183.1066	161.3526
0.3	188.2022	181.8044	180.4693
0.7	188.8215	183.2706	185.6084

Table 4.3: Chain Profit for Low Impact Recall

High impact recall- chain profit			
Crisis Likelihood (χ)	**QICS**	**CQCS**	**NCS**
0.05	99.2869	93.4773	83.4466
0.3	70.0033	67.181	68.7171
0.7	69.0765	66.5301	68.9245

Table 4.4: Chain Profit for High Impact Recall

Previous research (Rubel 2018; Mukherjee and Chauhan 2019) has found that under advertising competition, the profit of competing firms increase with a low crisis likelihood and decreases for high likelihood.

4.5. Conclusion and Managerial Implications

Cost-sharing in supply chains have evolved to be profitable and fruitful strategies for manufacturers and suppliers alike. In this paper, we have investigated two cost-sharing contracts - Quality Improvement Cost-sharing Contract and Comprehensive Quality Cost-sharing Contract offered by a manufacturer in the presence of a risky supplier, susceptible to recall a product. We emphasized the fact that a recall is sometimes unavoidable even if the manufacturer and supplier are aware of the possibility of a recall. Under such circumstances, we examine how the quality efforts of the suppliers, advertising effort of the manufacturer and the cost-sharing decisions of the manufacturer influence the equilibrium performance of the firms. We also draw insights on how the impact and likelihood of a recall influence the equilibrium decisions.

Our research highlights that during an impending product-recall, envisioned by supply chain partners, a cost-sharing contract can motivate the suppliers to increase quality effort in the pre-crisis regime and thereby lower the effect of the recall on goodwill, quality and consequently profit. Therefore, management of manufacturing firms might be interested in forming quality cost-sharing contracts even with risky suppliers.

However, for risky suppliers, a manufacturer should try to negotiate partial quality cost-sharing agreement instead of full quality cost-sharing as the former is more profitable for all players. A supplier's negligence of crisis should be proactively managed by all the chain members as the negligence results in "collateral damage" and is very detrimental to the profit of all players. If a supplier is prone to high impact recall, a manufacturer may not enter into a cost-sharing agreement with it. However, in presence of other contractual relationships which share chain profits, the manufacturer might enter into a cost-sharing contract because no cost sharing always result in worst chain profit irrespective of the magnitude of recall.

This research has some limitations and can be extended in several directions. We have not considered competition in our model. It will be interesting to test if the results of our research change in the presence of chain to chain competition or competition amongst multiple manufacturers with common suppliers. We have not assumed any decay for quality in our model. Such an assumption of quality decay may change some results of our study. It is also interesting to incorporate the effect of pricing decision along with quality cost-sharing in a model. The goodwill can also be affected by price (e.g.(Buratto, Cesaretto, and De Giovanni 2019)), and such a study would be very fruitful to provide insights on multiple decision making during a product recall. Since we found that even during product recalls, chain profit is significantly higher for cost-sharing contracts, investigating a cooperative game involving profit sharing might be an interesting research direction.

Chapter 5

Conclusions and Future Research

This dissertation investigates several important decision-making problems in the areas of advertising, pricing and quality investments when a product recall is envisioned by a firm, its competitors or other supply chain partners. We examined three key issues :

- Advertising in a duopoly setting during a product recall.
- Joint pricing and advertising to counter the spillover effect due to a product recall.
- Join collaboration on quality and its impact on the product recall.

Under differential game-theoretic framework, we proposed different models which augment the extant literature by validating the previous findings ((Rubel, Naik, and Srinivasan 2011; Borah and Tellis 2016; Gao et al. 2015; Chao, Iravani, and Savaskan 2009)) or proposed some new novel insights and results.

Our findings from the first essay mainly highlight that advertising effectiveness, crisis likelihood, crisis impact, brand forgetting, goodwill sensitivity of the consumers and competition are some of the critical factors determining the equilibrium advertising and pricing strategies of the firms competing during a product recall. The anticipation of a product recall and estimation of its impact are of utmost importance to a firm as well as its rival. Therefore, firms should try to collect information about the factors that affect recall chances for itself and the competitors alike. Our results establish a few key deciding factors and threshold values of crisis impact and likelihood-based on which competing firms can adjust pre-crisis and post-crisis advertising. We show how profits grow or decline under product recalls of different intensities.

Our model can be extended in several ways. First, recovery can be considered in the model where a firm recovers from the lost image after a short while instead of having a lingering negative effect. Second, a hazard rate following Weibull distribution with suitable parameters may be considered to incorporate that as time passes the chance of a recall decreases. Such a hazard distribution can potentially change our findings. Third, one may consider the diminishing effect of advertising returns. It will be an interesting exercise to verify if returns on advertising reduce with time, how the firms would adjust their level of advertising in the two regimes. Lastly, cross-firm effects of advertising can also be considered in the model to verify the findings of the previous literature.

In the second essay, we have considered two competing firms, one of which is the focal firm issuing a recall, and its competitor is the non-focal firm suffering from a negative spillover effect. We investigated the impact of the recall on the pricing and advertising decisions and the profit of the firms. We found that the immediate decrease or increase in pricing and advertising by the firms depends on the firms' goodwill just before the recall. We also showed how the equilibrium policies vary with time and state variables (goodwill of the firms) in the pre-crisis and post-crisis regimes. Our recommended policies depend on the likelihood and impact of the recall. We observed that the policies vary significantly in the two cases - (i) focal firm is a Stackelberg leader, (ii) two firms take simultaneous decisions under a Nash game.

The direct extension of our model is the examination of the positive spillover effect of a product recall which can potentially benefit a rival non-focal firm from a different country of origin. Moreover, a study can incorporate a short-lived spillover and three regimes of analysis - pre-crisis, spillover and post-crisis in a suitable model. However, in such case, it will be challenging to derive the feedback strategies given the random duration of the three epochs of the models.

The third essay is different from the first two, primarily because the unit of analyses are vertical members of a supply chain. We consider advertising and quality cost-sharing as the decisions of the manufacturer and quality efforts as the decision of the suppliers. In the presence of a risky supplier, who can trigger a product recall by supplying faulty parts, we found that a partial cost-sharing mechanism works better for the firms' profit most of the times. We also recommended various cost-sharing and quality effort policies for recalls of different likelihood and impact. A no cost-sharing may be a profitable option for a manufacturer when a risky supplier is present; however, a contractual agreement always enhances the chain profit.

Our work can be extended in several directions. One obvious extension is the inclusion of other contracts. A cooperative profit-sharing game would also be interesting because it would provide a mechanism of splitting the chain profit into fair shares in the presence of risky firms. Other extensions include the introduction of chain to competition or the presence of competing risky suppliers. Competition amongst suppliers can significantly change some of the results of our model.

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Appendices

A. Appendix

DERIVATION OF EQUATION (1).

The expectation in the following equation (6) can be expanded

$$\Pi(A_{i1},A_{i2}) = E[\int_0^{\infty} e^{-rs}\pi(s)ds + e^{-rt}\pi_{i2}(t)] \text{ where } i \in \{1,2\}$$

and the equation can be re written as (7)

$$\Pi(A_{i1},A_{i2},\chi) = \int_0^{\infty} e^{-(r+\chi)t}\{\pi(A_{i1}(t),\theta_1(t),\theta_2(t)) + \chi\pi_{i2}(A_{i2}(t))\}dt$$

This has been proved in (Rubel, Naik, and Srinivasan 2011) and hence we omit the proof. Euqtion (16) comes from using the same logic or derivation as above.
We show that the following two equations are equivalent:

$$\begin{aligned} V_{i1}(\theta_1,\theta_{2i}) = \underset{A_{i1}}{\text{Max}} \underset{R_i,R_j}{\mathbb{E}} \Big(\int_0^{\min\{t_i,t_j\}} & e^{-rt}\Pi_{i1}(t)dt + e^{-rt_i}V_{i2}(\theta_i,\theta_j).\boldsymbol{\Phi}[\boldsymbol{t_i < t_j}] \\ & + e^{-rt_j}\hat{V}_{i2}(\theta_i,\theta_j).\boldsymbol{\Phi}[\boldsymbol{t_j < t_i}]\Big) \\ V_{i1}(\theta_i,\theta_j) = \underset{A_{i1}}{\text{Max}} \int_0^{\infty} & e^{-(r+\chi_i+\chi_j)t}\{\Pi_{i1}(t)dt + \chi_i V_{i2}((1-\eta_i)\theta_i,\theta_j) \\ & + \chi_j V_{i2}(\theta_i,(1-\eta_j)\theta_j)\} \end{aligned}$$

Let $z = Min\{t_i,t_j\}$. Since the occurrence of the two recalls are assumed to be independent exponential stochastic processes,

$$\begin{aligned} f(t_i,\chi_i) &= \chi_i e^{-\chi_i t} \;\&\; f(t_j,\chi_j) = \chi_j e^{-\chi_j t} \\ \implies Pr(min\{t_i,t_j\} > t) &= Pr(t_i > t)Pr(t_j > t) \\ \implies Pr(min\{t_i,t_j\} > t &= e^{-t(\chi_i+\chi_j)} \end{aligned}$$

Using this result (A.5), equation (16) and after some algebraic manipulations, equation (A.3) reduces to equation (A.4). ■

PROOF OF PROPOSITION 1. We proof the proposition by solving the problem for post-crisis regime first and then for pre-crisis regime by using Hamilton-Jacobi-Bellman (HJB) equations.
(a) Using equation (13), the HJB equations for manufacturers M_1 and M_2 in the second regime

(post-crisis period), respectively, can be given by :

$$
\begin{aligned}
rV_{12}(\theta_1(t),\theta_2(t)(t)) &= \underset{A_{12}}{\mathrm{Max}}[(\alpha_1+\beta_1(\theta_1(t)-\theta_2(t)))m_{12}-\frac{\mu_1}{2}A_{12}^2+\frac{\partial V_{12}}{\partial\theta_1(t)}\dot{\theta}_1(t) \\
&\quad +\frac{\partial V_{12}}{\partial\theta_2(t)}\dot{\theta}_2(t)], \qquad (A.1) \\
rV_{22}(\theta_1(t),\theta_2(t)(t))] &= \underset{A_{22}}{\mathrm{Max}}[(\alpha_2+\beta_2(\theta_2(t)-\theta_1(t)))m_{22}-\frac{\mu_2}{2}A_{22}^2+\frac{\partial V_{22}}{\partial\theta_1(t)}\dot{\theta}_1(t) \\
&\quad +\frac{\partial V_{22}}{\partial\theta_2(t)}\dot{\theta}_2(t)]. \qquad (A.2)
\end{aligned}
$$

For maximization, differentiating the right side of $(A.1)$ with respect to (w.r.t.) A_{12} and equating to 0 we get, $-\mu_1 A_{12}+k_{12}\frac{\partial V_{12}}{\partial\theta_1}=0$, since $\dot{\theta}_1(t)=k_{12}A_{12}(t)-\delta_1\theta_1(t)$. Therefore $A_{12}^*=\frac{k_{12}}{\mu_1}\frac{\partial V_{12}}{\partial\theta_1}$. Similarly, differentiating the right side of $(A.2)$ w.r.t. A_{22} and equating the same to 0, we get $A_{22}^*=\frac{k_{22}}{\mu_2}\frac{\partial V_{22}}{\partial\theta_2}$.

We make an informed guess that the value function of the i^{th} manufacturer in the j^{th} regime is linear in the state variables θ_i and is of the form (method of undeterrmined coefficients):

$$
V_{ij}(\theta_1(t),\theta_2(t)(t)) = a_{ij}\theta_1(t)+b_{ij}\theta_2(t)+c_{ij} \qquad (A.3)
$$

Thus, $\frac{\partial V_{ij}}{\partial\theta_1}=a_{ij}$ and $\frac{\partial V_{ij}}{\partial\theta_2}=b_{ij}$. Therefore,

$$
\begin{aligned}
A_{12}^* &= \frac{k_{12}}{\mu_1}a_{12} \qquad (A.4) \\
A_{22}^* &= \frac{k_{22}}{\mu_2}b_{22} \qquad (A.5)
\end{aligned}
$$

Substituting $(A.3)$, $(A.4)$ and $(A.5)$ in equations $(A.1)$ and $(A.2)$, we do some algebraic manipulations and compare the coefficients of the state variables to find the values of the coefficients. This gives us two sets of linear equations, one set for M_1 and the other one for M_2. Thus,

$$
\begin{aligned}
ra_{12} &= m_{12}\beta_1 - a_{12}\delta_{12}, \\
rb_{12} &= -m_{12}\beta_1 - b_{12}\delta_{22}, \\
rc_{12} &= m_{12}\alpha_1 - \frac{\mu_1(k_{12}a_{12})^2}{2\mu_1^2}+\frac{(a_{12}k_{12})^2}{\mu_1}+\frac{b_{12}k_{22}b_{22}}{\mu_2}, \qquad (A.6)
\end{aligned}
$$

and

$$
\begin{aligned}
ra_{22} &= -m_{22}\beta_2 - a_{22}\delta_{12},\\
rb_{22} &= m_{22}\beta_1 - b_{22}\delta_{22},\\
rc_{22} &= \alpha_2 m_{22} - \frac{\mu_2(k_{22}b_{22})^2}{2\mu_2^2} + \frac{k_{12}^2 a_{12}a_{22}}{\mu_1} + \frac{(k_{22}b_{22})^2}{2\mu_2}. \qquad \text{(A.7)}
\end{aligned}
$$

Substituting a_{12} and b_{22} from (A.6) and (A.7) into in (A.4) and (A.5), respectively, we get the post-crisis equilibrium advertising efforts of manufacturers M_1 and M_2 as

$$
\begin{aligned}
A_{12}^* &= \frac{k_{12}m_{12}(\beta_1)}{\mu_1(r+\delta_{12})},\\
A_{22}^* &= \frac{k_{22}m_{22}(\beta_2)}{\mu_2(r+\delta_{22})}. \qquad \text{(A.8)}
\end{aligned}
$$

This completes the proof of Proposition 1 (a).

(b) Using equation (12), the pre-crisis HJB equations for the manufacturers M_1 and M_2 are, respectively, given by:

$$
\begin{aligned}
(r+\chi)V_{11}(\theta_1,\theta_2) &= \underset{A_{11}}{\text{Max}}[(\alpha_1+\beta_1(\theta_1-\theta_2))m_{11} - \frac{\mu_1}{2}A_{11}^2 + \frac{\partial V_{11}}{\partial \theta_1}\dot{\theta}_1(t)\\
&\quad + \frac{\partial V_{11}}{\partial \theta_2}\dot{\theta}_2(t) + \chi V_{12}((1-\eta)\theta_1,\theta_2)], \qquad \text{(A.9)}\\
(r+\chi)V_{21}(\theta_1,\theta_2) &= \underset{A_{21}}{\text{Max}}[(\alpha_2+\beta_2(\theta_2-\theta_1))m_{21} - \frac{\mu_2}{2}A_{21}^2 + \frac{\partial V_{21}}{\partial \theta_1}\dot{\theta}_1(t)\\
&\quad + \frac{\partial V_{21}}{\partial \theta_2}\dot{\theta}_2(t) + \chi V_{22}((1-\eta)\theta_1,\theta_2)]. \qquad \text{(A.10)}
\end{aligned}
$$

Differentiating with respect to the decision variables A_{11} and A_{21} and equating the result to 0, we get:

$$
A_{11}^* = \frac{k_{11}}{\mu_1}a_{11}, \qquad \text{(A.11)}
$$

$$
A_{21}^* = \frac{k_{21}}{\mu_2}b_{21}. \qquad \text{(A.12)}
$$

Using $(A.3)$, equations $(A.9)$ and $(A.10)$ can be rewritten as :

$$
\begin{aligned}
(r+\chi)(a_{11}\theta_1+b_{11}\theta_2+c_{11}) &= \operatorname*{Max}_{A_{11}}[(\alpha_1+\beta_1(\theta_1-\theta_2))m_{11}-\frac{\mu_1}{2}A_{11}^2+\frac{\partial V_{11}}{\partial\theta_1}\dot{\theta}_1(t) \\
&\quad +\frac{\partial V_{11}}{\partial\theta_2}\dot{\theta}_2(t)+\chi(a_{12}(1-\eta)\theta_1+b_{12}\theta_2 \\
&\quad +c_{12})], \qquad (A.13)\\
(r+\chi)(a_{21}\theta_1+b_{21}\theta_2+c_{21}) &= \operatorname*{Max}_{A_{21}}[(\alpha_2+\beta_2(\theta_2-\theta_1))m_{21}-\frac{\mu_2}{2}A_{21}^2+\frac{\partial V_{21}}{\partial\theta_1}\dot{\theta}_1(t) \\
&\quad +\frac{\partial V_{21}}{\partial\theta_2}\dot{\theta}_2(t)+\chi(a_{22}(1-\eta)\theta_1+b_{22}\theta_2 \\
&\quad +c_{22})]. \qquad (A.14)
\end{aligned}
$$

Comparing the coefficients of the state variables θ_i and ρ of the equation $(A.13)$ and $(A.14)$, we have six equations:

$$
\begin{aligned}
(r+\chi)a_{11} &= \beta_1 m_{11}-\delta_{11}a_{11}+\chi(1-\eta)a_{12},\\
(r+\chi)b_{11} &= -\beta_1 m_{11}-\delta_{21}b_{11}+\chi b_{12},\\
(r+\chi)c_{11} &= m_{11}\alpha_1-\frac{\mu_1(k_{11}a_{11})^2}{2\mu_1^2}+\frac{(a_{11}k_{11})^2}{\mu_1}+\frac{b_{11}k_{21}b_{21}}{\mu_2}+\chi c_{12},\\
(r+\chi)a_{21} &= -\beta_1 m_{21}-\delta_{11}a_{21}+\chi(1-\eta)a_{22},\\
(r+\chi)b_{21} &= \beta_1 m_{21}-\delta_{21}b_{21}+\chi b_{22},\\
(r+\chi)c_{21} &= \alpha_2 m_{21}-\frac{\mu_2(k_{21}b_{21})^2}{2\mu_2^2}+\frac{k_{11}^2a_{11}a_{21}}{\mu_1}+\frac{(k_{21}b_{21})^2}{2\mu_2}+\chi c_{22}. \qquad (A.15)
\end{aligned}
$$

Substituting the values of a_{11} and b_{21} we get the equilibrium advertising efforts in the first period (pre-crisis regime) as

$$
\begin{aligned}
A_{11}^* &= \frac{k_{11}\beta_1\big(m_{11}(r+\delta_{12})+m_{12}(1-\eta)\chi\big)}{\mu_1(r+\chi+\delta_{11})(r+\delta_{12})}, \qquad (A.16)\\
A_{21}^* &= \frac{k_{21}\beta_2\big(m_{21}(r+\delta_{22})+\chi m_{22}\big)}{\mu_2(r+\chi+\delta_{21})(r+\delta_{22})}.
\end{aligned}
$$

Hence, the Proposition 1 (b). This completes the proof of Proposition 1. ■

PROOF OF LEMMA 1. The expressions $(A.8)$ of Proposition 1 give the equilibrium advertising efforts of both manufacturers in post-crisis period. To establish this lemma, the first order condition is checked for each of the variables m_{i2}, β_i, δ_{i2} and r.

$$\frac{\partial A_{i2}^*}{\partial_{i2}} = \frac{k_{i2}\beta_i}{\mu_1(r+\delta_{i2})} > 0$$
$$\frac{\partial A_{i2}^*}{\partial \beta_i} = \frac{k_{i2}m_{i2}}{\mu_1(r+\delta_{i2})} > 0$$
$$\frac{\partial A_{i2}^*}{\partial k_{i2}} = \frac{m_{i2}\beta_i}{\mu_1(r+\delta_{i2})} > 0$$
$$\frac{\partial A_{i2}^*}{\partial r} = -\frac{k_{i2}m_{i2}\beta_i}{\mu_1(r+\delta_{i2})^2} < 0$$
$$\frac{\partial A_{i2}^*}{\partial \delta_1} = -\frac{k_{i2}m_{i2}(\beta_i)}{\mu_1(r+\delta_1)^2} < 0$$

Therefore, A_{i2} is decreasing with r and δ_i and increasing with the other parameters. This completes the proof of Lemma 1. ■

PROOF OF LEMMA 2.

$Expressing the first period advertising as a function of \chi$, we get,

$$A_{11}^* = \frac{k_{11}\beta_1\big(m_{11}(r+\delta_{12})+m_{12}(1-\eta)\chi\big)}{\mu_1(r+\chi+\delta_{11})(r+\delta_{12})}$$

We take the first derivative of the advertising with respect to χ and check when the derivative is negative i.e., ${A_{11}^*}'(\chi) < 0$.

$${A_{11}^*}'(\chi) = -\frac{k_{11}\beta_1\big(m_{11}(r+\delta_{12})-m_{12}(1-\eta)(r+\delta_{11})\big)}{\mu_1(r+\chi+\delta_{11})^2(r+\delta_{12})}$$

${A_{11}^*}'(\chi) < 0$ if $k_{11}\beta_1\big(m_{11}(r+\delta_{12})-m_{12}(1-\eta)(r+\delta_{11})\big) > 0$. After algebraic manipulations this gives,

$$\eta > 1 - \frac{m_{11}(r+\delta_{12})}{m_{12}(r+\delta_{11})}$$

(b) The first derivative of the pre-crisis optimal advertising of the non-focal firm yields

$$A_{21}^{'*(\chi)} = \frac{k_{21}\beta_2\big(m_{22}(r+\delta_{21}) - m_{21}(r+\delta_{22})\big)}{\mu_2(r+\chi+\delta_{21})^2(r+\delta_{22})}$$

The expression is surely negative if $m_{22}(r+\delta_{21}) < m_{21}(r+\delta_{22})$. ■

PROOF OF LEMMA 3. The first expression of $(A.16)$ in Proposition 1 gives the equilibrium advertising effort A_{11}^* for the focal firm in pre-crisis regime. Similar to Lemma 1, the first order partial differentiation of the equilibrium advertising, A_{11}^* with respect to η gives

$$\frac{A_{11}^*}{\partial \eta} = \frac{-k_{11}\beta_1 m_{12}\eta\chi}{\mu_1(r+\chi+\delta_{11})(r+\delta_{12})} \qquad < 0$$

(since all the parameters in the expression are positive). Thus, the equilibrium advertising effort of the focal firm in the pre-crisis regime is decreasing with the damaging effect η. ■

PROOF OF PROPOSITION 2. (a) From Proposition 1, the equilibrium advertising efforts of the focal firm M_1 for the post-crisis and pre-crisis regimes are A_{12}^* and A_{11}^* as given by (19) and (20), respectively. Let us define $\Gamma_1 = A_{12}^* - A_{11}^*$. Then after some simplification we have

$$\begin{aligned}
\Gamma_1 &= A_{12}^* - A_{11}^* \\
&= \frac{k_{12}m_{12}\beta_1}{\mu_1(r+\delta_{12})} - \frac{k_{11}\beta_1\big(m_{11}(r+\delta_{12}) + m_{12}(1-\eta)\chi\big)}{\mu_1(r+\chi+\delta_{11})(r+\delta_{12})} \\
&> 0 \text{ if } k_{12}m_{12} > \frac{k_{11}(m_{11}(r+\delta_{12}) + m_{12}(1-\eta)\chi)}{(r+\chi+\delta_{11})} \\
\implies \chi &> \frac{k_{11}m_{11}(r+\delta_{12}) - k_{12}m_{12}(r+\delta_{11})}{m_{12}\big(k_{12} - k_{11}(1-\eta)\big)}. \qquad (A.17)
\end{aligned}$$

This establishes the first part of Proposition 2.

(b) In a similar way, if we define $\Gamma_2 = A_{22}^* - A_{21}^*$, where A_{22}^* and A_{21}^* are the equilibrium advertising efforts of the non focal firm M_2, then after some simplification we have

$$\begin{aligned}
\Gamma_2 &= A_{22}^* - A_{21}^* \\
\implies \frac{k_{22}m_{22}\beta_2}{\mu_2(r+\delta_{22})} - \frac{k_{21}\beta_2\big(m_{21}(r+\delta_{22}) + \chi m_{22}\big)}{\mu_2(r+\chi+\delta_{21})(r+\delta_{22})} &> 0 \\
\implies \chi &> \frac{k_{21}m_{21}(r+\delta_{22}) - k_{22}m_{22}(r+\delta_{21})}{m_{22}(k_{21} - k_{22})}. \qquad (A.18)
\end{aligned}$$

This completes the proof of the proposition. ■

PROOF OF PROPOSITION 3. To establish the results of Proposition 3 we use the results which we derived in Proof of Proposition 1. The value function of manufacturer M_1 in the post-crisis (2nd regime) and pre-crisis (1st regime) periods can be expressed as given by the expression $(A.3)$ and is $V_{1j} = a_{1j}\theta_1(t) + b_{1j}\theta_2(t) + c_{1j}$ for $j = 1,2$, where V_{ij} satisfies the HJB equations as given by $(A.1)$ and $(A.9)$. Hence, comparing the corresponding coefficients of the state variables of the HJB equations $(A.1)$ and $(A.9)$ we obtain the relations as given in $(A.6)$, $(A.7)$ and $(A.15)$for manufacturer M_1 for post-crisis and pre-crisis regime, respectively. After algebraic manipulations, we get the solutions to a_{1j}, b_{1j}, c_{1j} for $j \in \{1,2\}$ as given in (21) and (22).
We show the algebraic manipulation to obtain a_{12}. The other coefficients can be similarly obtained.

$$\begin{aligned} ra_{12} &= m_{12}\beta_1 - a_{12}\delta_{12}, \qquad \text{(A.19)} \\ => a_{12} &= \frac{m_{12}(\beta_1)}{(r+\delta_{12})} \\ rb_{12} &= -m_{12}\beta_1 - b_{12}\delta_2, \\ rc_{12} &= \frac{m_{12}\left(2\alpha_1 + \beta_1\left(\frac{k_{12}^2 m_{12}\beta_1}{(r+\delta_{12})^2\mu_1} - \frac{2k_{22}^2 m_{22}\beta_2}{(r+\delta_{22})^2\mu_2}\right)\right)}{2}. \qquad \text{(A.20)} \end{aligned}$$

■

PROOF OF PROPOSITION 4. Proof of Proposition 4 is similar to the proof of Proposition 3. ■

PROOF OF PROPOSITION 5.

The proof of this proposition is similar to the proof of Proposition 1. The only difference is that in each period the firm visualizes an infinite horizon decision problem without considering the crisis likelihood. ■

PROOF OF PROPOSITION 6.

The proof of proposition 6 is tedious. The proof follows from the derivation of the HJB equations for M_i in the 3 regimes - regime before any firm recalls, regime between the two recalls and regime after the second recall. We proof the proposition in the following steps:

1. Find the HJB equations of the firms M_i, M_j in the third period.

2. From the first order conditions of the HJB equations with respect to the advertising decisions, find the equilibrium policy.

3. By inspection guess a linear value function structure.

4. Plug in the optimal policies in the firms' respective value functions and compare coefficients with the guessed linear value function.

5. Repeat the above steps 1-4 for HJB equations in Regime 2 when M_i recalls first.

6. Repeat the above steps 1-4 for HJB equations in Regime 2 when M_i recalls second.

7. Repeat the above steps 1-4 for HJB equations in Regime 1.

The above steps will give us the value functions and hence by finding the coefficients of the value functions we can determine the optimal advertising policies which depend on these coefficients.

Remark:*Here, to prove the propositions we need to solve for the first order condition with respect to advertising for both the firms M_i and M_j. However, for brevity we show only the calculations for M_i. The calculations for M_j are similar.*

The third regime HJB equations are identical to the second regime HJB equations in the case of one focal firm and the proof is therefore similar to the proof of proposition 1.
The HJB equation for M_i & M_j are given by :

$$
\begin{aligned}
rV_{i3}(\theta_i,\theta_j) = & \quad \underset{A_{i3}}{Max}\Big[\big(\alpha_i+\beta_i(\theta_i-\theta_j)\big)m_{i3}-\frac{\mu_i A_{i3}^2}{2}+\frac{\partial V_{i3}}{\partial \theta_i}\dot{\theta}_i+\frac{\partial V_{i3}}{\partial \theta_j}\dot{\theta}_j\Big], \\
rV_{j3}(\theta_i,\theta_j) = & \quad \underset{A_{j3}}{Max}\Big[\big(\alpha_j+\beta_j(\theta_j-\theta_i)\big)m_{j3}-\frac{\mu_j A_{j3}^2}{2}+\frac{\partial V_{j3}}{\partial \theta_i}\dot{\theta}_i+\frac{\partial V_{j3}}{\partial \theta_j}\dot{\theta}_j\Big].
\end{aligned}
$$

We assume the following form of the value functions:
We assume that the value functions are linear:

$$
\begin{aligned}
V_{i3}(\theta_i,\theta_j) &= a_{i3}\theta_i+b_{i3}\theta_j+c_{i3}, \\
V_{j3}(\theta_i,\theta_j) &= a_{j3}\theta_i+b_{j3}\theta_j+c_{j3}.
\end{aligned}
$$

Taking the first order condition of the HJB equations with respect to the advertising decisions, we find the equilibrium advertising decisions which on simplification gives:

$$A^*{}_{i3} = \frac{k}{\mu_i}\frac{\partial V_{i3}}{\partial \theta_i} = \frac{ka_{i3}}{\mu_i}.$$

We solve for the coefficients of the value functions. After algebraic manipulations and comparing the coefficients of θ_i, θ_j and ρ_i, we get the coefficients of the value functions of the third regime:

$$\begin{aligned} a_{i3} &= \frac{m_{i3}\beta_i}{(r+\delta_{i3})}, \\ b_{i3} &= \frac{-m_{i3}\beta_i}{(r+\delta_{j3})}, \\ c_{i3} &= \frac{m_{i3}\alpha}{r}, \\ . \end{aligned}$$

Consequently, the equilibrium advertising decision is

$$A^*_{i3} = \frac{ka_{i3}}{\mu_i} = \frac{k_{i3}m_{i3}\beta_i}{\mu_i(r+\delta_{i3})}.$$

For the second period the HJB equations will depend on who recalls first. At the beginning of the second period one of the recalls become a certainty and both firms know who has recalled. If M_i recalls first, the HJB equations of the two firms in the second regime are given by:

$$\begin{aligned} (r+\chi_j)V_{i2}(\theta_i,\theta_j) = \underset{A_{i2}}{Max}\Big[\big(\alpha_i+\beta_i(\theta_i-\theta_j)\big)m_{i2} - \frac{\mu_i A_{i2}^2}{2} + \frac{\partial V_{i2}}{\partial \theta_i}\dot{\theta}_i + \frac{\partial V_{i2}}{\partial \theta_j}\dot{\theta}_j \\ +\chi_j V_{i3}(\theta_i,(1-\eta_j)\theta_j)\Big], \\ (r+\chi_j)V_{j2}(\theta_i,\theta_j) = \underset{A_{j2}}{Max}\Big[\big(\alpha_j+\beta_j(\theta_j-\theta_i)\big)m_{j2} - \frac{\mu_j A_{j2}^2}{2} + \frac{\partial V_{j2}}{\partial \theta_i}\dot{\theta}_i + \frac{\partial V_{j2}}{\partial \theta_j}\dot{\theta}_j \\ +\chi_j V_{j3}(\theta_i,(1-\eta_j)\theta_j)\Big]. \end{aligned}$$

Proceeding similarly as the proofs above we let:

$$\begin{aligned} V_{i2}(\theta_i,\theta_j) &= a_{i2}\theta_i + b_{i2}\theta_j + c_{i2}, \\ V_{j2}(\theta_i,\theta_j) &= a_{j2}\theta_i + b_{j2}\theta_j + c_{j2}. \end{aligned}$$

Comparing the coefficients of the state variables and the constant term, we get:

$$\begin{aligned} a_{i2} &= \left(\beta_i m_{\text{i2}} + \chi_j a_{\text{i3}}\right) / \left(r + \chi_j + \delta_{i2}\right), \\ b_{i2} &= -\left(\beta_i m_{\text{i2}} - \chi_j \left(1-\eta_j\right) b_{\text{i3}}\right) / \left(r + \chi_j + \delta_{j2}\right), \\ c_{i2} &= \left(\alpha m_{\text{i2}} + \chi_j c_{\text{i3}}\right) / \left(r + \chi_j\right) \end{aligned}$$

Thus if M_i recalls first, the equlibrium decision is :

$$A^*_{\text{i2}} = \frac{k_{i2}\left(\beta_i m_{\text{i2}} + \chi_j a_{\text{i3}}\right)}{\left(\mu_i\left(r + \chi_j + \delta_{i2}\right)\right)}.$$

If M_i recalls second, the HJB equations of the two firms in the second regime are given by:

$$(r+\chi_i)\hat{V}_{i2}(\theta_i,\theta_j) = \underset{A_{i2}}{Max}\Big[\big(\alpha_i + \beta_i(\theta_i-\theta_j)\big)\hat{m}_{i2} - \frac{\mu_i\hat{A}^2_{i2}}{2} + \frac{\partial\hat{V}_{i2}}{\partial\theta_i}\dot{\theta}_i + \frac{\partial\hat{V}_{i2}}{\partial\theta_j}\dot{\theta}_j + \chi_j V_{i3}((1-\eta_i)\theta_i,\theta_j)\Big],$$

$$(r+\chi_i)\hat{V}_{j2}(\theta_i,\theta_j) = \underset{A_{j2}}{Max}\Big[\big(\alpha_j + \beta_j(\theta_j-\theta_i)\big)\hat{m}_{j2} - \frac{\mu_j\hat{A}^2_{j2}}{2} + \frac{\partial\hat{V}_{j2}}{\partial\theta_i}\dot{\theta}_i + \frac{\partial\hat{V}_{j2}}{\partial\theta_j}\dot{\theta}_j + \chi_j V_{j3}((1-\eta_i)\theta_i,\theta_j)\Big].$$

Proceeding similarly as the proofs above we let:

$$\begin{aligned} \hat{V}_{i2}(\theta_i,\theta_j) &= \hat{a}_{i2}\theta_i + \hat{b}_{i2}\theta_j + \hat{c}_{i2}, \\ \hat{V}_{j2}(\theta_i,\theta_j) &= \hat{a}_{j2}\theta_i + \hat{b}_{j2}\theta_j + \hat{c}_{j2}. \end{aligned}$$

Comparing the coefficients of the state variables and the constant term, we get:

$$\hat{a}_{i2} = \frac{(\beta_i \hat{m}_{i2} + \chi_i(1-\eta_i)a_{i3})}{(r+\chi_i+\hat{\delta}_{i2})},$$
$$\hat{b}_{i2} = \frac{-(\beta_i \hat{m}_{i2} - \chi_i b_{i3}}{(r+\chi_i+\hat{\delta}_{j2})},$$
$$\hat{c}_{i2} = \frac{(-\mu_i \frac{\hat{A}^{*\,2}_{i2}}{2} + \chi_i d_{i3} + \hat{a}_{i2}k_{i2}\hat{A}^*_{i2} + \hat{b}_{i2}k\hat{A}^*_{j2})}{(r+\chi_i)}.$$

Thus the equilibrium decision is :

$$\hat{A}^*_{i2} = \frac{k}{\mu_i(r+\chi_i+\hat{\delta}_{i2})}[\beta_i \hat{m}_{i2} + \chi_i(1-\eta_i)a_{i3}] \text{ if manufacturer } i \text{ recalls second}$$

From equations (17) and (18), the first period HJB equations of the two firms are:

$$(r+\chi_i+\chi_j)V_{i1}(\theta_i,\theta_j) = \underset{A_{i1}}{Max}\Big[\big(\alpha_i+\beta_i(\theta_i-\theta_j)\big)m_{i1} - \frac{\mu_i A^2_{i1}}{2} + \frac{\partial V_{i1}}{\partial \theta_i}\dot{\theta}_i + \frac{\partial V_{i1}}{\partial \theta_j}\dot{\theta}_j + \chi_i V_{i2}(\theta_i,\theta_j) + \chi_j \hat{V}_{i2}(\theta_i,\theta_j)\Big],$$
$$(r+\chi_i+\chi_j)V_{j1}(\theta_i,\theta_j) = \underset{A_{j1}}{Max}\Big[\big(\alpha_j+\beta_j(\theta_j-\theta_i)\big)m_{j2} - \frac{\mu_j A^2_{j1}}{2} + \frac{\partial V_{j1}}{\partial \theta_i}\dot{\theta}_i + \frac{\partial V_{j1}}{\partial \theta_j}\dot{\theta}_j + \chi_i V_{j2}(\theta_i,\theta_j) + \chi_j \hat{V}_{j2}(\theta_i,\theta_j)\Big].$$

Assuming the following form of the value functions,

$$V_{i1}(\theta_i,\theta_j) = a_{i1}\theta_i + b_{i1}\theta_j + c_{i1},$$
$$V_{j1}(\theta_i,\theta_j) = a_{j1}\theta_i + b_{j1}\theta_j + c_{j1},$$

and proceeding in the standard way we get:

$$
\begin{aligned}
c_{i1} &= \frac{(\alpha m_{i1} + \chi_i c_{i2} + \chi_j \hat{c}_{i2})}{(r + \chi_i + \chi_j)}, \\
a_{i1} &= \frac{(\beta_i m_{i1} + \chi_i a_{i2} + \chi_j \hat{a}_{i2})}{(r + \chi_i + \chi_j + \delta_i)}, \\
b_{i1} &= \frac{(-m_{i1} + \chi_i b_{i2} + \chi_j \hat{b}_{i2})}{(r + \chi_i + \chi_j + \delta_j)},
\end{aligned}
\tag{A.21}
$$

Thus, the equilibrium decision is

$$
A_{i1}^* = \frac{k_{i1} a_{i1}}{\mu_i} = \frac{k_{i1}(\beta_i m_{i1} + \chi_i a_{i2} + \chi_j \hat{a}_{i2})}{\mu_i (r + \chi_i + \chi_j + \delta_{i1})}.
$$

■

PROOF OF PROPOSITION 7.

(a) From Proposition 6, the first regime advertising of M_i, if M_i recalls first is given by :

$$
A_{i1}^* = \frac{k_{i1}\left[(\beta_i m_{i1} + \chi_j \hat{a}_{i2} + \chi_i a_{i2}\right]}{\mu_i (r + \chi_i + \chi_j + \delta_{i1})}
$$

Differentiating A_{i1}^* with respect to η_i, we get:

$$
\frac{\partial A_{i1}^*}{\partial \eta_i} = -\frac{k_{i1} m_{i3} \beta_i \chi_i \chi_j}{(r + \delta_{i3}) \mu_i (r + \hat{\delta}_{i2} + \chi_i)(r + \delta_{i1} + \chi_i + \chi_j)} < 0
$$

Hence the first regime advertising is decreasing in η_i.

(b)From Proposition 6, the second regime advertising of M_i, if M_i recalls first is given by :

$$
A_{i2}^* = \frac{k_{i2}}{\mu_i (r + \chi_j + \delta_i)} [\beta_i m_{i2} + \chi_j a_{i3}].
$$

Differentiating A^*_{i2} with respect to χ_j, we get:

$$\begin{aligned}\frac{\partial A^*_{i2}}{\partial \chi_j} &= \frac{\beta_i k_{\text{i2}} m_{\text{i3}}}{\mu_i\left(\delta_{\text{i3}}+r\right)\left(\delta_{\text{i2}}+\chi_j+r\right)} - \frac{k_{\text{i2}}\left(\beta_i m_{\text{i2}}+\frac{\beta_i m_{\text{i3}}\chi_j}{\delta_{\text{i3}}+r}\right)}{\mu_i\left(\delta_{\text{i2}}+\chi_j+r\right)^2} \\ &> 0 \text{ if } m_{i2} < m_{i3} \text{ (on simplification)}.\end{aligned}$$

(c) Taking the derivative of $\hat{A}^*_{i2}$ from Proposition 6,

$$\frac{\partial \hat{A}^*_{i2}}{\partial \eta_i} = -\frac{\hat{k}_{i2} m_{i3} \beta_i \chi_i}{(r+\delta_{i3})\mu_i(r+\chi_i+\hat{\delta}_{i2})} < 0.$$

Therefore, if M_i recalls second, the second regime advertisement is decreasing with the damage η_i.

■

PROOF OF PROPOSITION 8.

The proof of Proposition 8 follows from the proof of proposition 6. From equations (A.21), we have the coefficients of the value functions in the first period. Therefore, putting $t = 0$ and assuming the initial values of $\theta_i(0), \theta_j(0)$, we can find the long term expected profit of the firm.

B. Appendix

PROOF OF PROPOSITION 1. We prove the proposition by solving the problem for post-crisis regime first and then for pre-crisis regime by using Hamilton-Jacobi-Bellman (HJB) equations. Using equation (19), the HJB equations for supplier 1 and supplier 2 in the second regime (post-crisis period), are be given by :

$$\begin{aligned} rV_{11}(\theta_{12},\theta_{22}) &= \operatorname*{Max}_{a_{12},p_{12}}[(Q_{12})p_{12} - \frac{\mu_1}{2}a_{12}^2(t) + \frac{\partial V_{11}}{\partial \theta_{12}}\dot{\theta}_{12}(t) + \frac{\partial V_{11}}{\partial \theta_{22}}\dot{\theta}_{22}(t)] \\ rV_{m2}(\theta_{12},\theta_{22}) &= \operatorname*{Max}_{a_{22},p_{22}}[(Q_{22})p_{22} - \frac{\mu_2}{2}a_{22}^2(t) + \frac{\partial V_{m2}}{\partial \theta_{12}}\dot{\theta}_{12}(t) + \frac{\partial V_{m2}}{\partial \theta_{22}}\dot{\theta}_{22}(t)] \end{aligned} \tag{B.1}$$

where Q_{ij}s are the demand functions in equation (3.1). Similarly, from equations (16) and (17) the first period HJB equations of the players are given by:

$$\begin{aligned} (r+\chi)\hat{V}_{m1}(\theta_{11},\theta_{21}) &= \operatorname*{Max}_{a_{11},p_{11}}[(Q_{11})p_{11} - \frac{\mu_1}{2}a_{11}^2(t) + \frac{\partial \hat{V}_{m1}}{\partial \theta_{11}}\dot{\theta}_{11}(t) + \frac{\partial \hat{V}_{m1}}{\partial \theta_{21}}\dot{\theta}_{21}(t) \\ &\quad + \chi V_{11}((1-\eta)\theta_{11}(t),(1-\phi)\theta_{21}(t)] \\ (r+\chi)\hat{V}_{m2}(\theta_{11},\theta_{21}) &= \operatorname*{Max}_{a_{21},p_{21}}[(Q_{21})p_{21} - \frac{\mu_2}{2}a_{21}^2(t) + \frac{\partial \hat{V}_{m2}}{\partial \theta_{11}}\dot{\theta}_{11}(t) + \frac{\partial \hat{V}_{m2}}{\partial \theta_{21}}\dot{\theta}_{21}(t)] \\ &\quad + \chi V_{m2}((1-\eta)\theta_{11}(t),(1-\phi)\theta_{21}(t)] \end{aligned} \tag{B.2}$$

The solution procedure is to start solving the second period HJB equations.
Recall that

$$\begin{aligned} Q_{1j} &= \alpha_1 - p_{1j} + \beta_1 p_{2j} + \theta_{1j} - \gamma_1 \theta_{2j} \\ Q_{2j} &= \alpha_2 - p_{2j} + \beta_2 p_{1j} + \theta_{1j} - \gamma_2 \theta_{1j} \end{aligned} \tag{B.3}$$

and

$$\dot{\theta}_{ij}(t) = k_{ij}a_{ij}(t) - \delta_{ij}\theta_{ij}(t), \quad \forall t \in 0 \le t \le \infty, \theta_{ij}(0) = \tilde{\theta_{ij}} \tag{B.4}$$

Putting the demand functions and the state equations for post-crisis regime in equations (B.2) we get ,

$$rV_{11}(\theta_{12},\theta_{22}) = \underset{a_{12},p_{12}}{\text{Max}}[(\alpha_1 - p_{12} + \beta_1 p_{22} + \theta_{12} - \gamma_1\theta_{22})p_{12} - \frac{\mu_1}{2}a_{12}^2(t) + \frac{\partial V_{11}}{\partial \theta_{12}}(k_{12}a_{12}(t) - \delta_{12}\theta_{12}(t)) + \frac{\partial V_{11}}{\partial \theta_{22}}(k_{22}a_{22}(t) - \delta_{22}\theta_{22}(t))] \quad \text{(B.5)}$$

$$rV_{22}(\theta_{12},\theta_{22}) = \underset{a_{22},p_{22}}{\text{Max}}[(\alpha_2 - p_{22} + \beta_2 p_{12} + \theta_{12} - \gamma_2\theta_{12})p_{22} - \frac{\mu_2}{2}a_{22}^2(t) + \frac{\partial V_{22}}{\partial \theta_{12}}(k_{12}a_{12}(t) - \delta_{12}\theta_{12}(t)) + \frac{\partial V_{22}}{\partial \theta_{22}}(k_{22}a_{22}(t) - \delta_{22}\theta_{22}(t))] \quad \text{(B.6)}$$

The first order conditions of the right hand sides of (B.6) with respect to the decision variables a_{12} and p_{12} and the first order conditions of the right hand sides of (B.7) with respect to the decision variables a_{22} and p_{22} give:

$$\alpha_1 - 2p_{12} + \beta_1 p_{22} + \theta_{12} - \gamma_1\theta_{22} = 0 \quad \text{(B.7)}$$

$$-\mu_1 a_{12} + k_{12}\frac{\partial V_{11}}{\partial \theta_{12}} = 0 \quad \text{(B.8)}$$

$$\alpha_2 - 2p_{22} + \beta_2 p_{12} + \theta_{12} - \gamma_2\theta_{22} = 0 \quad \text{(B.9)}$$

$$-\mu_2 a_{22} + k_{22}\frac{\partial V_{22}}{\partial \theta_{22}} = 0 \quad \text{(B.10)}$$

Simultaneously solving (B.8) and (B.10) we get,

$$p_{12}(t) = \frac{(2\alpha_1 + \alpha_2\beta_1 + (2-\beta_1\gamma_2)\theta_{12}(t) + (\beta_1 - 2\gamma_1)\theta_{22}(t))}{(4-\beta_1\beta_2)}, \quad \text{(B.11)}$$

$$p_{22}(t) = \frac{(2\alpha_2 + \alpha_1\beta_2 + (2-\beta_2\gamma_1)\theta_{22}(t) + (\beta_2 - 2\gamma_2)\theta_{12}(t))}{(4-\beta_1\beta_2)}.$$

From (B.9) and (B.10) we get the equilibrium advertising efforts,

$$a_{12} = \frac{k_{12}}{\mu_1}\frac{\partial V_{12}}{\partial \theta_{12}} \quad \text{(B.12)}$$

$$a_{22} = \frac{k_{22}}{\mu_2}\frac{\partial V_{22}}{\partial \theta_{22}} \quad \text{(B.13)}$$

We have assumed that the value functions are given by the following equations:

$$\begin{aligned} V_{12}(\theta_{12},\theta_{22}) &= A_{12}\theta_{12}(t)^2+B_{i2}\theta_{22}(t)^2+C_{12}\theta_{12}(t)\theta_{22}(t)+D_{12}\theta_{12}(t)+E_{12}\theta_{22}(t)+F_{12} \\ V_{22}(\theta_{12},\theta_{22}) &= A_{22}\theta_{12}(t)^2+B_{22}\theta_{22}(t)^2+C_{22}\theta_{12}(t)\theta_{22}(t)+D_{22}\theta_{12}(t)+E_{22}\theta_{22}(t)+F_{22} \end{aligned} \tag{B.14}$$

$$\frac{\partial V_{12}}{\partial \theta_{12}} = (2A_{12}\theta_{12}(t)+C_{12}\theta_{22}(t)+D_{12}) \tag{B.15}$$

$$\frac{\partial V_{22}}{\partial \theta_{22}} = (2B_{22}\theta_{22}(t)+C_{22}\theta_{12}(t)+E_{22})$$

Combining (B.13),(B.14) and (B.16) we get :

$$a_{12} = \frac{k_{12}}{\mu_1}(2A_{12}\theta_{12}(t)+C_{12}\theta_{22}(t)+D_{12}) \tag{B.16}$$

$$a_{22} = \frac{k_{22}}{\mu_2}(2B_{22}\theta_{22}(t)+C_{22}\theta_{12}(t)+E_{22}) \tag{B.17}$$

For the derivation of the value functions we substitute the decision variables from (B.12),(B.17) and (B.18) in the equations (B.6) and (B.7). Then we compare and equate the coefficients of the state variables or their associations i.e the coefficients of $\theta_i, \theta_i^2, \theta_i\theta_{(3-i)}, i \in \{1,2\}$ and the constant terms which give us the following twelve equation needed to be solved simultaneously to get the coefficients $A_{i2}, B_{i2}, C_{i2}, D_{i2}, E_{i2}, F_{i2}$. The equations are:

$$
\begin{aligned}
-2A_{12}\delta_{12}+\frac{2A_{12}^2k_{12}^2}{\mu_1}+\frac{(\beta_1\gamma_2-2)^2}{(\beta_1\beta_2-4)^2}+\frac{C_{12}C_{22}k_{22}^2}{\mu_2}-rA_{12} &= 0\\
\frac{(\beta_1-2\gamma_1)^2}{(\beta_1\beta_2-4)^2}+B_{12}\left(\frac{4B_{22}k_{22}^2}{\mu_2}-2\delta_{22}\right)+\frac{C_{12}^2k_{12}^2}{2\mu_1}-rB_{12} &= 0\\
\frac{2A_{12}C_{12}k_{12}^2}{\mu_1}+\frac{2B_{22}C_{12}k_{22}^2}{\mu_2}+\frac{2B_{12}C_{22}k_{22}^2}{\mu_2}+&\\
\frac{\beta_1\left(-2\gamma_2\left(\beta_1-2\gamma_1\right)+\beta_2\left(8-\beta_1\beta_2\right)C_{12}\delta_{12}+4\right)}{(\beta_1\beta_2-4)^2}-&\\
\frac{8\left(\gamma_1+2C_{12}\delta_{12}\right)}{(\beta_1\beta_2-4)^2}-C_{12}\delta_{22}-rC_{12} &= 0\\
-\frac{2\alpha_2\beta_1^2\gamma_2}{(\beta_1\beta_2-4)^2}-\frac{4\alpha_1\left(\beta_1\gamma_2-2\right)}{(\beta_1\beta_2-4)^2}+&\\
\frac{4\alpha_2\beta_1}{(\beta_1\beta_2-4)^2}+\frac{2A_{12}D_{12}k_{12}^2}{\mu_1}+\frac{\left(E_{12}C_{22}+E_{22}C_{12}\right)k_{22}^2}{\mu_2}-\delta_{12}D_{12}-rD_{12} &= 0\\
-\frac{2\left(-\alpha_2\beta_1-2\alpha_1\right)\left(\beta_1-2\gamma_1\right)}{(\beta_1\beta_2-4)^2}+\frac{2E_{12}B_{22}k_{22}^2}{\mu_2}+&\\
\frac{2E_{22}B_{12}k_{22}^2}{\mu_2}+\frac{C_{12}D_{12}k_{12}^2}{\mu_1}-E_{12}\delta_{22}-rE_{12} &= 0\\
\frac{(\alpha_2\beta_1+2\alpha_1)^2}{(\beta_1\beta_2-4)^2}+\frac{D_{12}^2k_{12}^2}{2\mu_1}+\frac{E_{12}E_{22}k_{22}^2}{\mu_2}-rF_{12} &= 0\\
A_{22}\left(\frac{4A_{12}k_{12}^2}{\mu_1}-2\delta_{12}\right)+\frac{(\beta_2-2\gamma_2)^2}{(\beta_1\beta_2-4)^2}+\frac{C_{22}^2k_{22}^2}{2\mu_2}-rA_{22} &= 0\\
\frac{(\beta_2\gamma_1-2)^2}{(\beta_1\beta_2-4)^2}-2B_{22}\delta_{22}+\frac{2B_{22}^2k_{22}^2}{\mu_2}+\frac{C_{12}C_{22}k_{12}^2}{\mu_1}-rB_{22} &= 0\\
\frac{2A_{22}C_{12}k_{12}^2}{\mu_1}+\frac{2A_{12}C_{22}k_{12}^2}{\mu_1}+\frac{2B_{22}C_{22}k_{22}^2}{\mu_2}-&\\
\frac{\beta_2\left(2\gamma_1\left(\beta_2-2\gamma_2\right)+\beta_1\left(\beta_1\beta_2-8\right)C_{22}\delta_{12}-4\right)+(\beta_1\beta_2-4)^2C_{22}\delta_{22}+8\left(\gamma_2+2C_{22}\delta_{12}\right)}{(\beta_1\beta_2-4)^2}&\\
-rC_{22} &= 0\\
\frac{2\left(\alpha_1\beta_2+2\alpha_2\right)\left(\beta_2-2\gamma_2\right)}{(\beta_1\beta_2-4)^2}+D_{22}\left(\frac{2A_{12}k_{12}^2}{\mu_1}-\delta_{12}\right)+\frac{2A_{22}D_{12}k_{12}^2}{\mu_1}+\frac{C_{22}E_{22}k_{22}^2}{\mu_2}-rD_{22} &= 0\\
-\frac{2\left(\alpha_1\beta_2+2\alpha_2\right)\left(\beta_2\gamma_1-2\right)}{(\beta_1\beta_2-4)^2}+\frac{2B_{22}E_{22}k_{22}^2}{\mu_2}+\frac{k_{12}^2\left(C_{22}D_{12}+C_{12}D_{22}\right)}{\mu_1}-\delta_{22}E_{22}-rE_{22} &= 0\\
\frac{(\alpha_1\beta_2+2\alpha_2)^2}{(\beta_1\beta_2-4)^2}+\frac{D_{12}D_{22}k_{12}^2}{\mu_1}+\frac{E_{22}^2k_{22}^2}{2\mu_2}-rF_{22} &= 0
\end{aligned}
\tag{B.18}
$$

The solutions of the above 12 simultaneous non linear equations give us the coefficients of the value functions. Due to the obvious complexity of the set of linear equations, we derive the solutions numerically.

Proceeding similarly as in the second regime's derivation of the equilibrium value of the decision variables, for the first regime we get,

$$\begin{aligned}
p_{11}(t) &= \frac{(2\alpha_1+\alpha_2\beta_1+(2-\beta_1\gamma_2)\theta_{11}(t)+(\beta_1-2\gamma_1)\theta_{22}(t))}{(4-\beta_1\beta_2)}, \qquad \text{(B.19)}\\
p_{21}(t) &= \frac{(2\alpha_2+\alpha_1\beta_2+(2-\beta_2\gamma_1)\theta_{21}(t)+(\beta_2-2\gamma_2)\theta_{11}(t))}{(4-\beta_1\beta_2)}.
\end{aligned}$$

The equilibrium pre-crisis advertising efforts are,

$$a_{11} = \frac{k_{11}}{\mu_1}\frac{\partial V_{11}}{\partial \theta_{11}} \qquad \text{(B.20)}$$

$$a_{21} = \frac{k_{21}}{\mu_2}\frac{\partial V_{21}}{\partial \theta_{21}} \qquad \text{(B.21)}$$

We have assumed that the value functions are given by the following equations:

$$\begin{aligned}
V_{11}(\theta_{11},\theta_{21}) &= A_{11}\theta_{11}(t)^2+B_{11}\theta_{21}(t)^2+C_{11}\theta_{11}(t)\theta_{21}(t)+D_{11}\theta_{11}(t)+E_{11}\theta_{21}(t)+F_{11}\\
V_{21}(\theta_{11},\theta_{21}) &= A_{21}\theta_{11}(t)^2+B_{21}\theta_{21}(t)^2+C_{21}\theta_{11}(t)\theta_{21}(t)+D_{21}\theta_{11}(t)+E_{21}\theta_{21}(t)+F_{21}
\end{aligned} \qquad \text{(B.22)}$$

$$\begin{aligned}
\frac{\partial V_{11}}{\partial \theta_{11}} &= (2A_{11}\theta_{11}(t)+C_{11}\theta_{21}(t)+D_{11}) \qquad \text{(B.23)}\\
\frac{\partial V_{21}}{\partial \theta_{21}} &= (2B_{21}\theta_{21}(t)+C_{21}\theta_{11}(t)+E_{21})
\end{aligned}$$

Combining (B.20),(B.21) and (B.23) we get :

$$a_{11} = \frac{k_{11}}{\mu_1}(2A_{11}\theta_{11}(t)+C_{11}\theta_{21}(t)+D_{11}) \qquad \text{(B.24)}$$

$$a_{21} = \frac{k_{21}}{\mu_2}(2B_{21}\theta_{21}(t)+C_{21}\theta_{11}(t)+E_{21}) \qquad \text{(B.25)}$$

Again plugging in the values of the decision variables in the HJB equations of the first period and then comparing the coefficients of the state variables we get the following set of 12 non linear equations whose solution will render the coefficients of the value function in the first period.

$$
\begin{aligned}
-2A_{11}\delta_{11}+A_{12}(\eta-1)^2\chi+\frac{2A_{11}^2k_{11}^2}{\mu_1}+\frac{(\beta_1\gamma_2-2)^2}{(\beta_1\beta_2-4)^2}+\frac{C_{11}C_{21}k_{21}^2}{\mu_2}-(r+\chi) &= 0\\
\frac{(\beta_1-2\gamma_1)^2}{(\beta_1\beta_2-4)^2}+B_{11}\left(\frac{4B_{21}k_{21}^2}{\mu_2}-2\delta_{21}\right)+B_{12}\chi(\phi-1)^2+\frac{C_{11}^2k_{11}^2}{2\mu_1}-(r+\chi)B_{11} &= 0\\
\frac{2A_{11}C_{11}k_{11}^2}{\mu_1}-\frac{2(\beta_1-2\gamma_1)(\beta_1\gamma_2-2)}{(\beta_1\beta_2-4)^2}+\frac{2k_{21}^2(B_{21}C_{11}+B_{11}C_{21})}{\mu_2}-&\\
C_{11}(\delta_{11}+\delta_{21})+C_{12}(\eta-1)\chi(\phi-1)-(r+\chi)C_{11} &= 0\\
-\frac{2(\alpha_2\beta_1+2\alpha_1)(\beta_1\gamma_2-2)}{(\beta_1\beta_2-4)^2}+D_{11}\left(\frac{2A_{11}k_{11}^2}{\mu_1}-\delta_{11}\right)+&\\
\frac{k_{21}^2(C_{21}E_{11}+C_{11}E_{21})}{\mu_2}+D_{12}(\chi-\eta\chi)-(r+\chi)D_{11} &= 0\\
\frac{2(\alpha_2\beta_1+2\alpha_1)(\beta_1-2\gamma_1)}{(\beta_1\beta_2-4)^2}+&\\
\frac{2k_{21}^2(B_{21}E_{11}+B_{11}E_{21})}{\mu_2}+\frac{C_{11}D_{11}k_{11}^2}{\mu_1}-\delta_{21}E_{11}+E_{12}\chi(1-\phi)-(r+\chi)E_{11} &= 0\\
\frac{(\alpha_2\beta_1+2\alpha_1)^2}{(\beta_1\beta_2-4)^2}+\frac{D_{11}^2k_{11}^2}{2\mu_1}+\frac{E_{11}E_{21}k_{21}^2}{\mu_2}+F_{12}\chi-(r+\chi)F_{11} &= 0\\
A_{22}(\eta-1)^2\chi+A_{21}\left(\frac{4A_{11}k_{11}^2}{\mu_1}-2\delta_{11}\right)+\frac{(\beta_2-2\gamma_2)^2}{(\beta_1\beta_2-4)^2}+\frac{C_{21}^2k_{21}^2}{2\mu_2}-(r+\chi)A_{21} &= 0\\
\frac{(\beta_2\gamma_1-2)^2}{(\beta_1\beta_2-4)^2}-2B_{21}\delta_{21}+\frac{2B_{21}^2k_{21}^2}{\mu_2}+B_{22}\chi(\phi-1)^2+\frac{C_{11}C_{21}k_{11}^2}{\mu_1}-(r+\chi)B_{21} &= 0\\
\frac{2k_{11}^2(A_{21}C_{11}+A_{11}C_{21})}{\mu_1}-\frac{2(\beta_2\gamma_1-2)(\beta_2-2\gamma_2)}{(\beta_1\beta_2-4)^2}+&\\
\frac{2B_{21}C_{21}k_{21}^2}{\mu_2}-C_{21}(\delta_{11}+\delta_{21})+C_{22}(\eta-1)\chi(\phi-1)-(r+\chi)C_{21} &= 0\\
\frac{2(\alpha_1\beta_2+2\alpha_2)(\beta_2-2\gamma_2)}{(\beta_1\beta_2-4)^2}+D_{21}\left(\frac{2A_{11}k_{11}^2}{\mu_1}-\delta_{11}\right)+\frac{2A_{21}D_{11}k_{11}^2}{\mu_1}+&\\
\frac{C_{21}E_{21}k_{21}^2}{\mu_2}+D_{22}(\chi-\eta\chi)-(r+\chi)D_{21} &= 0\\
-\frac{2(\alpha_1\beta_2+2\alpha_2)(\beta_2\gamma_1-2)}{(\beta_1\beta_2-4)^2}+\frac{2B_{21}E_{21}k_{21}^2}{\mu_2}+&\\
\frac{k_{11}^2(C_{21}D_{11}+C_{11}D_{21})}{\mu_1}-\delta_{21}E_{21}+E_{22}\chi(1-\phi)-(r+\chi)E_{21} &= 0\\
\frac{(\alpha_1\beta_2+2\alpha_2)^2}{(\beta_1\beta_2-4)^2}+\frac{D_{11}D_{21}k_{11}^2}{\mu_1}+\frac{E_{21}^2k_{21}^2}{2\mu_2}+F_{22}\chi-(r+\chi)F_{21} &= 0
\end{aligned}
\tag{B.26}
$$

Due to the complexity of the non linear system of equations, we solve the equations numerically. The above concludes the proof of Proposition 1.

PROOF OF PROPOSITION 2. We prove the proposition by solving the problem for post-crisis regime first and then for pre-crisis regime by using Hamilton-Jacobi-Bellman (HJB) equations.
In the Stackelberg game, the leader focal firm, moves first and announce its price and advertising. Leader observes the reaction function of the follower and uses the same to find the equilibrium decisions.
The solution procedure is same as the Nash game in the sense that we have to take the first order condition of the HJB equations to find the equilibrium decisions. Therefore, proceeding as in the Nash game, taking the first order condition of (B.6) with respect to p_{22}, we get,

$$\begin{aligned} &\alpha_2 - 2p_{22} + \beta_2 p_{12} + \theta_{12} - \gamma_2\theta_{12} = 0 \\ \implies\ & p_{22} = \frac{\alpha_2 + \beta_2 p_{12} + \theta_{12} - \gamma_2\theta_{12}}{2}. \end{aligned} \tag{B.27}$$

Putting (B.27) in (B.5) we get:

$$\begin{aligned} rV_{11}(\theta_{12}, \theta_{22}) \;=\; & \underset{a_{12},p_{12}}{\text{Max}}\Big[\left(\alpha_1 - p_{12} + \beta_1\left(\frac{\alpha_2 + \beta_2 p_{12} + \theta_{12} - \gamma_2\theta_{12}}{2}\right) + \theta_{12} - \gamma_1\theta_{22}\right)p_{12} - \frac{\mu_1}{2}a_{12}^2(t) + \\ & \frac{\partial V_{11}}{\partial \theta_{12}}(k_{12}a_{12}(t) - \delta_{12}\theta_{12}(t)) + \frac{\partial V_{11}}{\partial \theta_{22}}(k_{22}a_{22}(t) - \delta_{22}\theta_{22}(t))\Big] \end{aligned} \tag{B.28}$$

The first order condition of (B.28) with respect to p_{11}, we get,

$$p_{12}^*(t) \;=\; \frac{(2\alpha_1 + \alpha_2\beta_1 + (2 - \beta_1\gamma_2)\theta_{12}(t) + (\beta_1 - 2\gamma_1)\theta_{22}(t))}{2(2 - \beta_1\beta_2)} \tag{B.29}$$

Putting the value of p_{12} from (B.29) in (B.27) and simplifying we get,

$$p_{22}^*(t) \;=\; \frac{(4\alpha_2 + 2\alpha_1\beta_2 - \alpha_2\beta_1\beta_2 + (2\beta_2 + \beta_1\beta_2\gamma_2 - 4\gamma_2)\theta_{12}(t) + (4 - \beta_1\beta_2 - 2\beta_2\gamma_1)\theta_{22}(t))}{4(2 - \beta_1\beta_2)} \tag{B.30}$$

The first regimes pricing decisions can similarly be derived by replacing the regime index 2 by 1 in the above procedure. Technically, the advertising decisions at equilibrium needs no different treatment for the Stackelberg game because of the mathematical structure of our game. In the Stackelberg game, the advertising efforts have same expressions in terms of the state variables and can be derived in the same manner in the Nash game. However, as mentioned earlier, the value function coefficients will be different in the two games and hence the advertising efforts will also vary. We assume the value functions have the following structure

$$\begin{aligned} V_{1j}(\theta_{1j},\theta_{2j}) &= A_{1j}\theta_{1j}(t)^2+B_{ij}\theta_{2j}(t)^2+C_{1j}\theta_{1j}(t)\theta_{2j}(t)+D_{1j}\theta_{1j}(t)+E_{1j}\theta_{2j}(t)+F_{1j} \\ V_{2j}(\theta_{1j},\theta_{2j}) &= A_{2j}\theta_{1j}(t)^2+B_{2j}\theta_{2j}(t)^2+C_{2j}\theta_{1j}(t)\theta_{2j}(t)+D_{2j}\theta_{1j}(t)+E_{2j}\theta_{2j}(t)+F_{2j} \end{aligned} \tag{B.31}$$

and present the set of equations for the two periods which give the solutions to the value function coefficients.

Proceeding similarly as in the case of Nash games, we compare the coefficients of state variables of the value functions with coefficients in the right hand side of the HJB equations after substituting the decisions variables with the solutions we obatined. Comparison yields the following equations for the second regime: The equations are:

$$\begin{aligned}
-2A_{12}\delta_{12}+\frac{2A_{12}^2k_{12}^2}{\mu_1}-\frac{(\beta_1\gamma_2-2)^2}{8(\beta_1\beta_2-2)}+\frac{C_{12}C_{22}k_{22}^2}{\mu_2}-rA_{12} &= 0 \\
-\frac{(\beta_1-2\gamma_1)^2}{8(\beta_1\beta_2-2)}+B_{12}\left(\frac{4B_{22}k_{22}^2}{\mu_2}-2\delta_{22}\right)+\frac{C_{12}^2k_{12}^2}{2\mu_1}-rB_{12} &= 0 \\
\frac{2A_{12}C_{12}k_{12}^2}{\mu_1}+\frac{2B_{22}C_{12}k_{22}^2}{\mu_2}+\frac{2B_{12}C_{22}k_{22}^2}{\mu_2}+ & \\
\frac{\beta_1^2\gamma_2+4\gamma_1-2\beta_1(\gamma_1\gamma_2+2\beta_2C_{12}(\delta_{12}+\delta_{22})+1)+8C_{12}(\delta_{12}+\delta_{22})}{4\beta_1\beta_2-8}-rC_{12} &= 0 \\
\frac{(\alpha_2\beta_1+2\alpha_1)(\beta_1\gamma_2-2)}{4\beta_1\beta_2-8}+D_{12}\left(\frac{2A_{12}k_{12}^2}{\mu_1}-\delta_{12}\right)+\frac{k_{22}^2(C_{22}E_{12}+C_{12}E_{22})}{\mu_2}-rD_{12} &= 0 \\
\frac{(\alpha_2\beta_1+2\alpha_1)(\beta_1-2\gamma_1)}{8-4\beta_1\beta_2}+\frac{2k_{22}^2(B_{22}E_{12}+B_{12}E_{22})}{\mu_2}+\frac{C_{12}D_{12}k_{12}^2}{\mu_1}-\delta_{22}E_{12}-rE_{12} &= 0 \\
-\frac{(\alpha_2\beta_1+2\alpha_1)^2}{8(\beta_1\beta_2-2)}+\frac{D_{12}^2k_{12}^2}{2\mu_1}+\frac{\mathrm{E12}E_{22}k_{22}^2}{\mu_2}-rF_{12} &= 0 \\
\frac{1}{16}\left(8A_{22}\left(\frac{8A_{12}k_{12}^2}{\mu_1}-4\delta_{12}\right)+\frac{(\beta_2(\beta_1\gamma_2+2)-4\gamma_2)^2}{(\beta_1\beta_2-2)^2}+\frac{8C_{22}^2k_{22}^2}{\mu_2}\right)-rA_{22} &= 0 \\
\frac{(\beta_2(\beta_1+2\gamma_1)-4)^2}{16(\beta_1\beta_2-2)^2}-2B_{22}\delta_{22}+\frac{2B_{22}^2k_{22}^2}{\mu_2}+\frac{C_{12}C_{22}k_{12}^2}{\mu_1}-rB_{22} &= 0
\end{aligned}$$

$$
\begin{aligned}
&\frac{2A_{22}C_{12}k_{12}^2}{\mu_1}+\frac{2A_{12}C_{22}k_{12}^2}{\mu_1}+\frac{2B_{22}C_{22}k_{22}^2}{\mu_2}- \\
&\frac{16\left(\gamma_2+2C_{22}\left(\delta_{12}+\delta_{22}\right)\right)-8\beta_2\left(\gamma_1\gamma_2+\beta_1\left(\gamma_2+4C_{22}\left(\delta_{12}+\delta_{22}\right)\right)+1\right)}{8\left(\beta_1\beta_2-2\right)^2}- \\
&\frac{\beta_2^2\left(4\gamma_1+\beta_1\left(2\gamma_1\gamma_2+\beta_1\left(\gamma_2+8C_{22}\left(\delta_{12}+\delta_{22}\right)\right)+2\right)\right)}{8\left(\beta_1\beta_2-2\right)^2} \\
&-rC_{22} \quad = \quad 0 \\
&-\frac{\left(\alpha_2\left(\beta_1\beta_2-4\right)-2\alpha_1\beta_2\right)\left(\beta_2\left(\beta_1\gamma_2+2\right)-4\gamma_2\right)}{8\left(\beta_1\beta_2-2\right)^2}+D_{22}\left(\frac{2A_{12}k_{12}^2}{\mu_1}-\delta_{12}\right)+ \\
&\frac{2A_{22}D_{12}k_{12}^2}{\mu_1}+\frac{C_{22}E_{22}k_{22}^2}{\mu_2}-rD_{22} \quad = \quad 0 \\
&\frac{\left(\alpha_2\left(\beta_1\beta_2-4\right)-2\alpha_1\beta_2\right)\left(\beta_2\left(\beta_1+2\gamma_1\right)-4\right)}{8\left(\beta_1\beta_2-2\right)^2}+ \\
&\frac{2B_{22}E_{22}k_{22}^2}{\mu_2}+\frac{k_{12}^2\left(C_{22}D_{12}+C_{12}D_{22}\right)}{\mu_1}-\delta_{22}E_{22}-rE_{22} \quad = \quad 0 \\
&\frac{1}{16}\left(\frac{\left(\alpha_2\left(\beta_1\beta_2-4\right)-2\alpha_1\beta_2\right)^2}{\left(\beta_1\beta_2-2\right)^2}+\frac{16D_{12}D_{22}k_{12}^2}{\mu_1}+\frac{8E_{22}^2k_{22}^2}{\mu_2}\right)-rF_{22} \quad = \quad 0
\end{aligned}
\tag{B.32}
$$

Clearly, the equations (B.32) are different from the set of equations (B.18). Hence these give us a different solution for the Stackelberg game. The Stackelberg game's first period equations for solving the value function coefficients are

$$
\begin{aligned}
&-2A_{11}\delta_{11}+A_{12}(\eta-1)^2\chi+\frac{2A_{11}^2k_{11}^2}{\mu_1}-\frac{\left(\beta_1\gamma_2-2\right)^2}{8\left(\beta_1\beta_2-2\right)}+\frac{C_{11}C_{21}k_{21}^2}{\mu_2}-(r+\chi) \quad = \quad 0 \\
&-\frac{\left(\beta_1-2\gamma_1\right)^2}{8\left(\beta_1\beta_2-2\right)}+B_{11}\left(\frac{4B_{21}k_{21}^2}{\mu_2}-2\delta_{21}\right)+B_{12}\chi(\phi-1)^2+\frac{C_{11}^2k_{11}^2}{2\mu_1}-(r+\chi)B_{11} \quad = \quad 0 \\
&\frac{2A_{11}C_{11}k_{11}^2}{\mu_1}+\frac{2k_{21}^2\left(B_{21}C_{11}+B_{11}C_{21}\right)}{\mu_2}+ \\
&\frac{\beta_1^2\gamma_2-2\beta_1\left(\gamma_1\gamma_2+2\beta_2C_{11}\left(\delta_{11}+\delta_{21}\right)+1\right)+4\left(\gamma_1+2C_{11}\left(\delta_{11}+\delta_{21}\right)\right)}{4\beta_1\beta_2-8}+ \\
&C_{12}(\eta-1)\chi(\phi-1)-(r+\chi)C_{11} \quad = \quad 0 \\
&-\frac{2\left(\alpha_2\beta_1+2\alpha_1\right)\left(\beta_1\gamma_2-2\right)}{\left(\beta_1\beta_2-4\right)^2}+D_{11}\left(\frac{2A_{11}k_{11}^2}{\mu_1}-\delta_{11}\right)+ \\
&\frac{k_{21}^2\left(C_{21}E_{11}+C_{11}E_{21}\right)}{\mu_2}+D_{12}(\chi-\eta\chi)-(r+\chi)D_{11} \quad = \quad 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{(\alpha_2\beta_1+2\alpha_1)(\beta_1\gamma_2-2)}{4\beta_1\beta_2-8}+D_{11}\left(\frac{2A_{11}k_{11}^2}{\mu_1}-\delta_{11}\right)+& \\
\frac{k_{21}^2(C_{21}E_{11}+C_{11}E_{21})}{\mu_2}+D_{12}\chi(1-\eta)-(r+\chi)E_{11} &= 0 \\
-\frac{(\alpha_2\beta_1+2\alpha_1)^2}{8(\beta_1\beta_2-2)}+\frac{D_{11}^2k_{11}^2}{2\mu_1}+\frac{\mathrm{E11}E_{21}k_{21}^2}{\mu_2}+F_{12}\chi-(r+\chi)F_{11} &= 0 \\
\frac{1}{16}\left(16A_{22}(\eta-1)^2\chi+8A_{21}\left(\frac{8A_{11}k_{11}^2}{\mu_1}-4\delta_{11}\right)+\frac{(\beta_2(\beta_1\gamma_2+2)-4\gamma_2)^2}{(\beta_1\beta_2-2)^2}+\frac{8C_{21}^2k_{21}^2}{\mu_2}\right)& \\
-(r+\chi)A_{21} &= 0 \\
\frac{(\beta_2(\beta_1+2\gamma_1)-4)^2}{16(\beta_1\beta_2-2)^2}-2B_{21}\delta_{21}+\frac{2B_{21}^2k_{21}^2}{\mu_2}+B_{22}\chi(\phi-1)^2+\frac{C_{11}C_{21}k_{11}^2}{\mu_1}-(r+\chi)B_{21} &= 0 \\
C_{22}(\eta-1)\chi(\phi-1)-\frac{(\beta_2(\beta_1+2\gamma_1)-4)(\beta_2(\beta_1\gamma_2+2)-4\gamma_2)}{8(\beta_1\beta_2-2)^2}& \\
+C_{21}\left(\frac{2A_{11}k_{11}^2}{\mu_1}-\delta_{11}\right)+\frac{2A_{21}C_{11}k_{11}^2}{\mu_1}+C_{21}\left(\frac{2B_{21}k_{21}^2}{\mu_2}-\delta_{21}\right)-(r+\chi)C_{21} &= 0 \\
\frac{2(\alpha_1\beta_2+2\alpha_2)(\beta_2-2\gamma_2)}{(\beta_1\beta_2-4)^2}+D_{21}\left(\frac{2A_{11}k_{11}^2}{\mu_1}-\delta_{11}\right)+\frac{2A_{21}D_{11}k_{11}^2}{\mu_1}+& \\
\frac{C_{21}E_{21}k_{21}^2}{\mu_2}+D_{22}(\chi-\eta\chi)-(r+\chi)D_{21} &= 0 \\
-\frac{(\alpha_2(\beta_1\beta_2-4)-2\alpha_1\beta_2)(\beta_2(\beta_1\gamma_2+2)-4\gamma_2)}{8(\beta_1\beta_2-2)^2}-\delta_{11}D_{21}+D_{22}(\chi-\eta\chi)& \\
+\frac{2k_{11}^2(A_{21}D_{11}+A_{11}D_{21})}{\mu_1}+\frac{C_{21}E_{21}k_{21}^2}{\mu_2}-(r+\chi)E_{21} &= 0 \\
\frac{1}{16}\left(\frac{(\alpha_2(\beta_1\beta_2-4)-2\alpha_1\beta_2)^2}{(\beta_1\beta_2-2)^2}+\frac{16D_{11}D_{21}k_{11}^2}{\mu_1}+\frac{8E_{21}^2k_{21}^2}{\mu_2}+16F_{22}\chi\right)-(r+\chi)F_{21} &= 0
\end{aligned}
\tag{B.33}
$$

Equations (B.33) give the solution of the first period coefficients of the value functions. Again clearly, the equations and hence the coefficients in general will be different from the Nash game. This concludes the proof of proposition 2.

C. Appendix

We have first assumed that the value functions are linear in the state variables G(t) and Q(t) and have later verified our claim. For the post-crisis period the value functions are assumed to take the form,

$$\begin{aligned} V_i(G(t),Q(t)) &= X_iG(t)+Y_iQ(t)+Z_i \\ &\text{in the post-crisis period} \\ \hat{V}_i(G(t),Q(t)) &= \hat{X}_iG(t)+\hat{Y}_iQ(t)+\hat{Z}_i \\ &\text{in the pre-crisis period.} \end{aligned} \tag{C.1}$$

where $i \in \{s1, s2, m\}$

The proofs of the Propositions 1,2,3 are similar. Also, we need to find the value functions in the two periods in order to solve for the equilibrium decisions of the players.

PROOF OF THE PROPOSITIONS. The proof of the Propositions 1,2,4,5,7,8,9 needs cross references as the decision variables are inter-related. We prove the propositions by solving the problems for each firm in the post-crisis regime first and then for pre-crisis regime by using Hamilton-Jacobi-Bellman (HJB) equations.

The HJB equations for supplier 1 and supplier 2 in the second regime (post-crisis period), are be given by :

$$\begin{aligned} rV_{s1}(G,Q) &= \underset{q_{12}}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_{s1}-C_{s1}+\frac{\partial V_{s1}}{\partial G}\dot{G}(t)+\frac{\partial V_{s1}}{\partial Q}\dot{Q}(t)] \\ &= \underset{q_{12}}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_{s1}-C_{s1} \\ &\quad +\frac{\partial V_{s1}}{\partial G}(k_{g2}A_2(t)+k_{12}q_{12}(t)+k_{22}q_{22}(t)-\delta_2 G(t)) \\ &\quad +\frac{\partial V_{s1}}{\partial Q}(l_{12}q_{12}(t)+l_{22}q_{22}(t))] \\ rV_{s2}(G,Q) &= \underset{q_{22}}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_{s2}-C_{s2}+\frac{\partial V_{s2}}{\partial G}\dot{G}(t)+\frac{\partial V_{s2}}{\partial Q}\dot{Q}(t)] \\ &= \underset{q_{22}}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_{s2}-C_{s2} \\ &\quad +\frac{\partial V_{s2}}{\partial G}(k_{g2}A_2(t)+k_{12}q_{12}(t)+k_{22}q_{22}(t)-\delta_2 G(t)) \\ &\quad +\frac{\partial V_{s2}}{\partial Q}(l_{12}q_{12}(t)+l_{22}q_{22}(t))] \end{aligned} \tag{C.2}$$

where

$$
\begin{aligned}
C_{s1} &= \frac{\big((1-\phi_2)\mu_{s1}+\omega+\xi\theta\big)q_{12}^2(t)}{2} && \text{(C.3)}\\
&\quad \text{for the QICS contract}\\
C_{s1} &= \frac{(1-\phi_2)(\mu_{s1}+\omega+\xi\theta)q_{12}^2(t)}{2} && \text{(C.4)}\\
&\quad \text{for the CQCS contract}\\
C_{s2} &= \frac{\big((1-\sigma_2)\mu_{s2}\big)q_{22}^2(t)}{2} && \text{(C.5)}
\end{aligned}
$$

For maximization of the value function, under the QICS contract differentiating the right side of $(C.2)$ with respect to (w.r.t.) q_{12} and equating to 0 we get,
$\frac{\partial V_{s1}}{\partial G}k_{12}+\frac{\partial V_{s1}}{\partial Q}l_{12}-((1-\phi_2)\mu_{s1}+\omega+\xi\theta\big)q_{12}(t)=0.$
Therefore,

$$
\begin{aligned}
q_{12}^*(t) &= \frac{\frac{\partial V_{s1}}{\partial G}k_{12}+\frac{\partial V_{s1}}{\partial Q}l_{12}}{((1-\phi_2)\mu_{s1}+\omega+\xi\theta\big)}\\
&= \frac{X_{s1}k_{12}+Y_{s1}l_{12}}{((1-\phi_2)\mu_{s1}+\omega+\xi\theta\big)} && \text{(C.6)}
\end{aligned}
$$

We get (C.6) substituting the value of V_{s1} from (C.1) . For maximization of the value function, under the CQCS contract differentiating the right side of $(C.2)$ with respect to (w.r.t.) q_{12}, using (C.4) as cost and equating to 0 we get,
$\frac{\partial V_{s1}}{\partial G}k_{12}+\frac{\partial V_{s1}}{\partial Q}l_{12}-(1-\phi_2)(\mu_{s1}+\omega+\xi\theta\big)q_{12}(t)=0.$
Simplifying we get,

$$
q_{12}^*(t) = \frac{X_{s1}k_{12}+Y_{s1}l_{12}}{(1-\phi_2)(\mu_{s1}+\omega+\xi\theta\big)} \quad \text{(C.7)}
$$

Proceeding similarly as in then case of finding $q_{12}^*(t)$ we use the maximizing technique and take the derivative of the HJB equation of Supplier 2 with respect to $q_{22}^*(t)$. Simplifying and equating to derivative to 0 gives, $\frac{\partial V_{s2}}{\partial G}k_{22}+\frac{\partial V_{s2}}{\partial Q}l_{22}-(1-\sigma_2)\mu_{s2}q_{22}(t)=0.$

$$
q_{22}^*(t) = \frac{X_{s2}k_{22}+Y_{s2}l_{22}}{(1-\sigma_2)\mu_{s2}} \quad \text{(C.8)}
$$

The HJB equation for manufacturer in the second regime (post-crisis period), is be given by :

$$\begin{aligned} rV_m(G,Q) &= \underset{A_2,\phi_2,\sigma_2}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_m - C_m + \frac{\partial V_m}{\partial G}\dot{G}(t) + \frac{\partial V_m}{\partial Q}\dot{Q}(t)] \\ &= \underset{A_2,\phi_2,\sigma_2}{\text{Max}}[(\alpha+\beta G+\gamma Q)M_m - C_m \\ &\quad + \frac{\partial V_m}{\partial G}(k_{g2}A_2(t) + k_{12}q_{12}(t) + k_{22}q_{22}(t) - \delta_2 G(t)) \\ &\quad + \frac{\partial V_m}{\partial Q}(l_{12}q_{12}(t) + l_{22}q_{22}(t))] \end{aligned} \tag{C.9}$$

where

$$Cm = \frac{\mu_m A_2^2(t)}{2} + \frac{\phi_2(t)\mu_{s1}q_{12}^2(t)}{2} + \frac{\sigma_2(t)\mu_{s2}q_{22}^2(t)}{2} \quad \text{for the QICS contract} \tag{C.10}$$

$$Cm = \frac{\mu_m A_2^2(t)}{2} + \frac{\phi_2(t)\Big(\mu_{s1}+\omega+\xi\theta\Big)q_{12}^2(t)}{2} + \frac{\sigma_2(t)\mu_{s2}q_{22}^2(t)}{2} \quad \text{for the CQCS contract} \tag{C.11}$$

For the manufacturer, we need to find the equilibrium advertising $A_2^*(t)$ and the cost-sharing proportions, $\phi_2^*(t)$ and $\sigma_2^*(t)$. Thus, $\frac{\partial V_{ij}}{\partial \theta_1} = a_{ij}$ and $\frac{\partial V_{ij}}{\partial \theta_2} = b_{ij}$. Therefore, differentiating the HJB equation of the manufacturer with respect to $A_2(t)$, and equating to zero we get , $k_{g2}X_m - \mu_m A_2(t) = 0$. Consequently,

$$A_2^*(t) = \frac{k_{g2}X_m}{\mu_m} \tag{C.12}$$

To find the equilibrium $\phi_2(t)$, under the QICS contract we use the equations (C.6) and (C.8) to input the values of the reaction functions of Supplier 1 and supplier 2, i.e $q_{i2}(t), i \in \{1,2\}$ in terms of $\phi_2(t)$ and $\sigma_2(t)$ in the HJB equation (C.9) of the manufacturer. Substituting the values of $q_{i2}^*(t)$ in (C.9), taking the derivative with respect to ϕ_2 and simplifying and equating the same to 0 we get:

$$\frac{\mu_{s1}(k_{12}X_{s1} + l_{12}Y_{s1})(\Lambda_1 - \Lambda_2)}{2(\theta\xi + \mu_{s1}(1-\phi_2) + \omega)^3} = 0, \tag{C.13}$$

where

$$\begin{aligned}\Lambda_1 &= (\theta\xi + (1-\phi_2)\mu_{s1} + \omega)(2k_{12}X_m + 2l_{12}Y_m) \\ \Lambda_2 &= (\theta\xi + (1+\phi_2)\mu_{s1} + \omega)(k_{12}X_{s1} + l_{12}Y_{s1}).\end{aligned} \tag{C.14}$$

Algebraic manipulations give,

$$\phi_2^*(t) = \frac{(\theta\xi + \mu_{s1} + \omega)(k_{12}(2X_m - X_{s1}) + l_{12}(2Y_m - Y_{s1}))}{\mu_{s1}(k_{12}(2X_m + X_{s1}) + l_{12}(2Y_m + Y_{s1}))} \tag{C.15}$$

Similarly, taking the derivative of the HJB equations (C.9) with respect to σ_2 and equating to 0 and doing the manipulations , we get:

$$\sigma_2^*(t) = \frac{(k_{22}(2X_m - X_{s2}) + l_{22}(2Y_m - Y_{s2}))}{\mu_{s1}(k_{22}(2X_m + X_{s2}) + l_{22}(2Y_m + Y_{s2}))} \tag{C.16}$$

We can reduce the equilibrium policies of the players in terms of the system parameters on simplification. However, to do so we need the coefficients X_i, Y_i and Z_i where $i \in \{s1, s2, m\}$. We show hoe to derive the same for the manufacturer. The suppliers' value functions' coefficients can be similarly derived.

For the QICS contract, from (C.3), (C.5) and (C.10) we note that

$$\begin{aligned}Cm &= \frac{\mu_m A_2^2(t)}{2} + \frac{\phi_2(t)\mu_{s1}q_{12}^2(t)}{2} + \frac{\sigma_2(t)\mu_{s2}q_{22}^2(t)}{2} \\ C_{s1} &= \frac{\big((1-\phi_2)\mu_{s1} + \omega + \xi\theta\big)q_{12}^2(t)}{2} \\ C_{s2} &= \frac{\big((1-\sigma_2)\mu_{s2}\big)q_{22}^2(t)}{2}\end{aligned} \tag{C.17}$$

Therefore the coefficients X_m, Y_m and Z_m can be derived by putting in the equilibrium solutions of the decision variables in the value function of the manufacturer we get:

$$\begin{aligned}r(X_mG + Y_mQ + Z_m) &= [(\alpha + \beta G + \gamma Q)M_m - \frac{\mu_m A_2^{*2}(t)}{2} + \frac{\phi_2(t)\mu_{s1}q_{12}^{*2}(t)}{2} \\ &\quad + \frac{\sigma_2(t)\mu_{s2}q_{22}^{*2}(t)}{2} + \frac{\partial V_m}{\partial G}(k_{g2}A_2^*(t) + k_{12}q_{12}^*(t) + k_{22}q_{22}^*(t) - \delta_2 G(t)) \\ &\quad + \frac{\partial V_m}{\partial Q}(l_{12}q_{12}^*(t) + l_{22}q_{22}^*(t))]\end{aligned} \tag{C.18}$$

Clearly, from equations (C.6), (C.7), (C.12), (C.15) and (C.16), the decision variables are expressions in terms of system parameters and the coefficients of value functions. Therefore, the state variables are not present directly in the expressions. Therefore, assuming the linear form of the value function, noting that $\frac{\partial V_m}{\partial G} = X_m$ and $\frac{\partial V_m}{\partial Q} = Y_m$, comparison of the the coefficients of $G(t)$ and $Q(t)$ from the two sides of (C.20) gives the solutions of X_m :

$$\begin{aligned} X_m &= \frac{\beta M_m}{r+\delta_2} \\ Y_m &= \frac{\gamma M_m}{r} \end{aligned} \tag{C.19}$$

Similarly, for S1 and S2 we get

$$\begin{aligned} X_{s1} &= \frac{\beta M_{s1}}{r+\delta_2} \\ Y_{s1} &= \frac{\gamma M_{s1}}{r} \\ X_{s2} &= \frac{\beta M_{s2}}{r+\delta_2} \\ Y_{s2} &= \frac{\gamma M_{s2}}{r} \end{aligned} \tag{C.20}$$

Putting (C.15) in (C.6) and (C.16) in (C.8) and substituting the values of X_m and Y_m from (C.19) we get the following solutions on simplification:

$$\begin{aligned} q_{12}^*(t) &= \frac{(2M_m + M_{s1})(k_{12}r\beta + l_{12}\gamma(r+\delta_2))}{2r(r+\delta_2)(\mu_{s1} + \theta\xi + \omega)} \\ q_{22}^*(t) &= \frac{(2M_m + M_{s2})(k_{22}r\beta + l_{22}\gamma(r+\delta_2))}{2r(r+\delta_2)\mu_{s2}} \\ \phi_2^{QICS*}(t) &= \frac{(2M_m - M_{s1})(\mu_{s1} + \theta\xi + \omega)}{(2M_m + M_{s1})\mu_{s1}} \\ \sigma_2^*(t) &= \frac{(2M_m - M_{s2})}{(2M_m + M_{s2})} \\ A_2^*(t) &= \frac{k_{g2}M_m\beta}{(r+\delta_2)\mu_m}. \end{aligned} \tag{C.21}$$

and

$$q_{22}^*(t) = \frac{X_{s2}k_{22} + Y_{s2}l_{22}}{(1-\sigma_2)\mu_{s2}} \tag{C.22}$$

The above concludes the proof for the quality decisions of the suppliers in the second regime. The

HJB equations of the three players in the pre-crisis regime are:

$$\begin{aligned}
(r+\chi)\hat{V}_{s1}(G,Q) &= \underset{q_{11}}{\text{Max}}[(\alpha+\beta G+\gamma Q)\hat{M}_{s1}-\hat{C}_{s1}+\frac{\partial \hat{V}_{s1}}{\partial G}\dot{G}(t)+\frac{\partial \hat{V}_{s1}}{\partial Q}\dot{Q}(t)] \\
&\quad +\chi V_{s1}((1-\eta)G,(1-\xi)Q)] \\
&= \underset{q_{11}}{\text{Max}}[(\alpha+\beta G+\gamma Q)\hat{M}_{s1}-C_{s1} \\
&\quad +\frac{\partial \hat{V}_{s1}}{\partial G}(k_{g1}A_1(t)+k_{11}q_{11}(t)+k_{21}q_{21}(t)-\delta_1 G(t)) \\
&\quad +\frac{\partial \hat{V}_{s1}}{\partial Q}(e^{-\lambda\chi}l_{11}q_{11}(t)+l_{21}q_{21}(t))] \\
&\quad +\chi V_{s1}((1-\eta)G,(1-\xi)Q)] \\
(r+\chi)\hat{V}_{s2}(G,Q) &= \underset{q_{21}}{\text{Max}}[(\alpha+\beta G+\gamma Q)\hat{M}_{s2}-\hat{C}_{s2}+\frac{\partial \hat{V}_{s2}}{\partial G}\dot{G}(t)+\frac{\partial \hat{V}_{s2}}{\partial Q}\dot{Q}(t)] \\
&\quad +\chi V_{s2}((1-\eta)G,(1-\xi)Q)] \\
&= \underset{q_{21}}{\text{Max}}[(\alpha+\beta G+\gamma Q)\hat{M}_{s2}-C_{s2} \\
&\quad +\frac{\partial \hat{V}_{s2}}{\partial G}(k_{g1}A_1(t)+k_{11}q_{11}(t)+k_{21}q_{21}(t)-\delta_1 G(t)) \\
&\quad +\frac{\partial \hat{V}_{s2}}{\partial Q}(e^{-\lambda\chi}l_{11}q_{11}(t)+l_{21}q_{21}(t))] \\
&\quad +\chi V_{s2}((1-\eta)G,(1-\xi)Q)] \\
(r+\chi)\hat{V}_{m}(G,Q) &= \underset{A_1,\phi_1,\sigma_1}{\text{Max}}[(\alpha+\beta G+\gamma Q)\hat{M}_{m}-\hat{C}_{m}+\frac{\partial \hat{V}_{s2}}{\partial G}\dot{G}(t)+\frac{\partial \hat{V}_{s2}}{\partial Q}\dot{Q}(t)] \\
&\quad +\chi V_{m}((1-\eta)G,(1-\xi)Q)] \\
&= \underset{A_1,\phi_1,\sigma_1}{\text{Max}}[(\alpha+\beta G+\gamma Q)\hat{M}_{m}-C_{m} \\
&\quad +\frac{\partial \hat{V}_{m}}{\partial G}(k_{g1}A_1(t)+k_{11}q_{11}(t)+k_{21}q_{21}(t)-\delta_1 G(t)) \\
&\quad +\frac{\partial \hat{V}_{m}}{\partial Q}(e^{-\lambda\chi}l_{11}q_{11}(t)+l_{21}q_{21}(t))] \\
&\quad +\chi V_{m}((1-\eta)G,(1-\xi)Q)]
\end{aligned} \tag{C.23}$$

We note that the pre-crisis costs $\hat{C}_i$ may be different than the post crisis cost. For the QICS contract:

$$\begin{aligned}
\hat{C}_{s1} &= (\frac{(1-\phi_1)\mu_{s1}}{2}+\frac{(1-\chi)\omega}{2})q_{11}^2(t) \\
\hat{C}_{s2} &= \frac{(1-\sigma_1)\mu_{s2}q_{21}^2(t)}{2} \\
\hat{C}_{m} &= \frac{\mu_m A_2^2(t)}{2}+\frac{\phi_1(t)\mu_{s1}q_{11}^2(t)}{2}+\frac{\sigma_1(t)\mu_{s2}q_{21}^2(t)}{2}
\end{aligned} \tag{C.24}$$

For the CQCS contract, the costs are:

$$\begin{aligned}
\hat{C}_{s1} &= \frac{(1-\phi_1)\big(\mu_{s1}+(1-\chi)\omega\big)}{2}q_{11}^2(t) \\
\hat{C}_{s2} &= \frac{(1-\sigma_1)\mu_{s2}q_{21}^2(t)}{2} \\
\hat{C}_m &= \frac{\mu_m A_2^2(t)}{2}+\frac{\phi_1(t)(\mu s1+(1-\chi)\omega)q_{11}^2(t)}{2}+\frac{\sigma_1(t)\mu_{s2}q_{21}^2(t)}{2} \qquad (C.25)
\end{aligned}$$

We proceed exactly in the similar manner as in the case of second regime and take the first order optimality condition for decision variables -
$q_{11}(t)$ for supplier 1, $q_{21}(t)$ for supplier 2 and $A_1(t), \phi_1(t), \sigma_1(t)$ for the manufacturer. Proceeding as in the case of (C.6) we get

$$q_{11}^*(t) = \frac{\hat{X}_{s1}k_{11}+\hat{Y}_{s1}l_{11}e^{-\lambda\chi}}{((1-\phi_1)\mu_{s1}+(1-\chi)\omega)}(\text{ for the QICS contract}), \qquad (C.26)$$

$$q_{11}^*(t) = \frac{\hat{X}_{s1}k_{11}+\hat{Y}_{s1}l_{11}e^{-\lambda\chi}}{(1-\phi_1)\big(\mu_{s1}+(1-\chi)\omega\big)}(\text{ for the CQCS contract}), \qquad (C.27)$$

$$q_{21}^*(t) = \frac{\hat{X}_{s2}k_{21}+\hat{Y}_{s2}l_{21}}{(1-\sigma_1)\mu_{s2})}(\text{ for any contract}), \qquad (C.28)$$

$$\phi_1^{QICS*}(t) = \frac{\Big(e^{\lambda\chi}k_{11}\big(2\hat{X}_m-\hat{X}_{s1}\big)+l_{11}\big(2\hat{Y}_m-\hat{Y}_{s1}\big)\Big)(\mu_{s1}+\omega(1-\chi))}{\Big(e^{\lambda\chi}k_{11}\big(2\hat{X}_m+\hat{X}_{s1}\big)+l_{11}\big(2\hat{Y}_m+\hat{Y}_{s1}\big)\Big)\mu_{s1}}, \qquad (C.29)$$

$$\phi_1^{CQCS*}(t) = \frac{\Big(e^{\lambda\chi}k_{11}\big(2\hat{X}_m-\hat{X}_{s1}\big)+l_{11}\big(2\hat{Y}_m-\hat{Y}_{s1}\big)\Big)}{\Big(e^{\lambda\chi}k_{11}\big(2\hat{X}_m+\hat{X}_{s1}\big)+l_{11}\big(2\hat{Y}_m+\hat{Y}_{s1}\big)\Big)}, \qquad (C.30)$$

$$\sigma_1^*(t) = \frac{k_{21}\big(2\hat{X}_m-\hat{X}_{s2}\big)+l_{21}\big(2\hat{Y}_m-\hat{Y}_{s2}\big)}{k_{21}\big(2\hat{X}_m+\hat{X}_{s2}\big)+l_{21}\big(2\hat{Y}_m+\hat{Y}_{s2}\big)}, \qquad (C.31)$$

$$A_1^*(t) = \frac{k_{g1}\hat{X}_m}{\mu_m}. \qquad (C.32)$$

At this stage we note that the simultaneous equations which will give us the solutions of the coefficients of the value function (determined the method of comparison of coefficients from equations

(C.22)) are:

$$
\begin{aligned}
(r+\chi)\hat{X}_{s1} &= \beta\hat{M}_{s1}-\delta_1\hat{X}_{s1}+\chi(1-\eta)X_{s1} && \text{(C.33)}\\
(r+\chi)\hat{Y}_{s1} &= \gamma\hat{M}_{s1}+\chi(1-\xi)Y_{s1} && \text{(C.34)}\\
(r+\chi)\hat{Z}_{s1} &= \Delta_{s1} && \text{(C.35)}\\
(r+\chi)\hat{X}_{s2} &= \beta\hat{M}_{s2}-\delta_1\hat{X}_{s2}+\chi(1-\eta)X_{s2} && \text{(C.36)}\\
(r+\chi)\hat{Y}_{s2} &= \gamma\hat{M}_{s2}+\chi(1-\xi)Y_{s2} && \text{(C.37)}\\
(r+\chi)\hat{Z}_{s2} &= \Delta_{s2} && \text{(C.38)}\\
(r+\chi)\hat{X}_{m} &= \beta\hat{M}_{m}-\delta_1\hat{X}_{m}+\chi(1-\eta)X_{m} && \text{(C.39)}\\
(r+\chi)\hat{Y}_{m} &= \gamma\hat{M}_{m}+\chi(1-\xi)Y_{m} && \text{(C.40)}\\
(r+\chi)\hat{Z}_{m} &= \Delta_{m} && \text{(C.41)}
\end{aligned}
$$

Δ_i is the constant term which we do not use for any derivation. However, we use the constant term only during the numerical analysis.Due to the size of the expression of the constant term, and because it is not used to prove any of our proposition, we omit the expression from this appendix. From the above equations, substituting the values of ϕ_1 and σ_1 from (C.29), (C.30) and (C.31) into (C.25), (C.26) and (C.27) we simplify and manipulate algebraically to get,

$$
\begin{aligned}
q^*_{11}(t) &= \frac{1}{2(\mu_{s1}+(1-\chi)\omega)}\left(\frac{\gamma l_{11}e^{-\lambda\chi}(\chi(2M_m+(1-\xi)M_{s1})+r(2\hat{M}_m+\hat{M}_{s1})}{r(r+\chi)}\right. && \text{(C.42)}\\
&\quad \left.+\frac{\beta k_{11}((2\hat{M}_m+\hat{M}_{s1})(r+\delta_2)+(1-\eta)\chi(2M_m+M_{s1}))}{(r+\delta_2)(r+\delta_1+\chi)}\right),\\
q^*_{21}(t) &= \frac{1}{\mu_{s2}}\left(\chi\left(\frac{k_{21}\beta(1-\eta)}{(r+\delta_2)(r+\delta_1+\chi)}+\frac{l_{21}\gamma(2M_m+(1-\xi)M_{s2})}{r(r+\chi)}\right)+\right.\\
&\quad \left.(2\hat{M}_m+\hat{M}_{s2})\left(\frac{k_{21}\beta}{r+\delta_1+\chi}+\frac{l_{21}\gamma}{r+\chi}\right)\right), && \text{(C.43)}\\
A^*_1(t) &= \frac{k_{g1}\beta(\hat{M}_m(r+\delta_2)+M_m\chi(1-\eta))}{\mu_m(r+\delta_2)(r+\delta_1+\chi)}. && \text{(C.44)}
\end{aligned}
$$

The Proof of **PROPOSITION 1** follows from the derivations of (C.21) and (C.42). ■
The proof of **PROPOSITION 2** follows from the derivations of (C.21) and (C.43). ■
The proof of Proposition 3 can be obtained by putting $\phi_j = \sigma_j = 0$ and just finding the equilibrium values of decision variables - $q_{ij}(t)$. ■
The Proof of Proposition 4 follows from the derivations of (C.21) and (C.29). ■
The proof of Proposition 5 follows from the derivations of (C.21) and (C.30). ■
The Proof of Proposition 7 follows from from the derivations of (C.21) and (C.31). ■

The proof of Proposition 8 follows from from the derivations of (C.21) and (C.44). ■

The Proof of Proposition 9 follows from the solutions of the equations (C.33)-(C.41). ■