Functorial Properties of the Reticulation of a Universal Algebra

George Georgescu
University of Bucharest
georgescu.capreni@yahoo.com

Leonard Kwuida
Bern University of Applied Sciences
leonard.kwuida@bfh.ch

Claudia Mureşan*
University of Bucharest
cmuresan@fmi.unibuc.ro, c.muresan@yahoo.com

Abstract

The *reticulation* of an algebra A is a bounded distributive lattice whose prime spectrum of ideals (or filters), endowed with the Stone topology, is homeomorphic to the prime spectrum of congruences of A, with its own Stone topology. The reticulation allows algebraic and topological properties to be transferred between the algebra A and this bounded distributive lattice, a transfer which is facilitated if we can define a *reticulation functor* from a variety containing A to the variety of (bounded) distributive lattices. In this paper, we continue the study of the reticulation of a universal algebra initiated in [27], where we have used the notion of prime congruence introduced through the term condition commutator, for the purpose of creating a common setting for the study of the reticulation, applicable both to classical algebraic structures and to the algebras of logics. We characterize morphisms which admit an image through the

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*Corresponding author.

reticulation and investigate the kinds of varieties that admit reticulation functors; we prove that these include semi–degenerate congruence–distributive varieties with the Compact Intersection Property and semi–degenerate congruence–distributive varieties with congruence intersection terms, as well as generalizations of these, and additional varietal properties ensure that the reticulation functors preserve the injectivity of morphisms. We also study the property of morphisms of having an image through the reticulation in relation to another property, involving the complemented elements of congruence lattices, exemplify the transfer of properties through the reticulation with conditions Going Up, Going Down, Lying Over and the Congruence Boolean Lifting Property, and illustrate the applicability of such a transfer by using it to derive results for certain types of varieties from properties of bounded distributive lattices.
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1 Introduction

The *reticulation* of an algebra A from a variety $\mathcal{C}$ is a bounded distributive lattice $\mathcal{L}(A)$ such that the spectrum of the prime congruences of A, endowed with the Stone topology, is homeomorphic to the spectrum of the prime ideals or the prime filters of $\mathcal{L}(A)$, endowed with its Stone topology. The construction for the reticulations of the members of $\mathcal{C}$ allows algebraic and topological properties to be transferred between $\mathcal{C}$ and the variety $\mathcal{D}\mathbf{01}$ of bounded distributive lattices. While a known property of bounded distributive lattices ensures the uniqueness of $\mathcal{L}(A)$ up to a lattice isomorphism (once we have chosen, for its construction, either its spectrum of prime ideals or that of its prime filters, since the reticulation constructed w.r.t. to one of these prime spectra is dually lattice isomorphic to the one constructed w.r.t. the other), prior to our construction for the setting of universal algebra from [27], the existence of the reticulation had only been proven for several concrete varieties $\mathcal{C}$, out of which we mention: commutative unitary rings [33, 48], unitary rings [10], MV algebras [9], BL algebras [20] and (bounded commutative integral) residuated lattices [39, 40, 42].

In [27], we have constructed the reticulation for any algebra whose one–class congruence is compact, whose term condition commutator is commutative and distributive w.r.t. arbitrary joins and whose set of compact congruences is closed w.r.t. this commutator operation. In particular, our construction can be applied to any algebra from a semi–degenerate congruence–modular variety having the set of the compact congruences closed w.r.t. the modular commutator, hence this construction generalizes all previous constructions of the reticulation for particular varieties, and can be further applied to other varieties, both of classical algebras and of algebras arising in the study of non–classical logics. Indeed, note, in the papers cited above, that the construction for the reticulations of residuated lattices, which generalizes that for BL algebras, which in turn generalizes that for MV algebras, relies on prime filters of residuated lattices, which are just the meet–prime elements of their lattices of filters, which are isomorphic to their lattices of congruences, whose meet–prime elements are exactly their prime congruences w.r.t. the commutator since the commutator equals the intersection in the congruence–distributive variety of residuated lattices, thus also in its subvarieties of BL algebras, respectively MV algebras; on the other hand, the construction of the reticulation for unitary rings, which generalizes that for commutative unitary rings, relies on prime ideals, i.e. the ideals which are prime w.r.t. the multiplication of ideals, which are taken by the isomorphism between the lattice of ideals and the lattice of congruences into the prime congruences w.r.t. the modular commutator; hence our choice for the definition of the reticulation in this general setting, relying on congruences which are prime w.r.t. the commutator. Out of the many definitions for the commutator that can be found in mathematical litterature, we have chosen to work with the *term condition commutator* defined in [37], that we often simply call *commutator*; recall that, if A is a member of a congruence–modular variety, then all notions of commutator define the same commutator operation on A, usually referred to as the *modular commutator* on A. Moreover, most of our results can be applied to individual algebras that satisfy the conditions we enforce on their term condition commutator, even without investigating the varieties they generate or any variety they belong to, which can turn out useful in the study of non–equational classes of algebras.

After a preliminaries section in which we remind some notions from universal algebra and establish several notations, we recall our construction from [27] for the reticulation in this universal algebra setting in Section 3. A very useful tool for transferring properties through the reticulation between $\mathcal{C}$ and $\mathcal{D}\mathbf{01}$ is a reticulation functor $\mathcal{L} : \mathcal{C} \to \mathcal{D}\mathbf{01}$, whose preservation properties can be used for such a transfer. In [27], we have defined an image through the reticulation for any surjective morphism between algebras satisfying the conditions above for the compact congruences and the term condition commutator. In Section 4 we introduce the *functoriality of the reticulation*, which essentially means, for an arbitrary morphism $f : A \to B$ in $\mathcal{C}$ between algebras A and B from $\mathcal{C}$ having the commutators with the properties above, that f admits an image $\mathcal{L}(f)$ through the reticulation, that is f induces a 0 and join–preserving function $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$. The reticulations of the members of $\mathcal{C}$ along with these images of the morphisms in $\mathcal{C}$ through the reticulation give us a functor from the category $\mathcal{C}$ to the category of (bounded) distributive lattices iff all morphisms in $\mathcal{C}$ satisfy the functoriality of the reticulation and their images through the reticulation also preserve the meet (and the 1); we call such a functor a *reticulation functor for the variety* $\mathcal{C}$. It turns out that the *admissible morphisms* we have studied in [26, 43], that is the morphisms with the property that the inverse images of prime congruences through those morphisms are again prime congruences, are exactly the morphisms satisfying the functoriality of the reticulation and whose images through the reticulation are lattice morphisms. Unfortunately, we have not been able to construct a reticulation functor in the most general case for which we have constructed the reticulation, but we have obtained reticulation functors for remarkable kinds of varieties, such as semi–degenerate congruence–distributive varieties with the Compact Intersection Property (CIP) and semi–degenerate congruence–modular varieties with *compact commutator terms*, a notion we have defined by analogy to the more restrictive one of a congruence–distributive variety with compact intersection terms. Varieties with stronger properties, such as semi–degenerate congruence–extensible congruence–distributive varieties with the CIP or semi–degenerate varieties with equationally definable principal congruences (EDPC) and the CIP turn out to have reticulation functors which preserve the injectivity of morphisms. We conclude this section by transferring properties Going Up, Going Down and Lying Over on admissible morphisms through the reticulation, and, as an illustration of the applicability of the reticulation, using this transfer to derive a result on varieties with EDPC, which states that admissible morphisms in varieties with EDPC satisfy Going Up and admissible morphisms in semi–degenerate varieties with EDPC also satisfy Lying Over, which generalizes the similar results on MV algebras from [11] and on BL algebras from [47] (see also [16] for MV algebras and [36] for BL algebras); of course, our general result can also be applied to l–groups, BCK algebras, MTL algebras, Heyting algebras (see [30] for BCK algebras and other algebras of logic) etc.. In Section 5 we study the functoriality of the reticulation in relation with another property of morphisms, that we call *functoriality of the Boolean center*, involving the complemented elements of the congruence lattice of an algebra A, which form a Boolean sublattice of the lattice of congruences of A, called the *Boolean center* of this congruence lattice, whenever A satisfies the conditions above on compact congruences and the term condition commutator and, additionally, has the property that the term condition commutator of any congruence α of A with the one–class congruence of A equals α, in particular whenever A is a member of a semi–degenerate congruence–modular variety and has the set of the compact congruences closed w.r.t. the modular commutator. The functoriality of the Boolean center on a morphism $f : A \to B$ in $\mathcal{C}$ between algebras with the commutators as above essentially means that f induces a Boolean morphism between the Boolean centers of the congruence lattices of A and B; if all morphisms in $\mathcal{C}$ have this property, then we can define a functor from the category $\mathcal{C}$ to the category of Boolean algebras. We also study another property related to these Boolean centers, namely the Congruence Boolean Lifting Property (CBLP), which turns out to be transferrable through the reticulation in the case when $\mathcal{C}$ is semi–degenerate and congruence–modular. We conclude our paper with Section 6, containing examples for the notions in the previous sections which also prove independence relations between these notions.

2 Preliminaries

We refer the reader to [1, 14, 29, 35] for a further study of the following notions from universal algebra, to [7, 13, 17, 28, 46] for the lattice–theoretical ones, to [1, 21, 35, 45] for the results on commutators and to [1, 18, 19, 26, 43, 31] for the Stone topologies.

All algebras will be non–empty and they will be designated by their underlying sets; by *trivial algebra* we mean one–element algebra. For brevity, we denote by $A \cong B$ the fact that two algebras A and B of the same type are isomorphic. We abbreviate by *CIP* and *PIP* the Compact Intersection Property and the Principal Intersection Property, respectively.

$\mathbb{N}$ denotes the set of the natural numbers, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, and, for any $a, b \in \mathbb{N}$, we denote by $\overline{a, b}$ the interval in the lattice $(\mathbb{N}, \leq)$ bounded by a and b, where $\leq$ is the natural order. Let M, N be sets and $S \subseteq M$. Then $\mathcal{P}(M)$ denotes the set of the subsets of M and $(\mathrm{Eq}(M), \vee, \cap, \Delta_M = \{(x, x) \mid x \in M\}, \nabla_M = M^2)$ is the bounded lattice of the equivalences on M. We denote by $i_{S,M} : S \to M$ the inclusion map and by $id_M = i_{M,M}$ the identity map of M. For any function $f : M \to N$, we denote by $\mathrm{Ker}(f)$ the kernel of f, by f^* the inverse image of $f^2 = f \times f : M^2 \to N^2$ and we use the common notation f for the direct image of f^2.

Let L be a lattice. Then $\mathrm{Cp}(L)$ denotes the set of the compact elements of L, and $\mathrm{Id}(L)$ and $\mathrm{Spec}_{\mathrm{Id}}(L)$ denote the set of the ideals and that of the prime ideals of L, respectively. Let $U \subseteq L$ and $u \in L$. Then $[U)$ and $[u)$ denote the filters of L generated by U and by u, respectively, while $(U]$ and $(u]$ denote the ideals of L generated by U and by u, respectively.

We denote by $\mathcal{L}_n$ the n–element chain for any $n \in \mathbb{N}^*$, by $\mathcal{M}_3$ the five–element modular non–distributive lattice and by $\mathcal{N}_5$ the five–element non–modular lattice. Recall that a *frame* is a complete lattice with the meet distributive w.r.t. arbitrary joins.

Throughout this paper, by *functor* we mean covariant functor. $\mathcal{B}$ denotes the functor from the variety of bounded distributive lattices to the variety of Boolean algebras which takes each bounded distributive lattice to its Boolean center and every morphism in the former variety to its restriction to the Boolean centers. If L is a bounded lattice, then we denote by $\mathcal{B}(L)$ the set of the complemented elements of L even if L is not distributive.

(H) Throughout the rest of this paper, τ will be a universal algebras signature, $\mathcal{C}$ an equational class of τ–algebras and A an arbitrary member of $\mathcal{C}$. Unless mentioned otherwise, by *morphism* we mean τ–morphism.

Everywhere in this paper, we will mark global assumptions as above, for better visibility.

$\mathrm{Con}(A)$, $\mathrm{Max}(A)$, $\mathrm{PCon}(A)$ and $\mathcal{K}(A)$ denote the sets of the congruences, maximal congruences, principal congruences and finitely generated congruences of A, respectively; note that $\mathcal{K}(A)$ is the set of the compact elements of the lattice $\mathrm{Con}(A)$. $\mathrm{Max}(A)$ is called the *maximal spectrum* of A. For any $X \subseteq A^2$ and any $a, b \in A$, $Cg_A(X)$ will be the congruence of A generated by X and we shall denote by $Cg_A(a, b) = Cg_A(\{(a, b)\})$.

For any $\theta \in \mathrm{Con}(A)$, $p_\theta : A \to A/\theta$ will be the canonical surjective morphism; given any $X \in A \cup A^2 \cup \mathcal{P}(A) \cup \mathcal{P}(A^2)$, we denote by $X/\theta = p_\theta(X)$. If L is a distributive lattice, so that we have the canonical lattice embedding $\iota_L : \mathrm{Id}(L) \to \mathrm{Con}(L)$, then we will denote, for every $I \in \mathrm{Id}(L)$, by $\pi_I = p_{\iota_L(I)} : L \to L/I$.

Recall that, if B is a member of $\mathcal{C}$ and $f : A \to B$ is a morphism, then, for any $\alpha \in \mathrm{Con}(A)$ and any $\beta \in \mathrm{Con}(B)$, we have $f^*(\beta) \in [\mathrm{Ker}(f)) \subseteq \mathrm{Con}(A)$, $f(f^*(\beta)) = \beta \cap f(A^2) \subseteq \beta$ and $\alpha \subseteq f^*(f(\alpha))$; if $\alpha \in [\mathrm{Ker}(f))$, then $f(\alpha) \in \mathrm{Con}(f(A))$ and $f^*(f(\alpha)) = \alpha$. Hence $\theta \mapsto f(\theta)$ is a lattice isomorphism from $[\mathrm{Ker}(f))$ to $\mathrm{Con}(f(A))$ and thus it sets an order isomorphism from $\mathrm{Max}(A) \cap [\mathrm{Ker}(f))$ to $\mathrm{Max}(f(A))$. For the next lemma, note that $\mathrm{Ker}(p_\theta) = \theta$ for any $\theta \in \mathrm{Con}(A)$, and that $Cg_A(Cg_S(X)) = Cg_A(X)$ for any subalgebra S of A and any $X \subseteq S^2$.

Lemma 2.1. [8, Lemma 1.11], [49, Proposition 1.2] *If B is a member of $\mathcal{C}$ and $f : A \to B$ is a morphism, then, for any $X \subseteq A^2$ and any $\alpha, \theta \in \mathrm{Con}(A)$:*

- *$f(Cg_A(X) \vee \mathrm{Ker}(f)) = Cg_{f(A)}(f(X))$, so $Cg_B(f(Cg_A(X))) = Cg_B(f(X))$ and $(Cg_A(X) \vee \theta)/\theta = Cg_{A/\theta}(X/\theta)$;*
- *in particular, $f(\alpha \vee \mathrm{Ker}(f)) = Cg_{f(A)}(f(\alpha))$, so $(\alpha \vee \theta)/\theta = Cg_{A/\theta}(\alpha/\theta)$.*

If B is a member of $\mathcal{C}$ and $f : A \to B$ is a morphism, then, for any non–empty family $(\alpha_i)_{i \in I} \subseteq [\mathrm{Ker}(f))$, we have, in $\mathrm{Con}(f(A))$: $f(\bigvee\limits_{i \in I} \alpha_i) = \bigvee\limits_{i \in I} f(\alpha_i)$. Indeed, by

Lemma 2.1, $f(\bigvee_{i\in I}\alpha_i) = f(Cg_A(\bigcup_{i\in I}\alpha_i)) = Cg_{f(A)}(f(\bigcup_{i\in I}\alpha_i)) = Cg_{f(A)}(\bigcup_{i\in I}f(\alpha_i)) = \bigvee_{i\in I}f(\alpha_i)$.

We use the following definition from [37] for the *term condition commutator*, that we simply call *commutator* from now on. Let $\alpha,\beta\in \mathrm{Con}(A)$. For any $\mu\in\mathrm{Con}(A)$, by $C(\alpha,\beta;\mu)$ we denote the fact that the following condition holds: for all $n,k\in\mathbb{N}$ and any term t over τ of arity $n+k$, if $(a_i,b_i)\in\alpha$ for all $i\in\overline{1,n}$ and $(c_j,d_j)\in\beta$ for all $j\in\overline{1,k}$, then $(t^A(a_1,\ldots,a_n,c_1,\ldots,c_k),t^A(a_1,\ldots,a_n,d_1,\ldots,d_k))\in\mu$ iff $(t^A(b_1,\ldots,b_n,c_1,\ldots,c_k),t^A(b_1,\ldots,b_n,d_1,\ldots,d_k))\in\mu$. We denote by $[\alpha,\beta]_A=\bigcap\{\mu\in\mathrm{Con}(A)\mid C(\alpha,\beta;\mu)\}$; we call $[\alpha,\beta]_A$ the *commutator of* α *and* β in A. The operation $[\cdot,\cdot]_A:\mathrm{Con}(A)\times\mathrm{Con}(A)\to\mathrm{Con}(A)$ is called the *commutator of* A.

Recall that $\mathcal{C}$ is said to be *congruence–modular*, respectively *congruence–distributive* iff the congruence lattices of all its members are modular, respectively distributive.

By [21], if $\mathcal{C}$ is congruence–modular, then, for each member M of $\mathcal{C}$, $[\cdot,\cdot]_M$ is the unique binary operation on $\mathrm{Con}(M)$ such that, for all $\alpha,\beta\in\mathrm{Con}(M)$, $[\alpha,\beta]_M=\min\{\mu\in\mathrm{Con}(M)\mid\mu\subseteq\alpha\cap\beta$ and, for any member N of $\mathcal{C}$ and any surjective morphism $h:M\to N$ in $\mathcal{C}$, $\mu\vee\mathrm{Ker}(h)=h^*([h(\alpha\vee\mathrm{Ker}(h)),h(\beta\vee\mathrm{Ker}(h))]_N)\}$. Therefore, if $\mathcal{C}$ is congruence–modular, $\alpha,\beta,\theta\in\mathrm{Con}(A)$ and f is surjective, then $[f(\alpha\vee\mathrm{Ker}(f)),f(\beta\vee\mathrm{Ker}(f))]_B=f([\alpha,\beta]_A\vee\mathrm{Ker}(f))$, thus $[(\alpha\vee\theta)/\theta,(\beta\vee\theta)/\theta]_B=([\alpha,\beta]_A\vee\theta)/\theta$, hence, if $\theta\subseteq\alpha\cap\beta$, then $[\alpha/\theta,\beta/\theta]_{A/\theta}=([\alpha,\beta]_A\vee\theta)/\theta$, and, if, moreover, $\theta\subseteq[\alpha,\beta]_A$, then $[\alpha/\theta,\beta/\theta]_{A/\theta}=[\alpha,\beta]_A/\theta$.

By [37, Lemma 4.6, Lemma 4.7, Theorem 8.3], the commutator is smaller than the intersection and increasing in both arguments. If $\mathcal{C}$ is congruence–modular, then the commutator is also commutative and distributive in both arguments with respect to arbitrary joins. By [32], if $\mathcal{C}$ is congruence–distributive, then, in each member of $\mathcal{C}$, the commutator coincides with the intersection of congruences. Clearly, if the commutator of A coincides with the intersection of congruences, then $\mathrm{Con}(A)$ is a frame, in particular it is congruence–distributive. Recall, however, that, since the lattice $\mathrm{Con}(A)$ is complete and algebraic, thus upper continuous, $\mathrm{Con}(A)$ is a frame whenever it is distributive.

By [21, Theorem 8.5, p. 85], if $\mathcal{C}$ is congruence–modular, then the following are equivalent:

- for any algebra M from $\mathcal{C}$, $[\nabla_M,\nabla_M]_M=\nabla_M$;
- for any algebra M from $\mathcal{C}$ and any $\theta\in\mathrm{Con}(M)$, $[\theta,\nabla_M]_M=\theta$;
- $\mathcal{C}$ has no skew congruences, that is, for any algebras M and N from $\mathcal{C}$, $\mathrm{Con}(M\times N)=\{\theta\times\zeta\mid\theta\in\mathrm{Con}(M),\zeta\in\mathrm{Con}(N)\}$.

Recall that $\mathcal{C}$ is said to be *semi–degenerate* iff no non–trivial algebra in $\mathcal{C}$ has one–element subalgebras. By [35], $\mathcal{C}$ is semi–degenerate iff, for all members M of $\mathcal{C}$, $\nabla_M \in \mathcal{K}(M)$. By [1, Lemma 5.2] and the fact that, in congruence–distributive varieties, the commutator coincides with the intersection, we have: if $\mathcal{C}$ is either congruence–distributive or both congruence–modular and semi–degenerate, then $\mathcal{C}$ has no skew congruences.

If $[\cdot,\cdot]_A$ is commutative and distributive w.r.t. the join (in particular if $\mathcal{C}$ is congruence–modular), then, if A has principal commutators, that is $[\mathrm{PCon}(A), \mathrm{PCon}(A)]_A \subseteq \mathrm{PCon}(A)$, then $[\mathcal{K}(A), \mathcal{K}(A)]_A \subseteq \mathcal{K}(A)$.

We denote the set of the *prime congruences* of A by $\mathrm{Spec}(A)$. As defined in [21], $\mathrm{Spec}(A) = \{\phi \in \mathrm{Con}(A) \setminus \{\nabla_A\} \mid (\forall\, \alpha, \beta \in \mathrm{Con}(A))\, ([\alpha,\beta]_A \subseteq \phi \Rightarrow \alpha \subseteq \phi \text{ or } \beta \subseteq \phi)\}$. $\mathrm{Spec}(A)$ is called the *(prime) spectrum* of A. Recall that $\mathrm{Spec}(A)$ is not necessarily non–empty. However, by [1, Theorem 5.3], if $\mathcal{C}$ is congruence–modular and semi–degenerate, then any proper congruence of A is included in a maximal congruence of A, and any maximal congruence of A is prime. Recall, also, that, if $\mathcal{C}$ is congruence–modular, B is a member of $\mathcal{C}$ and $f : A \to B$ is a morphism, then the map $\alpha \mapsto f(\alpha)$ is an order isomorphism from $\mathrm{Spec}(A) \cap [\mathrm{Ker}(f))$ to $\mathrm{Spec}(f(A))$, thus to $\mathrm{Spec}(B)$ if f is surjective, case in which its inverse is $f^* \mid_{\mathrm{Spec}(B)} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. In [26, 43], we have called f an *admissible morphism* iff $f^*(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(A)$.

Remark 2.2. By the above, if f is surjective, then f is admissible.

Assume that $[\cdot,\cdot]_A$ is commutative and distributive w.r.t. arbitrary joins and that $\mathrm{Spec}(A)$ is non–empty, which hold if $\mathcal{C}$ is congruence–modular and semi–degenerate and A is non–trivial. For each $\theta \in \mathrm{Con}(A)$, we denote by $V_A(\theta) = \mathrm{Spec}(A) \cap [\theta)$ and by $D_A(\theta) = \mathrm{Spec}(A) \setminus V_A(\theta)$. Then, by [1, 27], $(\mathrm{Spec}(A), \{D_A(\theta) \mid \theta \in \mathrm{Con}(A)\})$ is a topological space in which, for all $\alpha, \beta \in \mathrm{Con}(A)$ and any family $(\gamma_i)_{i \in I} \subseteq \mathrm{Con}(A)$, the following hold:

- $V_A([\alpha,\beta]_A) = V_A(\alpha \cap \beta) = V_A(\alpha) \cup V_A(\beta)$ and $V_A(\bigvee\limits_{i \in I} \gamma_i) = \bigcap\limits_{i \in I} V_A(\gamma_i)$;
- if $\mathcal{C}$ is congruence–modular and semi–degenerate, then: $V_A(\alpha) = \emptyset$ iff $\alpha = \nabla_A$.

$\{D_A(\theta) \mid \theta \in \mathrm{Con}(A)\}$ is called the *Stone topology* on $\mathrm{Spec}(A)$ and it has $\{D_A(Cg_A(a,b)) \mid a, b \in A\}$ as a basis. In the same way, but replacing congruences with ideals, one defines the Stone topology on the set of prime ideals of a bounded distributive lattice.

3 The Construction of the Reticulation of a Universal Algebra and Related Results

In this section, we recall the construction for the reticulation of A from [27] and point out its basic properties.

(H) Throughout this section, we shall assume that $[\cdot,\cdot]_A$ is commutative and distributive w.r.t. arbitrary joins and that $\nabla_A \in \mathcal{K}(A)$, which hold in the particular case when $\mathcal{C}$ is congruence–modular and semi–degenerate.

For every $\theta \in \mathrm{Con}(A)$, we denote by $\rho_A(\theta)$ the *radical* of θ: $\rho_A(\theta) = \bigcap\{\phi \in \mathrm{Spec}(A) \mid \theta \subseteq \phi\} = \bigcap_{\phi \in V_A(\theta)} \phi$. We denote by $\mathrm{RCon}(A)$ the set of the *radical congruences* of A: $\mathrm{RCon}(A) = \{\rho_A(\theta) \mid \theta \in \mathrm{Con}(A)\} = \{\theta \in \mathrm{Con}(A) \mid \theta = \rho_A(\theta)\} = \{\bigcap M \mid M \subseteq \mathrm{Spec}(A)\}$. If the commutator of A equals the intersection (so that A is congruence–distributive), in particular if $\mathcal{C}$ is congruence–distributive, then $\mathrm{Spec}(A)$ is the set of the prime elements of the lattice $\mathrm{Con}(A)$, thus its set of meet–irreducible elements, hence $\mathrm{RCon}(A) = \mathrm{Con}(A)$ since the lattice $\mathrm{Con}(A)$ is algebraic.

Note that, for any $\alpha, \beta, \theta \in \mathrm{Con}(A)$, the following equivalences hold: $\alpha \subseteq \rho_A(\beta)$ iff $\rho_A(\alpha) \subseteq \rho_A(\beta)$ iff $V_A(\alpha) \supseteq V_A(\beta)$; thus $\rho_A(\alpha) = \rho_A(\beta)$ iff $V_A(\alpha) = V_A(\beta)$. By the above and the properties of the Stone topology on $\mathrm{Spec}(A)$ recalled in Section 2, we have proven, in [27], that, for any $n \in \mathbb{N}^*$, any $\alpha, \beta \in \mathrm{Con}(A)$ and any $(\gamma_i)_{i\in I} \subseteq \mathrm{Con}(A)$, we have:

- $\rho_A(\rho_A(\alpha)) = \rho_A(\alpha)$; $\alpha \subseteq \rho_A(\beta)$ iff $\rho_A(\alpha) \subseteq \rho_A(\beta)$; $\rho_A(\alpha) = \alpha$ iff $\alpha \in \mathrm{RCon}(A) \supseteq \mathrm{Spec}(A)$;
- $\rho_A(\bigvee_{i\in I} \gamma_i) = \rho_A(\bigvee_{i\in I} \rho_A(\gamma_i)) = \bigvee_{i\in I} \rho_A(\gamma_i)$; $\rho_A([\alpha,\beta]_A^n) = \rho_A([\alpha,\beta]_A) = \rho_A(\alpha \wedge \beta) = \rho_A(\alpha) \wedge \rho_A(\beta)$;
- $\rho_A(\nabla_A) = \nabla_A$; if $\mathcal{C}$ is congruence–modular and semi–degenerate, then: $\rho_A(\alpha) = \nabla_A$ iff $\alpha = \nabla_A$;
- $\rho_{A/\theta}((\alpha \vee \theta)/\theta) = \rho_A(\alpha \vee \theta)/\theta$.

If we define $\equiv_A = \{(\alpha,\beta) \in \mathrm{Con}(A) \times \mathrm{Con}(A) \mid \rho_A(\alpha) = \rho_A(\beta)\}$, then, by the above, $\equiv_A$ is a lattice congruence of $\mathrm{Con}(A)$ that preserves arbitrary joins and fulfills $[\alpha,\beta]_A \equiv_A \alpha \cap \beta$ for all $\alpha, \beta \in \mathrm{Con}(A)$. By the above, if the commutator of A equals the intersection, in particular if $\mathcal{C}$ is congruence–distributive, then $\rho_A(\theta) = \theta$ for all $\theta \in \mathrm{Con}(A)$, hence $\equiv_A = \Delta_{\mathrm{Con}(A)}$. Recall that A is called a *semiprime algebra* iff $\Delta_A \in \mathrm{RCon}(A)$, that is iff $\rho_A(\Delta_A) = \Delta_A$. Therefore, if the commutator of A

equals the intersection, then A is semiprime, and, if $\mathcal{C}$ is congruence–distributive, then all members of $\mathcal{C}$ are semiprime. Of course, $\theta \subseteq \rho_A(\theta)$ for all $\theta \in \mathrm{Con}(A)$, so $\rho_A(\theta) = \Delta_A$ implies $\theta = \Delta_A$, hence, if A is semiprime, then $\Delta_A/{\equiv_A} = \{\Delta_A\}$. By the above, if $\mathcal{C}$ is congruence–modular and semi–degenerate, then $\nabla_A/{\equiv_A} = \{\nabla_A\}$.

Remark 3.1. Assume that A is semiprime and let $\alpha, \beta \in \mathrm{Con}(A)$. Then $\rho_A([\alpha, \beta]_A) = \rho_A(\alpha \cap \beta)$, hence, by the above: $[\alpha, \beta]_A = \Delta_A$ iff $\alpha \cap \beta = \Delta_A$.

We will often use the remarks in this paper without referencing them.

By the properties of the commutator, the quotient bounded lattice, $(\mathrm{Con}(A)/{\equiv_A}, \vee, \wedge, \mathbf{0}, \mathbf{1})$, is a frame. We denote by $\lambda_A : \mathrm{Con}(A) \to \mathrm{Con}(A)/{\equiv_A}$ the canonical surjective lattice morphism. The intersection ${\equiv_A} \cap (\mathcal{K}(A))^2 \in \mathrm{Eq}(\mathcal{K}(A))$ will also be denoted $\equiv_A$; $\mathcal{L}(A) = \mathcal{K}(A)/{\equiv_A}$ will be its quotient set and we will use the same notation for the canonical surjection: $\lambda_A : \mathcal{K}(A) \to \mathcal{L}(A)$.

(H) Throughout the rest of this section, we shall assume that $\mathcal{K}(A)$ is closed w.r.t. the commutator of A.

Then, by [27, Proposition 9], $\mathcal{L}(A)$ is a bounded sublattice of $\mathrm{Con}(A)/{\equiv_A}$, thus it is a bounded distributive lattice. Note that, in the particular case when the commutator of A coincides with the intersection, the fact that $\mathcal{K}(A)$ is closed w.r.t. the commutator means that $\mathcal{K}(A)$ is a sublattice of $\mathrm{Con}(A)$. So, if $\mathcal{C}$ is congruence–distributive, then: $\mathcal{C}$ has the CIP iff $\mathcal{K}(M)$ is a sublattice of $\mathrm{Con}(M)$ in each member M of $\mathcal{C}$.

Note from the above that, for any $\theta \in \mathrm{Con}(A)$, we have: $\lambda_A(\theta) = \mathbf{1}$ iff $\theta = \nabla_A$.

Let $\theta \in \mathrm{Con}(A)$. Then we denote by $\theta^* = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \alpha \subseteq \theta\}$. Of course, $\mathbf{0} = \lambda_A(\Delta_A) \in \theta^*$. Let $\alpha, \beta \in \mathcal{K}(A)$. Then clearly $\alpha \vee \beta \in \mathcal{K}(A)$, $\lambda_A(\alpha \vee \beta) = \lambda_A(\alpha) \vee \lambda_A(\beta)$ and, if $\alpha \subseteq \theta$ and $\beta \subseteq \theta$, then $\alpha \vee \beta \subseteq \theta$. Since $\mathcal{K}(A)$ is closed w.r.t. the commutator of A, we have $[\alpha, \beta]_A \in \mathcal{K}(A)$, and, if $\alpha \subseteq \theta$ and $\lambda_A(\beta) \leq \lambda_A(\alpha)$, then $[\alpha, \beta]_A \subseteq \alpha \subseteq \theta$ and $\lambda_A(\beta) = \lambda_A(\alpha) \wedge \lambda_A(\beta) = \lambda_A([\alpha, \beta]_A)$. Hence $\theta^* \in \mathrm{Id}(\mathcal{L}(A))$.

Proposition 3.2. [27, Proposition 10, (ii)] *The map $\theta \mapsto \theta^*$ from $\mathrm{Con}(A)$ to $\mathrm{Id}(\mathcal{L}(A))$ is surjective.*

Proposition 3.3. [27, Proposition 11] *If $\theta \in \mathrm{Spec}(A)$, then $\theta^* \in \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$, and the map $\phi \mapsto \phi^*$ is an order isomorphism from $\mathrm{Spec}(A)$ to $\mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$ and a homeomorphism w.r.t. the Stone topologies.*

The previous proposition allows us to define:

Definition 3.4. $\mathcal{L}(A)$ is called the *reticulation* of A.

By the above, if the commutator of A equals the intersection, in particular if $\mathcal{C}$ is congruence–distributive, then $\lambda_A : \mathrm{Con}(A) \to \mathrm{Con}(A)/\equiv_A$ is a lattice isomorphism, $\mathcal{K}(A)$ is a bounded sublattice of $\mathrm{Con}(A)$ (recall that we are under the hypotheses that $[\mathcal{K}(A), \mathcal{K}(A)]_A \subseteq \mathcal{K}(A)$ and $\nabla_A \in \mathcal{K}(A)$) and $\lambda_A : \mathcal{K}(A) \to \mathcal{L}(A)$ is a lattice isomorphism, therefore we may take $\mathcal{L}(A) = \mathcal{K}(A)$, hence, if, additionally, A is finite, so that $\mathcal{K}(A) = \mathrm{Con}(A)$, then we may take $\mathcal{L}(A) = \mathrm{Con}(A)$.

4 Functoriality of the Reticulation

(H) Throughout this section, B will be an arbitrary member of $\mathcal{C}$ and $f : A \to B$ shall be an arbitrary morphism in $\mathcal{C}$.

We define $f^\bullet : \mathrm{Con}(A) \to \mathrm{Con}(B)$ by: $f^\bullet(\alpha) = Cg_B(f(\alpha))$, so that $f^\bullet(\alpha) = f(\alpha \vee \mathrm{Ker}(f))$ by Lemma 2.1. Let us note that $f^\bullet$ and f^* are order–preserving and, of course, so is the direct image of f^2. Notice, also, that, for all $\alpha \in \mathrm{Con}(A)$, $f(\alpha) \subseteq f^\bullet(\alpha)$, and, if f is surjective and $\alpha \in [\mathrm{Ker}(f))$, then $f(\alpha) = f^\bullet(\alpha)$. Of course, $f^\bullet(\Delta_A) = \Delta_B$.

Remarks 4.1. *(i)* $f^\bullet$ is the unique left adjoint of f^*, that is, for all $\alpha \in \mathrm{Con}(A)$ and all $\beta \in \mathrm{Con}(B)$: $f^\bullet(\alpha) \subseteq \beta$ iff $\alpha \subseteq f^*(\beta)$.
Indeed, for the direct implication, notice that $f(\alpha) \subseteq f^\bullet(\alpha) \subseteq \beta$ implies $\alpha \subseteq f^*(f(\alpha)) \subseteq f^*(\beta)$. For the converse, note that $\alpha \subseteq f^*(\beta)$ implies $f(\alpha) \subseteq f(f^*(\beta)) \subseteq \beta \in \mathrm{Con}(B)$, hence $f^\bullet(\alpha) = Cg_B(f(\alpha)) \subseteq \beta$. Therefore $f^\bullet$ is a left adjoint of f^*, and it is unique by the properties of adjoint pairs of morphisms between posets.

(ii) $f^\bullet$ preserves arbitrary joins of congruences of A.
This follows from Lemma 2.1, but also from the properties of adjoint pairs of lattice morphisms between complete lattices and the fact that f^* preserves arbitrary intersections, since it is the inverse image of f^2.

(iii) If C is a member of $\mathcal{C}$ and $g : B \to C$ is a morphism in $\mathcal{C}$, then $(g \circ f)^\bullet = g^\bullet \circ f^\bullet$.
It is immediate that $g^\bullet \circ f^\bullet$ is the unique left adjoint of $(g \circ f)^* = f^* \circ g^*$, so the equality above follows by *(i)*.

By Lemma 2.1, we may consider the restrictions: $f^\bullet\mid_{\mathrm{PCon}(A)}: \mathrm{PCon}(A) \to \mathrm{PCon}(B)$ and $f^\bullet\mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$.

We recall the following definition from [6]: $\mathcal{C}$ is called a *variety with* $\vec{0}$ *and* $\vec{1}$ iff there exists an $n \in \mathbb{N}^*$ and constants $0_1, \ldots, 0_n, 1_1, \ldots, 1_n$ from τ such that, if we denote by $\vec{0} = (0_1, \ldots, 0_n)$ and $\vec{1} = (1_1, \ldots, 1_n)$, then $\mathcal{C} \vDash \vec{0} \approx \vec{1} \Rightarrow x \approx y$, that is, for any member M of $\mathcal{C}$, if $0_i^M = 1_i^M$ for all $i \in \overline{1, n}$, then M is the trivial algebra. For instance, any variety of bounded ordered structures is a variety with $\vec{0}$ and $\vec{1}$, with $n = 1$. Clearly, any variety with $\vec{0}$ and $\vec{1}$ is semi–degenerate.

Remark 4.2. If $\mathcal{C}$ is a variety with $\vec{0}$ and $\vec{1}$, with $n \in \mathbb{N}^*$ as in the definition above, then, for all $i \in \overline{1,n}$, $(0_i^B, 1_i^B) = (f(0_i^A), f(1_i^A)) \in f(\nabla_A) \subseteq f^\bullet(\nabla_A) = Cg_B(f(\nabla_A))$, hence $B/f^\bullet(\nabla_A) \vDash \vec{0} \approx \vec{1}$, thus $f^\bullet(\nabla_A) = \nabla_B$.

Remark 4.3. As shown in [43], $(f^*)^{-1}(\{\nabla_B\}) = \{\nabla_A\}$, otherwise written $f^*(\theta) \neq \nabla_A$ for all $\theta \in \mathrm{Con}(B) \setminus \{\nabla_B\}$, holds if $\mathcal{C}$ is semi–degenerate, in particular it holds if $\mathcal{C}$ is a variety with $\vec{0}$ and $\vec{1}$.

(H) Throughout the rest of this section, we shall assume that $[\cdot,\cdot]_A$ and $[\cdot,\cdot]_B$ are commutative and distributive w.r.t. arbitrary joins and that $\nabla_A \in \mathcal{K}(A)$ and $\nabla_B \in \mathcal{K}(B)$, all of which hold in the particular case when $\mathcal{C}$ is congruence–modular and semi–degenerate. We will also assume that $\mathcal{K}(A)$ and $\mathcal{K}(B)$ are closed w.r.t. the commutator.

Definition 4.4. We will say that f satisfies the *functoriality of the reticulation* (abbreviated *FRet*) iff there exists a function $\mathcal{L}(f)$ that closes the following diagram commutatively, that is iff the function $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ is well defined by: $\mathcal{L}(f)(\lambda_A(\alpha)) = \lambda_B(f^\bullet(\alpha))$ for all $\alpha \in \mathcal{K}(A)$.

$$\begin{tikzcd}
\mathcal{K}(A) \arrow[r, "f^\bullet\mid_{\mathcal{K}(A)}"] \arrow[d, "\lambda_A"'] & \mathcal{K}(B) \arrow[d, "\lambda_B"] \\
\mathcal{L}(A) \arrow[r, "\mathcal{L}(f)"] & \mathcal{L}(B)
\end{tikzcd}$$

Proposition 4.5. *There exists at most one function $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ that closes the diagram above commutatively, and such a function preserves the* $\mathbf{0}$ *and the join. Additionally:*

(i) if f is surjective or $\mathcal{C}$ is a variety with $\vec{0}$ and $\vec{1}$, then $\mathcal{L}(f)$ preserves the $\mathbf{1}$*;*

(ii) if f is surjective and $\mathcal{C}$ is congruence–modular, then $\mathcal{L}(f)$ is a bounded lattice morphism.

Proof. Let $\alpha, \beta \in \mathcal{K}(A)$. By the surjectivity of λ_A, if $\mathcal{L}(f)$ exists, then it is uniquely defined by: $\mathcal{L}(f)(\lambda_A(\theta)) = \lambda_B(f^\bullet(\theta))$ for all $\theta \in \mathcal{K}(A)$. Assume that this function is well defined. Then $\mathcal{L}(f)(\mathbf{0}) = \mathcal{L}(f)(\lambda_A(\Delta_A)) = \lambda_B(f^\bullet(\Delta_A)) = \lambda_B(Cg_B(f(\Delta_A))) = \lambda_B(\Delta_B) = \mathbf{0}$ and $\mathcal{L}(f)(\lambda_A(\alpha) \vee \lambda_A(\beta)) = \mathcal{L}(f)(\lambda_A(\alpha \vee \beta)) = \lambda_B(f^\bullet(\alpha \vee \beta)) = \lambda_B(f^\bullet(\alpha) \vee f^\bullet(\beta)) = \lambda_B(f^\bullet(\alpha)) \vee \lambda_B(f^\bullet(\beta)) = \mathcal{L}(f)(\lambda_A(\alpha)) \vee \mathcal{L}(f)(\lambda_A(\beta))$.

(i) If f is surjective or $\mathcal{C}$ is a variety with $\vec{0}$ and $\vec{1}$, then $\mathcal{L}(f)(\mathbf{1}) = \mathcal{L}(f)(\lambda_A(\nabla_A)) = \lambda_B(f^\bullet(\nabla_A)) = \lambda_B(f(\nabla_A)) = \lambda_B(\nabla_B) = \mathbf{1}$.
(ii) If f is surjective and $\mathcal{C}$ is congruence–modular, then, by Lemma 2.1, $\mathcal{L}(f)(\lambda_A(\alpha)\wedge \lambda_A(\beta)) = \mathcal{L}(f)(\lambda_A([\alpha,\beta]_A) = \lambda_B(f^\bullet([\alpha,\beta]_A)) = \lambda_B(Cg_B(f([\alpha,\beta]_A))) = \lambda_B(f([\alpha,\beta]_A \vee \mathrm{Ker}(f))) = \lambda_B([f(\alpha\vee\mathrm{Ker}(f)), f(\beta\vee\mathrm{Ker}(f))]_B) = \lambda_B(f(\alpha\vee\mathrm{Ker}(f)))\wedge \lambda_B(f(\beta\vee\mathrm{Ker}(f))) = \lambda_B(Cg_B(f(\alpha))) \wedge \lambda_B(Cg_B(f(\beta))) = \lambda_B(f^\bullet(\alpha)) \wedge \lambda_B(f^\bullet(\beta)) = \mathcal{L}(f)(\lambda_A(\alpha)) \wedge \mathcal{L}(f)(\lambda_A(\beta))$. □

Throughout the rest of this paper we keep the notation $\mathcal{L}(f)$ for the unique function associated as in Definition 4.4 to a morphism f that satisfies FRet.

Remark 4.6. Obviously, if f is an isomorphism, then f satisfies FRet and $\mathcal{L}(f)$ is a lattice isomorphism (in particular $\mathcal{L}(f)$ preserves the meet and the $\mathbf{1}$), but the converse does not hold, as shown by the case of the morphism $l : Q \to P$ in Example 6.4. Note that, in particular, $id_A^\bullet = id_{\mathrm{Con}(A)}$, thus $\mathcal{L}(id_A) = id_{\mathcal{L}(A)}$.

Remark 4.7. As shown by the morphism $v : V \to V$ in Example 6.5, f may fail FRet, while $f^\bullet$ preserves the meet and the commutator and $f^\bullet(\nabla_A) \equiv_B \nabla_B$.

Lemma 4.8.
- *If the commutator of A coincides with the intersection, then f fulfills FRet.*
- *In particular, if $\mathcal{C}$ is congruence–distributive and semi–degenerate and has the CIP, then all morphisms in $\mathcal{C}$ fulfill FRet.*
- *If the commutators of A and B coincide to the intersection, in particular if $\mathcal{C}$ is congruence–distributive, then f fulfills FRet and the following equivalences hold: $\mathcal{L}(f)$ preserves the meet iff $f^\bullet(\alpha \cap \beta) = f^\bullet(\alpha) \cap f^\bullet(\beta)$ for all $\alpha, \beta \in \mathcal{K}(A)$, $\mathcal{L}(f)$ preserves the $\mathbf{1}$ iff $f^\bullet(\nabla_A) = \nabla_B$, $\mathcal{L}(f)$ is injective or surjective iff $f^\bullet\mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ is injective or surjective, respectively.*

Proof. If the commutator of A coincides with the intersection, then $\rho_A = id_{\mathrm{Con}(A)}$, so, for all $\alpha, \beta \in \mathrm{Con}(A)$, $\lambda_A(\alpha) = \lambda_A(\beta)$ iff $\alpha = \beta$, thus, trivially, f fulfills FRet.

If, additionally, the commutator of B coincides with the intersection, then both $\lambda_A : \mathcal{K}(A) \to \mathcal{L}(A)$ and $\lambda_B : \mathcal{K}(B) \to \mathcal{L}(B)$ are lattice isomorphisms, so the equality $\mathcal{L}(f)\circ\lambda_A = \lambda_B \circ f^\bullet$ proves the equivalences in the enunciation. In fact, we may take $\mathcal{L}(A) = \mathcal{K}(A)$ and $\mathcal{L}(B) = \mathcal{K}(B)$, so that λ_A and λ_B become $id_{\mathcal{K}(A)} : \mathcal{K}(A) \to \mathcal{L}(A)$ and $id_{\mathcal{K}(B)} : \mathcal{K}(B) \to \mathcal{L}(B)$, respectively, and $\mathcal{L}(f) = f^\bullet$. □

Remark 4.9. If f fulfills FRet and $f^\bullet : \mathrm{Con}(A) \to \mathrm{Con}(B)$ preserves the intersection, then, clearly, $\mathcal{L}(f)$ preserves the meet. As shown by Example 6.5, the converse does not hold.

Proposition 4.10. *Let C be a member of $\mathcal{C}$ such that $[\cdot,\cdot]_C$ is commutative and distributive w.r.t. arbitrary joins, $\nabla_C \in \mathcal{K}(C)$ and $\mathcal{K}(C)$ is closed w.r.t. the commutator, and let $g : B \rightarrow C$ be a morphism. If f and g satisfy FRet, then $g \circ f$ satisfies FRet and $\mathcal{L}(g \circ f) = \mathcal{L}(g) \circ \mathcal{L}(f)$. Also:*

- *if, additionally, $\mathcal{L}(f)$ and $\mathcal{L}(g)$ preserve the $\mathbf{1}$, then $\mathcal{L}(g \circ f)$ preserves the $\mathbf{1}$;*
- *if, additionally, $\mathcal{L}(f)$ and $\mathcal{L}(g)$ preserve the meet, then $\mathcal{L}(g \circ f)$ preserves the meet.*

Proof. $\lambda_C \circ (g \circ f)^\bullet = \lambda_C \circ g^\bullet \circ f^\bullet = \mathcal{L}(g) \circ \lambda_B \circ f^\bullet = \mathcal{L}(g) \circ \mathcal{L}(f) \circ \lambda_A$, therefore $g \circ f$ satisfies FRet and, by the uniqueness stated in Proposition 4.5, $\mathcal{L}(g \circ f) = \mathcal{L}(g) \circ \mathcal{L}(f)$, hence the statements on the preservation of the $\mathbf{1}$ and the meet. □

By Propositions 4.5 and 4.10, if all morphisms in $\mathcal{C}$ satisfy FRet and are such that their images through the map $\mathcal{L}$ preserve the meet, so that these images are lattice morphisms, then $\mathcal{L}$ becomes a covariant functor from the category $\mathcal{C}$ to the category of distributive lattices, and, if, additionally, these images preserve the $\mathbf{1}$, then $\mathcal{L}$ is a functor from the category $\mathcal{C}$ to the category of bounded distributive lattices. In either of these cases, we call $\mathcal{L}$ the *reticulation functor* for $\mathcal{C}$.

Lemma 4.11. [26, 43] *If $\phi \in \mathrm{Con}(A) \setminus \{\nabla_A\}$, then the following are equivalent:*

(i) $\phi \in \mathrm{Spec}(A)$;

(ii) for all $\alpha, \beta \in \mathrm{PCon}(A)$, $[\alpha, \beta]_A \subseteq \phi$ implies $\alpha \subseteq \phi$ or $\beta \subseteq \phi$;

(iii) for all $\alpha, \beta \in \mathcal{K}(A)$, $[\alpha, \beta]_A \subseteq \phi$ implies $\alpha \subseteq \phi$ or $\beta \subseteq \phi$.

Lemma 4.12. *For all $\alpha, \beta \in \mathrm{Con}(A)$, $\rho_B(f^\bullet([\alpha, \beta]_A)) \subseteq \rho_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$.*

Proof. Let $\psi \in \mathrm{Spec}(B)$ such that $[f^\bullet(\alpha), f^\bullet(\beta)]_B \subseteq \psi$, so that $f^\bullet(\alpha) \subseteq \psi$ or $f^\bullet(\beta) \subseteq \psi$, so that $f^\bullet([\alpha, \beta]_A) \subseteq \psi$. Hence $V_B([f^\bullet(\alpha), f^\bullet(\beta)]_B) \subseteq V_B(f^\bullet([\alpha, \beta]_A))$, therefore $\rho_B(f^\bullet([\alpha, \beta]_A)) \subseteq \rho_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$. □

Theorem 4.13. *The following are equivalent:*

(i) f is admissible;

(ii) f satisfies FRet and $\mathcal{L}(f)$ preserves the meet (so that $\mathcal{L}(f)$ is a lattice morphism);

(iii) for all $\alpha, \beta \in \mathcal{K}(A)$, $\lambda_B(f^\bullet([\alpha, \beta]_A)) = \lambda_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$;

(iv) for all $\alpha, \beta \in \mathcal{K}(A)$, $\rho_B(f^\bullet([\alpha,\beta]_A)) = \rho_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$;

(v) for all $\alpha, \beta \in \mathcal{K}(A)$, $\rho_B(f^\bullet([\alpha,\beta]_A)) \supseteq [f^\bullet(\alpha), f^\bullet(\beta)]_B$.

Proof. (iii)$\Leftrightarrow$(iv): By the definition of $\equiv_B$.
(iv)$\Leftrightarrow$(v): By Lemma 4.12 and the fact that $\rho_B(f^\bullet([\alpha,\beta]_A)) \supseteq [f^\bullet(\alpha), f^\bullet(\beta)]_B$ iff $\rho_B(f^\bullet([\alpha,\beta]_A)) \supseteq \rho_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$.
(i)$\Rightarrow$(iii): Let $\alpha, \beta \in \mathcal{K}(A)$ and $\psi \in \mathrm{Spec}(B)$, so that $f^*(\psi) \in \mathrm{Spec}(A)$ since f is admissible, thus, since $(f^\bullet, f^*)$ is an adjoint pair: $f^\bullet([\alpha,\beta]_A) \subseteq \psi$ iff $[\alpha,\beta]_A \subseteq f^*(\psi)$ iff $\alpha \subseteq f^*(\psi)$ or $\beta \subseteq f^*(\psi)$ iff $f^\bullet(\alpha) \subseteq \psi$ or $f^\bullet(\beta) \subseteq \psi$ iff $[f^\bullet(\alpha), f^\bullet(\beta)]_B \subseteq \psi$. Therefore $V_B(f^\bullet([\alpha,\beta]_A)) = V_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$, so $\rho_B(f^\bullet([\alpha,\beta]_A)) = \rho_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$, thus $\lambda_B(f^\bullet([\alpha,\beta]_A)) = \lambda_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$.
(i),(iii)$\Rightarrow$(ii): Let $\alpha, \beta \in \mathcal{K}(A)$ such that $\lambda_A(\alpha) = \lambda_A(\beta)$, so that $\rho_A(\alpha) = \rho_A(\beta)$, thus $V_A(\alpha) = V_A(\beta)$.

Let $\psi \in \mathrm{Spec}(B)$, so that $f^*(\psi) \in \mathrm{Spec}(A)$ since f is admissible, thus, by the above and the fact that $(f^\bullet, f^*)$ is an adjoint pair: $f^\bullet(\alpha) \subseteq \psi$ iff $\alpha \subseteq f^*(\psi)$ iff $\beta \subseteq f^*(\psi)$ iff $f^\bullet(\beta) \subseteq \psi$, therefore $V_B(f^\bullet(\alpha)) = V_B(f^\bullet(\beta))$, so that $\rho_B(f^\bullet(\alpha)) = \rho_B(f^\bullet(\beta))$, thus $\mathcal{L}(f)(\lambda_A(\alpha)) = \lambda_B(f^\bullet(\alpha)) = \lambda_B(f^\bullet(\beta)) = \mathcal{L}(f)(\lambda_A(\beta))$, hence $\mathcal{L}(f)$ is well defined, that is f fulfills FRet.

Now let $\gamma, \delta \in \mathcal{K}(A)$, arbitrary. Then $\mathcal{L}(f)(\lambda_A(\gamma) \wedge \lambda_A(\delta)) = \mathcal{L}(f)(\lambda_A([\gamma,\delta]_A)) = \lambda_B(f^\bullet([\gamma,\delta]_A)) = \lambda_B([f^\bullet(\gamma), f^\bullet(\delta)]_B) = \lambda_B(f^\bullet(\gamma)) \wedge \lambda_B(f^\bullet(\delta)) = \mathcal{L}(f)(\lambda_A(\gamma)) \wedge \mathcal{L}(f)(\lambda_A(\delta))$.
(ii)$\Rightarrow$(iii): Let $\alpha, \beta \in \mathcal{K}(A)$, so that $[\alpha,\beta]_A \in \mathcal{K}(A)$ and $\lambda_B(f^\bullet([\alpha,\beta]_A)) = \mathcal{L}(f)(\lambda_A([\alpha,\beta]_A)) = \mathcal{L}(f)(\lambda_A(\alpha) \wedge \lambda_A(\beta)) = \mathcal{L}(f)(\lambda_A(\alpha)) \wedge \mathcal{L}(f)(\lambda_A(\beta)) = \lambda_B(f^\bullet(\alpha)) \wedge \lambda_B(f^\bullet(\beta)) = \lambda_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$.
(iii)$\Rightarrow$(i): Let $\alpha, \beta \in \mathcal{K}(A)$ and $\psi \in \mathrm{Spec}(B)$. Then $\lambda_B(f^\bullet([\alpha,\beta]_A)) = \lambda_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$, thus $\rho_B(f^\bullet([\alpha,\beta]_A)) = \rho_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$, so that $V_B(f^\bullet([\alpha,\beta]_A)) = V_B([f^\bullet(\alpha), f^\bullet(\beta)]_B)$, therefore, since $(f^\bullet, f^*)$ is an adjoint pair: $[\alpha,\beta]_A \subseteq f^*(\psi)$ iff $f^\bullet([\alpha,\beta]_A) \subseteq \psi$ iff $[f^\bullet(\alpha), f^\bullet(\beta)]_B \subseteq \psi$ iff $f^\bullet(\alpha) \subseteq \psi$ or $f^\bullet(\beta) \subseteq \psi$ iff $\alpha \subseteq f^*(\psi)$ or $\beta \subseteq f^*(\psi)$. By Lemma 4.11, it follows that $f^*(\psi) \in \mathrm{Spec}(A)$, hence f is admissible. □

Corollary 4.14. *If $f^\bullet([\alpha,\beta]_A) = [f^\bullet(\alpha), f^\bullet(\beta)]_B$ for all $\alpha, \beta \in \mathcal{K}(A)$, then f satisfies FRet and $\mathcal{L}(f)$ is a lattice morphism. The converse does not hold.*

Proof. By Theorem 4.13, the direct implication holds. Example 6.5 disproves the converse. □

Lemma 4.15. [26, Corollary 7.4] *If $\mathcal{C}$ is congruence–distributive and has the CIP, in particular if $\mathcal{C}$ is congruence–distributive and has the PIP, then every morphism in $\mathcal{C}$ is admissible.*

Proposition 4.16. *If $\mathcal{C}$ is congruence–distributive and has the CIP, in particular if $\mathcal{C}$ is congruence–distributive and has the PIP, then f fulfills FRet and $f^{\bullet} : \mathcal{K}(A) \to \mathcal{K}(B)$ and $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ are lattice morphisms, so that, if $\mathcal{C}$ is also semi–degenerate, then $\mathcal{L}$ is a functor from the category $\mathcal{C}$ to the category of distributive lattices.*

If, moreover, $\mathcal{C}$ is a congruence–distributive variety with $\vec{0}$ and $\vec{1}$ and the CIP, then $\mathcal{L}$ is a functor from the category $\mathcal{C}$ to the category of bounded distributive lattices.

Proof. By Lemma 4.15 and Theorem 4.13, f fulfills FRet and $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ is a lattice morphism, so that $f^{\bullet} : \mathcal{K}(A) \to \mathcal{K}(B)$ is a lattice morphism since, in this particular case, $\mathcal{K}(A)$ and $\mathcal{K}(B)$ are sublattices of $\mathrm{Con}(A)$ and $\mathrm{Con}(B)$, respectively, and $\lambda_A : \mathcal{K}(A) \to \mathcal{L}(A)$ and $\lambda_B : \mathcal{K}(B) \to \mathcal{L}(B)$ are lattice isomorphisms. ☐

Remark 4.17. If f satisfies FRet and $f^{\bullet} \mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ is surjective, then, by the surjectivity of $\lambda_B : \mathcal{K}(B) \to \mathcal{L}(B)$, it follows that $\mathcal{L}(f) \circ \lambda_A = \lambda_B \circ f^{\bullet}$ is surjective, hence $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ is surjective.

Lemma 4.18. *(i) If f is surjective, then f satisfies FRet and $\mathcal{L}(f)$ is a bounded lattice morphism.*

(ii) If f is surjective, then $f^{\bullet} : \mathrm{Con}(A) \to \mathrm{Con}(B)$, $f^{\bullet} \mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ and $f^{\bullet} \mid_{\mathrm{PCon}(A)}: \mathrm{PCon}(A) \to \mathrm{PCon}(B)$ are surjective.

(iii) If $f^{\bullet} : \mathrm{Con}(A) \to \mathrm{Con}(B)$ is surjective, then $f^{\bullet} \mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ is surjective, so, if, additionally, f satisfies FRet, then $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ is surjective.

(iv) If $\mathcal{C}$ is congruence–distributive and $f^{\bullet} : \mathrm{Con}(A) \to \mathrm{Con}(B)$ is surjective, then f satisfies FRet and $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ is surjective.

Proof. (i) By Proposition 4.5, (i), Theorem 4.13 and the fact that all surjective morphisms are admissible.

(ii) By Lemma 2.1, for all $a, b \in A$ and any $\beta \in \mathrm{Con}(B)$, we have $f^{\bullet}(Cg_A(a,b)) = Cg_B(f(a), f(b))$ and $\beta = \bigvee_{(x,y)\in\beta} Cg_B(x,y)$, which, along with the fact that $f^{\bullet}$ preserves arbitrary joins and the surjectivity of f, proves that $f^{\bullet}(\mathrm{Con}(A)) = \mathrm{Con}(B)$, $f^{\bullet}(\mathcal{K}(A)) = \mathcal{K}(B)$ and $f^{\bullet}(\mathrm{PCon}(A)) = \mathrm{PCon}(B)$.

(iii) Let $\beta \in \mathcal{K}(B)$. Since $f^{\bullet} : \mathrm{Con}(A) \to \mathrm{Con}(B)$ is surjective, it follows that there exists an $\alpha \in \mathrm{Con}(A)$ such that $\beta = f^{\bullet}(\alpha) = f^{\bullet}(\bigvee_{(a,b)\in\alpha} Cg_A(a,b)) = \bigvee_{(a,b)\in\alpha} f^{\bullet}(Cg_A(a,b))$, hence, for some $n \in \mathbb{N}^*$ and some $(a_1, b_1), \ldots, (a_n, b_n) \in \alpha$,

$\beta = \bigvee_{i=1}^{n} f^{\bullet}(Cg_A(a_i, b_i)) = f^{\bullet}(Cg_A(\{(a_1, b_1), \ldots, (a_n, b_n)\})) \in f^{\bullet}(\mathcal{K}(A))$. Therefore $f^{\bullet}\mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ is surjective.
(iv) By (iii) and Lemma 4.8. $\square$

Proposition 4.19. *$f^{\bullet}\mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ is surjective iff $f^{\bullet} : \mathrm{Con}(A) \to \mathrm{Con}(B)$ is surjective iff $f^{\bullet}\mid_{[\mathrm{Ker}(f))}: [\mathrm{Ker}(f)) \to \mathrm{Con}(B)$ is surjective.*

Proof. By Lemma 4.18, (iii), if $f^{\bullet} : \mathrm{Con}(A) \to \mathrm{Con}(B)$ is surjective, then $f^{\bullet}\mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ is surjective; the converse of this implication follows from the fact that $f^{\bullet}$ preserves arbitrary joins of congruences.

The fact that $f^{\bullet}(\alpha) = f(\alpha \vee \mathrm{Ker}(f)) = f^{\bullet}(\alpha \vee \mathrm{Ker}(f))$ for all $\alpha \in \mathrm{Con}(A)$ gives us the second equivalence. $\square$

Remark 4.20. By Lemma 4.18, (ii), if f is surjective, then, if $\mathcal{K}(A) = \mathrm{Con}(A)$ or $\mathrm{PCon}(A) = \mathrm{Con}(A)$ or $\mathrm{PCon}(A) = \mathcal{K}(A)$ or A is simple, then $\mathcal{K}(B) = \mathrm{Con}(B)$ or $\mathrm{PCon}(B) = \mathrm{Con}(B)$ or $\mathrm{PCon}(B) = \mathcal{K}(B)$ or B is simple, respectively.

Indeed, if $\mathcal{K}(A) = \mathrm{Con}(A)$, then $\mathcal{K}(B) = f(\mathcal{K}(A)) = f(\mathrm{Con}(A)) = \mathrm{Con}(B)$, and analogously for the next two statements. The fact that $f^{\bullet}(\Delta_A) = \Delta_B$ and, since f is surjective, $f^{\bullet}(\nabla_A) = \nabla_B$, gives us the last statement.

Remark 4.21. Recall that a complete lattice has all elements compact iff it satisfies the Ascending Chain Condition (ACC). Thus $\mathcal{K}(A) = \mathrm{Con}(A)$ iff $\mathrm{Cp}(\mathrm{Con}(A)) = \mathrm{Con}(A)$ iff $\mathrm{Con}(A)$ satisfies the Ascending Chain Condition, which holds, in particular, if $\mathrm{Con}(A)$ has finite height, in particular if $\mathrm{Con}(A)$ is finite, for instance if A is finite or simple.

If the commutator of A equals the intersection, in particular if $\mathcal{C}$ is congruence–distributive, then $\mathcal{K}(A) = \mathrm{Cp}(\mathrm{Con}(A))$ is a sublattice of $\mathrm{Con}(A)$ with all elements compact and $\mathcal{L}(A) \cong \mathcal{K}(A)$, thus $\mathcal{L}(A) = \mathrm{Cp}(\mathcal{L}(A))$, i.e. $\mathcal{L}(A)$ has all elements compact, that is $\mathcal{L}(A)$ satisfies the ACC, according to the above.

Proposition 4.22. *$\mathcal{L}$ preserves surjectivity; more precisely, if f is surjective, then f fulfills FRet and $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ is a surjective lattice morphism.*

Proof. By Lemma 4.18, (i), (ii) and (iii). $\square$

Remark 4.23. If the commutator of A equals the intersection, $\mathrm{Con}(A)$ is a chain and $(f^*)^{-1}(\{\nabla_B\}) = \{\nabla_A\}$, then f satisfies FRet and $\mathcal{L}(f)$ is a lattice morphism.

Indeed, this follows from Theorem 4.13 and the fact that, in this case, f is admissible, since $\mathrm{Spec}(A) = \mathrm{Con}(A) \setminus \{\nabla_A\}$. See also Lemma 4.8 and [43].

Let I be a non–empty set and, for each $i \in I$, p_i and q_i be terms over τ of arity 4. Recall that $(p_i, q_i)_{i\in I}$ is a *system of congruence intersection terms for* $\mathcal{C}$ iff, for any member M of $\mathcal{C}$ and any $a, b, c, d \in M$, $Cg_M(a, b) \cap Cg_M(c, d) = \bigvee_{i\in I} Cg_M(p_i^M(a, b, c, d), q_i^M(a, b, c, d))$ [1].

By analogy to the previous definition, let us introduce:

Definition 4.24. $(p_i, q_i)_{i\in I}$ is a *system of congruence commutator terms for* $\mathcal{C}$ iff, for any member M of $\mathcal{C}$ and any $a, b, c, d \in M$, $[Cg_M(a, b), Cg_M(c, d)]_M = \bigvee_{i\in I} Cg_M(p_i^M(a, b, c, d), q_i^M(a, b, c, d))$.

Remark 4.25. Clearly, if $\mathcal{C}$ is congruence–distributive and admits a finite system of congruence intersection terms, then, in each member M of $\mathcal{C}$, $\mathcal{K}(M)$ is closed w.r.t. the intersection.

More generally, if $\mathcal{C}$ admits a finite system of congruence commutator terms, then, in each member M of $\mathcal{C}$, $\mathcal{K}(M)$ is closed w.r.t. the commutator.

Proposition 4.26. *If $\mathcal{C}$ admits a system of congruence commutator terms, then $f^\bullet([\alpha, \beta]_A) = [f^\bullet(\alpha), f^\bullet(\beta)]_B$ for all $\alpha, \beta \in \mathrm{Con}(A)$, in particular f fulfills FRet and $\mathcal{L}(f)$ is a lattice morphism.*

Proof. Let $(p_i, q_i)_{i\in I}$ be a system of congruence commutator terms for $\mathcal{C}$.

We first prove that $f^\bullet$ preserves the commutator applied to principal congruences. Let $a, b, c, d \in A$. Then, since $f^\bullet$ preserves arbitrary joins:

$$f^\bullet([Cg_A(a, b), Cg_A(c, d)]_A) = f^\bullet(\bigvee_{i\in I} Cg_A(p_i^A(a, b, c, d), q_i^A(a, b, c, d))) =$$

$$\bigvee_{i\in I} f^\bullet(Cg_A(p_i^A(a, b, c, d), q_i^A(a, b, c, d))) = \bigvee_{i\in I} Cg_B(f(p_i^A(a, b, c, d)), f(q_i^A(a, b, c, d)))$$

$$= \bigvee_{i\in I} Cg_B(p_i^B(f(a), f(b), f(c), f(d))), q_i^B(f(a), f(b), f(c), f(d))))$$

$$= [Cg_B(f(a), f(b)), Cg_B(f(c), f(d))]_B = [f^\bullet(Cg_A(a, b)), f^\bullet(Cg_A(c, d))]_B.$$

Now let $\alpha, \beta \in \mathrm{Con}(A)$. Then $\alpha = \bigvee_{j\in J} \alpha_j$ and $\beta = \bigvee_{k\in K} \beta_k$ for some non–empty families $(\alpha_j)_{j\in J} \subseteq \mathrm{PCon}(A)$ and $(\beta_k)_{k\in K} \subseteq \mathrm{PCon}(A)$. From the above and the fact that $f^\bullet$ preserves arbitrary joins, we obtain:

$$f^\bullet([\alpha, \beta]_A) = f^\bullet([\bigvee_{j\in J} \alpha_j, \bigvee_{k\in K} \beta_k]_A) = f^\bullet(\bigvee_{j\in J} \bigvee_{k\in K} [\alpha_j, \beta_k]_A) = \bigvee_{j\in J} \bigvee_{k\in K} f^\bullet([\alpha_j, \beta_k]_A) =$$

$$\bigvee_{j\in J}\bigvee_{k\in K}[f^{\bullet}(\alpha_j),f^{\bullet}(\beta_k)]_B=[\bigvee_{j\in J}f^{\bullet}(\alpha_j),\bigvee_{k\in K}f^{\bullet}(\beta_k)]_B=[f^{\bullet}(\bigvee_{j\in J}\alpha_j),f^{\bullet}(\bigvee_{k\in K}\beta_k)]_B$$
$$=[f^{\bullet}(\alpha),f^{\bullet}(\beta)]_B.$$

Apply Theorem 4.13 for the last statement. □

In view of Remark 4.25, we obtain:

Corollary 4.27.
- *If $\mathcal{C}$ is semi–degenerate and admits a system of congruence commutator terms, then $\mathcal{L}$ is a functor from $\mathcal{C}$ to the variety of distributive lattices.*
- *If $\mathcal{C}$ is a variety with $\vec{0}$ and $\vec{1}$ that admits a system of congruence commutator terms, then $\mathcal{L}$ is a functor from $\mathcal{C}$ to the variety of bounded distributive lattices.*

Corollary 4.28.
- *If $\mathcal{C}$ is semi–degenerate and congruence–distributive and admits a system of congruence intersection terms, then $\mathcal{L}$ is a functor from $\mathcal{C}$ to the variety of distributive lattices.*
- *If $\mathcal{C}$ is a congruence–distributive variety with $\vec{0}$ and $\vec{1}$ that admits a system of congruence intersection terms, then $\mathcal{L}$ is a functor from $\mathcal{C}$ to the variety of bounded distributive lattices.*

Recall that a join–semilattice with smallest element $(L,\vee,0)$ is said to be *dually Brouwerian* iff there exists a binary operation $\dot{-}$ on L, called *dual relative pseudocomplementation*, such that, for all $a,b,c\in L$, $a\dot{-}b\leq c$ iff $a\leq b\vee c$. In particular, in a dually Brouwerian join–semilattice $(L,\vee,0)$, we have, for all $a,b\in L$: $a\dot{-}b=0$ iff $a\leq b$.

Let L and M be dually Brouwerian join–semilattices. We call $h:L\rightarrow M$ a *dually Brouwerian join–semilattice morphism* iff h preserves the 0, the join and the dual relative pseudocomplementation; if L and M are lattices and h also preserves the meet, then we call h a *dually Brouwerian lattice morphism.* Note that, if L is a lattice, then L is distributive, as one can easily derive from [32, Lemma 4.4].

Following [32], we say that $\mathcal{C}$ has *equationally definable principal congruences* (abbreviated *EDPC*) iff there exist an $n\in\mathbb{N}^*$ and terms $p_1,\ldots,p_n,q_1,\ldots,q_n$ of arity 4 over τ such that, for all members M of $\mathcal{C}$ and all $a,b\in M$, $Cg_M(a,b)=\{(c,d)\in M^2\mid(\forall\, i\in\overline{1,n})\,(p_i^M(a,b,c,d)=q_i^M(a,b,c,d))\}$.

Theorem 4.29. [12, 34]

(i) If $\mathcal{C}$ has EDPC, then $\mathcal{C}$ is congruence–distributive.

(ii) $\mathcal{C}$ *has EDPC if and only if, for any member* M *of* $\mathcal{C}$*, the semilattice* $(\mathcal{K}(M), \vee, \Delta_M)$ *is dually Browerian. In this case, if* $n \in \mathbb{N}^*$ *and* $p_1, q_1, \ldots, p_n, q_n$ *are as above, then, for any member* M *of* $\mathcal{C}$*, the operation* $\dot{-}$ *of the dually Brouwerian semilattice* $\mathcal{K}(M)$ *is defined on* $\mathrm{PCon}(M)$ *by:*

$$Cg_M(c,d) \dot{-} Cg_M(a,b) = \bigvee_{i=1}^{n} Cg_M(p_i^M(a,b,c,d), q_i^M(a,b,c,d))$$

for any $a,b,c,d \in M$*.*

Lemma 4.30. *If* $\mathcal{C}$ *has EDPC, then, for all* $\alpha, \beta \in \mathrm{PCon}(A)$*,* $f^\bullet(\alpha \dot{-} \beta) = f^\bullet(\alpha) \dot{-} f^\bullet(\beta)$*.*

Proof. Let $n \in \mathbb{N}^*$ and $p_1, q_1, \ldots, p_n, q_n$ be as in Theorem 4.29, and $a,b,c,d \in A$. Then, by Theorem 4.29 and Lemma 2.1:

$$f^\bullet(Cg_A(c,d) \dot{-} Cg_A(a,b)) = f^\bullet(\bigvee_{i=1}^{n} Cg_A(p_i^A(a,b,c,d), q_i^A(a,b,c,d))) =$$

$$\bigvee_{i=1}^{n} f^\bullet(Cg_A(p_i^A(a,b,c,d), q_i^A(a,b,c,d))) = \bigvee_{i=1}^{n} Cg_B(f(p_i^A(a,b,c,d)), f(q_i^A(a,b,c,d)))$$

$$= \bigvee_{i=1}^{n} Cg_B(p_i^B(f(a), f(b), f(c), f(d)), q_i^B(f(a), f(b), f(c), f(d)))$$

$$= Cg_B(f(c), f(d)) \dot{-} Cg_B(f(a), f(b)) = f^\bullet(Cg_A(c,d)) \dot{-} f^\bullet(Cg_A(a,b)).$$

□

Remark 4.31. [5] If $\mathcal{C}$ has EDPC, then, for all $\alpha, \beta, \gamma \in \mathcal{K}(A)$:

- $(\alpha \vee \beta) \dot{-} \gamma = (\alpha \dot{-} \gamma) \vee (\beta \dot{-} \gamma)$;
- $\alpha \dot{-} (\beta \vee \gamma) = (\alpha \dot{-} \beta) \dot{-} \gamma$.

Proposition 4.32. *If* $\mathcal{C}$ *has EDPC, then, for all* $\alpha, \beta \in \mathcal{K}(A)$*,* $f^\bullet(\alpha \dot{-} \beta) = f^\bullet(\alpha) \dot{-} f^\bullet(\beta)$*.*

Proof. Let $\theta \in \mathrm{PCon}(A)$ and $\alpha \in \mathcal{K}(A)$, so that $\alpha = \bigvee_{i=1}^{r} \alpha_i$ for some $r \in \mathbb{N}^*$ and some $\alpha_1, \ldots, \alpha_r \in \mathrm{PCon}(A)$. Then, by Lemma 4.30, $f^\bullet(\alpha \dot{-} \theta) = f^\bullet((\bigvee_{i=1}^{r} \alpha_i) \dot{-} \theta) =$

$f^\bullet(\bigvee_{i=1}^{r}(\alpha_i\dot{-}\theta)) = \bigvee_{i=1}^{r} f^\bullet(\alpha_i\dot{-}\theta) = \bigvee_{i=1}^{r}(f^\bullet(\alpha_i)\dot{-}f^\bullet(\theta)) = (\bigvee_{i=1}^{r} f^\bullet(\alpha_i))\dot{-}f^\bullet(\theta) = f^\bullet(\bigvee_{i=1}^{r} \alpha_i)\dot{-}f^\bullet(\theta) = f^\bullet(\alpha)\dot{-}f^\bullet(\theta)$.

Now let $\beta \in \mathcal{K}(A)$, so that $\beta = \bigvee_{j=1}^{s} \beta_j$ for some $s \in \mathbb{N}^*$ and some $\beta_1, \ldots, \beta_s \in \mathrm{PCon}(A)$. We apply induction on $t \in \overline{1, s}$. By the above, $f^\bullet(\alpha\dot{-}\beta_1) = f^\bullet(\alpha)\dot{-}f^\bullet(\beta_1)$. Now assume that, for some $t \in \overline{1, s-1}$, $f^\bullet(\alpha\dot{-}(\bigvee_{j=1}^{t} \beta_j)) = f^\bullet(\alpha)\dot{-}f^\bullet(\bigvee_{j=1}^{t} \beta_j)$. Then, since $\alpha\dot{-}(\bigvee_{j=1}^{t} \beta_j) \in \mathcal{K}(A)$, $f^\bullet(\alpha\dot{-}(\bigvee_{j=1}^{t+1} \beta_j)) = f^\bullet((\alpha\dot{-}(\bigvee_{j=1}^{t} \beta_j))\dot{-}\beta_{t+1}) = f^\bullet(\alpha\dot{-}(\bigvee_{j=1}^{t} \beta_j))\dot{-}f^\bullet(\beta_{t+1}) = (f^\bullet(\alpha)\dot{-}f^\bullet(\bigvee_{j=1}^{t} \beta_j))\dot{-}f^\bullet(\beta_{t+1}) = f^\bullet(\alpha)\dot{-}(f^\bullet(\bigvee_{j=1}^{t} \beta_j) \vee f^\bullet(\beta_{t+1})) = f^\bullet(\alpha)\dot{-}f^\bullet(\bigvee_{j=1}^{t+1} \beta_j)$. Thus $f^\bullet(\alpha\dot{-}\beta) = f^\bullet(\alpha\dot{-}(\bigvee_{j=1}^{s} \beta_j)) = f^\bullet(\alpha)\dot{-}f^\bullet(\bigvee_{j=1}^{s} \beta_j) = f^\bullet(\alpha)\dot{-}f^\bullet(\beta)$. □

Corollary 4.33. *If $\mathcal{C}$ has EDPC, then $\mathcal{L}(f) = f^\bullet : \mathcal{L}(A) = \mathcal{K}(A) \to \mathcal{L}(B) = \mathcal{K}(B)$ is a dually Brouwerian join–semilattice morphism.*

Proof. By Remarks 4.1, Proposition 4.32 and Theorem 4.29, (i). □

Remark 4.34. If $\mathcal{C}$ is a discriminator variety, then, by [32, Theorem 5.5], $\mathrm{PCon}(A) = \mathcal{K}(A) \cong \mathcal{L}(A)$ is a relatively complemented sublattice of $\mathrm{Con}(A)$; we set $\mathcal{K}(A) = \mathcal{L}(A)$, and the same for B. From [32, Lemma 5.3] it follows that $\mathcal{L}(f) = f^\bullet \mid_{\mathrm{PCon}(A)}$: $\mathrm{PCon}(A) \to \mathrm{PCon}(B)$ is a relatively complemented lattice morphism.

Remark 4.35. $\mathcal{L}$ reflects neither injectivity, nor surjectivity, as shown by the case of the morphism $l : Q \to P$ from Example 6.4. $\mathcal{L}$ does not preserve injectivity and does not reflect surjectivity even for congruence–distributive varieties, as shown by the case of the morphism $i_{\mathcal{L}_2^2,\mathcal{M}_3} : \mathcal{L}_2^2 \to \mathcal{M}_3$ from Example 6.3.

If the commutators of A and B coincide to the intersection, $\mathcal{K}(A) = \mathrm{Con}(A)$ and f is surjective, then $f^\bullet : \mathrm{Con}(A) \to \mathrm{Con}(B)$ is surjective, thus $\mathcal{K}(B) = \mathrm{Con}(B)$ and $f^\bullet : \mathcal{K}(A) \to \mathcal{K}(B)$ is surjective, hence $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ is surjective. In particular, in congruence–distributive varieties, the functor $\mathcal{L}$ preserves the surjectivity of morphisms defined on finite algebras.

Remark 4.36. If f is injective, then, for all $\theta \in \mathrm{Con}(A)$, we have: $f^{\bullet}(\theta) = \Delta_B$ iff $\theta = \Delta_A$. Indeed, $f(\Delta_A) \subseteq \Delta_B$, so $f^{\bullet}(\Delta_A) = \Delta_B$, while, since $f(\theta) \subseteq f^{\bullet}(\theta)$, $f^{\bullet}(\theta) = \Delta_B$ implies $f(\theta) \subseteq \Delta_B$, which implies $\theta = \Delta_A$ if f is injective.

Proposition 4.37. *If $\mathcal{C}$ is semi–degenerate and has EDPC and the CIP, then $\mathcal{L}$ is a functor from the category $\mathcal{C}$ to the category of distributive lattices which preserves injectivity.*

Proof. Assume that $\mathcal{C}$ has EDPC and the CIP, so that every morphism in $\mathcal{C}$ satisfies FRet and $\mathcal{L}$ is a functor from $\mathcal{C}$ to the variety of distributive lattices by Theorem 4.29, (i), and Proposition 4.16, and also assume that f is injective. Let $\alpha, \beta \in \mathcal{K}(A)$. Then, by Theorem 4.29, (ii), Proposition 4.32 and the injectivity of f: $f^{\bullet}(\alpha) \subseteq f^{\bullet}(\beta)$ iff $f^{\bullet}(\alpha) \dot{-} f^{\bullet}(\beta) = \Delta_B$ iff $f^{\bullet}(\alpha \dot{-} \beta) = \Delta_B$ iff $\alpha \dot{-} \beta = \Delta_A$ iff $\alpha \subseteq \beta$. Hence: $f^{\bullet}(\alpha) = f^{\bullet}(\beta)$ iff $\alpha = \beta$, therefore $f^{\bullet}$ is injective, thus so is $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$, since $\mathcal{C}$ is congruence–distributive. $\square$

Remark 4.38. Assume that f is injective and the canonical embedding of $f(A)$ into B satisfies the Congruence Extension Property. Then, for $\alpha \in \mathrm{Con}(A)$, $f^{\bullet}(\alpha) \cap f(A)^2 = f(\alpha)$, hence the map $f^{\bullet} : \mathrm{Con}(A) \to \mathrm{Con}(B)$ is injective, thus so are its restrictions $f^{\bullet} \mid_{\mathcal{K}(A)} : \mathcal{K}(A) \to \mathcal{K}(B)$ and $f^{\bullet} \mid_{\mathrm{PCon}(A)} : \mathrm{PCon}(A) \to \mathrm{PCon}(B)$.

Thus, if, additionally, the commutators of A and B coincide to the intersection, so that $\mathcal{K}(A)$ and $\mathcal{K}(B)$ are sublattices of $\mathrm{Con}(A)$ and $\mathrm{Con}(B)$, respectively, $\lambda_A : \mathcal{K}(A) \to \mathcal{L}(A)$ and $\lambda_B : \mathcal{K}(B) \to \mathcal{L}(B)$ are lattice isomorphisms and, as noted in Lemma 4.8, f satisfies FRet, it follows that $\mathcal{L}(f)$ is injective.

Therefore, in view of Proposition 4.16, we have:

Proposition 4.39. *If $\mathcal{C}$ is semi–degenerate, congruence–distributive and congruence –extensible and it has the CIP, then $\mathcal{L}$ is a functor from the category $\mathcal{C}$ to the category of distributive lattices which preserves injectivity.*

In what follows we apply the functoriality of the reticulation to the study of properties Going Up, Going Down and Lying Over in algebras whose semilattices of compact congruences and commutators are as above.

Definitions 4.40. We say that f fulfills property *Going Up* (abbreviated *GU*) if and only if, for any $\phi, \psi \in \mathrm{Spec}(A)$ and any $\phi_1 \in \mathrm{Spec}(B)$ such that $\phi \subseteq \psi$ and $f^*(\phi_1) = \phi$, there exists $\psi_1 \in \mathrm{Spec}(B)$ such that $\phi_1 \subseteq \psi_1$ and $f^*(\psi_1) = \psi$.

We say that f fulfills property *Going Down* (abbreviated *GD*) if and only if, for any $\phi, \psi \in \mathrm{Spec}(A)$ and any $\phi_1 \in \mathrm{Spec}(B)$ such that $\phi \supseteq \psi$ and $f^*(\phi_1) = \phi$, there exists $\psi_1 \in \mathrm{Spec}(B)$ such that $\phi_1 \supseteq \psi_1$ and $f^*(\psi_1) = \psi$.

We say that f fulfills property *Lying Over* (abbreviated *LO*) if and only if, for any $\phi \in \mathrm{Spec}(A)$ such that $\mathrm{Ker}(f) \subseteq \phi$, there exists $\phi_1 \in \mathrm{Spec}(B)$ such that $f^*(\phi_1) = \phi$.

Definitions 4.41. Let L, M be bounded lattices and $h : L \to M$ be a bounded lattice morphism.

We say that h fulfills property *Id–Going Up* (abbreviated *Id–GU*) if and only if, for any $P, Q \in \mathrm{Spec}_{\mathrm{Id}}(L)$ and any $P_1 \in \mathrm{Spec}_{\mathrm{Id}}(M)$ such that $P \subseteq Q$ and $h^{-1}(P_1) = P$, there exists $Q_1 \in \mathrm{Spec}_{\mathrm{Id}}(M)$ such that $P_1 \subseteq Q_1$ and $h^{-1}(Q_1) = Q$.

We say that h fulfills property *Id–Going Down* (abbreviated *Id–GD*) if and only if, for any $P, Q \in \mathrm{Spec}_{\mathrm{Id}}(L)$ and any $P_1 \in \mathrm{Spec}_{\mathrm{Id}}(M)$ such that $P \supseteq Q$ and $h^{-1}(P_1) = P$, there exists $Q_1 \in \mathrm{Spec}_{\mathrm{Id}}(M)$ such that $P_1 \supseteq Q_1$ and $h^{-1}(Q_1) = Q$.

We say that h fulfills property *Id–Lying Over* (abbreviated *Id–LO*) if and only if, for any $P \in \mathrm{Spec}_{\mathrm{Id}}(L)$ such that $h^{-1}(\{0\}) \subseteq P$, there exists $P_1 \in \mathrm{Spec}_{\mathrm{Id}}(M)$ such that $h^{-1}(P_1) = P$.

Remark 4.42. If L and M are bounded distributive lattices and $h : L \to M$ is a bounded lattice morphism, then $h^{-1}(\mathrm{Spec}_{\mathrm{Id}}(M)) \subseteq \mathrm{Spec}_{\mathrm{Id}}(L)$.

For the sake of completeness, we include here a proof for the next lemma:

Lemma 4.43. [27] *For any $\alpha \in \mathcal{K}(A)$ and any $\phi \in \mathrm{Spec}(A)$, we have: $\lambda_A(\alpha) \in \phi^*$ iff $\alpha \subseteq \phi$.*

Proof. If $\alpha \subseteq \phi$, then $\alpha \in \mathcal{K}(A) \cap (\phi]$, hence $\lambda_A(\alpha) \in \lambda_A(\mathcal{K}(A) \cap (\phi]) = \phi^*$.

If $\lambda_A(\alpha) \in \phi^* = \lambda_A(\mathcal{K}(A) \cap (\phi])$, then, for some $\beta \in \mathcal{K}(A)$ such that $\beta \subseteq \phi$, we have $\lambda_A(\alpha) = \lambda_A(\beta)$, that is $\rho_A(\alpha) = \rho_A(\beta)$, so that $\phi \in V_A(\beta) = V_A(\alpha)$, thus $\alpha \subseteq \phi$. $\square$

Lemma 4.44. *For any $\phi \in \mathrm{Spec}(A)$, we have: $\mathrm{Ker}(f) \subseteq \phi$ iff $\mathcal{L}(f)^{-1}(\{\mathbf{0}\}) \subseteq \phi^*$.*

Proof. Note that $\mathcal{L}(f)^{-1}(\{\mathbf{0}\}) = \mathcal{L}(f)^{-1}(\{\lambda_B(\Delta_B)\}) = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \mathcal{L}(f)(\lambda_A(\alpha)) = \lambda_B(\Delta_B)\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \lambda_B(f^\bullet(\alpha)) = \lambda_B(\Delta_B)\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \lambda_B(f^\bullet(\alpha)) = \lambda_B(\Delta_B)\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \rho_B(f^\bullet(\alpha)) = \rho_B(\Delta_B)\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), f^\bullet(\alpha) \subseteq \rho_B(\Delta_B)\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \alpha \subseteq f^*(\rho_B(\Delta_B))\} = \lambda_A(\mathcal{K}(A) \cap (f^*(\rho_B(\Delta_B))])$.

Now let $\phi \in \mathrm{Spec}(A)$, and recall that $\phi^* = \lambda_A(\mathcal{K}(A) \cap (\phi])$. Notice that, for any $\alpha \in \mathcal{K}(A)$, $\lambda_A(\alpha) \in \lambda_A(\mathcal{K}(A) \cap (\phi])$ implies that, for some $\beta \in \mathcal{K}(A) \cap (\phi]$, we have $\lambda_A(\alpha) = \lambda_A(\beta)$, so that $\alpha \subseteq \rho_A(\alpha) = \rho_A(\beta) \subseteq \rho_A(\phi) = \phi$, thus $\alpha \subseteq \phi$; hence: $\lambda_A(\alpha) \in \lambda_A(\mathcal{K}(A) \cap (\phi])$ iff $\alpha \in \mathcal{K}(A) \cap (\phi]$.

Therefore: $\mathcal{L}(f)^{-1}(\{\mathbf{0}\}) \subseteq \phi^*$ iff $\lambda_A(\mathcal{K}(A) \cap (f^*(\rho_B(\Delta_B))]) \subseteq \lambda_A(\mathcal{K}(A) \cap (\phi])$ iff $\mathcal{K}(A) \cap (f^*(\rho_B(\Delta_B))] \subseteq \mathcal{K}(A) \cap (\phi]$ iff $\mathcal{K}(A) \cap (f^*(\rho_B(\Delta_B))] \subseteq (\phi]$ iff every $\alpha \in \mathcal{K}(A)$ such that $\alpha \subseteq f^*(\rho_B(\Delta_B))$ satisfies $\alpha \subseteq \phi$ iff $\bigvee(\mathcal{K}(A) \cap (f^*(\rho_B(\Delta_B))]) \subseteq \phi$, that is $f^*(\rho_B(\Delta_B)) \subseteq \phi$.

Since $f^*(\Delta_B) \subseteq f^*(\rho_B(\Delta_B))$, by the above $\mathcal{L}(f)^{-1}(\{\mathbf{0}\}) \subseteq \phi^*$ implies $f^*(\Delta_B) \subseteq \phi$, that is $\mathrm{Ker}(f) \subseteq \phi$.

On the other hand, again since $\rho_A(\phi) = \phi$, we have: $f^*(\Delta_B) = \mathrm{Ker}(f) \subseteq \phi$ iff $\rho_A(f^*(\Delta_B)) \subseteq \phi$, that is $\bigcap(\mathrm{Spec}(A) \cap [f^*(\Delta_B))) \subseteq \phi$, which, since $f^*(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(A) \cap [f^*(\Delta_B))$, implies that $f^*(\rho_B(\Delta_B)) = f^*(\bigcap \mathrm{Spec}(B)) = \bigcap f^*(\mathrm{Spec}(B)) \subseteq \bigcap(\mathrm{Spec}(A) \cap [f^*(\Delta_B))) \subseteq \phi$, so that $\mathcal{L}(f)^{-1}(\{\mathbf{0}\}) \subseteq \phi^*$ by the above. $\square$

Proposition 4.45. *If f is admissible, then: f satisfies property GU, GD, LO iff $\mathcal{L}(f)$ satisfies Id–GU, Id–GD, Id–LO, respectively.*

Proof. By Proposition 3.3, the maps $u_A : \mathrm{Spec}(A) \to \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$ and $u_B : \mathrm{Spec}(B) \to \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(B))$ defined by $u_A(\phi) = \phi^*$ and $u_B(\psi) = \psi^*$ for any $\phi \in \mathrm{Spec}(A)$ and any $\psi \in \mathrm{Spec}(B)$ are order isomorphisms.

The following diagram is commutative:

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r, "u_A"] & \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A)) \\ \mathrm{Spec}(B) \arrow[u, "f^*"] \arrow[r, "u_B"] & \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(B)) \arrow[u, "\mathcal{L}(f)^*"'] \end{tikzcd}$$

Indeed, by Lemma 4.43 and the fact that $f^\bullet(\mathcal{K}(A)) \subseteq \mathcal{K}(B)$, for any $\psi \in \mathrm{Spec}(B)$, we have: $\mathcal{L}(f)^*(u_B(\psi)) = \mathcal{L}(f)^*(\psi^*) = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \mathcal{L}(f)(\lambda_A(\alpha)) \in \psi^*\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \lambda_B(f^\bullet(\alpha)) \in \psi^*\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), f^\bullet(\alpha) \subseteq \psi\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \alpha \subseteq f^*(\psi)\} = \lambda_A(\mathcal{K}(A) \cap (f^*(\psi)]) = f^*(\psi)^* = u_A(f^*(\psi))$.

Hence the statements in the enunciation on GU and GD versus Id–GU and Id–GD, respectively. By Lemma 4.44, we have, for every $\phi \in \mathrm{Spec}(A)$: $\mathcal{L}(f)^{-1}(\{\mathbf{0}\}) \subseteq u_A(\phi)$ iff $\mathrm{Ker}(f) \subseteq \phi$, which, along with the commutativity of the diagram above, yields the statement on LO versus Id–LO in the enunciation. $\square$

Proposition 4.46. *Any dually Brouwerian lattice morphism satisfies Id–GU.*

Proof. Let L and M be lattices with smallest element such that $(L, \vee, 0)$ and $(M, \vee, 0)$ are dually Brouwerian join–semilattices, and $h : L \to M$ be a dually Brouwerian lattice morphism.

Let $P, Q \in \mathrm{Spec}_{\mathrm{Id}}(L)$ and $P_1 \in \mathrm{Spec}_{\mathrm{Id}}(M)$ such that $P \subseteq Q$ and $h^{-1}(P_1) = P$.

Let us denote by $S = L \setminus P$ and $T = L \setminus Q$, so that $T \subseteq S$, so that $h^{-1}(P_1) \cap T = P \cap T = \emptyset$ and thus $P_1 \cap h(T) = \emptyset$. By Zorn's Lemma, it follows that there exists an ideal Q_1 of M such that $Q_1 \cap h(T) = \emptyset$ and Q_1 is maximal w.r.t. this property, so that

$P_1 \subseteq Q_1$. Since $Q \in \mathrm{Spec}_{\mathrm{Id}}(L)$, it follows that T is closed w.r.t. the meet, thus $h(T)$ is closed w.r.t. the meet, from which it immediately follows that $Q_1 \in \mathrm{Spec}_{\mathrm{Id}}(M)$.

$h^{-1}(Q_1) \cap T \subseteq h^{-1}(Q_1) \cap h^{-1}(h(T)) = h^{-1}(Q_1 \cap h(T)) = \emptyset$, thus $h^{-1}(Q_1) \setminus Q = h^{-1}(Q_1) \cap (L \setminus Q) = \emptyset$, therefore $h^{-1}(Q_1) \subseteq Q$.

Now let $x \in Q$ and assume by absurdum that $x \notin h^{-1}(Q_1)$, that is $h(x) \notin Q_1$, so that $Q_1 \subsetneq Q_1 \vee (h(x)]$ and thus $(Q_1 \vee (h(x)]) \cap h(T) \neq \emptyset$ by the choice of Q_1, so that, for some $t \in T$ and some $a \in Q_1$, $h(t) \leq h(x) \vee a$, thus $h(t \dot{-} x) = h(t) \dot{-} h(x) \leq a$, hence $h(t \dot{-} x) \in Q_1$, thus $t \dot{-} x \in h^{-1}(Q_1) \subseteq Q$, so that, since $t \dot{-} x \leq t \dot{-} x$, we have $t \leq (t \dot{-} x) \vee x \in Q$, thus $t \in Q = L \setminus T$, and we have a contradiction. Hence $Q \subseteq h^{-1}(Q_1)$, therefore $h^{-1}(Q_1) = Q$. □

The proof of the proposition above follows the lines of analogous results for MV algebras and BL algebras from [11] and [47], respectively. The two previous propositions yield the following result from [26] as a corollary:

Corollary 4.47. *If $\mathcal{C}$ has EDPC and f is admissible, then f satisfies GU.*

According to a result from [26], GU implies LO in semi–degenerate varieties, and, moreover, if a morphism satisfies GU and the one–class congruence of its codomain is compact, then that morphism also satisfies LO. Therefore, since we are under the assumption that $\nabla_B \in \mathcal{K}(B)$, we also get:

Corollary 4.48. *If $\mathcal{C}$ has EDPC and f is admissible, then f satisfies LO.*

5 Functoriality of the Boolean Center

(H) Throughout this section, B will be a member of $\mathcal{C}$, $f : A \to B$ will be a morphism and we will assume that: $\nabla_A \in \mathcal{K}(A)$, $\nabla_B \in \mathcal{K}(B)$, the commutators of A and B are commutative and distributive w.r.t. arbitrary joins, all of which hold in the particular case when $\mathcal{C}$ is congruence–modular and semi–degenerate. We will also assume that $\mathcal{K}(A)$ and $\mathcal{K}(B)$ are closed w.r.t. the commutators of A and B, respectively.

If $\mathcal{B}(\mathrm{Con}(A))$ and $\mathcal{B}(\mathrm{Con}(B))$ are Boolean sublattices of $\mathrm{Con}(A)$ and $\mathrm{Con}(B)$, respectively, then we say that f satisfies the *functoriality of the Boolean center* (abbreviated *FBC*) iff:

(FBC1) $f^{\bullet}(\mathcal{B}(\mathrm{Con}(A))) \subseteq \mathcal{B}(\mathrm{Con}(B))$;
(FBC2) $f^{\bullet} \mid_{\mathcal{B}(\mathrm{Con}(A))} : \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathrm{Con}(B))$ is a Boolean morphism.

(H) Throughout the rest of this section, we will also assume that $[\alpha, \nabla_A]_A = \alpha$ for all $\alpha \in \mathrm{Con}(A)$ and $[\beta, \nabla_B]_B = \beta$ for all $\beta \in \mathrm{Con}(B)$, which also hold in the particular case when $\mathcal{C}$ is congruence–modular and semi–degenerate.

Under the conditions above, by [27, Lemma 24], $\mathcal{B}(\mathrm{Con}(A))$ is a Boolean sublattice of $\mathrm{Con}(A)$, on which the commutator coincides with the intersection; moreover, by [27, Lemma 18, (iv)], for all $\sigma \in \mathcal{B}(\mathrm{Con}(A))$ and all $\theta \in \mathrm{Con}(A)$, we have $[\sigma, \theta]_A = \sigma \cap \theta$; also, for all $\alpha, \beta \in \mathrm{Con}(A)$ such that $\alpha \vee \beta = \nabla_A$, we have $[\alpha, \beta]_A = \alpha \cap \beta$. By [27, Proposition 19, (iv)], $\mathcal{B}(\mathrm{Con}(A)) \subseteq \mathcal{K}(A)$, so that $\lambda_A(\mathcal{B}(\mathrm{Con}(A))) \subseteq \mathcal{B}(\mathcal{L}(A))$ and $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathcal{L}(A))$ is a Boolean morphism.

Lemma 5.1. [27, Theorem 5, (i)] *If $\mathcal{C}$ is congruence–modular and semi–degenerate, then the Boolean morphism $\lambda_A \mid_{\mathcal{B}(\mathcal{L}(A))}: \mathcal{B}(\mathcal{L}(A)) \to \mathcal{B}(\mathcal{L}(B))$ is injective. If, furthermore, A is semiprime or its commutator is associative, then this restriction of λ_A is a Boolean isomorphism.*

Lemma 5.2. [27, Lemma 25] *If $\mathcal{C}$ is congruence–modular and semi–degenerate and A is semiprime, then, for all $\alpha \in \mathrm{Con}(A)$: $\lambda_A(\alpha) \in \mathcal{B}(\mathcal{L}(A))$ iff $\alpha \in \mathcal{B}(\mathrm{Con}(A))$.*

Remark 5.3. Since $\mathcal{B}(\mathrm{Con}(A)) \subseteq \mathcal{K}(A) \subseteq \mathrm{Con}(A)$, it follows that, if $\mathrm{Con}(A)$ is a Boolean lattice, in particular if A is simple, then $\mathcal{B}(\mathrm{Con}(A)) = \mathcal{K}(A) = \mathrm{Con}(A)$.

Since the same holds for B, we may notice that: f satisfies (FBC1) if $\mathcal{B}(\mathrm{Con}(B)) = \mathcal{K}(B)$, in particular if $\mathrm{Con}(B)$ is a Boolean lattice, in particular if B is simple.

Remark 5.4. If f satisfies (FBC1), $f^\bullet \mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ preserves the commutator and $f^\bullet(\nabla_A) = \nabla_B$, the latter holding if f is surjective or $\mathcal{C}$ is a variety with $\vec{0}$ and $\vec{1}$, then, since the commutators of A and B coincide to the intersection on $\mathcal{B}(\mathrm{Con}(A))$ and $\mathcal{B}(\mathrm{Con}(B))$, respectively, it follows that f satisfies FBC.

In particular, f satisfies FBC if $f^\bullet : \mathrm{Con}(A) \to \mathrm{Con}(B)$ is a bounded lattice morphism, that is if:

- $f^\bullet(\nabla_A) = \nabla_B$, in particular if f is surjective or $\mathcal{C}$ is a variety with $\vec{0}$ and $\vec{1}$,
  and:
- $f^\bullet$ preserves the intersection, in particular if f is surjective and the commutators of A and B coincide to the intersection, in particular if f is surjective and $\mathcal{C}$ is congruence–distributive.

Remark 5.5. If f fulfills FRet and $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ is a bounded lattice morphism, then f fulfills FBC and the image of $\mathcal{L}(f)$ through the functor $\mathcal{B}$ is $\mathcal{B}(\mathcal{L}(f)) = \mathcal{L}(f) \mid_{\mathcal{B}(\mathcal{L}(A))}: \mathcal{B}(\mathcal{L}(A)) \to \mathcal{B}(\mathcal{L}(B))$.

If all morphisms in $\mathcal{C}$ fulfill FRet and $\mathcal{L}$ is a functor from $\mathcal{C}$ to the variety of bounded distributive lattices, then $\mathcal{B}\circ\mathcal{L}$ is a functor from $\mathcal{C}$ to the variety of Boolean algebras.

Thus, in view of Proposition 4.16:

Corollary 5.6. *If $\mathcal{C}$ is a congruence–distributive variety with $\vec{0}$ and $\vec{1}$ and the CIP, then every morphism in $\mathcal{C}$ fulfills FBC.*

Remark 5.7. $\mathcal{B}\circ\mathcal{L}$ does not preserve surjectivity, as shown by the example of the surjective morphism $h:\mathcal{N}_5\rightarrow\mathcal{L}_2^2$ from Example 6.3. Note, also, that the bounded lattice morphism $\mathcal{L}(h)$ is surjective, but the Boolean morphism $\mathcal{B}(\mathcal{L}(h))$ is not surjective.

On the other hand, notice the bounded lattice embedding $i_{\mathcal{L}_2,\mathcal{N}_5}$ from Example 6.3, in whose case the Boolean morphism $\mathcal{B}(\mathcal{L}(i_{\mathcal{L}_2,\mathcal{N}_5}))$ is surjective, while the bounded lattice morphism $\mathcal{L}(i_{\mathcal{L}_2,\mathcal{N}_5})$ is not surjective.

Proposition 5.8. *If:*

- *$\mathcal{C}$ is congruence–modular and semi–degenerate,*
- *f fulfills FRet and $\mathcal{L}(f)$ preserves the* $\mathbf{1}$*,*
- *$\mathcal{L}(f)\mid_{\mathcal{B}(\mathcal{L}(A))}$ preserves the meet, in particular if $\mathcal{L}(f)$ preserves the meet,*
- *and B is semiprime,*

then f fulfills FBC.

Proof. Since $f^\bullet$ preserves the join and thus so does $\mathcal{L}(f)$, it follows that $\mathcal{L}(f)\mid_{\mathcal{B}(\mathcal{L}(A))}\colon \mathcal{B}(\mathcal{L}(A))\rightarrow\mathcal{L}(B)$ is a bounded lattice morphism, hence $\mathcal{L}(f)(\mathcal{B}(\mathcal{L}(A)))\subseteq\mathcal{B}(\mathcal{L}(B))$ and so $\mathcal{L}(f)\mid_{\mathcal{B}(\mathcal{L}(A))}\colon \mathcal{B}(\mathcal{L}(A))\rightarrow\mathcal{B}(\mathcal{L}(B))$ is a bounded lattice morphism, thus a Boolean morphism.

Let $\alpha\in\mathcal{B}(\mathrm{Con}(A))$. Then $\lambda_A(\alpha)\in\mathcal{B}(\mathcal{L}(A))$, thus, by the above, $\lambda_B(f^\bullet(\alpha))=\mathcal{L}(f)(\lambda_A(\alpha))\in\mathcal{B}(\mathcal{L}(B))$, so that $f^\bullet(\alpha)\in\mathcal{B}(\mathrm{Con}(B))$ by Lemma 5.2. Hence $f^\bullet(\mathcal{B}(\mathrm{Con}(A)))\subseteq\mathcal{B}(\mathrm{Con}(B))$.

Trivially, $f^\bullet(\Delta_A)=\Delta_B$. We have $\lambda_B(f^\bullet(\nabla_A))=\mathcal{L}(f)(\lambda_A(\nabla_A))=\mathcal{L}(f)(\mathbf{1})=\mathbf{1}=\lambda_B(\nabla_B)$, thus $f^\bullet(\nabla_A)=\nabla_B$ by Lemma 5.1. Let $\alpha,\beta\in\mathcal{B}(\mathrm{Con}(A))\subseteq\mathcal{K}(A)$. Then $\lambda_B(f^\bullet(\alpha\cap\beta))=\mathcal{L}(f)(\lambda_A(\alpha\cap\beta))=\mathcal{L}(f)(\lambda_A(\alpha)\wedge\lambda_A(\beta))=\mathcal{L}(f)(\lambda_A(\alpha))\wedge\mathcal{L}(f)(\lambda_A(\beta))=\lambda_B(f^\bullet(\alpha))\wedge\lambda_B(f^\bullet(\beta))=\lambda_B(f^\bullet(\alpha)\cap f^\bullet(\beta))$, so that $f^\bullet(\alpha\cap\beta)=f^\bullet(\alpha)\cap f^\bullet(\beta)$ by Lemma 5.1. Therefore $f^\bullet\mid_{\mathcal{B}(\mathrm{Con}(A))}\colon \mathcal{B}(\mathrm{Con}(A))\rightarrow\mathcal{B}(\mathrm{Con}(B))$ is a Boolean morphism. $\square$

Corollary 5.9. *If:*

- $\mathcal{C}$ *is semi–degenerate,*
- $f^\bullet(\nabla_A) = \nabla_B$ *and* $f^\bullet(\alpha \cap \beta) = f^\bullet(\alpha) \cap f^\bullet(\beta)$ *for all* $\alpha, \beta \in \mathcal{B}(\mathrm{Con}(A))$*,*
- $\mathcal{C}$ *is congruence–modular and the commutators of* A *and* B *coincide to the intersection, in particular if* $\mathcal{C}$ *is congruence–distributive,*

then f *fulfills FBC.*

Proposition 5.10.

- *FRet does not imply FBC, not even in congruence–distributive varieties.*
- *FBC does not imply FRet.*

Proof. The lattice morphism g in Example 6.3 fulfills the FRet, but fails the FBC.
The morphism h in Example 6.5 satisfies FBC, but fails the FRet. ☐

Remark 5.11. If f fulfills FBC and $f^\bullet(\nabla_A) = \nabla_B$, in particular if f fulfills FBC and FRet, then $\mathcal{L}(f)$ preserves the **1**, but, as shown by the case of the bounded lattice morphism k in Example 6.3, $\mathcal{L}(f)$ does not necessarily preserve the meet.

Remark 5.12. If the commutators of A and B coincide to the intersection and the lattices $\mathrm{Con}(A)$ and $\mathrm{Con}(B)$ are Boolean, then the following are equivalent:

- f fulfills FBC;
- f fulfills FRet and $\mathcal{L}(f)$ preserves the meet and the **1**.

Remark 5.13. If f fulfills FRet and FBC, then $\mathcal{L}(f)\mid_{\mathcal{B}(\mathcal{L}(A))}: \mathcal{B}(\mathcal{L}(A)) \to \mathcal{B}(\mathcal{L}(B))$ is a Boolean morphism that makes the following diagram commutative:

$$\begin{tikzcd}
\mathcal{B}(\mathrm{Con}(A)) \arrow[r, "f^\bullet \mid_{\mathcal{B}(\mathrm{Con}(A))}"] \arrow[d, "\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}"'] & \mathcal{B}(\mathrm{Con}(B)) \arrow[d, "\lambda_B \mid_{\mathcal{B}(\mathrm{Con}(B))}"] \\
\mathcal{B}(\mathcal{L}(A)) \arrow[r, "\mathcal{L}(f) \mid_{\mathcal{B}(\mathcal{L}(A))}"] & \mathcal{B}(\mathcal{L}(B))
\end{tikzcd}$$

Remark 5.14. Obviously, whenever $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$ is injective, it follows that $\mathcal{L}(f)\mid_{\mathcal{B}(\mathcal{L}(A))}: \mathcal{B}(\mathcal{L}(A)) \to \mathcal{B}(\mathcal{L}(B))$ is injective, as well.

Corollary 5.15.

- *If* $\mathcal{C}$ *has EDPC and* f *is injective, then* $\mathcal{L}(f)\mid_{\mathcal{B}(\mathcal{L}(A))}: \mathcal{B}(\mathcal{L}(A)) \to \mathcal{B}(\mathcal{L}(B))$ *is injective.*
- *If* $\mathcal{C}$ *is a variety with* $\vec{0}$ *and* $\vec{1}$*, EDPC and the CIP, then the functor* $\mathcal{B} \circ \mathcal{L}$ *preserves injectivity.*

Proof. By Remark 5.14 and Propositions 4.37 and 4.16. □

Proposition 5.16. *If $f^\bullet(\nabla_A) = \nabla_B$ and $f^\bullet\mid_{\mathcal{B}(\mathrm{Con}(A))}$ preserves the intersection, in particular if $f^\bullet$ preserves the commutator, then f fulfills the FBC.*

Proof. Let $\alpha \in \mathcal{B}(\mathrm{Con}(A))$, so that, for some $\beta \in \mathcal{B}(\mathrm{Con}(A))$, $\alpha \vee \beta = \nabla_A$ and $[\alpha, \beta]_A = \alpha \cap \beta = \Delta_A$. Then $f^\bullet(\alpha) \vee f^\bullet(\beta) = f^\bullet(\alpha \vee \beta) = f^\bullet(\nabla_A) = \nabla_B$ and thus $f^\bullet(\alpha) \cap f^\bullet(\beta) = [f^\bullet(\alpha), f^\bullet(\beta)]_B = f^\bullet([\alpha, \beta]_A) = f^\bullet(\Delta_A) = \Delta_B$, hence $f^\bullet(\alpha) \in \mathcal{B}(\mathrm{Con}(B))$, so f fulfills (FBC1). Also, $f^\bullet(\Delta_A) = \Delta_B$, $f^\bullet(\nabla_A) = \nabla_B$ and $f^\bullet$ preserves the join and the commutator, that is the intersection on $\mathcal{B}(\mathrm{Con}(A))$. □

Corollary 5.17. *If $\mathcal{C}$ is congruence–modular and f is surjective, then f fulfills the FBC.*

Definition 5.18. We say that a congruence θ of A fulfills the *Congruence Boolean Lifting Property* (abbreviated *CBLP*) iff the map $p_\theta^\bullet\mid_{\mathcal{B}(\mathrm{Con}(A))} = p_\theta\mid_{\mathcal{B}(\mathrm{Con}(A))}$: $\mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathrm{Con}(A/\theta))$ is surjective. We say that A fulfills the *Congruence Boolean Lifting Property* (*CBLP*) iff all congruences of A satisfy the CBLP.

For instance, if $\theta \in \mathrm{Con}(A)$ such that A/θ is simple, so that $\mathcal{B}(\mathrm{Con}(A/\theta)) = \mathrm{Con}(A/\theta) \cong \mathcal{L}_2$, then θ satisfies the CBLP, so, in particular, any maximal congruence of A has the CBLP.

(H) Throughout the rest of this section $\mathcal{C}$ will be congruence–modular.

Remark 5.19. Let $\theta \in \mathrm{Con}(A)$. Then, by Lemma 2.1, $p_\theta^\bullet : \mathrm{Con}(A) \to \mathrm{Con}(A/\theta)$ is defined by $p_\theta^\bullet(\alpha) = (\alpha \vee \theta)/\theta$ for all $\alpha \in \mathrm{Con}(A)$, and, by Corollary 5.17, the map $p_\theta^\bullet\mid_{\mathcal{B}(\mathrm{Con}(A))} = p_\theta\mid_{\mathcal{B}(\mathrm{Con}(A))}$: $\mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathrm{Con}(A/\theta))$ is well defined and it is a Boolean morphism.

Lemma 5.20. *Let $\alpha, \beta \in \mathrm{Con}(A)$ with $\beta \subseteq \alpha$.*

(i) If β and α/β have the CBLP, then α has the CBLP.

(ii) If α has the CBLP, then α/β has the CBLP.

Proof. By the Second Isomorphism Theorem, the map $\varphi_{\alpha,\beta} : A/\alpha \to (A/\beta)/(\alpha/\beta)$, defined by $\varphi_{\alpha,\beta}(a/\alpha) = (a/\beta)/(\alpha/\beta)$ for all $a \in A$, is an isomorphism in $\mathcal{C}$, so that $\varphi_{\alpha,\beta}^\bullet : \mathrm{Con}(A/\alpha) \to \mathrm{Con}((A/\beta)/(\alpha/\beta))$ is a lattice isomorphism and thus $\mathcal{B}(\varphi_{\alpha,\beta}^\bullet) : \mathcal{B}(\mathrm{Con}(A/\alpha)) \to \mathcal{B}(\mathrm{Con}((A/\beta)/(\alpha/\beta)))$ is a Boolean isomorphism. For all $\theta \in \mathrm{Con}(A)$, $\varphi_{\alpha,\beta}^\bullet(p_\alpha^\bullet(\theta)) = \varphi_{\alpha,\beta}^\bullet((\theta \vee \alpha)/\alpha) = ((\theta \vee \alpha)/\beta)/(\alpha/\beta) = ((\theta \vee \beta \vee \alpha)/\beta)/(\alpha/\beta) = ((\theta \vee \beta)/\beta \vee \alpha/\beta)/(\alpha/\beta) = p_{\alpha/\beta}^\bullet((\theta \vee \beta)/\beta) = p_{\alpha/\beta}^\bullet(p_\beta^\bullet(\theta))$, hence the diagram below on the left is commutative, thus so is the diagram below on the right, hence the implications in the enunciation:

$$\begin{tikzcd} \mathrm{Con}(A) \arrow[r, "p_\alpha^\bullet"] \arrow[d, "p_\beta^\bullet"'] & \mathrm{Con}(A/\alpha) \arrow[d, "\varphi_{\alpha,\beta}^\bullet"] \\ \mathrm{Con}(A/\beta) \arrow[r, "p_{\alpha/\beta}^\bullet"] & \mathrm{Con}((A/\alpha)/(\alpha/\beta)) \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B}(\mathrm{Con}(A)) \arrow[r, "p_\alpha^\bullet \mid_{\mathcal{B}(\mathrm{Con}(A))}"] \arrow[d, "p_\beta^\bullet \mid_{\mathcal{B}(\mathrm{Con}(A))}"'] & \mathcal{B}(\mathrm{Con}(A/\alpha)) \arrow[d, "\mathcal{B}(\varphi_{\alpha,\beta}^\bullet)"] \\ \mathcal{B}(\mathrm{Con}(A/\beta)) \arrow[r, "p_{\alpha/\beta}^\bullet \mid_{\mathcal{B}(\mathrm{Con}(A/\beta))}"] & \mathcal{B}(\mathrm{Con}((A/\alpha)/(\alpha/\beta))) \end{tikzcd}$$

□

Proposition 5.21. *A has the CBLP iff, for all $\theta \in \mathrm{Con}(A)$, A/θ has the CBLP.*

Proof. By Lemma 5.20, (ii), for the direct implication, and the fact that A is isomorphic to A/Δ_A, for the converse. □

Proposition 5.22. *Let $\theta \in \mathrm{Con}(A)$. Then: A/θ is semiprime iff $\theta \in \mathrm{RCon}(A)$.*

Proof. $\Delta_{A/\theta} = (\Delta_A \vee \theta)/\theta = \theta/\theta$ and $\rho_{A/\theta}(\Delta_{A/\theta}) = \rho_A(\Delta_A \vee \theta)/\theta = \rho_A(\theta)/\theta$. Hence A/θ is semiprime iff $\rho_{A/\theta}(\Delta_{A/\theta}) = \Delta_{A/\theta}$ iff $\rho_A(\theta)/\theta = \theta/\theta$ iff $\rho_A(\theta) = \theta$ iff $\theta \in \mathrm{RCon}(A)$. □

Corollary 5.23.

- *A/θ is semiprime for all $\theta \in \mathrm{Con}(A)$ iff $\mathrm{RCon}(A) = \mathrm{Con}(A)$.*
- *If the commutator of A equals the intersection, then A/θ is semiprime for all $\theta \in \mathrm{Con}(A)$.*

(H) Throughout the rest of this section $\mathcal{C}$ will be congruence–modular and semi–degenerate.

Recall that an ideal I of a bounded distributive lattice L is said to have the *Id–BLP* iff the Boolean morphism $\mathcal{B}(\pi_I) : \mathcal{B}(L) \to \mathcal{B}(L/I)$ is surjective [15], and L is said to have the *Id–BLP* iff all its ideals have the Id–BLP.

Recall from Section 3 that, for any $\theta \in \mathrm{Con}(A)$, we have $\theta^* \in \mathrm{Id}(\mathcal{L}(A))$.

Theorem 5.24. [27, Theorem 7] *For any $\theta \in \mathrm{Con}(A)$, the map $\varphi_\theta : \mathcal{L}(A/\theta) \to \mathcal{L}(A)/\theta^*$ defined by $\varphi_\theta(\lambda_{A/\theta}((\alpha \vee \theta)/\theta)) = \lambda_A(\alpha)/\theta^*$ for all $\alpha \in \mathcal{K}(A)$, is a lattice isomorphism.*

Lemma 5.25. *Let $\theta \in \mathrm{Con}(A)$.*

- *If $\lambda_{A/\theta} \mid_{\mathcal{B}(\mathrm{Con}(A/\theta))}: \mathcal{B}(\mathrm{Con}(A/\theta)) \to \mathcal{B}(\mathcal{L}(A/\theta))$ is surjective and θ has the CBLP, then θ^* has the Id–BLP.*

- *If* $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathcal{L}(A))$ *is surjective and* $\lambda_{A/\theta} \mid_{\mathcal{B}(\mathrm{Con}(A/\theta))}: \mathcal{B}(\mathrm{Con}(A/\theta)) \to \mathcal{B}(\mathcal{L}(A/\theta))$ *is bijective, then:* θ *has the CBLP iff* θ^* *has the Id–BLP (in* $\mathcal{L}(A)$*).*

Proof. By the definitions, θ has the CBLP iff the Boolean morphism $p_\theta^\bullet \mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathrm{Con}(A/\theta))$ is surjective, while θ^* has the Id–BLP iff the Boolean morphism $\mathcal{B}(\pi_{\theta^*}) : \mathcal{B}(\mathcal{L}(A)) \to \mathcal{B}(\mathcal{L}(A)/\theta^*)$ is surjective.

The definition of the lattice isomorphism φ_θ from Theorem 5.24 shows that the following diagram on the left is commutative, hence, by considering the restrictions of the maps in this diagram to the Boolean centers, we obtain the commutative diagram below on the right:

$$\begin{tikzcd}
\mathcal{K}(A) \arrow[rr, "p_\theta^\bullet \mid_{\mathcal{K}(A)}"] \arrow[d, "\lambda_A"'] & & \mathcal{K}(A/\theta) \arrow[d, "\lambda_{A/\theta}"] \\
\mathcal{L}(A) \arrow[rr, "\mathcal{L}(p_\theta)"] \arrow[dr, "\pi_{\theta^*}"'] & & \mathcal{L}(A/\theta) \arrow[dl, "\varphi_\theta"] \\
& \mathcal{L}(A)/\theta^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}(\mathrm{Con}(A)) \arrow[rr, "p_\theta^\bullet \mid_{\mathcal{B}(\mathrm{Con}(A))}"] \arrow[d, "\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}"'] & & \mathcal{B}(\mathrm{Con}(A/\theta)) \arrow[d, "\lambda_{A/\theta} \mid_{\mathcal{B}(\mathrm{Con}(A/\theta))}"] \\
\mathcal{B}(\mathcal{L}(A)) \arrow[rr, "\mathcal{L}(p_\theta) \mid_{\mathcal{B}(\mathcal{L}(A))}"] \arrow[dr, "\mathcal{B}(\pi_{\theta^*})"'] & & \mathcal{B}(\mathcal{L}(A/\theta)) \arrow[dl, "\mathcal{B}(\varphi_\theta)"] \\
& \mathcal{B}(\mathcal{L}(A)/\theta^*) &
\end{tikzcd}$$

Thus $\mathcal{L}(p_\theta) \mid_{\mathcal{B}(\mathcal{L}(A))} \circ \lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))} = \lambda_{A/\theta} \mid_{\mathcal{B}(\mathrm{Con}(A/\theta))} \circ p_\theta^\bullet \mid_{\mathcal{B}(\mathrm{Con}(A))}$, hence the statements in the enunciation. □

Remark 5.26. By a result in [21] recalled in Section 2, since $\mathcal{C}$ is congruence–modular, if the commutator of A is associative, then, for all $\theta \in \mathrm{Con}(A)$, the commutator of A/θ is associative.

Proposition 5.27. *Let* $\theta \in \mathrm{Con}(A)$*.*

- *If* $\theta \in \mathrm{RCon}(A)$ *and* θ *has CBLP, then* θ^* *has the Id–BLP.*
- *If* $\Delta_A, \theta \in \mathrm{RCon}(A)$*, then:* θ *has CBLP iff* θ^* *has the Id–BLP.*
- *If the commutator of* A/θ *is associative and* θ *has CBLP, then* θ^* *has the Id–BLP.*
- *If the commutator of* A *is associative, then:* θ *has CBLP iff* θ^* *has the Id–BLP.*

Proof. By Lemmas 5.25 and Lemma 5.1, Proposition 5.22 and Remark 5.26. □

Theorem 5.28.
- *If* $\mathrm{RCon}(A) = \mathrm{Con}(A)$*, then:* A *has the CBLP iff* $\mathcal{L}(A)$ *has the Id–BLP.*

- *If the commutator of A is associative, then: A has the CBLP iff $\mathcal{L}(A)$ has the Id–BLP.*

Proof. By Propositions 5.27 and 3.2 and Remark 5.26. □

Proposition 5.29. *Let $n \in \mathbb{N}^*$, $M_1, \ldots, M_n$ be members of $\mathcal{C}$ and $\theta_1 \in \mathrm{Con}(M_1)$, $\ldots, \theta_n \in \mathrm{Con}(M_n)$. Then:*

(i) $\theta_1 \times \ldots \times \theta_n$ has the CBLP iff $\theta_1, \ldots, \theta_n$ have the CBLP;

(ii) $M_1 \times \ldots \times M_n$ has the CBLP iff $M_1, \ldots, M_n$ have the CBLP.

Proof. (i) Let $M = M_1 \times \ldots \times M_n$ and $\theta = \theta_1 \times \ldots \times \theta_n \in \mathrm{Con}(M)$, and note that $M/\theta = M_1/\theta_1 \times \ldots \times M_n/\theta_n$. Since $\mathcal{C}$ is congruence–modular and semi–degenerate, the direct products $M_1 \times \ldots \times M_n$ and $M_1/\theta_1 \times \ldots \times M_n/\theta_n$ have no skew congruences, hence $\mathcal{B}(\mathrm{Con}(M)) = \mathcal{B}(\mathrm{Con}(M_1) \times \ldots \times \mathrm{Con}(M_n)) = \mathcal{B}(\mathrm{Con}(M_1)) \times \ldots \times \mathcal{B}(\mathrm{Con}(M_n))$ and $\mathcal{B}(\mathrm{Con}(M/\theta)) = \mathcal{B}(\mathrm{Con}(M_1/\theta_1) \times \ldots \times \mathrm{Con}(M_n/\theta_n)) = \mathcal{B}(\mathrm{Con}(M_1/\theta_1)) \times \ldots \times \mathcal{B}(\mathrm{Con}(M_n/\theta_n))$. For all $\alpha_1 \in \mathrm{Con}(M_1), \ldots, \alpha_n \in \mathrm{Con}(M_n)$, if $\alpha = \alpha_1 \times \ldots \times \alpha_n$, then $p^\bullet_\theta(\alpha) = (\alpha \vee \theta)/\theta = ((\alpha_1 \vee \theta_1)/\theta_1, \ldots, (\alpha_n \vee \theta_n)/\theta_n) = (p^\bullet_{\theta_1}(\alpha_1), \ldots, p^\bullet_{\theta_n}(\alpha_n))$, thus $p^\bullet_\theta = p^\bullet_{\theta_1} \times \ldots \times p^\bullet_{\theta_n}$. Hence $p^\bullet_\theta \mid_{\mathcal{B}(\mathrm{Con}(M))} \colon \mathcal{B}(\mathrm{Con}(M)) \to \mathcal{B}(\mathrm{Con}(M/\theta))$ is surjective iff $p^\bullet_{\theta_1} \mid_{\mathcal{B}(\mathrm{Con}(M_1))} \colon \mathcal{B}(\mathrm{Con}(M_1)) \to \mathcal{B}(\mathrm{Con}(M_1/\theta_1)), \ldots, p^\bullet_{\theta_n} \mid_{\mathcal{B}(\mathrm{Con}(M_n))} \colon \mathcal{B}(\mathrm{Con}(M_n)) \to \mathcal{B}(\mathrm{Con}(M_n/\theta_n))$ are surjective.
(ii) By (i). □

Remark 5.30. In Proposition 5.29, (i), instead of $\mathcal{C}$ being congruence–modular and semi–degenerate, it suffices for $\mathcal{C}$ to be congruence–modular and the direct product $M_1 \times \ldots \times M_n$ to have no skew congruences.

Recall that a bounded distributive lattice L is said to be *B–normal* iff, for all $x, y \in L$ such that $x \vee y = 1$, there exist $a, b \in \mathcal{B}(L)$ such that $x \vee a = y \vee b = 1$ and $a \wedge b = 0$. L is said to be *B–conormal* iff its dual is B–normal.

Definition 5.31. We say that the algebra A is *congruence B–normal* iff, for all $\phi, \psi \in \mathrm{Con}(A)$ such that $\phi \vee \psi = \nabla_A$, there exist $\alpha, \beta \in \mathcal{B}(\mathrm{Con}(A))$ such that $\phi \vee \alpha = \psi \vee \beta = \nabla_A$ and $[\alpha, \beta]_A = \Delta_A$.

Remark 5.32. If A is congruence–distributive, then A is congruence B–normal iff its congruence lattice is B–normal. More generally, if A is semiprime, then A is congruence B–normal iff its congruence lattice satisfies the B–normality condition excepting distributivity.

Congruence B–normal algebras generalize commutative exchange rings [44, Theorem 1.7], quasi–local residuated lattices [23, 41] and congruence–distributive B–normal algebras [24].

The following proofs are very similar to those of the analogous statements from [24, Theorem 4.28], but we introduce them here for the sake of completeness.

Lemma 5.33. *The following are equivalent:*

(i) A is congruence B–normal;

(ii) for all $\phi, \psi \in \mathcal{K}(A)$ such that $\phi \vee \psi = \nabla_A$, there exist $\alpha, \beta \in \mathcal{B}(\mathrm{Con}(A))$ such that $\phi \vee \alpha = \psi \vee \beta = \nabla_A$ and $[\alpha, \beta]_A = \Delta_A$.

Proof. (i)$\Rightarrow$(ii): Trivial.
(ii)$\Rightarrow$(i): Let $\phi, \psi \in \mathrm{Con}(A)$ such that $\phi \vee \psi = \nabla_A$, that is $\nabla_A = \bigvee\{Cg_A(a, b) \mid (a, b) \in \phi \cup \psi\}$. But $\nabla_A \in \mathcal{K}(A)$, thus, for some $n, k \in \mathbb{N}^*$, there exist $(a_1, b_1), \ldots, (a_n, b_n) \in \phi$ and $(c_1, d_1), \ldots, (c_k, d_k) \in \psi$ such that $\nabla_A = \varepsilon \vee \xi$, where $\varepsilon = \bigvee_{i=1}^{n} Cg_A(a_i, b_i) \in \mathcal{K}(A)$ and $\xi = \bigvee_{j=1}^{k} Cg_A(c_j, d_j) \in \mathcal{K}(A)$. Hence there exist $\alpha, \beta \in \mathrm{Con}(A)$ such that $[\alpha, \beta]_A = \Delta_A$ and $\varepsilon \vee \alpha = \xi \vee \beta = \nabla_A$, so that $\phi \vee \alpha = \psi \vee \beta = \nabla_A$ since $\varepsilon \subseteq \phi$ and $\xi \subseteq \psi$. $\square$

Proposition 5.34. *(i) If A is congruence B–normal, then $\mathcal{L}(A)$ is B–normal.*

(ii) If $\mathcal{C}$ is congruence–modular and semi–degenerate and the Boolean morphism $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \rightarrow \mathcal{B}(\mathcal{L}(A))$ is surjective, then: A is congruence B–normal iff $\mathcal{L}(A)$ is B–normal.

Proof. (i) Assume that A is congruence B–normal and let $\theta, \zeta \in \mathcal{K}(A)$ such that $\lambda_A(\theta) \vee \lambda_A(\zeta) = \mathbf{1}$, that is $\lambda_A(\theta \vee \zeta) = \mathbf{1}$, so that $\theta \vee \zeta = \nabla_A$, hence there exist $\alpha, \beta \in \mathcal{B}(\mathrm{Con}(A))$ such that $\theta \vee \alpha = \zeta \vee \beta = \nabla_A$ and $[\alpha, \beta]_A = \Delta_A$, thus $\lambda_A(\alpha), \lambda_A(\beta) \in \mathcal{B}(\mathcal{L}(A))$, $\lambda_A(\theta) \vee \lambda_A(\alpha) = \lambda_A(\theta \vee \alpha) = \mathbf{1} = \lambda_A(\zeta \vee \beta) = \lambda_A(\zeta) \vee \lambda_A(\beta)$ and $\lambda_A(\alpha) \wedge \lambda_A(\beta) = \lambda_A([\alpha, \beta]_A) = \mathbf{0}$. Therefore $\mathcal{L}(A)$ is B–normal.
(ii) Assume that $\mathcal{C}$ is congruence–modular and semi–degenerate and that this Boolean morphism is surjective, so that it is a Boolean isomorphism by Lemma 5.1. By (i), it suffices to prove the converse implication, so assume that $\mathcal{L}(A)$ is B–normal, and let $\phi, \psi \in \mathcal{K}(A)$ such that $\phi \vee \psi = \nabla_A$. Then $\lambda_A(\phi) \vee \lambda_A(\psi) = \lambda_A(\phi \vee \psi) = \mathbf{1}$, hence, by the surjectivity of λ_A restricted to the Boolean centers, there exist $\alpha, \beta \in \mathcal{B}(\mathrm{Con}(A))$ such that $\lambda_A(\phi \vee \alpha) = \lambda_A(\phi) \vee \lambda_A(\alpha) = \mathbf{1} = \lambda_A(\nabla_A) = \lambda_A(\psi) \vee \lambda_A(\beta) = \lambda_A(\psi \vee \beta)$ and $\lambda_A([\alpha, \beta]_A) = \lambda_A(\alpha) \wedge \lambda_A(\beta) = \mathbf{0} = \lambda_A(\Delta_A)$, therefore, by the injectivity of this Boolean morphism, $\phi \vee \alpha = \psi \vee \beta = \nabla_A$ and $[\alpha, \beta]_A = \Delta_A$. By Lemma 5.33, it follows that A is congruence B–normal. $\square$

Theorem 5.35. *If the Boolean morphism* $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))} : \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathcal{L}(A))$ *is surjective, then the following are equivalent:*

(i) A has the CBLP;

(ii) $\mathcal{L}(A)$ *has the Id–BLP;*

(iii) $\mathcal{L}(A)$ *is B–normal;*

(iv) A is congruence B–normal;

(v) the topological space $(\mathrm{Spec}(A), \{D_A(\theta) \mid \theta \in \mathrm{Con}(A)\})$ *is strongly zero–dimensional.*

Proof. By Lemma 5.1, $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))} : \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathcal{L}(A))$ is a Boolean isomorphism.
(i)⇔(ii): By Lemma 5.25 and Proposition 3.2.
(ii)⇔(iii): By [15, Proposition 13].
(iii)⇔(iv): By Proposition 5.34, (ii).
(iv)⇔(v): Analogously to the proof of the similar equivalence from [24, Theorem 4.28]. □

Remark 5.36. By [15], $\mathcal{L}(A)$ is B–normal iff $\mathrm{Id}(\mathcal{L}(A))$ is B–normal iff $\mathrm{Filt}(\mathcal{L}(A))$ is B–conormal.

Corollary 5.37. *If* $\mathcal{C}$ *is congruence–modular and semi–degenerate and either A is semiprime or its commutator is associative, then: A has the CBLP iff* $\mathcal{L}(A)$ *has the Id–BLP iff* $\mathcal{L}(A)$ *is B–normal iff A is congruence B–normal iff the topological space* $(\mathrm{Spec}(A), \{D_A(\theta) \mid \theta \in \mathrm{Con}(A)\})$ *is strongly zero–dimensional.*

Proof. By Theorem 5.35 and Lemma 5.1. □

Remark 5.38. Theorem 5.35 extends results such as: commutative unitary rings with the lifting property are exactly exchange rings [44], residuated lattices with the Boolean Lifting Property are exactly quasi–local residuated lattices [25], in semi–degenerate congruence–distributive varieties, algebras with CBLP are exactly B–normal algebras [24, Theorem 4.28].

6 Particular Cases and Examples

Remark 6.1. By [13, Theorem 8.11, p.126], the variety of distributive lattices has the PIP, thus also the CIP, since it is congruence–distributive. Therefore, by Proposition 4.16, $\mathcal{L}$ is a functor from the variety of distributive lattices to itself, as well as from the variety of bounded distributive lattices to itself.

Remarks 6.2. • Any Boolean algebra A is isomorphic to its reticulation, since $\mathrm{Id}(A) \cong \mathrm{Con}(A)$ and thus $\mathrm{Spec}_{\mathrm{Id}}(A)$ and $\mathrm{Spec}(A)$, endowed with the Stone topologies, are homeomorphic, A is a bounded distributive lattice and $\mathcal{L}(A)$ is unique up to a lattice isomorphism.

• A finite modular lattice L is isomorphic to its reticulation iff L is a Boolean algebra. Indeed, the converse implication follows from the above, while, for the direct implication, we may notice that, since L is congruence–distributive and finite, we have $\mathcal{L}(L) \cong \mathcal{K}(L) = \mathrm{Con}(L)$, which is a Boolean algebra [13, 28, 17].

• By Remark 4.21, a lattice without ACC can not be isomorphic to its reticulation.

• If A and B are algebras with the CIP and the commutators equalling the intersection having $\mathrm{Con}(A) \cong \mathrm{Con}(B)$, then $\mathcal{K}(A) = \mathrm{Cp}(\mathrm{Con}(A))$ and $\mathcal{K}(B) = \mathrm{Cp}(\mathrm{Con}(B))$ are sublattices of $\mathrm{Con}(A)$ and $\mathrm{Con}(B)$, respectively, so we have $\mathcal{L}(A) \cong \mathcal{K}(A) \cong \mathcal{K}(B) \cong \mathcal{L}(B)$.

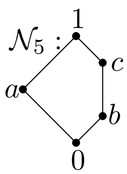
In particular, any lattice with the CIP, thus any finite or distributive lattice, has its reticulation isomorphic to the reticulation of its dual.

In the following examples, we have calculated the commutators using the method from [38]. Note that, in each of these examples, the commutator is distributive w.r.t. the join, hence, by [1, Proposition 1.2], the prime congruences of A are the meet–irreducible elements ϕ of $\mathrm{Con}(A)$ with the property that $[\alpha, \alpha]_A \subseteq \phi$ implies $\alpha \subseteq \phi$ for all $\alpha \in \mathrm{Con}(A)$.

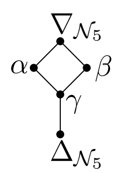
Example 6.3. By Lemma 4.8, all the algebras in this example are semiprime and all the morphisms in this example fulfill FRet, since we are in the congruence–distributive variety of lattices and the following algebras are finite, thus all their congruences are compact, so these algebras trivially satisfy the CIP. Bounded lattices form a congruence–distributive variety with $\vec{0}$ and $\vec{1}$, thus all bounded lattice morphisms in this example also satisfy the FBC, according to Proposition 5.8. For the bounded lattice morphisms between bounded distributive lattices we can even apply Corollary 5.6, since the variety of distributive lattices has the CIP, because it is filtral [2, Example 2.11]; moreover, the variety of distributive lattices has the PIP [13, Theorem 8.11].

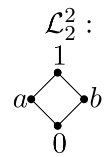
Let us consider the congruence–distributive variety of lattices, $\mathcal{L}_2^2 = \{0, a, b, 1\}$, $\mathcal{L}_2 = \{0, a\}$ and let us consider the lattice embedding $i_{\mathcal{L}_2,\mathcal{L}_2^2} : \mathcal{L}_2 \to \mathcal{L}_2^2$. Then we may take $\mathcal{L}(\mathcal{L}_2) = \mathcal{K}(\mathcal{L}_2) = \mathrm{Con}(\mathcal{L}_2) = \{\Delta_{\mathcal{L}_2}, \nabla_{\mathcal{L}_2}\} \cong \mathcal{L}_2$ and $\mathcal{L}(\mathcal{L}_2^2) = \mathcal{K}(\mathcal{L}_2^2) = \mathrm{Con}(\mathcal{L}_2^2) = \{\Delta_{\mathcal{L}_2^2}, \phi, \psi, \nabla_{\mathcal{L}_2^2}\} \cong \mathcal{L}_2^2$, where $\mathcal{L}_2^2/\phi = \{\{0, a\}, \{b, 1\}\}$ and $\mathcal{L}_2^2/\psi = \{\{0, b\}, \{a, 1\}\}$. Then $i_{\mathcal{L}_2,\mathcal{L}_2^2}$ fulfills FRet, with $\mathcal{L}(i_{\mathcal{L}_2,\mathcal{L}_2^2}) = i^{\bullet}_{\mathcal{L}_2,\mathcal{L}_2^2}$, which preserves the meet, but does not preserve the $\mathbf{1}$, since $i^{\bullet}_{\mathcal{L}_2,\mathcal{L}_2^2}(\nabla_{\mathcal{L}_2}) = Cg_{\mathcal{L}_2^2}(i_{\mathcal{L}_2,\mathcal{L}_2^2}(\nabla_{\mathcal{L}_2})) = \alpha \neq \nabla_{\mathcal{L}_2^2}$. Recall that, since we are in a congruence–distributive variety, $\rho_{\mathcal{L}_2^2} = id_{\mathrm{Con}(\mathcal{L}_2^2)}$.

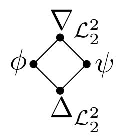
Here is an example of a morphism k in the congruence–distributive semi–degenerate variety of bounded lattices such that $\mathcal{L}(k)$ does not preserve the meet, or, equivalently, such that $k^{\bullet}$ does not preserve the intersection of congruences. Let $k : \mathcal{N}_5 \to \mathcal{N}_5$ be the bounded lattice morphism defined by the table below:

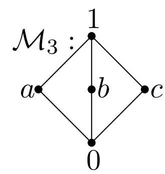


u	0	a	b	c	1
$k(u)$	0	a	b	b	1
$h(u)$	0	a	b	b	1









θ	$\Delta_{\mathcal{N}_5}$	α	β	γ	$\nabla_{\mathcal{N}_5}$
$k^{\bullet}(\theta)$	$\Delta_{\mathcal{N}_5}$	α	β	$\Delta_{\mathcal{N}_5}$	$\nabla_{\mathcal{N}_5}$
$h^{\bullet}(\theta)$	$\Delta_{\mathcal{L}_2^2}$	ϕ	ψ	$\Delta_{\mathcal{L}_2^2}$	$\nabla_{\mathcal{L}_2^2}$

θ	$\Delta_{\mathcal{L}_2^2}$	ϕ	ψ	$\nabla_{\mathcal{L}_2^2}$
$i^{\bullet}_{\mathcal{L}_2^2,\mathcal{M}_3}(\theta)$	$\Delta_{\mathcal{M}_3}$	$\nabla_{\mathcal{M}_3}$	$\nabla_{\mathcal{M}_3}$	$\nabla_{\mathcal{M}_3}$

$\mathcal{N}_5$ has the congruence lattice above, where $\mathcal{N}_5/\alpha = \{\{0, b, c\}, \{a, 1\}\}$, $\mathcal{N}_5/\beta = \{\{0, a\}, \{b, c, 1\}\}$ and $\mathcal{N}_5/\gamma = \{\{0\}, \{a\}, \{b, c\}, \{1\}\}$. We have $k^{\bullet}(\alpha) \cap k^{\bullet}(\beta) = \alpha \cap \beta = \gamma \neq \Delta_{\mathcal{N}_5} = k^{\bullet}(\gamma) = k^{\bullet}(\alpha \cap \beta)$.

Let us also consider $\mathcal{M}_3$ with the elements denoted as above and the bounded lattice embedding $i_{\mathcal{L}_2^2,\mathcal{M}_3} : \mathcal{L}_2^2 \to \mathcal{M}_3$. $\mathcal{B}(\mathrm{Con}(\mathcal{M}_3)) = \mathrm{Con}(\mathcal{M}_3) = \{\Delta_{\mathcal{M}_3}, \nabla_{\mathcal{M}_3}\} \cong \mathcal{L}_2$. $i_{\mathcal{L}_2^2,\mathcal{M}_3}$ is injective and not surjective, but, as shown by the table above, $i^{\bullet}_{\mathcal{L}_2^2,\mathcal{M}_3}$ is surjective and not injective, hence so is $\mathcal{L}(i_{\mathcal{L}_2^2,\mathcal{M}_3})$, since we are in a congruence–distributive variety.

Let $h : \mathcal{N}_5 \to \mathcal{L}_2^2$ be the surjective lattice morphism defined by the table above. Then $h^{\bullet} : \mathrm{Con}(\mathcal{N}_5) = \mathcal{K}(\mathcal{N}_5) \to \mathrm{Con}(\mathcal{L}_2^2) = \mathcal{K}(\mathcal{L}_2^2)$ is surjective, thus so is $\mathcal{L}(h) : \mathcal{L}(\mathcal{N}_5) \to \mathcal{L}(\mathcal{L}_2^2)$, and h fulfills the FBC, as announced above, but $h^{\bullet} \mid_{\mathcal{B}(\mathrm{Con}(\mathcal{N}_5))}: \mathcal{B}(\mathrm{Con}(\mathcal{N}_5)) = \{\Delta_{\mathcal{N}_5}, \nabla_{\mathcal{N}_5}\} \to \mathcal{B}(\mathrm{Con}(\mathcal{L}_2^2)) = \mathrm{Con}(\mathcal{L}_2^2)$ is not surjective, thus neither is $\mathcal{B}(\mathcal{L}(h)) : \mathcal{B}(\mathcal{L}(\mathcal{N}_5)) \to \mathcal{B}(\mathcal{L}(\mathcal{L}_2^2))$, since we are in a congruence–distributive variety and $\mathcal{N}_5$ and $\mathcal{L}_2^2$ are finite, so that we may take $\mathcal{L}(\mathcal{N}_5) = \mathcal{K}(\mathcal{N}_5) = \mathrm{Con}(\mathcal{N}_5)$, $\mathcal{L}(\mathcal{L}_2^2) = \mathcal{K}(\mathcal{L}_2^2) = \mathrm{Con}(\mathcal{L}_2^2)$ and $\mathcal{L}(h) = h^{\bullet} : \mathrm{Con}(\mathcal{N}_5) \to \mathrm{Con}(\mathcal{L}_2^2)$.

The bounded lattice embedding $i_{\mathcal{L}_2,\mathcal{N}_5}$ fulfills the FBC, as announced above, and, here as well, we may take $\mathcal{L}(\mathcal{L}_2) = \mathcal{K}(\mathcal{L}_2) = \mathrm{Con}(\mathcal{L}_2) = \{\Delta_{\mathcal{L}_2}, \nabla_{\mathcal{L}_2}\} = \mathcal{B}(\mathrm{Con}(\mathcal{L}_2))$ and $\mathcal{L}(i_{\mathcal{L}_2,\mathcal{N}_5}) = i^{\bullet}_{\mathcal{L}_2,\mathcal{N}_5} : \mathrm{Con}(\mathcal{L}_2) \to \mathrm{Con}(\mathcal{N}_5)$, so that $\mathcal{B}(\mathcal{L}(i_{\mathcal{L}_2,\mathcal{N}_5})) = \mathcal{L}(i^{\bullet}_{\mathcal{L}_2,\mathcal{N}_5}) =$

$i^\bullet_{\mathcal{L}_2,\mathcal{N}_5}\ |_{\mathcal{B}(\mathrm{Con}(\mathcal{L}_2))}$: $\mathcal{B}(\mathrm{Con}(\mathcal{L}_2)) \to \mathcal{B}(\mathrm{Con}(\mathcal{N}_5))$. Since $i^\bullet_{\mathcal{L}_2,\mathcal{N}_5}(\mathcal{B}(\mathrm{Con}(\mathcal{L}_2))) = i^\bullet_{\mathcal{L}_2,\mathcal{N}_5}(\mathrm{Con}(\mathcal{L}_2)) = \{\Delta_{\mathcal{N}_5}, \nabla_{\mathcal{N}_5}\} = \mathcal{B}(\mathrm{Con}(\mathcal{N}_5)) \subsetneq \mathrm{Con}(\mathcal{N}_5)$, it follows that $\mathcal{B}(\mathcal{L}(i_{\mathcal{L}_2,\mathcal{N}_5}))$ is surjective, while $\mathcal{L}(i_{\mathcal{L}_2,\mathcal{N}_5})$ is not surjective.

Here is a lattice morphism that fails FBC, and, since it is a morphism between finite lattices, it satisfies FRet, as all morphisms above: let $g : \mathcal{L}_2^2 \to \mathcal{N}_5$ be defined by the following table, so that $g^\bullet$ has this definition:

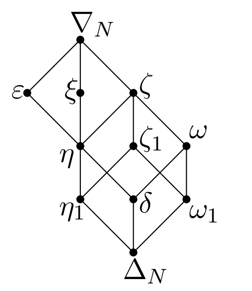
u	0	a	b	1
$g(u)$	0	0	b	b

θ	$\Delta_{\mathcal{L}_2^2}$	ϕ	ψ	$\nabla_{\mathcal{L}_2^2}$
$g^\bullet(\theta)$	$\Delta_{\mathcal{N}_5}$	$\Delta_{\mathcal{N}_5}$	α	α

We have $g^\bullet(\mathcal{B}(\mathrm{Con}(\mathcal{L}_2^2))) = g^\bullet(\mathrm{Con}(\mathcal{L}_2^2)) = \{\Delta_{\mathcal{N}_5}, \alpha\} \nsubseteq \{\Delta_{\mathcal{N}_5}, \nabla_{\mathcal{N}_5}\} = \mathcal{B}(\mathrm{Con}(\mathcal{N}_5))$, thus g fails (FBC1).

Example 6.4. Let $\tau = (2)$ and let us consider the following τ–algebra from [27, Example 4]: $(N, +^N)$, with $N = \{a, b, c, x, y\}$ and $+^N : N^2 \to N$ defined by the following table. Note that some of the congruences of N, as well as of the algebra M from the same example, have been omitted in [27]; here is the correct Hasse diagram of $\mathrm{Con}(N)$, where: $N/\delta = \{\{a, b\}, \{c\}, \{x\}, \{y\}\}$, $N/\eta_1 = \{\{a\}, \{b, c\}, \{x\}, \{y\}\}$, $N/\eta = \{\{a, b, c\}, \{x\}, \{y\}\}$, $N/\omega_1 = \{\{a\}, \{b\}, \{c\}, \{x, y\}\}$, $N/\omega_1 = \{\{a, b\}, \{c\}, \{x, y\}\}$, $N/\zeta_1 = \{\{a\}, \{b, c\}, \{x, y\}\}$, $N/\zeta = \{\{a, b, c\}, \{x, y\}\}$, $N/\varepsilon = \{\{a, b, c, x\}, \{y\}\}$ and $N/\xi = \{\{a, b, c, y\}, \{x\}\}$.

$+^N$	a	b	c	x	y
a	a	b	c	a	a
b	b	b	c	b	b
c	c	c	c	c	c
x	x	x	x	x	x
y	y	y	y	y	y



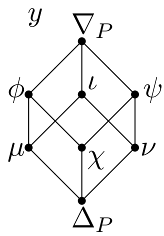
$[\cdot,\cdot]_N$ is given by the following table, so that $\mathrm{Spec}(N) = \{\omega\}$, thus $\mathrm{RCon}(N) = \{\omega, \nabla_N\}$, hence $\mathcal{L}(N) = \mathcal{K}(N)/\equiv_N = \mathrm{Con}(N)/\equiv_N = \{(\omega], [\omega)\} = \{\mathbf{0}, \mathbf{1}\} \cong \mathcal{L}_2$. By Proposition 5.22, since $\Delta_N \notin \mathrm{RCon}(N)$, while $\omega \in \mathrm{RCon}(N)$, N is not semiprime, but N/ω is semiprime.

$[\cdot,\cdot]_N$	Δ_N	δ	η_1	η	ω_1	ω	ζ_1	ζ	ε	ξ	∇_N
Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N
δ	Δ_N	δ	Δ_N	δ	Δ_N	δ	Δ_N	δ	δ	δ	δ
η_1	Δ_N	Δ_N	η_1	η_1	Δ_N	Δ_N	η_1	η_1	η_1	η_1	η_1
η	Δ_N	δ	η_1	η	Δ_N	δ	η_1	η	η	η	η
ω_1	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N
ω	Δ_N	δ	Δ_N	δ	Δ_N	δ	Δ_N	δ	δ	δ	δ
ζ_1	Δ_N	Δ_N	η_1	η_1	Δ_N	Δ_N	η_1	η_1	η_1	η_1	η_1
ζ	Δ_N	δ	η_1	η	Δ_N	δ	η_1	η	η	η	η
ε	Δ_N	δ	η_1	η	Δ_N	δ	η_1	η	η	η	η
ξ	Δ_N	δ	η_1	η	Δ_N	δ	η_1	η	η	η	η
∇_N	Δ_N	δ	η_1	η	Δ_N	δ	η_1	η	η	η	η

Note that $\mathcal{B}(\mathrm{Con}(N)) = \{\Delta_N, \omega_1, \varepsilon, \xi, \nabla_N\}$, which is not a sublattice of $\mathrm{Con}(N)$, since it is not closed w.r.t. the intersection. Note, also, that $\{a\}$ is a subalgebra of N, thus the variety generated by N is not semi–degenerate; the same holds for all the algebras in this example, as well as those in the following example, because each of these algebras has trivial subalgebras.

Let $(P, +^P)$ be the following τ–algebra: $P = \{a, b, x, y\}$, with $+^P : P^2 \to P$ defined by the table that follows:

$+^P$	a	b	x	y
a	a	b	y	y
b	b	b	y	y
x	x	x	x	x
y	y	y	y	y



$[\cdot,\cdot]_P$	Δ_P	χ	ϕ	μ	ψ	ν	ι	∇_P
Δ_P	Δ_P	Δ_P	Δ_P	Δ_P	Δ_P	Δ_P	Δ_P	Δ_P
χ	Δ_P	Δ_P	Δ_P	Δ_P	Δ_P	Δ_P	Δ_P	Δ_P
ϕ	Δ_P	Δ_P	μ	μ	Δ_P	Δ_P	μ	μ
μ	Δ_P	Δ_P	μ	μ	Δ_P	Δ_P	μ	μ
ψ	Δ_P	Δ_P	Δ_P	Δ_P	ν	ν	ν	ν
ν	Δ_P	Δ_P	Δ_P	Δ_P	ν	ν	ν	ν
ι	Δ_P	Δ_P	μ	μ	ν	ν	ι	ι
∇_P	Δ_P	Δ_P	μ	μ	ν	ν	ι	ι

$\mathrm{Con}(P) = \mathcal{B}(\mathrm{Con}(P)) = \{\Delta_P, \chi, \phi, \psi, \mu, \nu, \iota, \nabla_P\} \cong \mathcal{L}_2^3$, where $P/\chi = \{\{a\}, \{b\}, \{x, y\}\}$, $P/\phi = \{\{a, b\}, \{x, y\}\}$, $P/\psi = \{\{a\}, \{b, x, y\}\}$, $P/\mu = \{\{a, b\}, \{x\}, \{y\}\}$, $P/\nu = \{\{a\}, \{x\}, \{b, y\}\}$ and $P\iota = \{\{a, b, y\}, \{x\}\}$, as in the diagram above. The commutator of P has the table above, hence $\mathrm{Spec}(P) = \{\phi, \psi\}$, thus $\Delta_P \notin \{\phi, \psi, \chi, \nabla_P\} = \mathrm{RCon}(P)$, so P is not semiprime, and $\mathcal{L}(P) = \mathcal{B}(\mathcal{L}(P)) = \mathcal{B}(\mathcal{K}(P)/{\equiv_P}) = \mathcal{B}(\mathrm{Con}(P)/{\equiv_P}) = \mathrm{Con}(P)/{\equiv_P} = \{\{\Delta_P, \chi\}, \{\phi, \mu\}, \{\psi, \nu\}, \{\iota, \nabla_P\}\}$

$\cong \mathcal{L}_2^2$, hence $\lambda_P\mid_{\mathcal{B}(\mathrm{Con}(P))}$: $\mathcal{B}(\mathrm{Con}(P)) = \mathrm{Con}(P) \to \mathcal{B}(\mathcal{L}(P)) = \mathcal{L}(P)$ is a surjective Boolean morphism.

Let $g : P \to N$ and $h : N \to P$ be the following τ–morphisms:

u	a	b	x	y
$g(u)$	a	a	y	a

u	a	b	c	x	y
$h(u)$	x	x	x	y	x

θ	Δ_P	χ	ϕ	μ	ψ	ν	ι	∇_P
$g^\bullet(\theta)$	Δ_N	ξ	ξ	Δ_N	ξ	Δ_N	Δ_N	ξ

θ	Δ_N	δ	η_1	η	ω_1	ω	ζ_1	ζ	ε	ξ	∇_N
$h^\bullet(\theta)$	Δ_P	Δ_P	Δ_P	Δ_P	χ	χ	χ	χ	χ	Δ_P	χ

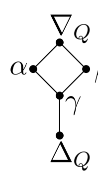
Then $g^\bullet$ and $h^\bullet$ have the tables above.

We have $\nabla_P \equiv_P \iota$, but $g^\bullet(\nabla_P) = \xi \not\equiv_N \Delta_N = g^\bullet(\iota)$, hence g fails FRet. Note that $g^\bullet$ preserves the intersection, but not the commutator, since $g^\bullet([\psi,\psi]_P) = g^\bullet(\nu) = \Delta_N \neq \eta = [\xi,\xi]_N = [g^\bullet(\psi), g^\bullet(\psi)]_N$.

Since $h^\bullet(\mathrm{Con}(N)) = \{\Delta_P, \chi\} = \lambda_P(\Delta_P)$ and $[\chi,\chi]_P = \Delta_P$, h satisfies FRet and $h^\bullet$ preserves the commutator. $h^\bullet(\varepsilon) \cap h^\bullet(\zeta) = \chi \cap \chi = \chi \neq \Delta_P = h^\bullet(\eta) = h^\bullet(\varepsilon \cap \zeta)$, thus $h^\bullet$ does not preserve the intersection, and $\mathcal{L}(h)(\mathbf{1}) = \mathcal{L}(h)(\lambda_N(\nabla_N)) = \lambda_P(h^\bullet(\nabla_N)) = \lambda_P(\chi) \neq \lambda_P(\nabla_P) = \mathbf{1}$.

Let $(Q, +^Q)$ be the following τ–algebra: $Q = \{a, b, x, y\}$, with $+^Q : Q^2 \to Q$ defined by the table below:

$+^Q$	a	b	x	y
a	a	b	x	x
b	b	b	y	y
x	x	x	x	x
y	y	y	y	y



$[\cdot,\cdot]_Q$	Δ_Q	α	β	γ	∇_Q	$\rho_Q(\cdot)$
Δ_Q	Δ_Q	Δ_Q	Δ_Q	Δ_Q	Δ_Q	∇_Q
α	Δ_Q	α	γ	Δ_Q	α	α
β	Δ_Q	γ	β	Δ_Q	β	β
γ	Δ_Q	Δ_Q	Δ_Q	Δ_Q	Δ_Q	γ
∇_Q	Δ_Q	α	β	Δ_Q	∇_Q	γ

Then Q has the congruence lattice represented above, with $Q/\alpha = \{\{a,b\}, \{x,y\}\}$, $Q/\beta = \{\{a\},\{b,x,y\}\}$ and $Q/\gamma = \{\{a\},\{b\},\{x,y\}\}$. The commutator of Q has the table above, hence $\mathrm{Spec}(Q) = \{\alpha,\beta\}$, so ρ_Q is as above and thus $\mathcal{L}(Q) = \mathcal{K}(Q)/\equiv_Q = \mathrm{Con}(Q)/\equiv_Q = \{\{\Delta_Q,\gamma\},\{\alpha\},\{\beta\},\{\nabla_Q\}\} = \{\mathbf{0}, \lambda_Q(\alpha), \lambda_Q(\beta), \mathbf{1}\} \cong \mathcal{L}_2^2$. $\mathcal{B}(\mathrm{Con}(Q)) = \{\Delta_Q, \nabla_Q\} \cong \mathcal{L}_2$, hence the Boolean morphism $\lambda_Q\mid_{\mathcal{B}(\mathrm{Con}(Q))}$: $\mathcal{B}(\mathrm{Con}(Q)) \to \mathcal{B}(\mathcal{L}(Q)) = \mathcal{L}(Q)$ is injective, but not surjective.

Let $k : Q \to N$ and $l : Q \to P$ be the following τ–morphisms:

u	a	b	x	y
$k(u)$	a	b	c	c
$l(u)$	a	b	y	y

θ	Δ_Q	α	β	γ	∇_Q
$k^\bullet(\theta)$	Δ_N	ξ_1	ψ_1	Δ_N	χ_1
$l^\bullet(\theta)$	Δ_P	μ	ν	Δ_P	ι

Then $h^\bullet$ has the table above, so h fulfills FRet and $\mathcal{L}(h)$ preserves the $\mathbf{1}$, although $h^\bullet(\nabla_Q) \neq \nabla_M$: $\mathcal{L}(h)(\mathbf{1}) = \mathcal{L}(h)(\lambda_Q(\nabla_Q)) = \lambda_M(h^\bullet(\nabla_Q)) = \lambda_M(\varepsilon) = \mathbf{1}$. But $\mathcal{L}(h)$

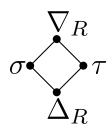
does not preserve the meet, because: $\mathcal{L}(h)(\lambda_Q(\alpha) \wedge \lambda_Q(\beta)) = \mathcal{L}(h)(\lambda_Q([\alpha,\beta]_Q)) = \mathcal{L}(h)(\lambda_Q(\Delta_Q)) = \mathcal{L}(h)(\mathbf{0}) = \mathbf{0} \neq \mathbf{1} = \mathbf{1} \wedge \mathbf{1} = \lambda_M(\varepsilon) \wedge \lambda_M(\varepsilon) = \lambda_M(h^\bullet(\alpha)) \wedge \lambda_M(h^\bullet(\beta)) = \mathcal{L}(h)(\lambda_Q(\alpha)) \wedge \mathcal{L}(h)(\lambda_Q(\beta))$. $h^\bullet$ preserves neither the intersection, nor the commutator: $h^\bullet(\alpha \cap \beta) = h^\bullet(\gamma) = \Delta_M \neq \varepsilon = \varepsilon \cap \varepsilon = h^\bullet(\alpha) \cap h^\bullet(\beta)$ and $h^\bullet([\alpha,\beta]_Q) = h^\bullet(\Delta_Q) = \Delta_M \neq \varepsilon = [\varepsilon,\varepsilon]_M = [h^\bullet(\alpha), h^\bullet(\beta)]_M$.

$k^\bullet$ has the table above, so k fulfills FRet and $\mathcal{L}(k)$ preserves the meet and the $\mathbf{1}$, although $k^\bullet(\nabla_Q) \neq \nabla_N$, and $k^\bullet$ preserves both the intersection and the commutator.

$l^\bullet$ is defined as above, so l fulfills FRet and $\mathcal{L}(l)$ preserves the meet and the $\mathbf{1}$, although $l^\bullet(\nabla_Q) \neq \nabla_P$, and $l^\bullet$ preserves both the intersection and the commutator. Note that $l^\bullet\mid_{\mathcal{B}(\mathrm{Con}(Q))}: \mathcal{B}(\mathrm{Con}(Q)) = \{\Delta_Q, \nabla_Q\} \to \mathcal{B}(\mathrm{Con}(P)) = \{\Delta_P, \mu, \nu, \nabla_P\}$ is an injective Boolean morphism, and that, while l is neither injective, nor surjective, $\mathcal{L}(l) : \mathcal{L}(Q) = \mathcal{B}(\mathcal{L}(Q)) \to \mathcal{L}(P) = \mathcal{B}(\mathcal{L}(P)) \cong \mathcal{L}_2^2$ is a Boolean isomorphism.

Now let $(R, +^R)$ be the τ–algebra defined by $R = \{a,b,c\}$ and the following table for the operation $+^R$:

$+^R$	a	b	c
a	a	b	b
b	b	b	b
c	c	c	c



$[\cdot,\cdot]_R$	Δ_R	σ	τ	∇_R
Δ_R	Δ_R	Δ_R	Δ_R	Δ_R
σ	Δ_R	σ	Δ_R	σ
τ	Δ_R	Δ_R	Δ_R	Δ_R
∇_R	Δ_R	σ	Δ_R	σ

Then R has the congruence lattice above, with $R/\sigma = \{\{a,b\},\{c\}\}$ and $R/\tau = \{\{a\},\{b,c\}\}$, and the commutator of R has the previous definition, so that $\mathrm{Spec}(R) = \{\tau\}$ and thus $\mathrm{RCon}(R) = \{\tau, \nabla_R\}$, so $\mathcal{L}(R) = \mathcal{K}(R)/\equiv_R = \mathrm{Con}(R)/\equiv_R = \{\{\Delta_R, \tau\}, \{\sigma, \nabla_R\}\} = \{\mathbf{0}, \mathbf{1}\} \cong \mathcal{L}_2$, hence the Boolean morphism $\lambda_R\mid_{\mathcal{B}(\mathrm{Con}(R))}: \mathcal{B}(\mathrm{Con}(R)) = \mathrm{Con}(R) \to \mathcal{B}(\mathcal{L}(R)) = \mathcal{L}(R)$ is surjective, but not injective.

Let $d : R \to N$, $e : R \to N$, $j : R \to N$ and $m : R \to P$ be the τ–morphisms defined as follows:

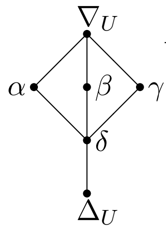
u	a	b	c
$d(u)$	a	b	b
$e(u)$	a	c	c
$j(u)$	y	y	a
$m(u)$	a	y	x

θ	Δ_R	σ	τ	∇_R
$d^\bullet(\theta)$	Δ_N	δ	Δ_N	δ
$e^\bullet(\theta)$	Δ_N	η	Δ_N	η
$j^\bullet(\theta)$	Δ_N	Δ_N	ξ	ξ
$m^\bullet(\theta)$	Δ_P	ι	χ	∇_P

Then $d^\bullet$, $e^\bullet$, $j^\bullet$ and $m^\bullet$ have the definitions above, so d, e and m fulfill FRet, while j fails FRet, since $\Delta_R \equiv_R \tau$, but $j^\bullet(\Delta_R) = \Delta_N \not\equiv_N \xi = j^\bullet(\tau)$. Note that $\mathcal{L}(d)$ preserves the meet and the intersection, but not the $\mathbf{1}$. $\mathcal{L}(e)$ and $\mathcal{L}(m)$ preserve the $\mathbf{1}$, $m^\bullet$ and $e^\bullet$ preserve the intersection and the commutator, while $j^\bullet$ preserves the intersection, but not the commutator, because $j^\bullet([\tau,\tau]_R) = j^\bullet(\Delta_R) = \Delta_N \neq \eta = [\xi,\xi]_N = [j^\bullet(\tau), j^\bullet(\tau)]_N$.

Example 6.5. Let $\tau = (2)$ and let us consider the following τ–algebra from [3, Example 6.3] and [4, Example 4.2]: $(U, +^U)$, with $U = \{0, a, b, c, d\}$ and $+^U : U^2 \to U$ defined by the following table, along with the subalgebra $T = \{0, a, b, c\}$ of U, the τ–embedding $i_{T,U} : T \to U$ and the τ–morphism $t : U \to T$ defined by the table below:

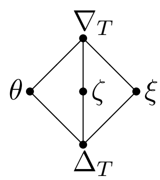
$+^U$	0	a	b	c	d
0	0	a	b	c	d
a	a	0	c	b	b
b	b	c	0	a	a
c	c	b	a	0	0
d	d	b	a	0	0



$[\cdot,\cdot]_U$	Δ_U	α	β	γ	δ	∇_U
Δ_U	Δ_U	Δ_U	Δ_U	Δ_U	Δ_U	Δ_U
α	Δ_U	δ	δ	δ	δ	δ
β	Δ_U	δ	δ	δ	δ	δ
γ	Δ_U	δ	δ	δ	δ	δ
δ	Δ_U	δ	δ	δ	Δ_U	δ
∇_U	Δ_U	δ	δ	δ	δ	δ

ϕ	Δ_T	θ	ζ	ξ	∇_T
$i^\bullet_{T,U}(\phi)$	Δ_U	α	β	γ	∇_U

u	0	a	b	c	d
$t(u)$	0	a	a	0	0



$[\cdot,\cdot]_T$	Δ_T	θ	ζ	ξ	∇_T
Δ_T	Δ_T	Δ_T	Δ_T	Δ_T	Δ_T
θ	Δ_T	Δ_T	Δ_T	Δ_T	Δ_T
ζ	Δ_T	Δ_T	Δ_T	Δ_T	Δ_T
ξ	Δ_T	Δ_T	Δ_T	Δ_T	Δ_T
∇_T	Δ_T	Δ_T	Δ_T	Δ_T	Δ_T

ϕ	Δ_U	α	β	γ	δ	∇_U
$t^\bullet(\phi)$	Δ_T	θ	θ	Δ_T	Δ_T	θ

$\mathrm{Con}(T) = \{\Delta_T, \theta, \zeta, \xi, \nabla_T\} \cong \mathcal{M}_3$, with the Hasse diagram above, where $T/\theta = \{\{0, a\}, \{b, c\}\}$, $T/\zeta = \{\{0, b\}, \{a, c\}\}$, $T/\xi = \{\{0, c\}, \{a, b\}\}$. Note that $\mathcal{B}(\mathrm{Con}(T))$ $= \mathrm{Con}(T)$, which is not a Boolean lattice. The commutator of T has the value Δ_T for every pair of congruences of T, so $\mathrm{Spec}(T) = \emptyset$, thus $\mathcal{L}(T) = \{\mathbf{0}\} \cong \mathcal{L}_1$, thus, trivially, t satisfies FRet. As shown by the table of $t^\bullet$ above, $t^\bullet$ preserves the commutator, but not the intersection, since $t^\bullet(\alpha \cap \beta) = t^\bullet(\delta) = \Delta_T \neq \theta = \theta \cap \theta = t^\bullet(\alpha) \cap t^\bullet(\beta)$.

U has the congruence lattice represented above, where $U/\alpha = \{\{0, a\}, \{b, c, d\}\}$, $U/\beta = \{\{0, b\}, \{a, c, d\}\}$, $U/\gamma = \{\{0, c, d\}, \{a, b\}\}$ and $U/\delta = \{\{0\}, \{a\}, \{b\}, \{c, d\}\}$. As shown by the table of $[\cdot,\cdot]_U$ above, calculated in [27, Example 3], we have $\mathrm{Spec}(U) = \emptyset$, thus $\rho_U(\sigma) = \nabla_U$ for all $\sigma \in \mathrm{Con}(U)$, and hence $\mathcal{L}(U) = \{\mathbf{0}\} \cong \mathcal{L}_1$, therefore, trivially, $i_{T,U}$ fulfills FRet. Also, trivially, $\mathcal{L}(i_{T,U})$ and $\mathcal{L}(t)$ are lattice isomorphisms. $[i^\bullet_{T,U}(\theta), i^\bullet_{T,U}(\theta)]_U = [\alpha, \alpha]_U = \delta \notin i^\bullet_{T,U}(\mathrm{Con}(T))$, in particular $[i^\bullet_{T,U}(\theta), i^\bullet_{T,U}(\theta)]_U \neq i^\bullet_{T,U}([\theta, \theta]_T)$. So $i^\bullet_{T,U}$ does not preserve the commutator, and, despite $i_{T,U}$ being injective, $i^\bullet_{T,U}$ does not preserve the intersection, either, since $i^\bullet_{T,U}(\theta \cap \zeta) = i^\bullet_{T,U}(\Delta_T) = \Delta_U \neq \delta = \alpha \cap \beta = i^\bullet_{T,U}(\theta) \cap i^\bullet_{T,U}(\zeta)$.

$\mathcal{B}(\mathrm{Con}(U)) = \{\Delta_U, \nabla_U\} \cong \mathcal{L}_2$, hence the Boolean morphism $\lambda_U \mid_{\mathcal{B}(\mathrm{Con}(U))}$: $\mathcal{B}(\mathrm{Con}(U)) \to \mathcal{B}(\mathcal{L}(U)) = \mathcal{L}(U)$ is surjective, but not injective. Note that $[\phi, \nabla_U]_U =$

ϕ for all $\phi \in \mathrm{Con}(U)$, which proves that the stronger assumption that $\mathcal{C}$ is congruence–modular and semi–degenerate is necessary for the properties of $\mathcal{B}(\mathrm{Con}(U))$ and this restriction of λ_U recalled above.

Let us also consider the τ–algebra $(V, +^V)$, with $V = \{0, s, t\}$ and $+^V$ defined by the following table:

$+^V$	0	s	t
0	0	s	t
s	s	0	t
t	t	t	0

$[\cdot,\cdot]_V$	Δ_V	σ	∇_V
Δ_V	Δ_V	Δ_V	Δ_V
σ	Δ_V	Δ_V	σ
∇_V	Δ_V	σ	σ

u	0	a	b	c	d
$h(u)$	0	0	t	t	t

ϕ	Δ_U	α	β	γ	δ	∇_U
$h^\bullet(\phi)$	Δ_V	Δ_V	∇_V	∇_V	Δ_V	∇_V

Notice that $\mathrm{Con}(V) = \{\Delta_V, \sigma, \nabla_V\} \cong \mathcal{L}_3$, with $\sigma = eq(\{0, s\}, \{t\})$, and that the commutator of V has the table above, so that $\mathrm{Spec}(V) = \{\Delta_V\}$ and hence $\mathcal{L}(V) = \{\{\Delta_V\}, \{\sigma, \nabla_V\}\} \cong \mathcal{L}_2$. The map $h : U \to V$ defined by the table above is a τ–morphism and $h^\bullet$ is defined as above, hence $h^\bullet(\mathcal{B}(\mathrm{Con}(U))) = h^\bullet(\{\Delta_U, \nabla_U\}) = \{\Delta_V, \nabla_V\} = \mathcal{B}(\mathrm{Con}(V))$ and $h^\bullet \mid_{\mathcal{B}(\mathrm{Con}(U))}$ is a Boolean isomorphism between $\mathcal{B}(\mathrm{Con}(U))$ and $\mathcal{B}(\mathrm{Con}(V))$, thus h satisfies the FBC, but $\Delta_U \equiv_U \nabla_U$, while $(h^\bullet(\Delta_U), h^\bullet(\nabla_U)) = (\Delta_V, \nabla_V) \notin \equiv_V$, thus h fails FRet.

Now let us consider the map $v : V \to V$ defined by the following table. Then $v^\bullet$ has the following definition, thus v fails FRet since $\sigma \equiv_V \nabla_V$, but $v^\bullet(\sigma) = \Delta_V \not\equiv_V \sigma = v^\bullet(\nabla_V)$, despite the fact that $v^\bullet$ preserves the commutator and the intersection and $v^\bullet(\nabla_V) \equiv_V \nabla_V$.

u	0	s	t
$v(u)$	0	0	s

ϕ	Δ_V	σ	∇_V
$v^\bullet(\phi)$	Δ_V	Δ_V	σ

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