Forthcoming at *Operations Research*

Parallel Innovation Contests

Ersin Körpeoğlu
University College London, School of Management, London, UK, E14 5AA, e.korpeoglu@ucl.ac.uk

Gizem Korpeoglu
Department of Industrial Engineering and Innovative Sciences, Eindhoven University of Technology, c.g.korpeoglu@tue.nl

İsa E. Hafalır
University of Technology Sydney Business School, Sydney, Australia, Isa.Hafalir@uts.edu.au

We study multiple parallel contests where contest organizers elicit solutions to innovation-related problems from a set of solvers. Each solver may participate in multiple contests and exert effort to improve her solution for each contest she enters, but the quality of her solution in each contest also depends on an output uncertainty. We first analyze whether an organizer's profit can be improved by discouraging solvers from participating in multiple contests. We show, interestingly, that organizers benefit from solvers participating in multiple contests when the solver's output uncertainty in these contests is sufficiently large. A managerial insight from this result is that when all organizers are eliciting innovative solutions rather than low-novelty solutions, they may benefit from solvers participating in multiple contests. We also show that organizers' average profit increases when solvers participate in multiple contests even when some contests seek low-novelty solutions, as long as other contests seek cutting-edge innovation. We further show that an organizer's profit is unimodal in the number of contests, and the optimal number of contests increases with the solver's output uncertainty. This finding may explain why many organizations run multiple contests in practice, and it suggests running a larger number of contests when the majority of these organizations are seeking innovative solutions rather than low-novelty solutions.

Key words: Competition, Crowdsourcing, Platform, Tournament.

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1. Introduction

With the advancements in information technology and the Internet, organizations have started to look beyond their boundaries in their search for innovation (Chesbrough 2003). For example, 85% of top global brands have used crowdsourcing in the last ten years (Chen et al. 2020). A popular and cost-effective method used for crowdsourcing is an *innovation contest*. In an innovation contest, an organizer elicits innovative solutions to a challenging problem from a group of solvers and gives an award to the solver who submits the best solution.

Contests are becoming increasingly popular; crowdsourcing platforms such as InnoCentive and Topcoder now organize numerous contests each year and generate $1 billion in revenue with an annual growth rate of 37.1% (Chen et al. 2020). InnoCentive, for example, organizes around 200 contests annually for its customers, and these contests are often run in parallel. InnoCentive members often participate in multiple contests and may win cash awards ranging from \$5,000 to \$1 million.[1] Similarly, Topcoder organizes around 6,000 software contests annually, and Topcoder members compete for awards of around \$10,000. Interviews we conducted with practitioners at InnoCentive and Topcoder as part of our research revealed that a crowdsourcing platform either determines the rules for the contests (such as the awards given to winners) on behalf of its customers, or instructs its customers in how to set these rules.[2] These interviews also revealed that a contest platform may encourage or discourage solvers from participating in multiple contests by setting its terms and conditions accordingly.[3]

Besides contest platforms, many other organizations also run multiple contests in parallel. For instance, Elanco, an animal healthcare company, has organized five contests in 2016, eliciting innovative solutions to animal healthcare problems (Elanco 2017). Similarly, the Bill and Melinda Gates Foundation (hereafter, the Gates Foundation) has organized 14 contests in 2016 within the Grand Challenges Explorations initiative, where solvers develop innovative solutions to challenging healthcare problems. Most of these contests are run in parallel, providing solvers with several problems to work on, yet, some of the organizations running them discourage solvers from entering multiple contests. For instance, the Gates Foundation allows submission only to a single contest (GrandChallenges 2017).

Practitioners who run multiple contests need to make several important decisions, and one of these decisions is whether to discourage solvers from participating in multiple contests. If solvers are entering only one contest at a time, their effort will not be divided between multiple contests, so they will be able to devote more effort and attention to the particular contest they are entering. Indeed, most of the scant literature on multiple contests assumes that each solver enters only one contest (e.g., Azmat and Möller 2009). In practice, however, platforms such as InnoCentive allow solvers to enter multiple contests. Another important decision for practitioners is how many contests to run in parallel, because this affects solvers' incentive to exert effort and the quality of

[1] Statistical analysis at InnoCentive reveals that in theoretical challenges where solvers develop solutions with no requirement for implementation, they often work on multiple contests in parallel. Specifically, on four random days within the past twelve months, 57.4% of solvers opened more than one project room to work on multiple contests in live theoretical challenges in a day. Note that this number is likely to be a significant underestimation of the actual percentage of solvers working on multiple contests because this analysis does not take into account solvers who work on some contests offline and those who allocate one day to one contest and the next day to another contest. We thank Graham Buchanan, director of marketing at InnoCentive, for sharing this statistic.

[2] We thank John Elliott, former business development manager at InnoCentive, Greg Bell, former head of marketing and community at Topcoder, and Clinton Bonner, director of marketing and crowdsourcing strategy at Topcoder for providing insights into their operations.

[3] During our interviews with managers at Topcoder, we learned that development challenges that seek low-novelty solutions are designed to focus solvers' efforts on a single contest. In algorithm challenges that seek innovative solutions, solvers quite often work on multiple contests in parallel.

their solutions in each contest. In this paper, we provide insights into these decisions by answering the following research questions. (Q1) When should solvers be discouraged from participating in multiple contests? (Q2) How does the number of contests affect an organizer's profit?

To answer these questions, we build a game-theoretic model of innovation contests where multiple contest organizers elicit solutions from a set of solvers. After all the awards are announced, each solver exerts effort to improve her solution in each contest she enters, where the quality of her solution also depends on her output uncertainty. Our model offers a number of new features that contribute to the theory on innovation contests. First, while previous studies have focused only on a single contest, we analyze multiple parallel contests. This analysis requires us to characterize a multidimensional optimization problem for each solver who decides how much effort to invest in each contest by assessing the total cost of the effort required. This technical contribution is even more pronounced when considering heterogeneous contests with different characteristics. Second, while previous studies assume that a solver can exert unbounded effort and incur an unbounded cost, we consider the solver's budget constraint, as this is more consistent with what happens in practice. Third, building on the economics and operations literature, we factor in two effects that determine the shape of a solver's cost function: (i) each contest exhibits diseconomies of scale, as it may be increasingly difficult for a solver to improve the quality of her solution for a certain contest (e.g., Mihm and Schlapp 2019), and (ii) there are potential economies of scope across contests, as exerting effort in one contest may reduce the cost of effort in another (e.g., Willig 1979, Panzar and Willig 1981).[4] While these novelties increase the complexity of our analysis and require special technical attention, they allow us to capture important aspects of innovation contests in practice.

We answer our first research question by comparing an "exclusive" case where each solver can participate in only one contest with a "non-exclusive" case where each solver can participate in multiple contests. We show that when solvers face sufficiently large output uncertainty, an organizer's profit in the non-exclusive case is larger than in the exclusive case. The intuition is as follows. While an exclusive contest incentivizes solvers to exert more effort, a non-exclusive contest attracts a larger number of solvers and hence a more diverse set of solutions. The diversity effect outweighs the incentive effect when solvers face sufficiently large output uncertainty. This result suggests that practitioners should run non-exclusive contests when seeking innovative solutions and exclusive contests when seeking low-novelty solutions. For example, InnoCentive can achieve the best outcome from theoretical challenges that seek innovative solutions (e.g., finding solutions to increase

[4] Several factors can contribute to economies of scope. For instance, Sutton (2001) mentions the following factors that lead to economies of scope in R&D: "There may be some common elements in the technologies employed along two different [research] trajectories, and know-how accumulated along one trajectory may benefit the firm in its advance along some other trajectory" (page 24). For example, a solver at Topcoder can use the same programming language or the same code fragment in different contests.

the literacy rate of deaf children in developing countries) by encouraging solvers to participate in multiple contests. In contrast, Topcoder can achieve the best outcome from development challenges that seek low-novelty solutions (e.g., finding bugs in a software) by discouraging solvers from participating in multiple contests (e.g., by restricting the number of contests a solver can submit a solution to). We also show that when multiple contests have different characteristics – for example, some seeking low-novelty solutions, whereas others innovative solutions – the non-exclusive case yields a larger average or total profit, although organizers that seek low-novelty solutions may be worse off. Thus, in this case, practitioners should weigh in the overall benefit against the individual loss for some organizers to determine whether to run exclusive or non-exclusive contests.

We next analyze how the number of contests affects an organizer's profit and show that an organizer's profit can increase up to an optimal number of contests. This result holds regardless of whether contests have similar or different characteristics. The intuition of this result depends on the solver's output uncertainty. When the solver's output uncertainty is large, as discussed above, running non-exclusive contests maximizes each organizer's profit, and there is an optimal number of non-exclusive contests. This is because running a greater number of non-exclusive contests may benefit organizers due to the economies-of-scope effect, but it may also be detrimental to them, because solvers may split their efforts among more contests or they may even refrain from participating in some of these contests. We also show, interestingly, that the optimal number of contests increases with the solver's output uncertainty. This finding (along with its intuition) suggests that practitioners who seek innovative solutions may benefit from organizing multiple contests that exhibit economies of scope. When the solver's output uncertainty is small, running exclusive contests maximizes each organizer's profit. In the exclusive case, because each solver enters only one contest, the economies-of-scope effect disappears, but a different trade-off arises. As the number of contests increases, the number of solvers in each contest decreases, thereby incentivizing each solver to exert more effort, but reducing the diversity of solutions. Thus, when the solver's output uncertainty is small (e.g., when organizers seek low-novelty solutions), the incentive effect outweighs the diversity effect, so running multiple contests improves the profit for each organizer.

We extend our main insights to several interesting cases. First, although it is common in the innovation contest literature to assume that solvers are identical, we consider heterogeneity among solvers. Second, while it is also standard to assume that the quality of a solution is an additive function of a solver's effort and output uncertainty, we show that our results still hold when the solution quality is a multiplicative function of effort and output uncertainty. Although these extensions yield the same insights as our main analysis, they contribute to the contest theory because they require both a novel analysis and special technical attention. We hope our analysis can guide future work that aims to incorporate these model components.

Related Literature. Our paper contributes to the literature on innovation contests and in particular helps to extend the scant literature on multiple contests.

Research on innovation contests was pioneered by Taylor (1995) and Fullerton and McAfee (1999), who show that it is optimal to restrict entry to a contest. Terwiesch and Xu (2008) also did pioneering work by proposing a modeling framework and in showing that a free-entry open-innovation contest is optimal. By generalizing the findings of Terwiesch and Xu (2008), Boudreau et al. (2011) show empirically and Ales et al. (2020) show analytically that free entry is optimal only when the solver's output uncertainty is sufficiently large.[5] Various other authors also build on the modeling framework of Terwiesch and Xu (2008). For example, Nittala and Krishnan (2016) examine the design of innovation contests within firms, Ales et al. (2017) study the optimal set of awards in a contest, Mihm and Schlapp (2019) analyze whether and how to give feedback to solvers, Hu and Wang (2021) examine whether to run a single-stage or a sequential contest when solvers' solutions depend on multiple attributes, and Korpeoğlu et al. (2018) look at the optimal duration and award scheme of a contest.[6] Building on the modeling framework of these studies, we contribute to this literature in two ways. First, while these studies look only at a single contest, we consider multiple contests. This multiple-contest environment helps us bridge the gap between theory and practice and contribute to the theory on innovation contests by capturing novel features such as a solver's capacity constraint and economies of scope across contests. Second, we analyze novel research questions relating to when an organizer should discourage solvers from participating in multiple contests and how the number of contests affects an organizer's profit.

Of the relatively few studies that have examined multiple contests, DiPalantino and Vojnović (2009) study multiple all-pay contests with exogenously given awards and characterize equilibria for solvers, yet they do not analyze the optimal decisions for organizers. Azmat and Möller (2009) consider two identical Tullock contests and analyze the optimal award structure for organizers who

[5] Our paper has some similarities to studies that have examined when a free-entry open-innovation contest is optimal. This is because discouraging solvers from participating in multiple contests leads to fewer solvers in each contest, and this then leads to a tradeoff between eliciting greater effort from solvers (by reducing the number of solvers) and obtaining a more diverse set of solutions. Our paper, however, significantly differs from this literature. First, our paper also looks at questions that were outside the scope of previous studies, such as the impact of the number of contests on an organizer's profit. Second, we show our results by considering several aspects that these papers do not, such as a multiplicative output function, heterogeneous solvers, and multiple (possibly asymmetric) contests. Finally, our paper considers other drivers that affect solvers' incentive to exert effort such as economies of scope and splitting effort across multiple contests. Thus, our paper also finds directionally different results from these studies.

[6] For a detailed review of this literature and other types of contests, we refer the reader to Ales et al. (2019). Our paper is broadly related to studies that consider heterogeneous solvers by ignoring uncertainty (e.g., Moldovanu and Sela 2001, Körpeoğlu and Cho 2018, Stouras et al. 2017), to studies that analyze other types of contests (e.g., dynamic contests by Bimpikis et al. 2019), to empirical studies on crowdsourcing (e.g., Jiang et al. 2016, Hwang et al. 2019, Aggarwal et al. 2020), and to theoretical studies on new product development (e.g., Mihm 2010, Lobel et al. 2016).

are competing to attract a set of identical solvers.[7] Büyükboyacı (2016) considers two solvers where each solver exerts a large or small amount of effort and compares running two parallel contests (potentially one solver in each contest) with running a single contest. Hafalır et al. (2018) compare running two all-pay contests with running a single all-pay contest and focus on the equilibrium among solvers without analyzing the optimal decisions for organizers.

It is noteworthy that the scant literature on multiple contests has provided only preliminary answers to some aspects of multiple contests. First, the papers referred to above pay attention only to exclusive contests and overlook non-exclusive contests, and hence they cannot compare the two. Our results, however, confirm the benefits of non-exclusive contests for innovative settings. Second, while these papers assume that an organizer is interested in all solutions, we assume that the organizer is interested in the best solution—an objective more typical of innovation settings (cf. Terwiesch and Xu 2008). Third, while the papers above consider the impact of the solver's effort, we consider how the solver's effort and output uncertainty affect the quality of the solution submitted and thus the organizer's profit. It is well established in the literature that uncertainty plays a prominent role in real-world innovation contests (cf. Boudreau et al. 2011). Finally, our paper considers model components that those papers do not, such as asymmetry across contests and heterogeneous solvers. These aspects of our paper contribute to the theory on innovation contests and help us to generate managerial insights.

Our paper is also related to the economics literature on games with multiple battlefields (i.e., Colonel Blotto games). The seminal paper in this literature is that of Roberson (2006), who characterizes the equilibrium in a game where two colonels simultaneously distribute forces across n battlefields, and within each battlefield the colonel that allocates more forces wins. Kovenock and Roberson (2010) consider more general success functions (where a colonel with more forces does not necessarily win on a battlefield), cost functions, and utility functions for colonels. Hortala-Vallve and Llorente-Saguer (2012) consider opposing parties with different relative intensities and characterize the colonels' payoffs that sustain a pure-strategy Nash equilibrium. Roberson and Kvasov (2012) consider the case where the budget (in terms of total forces) do not have the "use it or lose it" feature. Konrad and Kovenock (2012) characterize equilibria in a model where solvers

[7] Our paper is broadly related to the literature on multiple auctions. The pioneering paper by McAfee (1993) shows that, in equilibrium, sellers hold identical auctions and buyers randomize the sellers they visit. Peters and Severinov (1997) extend the McAfee (1993) model and analyze how reserve prices are determined. In the operations literature, Beil and Wein (2009) consider two competing auctioneers and settings where bidders can participate in both auctions or only a single auction. They show that for multi-item auctions, only the auctioneer with the smaller ratio of bidders per item benefits from the existence of bidders that participate in both auctions. Not only do these papers address different research questions than ours, but there are also three fundamental differences between auctions and contests. First, while auction settings typically involve private information, in contest settings there is moral hazard, because the solver's efforts cannot be observed. Second, an auctioneer maximizes the total bid from bidders, whereas a contest organizer maximizes the quality of the best solution less of the total award. Finally, while the bids in an auction are deterministic, the quality of a solver's solution in a contest depends on the level of output uncertainty.

first choose contests (lifeboats) and then compete in all-pay contests with multiple identical prizes. The main difference between our paper and the aforementioned work is that these papers do not consider organizers; and they just characterize the equilibrium among solvers (colonels), whereas our work analyzes the impact of the solvers' competition on organizers who benefit from solvers' efforts and output uncertainty.

2. The Model

Consider M innovation contests where M contest organizers elicit solutions to innovation-related problems from a set of N solvers ("she"). Below, we describe our model of solvers and organizers and then present the equilibrium.

Solvers. Each solver $i \in \{1,2,...,N\}$ develops a solution for each contest $m \in \{1,2,...,M\}$ she participates in and generates an output $y_{im} \subseteq \mathbb{R} \cup \{-\infty, \infty\}$. The output y_{im} represents the quality of solver i's solution in contest m or its monetary value to organizer m. The output y_{im} is determined by solver i's effort e_{im} in contest m and solver i's output shock $\widetilde{\xi}_{im}$ in contest m, and it takes the following additive form: $y_{im} = y(e_{im}, \widetilde{\xi}_{im}) = r(e_{im}) + \widetilde{\xi}_{im}$. We next elaborate on these two terms.

First, each solver i can improve her output by exerting effort $e_{im} \subseteq \mathbb{R}_+$ in contest m. A solver's effort may represent the set of actions she takes to improve her output, such as "conducting a thorough patent search and literature review, or implementing rigorous quality control systems with high standards" (Terwiesch and Xu 2008, page 1532). For example, a logo designer may exert effort by drawing multiple sketches before choosing the best one to submit (Ales et al. 2017). The effort e_{im} leads to a deterministic improvement $r(e_{im})$ of the output, where r is an increasing and concave function of e_{im}, and r' is homogenous of degree $-k$, where $k \geq 0$. This mild assumption is satisfied by functional forms that are commonly used in the literature, such as linear and logarithmic forms. We assume that all functions in the paper are thrice continuously differentiable.

Second, each solver faces uncertainty while developing her solution, and we capture this uncertainty with an output shock $\widetilde{\xi}_{im}$, which is independent for each solver i and for each contest m.[8] We allow for asymmetry across contests. Specifically, the output shock $\widetilde{\xi}_{im}$ in contest m follows a cumulative distribution function H_m and a density function h_m with $E[\widetilde{\xi}_{im}] = 0$ over support $\Xi_m = [\underline{s}_m, \overline{s}_m]$, where $\underline{s}_m < \overline{s}_m$, $\underline{s}_m \in \mathbb{R} \cup \{-\infty\}$, and $\overline{s}_m \in \mathbb{R} \cup \{\infty\}$. We assume that h_m is log-concave, i.e., $\log(h_m)$ is concave for all $m \in \{1,2,...,M\}$. This property is satisfied by most commonly-used distributions such as the Gumbel distribution used by Terwiesch and Xu (2008), the uniform distribution used by Mihm and Schlapp (2019), and normal, exponential, and logistic distributions.

[8] In practice, there may be some contest-specific dependence due to the uncertainty of the evaluation process. In this case, each solver i's output shock in contest m can be modeled as $\widetilde{\xi}_{im} + \widetilde{\epsilon}_m$ where $\widetilde{\epsilon}_m$ is a shock that is specific to contest m. Because $\widetilde{\epsilon}_m$ terms would appear in all solvers' outputs, they would not affect solvers' rankings or our analysis, and hence we omit them.

Throughout the paper, we analyze the impact of the solver's output uncertainty by changing the spread of the density h_m. For that purpose, we use the notion of a scale transformation (e.g., Rothschild and Stiglitz 1978). When the output shock $\widetilde{\xi}_{im}$ is transformed by a scale transformation with parameter α_m, the transformed random variable $\widehat{\xi}_{im} = \alpha_m \widetilde{\xi}_{im}$ has mean 0, and variance $\alpha_m^2 Var(\widetilde{\xi}_{im})$. Thus, when $\alpha_m > 1$, the transformed density is more spread out. Let $\widetilde{\xi}_m^N$ be a random variable that represents the largest output shock among $\{\widetilde{\xi}_{1m}, \widetilde{\xi}_{2m}, ..., \widetilde{\xi}_{Nm}\}$, and let $\mu_{N,m} = E[\widetilde{\xi}_m^N]$.

Solver i's utility $U_i = U(e_i, x_i) : \mathbb{R}_+^{2M} \to \mathbb{R}$ is defined over the vector of efforts $e_i \equiv (e_{i1}, e_{i2}, ..., e_{iM})$ she exerts and the vector of awards $x_i \equiv (x_{i1}, x_{i2}, ..., x_{iM})$ she receives. Solver i's utility takes the form $U_i = \sum_{m=1}^{M} x_{im} - \psi(e_{i1}, e_{i2}, ..., e_{iM})$, and ψ represents the solver's disutility or cost associated with her effort. We assume that each solver has limited resources with a monetary budget $\overline{B}$ that she can use to cover her total cost. We also assume that ψ has the following properties that seem consistent with contest practice. First, each contest exhibits diseconomies of scale because a solver may have to allocate more time, effort, or money to improve her output at a certain contest. Thus, ψ is increasing in e_{im} with positive second partial derivatives; i.e., $\frac{\partial \psi}{\partial e_{im}} > 0$ and $\frac{\partial^2 \psi}{\partial e_{im}^2} \geq 0$. This property is in line with studies that assume a convex cost of effort in a single contest (e.g., Mihm and Schlapp 2019). Second, as discussed in §1, there are potential economies of scope across contests because when a solver exerts more effort in one contest, the cost of her effort in another contest may decrease due to factors such as common investments (e.g., Willig 1979, Panzar and Willig 1981). For example, a solver who conducts a literature review for a contest at InnoCentive or Topcoder may find it easier to conduct literature reviews for other contests in the same subject category. Thus, ψ has negative cross-partial derivatives; i.e., $\frac{\partial^2 \psi}{\partial e_{il} \partial e_{im}} < 0$ for all $l \neq m$.

As the tractability of the general cost function ψ is limited, we assume the following form:

$$\psi(e_{i1}, e_{i2}, ..., e_{iM}) = \eta \left(\sum_{m=1}^{M} \phi(e_{im}) \right), \tag{1}$$

where η is an increasing and homogenous "scope" function of degree b (< 1) and ϕ is an increasing and homogenous "scarcity" function of degree p (> 1). We further assume that $bp \geq 1$ to ensure that $\eta \circ \phi$ is convex, and that either $bp > 1$ or $k > 0$ (where the derivative of the effort function r' is homogenous of degree $-k$). Lemma EC.4 of Online Appendix shows that the cost function ψ in (1) exhibits both diseconomies of scale and economies of scope as discussed above. Note that when there is a single contest (i.e., $M = 1$), ψ in (1) boils down to a convex cost function that subsumes the cost functions used in the literature, such as $\psi(e) = ce$ used by Terwiesch and Xu (2008), $\psi(e) = ce^{bp}$, where $bp \geq 1$ used by Ales et al. (2020, 2017), and $\psi(e) = ce^2$ used by Mihm and Schlapp (2019). We summarize all of our assumptions below.

Assumption 1. *The cost function* $\psi(e_{i1}, e_{i2}, ..., e_{iM}) = \eta\left(\sum_{m=1}^{M} \phi(e_{im})\right)$*, where the scope function* η *is increasing and homogenous of degree* $b(<1)$*, the scarcity function* ϕ *is increasing and homogenous of degree* $p(>1)$*, and* $bp \geq 1$*. The derivative of the effort function* r' *is homogenous of degree* $-k$ $(k \geq 0)$*, and either* $bp > 1$ *or* $k > 0$*. Density* h_m *is log-concave for all* $m \in \{1, 2, ..., M\}$*.*

Organizers. As is common in practice and in the literature discussed in §1, we assume a winner-take-all award scheme. Specifically, each organizer m gives an award A_m to the solver with the largest output, i.e., the winner of contest m. The winner-take-all award scheme is proven to be optimal in a single contest where the output shock density h_m is log-concave as in our setting (see Proposition 3 of Ales et al. 2017). Under this award scheme, if solver i wins contest m, her award is $x_{im} = A_m$; otherwise, $x_{im} = 0$. Consistent with the innovation contest literature (Terwiesch and Xu 2008, Mihm and Schlapp 2019), we assume that organizers are interested in the largest output in their contests. For example, in a contest for a logo design, an organizer is interested in finding the best logo, which will eventually be implemented. Thus, organizer m's profit Π_m consists of the largest output minus the award given in contest m, i.e., $\Pi_m = \max_i \ y_{im} - A_m$.

The sequence of events is as follows. First, the awards $(A_1, A_2, ..., A_M)$ for all contests are announced, then each solver i determines her effort e_{im} in each contest m she participates in, while considering her total cost of effort ψ. Afterwards, each solver i observes her output shock $\widetilde{\xi}_{im}$, and generates an output y_{im} in each contest m. Finally, each organizer m collects solutions from solvers participating in contest m, and gives the award A_m to the winner with the largest output.

Equilibrium among solvers. We next define and characterize a Nash equilibrium of the subgame among solvers. As is common in the innovation contest literature, we focus on a symmetric pure-strategy Nash equilibrium (hereafter, symmetric equilibrium), and denote each solver's equilibrium effort in contest m by e_m^*. To solve for the equilibrium, we first derive solver i's probability of winning contest m by exerting effort e_{im}, given that all other solvers exert effort e_m^* in contest m:

$$P_m(e_{im}, e_m^*) = \int_{s \in \Xi_m} H_m(s + r(e_{im}) - r(e_m^*))^{N-1} h_m(s) ds. \tag{2}$$

Solver i chooses her effort e_{im} in each contest m to maximize her expected utility subject to a budget constraint by solving the following problem:

$$\max_{(e_{i1}, e_{i2}, ..., e_{iM})} \sum_{m=1}^{M} A_m P_m(e_{im}, e_m^*) - \psi\left(e_{i1}, e_{i2}, ..., e_{iM}\right) \text{ s.t. } \psi\left(e_{i1}, e_{i2}, ..., e_{iM}\right) \leq \overline{B}. \tag{3}$$

In a symmetric equilibrium, each solver exerts the equilibrium effort e_m^* in contest m that solves (3). We solve for the symmetric equilibrium by solving the Kuhn–Tucker conditions of (3). In §EC.1 of the Online Appendix, we investigate when a symmetric equilibrium exists. Because our model is quite general, it is not analytically tractable to characterize precise conditions on our model primitives to ensure the existence of a symmetric equilibrium. However, in Lemma EC.1 and Corollary

Table 1 Summary of key notation in the main body.

M	Number of contests and organizers	N	Number of solvers
e_{im}	Effort of solver i in contest m	r	Effort function; r' is homogenous of degree $-k$
y_{im}	Output of solver i in contest m	ϕ	Scarcity function, homogenous of degree p
ψ	Cost function; $=\eta\left(\sum_{m=1}^{M}\phi(e_{im})\right)$	η	Scope function, homogenous of degree b
$\widetilde{\xi}_{im}$	Shock of solver i in contest m	h_m,H_m	Density and distribution functions of $\widetilde{\xi}_{im}$
$\widetilde{\xi}_m^N$	$=\max\{\widetilde{\xi}_{1m},\widetilde{\xi}_{2m},...,\widetilde{\xi}_{Nm}\}$	$\mu_{N,m}$	$=E[\widetilde{\xi}_m^N]$
g	$=((\eta\circ\phi)'/r')^{-1}$	$\overline{B}$	Solver's budget
A_m	Award in contest m	Π_m, $\overline{\Pi}$	Profit of organizer m and the average profit

EC.1, we provide a limiting sufficient condition for the existence of a symmetric equilibrium that requires the output uncertainty to be sufficiently large and the cost parameter b to be above a certain threshold. Furthermore, in Lemma EC.2, we present precise sufficient conditions on our model primitives by considering the specific settings that are used in the innovation contest literature. Lastly, we numerically show that a symmetric equilibrium exists for a very broad class of distributions and parameter settings. Throughout the paper, we limit our attention to parameter settings that allow a symmetric equilibrium to exist. It is common in the innovation contest literature to assume such parameter settings (e.g., Terwiesch and Xu 2008, Mihm and Schlapp 2019) or to offer limiting sufficient conditions for the existence of equilibrium (e.g., Ales et al. 2017, Hu and Wang 2021). It is worth noting that all our insights are generated from situations where a symmetric equilibrium exists under a nonempty set of parameters and that all our figures illustrate settings in which there is a symmetric equilibrium.

We next characterize the symmetric equilibrium among solvers. In preparation, we let $g(x) = ((\eta\circ\phi)'/r')^{-1}(x)$. All proofs are provided in the Appendix and Table 1 summarizes our key notation.

Proposition 1. *Let* $I_{N,m} \equiv \int_{s\in\Xi_m}(N-1)H_m(s)^{N-2}h_m(s)^2ds$, $\overline{e}_m \equiv \phi^{-1}\left(\frac{\left(A_m I_{N,m}\right)^{\frac{p}{k+p-1}}\eta^{-1}(\overline{B})}{\sum_{l=1}^{M}\left(A_l I_{N,l}\right)^{\frac{p}{k+p-1}}}\right)$, *and* $\widehat{e}_m \equiv g\left(\left(A_m I_{N,m}\right)^{\frac{k+bp-1}{k+p-1}}\left(\sum_{l=1}^{M}\left(A_l I_{N,l}\right)^{\frac{p}{k+p-1}}\right)^{1-b}\right)$. *Suppose Assumption 1 holds. Then, either* $\widehat{e}_m \le \overline{e}_m$ *for all* $m\in\{1,2,...,M\}$ *or* $\widehat{e}_m > \overline{e}_m$ *for all* $m\in\{1,2,...,M\}$*. Furthermore, the unique symmetric equilibrium satisfies the following properties:*

(a) When $\widehat{e}_m \le \overline{e}_m$ *for all* $m\in\{1,2,...,M\}$*, the equilibrium effort* $e_m^* = \widehat{e}_m$ *for all* $m\in\{1,2,...,M\}$*.*

(b) When $\widehat{e}_m > \overline{e}_m$ *for all* $m\in\{1,2,...,M\}$*, the equilibrium effort* $e_m^* = \overline{e}_m$ *for all* $m\in\{1,2,...,M\}$*.*

Proposition 1 has some interesting implications. First, note that a solver's equilibrium effort levels in all contests are interlinked via the common cost function ψ (embedded in the function g) or via the solver's budget $\overline{B}$. Specifically, when the solver's budget constraint does not bind, she determines her effort by balancing "the marginal benefit of additional effort," which is the increase in her expected award, with "the marginal cost of additional effort." In this case, interestingly, a solver's effort in contest m increases with the awards offered in other contests, because larger awards in other contests lead the solver to exert more effort in those contests. Through economies of scope, this reduces the solver's marginal cost of effort, and hence increases her equilibrium effort in contest m. On the other hand, when the solver's budget constraint binds, the solver starts to split her budget across multiple contests, and her effort in contest m decreases as the awards offered in other contests increase because she shifts her budget to those contests that offer larger awards.

While our main model considers no fixed cost of participation and assumes that each solver participates in M contests, in §3.3, we incorporate a fixed cost of participation and also consider the case where each solver participates in a limited number of contests.

Coordinator. In our main analysis, we assume that a *coordinator* determines the awards in all contests. This assumption is consistent with practice for two reasons. First, as discussed in §1, many organizations such as Elanco and the Gates Foundation run multiple contests in parallel, and such organizations will determine the awards for all of their contests. Second, as we discuss in §1, our interviews with practitioners at InnoCentive and Topcoder reveal that a platform of this kind acts as a coordinator either by determining all awards on behalf of its customers or by instructing its customers how to set awards. We assume that the coordinator aims to maximize the expected average profit for organizers (hereafter, average profit), which is given by $\overline{\Pi} \equiv (1/M)(E[\sum_{m=1}^{M} \max_i \; y_{im}] - \sum_{m=1}^{M} A_m)$. Given the equilibrium effort e_m^*, we write $\max_i \; y_{im} = \max_i \; \{r(e_m^*) + \widetilde{\xi}_{im}\} = r(e_m^*) + \max_i \; \widetilde{\xi}_{im} = r(e_m^*) + \widetilde{\xi}_m^N$. Thus, the coordinator's objective is to maximize the average profit, which can be written as:

$$\overline{\Pi} = \frac{\sum_{m=1}^{M} r(e_m^*)}{M} + \frac{E\left[\sum_{m=1}^{M} \widetilde{\xi}_m^N\right]}{M} - \frac{\sum_{m=1}^{M} A_m}{M}. \quad (4)$$

The objective function in (4) may be suitable for a platform because a platform aims to increase the value created for each customer; and in our model, this value is captured by an organizer's profit. The objective function in (4) also seems suitable for an organization such as Elanco or the Gates Foundation while deciding whether to run contests in parallel.[9] On the other hand, when such an organization is determining whether to run a new contest in parallel with others or not to run it at all (and hence to forgo the potential profit), a more suitable objective could be to maximize the total profit $\Pi^{\Sigma} = \sum_{m=1}^{M} \Pi_m$ from contests. We analyze this alternative objective in §EC.2.1 of the Online Appendix. We also show that our main results hold in a decentralized case where organizers determine their own awards; see §EC.2.2 of the Online Appendix.

[9] In practice, another plausible case is that such an organization will determine whether to run contests in parallel or sequentially. As long as parallel contests create larger economies of scope than sequential contests, the results obtained when using a sequential model would be qualitatively similar to our results.

Contest asymmetry. We capture the asymmetry across contests as follows. We suppose that there are J $(\in \{1,2,...,M\})$ contest types and we use the subscript in parentheses to denote type-specific parameters. Specifically, each contest of the same type j $(\in \{1,2,...,J\})$ gives the same award $A_{(j)}$, and has the same output shock distribution $H_{(j)}$ and density $h_{(j)}$ over the same support $\Xi_{(j)}$, and hence the same $I_{N,(j)} = \int_{s\in\Xi_{(j)}}(N-1)H_{(j)}(s)^{N-2}h_{(j)}(s)^2 ds$ and the same equilibrium effort $e^*_{(j)}$. We let $M_{(j)}$ be the number of contests of type j, where $\sum_{j=1}^{J} M_{(j)} = M$.

We conduct our analysis in three stages. In §3, we analyze symmetric contests (i.e., $J=1$) to generate clean insights relating to our research questions. In this case, for notational convenience, we drop type-specific notation (i.e., subscript in parentheses). In §4, we show our main results and generate new insights when contests are asymmetric. In §5 and §EC.2 of the Online Appendix, we consider various extensions to show the robustness of our main insights.

3. Analysis of Symmetric Contests

In this section, we focus on symmetric contests (i.e., $J=1$). We start our analysis by characterizing the optimal set of awards. To do so, we make two assumptions (similar assumptions are common in the literature reviewed in §1). First, we assume that $r'(g(x))g'(x)$ is decreasing in x, which holds if and only if $2-2k-bp<0$, noting that the derivative of the effort function r' is homogenous of degree $-k$, the scope function η is homogenous of degree b, scarcity function ϕ is homogenous of degree p, and $g=((\eta\circ\phi)'/r')^{-1}$. This assumption is so that organizer m's profit Π_m is concave in award A_m. This concavity, along with the solvers' budget constraint, ensures that the coordinator always sets finite awards. Second, we assume that the effort function r is sufficiently concave or the cost function ψ is sufficiently convex (e.g., $b>0$ and $k\geq 1$, or $k>0$ and b is close to 1) so that solvers face sufficiently diminishing marginal returns compared to the cost they incur. Note that all assumptions we make regarding r and ψ are satisfied by effort and cost functions that are commonly used in studies focusing on a single contest. For example, our assumptions hold under the Terwiesch and Xu (2008) model, where $r(e)=\theta\log(e)$, $\psi(e)=ce$, and $\theta,c>0$; under the Ales et al. (2020, 2017) model, where $r(e)=\theta(e^{1-k}-1)/(1-k)$, $\psi(e)=ce^{pb}$, $k\geq 1$, $pb\geq 1$, and $b\in(0,1)$; and under the Mihm and Schlapp (2019) model, where $r(e)=\theta e$, $\psi(e)=ce^{pb}$, $\theta,c>0$, $pb=2$, and b is sufficiently close to 1. We summarize these assumptions below.

Assumption 2. *$2-2k-bp<0$ and the effort function r is sufficiently concave or the cost function ψ is sufficiently convex (e.g., $b>0$ and $k\geq 1$, or $k>0$ and b is close to 1).*

The following lemma characterizes the optimal set of awards.

Lemma 1. *Suppose Assumptions 1 and 2 hold. Let $\Phi(A) = r'(e^*)g'(AI_NM^{1-b})I_NM^{1-b} - 1$ and $\overline{A} = M^{b-1}g^{-1}\left(\phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)\right)/I_N$.*

(a) *If* $\Phi(\overline{A}) \geq 0$*, then the average profit* $\overline{\Pi}$ *is maximized at the optimal award* $A_m^* = A^* = \overline{A}$ *and equilibrium effort* $e_m^* = e^* = \phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)$*. If* $\Phi(\overline{A}) < 0$*, then there exists a unique* $\widehat{A}$ *such that* $\Phi(\widehat{A}) = 0$*, and* $\overline{\Pi}$ *is maximized at* $A_m^* = A^* = \widehat{A}$ *and* $e_m^* = e^* = g(A^* I_N M^{1-b})$*.*

(b) $\overline{A}$ *is decreasing in the number of contests* M*, and* $\widehat{A}$ *is increasing, constant, or decreasing in* M *when the degree of the derivative of the effort function* r'*,* $k < 1$*,* $k = 1$*, or* $k > 1$*, respectively.*

Lemma 1(a) shows that the average profit $\overline{\Pi}$ is maximized when the award in each contest is A^*. This is because solvers face diminishing marginal returns in their efforts, so balancing solvers' efforts using identical awards improve the average of best outputs across all contests $\left(\text{i.e., } \frac{1}{M}\sum_{m=1}^{M}(r(e_m^*) + \mu_N)\right)$, and in turn improves $\overline{\Pi}$. Let Π^* be an organizer's profit when the award in each contest is A^*. Under the optimal award A^* in Lemma 1, the average profit can be written as:

$$\overline{\Pi} = \Pi^* = r(e^*) + \mu_N - A^*. \tag{5}$$

Lemma 1(a) also shows that the optimal award A^* depends on whether or not the solver's budget constraint is binding. When it is not binding, it is optimal for the coordinator to set the awards to balance the marginal benefit and the marginal cost of an award on the average profit. However, when the solver's budget constraint is binding, it is optimal for the coordinator to set the awards at $\overline{A}$, which is just enough to induce each solver to incur a cost of $\overline{B}$, because a larger award cannot improve a solver's effort due to the budget constraint.

Lemma 1(b) shows that when the budget constraint binds, increasing the number of contests M reduces the optimal award $A^* = \overline{A}$. This is intuitive because increasing M leads solvers to split their efforts more, so the equilibrium effort e^* decreases, and hence the incentive effect of the award in terms of eliciting effort decreases with M. Lemma 1(b) also shows that when the budget constraint does not bind, the optimal award $A^* = \widehat{A}$ can be increasing, constant, or decreasing in M depending on the parameter k. We explain the intuition for the case where $k < 1$ (i.e., A^* increases with M) but the same idea applies to $k = 1$ and $k > 1$. When the marginal contribution of award A to the average profit increases, the optimal award A^* increases. By Lemma 1(a), A^* increases when increasing M raises $r'(e^*)\frac{\partial e^*}{\partial A} = r'(e^*)g'(AI_N M^{1-b})I_N M^{1-b}$. Because the equilibrium effort increases with M, and the effort function r is concave, increasing M decreases the marginal contribution of effort to output r (i.e., $r'(e^*)$), but increases the marginal contribution of the award on eliciting effort (i.e., $\frac{\partial e^*}{\partial A}$). When the marginal contribution of effort on increasing output is inelastic to a change in effort (i.e., $\left|\frac{d\log(r'(e))}{d\log(e)}\right| = -\frac{r''(e)e}{r'(e)} = k < 1$), the latter positive effect of larger M dominates the former negative effect (so increasing M reduces $r'(e^*)\frac{\partial e^*}{\partial A}$). Thus, A^* increases with M.

The rest of this section proceeds as follows. In §3.1, we compare exclusive and non-exclusive contests. In §3.2, we analyze how an organizer's profit changes with the number of contests. In §3.3, we enrich our analysis by first incorporating a fixed cost of participation and then considering each solver's participation in a limited number of contests. In §3.4, we discuss managerial insights.

3.1. Exclusive versus Non-Exclusive Contests

In this section, we analyze when solvers should be discouraged from participating in multiple contests. In practice, an organization such as the Gates Foundation or a platform such as Topcoder can discourage solvers from participating in multiple contests, for example, by allowing submission only to a single contest. We refer to the case where each solver can participate in only one contest as the exclusive case, and the case where each solver can participate in multiple contests as the non-exclusive case. Note that in our model and in Lemmas 1 and 2, $M \geq 1$ characterizes equilibrium and optimal awards under M non-exclusive contests, and $M = 1$ characterizes equilibrium and optimal awards under a single exclusive contest. We next compare exclusive and non-exclusive cases.[10]

Theorem 1. *Suppose Assumption 1 holds. Let $\overline{\Pi}^X$ be the average profit when the coordinator allocates solvers and awards optimally in the exclusive case. Suppose the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha > 0$. Then, there exists α_0 such that the average profit in the non-exclusive case $\overline{\Pi}$ is greater than that in the exclusive case $\overline{\Pi}^X$ for any $\alpha > \alpha_0$.*

Theorem 1 shows that when the solver's output uncertainty is sufficiently large, the non-exclusive case yields a larger average profit than the exclusive case; see Figure 1. To generate further insights, we use the following effort and cost functions that subsume the effort and cost functions that are commonly used in the literature (e.g., Terwiesch and Xu 2008, Körpeoğlu and Cho 2018).

Assumption 3. *$r(e) = \theta\log(e)$, $\eta(e) = ce^b$, and $\phi(e) = e^p$, where $\theta, c > 0$, $b \in (0,1)$, and $p \geq 1/b$.*

The following corollary shows that Theorem 1 is not an asymptotic result, and it characterizes α_0.

Corollary 1. *Consider two exclusive contests with N_1 and N_2 solvers, and let $\overline{\Pi}^X$ be the average profit in this case. Suppose Assumptions 1 and 3 hold, and that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with $\alpha > 0$. Let $\alpha_1 \equiv \frac{\theta^2 \max\{I_{N_1}, I_{N_2}, 2I_{N_1+N_2}\}}{p^2 b^2 \overline{B}}$, $\alpha_2 \equiv \frac{\theta}{bp} \frac{\log\left(I_{N_1} I_{N_2}\right) - 2\log\left(2^{1-b} I_{N_1+N_2}\right)}{2\mu_{N_1+N_2} - \mu_{N_1} - \mu_{N_2}}$, $\alpha_3 \equiv \frac{\theta^2 \min\{I_{N_1}, I_{N_2}, 2I_{N_1+N_2}\}}{p^2 b^2 \overline{B}}$, and $\alpha_4 \equiv \frac{\theta}{p} \frac{\log(2)}{\mu_{N_1+N_2} - \frac{\mu_{N_1}+\mu_{N_2}}{2} + \frac{bp\overline{B}}{2\theta}\left[\frac{1}{I_{N_1}} + \frac{1}{I_{N_2}} - \frac{1}{I_{N_1+N_2}}\right]}$.*
(i) When $\overline{B}$ is sufficiently large, $\overline{\Pi}$ is greater than $\overline{\Pi}^X$ if and only if $\alpha \geq \alpha_0 = \alpha_2$.
(ii) $\overline{\Pi}$ is greater than $\overline{\Pi}^X$ if $\alpha \geq \alpha_0 \equiv \max\{\alpha_1, \alpha_2\}$.
(iii) $\overline{\Pi}$ is less than $\overline{\Pi}^X$ if $\alpha \leq \min\{\alpha_3, \alpha_4\}$.

We next discuss the intuition of Theorem 1 and Corollary 1 using Figure 1. The average profit $\overline{\Pi}$ depends on the effort term $r(e^*)$, the shock term $\widehat{\mu}_N$ $(= E[\widehat{\xi}_m^N] = E[\alpha\widetilde{\xi}_{(1)m}^N])$, and the award term A^*. Figure 1 compares these three terms and the average profit in exclusive and non-exclusive cases as a function of the scale parameter α under Assumption 3 and under the assumption that the

[10] In the exclusive case, we assume that the coordinator determines awards and allocates solvers to contests optimally. Note that the average profit in this case is an upper bound for the average profit when each solver endogenously selects which contest to enter. Thus, our result applies directly to the case with endogenous entry as well.

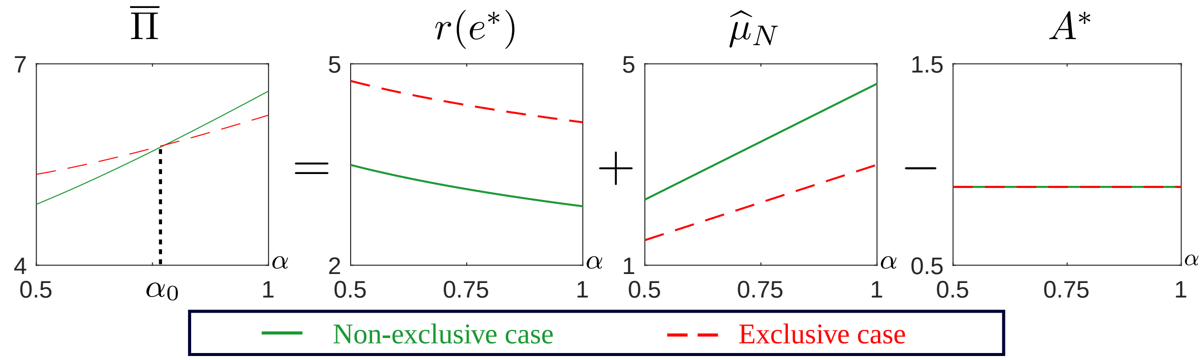


Figure 1 The average profit $\overline{\Pi}$ and the effort, shock, and award terms, respectively, in exclusive and non-exclusive cases as a function of the scale parameter α. Setting: $\widetilde{\xi}_{im} \sim$ Gumbel with mean 0 and scale parameter 1, $M = 5$, $N = 100$, $\overline{B} = 5$, $r(e) = 2\log(e)$, $\eta(e) = 0.1e^{0.9}$, and $\phi(e) = e^{2.5}$.

solver's budget $\overline{B}$ is sufficiently large (as in Corollary 1(i)). Because the award term in this figure is the same in both cases, whether the average profit is larger in the exclusive or the non-exclusive case depends only on effort and shock terms. On one hand, the shock term $\widehat{\mu}_N$ is greater in the non-exclusive case than in the exclusive case, because a non-exclusive contest attracts a larger number of solvers and a more diverse set of solutions can thus be obtained. On the other hand, the effort term $r(e^*)$ is larger in the exclusive case than in the non-exclusive case, because in an exclusive contest, a smaller number of solvers are competing, so each solver will exert more effort. In Figure 1, as the solver's output uncertainty (measured by α) increases, the difference between shock terms in non-exclusive and exclusive cases increases, whereas the difference between effort terms stays the same. Thus, when α is above a threshold α_0, the difference between shock terms outweighs the difference between effort terms, so the average profit is larger in the non-exclusive case than in the exclusive case. Note that, in the general setting of Theorem 1, the difference between effort terms and award terms in exclusive and non-exclusive cases can also increase with the scale parameter α, yet we show that when α is sufficiently large, the difference between shock terms outweighs the difference between effort terms and between award terms.

Corollary 1 shows that when the solver's budget constraint binds in both the exclusive and the non-exclusive cases (i.e., when $\alpha < \alpha_3$), the exclusive case benefits from solvers' focused efforts. Specifically, in the non-exclusive case, each solver splits her budget among multiple contests, whereas in the exclusive case she can allocate all her budget to a single contest. Thus, the exclusive case elicits greater effort. When the output uncertainty is small (when $\alpha < \max\{\alpha_3, \alpha_4\}$), the diversity effect is also small, so the exclusive case yields a larger average profit than the non-exclusive case. However, when the output uncertainty is sufficiently large (i.e., $\alpha > \alpha_2$), the total effort is small, so the budget constraint no longer binds, and a non-exclusive contest will generate a more diverse set of solutions. Thus, when the output uncertainty is sufficiently large (when $\alpha > \min\{\alpha_1, \alpha_2\}$), the non-exclusive case yields a larger average profit than the exclusive case.

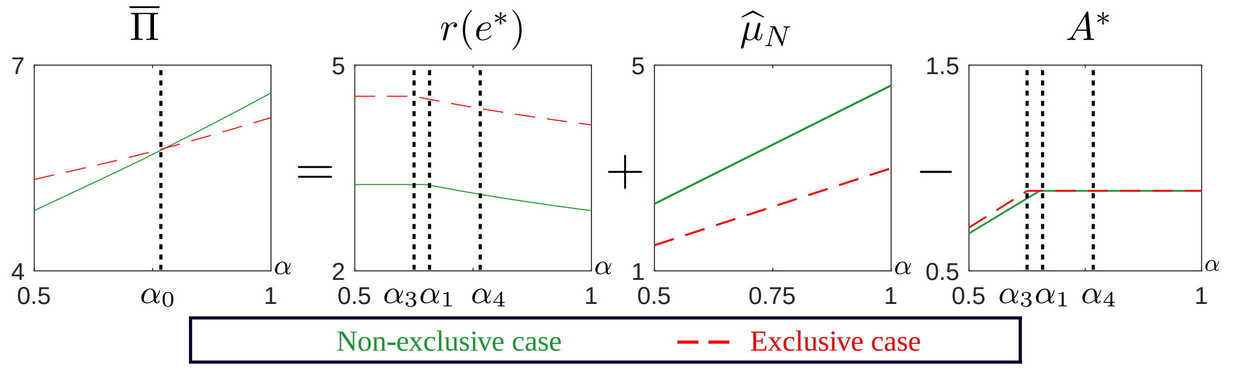


Figure 2 The average profit $\overline{\Pi}$ and effort, shock, and award terms, respectively, in exclusive and non-exclusive cases as a function of the scale parameter α. The setting is the same as Figure 1 except that $\overline{B} = 0.6$.

Theorem 1 and Corollary 1 have important implications for the contest theory and practice. First, these results suggest that in practice, organizers may benefit from running non-exclusive contests when they are seeking innovative solutions rather than low-novelty solutions.[11] For example, InnoCentive could maximize the outcome of theoretical challenges that seek innovative solutions by encouraging solvers to participate in multiple contests. In contrast, Topcoder could maximize the outcome of development challenges that seek low-novelty solutions by discouraging solvers from participating in more than one of these contests (e.g., by restricting the number of contests a solver can submit solutions to at any given time frame). Second, although many studies assume contests to be exclusive, solvers participating in multiple contests is not only common in practice (see §1) but is also often beneficial to organizers, as Theorem 1 and Corollary 1 show. Thus, although assuming exclusive contests may be reasonable for the specific examples considered in previous studies, relaxing this assumption is essential for studying multiple innovation contests. Therefore, in the following section we analyze non-exclusive contests, and address exclusive contests in §3.3.

3.2. Optimal Number of Contests

In this section, we assume contests to be non-exclusive and analyze how the average profit $\overline{\Pi}$, as well as each organizer's profit Π^*, changes with the number of contests M.

Theorem 2. *Suppose Assumption 1 holds. The average profit $\overline{\Pi}$ and an organizer's profit Π^* are unimodal in the number of contests M, i.e., there exists $M^* \in [1, \infty)$ such that $\frac{\partial \overline{\Pi}}{\partial M} > 0$ and $\frac{\partial \Pi^*}{\partial M} > 0$ for all $M < M^*$; and $\frac{\partial \overline{\Pi}}{\partial M} < 0$ and $\frac{\partial \Pi^*}{\partial M} < 0$ for all $M > M^*$.*

[11] In the innovation contest literature, the solver's output uncertainty is often associated with the novelty of solutions (e.g., Terwiesch and Xu 2008). In particular, solvers face little uncertainty in contests that seek low-novelty solutions, whereas they face much greater uncertainty in contests that seek innovative solutions. Nittala and Krishnan (2016) relate the solver's output uncertainty to how broadly an organizer defines a problem, which may be linked to how greater a degree of novelty an organizer is seeking in solvers' solutions.

Theorem 2 shows that there is an optimal number of contests M^* that maximizes both the average profit $\overline{\Pi}$ and each organizer's profit Π^*; see Figure 3.[12] The intuition is as follows. Until the solver's budget constraint binds, her total effort increases with the number of contests M because it is optimal for her to participate in more contests (see the proof of Theorem 2). Increasing the number of contests M has two effects. First, when a solver's budget constraint binds, she splits her total effort among more contests, and hence exerts less effort in each contest. This "scarcity effect" reduces Π^*. Second, as M increases, each solver enjoys larger economies of scope, which can be utilized by the coordinator to reduce the award A^* in each contest. This "scope effect" improves Π^*. When the number of contests is small (see region A in Figure 3, where $M \leq 5$), the solver's budget constraint does not bind, so there is no scarcity effect. Hence, the scope effect leads to a larger profit for each organizer. When the number of contests is large (see regions B and C in Figure 3, where $M > 5$), the solver's budget constraint binds, so the scarcity effect is positive. However, the benefit derived from the scope effect mitigates the reduced effort due to the scarcity effect, so each organizer's profit increases up to the optimal number of contests M^* (see region B in Figure 3, where $M^* = 8$). When the number of contests is above M^*, the benefit derived from the scope effect no longer mitigates the reduced effort due to the scarcity effect, so each organizer's profit decreases (see region C in Figure 3). Thus, each organizer's profit Π^* is unimodal in M, and there is an optimal number of contests M^*. This result suggests that an organization such as Elanco or the Gates Foundation may benefit from running multiple contests that exhibit economies of scope (due to common investment), but only up to the optimal number of contests M^*. The following corollary shows, interestingly, that M^* increases with the solver's output uncertainty.

Corollary 2. *Suppose that Assumption 1 holds, the optimal number of contests $M^* > 1$, and the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with parameter $\alpha > 0$. Then, M^* is increasing in α.*

Corollary 2 shows that the optimal number of contests M^* is closely related to the spread of the output shock $\widetilde{\xi}_{im}$. Specifically, when the spread of the output shock $\widetilde{\xi}_{im}$ increases via a scale transformation with $\alpha > 1$, the optimal number of contests M^* increases. The intuition is as follows. As the solver's output uncertainty increases, the marginal impact of her effort on her expected total award decreases, so she reduces her effort. Less effort reduces both the scarcity effect and the scope

[12] It is worth noting that the solver's probability of winning in our model boils down to the Tullock contest success function $e_{im}/(\sum_{j=1}^{N} e_{jm})$ (cf. Azmat and Möller 2009) when the effort function $r(e) = \log(e)$ and the output shock $\widetilde{\xi}_{im}$ follows a Gumbel distribution with mean zero and scale parameter 1. Even in that case, an innovation contest differs from a Tullock contest because in a Tullock contest, the organizer is interested in the total effort of solvers, whereas in an innovation contest, the organizer is interested in the best output of solvers, which consists of both the equilibrium effort and the maximum of output shocks (i.e., $\max_{i\in\{1,2,\ldots,N\}} \widetilde{\xi}_{im}$). Because of this difference, in Tullock contests the exclusive case always yields a larger average profit than the non-exclusive case, so Theorem 1 does not hold. Theorem 2, on the other hand, directly applies to Tullock contests.

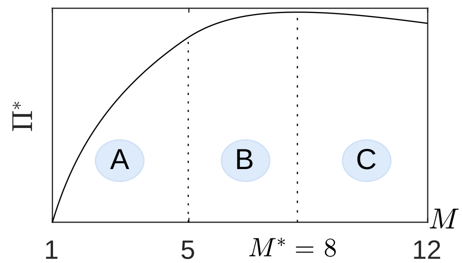


Figure 3 M values where there is no scarcity effect (A), there is a scarcity effect but it is dominated by the scope effect (B), and the scope effect is dominated by the scarcity effect (C). Setting: $\widetilde{\xi}_{im} \sim$ Gumbel with mean 0 and scale parameter 1, $N = 100$, $\overline{B} = 0.15$, $r(e) = \log(e)$, $\eta(e) = 0.1e^{0.6}$, and $\phi(e) = e^3$.

effect. However, as we show in Corollary 2, the scarcity effect decreases with the solver's output uncertainty more than the scope effect. Hence, the scope effect outweighs the scarcity effect up to a larger number of contests M^*. This finding suggests that organizers will benefit from a larger number of contests when they are seeking innovative solutions rather than low-novelty solutions.

3.3. Fixed Cost of Participation and Participation in a Limited Number of Contests

In this section, we enrich our analysis by first incorporating a fixed cost of participation and then considering a case in which each solver participates in a limited number of contests.

Fixed cost of participation. We consider a case where each solver incurs a fixed cost c_f for each contest she participates in and analyze the impact of the fixed cost on the solver's participation in multiple contests. As setting equal awards for all contests is optimal (see Lemma EC.5 of Online Appendix), we assume that the award for each contest is A. To isolate the impact of the fixed cost, we omit the solver's budget constraint, so her utility from participating in M contests is

$$U[M] = \frac{AM}{N} - M^b \eta(\phi(e^*)) - Mc_f, \tag{6}$$

where e^* is the equilibrium effort as given in Lemma 1. If the solver's participation condition holds (i.e., $U[M] \geq 0$), the solver finds it beneficial to participate in M contests.[13] We assume that the fixed cost c_f is not prohibitively high so that under award A, each solver participates in at least one contest (i.e., $U[1] \geq 0$). The following proposition characterizes the relationship between the solver's participation and the number of contests M.

Proposition 2. *Suppose Assumption 1 holds.*
(a) Suppose $k \geq 1$. The solver's participation condition holds for any M.
(b) Suppose $k < 1$, and that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha > 0$. Then, there exists a unique $\overline{M}$ such that the solver's participation condition is violated when $M > \overline{M}$. Also, $\overline{M}$ is increasing in α.

[13] Alternatively, one may define the solver's participation condition as $U[M] = \max_{m\in\{1,2,\ldots,M\}}\{U[m]\}$. Proposition 2 holds under this definition as well.

Proposition 2(a) shows, interestingly, that having a fixed cost does not necessarily limit the number of contests that each solver participates in. The intuition is as follows. The solver's participation condition (i.e., $U[M] \geq 0$) depends on the solver's utility $U[M]$, and the number of contests M has two opposing effects on $U[M]$. On one hand, as M increases, the solver can improve her expected total award by participating in more contests, and this raises the solver's utility $U[M]$. On the other hand, each solver increases her effort e^* to compete in more contests and to benefit from economies of scope, and this reduces $U[M]$. Depending on which effect is more dominant, the solver's utility can increase or decrease. When $k \geq 1$ (where r' is homogenous of degree $-k$), the marginal impact of the solver's effort on her output decreases quickly. Hence, as M increases, she does not increase her total effort significantly, leading to a small increase in her cost of effort. Larger expected total award dominates larger cost of effort, so the solver's utility increases with M (see the proof of Proposition 2). Thus, the solver's participation condition holds for any M.

Proposition 2(b) shows that when $k < 1$, the solver's participation condition holds for a limited number of contests. The intuition is as follows. When $k < 1$, the marginal impact of the solver's effort on her output decreases slowly, so as M increases, she increases her total effort significantly, leading to a substantial increase in her cost of effort. The increased cost of effort dominates the increased expected total award, eventually leading her utility to decrease. Thus, when a solver participates in more than $\overline{M}$ contests, her participation condition is violated. Proposition 2(b) further shows that $\overline{M}$ increases with the solver's output uncertainty. This result is in line with Corollary 2, which shows that the optimal number of contests M^* increases with the solver's output uncertainty. Thus, these results suggest that even when there is a fixed cost of participation, both organizers and solvers benefit from a larger number of contests when organizers are seeking innovative solutions rather than low-novelty solutions.

Solver's participation in a limited number of contests. In practice, a solver may participate in a limited number of contests, either because these contests are exclusive as in §3.1 or because her participation condition prevents her from entering all contests (even though these contests are non-exclusive) as discussed above. We next analyze this case. For tractability, we consider a setting with N solvers where each solver enters a single contest. We compare the average profit when N solvers enter a single contest with the average profit in a two-contest setting where N_1 solvers enter one contest and $N_2 (= N - N_1)$ solvers enter the other contest. To isolate the impact of a solver's participation in a limited number of contests, we again omit the budget constraint.

Proposition 3. *Suppose that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha \widetilde{\xi}_{im}$ with a parameter $\alpha > 0$. Under Assumptions 1 and 3, two contests with N_1 and N_2 solvers yield a larger average profit $\overline{\Pi}^L$ than a single contest with $N_1 + N_2$ solvers if and only if $\alpha < \alpha_L \equiv \frac{\theta}{bp} \frac{\log(I_{N_1} I_{N_2}) - 2\log(I_{N_1+N_2})}{2\mu_{(1)}^{N_1+N_2} - \mu_{N_1} - \mu_{N_2}}$.*

Proposition 3 shows that when each solver participates in a limited number of contests, the average profit $\overline{\Pi}^L$ increases with more contests if and only if the solver's output uncertainty is sufficiently small. This is because each organizer's profit (denoted by $\Pi_m^{*,L}$ for $m = 1, 2$) increases with the number of contests M if and only if the solver's output uncertainty is sufficiently small. The intuition is as follows. Let N_m be the number of solvers in contest m. When each solver participates in a subset of contests, as M increases, solvers are split among more contests, so the number of solvers N_m in each contest m decreases. In contest m, this decrease in N_m can affect the organizer's profit $\Pi_m^{*,L} = r(e^*) + \widehat{\mu}_{N_m} - A^*$ through the effort term $r(e^*)$, the shock term $\widehat{\mu}_{N_m}$, and the award term A^*. First, the award term $A^* = \theta/(bp)$ in the setting of Proposition 3, so A^* does not change with N_m. Second, as N_m decreases, fewer solvers compete in contest m, and the impact of each solver's effort on her expected total award is generally larger, so each solver in contest m generally exerts more effort.[14] Thus, the effort term $r(e^*)$ generally increases as N_m decreases. Third, as N_m decreases, organizer m receives a less diverse set of solutions; i.e., the shock term $\widehat{\mu}_{N_m}$ decreases. When the solver's output uncertainty is small, the increase in the effort term $r(e^*)$ outweighs the decrease in the shock term $\widehat{\mu}_{N_m}$, so each organizer's profit $\Pi^{*,L}$ increases with more contests. In contrast, when the solver's output uncertainty is large, the decrease in the shock term outweighs the increase in the effort term, so each organizer's profit decreases with more contests.

3.4. Managerial Insights

In this section, we discuss the key managerial insights from our results. We classify these insights based on the solvers' output uncertainty and summarize them in Table 2.

When the solvers' output uncertainty is small, Theorem 1 and Corollary 1 show that each organizer's profit is maximized if solvers are discouraged from participating in multiple contests; i.e., exclusive contests are optimal. Proposition 3 builds on this result and shows that it is optimal to run multiple exclusive contests where each solver participates in a single contest. Thus, we advise practitioners who seek low-novelty solutions to run multiple contests in parallel but to discourage solvers from participating in more than one contest. This insight seems consistent with practice. For instance, as discussed in §1, Topcoder organizes multiple parallel development challenges that seek low-novelty solutions but aims to focus each solver's effort on just one of these contests.

When solvers' output uncertainty is large, Theorem 1 and Corollary 1 show that each organizer's profit is maximized if solvers are encouraged to participate in multiple contests; i.e., non-exclusive contests are optimal. Theorem 2 builds on this result, and shows that each organizer's profit

[14] We use the word "generally" because Ales et al. (2020) show that the equilibrium effort e^* decreases with the number of solvers N in a contest for most commonly used distributions for the output shock (e.g., exponential, Gumbel, logistic, or normal distribution).

Table 2 Summary of key results and managerial insights.

	Low level of uncertainty (e.g., when seeking low-novelty solutions)	High level of uncertainty (e.g., when seeking innovative solutions)
Exclusive vs non-exclusive	Exclusive contests are optimal (Theorem 1 and Corollary 1).	Non-exclusive contests are optimal (Theorem 1 and Corollary 1).
Multiple contests or not	Running more contests than any one solver can participate in improves each organizer's profit (Proposition 3).	Each organizer's profit increases with the number of contests up to an optimal number of contests (Theorem 2).
Managerial insights	Advisable to run multiple contests in parallel (up to a certain number) but discourage solvers from participating in more than one of these contests.	Advisable to run multiple contests in parallel (up to a certain number) and encourage solvers to participate in several of these contests.

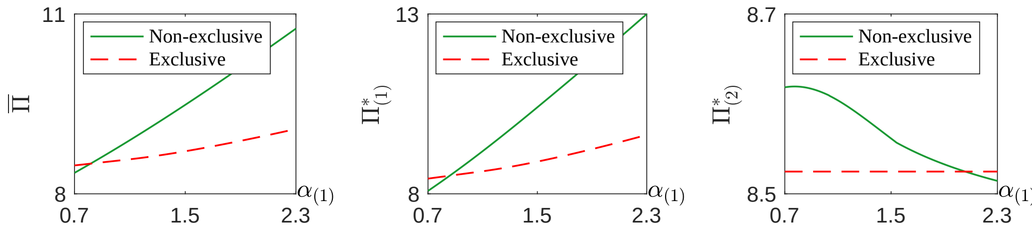
increases with the number of contests M only up to an optimal number of contests M^*. Consistent with this finding, Proposition 2(b), together with Proposition 3, suggests that each organizer's profit decreases as M exceeds the threshold $\overline{M}$ over which the solver's participation condition is violated. These results together show that each organizer's profit increases with M only up to $\min\{M^*, \overline{M}\}$. Interestingly, Corollary 2 and Proposition 2(b) show that $\min\{M^*, \overline{M}\}$ increases with the solver's output uncertainty. Combining all these findings, we advise practitioners who are seeking innovative solutions to run multiple parallel contests up to a certain threshold and to encourage solvers to participate in multiple contests. This insight seems to be consistent with practice. For instance, as discussed in §1, InnoCentive organizes multiple parallel theoretical challenges that seek innovative solutions, and solvers are encouraged to participate in several of these contests.

4. Analysis of Asymmetric Contests

In this section, we show that our main results hold for asymmetric contests and we analyze various aspects of such contests. We first provide a generalized version of Theorem 1.

THEOREM 3. *Suppose that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with parameter $\alpha > 0$ for a subset of contests $\mathcal{M}_I$. Under Assumption 1, there exists α_0 such that the average profit $\overline{\Pi}$ in the non-exclusive case is greater than the average profit $\overline{\Pi}^X$ in the exclusive case for any $\alpha > \alpha_0$.*

Theorem 3 not only extends Theorem 1 to asymmetric contests, but also shows a stronger result. Specifically, when the output uncertainty is sufficiently large for a subset of contests, the non-exclusive case yields a larger average profit than the exclusive case. Thus, when only a subset of contests seek cutting-edge innovation, even if other contests seek low-novelty solutions, the non-exclusive case yields a larger average profit than the exclusive case. The intuition is similar to Theorem 1. Whereas exclusive contests can elicit greater effort, non-exclusive contests benefit from a more diverse set of solutions. When a subset of contests are seeking cutting-edge innovation, the



(a) Average profit. (b) Type 1 organizer's profit. (c) Type 2 organizer's profit.

Figure 4 Comparison of (a) the average profit $\overline{\Pi}$ and (b)–(c) the organizer's profit $\Pi^*_{(j)}$ for contests of type j ($\in \{1,2\}$) under non-exclusive and exclusive cases as the output shock in type 1 contests is scale-transformed with a scale parameter $\alpha_{(1)}$. The setting is as in Example 1 where $\nu_{(1)} = 1$, $\nu_{(2)} = 0.7$, $M_{(1)} = M_{(2)} = 1$, $N = 40$, $\overline{B} = 0.05$, $r(e) = 2\log(e)$, $\eta(e) = 0.01e^{0.75}$, and $\phi(e) = e^{2.5}$.

diversity benefit in these contests outweighs potentially lower level of efforts in all contests. To generate further insights, we consider the following example.

Example 1. Suppose the setting in Assumption 3 and that $J = 2$, where the output shock $\widetilde{\xi}_{im}$ of solver i in a contest of type j follows Gumbel distribution with scale parameter $\nu_{(j)}$, $j \in \{1,2\}$.

Figure 4(a) depicts the average profit for exclusive and non-exclusive cases under the setting of Example 1 when the output uncertainty in type 1 contests increases via a scale transformation with a parameter $\alpha_{(1)}$. The figure illustrates Theorem 3 by showing that the non-exclusive case yields a larger average profit than the exclusive case when the output uncertainty in type 1 contests (measured by $\alpha_{(1)}$) increases. This does not necessarily mean, however, that the non-exclusive case leads to larger profits for both types of organizers. Specifically, an increase in $\alpha_{(1)}$ may increase or decrease the profits of type 2 organizers under the non-exclusive case, whereas it has no effect on the profits of type 2 organizers under the exclusive case; see Figure 4(c). The reason is as follows. An increase in $\alpha_{(1)}$ reduces the equilibrium effort in any type 1 contest. When the solver's budget constraint binds, reduced effort in type 1 contests leads to greater effort in type 2 contests. Thus, the profits of type 2 organizers increase with the uncertainty in type 1 contests. When the solver's budget constraint does not bind, reduced effort in type 1 contests leads to reduced effort in type 2 contests, due to lower economies of scope. Thus, the profits of type 2 organizers decrease with the uncertainty in type 1 contests. The exclusive case may therefore lead to larger profits for type 2 contests, although the average profit is larger under the non-exclusive case.

The following theorem analyzes the impact of the number of contests on organizers' profits.

Theorem 4. *Suppose Assumption 1 holds.*

(a) *For any set of awards $(A_{(1)}, A_{(2)}, ..., A_{(J)})$, an organizer's profit $\Pi^*_{(l)}$ in any contest of type $l \in \{1,2,...,J\}$ is unimodal in the number of contests $M_{(j)}$ of any contest type $j \in \{1,2,...,J\}$; i.e., there exists $M^*_{(j)}$ such that $\frac{\partial \Pi^*_{(l)}}{\partial M_{(j)}} > 0$ for all $M_{(j)} < M^*_{(j)}$; and $\frac{\partial \Pi^*_{(l)}}{\partial M_{(j)}} < 0$ for all $M_{(j)} > M^*_{(j)}$.*

(b) *Suppose that the output shock $\widetilde{\xi}_{im}$ in each contest of type l is transformed to $\widehat{\xi}_{im} = \alpha_{(l)}\widetilde{\xi}_{im}$ with a scale parameter $\alpha_{(l)} > 0$. $M^*_{(j)}$ is non-decreasing in $\alpha_{(l)}$.*

Theorem 4(a) shows that an organizer's profit $\Pi^*_{(l)}$ in any type l contest is unimodal in the number of any type j contests with a mode $M^*_{(j)}$. Similar to Theorem 2, this result stems from the tradeoff between the economies of scope across contests and the scarcity of resources due to the budget constraint. Although Theorem 4 considers a fixed set of awards, Figure 5(a) illustrates this result with the optimal set of awards using the setting in Example 1. An important observation is that different contests yield different profits. In the setting of Figure 5(a), type 1 contests have higher profit potential than type 2 contests. Thus, when the number of type 2 contests increases, profits in individual contests increase as shown in Figure 5(a), but the average profit can decrease due to the addition of low-profit contests; see Figure 5(b). Theorem 4(b) shows that the mode $M^*_{(j)}$ for any type j contest increases with the output uncertainty for any type l contest. This result suggests that it is better to run a larger number of contests in parallel when some of these contests are seeking cutting-edge innovation. This result corroborates Corollary 2 and has the same intuition.

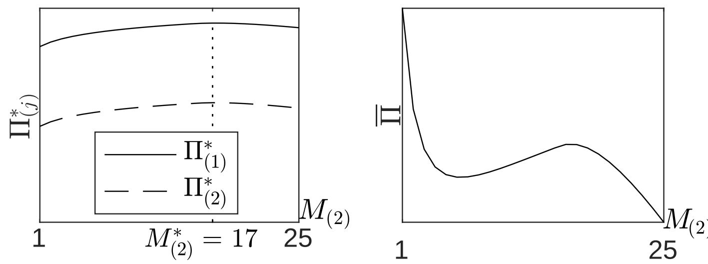
In practice, crowdsourcing platforms are often stratified so that some platforms like InnoCentive focus on obtaining highly innovative solutions, whereas others like Topcoder focus on less innovative solutions to more standard problems. We also observe that platforms tend to specialize in terms of the subject area. For instance, Topcoder focuses on software solutions, Kaggle on data science, Ennomotive on engineering solutions, and 99designs on design solutions. In a similar spirit, InnoCentive divides its contests into subject categories such as biology, chemistry, and business. Our results indicate that such specialization can provide two advantages. First, they show that having a large gap between the level of novelty sought in different contests can sometimes negatively affect some contests under the non-exclusive case (and can negatively affect others under the exclusive case). Thus, bringing together contests of similar types may help eliminate this negative effect. Second, specialization may increase economies of scope across contests in these platforms and hence increase the optimal number of parallel contests.

5. Extensions

We next show the robustness of our main results to cases with heterogeneous solvers (§5.1) and a multiplicative output function (§5.2). To tease out the impact of these model components, and for tractability purposes, we focus on symmetric contests where each contest offers a winner award A.

5.1. Contests with Heterogeneous Solvers

In §3, we assume that solvers are ex-ante symmetric consistent with the innovation contest literature reviewed in §1. In this section, we consider a case where solvers are heterogeneous with respect



(a) $\Pi^*_{(j)}$ as a function of $M_{(2)}$. (b) $\overline{\Pi}$ as a function of $M_{(2)}$.

Figure 5 (a) An organizer's profit $\Pi^*_{(j)}$ for contests of type $j \in \{1,2\}$ and (b) the average profit $\overline{\Pi}$ as a function of the number of contests of type 2. The setting is as in Example 1 where $\nu_{(1)} = 1.1$, $\nu_{(2)} = 1$, $M_{(1)} = 1$, $N = 100$, $\overline{B} = 0.05$, $r(e) = \log(e)$, $\eta(e) = 0.1e^{0.9}$, and $\phi(e) = e^2$.

to their cost of effort. Specifically, we assume that solver i has a cost of effort $c_i\eta\left(\sum_{m=1}^{M}\phi(e_{im})\right)$ when she exerts effort e_{im} in contest $m \in \{1,2,...,M\}$. We assume that c_i is common knowledge and $c_1 \geq c_2 \geq \cdots \geq c_N$ without loss of generality. For analytical tractability throughout the section, we assume the setting used by Terwiesch and Xu (2008), where the effort function $r(e) = \theta\log(e)$ and the output shock $\widetilde{\xi}_{im}$ follows a Gumbel distribution with mean zero and scale parameter α. We summarize our assumptions below.

Assumption 4. *$r(e) = \theta\log(e)$, the output shock $\widetilde{\xi}_{im}$ follows a Gumbel distribution with mean zero and scale parameter α, and $bp\alpha \geq \theta$.*

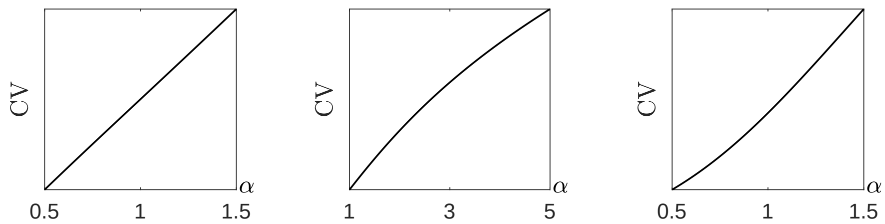
The following proposition characterizes the equilibrium effort and extends Theorems 1 and 2.

Proposition 4. *Let $\gamma_i = \frac{(e_i^*)^{\theta/\alpha}}{\sum_{j=1}^{N}(e_j^*)^{\theta/\alpha}}$ and $q(\gamma) = \frac{\gamma(1-\gamma)}{g^{-1}\left(\gamma^{\alpha/\theta}\right)}$. Suppose Assumptions 1 and 4 hold.*
(a) *When no solver's budget constraint binds, the equilibrium effort e_i^* satisfies*

$$e_i^* = g\left(\frac{AM^{1-b}\gamma_i(1-\gamma_i)}{\alpha c_i}\right) \text{ and } \sum_{j=1}^{N} q^{-1}\left(c_j\frac{q(\gamma_i)}{c_i}\right) = 1 \text{ for all } i = 1,2,...,N. \tag{7}$$

When all solvers' budget constraints bind, $e_i^* = \phi^{-1}\eta^{-1}\left(\frac{\overline{B}}{c_iM^b}\right)$ for $i \in \{1,2,...,N\}$.
(b) *There exists α_0 such that the average profit $\overline{\Pi}$ in the non-exclusive case is greater than the average profit $\overline{\Pi}^X$ in the exclusive case for any $\alpha > \alpha_0$.*
(c) *$\overline{\Pi}$ and an organizer's profit Π^* are increasing in any M such that no solver's budget constraint binds; and $\overline{\Pi}$ and Π^* are decreasing in any M such that all solvers' budget constraints bind.*

Proposition 4(a) characterizes the equilibrium when solvers are heterogeneous in their cost of effort. We can make interesting observations from (7). First, γ_i, which is related to the magnitude of solver i's effort relative to others, depends on the relative cost of solver i compared to the costs of other solvers. For instance, a solver with a lower cost of effort intuitively exerts more effort. Second, when no solver's budget constraint binds, all solvers exert more effort as the award A increases or



(a) $\widetilde{\xi}_{im} \sim$ Exponential(1). (b) $\widetilde{\xi}_{im} \sim$ Uniform(-1,1). (c) $\widetilde{\xi}_{im} \sim$ Gumbel(1).

Figure 6 Coefficient of variation (CV) as the shock $\widetilde{\xi}_{im}$ is scale transformed with scale parameter α.

as the number of contests M increases because γ_i does not depend on A or M. When all solvers' budget constraints bind, a solver's effort does not change with A but decreases with M. These results drive Proposition 4(c). Proposition 4(b) extends Theorem 1 and it has the same intuition.

5.2. Multiplicative Output Function

In this section, we show that our main results are not driven by the additive form we use for the output function. Specifically, we assume that solver i's output in contest m takes the multiplicative form $y_{im} = r(\widetilde{a}_{im}e_{im})$, where r is an increasing and concave function as in §2 and $\widetilde{a}_{im}$ is a positive valued random productivity shock that determines how effective a solver's effort is.[15] As in our main analysis in §3, we are interested in the impact of the output uncertainty. In line with the studies on other topics using multiplicative forms (e.g., Deo and Corbett 2009, Arifoglu et al. 2012), we utilize the coefficient of variation (i.e., standard deviation over mean) to measure the output uncertainty. To avoid assuming a specific distribution for $\widetilde{a}_{im}$ when capturing the change in the coefficient of variation, we define $\widetilde{a}_{im} = \exp(\widetilde{\xi}_{im})$, where $\widetilde{\xi}_{im}$ is a random variable defined as in §2, and we consider a scale transformation of $\widetilde{\xi}_{im}$. For instance, if $\widetilde{\xi}_{im}$ follows a normal distribution with mean 0 and standard deviation σ, and is transformed with scale parameter α, then $\widetilde{a}_{im}$ follows a lognormal distribution with mean $\exp(\alpha^2\sigma^2/2)$ and variance $[\exp(\alpha^2\sigma^2) - 1]\exp(\alpha^2\sigma^2)$. Thus, the coefficient of variation of lognormal $\sqrt{\exp(\alpha^2\sigma^2) - 1}$ increases with α. Our numerical analysis shows that when $\widetilde{\xi}_{im}$ follows a uniform, exponential, or Gumbel distribution, and $\widetilde{\xi}_{im}$ is transformed with a scale parameter α, the coefficient of variation of $\widetilde{a}_{im}$ increases with α; see Figure 6 for an illustration.[16] Thus, we use α to measure the output uncertainty.

The following proposition characterizes the equilibrium effort and extends Theorems 1 and 2.

[15] We are not aware of any paper on contests that uses a multiplicative model to capture effort and output uncertainty. Thus, we adopt the multiplicative model that Körpeoğlu and Cho (2018) use to capture effort and solver heterogeneity, although they did not consider output uncertainty.

[16] We have numerically shown that the coefficient of variation increases with the scale parameter α. We have randomly generated 10,000 instances each for uniform($-d$,d), exponential(λ), and Gumbel(μ) distributions. At each instance, we have randomly selected two α values from uniform(0,50) and checked whether a larger α leads to a larger coefficient of variation. We have randomly selected parameter values d, λ, and μ from uniform(0.5,5).

Proposition 5. *Suppose Assumption 1 holds.*

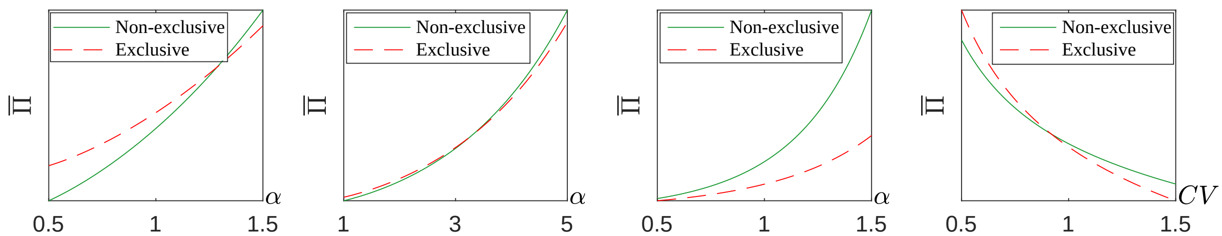
(a) *Let $\overline{g}$ be the increasing function such that $\overline{g}^{-1}(x) = (\eta \circ \phi)'(x)x$. The equilibrium effort $e_m^* = \min\left\{\overline{g}\left(AI_N M^{1-b}\right), \phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)\right\}$ where I_N is as in §2.*

(b) *Let $\overline{\Pi}^X$ be the average profit in the exclusive case. Suppose that the shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha > 0$. There exists α_0 such that the average profit $\overline{\Pi}$ in the non-exclusive case is greater than the average profit $\overline{\Pi}^X$ in the exclusive case for any $\alpha > \alpha_0$.*

(c) *The average profit $\overline{\Pi}$ and an organizer's profit Π^* are unimodal in the number of contests M.*

Proposition 5(a) shows that when the equilibrium effort is written in terms of the I_N term of the shock $\widetilde{\xi}_{im}$, it has a very similar structure to the equilibrium effort in the additive model. However, an organizer's profit $\Pi_m = r(\widetilde{a}_{im}e_m^*) - A$ is significantly different from an organizer's profit in the additive model. For example, we cannot decompose Π_m into additive effort and shock terms as in §3, and due to the multiplicative form, exerting more effort increases both the mean and the variance of the solver's output. Thus, one may expect that Theorem 1 will no longer hold because increasing uncertainty leads to a reduction in effort, which in turn decreases the variance of the solver's output. However, Proposition 5(b) shows that as the output uncertainty measured by the scale parameter α increases, the non-exclusive case yields a larger average profit than the exclusive case; see Figure 7(a)-(c) for an illustration of the results with different distributions of $\widetilde{\xi}_{im}$. Note that Proposition 5(b) holds when $\widetilde{a}_{im}$ follows a lognormal distribution (and hence $\widetilde{\xi}_{im}$ follows a normal distribution). We also numerically obtain the same result when $\widetilde{a}_{im}$ follows a Gamma distribution and we capture the uncertainty using the coefficient of variation, without using the approach above; see Figure 7(d) for an illustration.[17] The intuition of this seemingly counterintuitive result is similar to that of Theorem 1. A non-exclusive contest benefits from the best of a larger number of solutions, whereas an exclusive contest elicits greater effort. As the output uncertainty measured by α increases, the equilibrium effort decreases, thereby reducing the advantage of an exclusive contest. Although lower effort also leads to a decrease in the variance of the solver's output $y_i = r(\widetilde{a}_{im}e_m^*)$, increasing α leads to an increase in the variance of y_i by increasing the variance of $\widetilde{a}_{im}$. Because the latter effect outweighs the former effect, the variance of y_i increases with α, and so does the advantage of a non-exclusive contest. Thus, when the output

[17] We have numerically tested whether the non-exclusive case yields a larger average profit than the exclusive case when the coefficient of variation (CV) of Gamma distribution is sufficiently large. We have randomly generated 10,000 instances from Gamma distribution with a scale parameter drawn from Uniform(0.5,5). In all of these instances, we have checked CV values of 5, 10, ..., 50, and shown that there exists a CV value above which the non-exclusive case with $N = N_1 + N_2$ solvers yields a larger profit than the exclusive case with N_1 and N_2 solvers (N_1 and N_2 are randomly selected from discrete uniform distribution between 2 and 50). We let $M = 2$, $r(e) = \theta\frac{e^{1-a}-1}{1-a}$, and randomly selected parameter values $\theta \sim$ Uniform(0,10), $a \sim$ Uniform(0,1), $\overline{B} \sim$ Uniform(0,1), $bp \sim$ Uniform(2,5), $b \sim$ Uniform(0,1), $p = bp/b$, $c \sim$ Uniform(0,1).



(a) $\widetilde{\xi}_{im} \sim$ Exponential(1). (b) $\widetilde{\xi}_{im} \sim$ Uniform(-1,1). (c) $\widetilde{\xi}_{im} \sim$ Gumbel(1). (d) $\widetilde{a}_{im} \sim \text{Gamma}(1, 1/(CV)^2)$.

Figure 7 The average profit from five non-exclusive contests with 100 solvers versus the average profit from five exclusive contests with 20 solvers each when (a)–(c) $\widetilde{\xi}_{im}$ follows an exponential, uniform, or Gumbel distribution and is transformed with scale parameter α and (d) $\widetilde{a}_{im}$ follows a Gamma distribution and its coefficient of variation (CV) increases. Setting: $r(e) = 2\frac{e^{0.5}-1}{0.5}$ $\overline{B} = 0.6$, $\eta(e) = 0.1e^{0.9}$, and $\phi(e) = e^3$.

uncertainty, as measured by α, is sufficiently large, the non-exclusive case yields a larger average profit than the exclusive case. Proposition 5(c) extends Theorem 2 and has the same intuition.[18]

6. Conclusion

In recent years, contests have grown in popularity as a tool for outsourcing innovation from independent solvers. Each year, organizations such as Elanco and the Gates Foundation and platforms such as InnoCentive and Topcoder run numerous contests, providing solvers with several problems to work on. This multiple-contest environment leads to tensions that do not arise in a single-contest environment. Specifically, solvers may benefit from economies of scope by working on multiple contests, but due to limited resources they may have to split their efforts among multiple contests or even refrain from participating in some of these contests; this potentially reduces the profits for organizers. Discouraging solvers from participating in multiple contests may focus solvers' efforts but may hinder the diversity of solutions produced in each contest. These trade-offs raise two important questions for practitioners that the academic literature has yet to answer: When should solvers be discouraged from participating in multiple contests, and how does the number of contests affect an organizer's profit? In this paper, we take the first step towards answering these questions.

We analyze these questions by building a model of innovation contests, and our analysis yields the following results. First, we show that when solvers face a high degree of output uncertainty, holding non-exclusive contests where each solver can enter multiple contests generates larger profits

[18] Proposition 5, along with our numerical analyses, indicates that our insights are not driven by the functional form of the solver's output and hints that our main insights would hold when using even more general versions of the output function. For instance, a good candidate for a general model is where the solver's output follows a general distribution $F(y_{im}|e_{im})$ where F is decreasing in e_{im}. Our additive and multiplicative models are special cases of this general model, where $F(y_{im}|e_{im}) = H(y_{im} - r(e_{im}))$ and $F(y_{im}|e_{im}) = H(r^{-1}(y_{im})/e_{im})$, respectively. We leave the consideration of such general output functions to future research. An important issue in any such endeavor would be to consider how to measure the output uncertainty under such a general output function. Our work initiates this discussion and contributes to the innovation contest literature by offering an effective way to capture the output uncertainty in a multiplicative model while preserving analytical tractability.

for organizers than exclusive contests where solvers are permitted to participate in only one contest at any one time. In contrast, when the degree of uncertainty solvers face is low, holding exclusive contests generates larger profits for organizers. Second, we show that an organizer's profit can increase up to an optimal number of contests, and that the drivers of the optimal number of contests depend on the solvers' output uncertainty. Taken together, our results provide two managerial insights. First, practitioners seeking innovative solutions could run multiple parallel contests that exhibit economies of scope, and encourage solvers to participate in several of these contests at a time. Second, those seeking low-novelty solutions could run multiple parallel contests but discourage solvers from participating in more than one contest at a time.

In addition to providing some key managerial insights, we make several technical contributions to innovation contest theory. First, while previous studies have focused on a single contest, we study multiple contests and the resulting multidimensional optimization problem for individual solvers, who decide how much effort to exert in each contest they are entering by considering their total cost of effort. This technical contribution is even more pronounced when heterogeneous contests are considered. Second, while it is standard in the innovation contest literature to assume identical solvers and additive output functions, we consider heterogeneous solvers and multiplicative output functions. Our approaches to tackling these technically difficult cases can guide future studies that seek to capture these model components. Third, while it has mainly been assumed in prior studies that there is no fixed cost of participation or a bound on solvers' costs, we do consider a fixed cost and a budget constraint. Fourth, we propose a cost function that captures both diseconomies of scale in each contest and economies of scope across contests. While these features require special technical attention, they help our paper to capture a richer set of environments in practice.

Our model has the following limitations that can lead to new research opportunities. First, as is common in the literature, we use a static model when analyzing the impact of multiple parallel contests (see §1 for a detailed discussion). Consequently, our model does not take into account the organizer's decision of whether to run multiple contests in parallel or to run them sequentially. However, it captures a critical trade-off that may arise in a sequential setting. Specifically, running contests in parallel may lead to larger economies of scope, but may also lead solvers to split their effort. Future studies could consider how to schedule multiple contests dynamically, and potentially what the duration of each contest should be, to maximize the average or total profit. Second, while information asymmetry is not considered in our model, incorporating this into future models would be an important, albeit technically challenging, area for further research.

Acknowledgments

We sincerely appreciate helpful comments from the department editor Tava Olsen, the associate editor, and two anonymous reviewers. We are also grateful for helpful comments from seminar participants at Bilkent

University, Frankfurt School of Finance and Management, Hong Kong University of Science and Technology, Johns Hopkins University, Koc University, University of Cambridge, University of Connecticut, University of Virginia, University of Washington at Seattle, INFORMS Annual Meeting, MSOM Conference, POMS Annual Conference, and SAET Conference.

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Appendix. Proofs

Proof of Proposition 1. Let $I_{N,m} \equiv \int_{s\in\Xi_m}(N-1)H_m(s)^{N-2}h_m(s)^2ds$. The solver's budget constraint is $\sum_{m=1}^M \phi(e_m^*) \leq \eta^{-1}(\overline{B})$. Let λ be the Lagrange multiplier of this constraint. By Lemma EC.3 of Online Appendix, an equilibrium solves the following Kuhn-Tucker conditions:

$$A_m r'(e_m^*)I_{N,m} - \left(\frac{\phi(e_m^*)}{\sum_{l=1}^M \phi(e_l^*)}\right)^{1-b} \eta'\left(\phi(e_m^*)\right)\phi'(e_m^*) = \lambda^*\phi'(e_m^*),\ m\in\{1,2,...,M\}. \tag{8}$$

$$\lambda^*\left(\eta^{-1}(\overline{B}) - \sum_{m=1}^M \phi(e_m^*)\right) = 0,\ \sum_{m=1}^M \phi(e_m^*) \leq \eta^{-1}(\overline{B}),\text{ and } e_m^*,\lambda^* \geq 0,\ m\in\{1,2,...,M\}. \tag{9}$$

Case 1: Suppose $\lambda^* = 0$. Then, the equilibrium $(e_1^*, e_2^*, ..., e_M^*)$ solves (for all $m\in\{1,2,...,M\}$):

$$A_m r'(e_m^*)I_{N,m} = \eta'\left(\sum_{l=1}^M \phi(e_l^*)\right)\phi'(e_m^*). \tag{10}$$

We show that there exists a unique vector $(e_1^*, e_2^*, ..., e_M^*)$ that solves (10). We first convert (10) into M equations, each of which consists of a single variable. From (10), we have $A_m\varphi(e_m^*)I_{N,m} = \eta'(\sum_{l=1}^M \phi(e_l^*))$ for all $m\in\{1,2,...,M\}$ where $\varphi(x) = (r'/\phi')(x)$. Thus,

$$A_m I_{N,m}\varphi(e_m^*) = A_l I_{N,l}\varphi(e_l^*) \text{ for all } m,l\in\{1,2,...,M\},$$

From this relationship, we obtain

$$e_l^* = \varphi^{-1}\left(\frac{A_m I_{N,m}\varphi(e_m^*)}{A_l I_{N,l}}\right). \tag{11}$$

By plugging (11) back into (10), we obtain

$$\Omega_m(e_m^*, A_1, A_2, ..., A_m) \equiv A_m r'(e_m^*)I_{N,m} - \eta'\left(\sum_{l=1}^M \phi\left(\varphi^{-1}\left(\frac{A_m I_{N,m}\varphi(e_m^*)}{A_l I_{N,l}}\right)\right)\right)\phi'(e_m^*) = 0. \tag{12}$$

We next characterize $(e_1^*, e_2^*, ..., e_M^*)$. Because φ^{-1} is homogenous of degree $-1/(k+p-1)$, ϕ is homogenous of degree p, and η' is homogenous of degree $(b-1)$, we can write (12) as:

$$A_m r'(e_m^*) I_{N,m} - \left(\sum_{l=1}^{M} \left(\frac{A_m I_{N,m}}{A_l I_{N,l}} \right)^{-\frac{p}{k+p-1}} \right)^{b-1} \eta'\left(\phi\left(e_m^*\right)\right) \phi'(e_m^*) = 0. \tag{13}$$

Letting $g = ((\eta \circ \phi)'/r')^{-1}$, we can rewrite (13) as:

$$e_m^* = g\left((A_m I_{N,m})^{\frac{k+bp-1}{k+p-1}} \left(\sum_{l=1}^{M} (A_l I_{N,l})^{\frac{p}{k+p-1}} \right)^{1-b} \right) \text{ for all } m \in \{1, 2, ..., M\}. \tag{14}$$

Therefore, $(e_1^*, e_2^*, ..., e_M^*)$ is the unique symmetric equilibrium if and only if $\sum_{m=1}^{M} \phi(e_m^*) \leq \eta^{-1}(\overline{B})$.

Case 2: Suppose $\lambda^* > 0$. In this case, the unique candidate for the symmetric equilibrium effort e_m^* in contest $m \in \{1, 2, ..., M\}$ satisfies (8)-(9), and these conditions boil down to

$$A_m \varphi(e_m^*) I_{N,m} = A_l \varphi(e_l^*) I_{N,l} \text{ for all } m, l \in \{1, 2, ..., M\} \text{ and } \sum_{l=1}^{M} \phi(e_l^*) = \eta^{-1}(\overline{B}). \tag{15}$$

Then, plugging (11) into (15) gives $\sum_{l=1}^{M} \phi\left(\varphi^{-1}\left(\frac{A_m I_{N,m} \varphi(e_m^*)}{A_l I_{N,l}} \right) \right) = \eta^{-1}(\overline{B})$, and hence

$$e_m^* = \phi^{-1}\left(\frac{(A_m I_{N,m})^{\frac{p}{k+p-1}} \eta^{-1}(\overline{B})}{\sum_{l=1}^{M} (A_l I_{N,l})^{\frac{p}{k+p-1}}} \right). \tag{16}$$

Finally, we need to derive the condition under which $\lambda^* > 0$. The left-hand side of (8) is decreasing in e^* because r is concave (i.e., r' is decreasing), $\eta \circ \phi$ is convex (i.e., $(\eta' \circ \phi)\phi'$ is increasing), and $\left(\frac{\phi(e_m^*)}{\sum_{l=1}^{M} \phi(e_l^*)} \right)^{1-b}$ is increasing. The right-hand side of (8) is increasing because ϕ is convex. Thus, in order to have $\lambda^* > 0$, we need e_m^* in (14) to be strictly greater than e_m^* in (16).

Let $\widehat{e}_m \equiv g\left((A_m I_{N,m})^{\frac{k+bp-1}{k+p-1}} \left(\sum_{l=1}^{M} (A_l I_{N,l})^{\frac{p}{k+p-1}} \right)^{1-b} \right)$ and $\overline{e}_m = \phi^{-1}\left(\frac{\left(A_m I_{N,m}\right)^{\frac{p}{k+p-1}} \eta^{-1}(\overline{B})}{\sum_{l=1}^{M} \left(A_l I_{N,l}\right)^{\frac{p}{k+p-1}}} \right)$ for $m \in \{1, 2, ..., M\}$. Then, because $\varphi(x) = (r'/\phi')(x)$ is decreasing, we can deduce from (9) and (15) that either $\widehat{e}_m \leq \overline{e}_m$ for all $m \in \{1, 2, ..., M\}$ or $\widehat{e}_m > \overline{e}_m$ for all $m \in \{1, 2, ..., M\}$.

(a) Suppose $\widehat{e}_m \leq \overline{e}_m$ for all $m \in \{1, 2, ..., M\}$. Then the condition $\sum_{m=1}^{M} \phi(e_m^*) \leq \eta^{-1}(\overline{B})$ in case 1 is satisfied and the condition for $\lambda^* > 0$ in case 2 is violated. Thus, the unique symmetricequilibrium effort $e_m^* = \widehat{e}_m$ for all $m \in \{1, 2, ..., M\}$.

(b) Suppose $\widehat{e}_m > \overline{e}_m$ for all $m \in \{1, 2, ..., M\}$. Then the condition $\sum_{m=1}^{M} \phi(e_m^*) \leq \eta^{-1}(\overline{B})$ in case 1 is violated and the condition for $\lambda^* > 0$ in case 2 is satisfied. Thus, $e_m^* = \overline{e}_m$ for all $m \in \{1, 2, ..., M\}$.■

Proof of Lemma 1. **(a)** Because the coordinator optimally sets equal awards in all contests by Lemma EC.5 of Online Appendix, without loss of optimality, the coordinator's problem can be rewritten as follows (where A is the award given in each contest, and $\mu_{(j)}^N = E[\widetilde{\xi}_{(j)m}^N]$):

$$\max_{A} \; r(e^*) + \mu_{(1)}^N - A \text{ s.t. } e^* = \min\{g(A I_N M^{1-b}), \phi^{-1}\left(\eta^{-1}(\overline{B} M^{-b})\right)\}. \tag{17}$$

Note that the coordinator never sets A such that $g(A I_N M^{1-b}) > \phi^{-1}\left(\eta^{-1}(\overline{B} M^{-b})\right)$ because if that were the case, reducing A would improve the objective function in (17). Thus, without loss of optimality, (17) can be written as:

$$\max_{A}\ r(g(AI_NM^{1-b}))+\mu_{(1)}^{N}-A \text{ s.t. } g(AI_NM^{1-b})\le \phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right). \tag{18}$$

Let $\Phi(A)=r'\left(g\left(AI_NM^{1-b}\right)\right)g'\left(AI_NM^{1-b}\right)I_NM^{1-b}-1$. Note that Φ is the first derivative of the objective function in (18) with respect to A. Next, let $\overline{A}=M^{b-1}g^{-1}\left(\phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)\right)/I_N$, and suppose that $\Phi(\overline{A})\ge 0$. Because $r'(g(x))g'(x)$ is decreasing in x (as assumed in §2), the objective function in (18) is concave in A, and hence $\Phi(A)$ is decreasing in A. Moreover, because $A>\overline{A}$ violates the constraint in (18), and because Φ is decreasing, $A^*=\overline{A}$ solves (18). Thus, $A_m=A^*=\overline{A}$ maximizes the average profit $\overline{\Pi}$, and $e_m^*=e^*=\phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)$ is the corresponding equilibrium effort. Suppose that $\Phi(\overline{A})<0$. Then, because $\lim_{x\to 0}r'(g(x))g'(x)=\infty$, $\Phi(0)>0$ (which follows from $(r'\circ g)g'$ being homogenous of degree $\frac{2-2k-bp}{bp+k-1}<0$), and by the Intermediate Value Theorem, there exists $\widehat{A}$ such that $\Phi(\widehat{A})=0$. Note that $\widehat{A}$ is unique because Φ is decreasing. Hence, in this case, $A^*=\widehat{A}$ solves (18). Thus, we can conclude that $A_m=A^*=\widehat{A}$ maximizes $\overline{\Pi}$, and $e_m^*=e^*=g(A^*I_NM^{1-b})$ is the equilibrium effort.

(b) $\overline{A}=M^{b-1}g^{-1}\left(\phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)\right)/I_N$ is decreasing in M because $b<1$ and g^{-1}, ϕ^{-1}, and η^{-1} are decreasing functions. Because $(r'\circ g)g'$ is homogenous of degree $\frac{2-2k-bp}{bp+k-1}<0$, $\Phi(A)$ is decreasing in A. Thus, $\widehat{A}$ increases with M if and only if $\Phi(A)$ increases with M. We can rewrite $\Phi(A)=M^{(1-b)\frac{2-2k-bp}{bp+k-1}+1-b}r'\left(g\left(AI_N\right)\right)g'\left(AI_N\right)I_N-1$, which is decreasing in M if and only if $(1-b)\frac{2-2k-bp}{bp+k-1}+1-b=(1-b)\frac{2-2k-bp+bp+k-1}{bp+k-1}=\frac{(1-k)(1-b)}{bp+k-1}$. The result follows because $b<1$ and $bp>1$. ■

Proof of Theorem 1. We compare the average profit in exclusive and non-exclusive cases. In the exclusive case, let $N_m^{*,X}$ be the optimal number of solvers and $A_m^{*,X}$ be the optimal award in contest $m\in\{1,2,...,M\}$. Let $e_m^{*,X}$ be the corresponding equilibrium effort in contest $m\in\{1,2,...,M\}$. Suppose that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im}=\alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha>0$. Note that it is never optimal for the coordinator to set awards such that $e_m^{*,X}>\phi^{-1}\left(\eta^{-1}(\overline{B})\right)$ because the coordinator can improve the average profit by reducing the award in contest $m\in\{1,2,...,M\}$. Thus, by Lemma 1, the equilibrium effort in the exclusive case is $e_m^{*,X}=g\left(A_m^{*,X}I_{N_m^{*,X}}/\alpha\right)$ in contest m. Without loss of generality, assume that $e_1^{*,X}\ge e_m^{*,X}$ for all m. After incorporating the optimal solution, the average profit in the exclusive case becomes

$$\overline{\Pi}^X=\frac{1}{M}\sum_{m=1}^{M}(r(e_m^{*,X})+\alpha\mu^{N_m^{*,X}}-A_m^{*,X}). \tag{19}$$

In the non-exclusive case, suppose that the coordinator offers an award A in each contest so that the equilibrium effort in each contest m is $e^*=(\sum_{l=1}^{M}e_l^{*,X})/M$. From Lemma 1, we can see that this requires $e^*=g\left(AI_NM^{1-b}/\alpha\right)$. Note that because ϕ is increasing an convex, for sufficiently small α, $\sum_{m=1}^{M}\phi(e^*)\le\sum_{l=1}^{M}\phi(e_l^{*,X})\le\eta^{-1}(\overline{B})$; and note that g^{-1} is increasing. Thus, from (10),

$$A=\frac{\alpha M^{b-1}}{I_N}g^{-1}\left(\frac{\sum_{l=1}^{M}e_l^{*,X}}{M}\right)\le\frac{\alpha M^{b-1}}{I_N}g^{-1}\left(e_1^{*,X}\right)=\frac{M^{b-1}}{I_N}A_1^{*,X}I_{N_1^{*,X}}.$$

Then, the average profit in the non-exclusive case can be written as:

$$\overline{\Pi} = r\left(\frac{\sum_{l=1}^{M} e_l^{*,X}}{M}\right) + \frac{1}{M}\sum_{m=1}^{M}\alpha\mu_N - \frac{1}{M}\sum_{m=1}^{M} A \geq \frac{1}{M}\sum_{m=1}^{M} r(e_m^{*,X}) + \alpha\mu_N - \frac{M^{b-1}}{I_N}A_1^{*,X} I_{N_1^{*,X}}. \quad (20)$$

Subtracting (19) from (20) yields the following inequality

$$\overline{\Pi} - \overline{\Pi}^X \geq \alpha\left(\mu_N - \frac{1}{M}\sum_{m=1}^{M}\mu_{N_m^{*,X}} + \frac{1}{M}\sum_{m=1}^{M}\frac{A_m^{*,X}}{\alpha} - \frac{M^{b-1}A_1^{*,X} I_{N_1^{*,X}}}{\alpha I_N}\right). \quad (21)$$

By Lemma EC.6 of Online Appendix, $\lim_{\alpha\to\infty} A_m^{*,X}/\alpha = 0$ for each $m \in \{1,2,..,M\}$. Also, because $N > N_m^{*,X}$ for some $m \in \{1,2,...,M\}$, $\widetilde{\xi}_m^N$ first-order stochastically dominates $\widetilde{\xi}_m^{N_m}$ (and not vice versa), so $\mu_N > \frac{1}{M}\sum_{m=1}^{M}\mu_{N_m^{*,X}}$. Thus, there exists α_0 such that $\overline{\Pi} - \overline{\Pi}^X > 0$ for any $\alpha > \alpha_0$. ■

Proof of Corollary 1. Consider the exclusive case where N_1 and N_2 solvers participate in contest 1 and 2, respectively. In the non-exclusive case, all $N(= N_1 + N_2)$ solvers participate in both contests. Let $A^{*,M,N} = \min\{\frac{\theta}{bp}, \frac{bp\overline{B}}{\theta M I_N}\}$. In the non-exclusive case, by incorporating Assumption 3 to Proposition 1 and Lemma 1, we obtain the equilibrium effort $e^* = \left(\frac{\theta A^* I_N 2^{1-b}}{cbp}\right)^{\frac{1}{bp}}$, where $A^* = A^{*,2,N}$. Then, the average profit in the non-exclusive case is

$$\overline{\Pi} = r(e^*) + \mu_N - A^* = \frac{\theta}{bp}\log\left(\frac{A^{*,2,N}\theta I_N 2^{1-b}}{cbp}\right) + \mu_N - A^{*,2,N}.$$

Moreover, the average profit in the exclusive case with two contests is

$$\overline{\Pi}^X = \left[\frac{\theta}{bp}\log\left(\frac{\theta\sqrt{A^{*,1,N_1}I_{N_1}A^{*,1,N_2}I_{N_2}}}{cbp}\right) + \frac{\mu_{N_1}+\mu_{N_2}}{2} - \frac{A^{*,1,N_1}+A^{*,1,N_2}}{2}\right].$$

When $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with $\alpha > 0$, we have $A^{*,M,N} = \min\{\frac{\theta}{bp}, \frac{\alpha bp\overline{B}}{\theta M I_N}\}$.

(i) When $\alpha \geq \alpha_1 \equiv \frac{\theta^2}{p^2 b^2 \overline{B}}\max\{I_{N_1}, I_{N_2}, 2I_{N_1+N_2}\}$, we have $A^{*,2,N} = A^{*,1,N_m} = \frac{\theta}{pb}$. The difference between the average profit in the non-exclusive and the exclusive case is

$$\overline{\Pi} - \overline{\Pi}^X = \frac{\theta}{2bp}\log\left(\frac{2^{2-2b}I_{N_1+N_2}^2}{I_{N_1}I_{N_2}}\right) + \alpha\mu_{N_1+N_2} - \alpha\frac{\mu_{N_1}+\mu_{N_2}}{2}. \quad (22)$$

Because $\widetilde{\xi}_{N_1+N_2}$ first-order stochastically dominates $\widetilde{\xi}_{N_m}$ for $m \in \{1,2\}$ (and not vice versa), we have $\mu_{N_1+N_2} - \mu_{N_m} > 0$ for $m \in \{1,2\}$, so $\overline{\Pi} - \overline{\Pi}^X \geq 0$ if $\alpha \geq \alpha_0 \equiv \max\{\alpha_1, \alpha_2\}$, where

$$\alpha_2 \equiv \frac{\theta}{bp}\frac{\log(I_{N_1}I_{N_2}) - 2\log(2^{1-b}I_{N_1+N_2})}{2\mu_{N_1+N_2} - \mu_{N_1} - \mu_{N_2}}.$$

(ii) When $\alpha \leq \alpha_3 \equiv \frac{\theta^2}{p^2 b^2 \overline{B}}\min\{I_{N_1}, I_{N_2}, 2I_{N_1+N_2}\}$, we have $A^{*,2,N} = \frac{\alpha bp\overline{B}}{2\theta I_N}$ and $A^{*,1,N_m} = \frac{\alpha bp\overline{B}}{\theta I_{N_m}}$. Thus, the equilibrium effort under non-exclusive and exclusive cases are $e^* = \left(\overline{B}2^{-b}\right)^{\frac{1}{bp}}$ and $e^{*,X} = \left(\overline{B}\right)^{\frac{1}{bp}}$ respectively. The difference between the average profits is

$$\overline{\Pi} - \overline{\Pi}^X = -\frac{\theta}{p}\log(2) + \alpha\mu_{N_1+N_2} - \alpha\frac{\mu_{N_1}+\mu_{N_2}}{2} - \frac{\alpha bp\overline{B}}{2\theta I_{N_1+N_2}} + \frac{\alpha bp\overline{B}}{2\theta I_{N_1}} + \frac{\alpha bp\overline{B}}{2\theta I_{N_2}}. \quad (23)$$

Because $\widetilde{\xi}_{N_1+N_2}$ first-order stochastically dominates $\widetilde{\xi}_{N_m}$ for $m \in \{1,2\}$ (and not vice versa), we have $\mu_{N_1+N_2} - \mu_{N_m} > 0$ for $m \in \{1,2\}$. Furthermore, Lemma EC.7 of Online Appendix shows that

$\frac{1}{I_{N_1}}+\frac{1}{I_{N_2}}\geq\frac{1}{I_{N_1+N_2}}$, so $\overline{\Pi}-\overline{\Pi}^X\leq 0$ if $\alpha\leq\alpha_0\equiv\min\{\alpha_3,\alpha_4\}$, where

$$\alpha_4\equiv\frac{\theta}{p}\frac{\log(2)}{\mu_{N_1+N_2}-\frac{\mu_{N_1}+\mu_{N_2}}{2}+\frac{bp\overline{B}}{2\theta}\left[\frac{1}{I_{N_1}}+\frac{1}{I_{N_2}}-\frac{1}{I_{N_1+N_2}}\right]}.$$

(iii) Suppose that $\overline{B}$ is sufficiently large so that $\alpha_1<\alpha_2$. From the above discussion, we have $\overline{\Pi}-\overline{\Pi}^X\geq 0$ if $\alpha\geq\max\{\alpha_1,\alpha_2\}=\alpha_2$. Furthermore, from (23), we see that if $\alpha<\alpha_2$, $\overline{\Pi}-\overline{\Pi}^X<0$. Therefore, $\overline{\Pi}-\overline{\Pi}^X\geq 0$ if and only if $\alpha\geq\alpha_0=\alpha_2$. ■

Proof of Theorem 2. Let $\Phi(A)$ be defined as in Lemma 1. Let $\overline{e}=\phi^{-1}\left(\eta^{-1}(M^{-b}\overline{B})\right)$. Note that

$$\Phi(\overline{A})=r'\left(g\left(\overline{A}I_NM^{1-b}\right)\right)g'\left(\overline{A}I_NM^{1-b}\right)I_NM^{1-b}-1=r'\left(\overline{e}\right)g'\left(g^{-1}(\overline{e})\right)I_NM^{1-b}-1 \tag{24}$$

is increasing in M because $r'(g(x))g'(x)$ is decreasing in x and M^{1-b} is increasing in M. Because $\Phi(\overline{A})$ is increasing in M and $\lim_{x\to 0}r'(g(x))g'(x)=\infty$, there exists $M_0\in[1,\infty)$ such that $\Phi(\overline{A})<0$ for any $M<M_0$, and $\Phi(\overline{A})\geq 0$ for any $M\geq M_0$. We next show that the average profit $\overline{\Pi}=\Pi^*=r(e^*)+\mu_{(1)}^N-A^*$ is increasing in the number of contests M up to some M^* and decreasing afterwards. When $M<M_0$, from Lemma 1 and the above discussion, the constraint in (18) can be relaxed. Applying the Envelope Theorem to $\overline{\Pi}\equiv\max_A r(e^*)+\mu_{(1)}^N-A$, we obtain

$$\frac{\partial\overline{\Pi}}{\partial M}=r'(e^*)\frac{\partial e^*}{\partial M}=(1-b)r'(e^*)g'\left(A^*I_NM^{1-b}\right)A^*I_NM^{-b}. \tag{25}$$

Because g is increasing, $g'>0$, and because r is increasing, $r'>0$. Thus, from (25), $\overline{\Pi}$ is increasing in M when $M<M_0$. When $M\geq M_0$, $A^*=\overline{A}$, so the average profit becomes

$$\overline{\Pi}=\ r\left(\overline{e}\right)+\mu_{(1)}^N-\frac{1}{I_NM^{1-b}}g^{-1}\left(\overline{e}\right). \tag{26}$$

Noting that $\overline{e}=M^{-1/p}\phi^{-1}\left(\eta^{-1}(\overline{B})\right)$, and hence $\frac{\partial\overline{e}}{\partial M}=-(1/p)M^{-1/p-1}\phi^{-1}\left(\eta^{-1}(\overline{B})\right)=-\overline{e}/(pM)$, the derivative of the average profit $\overline{\Pi}$ with respect to M can be written as:

$$\frac{\partial\overline{\Pi}}{\partial M}=\ -r'\left(\overline{e}\right)\frac{\overline{e}}{pM}+\frac{1}{I_NM^{1-b}}(g^{-1})'\left(\overline{e}\right)\frac{\overline{e}}{pM}+\frac{1-b}{I_NM^{2-b}}g^{-1}\left(\overline{e}\right). \tag{27}$$

As r', η, and ϕ are homogenous of degree $-k$, b, and p, respectively, $g^{-1}=\left(\frac{(\eta\circ\phi)'}{r'}\right)$ is homogenous of degree $pb+k-1$. Noting that $(g^{-1})'(x)=(pb+k-1)g^{-1}(x)/x$, we can write (27) as

$$\frac{\partial\overline{\Pi}}{\partial M}=-\frac{r'\left(\overline{e}\right)\phi^{-1}\left(\eta^{-1}(\overline{B})\right)}{pM^{1/p+1}}+\frac{p+k-1}{pI_NM^{2-b}}g^{-1}\left(\overline{e}\right)=\frac{r'\left(\overline{e}\right)}{pM^{1/p+1}}\left(-\phi^{-1}\left(\eta^{-1}(\overline{B})\right)+\frac{p+k-1}{pI_NM^{1-b-1/p}}\frac{g^{-1}}{r'}\left(\overline{e}\right)\right).$$

Note that $\frac{\partial\overline{\Pi}}{\partial M}$ has the same sign as

$$\varsigma\equiv-\phi^{-1}\left(\eta^{-1}(\overline{B})\right)+\frac{p+k-1}{pI_NM^{(p+2k-2)/p}}\frac{g^{-1}}{r'}\left(\phi^{-1}\left(\eta^{-1}(\overline{B})\right)\right), \tag{28}$$

which is always decreasing in M because $pb+k-b>0$ and $p+2k-2>0$ (note that $pb+2k-2\geq 0$). Thus, there exists $M^*\in[M_0,\infty)$ such that $\varsigma>0$ and hence $\frac{\partial\overline{\Pi}}{\partial M}>0$ for all $M\in[M_0,M^*)$; and $\varsigma<0$ and hence $\frac{\partial\overline{\Pi}}{\partial M}<0$ for all $M>M^*$. Finally, since we also established above that $\frac{\partial\overline{\Pi}}{\partial M}>0$ for all $M<M_0$, we have $\frac{\partial\overline{\Pi}}{\partial M}>0$ for all $M<M^*$ and $\frac{\partial\overline{\Pi}}{\partial M}<0$ for all $M>M^*$.■

Proof of Corollary 2. Suppose that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a parameter $\alpha > 0$. After the transformation, $\widehat{I_N} = I_N/\alpha$. Thus, for any M, $\Phi(\overline{A})$ in (24) is decreasing in α, so M_0 in the proof of Theorem 2 is non-decreasing in α (increasing in α if $M_0 > 1$). Because ς in (28) is also increasing in α, $M^*(> 1)$ is increasing in α. ■

Proof of Proposition 2. Because r', η, and ϕ are homogenous of degree $-k$, b, and p, respectively, $g = ((\eta \circ \phi)'/r')^{-1}$ is homogenous of degree $1/(bp + k - 1)$. Thus, we can rewrite a solver's utility when she participates in M contests as (note that since we assume $\overline{B}$ is sufficiently large, the equilibrium effort $e^* = g(AI_N M^{1-b})$)

$$\begin{aligned} U[M] &= \frac{AM}{N} - \eta(M\phi(e^*)) - Mc_f = \frac{AM}{N} - \eta(M\phi(g(M^{1-b}AI_N))) - Mc_f \\ &= \frac{AM}{N} - M^{b+\frac{(1-b)bp}{bp+k-1}}\eta(\phi(g(AI_N))) - Mc_f = \frac{AM}{N} - M^{\frac{bp+b(k-1)}{bp+k-1}}\eta(\phi(g(AI_N))) - Mc_f. \end{aligned}$$

The derivative of $U[M]$ with respect to M

$$\begin{aligned} \frac{\partial U[M]}{\partial M} &= \frac{A}{N} - \frac{bp + b(k-1)}{bp + k - 1} M^{\frac{(b-1)(k-1)}{bp+k-1}} \eta(\phi(g(AI_N))) - c_f \\ &= U[1] - \left(\frac{bp + b(k-1)}{bp + k - 1} M^{\frac{(1-b)(1-k)}{bp+k-1}} - 1 \right) \eta(\phi(g(AI_N))). \end{aligned}$$

(a) Because $U[1] > 0$, when $k \geq 1$, we have $\frac{bp+b(k-1)}{bp+k-1} < 1$ and $\frac{(1-b)(1-k)}{bp+k-1} < 0$. Because $M \geq 1$, we have $\partial U[M]/\partial M > 0$. Thus, $U[M] > 0$ for all M.

(b) Suppose that $k < 1$. Then, $\frac{bp+b(k-1)}{bp+k-1} > 1$, and in turn, $\lim_{M\to\infty} U[M]/M = -\infty$ and $U[M]/M$ is decreasing in M. Thus, there exists a unique $\overline{M}$ such that $U[\overline{M}] = 0$, and $U[M] < 0$ for all $M > \overline{M}$. Furthermore, when the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a parameter $\alpha > 0$, the solver's utility becomes $U[M] = \frac{AM}{N} - M^{\frac{bp+b(k-1)}{bp+k-1}}\eta(\phi(g(AI_N/\alpha))) - Mc_f$. Thus, the solver's utility is increasing in α, which means that $\overline{M}$ is increasing in α. ■

Proof of Proposition 3. Consider two contests with N_1 and N_2 solvers and suppose that $\overline{B}$ is sufficiently large. Each organizer's profit is

$$\Pi_1^{*,L} = \frac{\theta}{bp}\log\left(\frac{\theta^2 I_{N_1}}{cb^2p^2}\right) + \mu_{N_1} - \frac{\theta}{bp} \text{ and } \Pi_2^{*,L} = \frac{\theta}{bp}\log\left(\frac{\theta^2 I_{N_2}}{cb^2p^2}\right) + \mu_{N_2} - \frac{\theta}{bp}.$$

The average profit under two contests

$$\overline{\Pi}^{L,II} = \frac{1}{2}\left[\frac{\theta}{bp}\log\left(\frac{\theta^4 I_{N_1} I_{N_2}}{c^2b^4p^4}\right) + \mu_{N_1} + \mu_{N_2} - \frac{2\theta}{bp}\right].$$

The average profit under a single contest with $N_1 + N_2$ solvers

$$\overline{\Pi}^{L,I} = \frac{\theta}{bp}\log\left(\frac{\theta^2 I_{N_1+N_2}}{cb^2p^2}\right) + \mu_{N_1+N_2} - \frac{\theta}{bp}.$$

When the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a parameter $\alpha > 0$, the difference between the average profit under two contests and that under a single contest is

$$\overline{\Pi}^{L,II} - \overline{\Pi}^{L,I} = \frac{\theta}{2bp}\log\left(\frac{I_{N_1}I_{N_2}}{I^2_{N_1+N_2}}\right) + \alpha\frac{\mu_{N_1} + \mu_{N_2}}{2} - \alpha\mu_{(1)}^{N_1+N_2}.$$

Noting that $\mu_{(1)}^{N_1+N_2} - \mu_{(1)}^{N_m} > 0$ for $m \in \{1,2\}$ because $\widetilde{\xi}_{(1)m}^{N_1+N_2}$ first-order stochastically dominates $\widetilde{\xi}_{(1)m}^{N_m}$ (and not vice versa), $\overline{\Pi}^{L,II} - \overline{\Pi}^{L,I} > 0$ if and only if $\alpha < \alpha_L$, where

$$\alpha_L \equiv \frac{\theta}{bp} \frac{\log\left(I_{N_1} I_{N_2}\right) - 2\log\left(I_{N_1+N_2}\right)}{2\mu_{N_1+N_2} - \mu_{N_1} - \mu_{N_2}}. \blacksquare$$

Proof of Theorem 3. We compare the average profit in exclusive and non-exclusive cases when a subset $\mathcal{M}_I$ of contests have sufficiently large uncertainty. Let $\mathcal{M}_S = \{1,2,...,M\} \setminus \mathcal{M}_I$, $M_S = |\mathcal{M}_S|$ and $M_I = |\mathcal{M}_I|$. Also, let I_m^N ($m \in \{1,2,...,M\}$) denote the I_m in Lemma 1 under N solvers. In the exclusive case, let $N_m^{*,X}$ be the optimal number of solvers and $A_m^{*,X}$ be the optimal award in contest $m \in \{1,2,...,M\}$. Let $e_m^{*,X}$ be the corresponding equilibrium effort in contest $m \in \{1,2,...,M\}$. Note that it is never optimal for the coordinator to set awards such that $e_m^{*,X} > \phi^{-1}\left(\eta^{-1}(M^{-b}\overline{B})\right)$ because the coordinator can improve the average profit by reducing the award in contest $m \in \{1,2,...,M\}$. Thus, by Proposition 1, the equilibrium effort in the exclusive case $e_m^{*,X} = g\left(A_m^{*,X} I_m^{N_m^{*,X}}\right)$ in contest m. After incorporating the optimal solution, the average profit in the exclusive case becomes

$$\overline{\Pi}^X = \frac{1}{M}\sum_{m=1}^{M}(r(e_m^{*,X}) + \mu_{N_m^{*,X},m} - A_m^{*,X}). \tag{29}$$

In the non-exclusive case, suppose that the coordinator offers an award $A_m = A_m^{*,X}$ in each contest $m \in \{1,2,...,M\}$ and let e_m^* be the corresponding equilibrium effort. Then, the average profit is

$$\overline{\Pi} = \frac{1}{M}\sum_{m=1}^{M} r(e_m^*) + \frac{1}{M}\sum_{m=1}^{M}\mu_{N,m} - \frac{1}{M}\sum_{m=1}^{M} A_m^{*,X}. \tag{30}$$

Suppose that the output shock $\widetilde{\xi}_{im}$ at each contests $m \in \mathcal{M}_I$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha > 0$, while keeping the output shocks in other contests the same. Then, the difference between the average profit in non-exclusive and exclusive cases is

$$\overline{\Pi} - \overline{\Pi}^X = \frac{1}{M}\sum_{m=1}^{M}(r(e_m^*) - r(e_m^{*,X})) + \frac{\alpha}{M}\sum_{m\in\mathcal{M}_I}(\mu_{N,m} - \mu_{N_m^{*,X},m}) + \frac{1}{M}\sum_{m\in\mathcal{M}_S}(\mu_{N,m} - \mu_{N_m^{*,X},m}). \tag{31}$$

We want to show that $\lim_{\alpha\to\infty}(\overline{\Pi} - \overline{\Pi}^X)/\alpha > 0$ so that $\overline{\Pi} > \overline{\Pi}^X$ for a sufficiently large α. As Lemma EC.6 of Online Appendix shows, $\lim_{\alpha\to\infty} A_m^{*,X}/\alpha = 0$ for all $m \in \mathcal{M}_I$, so by Proposition 1, $\lim_{\alpha\to\infty} e_m^* = g\left((A_m^{*,X} I_m^N)^{\frac{k+bp-1}{k+p-1}}\left(\sum_{l\in\mathcal{M}_S}(A_m^{*,X} I_l^N)^{\frac{p}{k+p-1}}\right)^{1-b}\right)$ in contest $m \in \mathcal{M}_S$. Thus, $\lim_{\alpha\to\infty}\frac{1}{\alpha}\sum_{m\in\mathcal{M}_S}(r(e_m^*) - r(e_m^{*,X})) = 0$. We also have $\lim_{\alpha\to\infty}\frac{1}{\alpha}\sum_{m\in\mathcal{M}_S}(\mu_{N,m} - \mu_{N_m^{*,X},m}) = 0$ because $\mu_{N,m}$ and $\mu_{N_m^{*,X},m}$ do not depend on α for any $m \in \mathcal{M}_S$. Furthermore,

$$\begin{aligned}
&\lim_{\alpha\to\infty}\left(\frac{1}{\alpha}(r(e_m^*) - r(e_m^{*,X}))\right) \\
&= \lim_{\alpha\to\infty}\frac{1}{\alpha}\left(r\left(g\left(\left(\frac{A_m^{*,X} I_m^N}{\alpha}\right)^{\frac{k+bp-1}{k+p-1}}\left(\sum_{m\in\mathcal{M}_S}(A_m^{*,X} I_l^N)^{\frac{p}{k+p-1}}\right)^{1-b}\right)\right) - r\left(g\left(\frac{A_m^{*,X} I_m^{N_m^{*,X}}}{\alpha}\right)\right)\right).
\end{aligned}$$

Case 1: $\lim_{\alpha\to\infty}(r(e_m^*) - r(e_m^{*,X})) > -\infty$. Then $\lim_{\alpha\to\infty}\left(\frac{1}{\alpha}(r(e_m^*) - r(e_m^{*,X}))\right) \geq 0$.

Case 2: $\lim_{\alpha\to\infty}((r(e_m^*) - r(e_m^{*,X}))) = -\infty$. Let $K_1 \equiv \left(\frac{A_m^{*,X} I_m^N}{\alpha}\right)^{\frac{k+bp-1}{k+p-1}}\left(\sum_{m\in\mathcal{M}_S}(A_m^{*,X} I_l^N)^{\frac{p}{k+p-1}}\right)^{1-b}$

and $K_2 = g(A_m^{*,X} I_m^{N_m^{*,X}})$. Then we have

$$\begin{aligned}
&\lim_{\alpha\to\infty}\left(\frac{1}{\alpha}(r(e_m^*) - r(e_m^{*,X}))\right) = \lim_{\alpha\to\infty}\frac{1}{\alpha}\left(r\left(\alpha^{\frac{-1}{k+p-1}}K_1\right) - r\left(\alpha^{\frac{-1}{k+bp-1}}K_2\right)\right).\\
&= \lim_{\alpha\to\infty}\left(\frac{-K_1}{k+p-1}r'\left(\alpha^{\frac{-1}{k+p-1}}K_1\right)\alpha^{\frac{-1}{k+p-1}-1} + \frac{K_2}{k+bp-1}r'\left(\alpha^{\frac{-1}{k+bp-1}}K_2\right)\alpha^{\frac{-1}{k+bp-1}-1}\right).\\
&= \lim_{\alpha\to\infty}\left(\frac{K_1}{k+p-1}r'(K_1)\,\alpha^{\frac{-p}{k+p-1}} + \frac{K_2}{k+bp-1}r'(K_2)\,\alpha^{\frac{-bp}{k+bp-1}}\right) = 0,
\end{aligned}$$

where the equalities follow from L'Hopital's Rule. Either case, $\lim_{\alpha\to\infty}\left(\frac{1}{\alpha}(r(e_m^*) - r(e_m^{*,X}))\right) \geq 0$, so

$$\lim_{\alpha\to\infty}\frac{\overline{\Pi} - \overline{\Pi}^X}{\alpha} = \lim_{\alpha\to\infty}\left(\frac{1}{\alpha M}\sum_{m\in\mathcal{M}_I}(r(e_m^*) - r(e_m^{*,X})) + \frac{1}{M}\sum_{m\in\mathcal{M}_I}(\mu_{N,m} - \mu_{N_m^{*,X},m})\right) > 0, \tag{32}$$

because under the assumption that $N > N_m^{*,X}$ for some $m \in \mathcal{M}_I$, we also have $N \geq N_l^{*,X}$ for all $l \in \mathcal{M}_I \setminus \{m\}$, so $\widetilde{\xi}_{(1)m}^N$ first-order stochastically dominates $\widetilde{\xi}_{(1)m}^{N_m}$ (and not vice versa), and hence we have $\mu_{N,m} > \mu_{N_m^{*,X},m}$ and similarly we have $\mu_{N,l} \geq \mu_{N_l^{*,X},l}$ for all $l \in \mathcal{M}_I \setminus \{m\}$. Thus, there exists α_0 such that for any $\alpha > \alpha_0$, we have $\overline{\Pi} - \overline{\Pi}^X > 0$. The only case where the assumption $N > N_m^{*,X}$ does not hold for any $m \in \mathcal{M}_I$ is the case where $\mathcal{M}_I$ has a single element m and $N_m^{*,X} = N$. Because contest m has the same profit as the average profit in the non-exclusive case under $M = 1$, and the budget constraint does not bind for a sufficiently large α, Theorem 4(a) implies that the average profit under the non-exclusive case is larger than that under the exclusive case. ■

Proof of Theorem 4. **(a)** The derivative of the organizer's profit in a contest of type l with respect to the number of contests of type j is

$$\frac{\Pi_{(l)}}{\partial M_{(j)}} = r'(e_{(l)}^*)\frac{\partial e_{(l)}^*}{\partial M_{(j)}}. \tag{33}$$

Because $r' > 0$, we need to show that there exists $M_{(j)}^*$ such that $\frac{\partial e_{(l)}^*}{\partial M_{(j)}} > 0$ for all $M_{(j)} < M_{(j)}^*$ and $\frac{\partial e_{(l)}^*}{\partial M_{(j)}} < 0$ for all $M_{(j)} > M_{(j)}^*$. Let $\widehat{e}_{(l)} \equiv g\left((A_{(l)}I_{(l)})^{\frac{k+bp-1}{k+p-1}}\left(\sum_{j=1}^J M_{(j)}\left(A_{(j)}I_{(j)}\right)^{\frac{p}{k+p-1}}\right)^{1-b}\right)$ and $\overline{e}_{(l)} = \phi^{-1}\left(\frac{\left(A_{(l)}I_{(l)}\right)^{\frac{p}{k+p-1}}\eta^{-1}(\overline{B})}{\sum_{j=1}^J M_{(j)}\left(A_{(j)}I_{(j)}\right)^{\frac{p}{k+p-1}}}\right)$ for $l \in \{1,2,...,J\}$. From Proposition 1, we can deduce that $e_{(l)}^* = \min\{\widehat{e}_{(l)}, \overline{e}_{(l)}\}$ for all $l \in \{1,2,...,J\}$. Since g and ϕ^{-1} are increasing and homogenous, $\widehat{e}_{(l)}$ is increasing and unbounded in $M_{(j)}$ and $\overline{e}_{(l)}$ is decreasing in $M_{(j)}$. Thus, there should exists $M_{(j)}^*$ such that $e_{(l)}^* = \widehat{e}_{(l)}$ for all $M_{(j)} < M_{(j)}^*$ and $e_{(l)}^* = \overline{e}_{(l)}$ for all $M_{(j)} > M_{(j)}^*$. Because $\varphi(x) = (r'/\phi')(x)$ is decreasing, we can deduce from (9) and (15) that either $\widehat{e}_{(l)} \leq \overline{e}_{(l)}$ for all $l \in \{1,2,...,J\}$ or $\widehat{e}_{(l)} > \overline{e}_{(l)}$ for all $l \in \{1,2,...,J\}$. Thus, for any $l \in \{1,2,...,J\}$, $\frac{\partial e_{(l)}^*}{\partial M_{(j)}} > 0$ and hence $\frac{\partial \Pi_{(l)}}{\partial M_{(j)}} > 0$ for all $M_{(j)} < M_{(j)}^*$; and $\frac{\partial e_{(l)}^*}{\partial M_{(j)}} < 0$ and $\frac{\partial \Pi_{(l)}}{\partial M_{(j)}} < 0$ for all $M_{(j)} > M_{(j)}^*$.

(b) Suppose that the output shock $\widetilde{\xi}_{im}$ in each contest of type $l \in \{1,2,...,J\}$ is transformed to $\widehat{\xi}_{im} = \alpha_{(l)}\widetilde{\xi}_{im}$ with a parameter $\alpha_{(l)} > 0$. Then, $\widehat{e}_{(l)} \equiv g\left(\left(\frac{A_{(l)}I_{(l)}}{\alpha_{(l)}}\right)^{\frac{k+bp-1}{k+p-1}}\left(\sum_{j=1}^J M_{(j)}\left(\frac{A_{(j)}I_{(j)}}{\alpha_{(j)}}\right)^{\frac{p}{k+p-1}}\right)^{1-b}\right)$

$\overline{e}_{(l)} = \phi^{-1}\left(\frac{\left(\frac{A_{(l)}I_{(l)}}{\alpha_{(l)}}\right)^{\frac{p}{k+p-1}}\eta^{-1}(\overline{B})}{\sum_{j=1}^{J} M_{(j)}\left(\frac{A_{(j)}I_{(j)}}{\alpha_{(j)}}\right)^{\frac{p}{k+p-1}}}\right)$. Then, we have

$$\frac{\widehat{e}_{(l)}}{\overline{e}_{(l)}} = \frac{\left(\frac{A_{(l)}I_{(l)}}{\alpha_{(l)}}\right)^{\frac{1}{k+p-1}} g\left(\left(\sum_{j=1}^{J} M_{(j)}\left(\frac{A_{(j)}I_{(j)}}{\alpha_{(j)}}\right)^{\frac{p}{k+p-1}}\right)^{1-b}\right)}{\left(\frac{A_{(l)}I_{(l)}}{\alpha_{(l)}}\right)^{\frac{1}{k+p-1}} \phi^{-1}\left(\frac{\eta^{-1}(\overline{B})}{\sum_{j=1}^{J} M_{(j)}\left(\frac{A_{(j)}I_{(j)}}{\alpha_{(j)}}\right)^{\frac{p}{k+p-1}}}\right)} = \frac{g\left(\left(\sum_{j=1}^{J} M_{(j)}\left(\frac{A_{(j)}I_{(j)}}{\alpha_{(j)}}\right)^{\frac{p}{k+p-1}}\right)^{1-b}\right)}{\phi^{-1}\left(\frac{\eta^{-1}(\overline{B})}{\sum_{j=1}^{J} M_{(j)}\left(\frac{A_{(j)}I_{(j)}}{\alpha_{(j)}}\right)^{\frac{p}{k+p-1}}}\right)}$$

is decreasing in $\alpha_{(n)}$ for any $n \in \{1,2,...,J\}$. Thus, because $\widehat{e}_{(l)}$ is increasing in $M_{(j)}$ and $\overline{e}_{(l)}$ is decreasing in $M_{(j)}$, $M^*_{(j)}$ is non-increasing with $\alpha_{(n)}$ for any $n \in \{1,2,...,J\}$. ■

Proof of Proposition 4. (a) Suppose that $\widetilde{\xi}_{im}$ are i.i.d Gumbel with scale parameter α, and let d_{im} ($i \in \{1,2,...,N\}$) be scalars. Then we have the following property (cf. Terwiesch and Xu 2008):

$$\Pr\left\{d_{im} + \widetilde{\xi}_{im} = \max_{j\in\{1,2,...,N\}}\left\{d_{jm} + \widetilde{\xi}_{jm}\right\}\right\} = \frac{\exp\left\{\frac{d_{im}}{\alpha}\right\}}{\sum_{j=1}^{N}\exp\left\{\frac{d_{jm}}{\alpha}\right\}}.$$

Let $d_{im} = r(e_{im})$. Then, solver i's utility can be written as:

$$U_i(e_{im}) = \sum_{m=1}^{M} A_m \frac{\exp\left\{\frac{r(e_{im})}{\alpha}\right\}}{\sum_{j=1}^{N}\exp\left\{\frac{r(e_{jm})}{\alpha}\right\}} - c_i\eta\left(\sum_{m=1}^{M}\phi(e_{im})\right).$$

Ignoring the solver's budget constraint, we obtain the first-order conditions with respect to e_{im} as:

$$\frac{A_m}{\alpha} r'(e_{im}) \frac{\exp\left\{\frac{r(e_{im})}{\alpha}\right\}\sum_{j\neq i}\exp\left\{\frac{r(e_{jm})}{\alpha}\right\}}{\left(\sum_{j=1}^{N}\exp\left\{\frac{r(e_{jm})}{\alpha}\right\}\right)^2} - c_i\eta'\left(\sum_{l=1}^{M}\phi(e_{il})\right)\phi'(e_{im}) = 0. \tag{34}$$

Evaluating (34) at $e_{im} = e^*_i$, $e_{jm} = e^*_j$, and $A_m = A$, and letting $f = g^{-1} = ((\eta\circ\phi)'/r')$, we obtain

$$\frac{A}{\alpha} \frac{\exp\left\{\frac{r(e^*_i)}{\alpha}\right\}\sum_{j\neq i}\exp\left\{\frac{r(e^*_j)}{\alpha}\right\}}{\left(\sum_{j=1}^{N}\exp\left\{\frac{r(e^*_j)}{\alpha}\right\}\right)^2} = c_i M^{b-1} f(e^*_i). \tag{35}$$

Note that f is homogenous of degree $pb+k-1>0$ and $pb+k-1>1-k$ from §2. Let $z_i = \exp\left\{\frac{r(e^*_i)}{\alpha}\right\}$ and $Z = \sum_{j=1}^{N} z_j$. Note that $e^*_i = r^{-1}(\alpha\log(z_i))$. Then, we can write (35) as:

$$\frac{A}{\alpha}\frac{z_i(Z-z_i)}{Z^2} = c_i M^{b-1} f(r^{-1}(\alpha\log z_i)) \text{ for all } i = 1,2,...,N.$$

Under $r(e_i) = \theta\log(e_i)$, we have $f(r^{-1}(\alpha\log(z_i)))/z_i = f(z_i^{\alpha/\theta})/z_i$, which is increasing in z_i given $pb\alpha/\theta \geq 1$ since f is homogenous of degree $pb+k-1 = pb$. Let $\gamma_i = \frac{z_i}{Z}$ for all $i \in \{1,2,...,N\}$. Then,

$$\gamma_i(1-\gamma_i) = c_i\frac{\alpha}{A}M^{b-1}f(z_i^{\alpha/\theta}) = c_i\frac{\alpha}{A}M^{b-1}f\left(\frac{z_i^{\alpha/\theta}}{Z^{\alpha/\theta}}\right)Z^{pb\alpha/\theta} = c_i\frac{\alpha}{A}M^{b-1}f\left(\gamma_i^{\alpha/\theta}\right)Z^{pb\alpha/\theta}. \tag{36}$$

Let $q(\gamma) = \frac{\gamma(1-\gamma)}{f(\gamma^{\alpha/\theta})}$, which is a decreasing function. Then, we can obtain from (36) that

$$\frac{q(\gamma_i)}{c_i} = \frac{\alpha}{A}M^{b-1}Z^{pb\alpha/\theta} = \frac{q(\gamma_j)}{c_j} \text{ for all } i,j \in \{1,2,...,N\}. \tag{37}$$

From (37), we obtain $\gamma_j = q^{-1}\left(c_j \frac{q(\gamma_i)}{c_i}\right)$. Thus, the following equations characterize γ_i:

$$\sum_{j=1}^{N} q^{-1}\left(c_j \frac{q(\gamma_i)}{c_i}\right) = 1 \text{ for all } i \in \{1,2,...,N\}. \tag{38}$$

Plugging $\gamma_i = \frac{z_i}{Z} = \frac{\exp\left\{\frac{r(e_i^*)}{\alpha}\right\}}{\sum_{j=1}^{N}\exp\left\{\frac{r(e_j^*)}{\alpha}\right\}} = \frac{(e_i^*)^{\theta/\alpha}}{\sum_{j=1}^{N}(e_j^*)^{\theta/\alpha}}$ in (35), we obtain $e_i^* = g\left(\frac{AM^{1-b}\gamma_i(1-\gamma_i)}{c_i\alpha}\right)$.

When all solvers' budget constraints bind, we obtain $e_i^* = \phi^{-1}\eta^{-1}\left(\frac{\overline{B}}{c_i M^b}\right)$ for $i \in \{1,2,...,N\}$.

(b) From (37), we can deduce that $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_N$ because q is decreasing and $c_1 \ge c_2 \ge \cdots \ge c_N$. By applying logarithmic transformation on (37) and using homogeneity of f, we obtain

$$\frac{\log(1-\gamma_j) - \log(1-\gamma_i) + \log c_i - \log c_j}{\log\gamma_j - \log\gamma_i} = \frac{pb\alpha}{\theta} - 1 \text{ for all } i \in \{1,2,...,N-1\}, j > i. \tag{39}$$

Thus, as α approaches infinity, γ_i approaches $1/N$. Thus, $e_i^* = g\left(\frac{AM^{1-b}\gamma_i(1-\gamma_i)}{c_i\alpha}\right)$ is asymptotically equivalent to $g\left(\frac{AM^{1-b}(N-1)}{c_i\alpha N^2}\right)$, which clearly approaches zero as α approaches infinity. Thus, for sufficiently large α, no solver's budget constraint binds. For any split of solvers in the exclusive case with N_m solvers in contest m, an upper bound on the average profit can be written as

$$\overline{\Pi}^X \le \frac{1}{M}\sum_{m=1}^{M}(\theta\log(e_{N_m,m}^{*,X}) + \mu_{N_m} - A), \tag{40}$$

where $e_{N_m,m}^{*,X}$ is the equilibrium effort of the solver with the lowest cost (and hence the highest effort by part (a)) in contest m. Note that $e_{i,m}^{*,X} = g\left(\frac{A\gamma_{N_m,i}(1-\gamma_{N_m,i})}{c_i\alpha}\right)$ where $\gamma_{N_m,i} = \frac{(e_{i,m}^{*,X})^{\theta/\alpha}}{\sum_{j=1}^{N_m}(e_{j,m}^{*,X})^{\theta/\alpha}}$. In the non-exclusive case, a lower bound on the average profit can be written as

$$\overline{\Pi} \ge \frac{1}{M}\sum_{m=1}^{M}(\theta\log(e_1^*) + \mu_N - A). \tag{41}$$

Thus, noting that g is homogenous of degree pb, the difference between the average profit in non-exclusive and exclusive cases satisfies

$$\overline{\Pi} - \overline{\Pi}^X \ge \alpha\left(\sum_{m=1}^{M}\frac{\theta}{pb\alpha}\log\left(\frac{M^{1-b}\gamma_1(1-\gamma_1)}{\gamma_{N_m,N_m}(1-\gamma_{N_m,N_m})}\right) + \mu_N - \frac{1}{M}\sum_{m=1}^{M}\mu_{N_m}\right). \tag{42}$$

As discussed above, as α approaches infinity, γ_i approaches $1/N$ and by the same reasoning, $\gamma_{N_m,i}$ approaches $1/N_m$. Also, when $N > N_m$, $\widetilde{\xi}_m^N$ first-order stochastically dominates $\widetilde{\xi}_m^{N_m}$ for $m \in \{1,2,...,M\}$ (and not vice versa), so by the same reasoning as Theorem 1, we have $\mu_N > \frac{1}{M}\sum_{m=1}^{M}\mu_{N_m}$. Thus, there exists α_0 such that for any $\alpha > \alpha_0$, we have $\overline{\Pi} - \overline{\Pi}^X > 0$.

(c) From (38), we can see that γ_i does not depend on M. Thus, when no solver's budget constraint binds, the equilibrium effort $e_i^* = g\left(\frac{AM^{1-b}\gamma_i(1-\gamma_i)}{c_i\alpha}\right)$ increases with M for all $i \in \{1,2,...,N\}$. Thus, an organizer's profit at any contest m, $\Pi_m^* = E[\max_{i\in\{1,2,...,N\}}\{\theta\log(e_i^*) + \widetilde{\xi}_{im}\}] - A$, as well as the average profit $\overline{\Pi} = \frac{1}{M}\sum_{m=1}^{M}\Pi_m^*$ increases with M. When all solvers' budget constraints bind, the equilibrium effort $e_i^* = \phi^{-1}\eta^{-1}\left(\frac{\overline{B}}{c_i M^b}\right)$ decreases with M for all $i \in \{1,2,...,N\}$. Thus, an organizer's profit Π_m at any contest m as well as the average profit $\overline{\Pi}$ decreases with M. ■

Proof of Proposition 5. **(a)** Solver i's output in contest m is $y_{im} = r(\widetilde{a}_{im} e_{im})$ where $\widetilde{a}_{im} = \exp(\widetilde{\xi}_{im})$. Suppose that the shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha > 0$. Then, after the transformation, $\widetilde{a}_{im} = \exp(\widetilde{\xi}_{im})$. Solver i's probability of winning is

$$
\begin{aligned}
&P(e_{im}, e_m^*) = Pr\{y_{im} \geq y_{jm},\ j \neq i\} = Pr\{r(e_{im}\exp(\alpha\widetilde{\xi}_{im})) \geq r(e_m^*\exp(\alpha\widetilde{\xi}_{jm})),\ j \neq i\} \\
&= \int_{s\in\Xi} Pr\left\{\frac{1}{\alpha}\log\left(\frac{e_{im}\exp(\alpha s)}{e_m^*}\right) \geq \widetilde{\xi}_{jm},\ j \neq i\right\} h(s)ds \\
&= \int_{s\in\Xi} Pr\left\{\widetilde{\xi}_{jm} \leq \frac{1}{\alpha}\log\left(\frac{e_{im}\exp(\alpha s)}{e_m^*}\right)\right\}^{N-1} h(s)ds = \int_{s\in\Xi} H\left(\frac{1}{\alpha}\log\left(\frac{e_{im}\exp(\alpha s)}{e_m^*}\right)\right)^{N-1} h(s)ds.
\end{aligned}
$$

The derivative of the probability of winning evaluated at symmetric equilibrium is:

$$
\left.\frac{\partial P(e_{im}, e^*)}{\partial e_{im}}\right|_{e_{im}=e^*} = \int_{s\in\Xi} (N-1)H(s)^{N-2}\left(\frac{1}{\alpha e_m^*}\right) h(s)^2 ds.
$$

Then, when the budget constraint does not bind, solver i's first-order condition is

$$
\left(\frac{A}{\alpha e_m^*}\right) I_N - \eta'\left(\sum_{m=1}^{M}\phi(e_m^*)\right)\phi'(e_m^*) = 0 \text{ where } I_N = \int_{s\in\Xi}(N-1)H(s)^{N-2} h(s)^2 ds.
$$

Then, letting $\overline{g}$ be the increasing function such that $(\overline{g}(x))^{-1} = (\eta\circ\phi)'(x)x$, we can write the solution to the above conditions as $e_m^* = \overline{g}\left(\frac{AI_N M^{1-b}}{\alpha}\right)$. Note that $\overline{g}$ is homogenous of degree $\frac{1}{bp}$. When $e_m^* \leq \phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)$, the budget constraint holds so $e_m^* = \overline{g}\left(\frac{AI_N M^{1-b}}{\alpha}\right)$ is the equilibrium and otherwise $e_m^* = \phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)$ is the equilibrium. Thus, the equilibrium effort satisfies $e_m^* = \min\left\{\overline{g}\left(AI_N M^{1-b}\right), \phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)\right\}$.

(b) An organizer's profit is $\Pi_m = E[\max_{i\in\{1,2,\ldots,N\}} r(e_m^*\exp(\alpha\widetilde{\xi}_{im}))] - A$. Because both r and exp are positive and increasing functions, and α and e_m^* are constants in i, for any $s_j > s_i$, we have $r(e_m^*\exp(\alpha s_j)) > r(e_m^*\exp(\alpha s_i))$. Thus, we can write $E[\max_{i\in\{1,2,\ldots,N\}} r(e_m^*\exp(\alpha\widetilde{\xi}_{im}))]$ as follows:

$$
\begin{aligned}
E[\max_{i\in\{1,2,\ldots,N\}} r(e_m^*\exp(\alpha\widetilde{\xi}_{im}))] &= \int_{s\in\Xi}\int_{s\in\Xi}\cdots\int_{s\in\Xi}\max_{i\in\{1,2,\ldots,N\}} r(e_m^*\exp(\alpha s_i))\prod_{i=1}^{N}(h(s_i)ds_i) \\
&= \int_{s\in\Xi}\int_{s\in\Xi}\cdots\int_{s\in\Xi} r(e_m^*\exp(\alpha\max_{i\in\{1,2,\ldots,N\}} s_i))\prod_{i=1}^{N}(h(s_i)ds_i) \\
&= E[r(e_m^*\exp(\alpha\max_{i\in\{1,2,\ldots,N\}}\widetilde{\xi}_{im}))] \\
&= E[r(\exp(\log(e_m^*) + \alpha\max_{i\in\{1,2,\ldots,N\}}\widetilde{\xi}_{im}))]
\end{aligned}
$$

For a sufficiently large α, $e_m^* = \overline{g}\left(\frac{AI_N M^{1-b}}{\alpha}\right)$. Thus, for any split of solvers in the exclusive case with N_m (≥ 2) solvers in contest m, the average profit under the exclusive case can be written as:

$$
\overline{\Pi}^X = \frac{1}{M}\sum_{m=1}^{M} E\left[r\Big(\exp\left(\log(\overline{g}(AI_{N_m}M^{1-b}\alpha^{-1})) + \alpha\widetilde{\xi}_m^{N_m}\right)\Big)\right] - A. \tag{43}
$$

In the non-exclusive case, the average profit can be written as:

$$
\overline{\Pi} = \frac{1}{M}\sum_{m=1}^{M} E\left[r\Big(\exp\left(\log(\overline{g}(AI_{N}M^{1-b}\alpha^{-1})) + \alpha\widetilde{\xi}_m^{N}\right)\Big)\right] - A. \tag{44}
$$

We have $\lim_{\alpha\to\infty}\frac{1}{\alpha}\log(e_m^*)=\lim_{\alpha\to\infty}\frac{1}{\alpha}\log(\alpha^{\frac{-1}{bp}}\overline{g}\,(AI_N M^{1-b}))=0$. Then, in the limit, $\frac{1}{\alpha}\log(e_m^*)+\max_{i\in\{1,2,\dots,N\}}\widetilde{\xi}_{im}$ first-order stochastically dominates $\frac{1}{\alpha}\log(e_m^*)+\max_{i\in\{1,2,\dots,N_m\}}\widetilde{\xi}_{im}$, and not vice versa. Thus, because r and exp are increasing functions, by Theorem 1.A.3 of Shaked and Shanthikumar (2007), for sufficiently large α, we have

$$E\left[r\left(\exp\left(\log(\overline{g}(AI_N M^{1-b}\alpha^{-1}))+\alpha\widetilde{\xi}_m^N\right)\right)\right]>E\left[r\left(\exp\left(\log(\overline{g}(AI_{N_m} M^{1-b}\alpha^{-1}))+\alpha\widetilde{\xi}_m^{N_m}\right)\right)\right].$$

Thus, there exists α_0 such that for any $\alpha>\alpha_0$, we have $\overline{\Pi}-\overline{\Pi}^X>0$.

(c) $\overline{g}\,(AI_N M^{1-b})$ is increasing in M and $\phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)$ is decreasing in M so there exists M_0 such that for any $M<M_0$, $e^*=\overline{g}\,(AI_N M^{1-b})$, and for any $M>M_0$, $e^*=\phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)$. Thus, the result holds because $E[r(\exp(\log(e_m^*)+\alpha\max_{i\in\{1,2,\dots,N\}}\widetilde{\xi}_{im}))]$ is increasing in $M<M_0$ and decreasing in $M>M_0$. ■

Online Appendix

EC.1. Existence of Equilibrium

In this section, we discuss the existence of a symmetric pure strategy Nash equilibrium. The section proceeds as follows. We first present Lemma EC.1 that provides a generic sufficient condition for the concavity of solver i's utility function U_i under her effort levels $(e_{i1}, e_{i2}, ..., e_{iM})$. In Lemma EC.2, we present more specific and precise sufficient conditions under special cases of our general model. In such cases, the set of efforts $(e_1^*, e_2^* ..., e_M^*)$ (characterized by Proposition 1) solves the solver's utility maximization problem in (3), and a symmetric equilibrium exists. Finally, we present a numerical analysis to show that a symmetric equilibrium exists under a broad set of parameters.

LEMMA EC.1. *Suppose* $b \geq \underline{b} \equiv 1 - \kappa\frac{p-1}{p}$ *where* $\kappa \equiv \inf_{(e_{i1}, e_{i2}, ..., e_{iM})} \frac{\sum_{l=1}^M e_{il}\phi'(e_{il})}{\sum_{l=1}^M e_{im}\phi'(e_{il})} \geq 1/M$. *Suppose that the output shock* $\widetilde{\xi}_{im}$ *is transformed to* $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ *with a scale parameter* $\alpha > 0$. *There exist* $\underline{\alpha} > 0$ *such that for any* $\alpha > \underline{\alpha}$, *solver i's utility function* $U_i = \sum_{m=1}^M A_m P_m(e_{im}, e_m^*) - \psi(e_{i1}, e_{i2}, ..., e_{iM})$ *is concave under her effort levels* $(e_{i1}, e_{i2} ..., e_{iM})$, *where* $P_m(e_{im}, e_m^*)$ *is as in (2).*

Proof. For notational convenience, we drop e_m^* from $P_m(e_{im}, e_m^*)$. The Hessian matrix of U_i is

$$D^2U_i = \begin{bmatrix} B_1 - \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)(\phi'(e_{i1}))^2 & \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{i1})\phi'(e_{i2}) & \cdots & \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{i1})\phi'(e_{iM}) \\ \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{i2})\phi'(e_{i1}) & B_2 - \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)(\phi'(e_{i2}))^2 & \cdots & \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{i2})\phi'(e_{iM}) \\ \vdots & \vdots & \ddots & \vdots \\ \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{iM})\phi'(e_{i1}) & \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{iM})\phi'(e_{i2}) & \cdots & B_M - \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)(\phi'(e_{iM}))^2 \end{bmatrix},$$

where $B_m = A_m P_m''(e_{im}) - \eta'\left(\sum_{l=1}^M \phi(e_{il})\right)\phi''(e_{im})$. U_i is concave if and only if D^2U_i is negative semi-definite. This holds if and only if D^2U_i has non-positive eigenvalues because D^2U_i is symmetric (so it has real eigenvalues). By Gershgorin Circle Theorem, D^2U_i has non-positive eigenvalues if

$$B_m - \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)(\phi'(e_{im}))^2 \leq 0 \text{ for } m \in \{1, 2, ..., M\}, \tag{EC.1}$$

and D^2U_i is diagonally dominant; i.e.,

$$\left|B_m - \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)(\phi'(e_{im}))^2\right| \geq \left|\sum_{n\in\{1,2,...,M\}\setminus\{m\}} \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{im})\phi'(e_{in})\right|. \tag{EC.2}$$

When the condition in (EC.1) holds, since $\eta'' < 0$, the condition in (EC.2) boils down to:

$$A_m P_m''(e_{im}) - \eta'\left(\sum_{l=1}^M \phi(e_{il})\right)\phi''(e_{im}) - \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{im})\left(\sum_{l=1}^M \phi'(e_{il})\right) \leq 0. \tag{EC.3}$$

We first show that (EC.1) holds when the level of uncertainty is sufficiently large. $\eta'\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{im}) = \left(\frac{\phi(e_{im})}{\sum_{l=1}^M \phi(e_{il})}\right)^{1-b} \eta'(\phi(e_{im}))\phi'(e_{im})$ is increasing in e_{im}, so when $e_{im} > 0$, $\eta'\left(\sum_{l=1}^M \phi(e_{il})\right)\phi''(e_{im}) + \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)(\phi'(e_{im}))^2 > 0$. Thus, it suffices to show that $P_m''(e_{im}) \leq 0$. After a scale transformation of the output shock $\widetilde{\xi}_{im}$ to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha >$

0, we have $P_m(e_{im}) = \int_{s\in\Xi_m} H_m\left(s+\frac{r(e_{im})-r(e^*_m)}{\alpha}\right)^{N-1} h_m(s)ds = E\left[H_m^{N-1}\left(\widetilde{\xi}_{im}+\frac{r(e_{im})-r(e^*_m)}{\alpha}\right)\right]$. Its first derivative $P'_m(e_{im}) = \frac{r'(e_{im})}{\alpha}E\left[h_m^{N-1}\left(\widetilde{\xi}_{im}+\frac{r(e_{im})-r(e^*_m)}{\alpha}\right)\right]$. Then, letting $r^*_m = r(e^*_m)$), we have

$$P''_m(e_{im}) = \frac{r'(e_{im})^2}{\alpha^2}E\left[(h_m^{N-1})'\left(\widetilde{\xi}_{im}+\frac{r(e_{im})-r^*_m}{\alpha}\right)\right] + \frac{r''(e_{im})}{\alpha}E\left[h_m^{N-1}\left(\widetilde{\xi}_{im}+\frac{r(e_{im})-r^*_m}{\alpha}\right)\right]. \quad \text{(EC.4)}$$

As α approaches infinity, both expectations in (EC.4) converge, and $E\left[h_{(1)}^{N-1}\left(\widetilde{\xi}_{im}+\frac{r(e_{im})-r^*_m}{\alpha}\right)\right]$ converges to a positive constant. Furthermore, since r is increasing and concave, $r'(e_{im})^2/\alpha^2$ (>0) approaches 0 faster than $r''(e_{im})/\alpha$ (<0). Thus, there exists $\underline{\alpha}$ such that for all $\alpha > \underline{\alpha}$, $P''_m(e_{im}) \le 0$.

To show that (EC.3) holds when $P''_m(e_{im}) \le 0$, it suffices to show the following property:

$$\begin{aligned}
&\eta'\left(\sum_{l=1}^M \phi(e_{il})\right)\phi''(e_{im}) + \eta''\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{im})\left(\sum_{l=1}^M \phi'(e_{il})\right)\\
&= \eta'\left(\sum_{l=1}^M \phi(e_{il})\right)\phi''(e_{im}) + \frac{b-1}{\left(\sum_{l=1}^M \phi(e_{il})\right)}\eta'\left(\sum_{l=1}^M \phi(e_{il})\right)\phi'(e_{im})\left(\sum_{l=1}^M \phi'(e_{il})\right)\\
&= \frac{\eta'\left(\sum_{l=1}^M \phi(e_{il})\right)}{\left(\sum_{l=1}^M \phi(e_{il})\right)}\left[\left(\sum_{l=1}^M \phi(e_{il})\right)\phi''(e_{im}) - (1-b)\phi'(e_{im})\left(\sum_{l=1}^M \phi'(e_{il})\right)\right] \ge 0. \qquad \text{(EC.5)}
\end{aligned}$$

Noting that $\phi'(e_{im}) = \frac{e_{im}}{p-1}\phi''(e_{im})$ and $\phi(e_{im}) = \frac{e_{im}}{p}\phi'(e_{im})$, the above condition is satisfied if and only if the following inequality is satisfied

$$\frac{\sum_{l=1}^M e_{il}\phi'(e_{il})}{\sum_{l=1}^M e_{im}\phi'(e_{il})}\frac{p-1}{p} \ge (1-b). \quad \text{(EC.6)}$$

This inequality is satisfied by any $(e_{i1}, e_{i2}, ..., e_{iM})$ when $b \ge \underline{b} \equiv 1 - \kappa\frac{p-1}{p}$, where

$$\kappa = \inf_{e_{i1},e_{i2},\dots,e_{iM}} \frac{\sum_{l=1}^M e_{il}\phi'(e_{il})}{\sum_{l=1}^M e_{im}\phi'(e_{il})}. \quad \text{(EC.7)}$$

Observe that when $e_{i1} \ge e_{im}$, $\frac{\sum_{l=1}^M e_{il}\phi'(e_{il})}{\sum_{l=1}^M e_{im}\phi'(e_{il})}$ is increasing in e_{i1}. Thus, when deriving κ we can restrict attention to e_{il} such that $e_{il} \le e_{im}$. Under this condition, we have

$$\frac{\sum_{l=1}^M e_{il}\phi'(e_{il})}{\sum_{l=1}^M e_{im}\phi'(e_{il})} \ge \frac{\sum_{l=1}^M e_{il}\phi'(e_{il})}{\sum_{l=1}^M e_{im}\phi'(e_{im})} \ge \frac{e_{im}\phi'(e_{im})}{Me_{im}\phi'(e_{im})} = \frac{1}{M}. \quad \text{(EC.8)}$$

Thus, $\kappa \ge 1/M$. Therefore, as a sufficient condition, whenever $b \ge 1 - \frac{p-1}{Mp}$, (EC.5) holds. ■

Using the proof of Lemma EC.1, we can obtain the following corollary.

Corollary EC.1. *Suppose $b \ge 1-\kappa\frac{p-1}{p}$, where $\kappa = \inf_{(e_{i1},e_{i2},\dots,e_{iM})}\frac{\sum_{l=1}^M e_{il}\phi'(e_{il})}{\sum_{l=1}^M e_{im}\phi'(e_{il})}$. In any setting where P_m is concave for all $m \in \{1,2,...,M\}$, U_i is concave. Therefore, under this condition, $(e^*_1, e^*_2..., e^*_M)$ in Proposition 1 solves the solver's utility-maximization problem in (3).*

Lemma EC.1 along with Corollary EC.1 imply that when the output uncertainty is sufficiently large, U_i is concave. It is important to note that some uncertainty in solvers' output is necessary for the existence of pure-strategy Nash equilibrium. To illustrate, consider an extreme case where solvers face no uncertainty. Then, given other solvers' efforts, a solver can set her efforts to yield marginally larger outputs than other solvers and win all contests. This leads to a discontinuity in the solver's utility, and hence prevents the existence of a pure-strategy Nash equilibrium. Thus, there should be sufficient uncertainty in solvers' outputs to ensure a pure-strategy Nash equilibrium.

Although it is analytically intractable to characterize a precise bound on the output uncertainty that is sufficient for concavity under our general setting, Corollary EC.1 helps us derive such bounds under some specific settings. Specifically, the innovation contest literature often assumes conditions under which a solver's utility function is concave for a single contest; and if these conditions are satisfied in all contests of a multiple-contest environment and $b \geq 1-\kappa\frac{p-1}{p}$, Corollary EC.1 ensures that a solver's utility function is concave under multiple contests as well. Along this line, the following lemma depicts two cases, where the concavity of a solver's utility is ensured under $b \geq 1-\kappa\frac{p-1}{p}$.

Lemma EC.2. *A solver's probability of winning P_m in any contest $m \in \{1,2,...,M\}$ is concave when*
(i) The effort function $r(e_{im}) = \theta \log(e_{im})$ and the output shock $\widetilde{\xi}_{im}$ follows Gumbel distribution with scale parameter $\nu_m \geq \theta$ (as in Terwiesch and Xu 2008);
(ii) $N=2$ and the output shock $\widetilde{\xi}_{im}$ follows uniform distribution (as in Mihm and Schlapp 2019) or any distribution with a decreasing density function (e.g., exponential).

Proof. (i) When the effort function and the output shock distribution is the same as that in Terwiesch and Xu (2008), the probability of winning P_m in a contest m is the same as that in a single contest setting of Terwiesch and Xu (2008). Thus, from the proof of Theorem 1A in Terwiesch and Xu (2008), we can see that a sufficient condition for P_m to be concave is $\nu_m \geq \theta$.
(ii) Plugging $N=2$ in (EC.4) yields the second derivative of the probability of winning as:

$$P_m''(e_{im}) = \frac{r'(e_{im})^2}{\alpha^2} E\left[h_m'\left(\widetilde{\xi}_{im} + \frac{r(e_{im}) - r_m^*}{\alpha}\right)\right] + \frac{r''(e_{im})}{\alpha} E\left[h_m\left(\widetilde{\xi}_{im} + \frac{r(e_{im}) - r_m^*}{\alpha}\right)\right].$$

When h is constant (as in uniform distribution) or decreasing, $h' \leq 0$ so $E\left[h_m'\left(\widetilde{\xi}_{im} + \frac{r(e_{im})-r_m^*}{\alpha}\right)\right] \leq 0$. Because $r' > 0$ and $r'' < 0$, we have $P_m'' \leq 0$, so P_m is concave. ■

Lemma EC.2 illustrates that the solver's output uncertainty need not be very large to guarantee the concavity of a solver's utility function. Although Lemmas EC.1 and EC.2 offer sufficient conditions for the concavity of a solver's utility, concavity is sufficient but not necessary for the existence of a symmetric pure-strategy Nash equilibrium. In fact, as pointed out in the following lemma, $(e_1^*, e_2^*..., e_M^*)$ characterized in Proposition 1 is the symmetric equilibrium in any setting where $(e_1^*, e_2^*..., e_M^*)$ solves the solver's utility-maximization problem in (3).

Lemma EC.3. *If* $(e_1^*, e_2^* ..., e_M^*)$ *in Proposition 1 solves the solver's utility-maximization problem in (3), then it is the symmetric pure-strategy Nash equilibrium in solvers' subgame.*

Proof. Suppose $(e_1^*, e_2^* ..., e_M^*)$ solves the solver's utility maximization problem in (3). Then, exerting efforts $(e_1^*, e_2^* ..., e_M^*)$ is a solver's best response when all other solvers exert efforts $(e_1^*, e_2^* ..., e_M^*)$. Thus, $(e_1^*, e_2^* ..., e_M^*)$ is a symmetric pure-strategy Nash equilibrium. ■

One may wonder how restrictive the condition in Lemma EC.3 is. To address this question, we numerically check the settings under which this condition is satisfied, and hence a symmetric equilibrium exists. For this analysis, we consider Gumbel, normal, and uniform distributions for the output uncertainty, and under each distribution, we randomly generate 10,000 instances to check whether a symmetric equilibrium exists. At each instance, we randomly draw N from $\{2, 3, ..., 20\}$, M from $\{1, 2, ..., 10\}$, b from Uniform$[0.5, 1]$, pb from Uniform$[1, 5]$, c, k, and θ from Uniform$[0.5, 1.5]$, and A_m from Uniform$[1, 10]$ for each contest $m \in \{1, 2, ..., M\}$. In simulations with Gumbel distribution, we use mean 0 and scale parameter ν_m for each contest $m \in \{1, 2, ..., M\}$, where ν_m is drawn from Uniform$[1, 10]$. In this case, 98.3% of 10,000 instances contain a symmetric equilibrium. In simulations with normal distribution, we use mean zero and standard deviation σ_m for each contest $m \in \{1, 2, ..., M\}$, where σ_m is drawn from Uniform$[1, 10]$. In this case, 98.5% of 10,000 instances contain a symmetric equilibrium. In simulations with uniform distribution, we use bounds $-a_m$ and a_m for each contest $m \in \{1, 2, ..., M\}$, where a_m is drawn from Uniform$[1, 10]$. In this case, 69.4% of 10,000 instances contain a symmetric equilibrium. The fact that uniform distribution yields fewer settings that contain a symmetric equilibrium is not surprising because under the parameter settings we use, uniform distribution has less variance (which corresponds to smaller α) than Gumbel or normal distributions. If we instead draw a_m from Uniform$[\sqrt{3}, 10\sqrt{3}]$, which corresponds to the same standard deviation as σ_m drawn from Uniform$[1, 10]$, we have 86.8% of 10,000 instances containing a symmetric equilibrium. Lastly, when we ensure sufficient curvature for the solver's cost function by assuming $pb \geq 2$ (i.e., drawing pb from Uniform$[2, 5]$), the percentage of settings with symmetric equilibrium for uniform distribution increases to 94.9%. These results indicate that symmetric equilibrium exists under a very broad set of parameters.

EC.2. Further Extensions

In this section, we provide further extensions of our main results. In §EC.2.1, we consider the total profit of organizers (instead of the average profit) as the coordinator's objective. In §EC.2.2, we consider the decentralized case where organizers set the awards at their contests and compete for solvers' efforts. In §EC.2.3, we consider an alternative way of modeling economies of scope. To focus on the isolated impact of these different aspects, we restrict attention to symmetric contests.

EC.2.1. Alternative Objective for Coordinator

Our main model in §2 assumes that the coordinator maximizes the average profit. As discussed in §2, this objective seems to be aligned with the objective of a contest platform, and with the objective of an organization such as Elanco or Gates Foundation when it determines whether to run contests in parallel. In this section, we analyze the case where the coordinator maximizes the total profit of organizers (hereafter, total profit). This alternative objective for the coordinator complements the one in §2, and provides insights for an organization that considers whether to run a new contest in parallel with others or to never run it (and hence lose the potential profit). We first discuss when the coordinator should run exclusive contests.

Corollary EC.2. *Theorem 1 holds when the coordinator maximizes the total profit.*

Corollary EC.2 shows that when the solver's output uncertainty is sufficiently large, the non-exclusive case yields a larger total profit than the exclusive case, so solvers should be encouraged to participate in multiple contests. Corollary EC.2 has exactly the same intuition as Theorem 1.

We next discuss the optimal number of contests. Consistent with practice, we restrict attention to contests with non-zero awards. Before presenting the main result of this section, we make the following assumption.

Assumption EC.1. *For $M=1$, $\Pi^* = r(e^*) + \mu_{(1)}^N - A^* > 0$, where e^* and A^* are as in Lemma 1.*

Assumption EC.1 states that when there is a single contest (i.e., $M=1$), an organizer can make positive profit by giving the optimal award A^*. We make this mild assumption because otherwise, increasing the number of contests may add up negative profits. The following proposition extends Theorem 2 by showing that the coordinator's objective is unimodal in the number of contests.

Proposition EC.1. *Suppose that the coordinator sets non-zero awards at M contests. Under Assumption EC.1, $\Pi^{*,\Sigma} \equiv \sum_{m=1}^{M} \Pi^*$ is unimodal in M, i.e., there exists $M^{*,\Sigma}$ such that $\frac{\partial \Pi^{*,\Sigma}}{\partial M} > 0$ for all $M < M^{*,\Sigma}$ and $\frac{\partial \Pi^{*,\Sigma}}{\partial M} < 0$ for all $M > M^{*,\Sigma}$.*

Proposition EC.1 shows that even when the coordinator maximizes the total profit, there is an optimal number of contests. To explain the intuition, we first discuss how each organizer's profit Π^* changes with the number of contests M, and then discuss the impact of M on the total profit. When M increases, as discussed in §3.2, each organizer's profit Π^* increases as long as the scope effect outweighs the scarcity effect. Yet, when M is above a threshold M^*, the scarcity effect outweighs the scope effect, so each organizer's profit Π^* decreases with M. When the coordinator maximizes the total profit, even if each organizer's profit Π^* decreases with M, the total profit $M\Pi^*$ may still increase with M, and hence it may be optimal to run more contests than M^*. However, Proposition EC.1 shows that as M increases, the decrease in each organizer's profit due to the scarcity effect becomes so large that the total profit decreases as well. Thus, in line with Theorem 2, there is an optimal number of contests $M^{*,\Sigma}$ even when the coordinator maximizes the total profit.

EC.2.2. Decentralized Contests

In this section, we consider the decentralized case where organizers set the awards at their contests and compete for solvers' efforts. Given that other organizers give the award $A_{j\neq m} = A^{*,D}$, and that each solver exerts the equilibrium effort e_m^* in contest m as in Lemma 1, each organizer m chooses award A_m to maximize expected profit by solving the following problem:

$$\max_{A_m} \; r(e_m^*) + \mu_N - A_m. \tag{EC.9}$$

We refer to $A^{*,D}$ that solves (EC.9) as the equilibrium award in the decentralized case. As in §2, we focus on symmetric pure-strategy Nash equilibria for both organizers and solvers.

Proposition EC.2. *In the decentralized case, under Assumption 3, the following results hold.*
(a) *Let $\overline{\Pi}^X$ be the average profit in the exclusive case. Suppose that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha > 0$. Then, there exists α_0 such that the average profit in the non-exclusive decentralized case $\overline{\Pi}^D$ is greater than $\overline{\Pi}^X$ for any $\alpha > \alpha_0$.*
(b) *There exist $M_1^{*,D}$ and $M_2^{*,D}$ such that $\frac{\partial\overline{\Pi}^D}{\partial M} > 0$ for all $M < M_1^{*,D}$ and $\frac{\partial\overline{\Pi}^D}{\partial M} < 0$ for all $M > M_2^{*,D}$.*

Proposition EC.2(a) extends Theorem 1 to the decentralized case, and has the same intuition as Theorem 1. The average profit in the exclusive centralized case $\overline{\Pi}^X$ is an upper bound for the average profit in the exclusive decentralized case where solvers determine which contest(s) to participate in and their efforts, organizers determine their awards. We use the upper bound $\overline{\Pi}^X$ because in the exclusive decentralized case, a pure-strategy Nash equilibrium among organizers may not exist. Note that whenever the average profit in the non-exclusive decentralized case $\overline{\Pi}^D$ is larger than the average profit in the exclusive centralized case $\overline{\Pi}^X$, each organizer's profit in the non-exclusive decentralized case is larger than that in the exclusive decentralized case.

Proposition EC.2(b) extends Theorem 2 to the decentralized case, and has the same intuition as Theorem 2. The only difference is that Theorem 2 shows that the average profit is unimodal in the number of contests M with a peak M^*, yet Proposition EC.2(b) shows two thresholds $M_1^{*,D}$ and $M_2^{*,D}$ such that each organizer's profit increases with M when $M < M_1^{*,D}$ and decreases with M when $M > M_2^{*,D}$. Nevertheless, this result corroborates the insight of Theorem 2 that multiple contests are beneficial to organizers only up to the optimal number of contests.

EC.2.3. Alternative Model for Economies of Scope

Consistent with the innovation contest literature (e.g., Terwiesch and Xu 2008, Ales et al. 2017), our main model in §2 interprets a solver's effort as the set of actions she takes to improve her output, such as conducting literature review. Alternatively, effort can be interpreted as deterministic improvement a solver makes to her solution quality (e.g., Moldovanu and Sela 2001). These two interpretations lead to modeling economies of scope through the solver's cost function ψ, and this is consistent with the traditional definition of economies of scope (e.g., Willig 1979, Panzar and Willig 1981). In this section, we consider a third interpretation of effort as the time a solver spends on a contest. To do so, we consider spillover in the solver's output function rather than economies of scope in the solver's cost function. Specifically, the time solver i spends on one contest may improve her output at another contest, so her output in contest m is $y_{im} = \theta\left(e_{im} + a\sum_{l\neq m} e_{il}\right) + \widetilde{\xi}_{im}$, where $a \in (0,1)$.[19] This model builds on the Sutton (2001) model of output spillover. The innovation contest literature that focuses on a single contest commonly uses this type of a linear effort function with a convex cost function (e.g., Mihm and Schlapp 2019, Hu and Wang 2021). Consistent with Sutton (2001) and the innovation contest literature, we assume that solver i's total cost of effort is $\sum_{l=1}^{M} \phi(e_{il})$, where ϕ is an increasing, convex, and homogenous function of degree p (>2). The cost function ϕ may represent the solver's disutility from spending time on a contest. To capture the impact of solvers' limited resources as in our main model, we assume that each solver's total effort cannot exceed $\overline{E}$.

Proposition EC.3. ***(a)*** *Let $\overline{\Pi}^X$ be the average profit when the coordinator optimally allocates solvers and awards in the exclusive case. Suppose that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha > 0$. Then, there exists α_0 such that the average profit in the non-exclusive case $\overline{\Pi}$ is greater than that in the exclusive case $\overline{\Pi}^X$ for any $\alpha > \alpha_0$.*
(b) *The average profit $\overline{\Pi}$ is unimodal in the number of contests M, i.e., there exists M^* such that $\frac{\partial\overline{\Pi}}{\partial M} > 0$ for all $M < M^*$ and $\frac{\partial\overline{\Pi}}{\partial M} < 0$ for all $M > M^*$.*

Proposition EC.3 extends Theorems 1 and 2, and presents somewhat expected results as there is a strong correlation between the three interpretations of effort and between output spillover and economies of scope. Specifically, when a solver spends more time on a contest, the deterministic part of her output, i.e., $\theta\left(e_{im} + a\sum_{l\neq m} e_{il}\right)$, at another contest also improves. Thus, when a solver improves her output at one contest, it is less costly to improve her output at another contest, leading to economies of scope across contests. This strong correlation among different interpretations of effort explains the analogous results in Proposition EC.3 and Theorems 1 and 2.

EC.3. Proofs of Further Extensions

Proof of Corollary EC.2. Because the number of contests is fixed in Theorem 1, whenever the average profit is maximized, the total profit is also maximized. Thus, Theorem 1 directly extends to the case where the coordinator maximizes the total profit. ■

[19] It is plausible that the spillover from one contest may diminish with the spillover from other contests. This case can be modeled by taking the coefficient a as a decreasing function of M. Our results extend to this case as well.

Proof of Proposition EC.1. Let $\overline{e} = \phi^{-1}\left(\eta^{-1}(M^{-b}\overline{B})\right)$. The coordinator's problem is

$$\max_A \; Mr(e^*) + M\mu_{(1)}^N - MA, \text{ where } e^* = \min\{g(AI_N M^{1-b}), \overline{e}\}. \tag{EC.10}$$

From the above problem, we can deduce that the coordinator never sets A such that $g(AI_NM^{1-b}) > \overline{e}$ because otherwise the coordinator can improve the total profit by reducing A. Thus, without loss of optimality, the coordinator's problem can be rewritten as follows:

$$\max_A \; Mr(g(AI_N M^{1-b})) + M\mu_{(1)}^N - MA, \text{ where } g(AI_NM^{1-b}) \le \overline{e}. \tag{EC.11}$$

Let $\Phi(A) = Mr'\left(g\left(AI_NM^{1-b}\right)\right) g'\left(AI_NM^{1-b}\right) I_NM^{1-b} - M$ and $\overline{A} = M^{b-1}g^{-1}\left(\overline{e}\right)/I_N$. Note that Φ is the first derivative of (EC.10) with respect to A. Suppose that $\Phi(\overline{A}) \ge 0$. Because $r'(g(x))g'(x)$ is decreasing in x (as assumed in §2), the objective function in (EC.11) is concave in A, and hence $\Phi(A)$ is decreasing in A. Because $A > \overline{A}$ violates the constraint in (EC.11), and Φ is decreasing, $A^* = \overline{A}$ solves (EC.11). Thus, $A_m = A^*$ maximizes the total profit $\Pi^{*,\Sigma}$, and $e_m^* = e^* = \overline{e}$ is the corresponding equilibrium effort. Suppose that $\Phi(\overline{A}) < 0$. Then, as $\lim_{x\to 0} r'(g(x))g'(x) = \infty$, we have $\Phi(0) > 0$, so by the Intermediate Value Theorem, there exists $\widehat{A}$ such that $\Phi(\widehat{A}) = 0$. Note that $\widehat{A}$ is unique because Φ is decreasing. In this case, $A^* = \widehat{A}$ solves (EC.11). Thus, $A_m = A^* = \widehat{A}$ maximizes the total profit $\Pi^{*,\Sigma}$, and $e_m^* = e^* = g(A^*I_NM^{1-b})$ is the corresponding equilibrium effort.

$\Phi(\overline{A})/M = r'\left(g\left(\overline{A}I_NM^{1-b}\right)\right) g'\left(\overline{A}I_NM^{1-b}\right) I_NM^{1-b} - 1 = r'\left(\overline{e}\right) g'\left(g^{-1}(\overline{e})\right) I_NM^{1-b} - 1$ is increasing in M because $\overline{e}$ is decreasing in M, $r'(g(x))g'(x)$ is decreasing in x, and M^{1-b} is increasing in M. Because $\Phi(\overline{A})/M$ is increasing in M and $\lim_{x\to 0} r'(g(x))g'(x) = \infty$ and hence $\lim_{M\to\infty}(\Phi(\overline{A})/M) > 0$, there exists $M_0 \in [1,\infty)$ such that for any $M < M_0$, $\Phi(\overline{A}) < 0$ and for any $M \ge M_0$, $\Phi(\overline{A}) \ge 0$.

Let $\Pi^{*,M}$ be an organizer's profit when there are M contests and the coordinator optimally chooses the award as A^*. We next show that the total profit $\Pi^{*,\Sigma} = M\Pi^{*,M} = Mr(e^*) + M\mu_{(1)}^N - MA^*$ is increasing in the number of contests M up to some M^* and decreasing afterwards. When $M < M_0$ as in the proof of Theorem 2, the constraint in (EC.11) can be relaxed. Applying the Envelope Theorem to $\Pi^{*,\Sigma} \equiv \max_A \; Mr(e^*) + M\mu_{(1)}^N - MA$, we obtain

$$\frac{\partial \Pi^{*,\Sigma}}{\partial M} = \Pi^{*,M} + Mr'(e^*)\frac{\partial e^*}{\partial M} = \Pi^{*,M} + M(1-b)r'(e^*)g'\left(A^*I_NM^{1-b}\right)A^*I_NM^{-b}. \tag{EC.12}$$

A^* that maximizes $\Pi^{*,\Sigma}$ also maximizes $\Pi^{*,M}$, and we have $\frac{\partial \Pi^{*,M}}{\partial M} > 0$ for all $M < M_0$ (see proof of Theorem 2), so under Assumption EC.1, we have $\Pi^{*,M} > \Pi^{*,1} > 0$. As $g' > 0$ and $r' > 0$, from (EC.12), $\Pi^{*,\Sigma}$ is increasing in M when $M < M_0$. When $M \ge M_0$, $A^* = \overline{A}$ so the objective function in (EC.11) can be written as:

$$\Pi^{*,\Sigma} = \; Mr\left(\overline{e}\right) + M\mu_{(1)}^N - \frac{M}{I_NM^{1-b}}g^{-1}\left(\overline{e}\right). \tag{EC.13}$$

The derivative of the coordinator's objective with respect to M

$$\frac{\partial \Pi^{*,\Sigma}}{\partial M} = \; \Pi^{*,M} + M\frac{\partial \Pi^{*,M}}{\partial M} = \; \Pi^{*,M} - Mr'\left(\overline{e}\right)\frac{\overline{e}}{pM} + \frac{1}{I_NM^{-b}}(g^{-1})'\left(\overline{e}\right)\frac{\overline{e}}{pM} + \frac{1-b}{I_NM^{1-b}}g^{-1}\left(\overline{e}\right).$$

As r', ϕ, and η are homogenous of degree $-k$, p, and b, respectively, $g^{-1} = \left(\frac{(\eta\circ\phi)'}{r'}\right)$ is homogenous of degree $pb+k-1$. Noting that $(g^{-1})'(x) = (pb+k-1)g^{-1}(x)/x$, we have

$$\frac{\partial \Pi^{*,M}}{\partial M} = -\frac{r'(\overline{e})\,\phi^{-1}\left(\eta^{-1}(\overline{B})\right)}{pM^{1/p+1}} + \frac{p+k-1}{pI_N M^{2-b}} g^{-1}(\overline{e}) = \frac{r'(\overline{e})}{pM^{1/p+1}}\left(-\phi^{-1}\left(\eta^{-1}(\overline{B})\right) + \frac{p+k-1}{pI_N M^{1-b-1/p}}\frac{g^{-1}}{r'}(\overline{e})\right).$$

Note that $\frac{\partial \Pi^{*,M}}{\partial M}$ has the same sign as $\varsigma \equiv -\phi^{-1}\left(\eta^{-1}(\overline{B})\right) + \frac{p+k-1}{pI_N M^{(p+2k-2)/p}}\frac{g^{-1}}{r'}\left(\phi^{-1}\left(\eta^{-1}(\overline{B})\right)\right)$, which is always decreasing in M because $pb+k-b>0$ and $p+2k-2>0$ (note that $pb+2k-2\geq 0$). Thus, there exists $M_1 \in [M_0,\infty)$ such that $\varsigma>0$ and hence $\frac{\partial \Pi^{*,M}}{\partial M}>0$ for all $M\in[M_0,M_1)$; and $\varsigma<0$ and hence $\frac{\partial \Pi^{*,M}}{\partial M}<0$ for all $M>M_1$. Then, as $\Pi^{*,M}>0$ for all $M<M_0$, and $\frac{\partial \Pi^{*,M}}{\partial M}>0$ for all $M\in[M_0,M_1)$, we have $\Pi^{*,M}>0$ for all $M<M_1$. For $M>M_1$, $\frac{\partial \Pi^{*,M}}{\partial M}<0$, and hence $\Pi^{*,M}$ is decreasing in M. Thus, there exists $M^{*,\Sigma}$ such that $\frac{\partial \Pi^{*,\Sigma}}{\partial M}>0$ for all $M<M^{*,\Sigma}$ and $\frac{\partial \Pi^{*,\Sigma}}{\partial M}<0$ for all $M>M^{*,\Sigma}$. Also, because $r(e^*) = r(1)+\int_1^{e^*} r'(e)de = r(1)+\int_1^{e^*} e^{-k}r'(1)de = r(1)+r'(1)\frac{(e^*)^{1-k}-1}{1-k}$, for $k\geq 1$, we have $\lim_{M\to\infty}\Pi^{*,M} = \lim_{M\to\infty}\left(r(1)+r'(1)\frac{(\overline{e})^{1-k}-1}{1-k}+\mu_{(1)}^N - \frac{1}{I_N M^{1-b}}g^{-1}(\overline{e})\right) = -\infty$, so $M^{*,\Sigma}\in\mathbb{R}_+$. ■

Proof of Proposition EC.2. We find the symmetric equilibrium in the decentralized case, and then prove parts (a) and (b), respectively.

We first find the unconstrained decentralized award by relaxing the solver's budget constraint, which we denote by $\widehat{A}$. Suppose that each organizer $k\neq m$ chooses $\widehat{A}$ and organizer m chooses A. Let e be the solver's effort in contest m and let $\widehat{e}$ be the unconstrained equilibrium effort at other contests. In this case, using (10), the first-order conditions for the solver can be written as:

$$\widehat{A}r'(\widehat{e})I_N - \eta'\left((M-1)\phi(\widehat{e})+\phi(e)\right)\phi'(\widehat{e}) = 0,$$
$$Ar'(e)\,I_N - \eta'\left((M-1)\phi(\widehat{e})+\phi(e)\right)\phi'(e) = 0.$$

Under Assumption 3, from the above equalities, we can derive the following equalities:

$$A\frac{\theta}{e^p}I_N = \widehat{A}\frac{\theta}{(\widehat{e})^p}I_N = cbp\left((M-1)(\widehat{e})^p+e^p\right)^{b-1}.$$

From the first equality, we get $e^p = \frac{A}{\widehat{A}}(\widehat{e})^p$, and by plugging this into the second equality, we get

$$\widehat{A}\frac{\theta}{(\widehat{e})^p}I_N = b\left((M-1)(\widehat{e})^p + \frac{A}{\widehat{A}}(\widehat{e})^p\right)^{b-1} = cbp(\widehat{e})^{(b-1)p}\left(\frac{(M-1)\widehat{A}+A}{\widehat{A}}\right)^{b-1},$$

which yields $\widehat{e} = (\widehat{A})^{\frac{1}{p}}\left(\frac{\theta I_N}{cbp\left((M-1)\widehat{A}+A\right)^{b-1}}\right)^{\frac{1}{pb}}$ and $e = A^{\frac{1}{p}}\left(\frac{\theta I_N}{cbp\left((M-1)\widehat{A}+A\right)^{b-1}}\right)^{\frac{1}{pb}}$.

While other organizers choose $\widehat{A}$, organizer m's profit (when choosing A) can be written as:

$$\Pi_m(A,\widehat{A}) = \frac{\theta}{p}\log(A) + \frac{\theta}{bp}\log\left(\frac{\theta I_N}{cbp}\right) + \frac{\theta(1-b)}{bp}\log\left((M-1)\widehat{A}+A\right) + \mu_{(1)}^N - A.$$

A necessary condition for $\widehat{A}$ to be unconstrained equilibrium is

$$\left.\frac{\partial \Pi_m(A,\widehat{A})}{\partial A}\right|_{A=\widehat{A}} = \frac{\theta}{p}\left(\frac{1}{\widehat{A}} + \frac{1-b}{b}\frac{1}{A+(M-1)\widehat{A}}\right)\Bigg|_{A=\widehat{A}} - 1 = \frac{\theta}{p}\left(\frac{1}{\widehat{A}} + \frac{1-b}{b}\frac{1}{M\widehat{A}}\right) - 1 = 0,$$

which yields

$$\widehat{A} = \frac{\theta\,(1-b+Mb)}{Mbp} = \frac{\theta\left(M+\frac{1-b}{b}\right)}{Mp}.$$

Let $\overline{A} = M^{b-1} g^{-1}\left(\phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)\right)/I_N = \frac{bp\overline{B}}{\theta M I_N}$. Note that if all organizers give award $\overline{A}$, then the equilibrium effort in each contest is $\phi^{-1}\left(\eta^{-1}(\overline{B}M^{-b})\right)$. Suppose that $\widehat{A} < \overline{A}$. The solver's budget constraint is satisfied by the unconstrained equilibrium, and hence the equilibrium award in the decentralized case is $A^{*,D} = \widehat{A}$. Note that all organizers giving awards $\overline{A}$ is not an equilibrium as

$$\frac{\theta}{p}\left(\frac{1}{\overline{A}} + \frac{1-b}{b}\frac{1}{M\overline{A}}\right) - 1 < \frac{\theta}{p}\left(\frac{1}{\widehat{A}} + \frac{1-b}{b}\frac{1}{M\widehat{A}}\right) - 1 = 0,$$

which shows that organizers can improve profits by reducing their awards.

Suppose that $\widehat{A} \geq \overline{A}$. In this case, any award $A < \overline{A}$ cannot be an equilibrium award because

$$\frac{\theta}{p}\left(\frac{1}{A} + \frac{1-b}{b}\frac{1}{MA}\right) - 1 > \frac{\theta}{p}\left(\frac{1}{\overline{A}} + \frac{1-b}{b}\frac{1}{M\overline{A}}\right) - 1 \geq \frac{\theta}{p}\left(\frac{1}{\widehat{A}} + \frac{1-b}{b}\frac{1}{M\widehat{A}}\right) - 1 = 0,$$

which indicates that an organizer has an incentive to increase the award above A. Suppose all other organizers give award $\check{A}$, where $\check{A} \geq \overline{A}$, and let $\check{e}$ be the corresponding equilibrium effort. In this case, when an organizer selects award A such that the solver's budget constraint binds, we have $\check{e} \equiv \phi^{-1}\left(\frac{\check{A}^{\frac{p}{k+p-1}}\eta^{-1}(\overline{B})}{(M-1)A^{\frac{p}{k+p-1}} + (M-1)\check{A}^{\frac{p}{k+p-1}}}\right)$ and $e \equiv \phi^{-1}\left(\frac{A^{\frac{p}{k+p-1}}\eta^{-1}(\overline{B})}{(M-1)A^{\frac{p}{k+p-1}} + (M-1)\check{A}^{\frac{p}{k+p-1}}}\right)$. Under Assumption 3, the equilibrium efforts become

$$\check{e} \equiv \left(\frac{\check{A}(\overline{B}/c)^{1/b}}{A + (M-1)\check{A}}\right)^{1/p} \text{ and } e \equiv \left(\frac{A(\overline{B}/c)^{1/b}}{A + (M-1)\check{A}}\right)^{1/p}.$$

Then,

$$\left.\frac{de}{dA}\right|_{A=\check{A}} = \frac{1}{p}\left(\frac{(\overline{B}/c)^{1/b}}{M}\right)^{1/p-1} \frac{(M-1)(\overline{B}/c)^{1/b}}{M^2\check{A}}. \tag{EC.14}$$

Organizer m's first-order condition evaluated at $A = \check{A}$ can be written as:

$$\left.\frac{\partial \Pi_m(A, \check{A})}{\partial A}\right|_{A=\check{A}} = \frac{\theta M^{1/p}}{(\overline{B}/c)^{1/pb}}\frac{1}{p}\left(\frac{(\overline{B}/c)^{1/b}}{M}\right)^{1/p-1} \frac{(M-1)(\overline{B}/c)^{1/b}}{M^2\check{A}} - 1 = 0, \tag{EC.15}$$

which yields the equilibrium award in the decentralized case $A^{*,D}$ as:

$$A^{*,D} = \check{A} = \frac{\theta}{p}\frac{(M-1)}{M}. \tag{EC.16}$$

(**a**) We next compare the average profit in the non-exclusive decentralized case with that in the exclusive case. Suppose that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with $\alpha > 0$. Note that $\overline{A}$ increases with the parameter α as $\widehat{I}_N = I_N/\alpha$ decreases with α. As $\widehat{A}$ does not depend on α, there exists $\overline{\alpha}$ such that for all $\alpha > \overline{\alpha}$, the equilibrium award in the decentralized case is $A^{*,D} = \widehat{A} = \frac{\theta\left(M+\frac{1-b}{b}\right)}{Mp}$. The average profit in the non-exclusive decentralized case is

$$\overline{\Pi}^D = \frac{\theta}{bp}\log\left(\frac{\theta^2 I_N M^{1-b}(Mb-b+1)}{\alpha c b^2 p^2 M}\right) + \alpha\mu_N - \frac{\theta(Mb-b+1)}{Mbp}. \tag{EC.17}$$

The equilibrium effort in the exclusive case is $e_m^{*,X} = \left(\frac{\theta A_m^{*,X} I_{N_m^{*,X}}}{cbp}\right)^{\frac{1}{bp}}$, where the optimal award $A_m^{*,X} = \frac{\theta}{bp}$, for $m \in \{1,2\}$. Then, the average profit in the exclusive case is

$$\overline{\Pi}^X = \frac{1}{2}\sum_{m=1}^{2}\left(\frac{\theta}{bp}\log\left(\frac{\theta^2 I_{N_m^{*,X}}}{\alpha c b^2 p^2}\right) + \alpha\mu_{N_m^{*,X}} - \frac{\theta}{bp}\right). \tag{EC.18}$$

The difference between the average profit in non-exclusive decentralized and exclusive cases is

$$\overline{\Pi}^D - \overline{\Pi}^X = \frac{\theta}{bp}\log\left(\frac{I_N\,(2b-b+1)}{2^b(I_{N_1^{*,X}} I_{N_2^{*,X}})^{1/2}}\right) + \alpha\left(\mu_N - \frac{1}{2}\sum_{m=1}^{2}\mu_{N_m^{*,X}}\right) - \frac{\theta(b-1)}{2b}. \tag{EC.19}$$

Using the same argument as in the proof of Theorem 1, we have $\mu_N > \frac{1}{2}\sum_{m=1}^2 \mu_{N_m^{*,X}}$ for $m \in \{1,2\}$. Thus, for a sufficiently large α, $\overline{\Pi}^D - \overline{\Pi}^X > 0$, so there exists α_0 $(\geq \overline{\alpha})$ such that for any scale transformation $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ of the output shock $\widetilde{\xi}_{im}$ with $\alpha > \alpha_0$, $\overline{\Pi}^D$ is greater than $\overline{\Pi}^X$.

(b) From part **(a),** for $\overline{A} = \frac{bp\overline{B}}{\theta M I_N}$, we know that the equilibrium award in the decentralized case is $A^{*,D} = \widehat{A}$ if $\widehat{A} < \overline{A}$ and $A^{*,D} = \check{A}$ if $\check{A} > \overline{A}$. First, we analyze when $\widehat{A} < \overline{A}$ holds and then we analyze when $\check{A} > \overline{A}$ holds. By rearranging, we obtain $\widehat{A} < \overline{A}$ if and only if

$$\frac{\theta\left(M + \frac{1-b}{b}\right)}{p} < \frac{bp\overline{B}}{\theta I_N}.$$

Thus, when M is sufficiently small, the equilibrium award $A^{*,D} = \widehat{A}$. Thus, there exists a threshold $M_1^{*,D}$ such that for all $M < M_1^{*,D}$, we have $A^{*,D} = \widehat{A}$. To have $M_1^{*,D} > 1$, we need $\theta < \left(\frac{bp^2\overline{B}}{(M+\frac{1-b}{b})I_N}\right)^{\frac{1}{2}}$.

Similarly to above, by rearranging, we obtain $\check{A} > \overline{A}$ if and only if $\frac{\theta(M-1)}{p} > \frac{bp\overline{B}}{\theta I_N}$. Thus, when M is sufficiently large, the equilibrium award $A^{*,D} = \check{A}$. Therefore, there exists $M_2^{*,D}$ such that for all $M > M_2^{*,D}$, we have $A^{*,D} = \check{A}$.

We next show that when $M < M_1^{*,D}$, so $A^{*,D} = \widehat{A}$, we have $\frac{\partial\overline{\Pi}^D}{\partial M} > 0$. Note that we have

$$\overline{\Pi}^D = \frac{\theta}{bp}\log\left(\frac{\theta^2 I_N M^{1-b}(Mb-b+1)}{\alpha c b^2 p^2 M}\right) + \mu_N - \frac{\theta(Mb-b+1)}{Mbp}.$$

As $b < 1$, $-\frac{\theta(Mb-b+1)}{Mbp}$ is increasing in M. Also, $\frac{M^{1-b}(Mb-b+1)}{M}$ is increasing in M for $M \geq 1$. Thus, we have $\frac{\partial\overline{\Pi}^D}{\partial M} > 0$ for all $M < M_1^{*,D}$. We next show that when $M > M_2^{*,D}$, we have $A^{*,D} = \check{A}$, and hence $\frac{\partial\overline{\Pi}^D}{\partial M} < 0$. Note that we have $e^* = \check{e} = ((\overline{B}/c)^{1/b}/M)^{(1/p)}$. Thus, $\overline{\Pi}^D = \frac{\theta}{p}\log\left(\frac{(\overline{B}/c)^{1/b}}{M}\right) + \mu_N - \check{A}$, which is decreasing in M, because $\check{A}$ in (EC.16) is increasing in M. ∎

Proof of Proposition EC.3. We first characterize the solver's equilibrium effort, and then prove the two parts of the proposition. Solver i solves the following problem:

$$\max_{e_{i1},e_{i2},\dots,e_{iM}} \quad \sum_{m=1}^{M} A_m \int H\left(s + (1-a)e_{im} + \sum_{l=1}^{M} a e_{il} - (1-a)e_m^* - \sum_{l=1}^{M} a e_l^*\right)^{N-1} h(s)ds - \sum_{m=1}^{M}\phi(e_{im}),$$
$$\text{s.t.} \quad \sum_{m=1}^{M} e_{im} \leq \overline{E}.$$

When the solver's constraint is relaxed, the first-order conditions of the above problem evaluated at symmetric equilibrium yields $\widehat{e}_m = (\phi')^{-1}\left(\left((1-a)A_m + a\sum_{l=1}^M A_l\right) I_N\right)$. When all contests give award A, the solver's equilibrium effort considering her constraint is

$$e_m^* = \min\left\{(\phi')^{-1}\left(A\left(1+a(M-1)\right)I_N\right), \frac{\overline{E}}{M}\right\}. \tag{EC.20}$$

(a) We prove the first part of the result for two contests but the result can be generalized to any number of contests $M > 2$. We compare the average profit in exclusive and non-exclusive cases. In the exclusive case, let $N_m^{*,X}$ be the optimal number of solvers and $A_m^{*,X}$ be the optimal award in contest $m \in \{1,2\}$, and let $e_m^{*,X}$ be the corresponding equilibrium effort in contest $m \in \{1,2\}$. Suppose that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha > 0$. Note that it is never optimal for the coordinator to set awards such that $e_m^{*,X} > \overline{E}$ (for $m \in \{1,2\}$) because the coordinator can improve the average profit by reducing the award in contest $m \in \{1,2\}$. Thus, the equilibrium effort in the exclusive case is $e_m^{*,X} = e_m^{*,X} = (\phi')^{-1}(A_m^{*,X} I_{N_m^{*,X}}/\alpha)$ in contest $m \in \{1,2\}$. Without loss of generality, suppose that $e_1^{*,X} \le e_2^{*,X}$. After incorporating the optimal solution, the average profit in the exclusive case becomes

$$\overline{\Pi}^X = \frac{1}{2}\sum_{m=1}^2 (\theta e_m^{*,X} + \alpha\mu_{(1)}^{N_m^{*,X}} - A_m^{*,X}). \tag{EC.21}$$

In the non-exclusive case, suppose that the coordinator offers an award A in each contest so that the equilibrium effort in each contest $m \in \{1,2\}$ is $e_m^* = (e_1^{*,X} + e_2^{*,X})/2$. Under sufficiently large α, $\sum_{m=1}^2 e_m^* = e_1^{*,X} + e_2^{*,X} \le \overline{E}$, so from (EC.20), A satisfies

$$A = \frac{\alpha}{(1+a)\,I_N}\phi'\left(\frac{e_1^{*,X}+e_2^{*,X}}{2}\right) \le \frac{\alpha}{(1+a)\,I_N}\phi'(e_2^{*,X}) = \frac{A_2^{*,X} I_{N_2^{*,X}}}{(1+a)\,I_N}.$$

Using the above inequality, the average profit in the non-exclusive case becomes

$$\overline{\Pi} = (1+a)\,\theta\left(\frac{e_1^{*,X}+e_2^{*,X}}{2}\right) + \alpha\mu_{(1)}^N - A \ge \frac{1}{2}\sum_{m=1}^2 (1+a)\,\theta e_m^{*,X} + \alpha\mu_{(1)}^N - \frac{A_2^{*,X} I_{N_2^{*,X}}}{(1+a)\,I_N}. \tag{EC.22}$$

The difference between the average profit in non-exclusive and exclusive cases satisfies

$$\overline{\Pi} - \overline{\Pi}^X \ge \alpha\left(\mu_{(1)}^N - \frac{1}{2}\sum_{m=1}^2 \mu_{(1)}^{N_m^{*,X}} + \frac{1}{2}\sum_{m=1}^2 \frac{A_m^{*,X}}{\alpha} - \frac{A_2^{*,X} I_{N_2^{*,X}}}{\alpha(1+a)I_N}\right). \tag{EC.23}$$

Using the same argument as in the proof of Theorem 1, we have $\mu_{(1)}^N > \frac{1}{2}\sum_{m=1}^2 \mu_{(1)}^{N_m^{*,X}}$ for $m \in \{1,2\}$. Thus, there exists α_0 such that for any $\alpha > \alpha_0$, we have $\overline{\Pi} - \overline{\Pi}^X > 0$.

(b) The average profit can be written as:

$$\begin{aligned}
\overline{\Pi} &= \frac{1}{M}\left[\sum_{m=1}^M\left((1-a)e_m^* + \sum_{l=1}^M a e_l^*\right) + \sum_{m=1}^M E[\widetilde{\xi}_{(1)m}^N] - \sum_{m=1}^M A_m\right] \\
&= \frac{1}{M}\left[\sum_{m=1}^M (1+(M-1)a)e_m^* + \sum_{m=1}^M \mu_{(1)}^N - \sum_{m=1}^M A_m\right]
\end{aligned}$$

$$= \frac{1}{M}\left[\sum_{m=1}^{M}(1+(M-1)a)(\phi')^{-1}\left(\left((1-a)A_m + a\sum_{l=1}^{M}A_l\right)I_N\right) + \sum_{m=1}^{M}\mu_{(1)}^N - \sum_{m=1}^{M}A_m\right].$$

Due to the symmetry with respect to all contests and the concavity of $(\phi')^{-1}$ (which is guaranteed because $p>2$), the coordinator sets the same award in each contest (otherwise the average profit can be improved by a perturbation that makes the awards equal with the same total award). Let $\overline{A} \equiv \frac{1}{I_N(1+(M-1)a)}\phi'\left(\frac{\overline{E}}{M}\right)$. Note from (EC.20) that when the coordinator offers an award $\overline{A}$ in each contest, then the solver's total equilibrium effort is $\overline{E}$. The coordinator never chooses an award $A>\overline{A}$ because otherwise, the average profit can be improved by reducing awards marginally (and keeping the total effort as $\overline{E}$). Thus, the coordinator solves the following problem:

$$\overline{\Pi}(A^*) = \max_A \left[(1+(M-1)a)(\phi')^{-1}\left((1+(M-1)a)AI_N\right) + \mu_{(1)}^N - A\right] \text{ s.t. } A\le \overline{A}.$$

Let $\widehat{A}$ be the solution to the above problem when the constraint is relaxed. Note that because $(\phi')^{-1}$ is increasing, when ignoring the constraint, the Envelope Theorem implies that

$$\frac{\partial\overline{\Pi}(\widehat{A})}{\partial M} = a(\phi')^{-1}\left((1+(M-1)a)\widehat{A}I_N\right) + (1+(M-1)a)((\phi')^{-1})'\left((1+(M-1)a)\widehat{A}I_N\right) > 0.$$

Thus, the coordinator's objective improves with M if $\widehat{A}<\overline{A}$. Also, we can derive $\widehat{A}$ as:

$$\widehat{A} = \frac{1}{(1+(M-1)a)I_N}(((\phi')^{-1})')^{-1}\left(\frac{1}{(1+(M-1)a)^2 I_N}\right) = (1+(M-1)a)^{\frac{p}{p-2}}(((\phi')^{-1})')^{-1}\left(\frac{1}{I_N}\right)\frac{1}{I_N},$$

which is increasing and unbounded in M because $p>2$ and $\widehat{A}<\overline{A}$. Thus, there exists M_0 such that for all $M\ge M_0$, $\widehat{A}\ge\overline{A}$, and hence it is optimal for the coordinator to set $A^*=\overline{A}$. Therefore, for $M\ge M_0$, the coordinator's objective under the optimal award becomes

$$\overline{\Pi}(A^*) = (1+(M-1)a)\frac{\overline{E}}{M} + \mu_{(1)}^N - \frac{1}{I_N(1+(M-1)a)}\phi'\left(\frac{\overline{E}}{M}\right).$$

The derivative of the coordinator's objective with respect to M

$$\frac{\partial\overline{\Pi}(A^*)}{\partial M} = -(1-a)\frac{\overline{E}}{M^2} + \frac{a}{I_N(1+(M-1)a)^2}\phi'\left(\frac{\overline{E}}{M}\right) + \frac{1}{I_N(1+(M-1)a)}\phi''\left(\frac{\overline{E}}{M}\right)\frac{\overline{E}}{M^2}.$$

Note that $\frac{\partial\overline{\Pi}(A^*)}{\partial M}$ has the same sign as:

$$\frac{M^2\partial\overline{\Pi}(A^*)}{\partial M} = -(1-a)\overline{E} + \frac{a\phi'(\overline{E})M}{I_N(1+(M-1)a)^2}M^{2-p} + \frac{\phi''(\overline{E})\overline{E}}{I_N(1+(M-1)a)}M^{2-p},$$

which is decreasing in M because $\frac{M}{(1+(M-1)a)^2}$ decreases with M and $p>2$. Furthermore, $\lim_{M\to\infty}\frac{\partial\overline{\Pi}(A^*)M^2}{\partial M} = -(1-a)\overline{E}$, which means that there exists M^* such that for all $M>M^*$, we have $\frac{\partial\overline{\Pi}(A^*)}{\partial M}<0$ and for all $M<M^*$ (where M^* can be equal to M_0), we have $\frac{\partial\overline{\Pi}(A^*)}{\partial M}>0$. ■

EC.4. Additional Results

LEMMA EC.4. *(i) The cost function $\psi=\eta\left(\sum_{m=1}^{M}\phi(e_{im})\right)$ exhibits diseconomies of scale for each contest m; i.e., $\frac{\partial^2\psi}{\partial e_{im}^2}\ge 0$ for all $m\in\{1,2,...,M\}$. When there is a single contest, i.e., $M=1$, ψ is a convex function. (ii) ψ exhibits economies of scope across contests; i.e., $\frac{\partial^2\psi}{\partial e_{im}e_{ij}}<0$ for all $j\ne m$.*

Proof. (i) The partial derivative of ψ with respect to e_{im}

$$\frac{\partial \psi}{\partial e_{im}} = \eta'\left(\sum_{l=1}^{M}\phi(e_{il})\right)\phi'(e_{im}). \tag{EC.24}$$

Because η' is homogenous of degree $(b-1)$, we have

$$\frac{\partial \psi}{\partial e_{im}} = \left(\frac{\sum_{l=1}^{M}\phi(e_{il})}{\phi(e_{im})}\right)^{b-1}\eta'\left(\phi(e_{im})\right)\phi'(e_{im}).$$

$\left(\frac{\sum_{l=1}^{M}\phi(e_{il})}{\phi(e_{im})}\right)^{b-1}$ is positive and increasing in e_{im} as $b<1$. Also, as $\eta\circ\phi$ is a convex function, $\eta'\left(\phi(e_{im})\right)\phi'(e_{im})$ is positive and increasing in e_{im}. Thus, $\frac{\partial\psi}{\partial e_{im}}$ is increasing in e_{im}, which means that $\frac{\partial^2\psi}{\partial e_{im}^2}>0$. When $M=1$, $\psi(e_{i1})=\eta\left(\phi(e_{i1})\right)$, which is convex because $\eta\circ\phi$ is convex by assumption.
(ii) Then, the cross partial derivative of ψ

$$\frac{\partial^2 \psi}{\partial e_{im}\partial e_{ij}} = \eta''\left(\sum_{l=1}^{M}\phi(e_{il})\right)\phi'(e_{im})\phi'(e_{ij}).$$

Because $\phi'>0$ and η is concave (i.e., $\eta''<0$), $\frac{\partial^2\psi}{\partial e_{im}\partial e_{ij}}<0$. ■

Lemma EC.5. *In an optimal award scheme $(A_1^*, A_2^*, ..., A_M^*)$ that maximizes the average or total profit, there exist no contests m and l such that $A_m^* > A_l^* > 0$.*

Proof. For ease of illustration, we prove this result for two contests, but the proof can be extended to any number of contests. While we prove this result for the average profit objective, the same steps can be applied to prove the result for the total profit objective. Suppose to the contrary that it is optimal for the coordinator to give different awards at different contests. Without loss of generality, we label the contest with the largest award as contest 1 and the contest with the smallest award as contest 2. Then, in the optimal award scheme (A_1^*, A_2^*), $A_1^* > A_2^*$. Let e_1^* and e_2^* be the corresponding equilibrium effort in contest 1 and 2, respectively. It is never optimal to set an award such that the Lagrange multiplier λ in (8)-(9) is strictly positive because the average profit can be improved by marginally reducing awards. Thus, e_1^* and e_2^* should satisfy (11), which means that $e_1^* > e_2^*$ because φ is decreasing. Consider a perturbation with an alternative set of awards (A_1, A_2) such that $r(e_1) = r(e_1^*) - \epsilon$ and $r(e_2) = r(e_2^*) + \epsilon$ (with a sufficiently small $\epsilon > 0$ such that $\sum_{m=1}^{2}\phi(e_m) \le \eta^{-1}(\overline{B})$ due to the concavity of r). Because the total effort $\sum_{m=1}^{2}\phi(e_m) \le \eta^{-1}(\overline{B})$, we have $A_m = \frac{g^{-1}(e_m)}{I_N}\left(\frac{\phi(e_m)}{\sum_{l=1}^{2}\phi(e_l)}\right)^{1-b}$ from (10). Then, the change in the average profit $\overline{\Pi}$ after the perturbation is (note that $e_1^* = r^{-1}(r(e_1)+\epsilon)$, $e_2^* = r^{-1}(r(e_2)-\epsilon)$, $\sum_{m=1}^{2} r(e_m) = \sum_{m=1}^{2} r(e_m^*)$, and $E\left[\sum_{m=1}^{2}\widetilde{\xi}_{(1)m}^{N}\right]$ does not change after perturbation)

$$\begin{aligned}
\Delta &\equiv (-A_1 + A_1^* - A_2 + A_2^*)/2 \\
&= -\frac{g^{-1}(e_1)}{2I_N}\left(\frac{\phi(e_1)}{\phi(e_1)+\phi(e_2)}\right)^{1-b} + \frac{g^{-1}(e_1)}{2I_N}\left(\frac{\phi(e_1)}{\phi(e_1)+\phi(r^{-1}(r(e_2)-\epsilon))}\right)^{1-b} \\
&\quad -\frac{g^{-1}(e_1)}{2I_N}\left(\frac{\phi(e_1)}{\phi(e_1)+\phi(r^{-1}(r(e_2)-\epsilon))}\right)^{1-b} + \frac{g^{-1}(r^{-1}(r(e_1)+\epsilon))}{2I_N}\left(\frac{\phi(r^{-1}(r(e_1)+\epsilon))}{\phi(r^{-1}(r(e_1)+\epsilon))+\phi(r^{-1}(r(e_2)-\epsilon))}\right)^{1-b}
\end{aligned}$$

$$-\frac{g^{-1}(e_2)}{2I_N}\left(\frac{\phi(e_2)}{\phi(e_1)+\phi(e_2)}\right)^{1-b}+\frac{g^{-1}(e_2)}{2I_N}\left(\frac{\phi(e_2)}{\phi(r^{-1}(r(e_1)+\epsilon))+\phi(e_2)}\right)^{1-b}$$
$$-\frac{g^{-1}(e_2)}{2I_N}\left(\frac{\phi(e_2)}{\phi(r^{-1}(r(e_1)+\epsilon))+\phi(e_2)}\right)^{1-b}+\frac{g^{-1}(r^{-1}(r(e_2)-\epsilon))}{2I_N}\left(\frac{\phi(r^{-1}(r(e_2)-\epsilon))}{\phi(r^{-1}(r(e_1)+\epsilon))+\phi(r^{-1}(r(e_2)-\epsilon))}\right)^{1-b}.$$

Taking the limit $\lim_{\epsilon\to 0}\frac{2I_N\Delta}{\epsilon}$, and noting that $e_m=r^{-1}(r(e_m))$ and $\varphi(e_1^*)A_1^*=\varphi(e_2^*)A_2^*$, we obtain

$$\begin{aligned}\delta\equiv\ &(1-b)\left(\frac{\phi(e_1)}{\phi(e_1)+\phi(e_2)}\right)^{-b}\frac{\phi(e_1)g^{-1}(e_1)}{(\phi(e_1)+\phi(e_2))^2}\frac{\phi'(e_2)}{r'(e_2)}-(1-b)\left(\frac{\phi(e_2)}{\phi(e_1)+\phi(e_2)}\right)^{-b}\frac{\phi(e_2)g^{-1}(e_2)}{(\phi(e_1)+\phi(e_2))^2}\frac{\phi'(e_1)}{r'(e_1)}\\ &+(1-b)\left(\frac{\phi(e_1)}{\phi(e_1)+\phi(e_2)}\right)^{-b}\frac{\phi(e_2)g^{-1}(e_1)}{(\phi(e_1)+\phi(e_2))^2}\frac{\phi'(e_1)}{r'(e_1)}+\left(\frac{\phi(e_1)}{\phi(e_1)+\phi(e_2)}\right)^{1-b}\frac{1}{g'(g^{-1}(e_1))}\frac{1}{r'(e_1)}\\ &-(1-b)\left(\frac{\phi(e_2)}{\phi(e_1)+\phi(e_2)}\right)^{-b}\frac{\phi(e_1)g^{-1}(e_2)}{(\phi(e_1)+\phi(e_2))^2}\frac{\phi'(e_2)}{r'(e_2)}-\left(\frac{\phi(e_2)}{\phi(e_1)+\phi(e_2)}\right)^{1-b}\frac{1}{g'(g^{-1}(e_2))}\frac{1}{r'(e_2)}.\end{aligned}$$

Note that whenever $\delta>0$, the average profit improves after the perturbation, so we prove that when k and b are sufficiently large, $\delta>0$. Note that the first line in δ is equal to zero because $\phi'(e_m)=p\phi(e_m)/e_m$ and $g^{-1}=\eta'(\phi)\phi'/r'$. Furthermore, because $2-2k-bp\le 0$ (as assumed in §2),

$$\left(\frac{\phi(e_1)}{\phi(e_1)+\phi(e_2)}\right)^{1-b}\frac{1}{g'(g^{-1}(e_1))}\frac{1}{r'(e_1)}>\left(\frac{\phi(e_2)}{\phi(e_1)+\phi(e_2)}\right)^{1-b}\frac{1}{g'(g^{-1}(e_2))}\frac{1}{r'(e_2)}. \tag{EC.25}$$

$\Upsilon(e_1,e_2)\equiv(1-b)\left(\frac{\phi(e_1)}{\phi(e_1)+\phi(e_2)}\right)^{-b}\frac{\phi(e_2)g^{-1}(e_1)}{(\phi(e_1)+\phi(e_2))^2}\frac{\phi'(e_1)}{r'(e_1)}$ approaches 0 as b approaches 1. Thus, when b is sufficiently close to 1, $\delta>0$ from (EC.25). Furthermore, we have $\Upsilon(e_1,e_2)-\Upsilon(e_2,e_1)>0$, and hence $\delta>0$ whenever

$$\frac{\phi(e_1)^{-b}g^{-1}(e_1)}{\phi(e_1)}\frac{\phi'(e_1)}{r'(e_1)}>\frac{\phi(e_2)^{-b}g^{-1}(e_2)}{\phi(e_2)}\frac{\phi'(e_2)}{r'(e_2)}. \tag{EC.26}$$

As $\frac{\phi^{-b}g^{-1}}{\phi}\frac{\phi'}{r'}$ is homogenous of degree $-bp+bp+k-1+p-1-p+k=2k-2$, (EC.26) holds when $k\ge 1$. In either case, $\delta>0$, which contradicts the optimality of $A_1^*>A_2^*$. ■

Lemma EC.6. *(Adopted from Lemma EC.7 of Ales et al. 2020) Suppose that $M=1$, and that the output shock $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im}=\alpha\widetilde{\xi}_{im}$ with a scale parameter $\alpha>0$. Then, $\lim_{\alpha\to\infty}\frac{A^*}{\alpha}=0$.*

Proof. When $\widetilde{\xi}_{im}$ is transformed to $\widehat{\xi}_{im}=\alpha\widetilde{\xi}_{im}$ with $\alpha>0$, I_N is converted to $\widehat{I}_N=I_N/\alpha$. Note that when $M=1$, relaxing the solver's budget constraint, the optimal award $\widehat{A}[\alpha]$ satisfies

$$r'\left(g\left(\frac{\widehat{A}[\alpha]I_N}{\alpha}\right)\right)g'\left(\frac{\widehat{A}[\alpha]I_N}{\alpha}\right)\frac{I_N}{\alpha}-1=0. \tag{EC.27}$$

Because $r'(g(x))g'(x)$ is decreasing in x, and I_N/α is decreasing in α, for $\widehat{A}[\alpha]$ to satisfy (EC.27), $\widehat{A}[\alpha]/\alpha$ should be decreasing in α. Since $\widehat{A}[\alpha]/\alpha$ is decreasing in α, and $\widehat{A}[\alpha]\ge 0$, $\widehat{A}[\alpha]/\alpha$ converges. Furthermore, because $\lim_{\alpha\to\infty}\frac{I_N}{\alpha}=0$, we need $\lim_{\alpha\to\infty}\frac{\widehat{A}[\alpha]}{\alpha}=0$ to satisfy (EC.27). Under $\widehat{A}$, the equilibrium effort $e^*=g\left(\frac{\widehat{A}[\alpha]I_N}{\alpha}\right)$. Because $\lim_{\alpha\to\infty}\frac{\widehat{A}[\alpha]}{\alpha}=0$, for a sufficiently large α, we have $\eta(\phi(e^*))\le\overline{B}$, so $A^*=\widehat{A}$. Thus, $\lim_{\alpha\to\infty}\frac{A^*[\alpha]}{\alpha}=0$. ■

Lemma EC.7. *For any $N_1,N_2\in\mathbb{Z}_+\setminus\{0,1\}$, $1/I_{N_1}+1/I_{N_2}\ge 1/I_{N_1+N_2}$.*

Proof. By Lemma EC.6 in Online Appendix of Ales et al. (2020), $(N_1+N_2)I_{N_1+N_2} \geq N_1 I_{N_1}$ and $(N_1+N_2)I_{N_1+N_2} \geq N_2 I_{N_2}$. Thus, $\frac{1}{I_{N_1}} \geq \frac{N_1}{(N_1+N_2)I_{N_1+N_2}}$ and $\frac{1}{I_{N_2}} \geq \frac{N_2}{(N_1+N_2)I_{N_1+N_2}}$. Adding these inequalities, we obtain $\frac{1}{I_{N_1}} + \frac{1}{I_{N_2}} \geq \frac{1}{I_{N_1+N_2}}$. ■