Time Delay Control and Frequency Splitting in the forced Kadomtsev-Petviashvili Equation

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Abstract

A time delay control is applied to the forced Kadomtsev-Petviashvili (KP) equation Using an appropriate perturbation method, we derive nonlinear equations describing amplitude and phase of the response and discuss in some detail external force-response and frequency-response curves for the fundamental resonance. For the uncontrolled system, we find a frequency splitting, a second frequency aappears in addition to the forcing one. Saddle-center bifurcation, jumps and hysteresis phenomena are observed together with closed orbits of the slow flow equations. There are stable two-period quasi-periodic modulated motion for the KP equation with amplitudes depending on the initial conditions. Subsequently, we study the controlled system finding sufficient conditions for a periodic behavior. We can accomplish a successful control because the amplitude peak of the fundamental resonance can be reduced and the saddle-center bifurcations and two-period quasi-periodic motions can be removed by adequate choices for delay and feedback gains.

Keywords: Kadomtsev-Petviashvili equation, vibration control, frequency splitting, fundamental resonance, feedback control.

1. Introduction

Vibration control theory is an important growing field in applied physics and a lot of papers consider resonant or non-resonant forced systems control. It is well known that dangerous nonlinear bifurcations and high-amplitude vibrations may happen and bring modifications or catastrophic failure in the systems under consideration. External excitations are widely employed to control these vibrations Berkovski (1995) studied nonlinear ion acoustic waves in a strongly coupled plasma subject to external electric field. He used the frame of the kinetic equation for sound waves and found that effects of nonlinear mode coupling can cause the instability saturation. Wang et al (2018) considered the axial vibrations suppression in a partial differential equation model of ascending mining cable elevator. They successfully used an observer-based output-feedback control law for the suppression of the axial vibration

Ge et al. (2011) performed a vibration control of an Euler–Bernoulli beam under unknown spatiotemporally varying disturbance and accomplished the vibration suppression using the adaptive boundary control technique Wah et al. (1996) studied the synchronization of two coupled nonlinear, in particular chaotic, systems which are not identical. They show how adaptive controllers can be used to adjust the parameters of the systems such that the two systems will synchronize. Oueni et al. (1999) have extensively studied a nonlinear active vibration absorber connected with the plant through control cubic nonlinearities. When the plant vibrates at primary resonance and the absorber frequency is equal to the plant natural frequency, they find that when the forcing amplitude increases beyond a certain threshold, high-amplitude dangerous vibrations disappeared.

Ji and Leung (2002) considered parametrically excited Duffing systems and found that using an appropriate feedback the stable region of the trivial solution can be broadened, a discontinuous bifurcation can be transformed into a continuous one and the response jump phenomenon can be removed

Delay state feedbacks have been used in the past years. (Maccari, 2001, 2003) proposed a time-delayed feedback control method for nonlinear oscillators. The author find that we can suppress high-amplitude response and two-period quasi-periodic motion of a parametrically or externally excited van der Pol oscillator. In particular, it has been shown that vibration control and quasi-periodic motion suppression are possible for appropriate choices of time delay and feedback gains. Elnaggar et al. (2016) considered the response of nonlinear controlled system with an external excitation using a delay state feedback. J. Cantisan et al., (2019) studied delay-induced resonancecs in time-delayed Duffing oscillators. They found the conjugate phenomenon, that is the oscillation caused by the time delay may be enhanced with the forcing without changing their frequencies

In this paper we cnsider the Kadomtsev-Petviashvili equation (Kadomtsev, Petviashvili, V., 1970; Petviashvili, V., Yan'kov, V., 1989),

$$u_{tx} + A u_{xxxx} + B u_{yy} + C (u^2)_{xx} = 0, \tag{1}$$

where $u=u(x,y,t)$ is a real function in 2+1 dimensions, A, B and C are the x-dissipative, y-dissipative and nonlinear coefficients, is a model for shallow long waves in the x-direction, with some not strong dispersion in the y-direction . If $B<0$, the equation is known as the KP-I equation (used when surface tension is strong) and if $B>0$ as the KP-II equation (surface tension is weak). The KP equation is integrable by the inverse scattering transform and moreover there are an infinite number of conserved quantities and a large family of quasiperiodic solutions with N independent phases. In particular, two-phase solutions describe shallow water waves with good accuracy. (Zhou, 1990; Ablowitz and Clarkson, 1991; Alexander et al.,1997), In the last years some effort has been devoted to the control of nonlinear evolution equations: for example has investigated the bifurcation control for the Zakharov-Kusnetsov equation in 2+1 dimensions (Maccari, 2010) and the Burgers-KdV equation in 1+1 dimensions (Maccari, 2008). We consider a forced KP equation,

$$u_{tx} + \tilde{A}u_{xxxx} + \tilde{B}u_{yy} + \tilde{C}u_{xx}^2 = 2\tilde{f}_0\cos(K_x x + K_y y + \omega t) \tag{2}$$

where $\tilde{f}_0$ is the external excitation coefficient, ω the (circular) frequency and K_x, K_y the wave numbers. We apply the transformation

$$u(x,y,t) = u(z), \quad z = K_x x + K_y y + \omega t, \tag{3}$$

and introduce two delay feedback control linear terms into equation (2),

$$\omega K_x u''(z) + \tilde{A}K_x^4 u''''(z) + \tilde{B}K_y^2 u''(z) + 2\tilde{C}K_x^2\left(u(z)u''(z) + \left(u'(z)\right)^2\right) = 2\tilde{f}_0 cos(z) + \tilde{E}_1 u''(z - z_0) + \tilde{E}_2 u'''(z - z_0) \tag{4}$$

where $\tilde{E}_1$ and $\tilde{E}_2$ are the control coefficients (feedback gains) and z_o the delay.

With two simple integration (and after setting to zero the integration constants), we obtain the nonlinear ordinary difference-differential equation,

$$u''(z) + \Omega^2 u(z) + Cu^2(z) = -2f_0\cos(z) + E_1 u(z - z_0) + E_2 u'(z - z_0), \tag{5a}$$

where

$$\Omega^2 = \frac{\omega K_x + \tilde{B}K_y^2}{\tilde{A}K_x^4},\ C = \frac{\tilde{C}}{\tilde{A}K_x^2}, \tag{5b}$$

$$f_0 = \frac{\tilde{f}_0}{\tilde{A}K_x^4},\ E_1 = \frac{\tilde{E}_1}{AK_x^4}, E_2 = \frac{\tilde{E}_2}{\tilde{A}K_x^4}. \tag{5c}$$

In order to apply a perturbation method, we assume weak nonlinearity, forcing term and feedback control and scale the coefficients,

$$(f_0, E_1, E_2) \to \varepsilon^2(f_0, E_1, E_2), \quad (C) \to \varepsilon(C), \tag{6}$$

where e is a small nondimensional parameter that is artificially introduced to serve as bookkeeping device and will be set equal to unity in the final analysis.

Equation (5) yields

$$u''(z) + \Omega^2 u(z) + \varepsilon C(u^2(z)) + \varepsilon^2(2f_0\cos(z) - E_1 u(z - z_0) - E_2 u'(z - z_0)) = 0, \tag{7}$$

that, in the following, will be studied from the viewpoint of bifurcation control, near the fundamental resonance (***Ω*** ≃*1*).

In the following, we will study the equation (5) from the viewpoint of bifurcation control, near the fundamental resonance ($\Omega \simeq 1$).

The paper is arranged as follows. In Sect. 2 we consider the fundamental resonance and use a perturbation method in order to obtain analytic approximate solutions. First of all, we introduce a slow time scale and obtain the solutions in terms of harmonic components. We can increase accurate solutions using higher orders of approximation in terms of the small parameter ε. However, for the first-order approximate solution, results are identical to those obtainable with the other perturbation methods. Obviously, there may be other solutions, for example large-amplitude quasi-periodic motion or chaotic behavior, which the slow flow equations do not describe.

After some calculations, we can get two slow-flow equations on the amplitude and the phase. Steady state solutions (corresponding to periodic motion) and their stability as well as external excitation-response and frequency-response curves are discussed. The feedback gains and the delay are chosen by analyzing the modulation equations of the amplitude and the phase. We find the occurrence of a saddle-centre bifurcation, jumps and hysteresis phenomena in the KP equation (7) under suitable conditions.

Moreover, it is demonstrated a frequency splitting occurrence and the existence of closed orbits that correspond to two-period modulated motions for the starting system (7). We show that that under appropriate conditions the motion is quasi-periodic, with two frequencies, because a second frequency adds to the forcing frequency and then stable doubly periodic motions are present with amplitudes depending on the initial conditions. The value of the second frequency depends on the amplitude of the external excitation.

In Sect.3 we apply a bifurcation control method. We demonstrate that, from the viewpoint of vibration control, a correct choice of the feedback gains and delay can enhance the control performance, delay or remove the saddle-center bifurcation in the forced KP equation, reduce the amplitude peak of the resonant response and suppress two-period quasi periodic behavior.

Finally, in the last section, we summarize the most important results and indicate some possible research developments.

2. Splitting frequency in the forced KP equation

In this section we study the forced KP equation (7) near the fundamental resonance we define a detuning parameter σ through the relation

$$\Omega = 1 + \varepsilon^2 \sigma. \tag{8}$$

taking into account the nearness between the natural frequency and the forcing one. We introduce the slow variable

$$\xi = \varepsilon^2 z, \tag{9a}$$

because we need to look on larger scales, if we want to get an adequate contribution by nonlinear and forcing terms.

Note that the variable change (9a) implies that differentiation with respect to the fast variable *z* must be substituted in the following way

$$\partial_z \to \varepsilon^2 \partial_\xi + \mathrm{inK}. \tag{9b}$$

The solution $u(z)$ of equation (7) can be expressed by means of a power series in the expansion parameter ε,

$$u(z) = \sum_{m=-\infty}^{+\infty} \varepsilon^{\gamma_m} \psi_m(\xi;\varepsilon) \exp(-\mathrm{imz}), \tag{10}$$

where $\gamma_n = |n|$ for $n \neq 0$, $\gamma_0 = 1$ and $\psi_m(\xi,\varepsilon) = c.c.\left(\psi_{-m}(\xi,\varepsilon)\right)$, where c.c. stands for complex conjugate.

Equation (10) can be written more explicitly

$$u(z) = \varepsilon\psi_0(\xi;\varepsilon) + (\psi_1(\xi;\varepsilon)\exp(-\mathrm{iz}) + \varepsilon\psi_2(\xi;\varepsilon)\exp(-2\mathrm{iz}) + c.c.) + h.o.t., \tag{11}$$

where *h.o.t.* = higher order terms and *c.c.* stands for complex conjugate of the preceding terms.

The functions $\psi_m(\xi,\varepsilon)$ depend on the parameter ε and we suppose that the limit for $\varepsilon \to 0$ exists and is finite and moreover they can be expanded in power series of ε, i.e.

$$\psi_m(\xi;\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \psi_m^{(i)}(\xi). \tag{12}$$

In the following for simplicity we use the abbreviations $\psi_n^{(0)} = \psi_n$ for $n = 0$ and $\psi_1^{(0)} = \psi$ for $n = 1$. In the lowest order calculations, only the functions corresponding to *i=0* appear.

We can state that the sought solution is a Fourier expansion in which the coefficients vary slowly in time and the lowest order terms correspond to the harmonic solution of the linear problem. Substituting the expression of the solution into the original equations and projecting onto each Fourier mode, we are able to fid the nonlinear evolution equations for the harmonic modes amplitudes.

For *m=1* we obtain the linear equation (order ε^2)

$$-2i\psi_\xi + 2\sigma\psi + 2C(\psi_0\psi + \psi_2\psi \;\;) + f_0 - (E_1 - \mathrm{i}E_2)e^{\mathrm{i}kz_0}\psi = 0, \tag{13}$$

Considering equation (7) for *m=2* and *m=0* yields (order ε)

$$\psi_0 = -2C|\psi|^2, \tag{14}$$

$$\psi_2 = \frac{C\psi^2}{3}. \tag{15}$$

Inserting the expressions (14) and (15) in (13), we can derive a differential equation for the evolution of the complex amplitude$y$,

$$\psi_\xi + (\alpha_1 + i\alpha_2)\psi + i\beta|\psi|^2\psi + i\gamma = 0, \tag{16}$$

where

$$\psi_\xi = \frac{d\psi}{d\xi}, \tag{17a}$$

$$\alpha_1 = -E\cos(z_0 + \phi), \quad \alpha_2 = \sigma - C_3\sin(\omega z_0 + \phi), \tag{17b}$$

$$\beta = -\frac{5C^2}{3}, \quad \gamma = \frac{f_0}{2}, \tag{17c}$$

$$E = \sqrt{\left(\frac{E_1}{2}\right)^2 + \left(\frac{E_2}{2}\right)^2}, \qquad \sin\phi = \frac{E_1}{2E}, \qquad \cos\phi = \frac{E_2}{2E}. \tag{17d}$$

Expressing the complex-valued function ψ into polar coordinates, we obtain

$$\Psi(\xi) = \rho(\xi)\exp\big(i\vartheta(\xi)\big), \tag{18}$$

and arrive at the model equations

$$\frac{d\rho}{d\xi} + \alpha_1\rho + \gamma\sin\vartheta = 0, \tag{19}$$

$$\rho\frac{d\vartheta}{d\xi} + \alpha_2\rho + \beta\rho^3 + \gamma\cos\vartheta = 0. \tag{20}$$

From equations (11), (14) and (15) we can express the field *u(z)* to the second approximation as

$$u(z) = 2\rho\cos(z - \vartheta) - 2C\rho^2\left(1 - \frac{\cos(2z - 2\vartheta)}{3}\right), \tag{21}$$

where ρ and J are given by Equations (19-20).

The validity of the approximate solution should be expected to be restricted on bounded intervals of the ξ-variable and on z-scale $z = O(1/\varepsilon^2)$. If one wishes to construct solutions on intervals such that $\xi = O(1/\varepsilon)$, then the higher terms must be included, because they will in general affect the solution.

Periodic solutions of the complete system described by equation (7) correspond to the fixed points of equations (19-20), which are obtained by the conditions $d\rho/d\xi = d\vartheta/d\xi = 0$.

The trivial solution is not possible, but steady-state finite-amplitude responses exist and the equilibrium points ρ_0, ϑ_0 are given by the external excitation –response curve

$$\alpha_2\rho_0 + \beta\rho_0^3 + \gamma\cos\vartheta_0 = 0 \tag{22}$$

and by

$$\sin\vartheta_0 = -\frac{\alpha_1}{\gamma}\rho_0. \tag{23}$$

In a similar way, we can derive the frequency-response equation

$$\sigma = E\sin\Phi + \left(\frac{5C^2}{3}\right)\rho_0^2 \pm \sqrt{\left(\frac{f_0}{2\rho_0}\right)^2 -}, \tag{24}$$

where $\Phi = \omega z_0 + \phi$.

In order to establish the stability of steady state solutions, we suppose small perturbations in the amplitudes and the phases on the steady state solutions and the resulting equations are then linearized. Subsequently we consider the eigenvalues of the corresponding system of first order differential equations with constant coefficients (the Jacobian matrix). A positive real root indicates an unstable solution, whereas if the real parts of the eigenvalues are all negative then the steady state solution is local stable.

The eigenvalue equation is

$$\lambda^2 + P\lambda + Q = 0. \tag{25}$$

where

$$P = -2\alpha_1, \quad Q = \alpha_1^2 + (\alpha_2 + \beta\rho_0^2)(\alpha_2 + 3\beta\rho_0^2). \tag{26}$$

and then the condition *P>0, Q>0* is requested for the stability of solutions (22-23).

Results of stability analysis are shown in Fig. 1 (external force-response curve) and Fig. 2 (frequency-response curve) for the uncontrolled system ($E_1=E_2=0$) where solid lines stand for stable solutions and dashed lines for unstable solutions. In all the figures saddle-center fold bifurcations are present. They correspond to a vertical tangency in the external excitation-response space, where the derivative of the response with respect to the control parameter is infinite. Moreover, a jump phenomenon is clearly observable. For increasing values of the external excitation, we observe a discontinue transition between two stable solutions. As the external excitation is gradually increased, the stable fixed point disappears. A jump from the lower branch to the upper branch (hysteresis) exists, because as the external excitation is decreased, the fixed point remains stable until a saddle-center bifurcation occurs and the system jumps to the lower branch.

We now perform a global analysis of the model system (19-20) and determine a condition for the existence of closed orbits. They correspond to two-period quasi-periodic solutions of the forced KP equation (7).

We consider the energy-like function,

$$H(\rho,\vartheta) = \rho\left(\alpha_2\rho + \frac{\beta}{2}\rho^3 + 2\gamma\cos\vartheta\right), \tag{27}$$

and suppose that there is a periodic solution *X(t)* of period *T*. After one cycle, ρ and J return to their starting values, and therefore ΔH=0 around any closed orbit.

On the other hand,

$$\frac{\mathrm{dH}}{d\xi} = \left(\alpha\rho + \frac{\beta\rho^3}{2} + 2\gamma\cos\vartheta\right)\frac{d\rho}{d\xi} + \left(\frac{3}{2}\beta\rho^2\frac{d\rho}{d\xi} + \alpha_2\frac{d\rho}{d\xi} - 2\gamma\sin\vartheta\frac{d\vartheta}{d\xi}\right)\rho \tag{28}$$

and with trivial manipulations

$$\frac{\mathrm{dH}}{d\xi} = 2\alpha_1\rho^2\frac{d\vartheta}{d\xi}. \tag{29}$$

We find that, around any closed orbit,

$$\Delta H = \int\frac{\mathrm{dH}}{d\xi}d\xi = 2\alpha_1\int\rho^2 d\vartheta, \tag{30}$$

where $\bar{T}$ is the period corresponding to the limit cycle in the (ρ,ϑ) plane. Note that ΔH is zero only if $\alpha_1 = 0$. As a consequence, $\alpha_1 = 0$ is a necessary condition for the existence of closed orbits and the energy-like function $H(\rho,J)$ is constant along the solution curves.

In order to determine the equilibrium points for the uncontrolled system, we observe that in the modulation Equations (19-20) $\alpha_1 = 0$ and $\alpha_2 = \sigma$ and there are only two independent parameters, because trough the rescaling

$$\rho \to L\rho,\ \ \xi \to T\xi,\ \ f_0 \to \frac{\mathrm{L}f_0}{T},\ \ L = \sqrt{\frac{|\alpha_2|}{|\beta|}},\ \ T = \frac{1}{|2\alpha|}, \tag{31}$$

we can always set $\alpha_2 = \pm 1$, $\beta = \pm 1$. We distinguish four cases:

i) α_2=*1*,β=*1*: there is only a center (elliptic equilibrium point), given by

$$(\rho_E,\vartheta_E) = \left(\sqrt[3]{\frac{F}{2} + \sqrt{\frac{F^2}{4} + \frac{1}{27}}} + \sqrt[3]{\frac{F}{2} - \sqrt{\frac{F^2}{4} + \frac{1}{27}}},\pi\right) \tag{32}$$

ii) α_2=*-1*, β=*1*: if $f_0 > \frac{4}{3\sqrt{3}}$ there is only a center, given by

$$(\rho_E,\vartheta_E) = \left(\sqrt[3]{\frac{F}{2} + \sqrt{\frac{F^2}{4} - \frac{1}{27}}} + \sqrt[3]{\frac{F}{2} - \sqrt{\frac{F^2}{4} - \frac{1}{27}}},\pi\right), \tag{33}$$

if $f_0 < \frac{4}{3\sqrt{3}}$, then we obtain three equilibrium points, given by

$$(\rho_{1E},\vartheta_{1E}) = \left(2\sqrt[3]{R}\cos\left(\frac{\pi-\phi}{3}\right),\pi\right)\ , \tag{34}$$

$$(\rho_{2E},\vartheta_{2E}) = \left(2\sqrt[3]{R}\cos\left(\frac{\phi}{3}\right),0\right), \tag{35}$$

$$(\rho_{3E},\vartheta_{3E}) = \left(2\sqrt[3]{R}\cos\left(\frac{\phi}{3} + \frac{4}{3}\pi\right),0\right)$$

where

$$\cos\phi = -\frac{3\sqrt{3}f_0}{4},\qquad \cos j = -\frac{3\sqrt{3}F}{2}, R = \frac{1}{3\sqrt{3}}. \tag{37}$$

The equilibrium point given by equation (35) is a saddle point, while the other two equilibrium points given by equations (34) and (36) are elliptic points. In this case, we observe jumps and hysteresis phenomena. If we consider the smaller elliptic point and increase the amplitude of the external force, suddenly in correspondence with the critical value $f_0 = \frac{4}{3\sqrt{3}}$, the solution jumps into the second elliptic point.

In Figs. 3-4 we show different solution curves in the plane (ρ, J): along each solution curve the energy-like function $H(\rho, J)$ of the equation (27) is constant. The case $F < \frac{2}{3\sqrt{3}}$ is represented in Fig. 3 and the case $f_0 > \frac{4}{3\sqrt{3}}$ in Fig. 4

iii) α_2=-1, β=-1: there is only an elliptic equilibrium point, given by equation (32) but with $\vartheta_E=\pi$.

iv) α_2=1, β=-1: if $f_0 > \frac{4}{3\sqrt{3}}$ there is only an ellipticϑ point, given by equation (33) but with $\vartheta_E=0$;

if $f_0 < \frac{4}{3\sqrt{3}}$ there are three equilibrium points, the first (saddle) and the second (elliptic) given by Equations (35) and (36) but with $\vartheta_E=\pi$, and the third, which is always elliptic, given by Equation (34) but with $\vartheta_E=0$.

The last two situations are connected by a simple ξ –inversion (and the transformation $J \to p + J$) to the first two cases.

The frequency of the small oscillations around the elliptic point in the plane (ρ, J) is

$$\Omega = \sqrt{\frac{f_0|\sigma - 5C^2\rho^2 E|}{2\rho_E}}. \tag{38}$$

The periodic motion around the elliptic point for the equations (19) and (20) corresponds to a two period quasi-periodic motion for the forced KP equation (7).

If we perform a linearization of equations (19) and (20) near the elliptic point, we can write explicitly the solution for the small oscillations

$$\rho(\xi) = \rho_E + (\rho_I - \rho_E)\cos(\Omega\xi) - \frac{f_0\cos\vartheta_E(\vartheta_I - \vartheta_E)}{2\Omega}\sin(\Omega\xi), \tag{39}$$

$$\vartheta(\xi) = \vartheta_E + (\vartheta_I - \vartheta_E)\cos(\Omega\xi) - \frac{(6\beta\rho^2 E + \sigma)(\rho_I - \rho_E)}{2\Omega\rho_E}\sin(\Omega\xi), \tag{40}$$

where ρ_I, J_I are connected to the initial conditions trough the following relations

$$u(0) = u_0 = -2C\rho_I^2 + 2\rho_I\cos\vartheta_I + \frac{2}{3}C\rho_I^2\cos 2\vartheta_I, \tag{41a}$$

$$\frac{\mathrm{du(0)}}{\mathrm{dz}} = u_0 = 2\rho_I\sin\vartheta_I + \frac{4}{3}C\rho_I^2\sin 2\vartheta_I. \tag{41b}$$

The resulting modulated motion is then characterized by a fundamental oscillation with frequency *1* slowly modulated in amplitude with frequency Ω. The solution is then characterized by two frequencies and we expect that in the Fourier spectrum the peak at the frequency of the excitation spreads out with a width Ω. Only if $\rho_I = \rho_E, J_I = J_E$, the solution is simply periodic and corresponds to the elliptic equilibrium point.

iii. Vibration control of the forced KP equation

If we want to modify or remove the saddle-center bifurcation, then we have to consider the bifurcation curves with the control terms and study three cases: (i) the delay feedback control with $E_1 \neq 0$, $E_2=0$, $\Phi = \frac{\pi}{2} + n\pi$, $z_0 = m\pi$, where *n, m=0,* ±1,..; (ii) the delay feedback control with $E_1=0$, $E_2 \neq 0$, $\Phi = n\pi$, $z_0 = m\pi$, where *n, m=0,* ±1,.. and (iii) the generic delay feedback control with E_1, E_2 different from zero and where Φ and the delay z_0 are different from the previous ones.

(iii) the delay feedback control with$E_1 \neq 0$, *E2=0.* We observe that the control term can shift the frequency of the occurrence of the saddle-center bifurcation, because of the curve modification, but the shape of the curve and the maximum value in the response and jump phenomena remain unchanged (Fig. 5).

(ii) the delay feedback control with $E_1=0$, $E_2 \neq 0$. The amplitude of the hysteresis cycle can be modified and for appropriate values of the control term the saddle-center bifurcation can be eliminated. The amplitude peak of the response is reduced and a successful strategy of response control can be accomplished (Fig. 6).

(iii) the generic delay feedback control with E_1, E_2 different from zero. In this case, we observe both the curve shift and the modification of the hysteresis curve (Fig. 7). Well-suited choices for the feedback gains can accomplish a good control strategy for the saddle-center bifurcation of the forced KP equation.

Checking carefully equations (26) and (30) we an conclude that if we want to eliminate the two-period quasi-periodic motion the feedback gains must be chosen in such a way that

$$\alpha_1 < 0, \qquad \cos\Phi > 0, \qquad \alpha_1^2 + (\alpha_2 + \beta\rho_0^2)(\alpha_2 + 3\beta\rho_0^2) > 0, \tag{42}$$

Centers disappear and are substituted with stable nodes or spirals.

The response is strongly influenced by the delay (recall that $z_0 = \Phi - \phi$) and if we consider equation (22) and set $d\rho/\mathrm{d}z_0 = 0$, we find for the time delay z_0 the condition

$$(\sigma + \beta\rho_0^2)\cos\Phi = 0, \tag{43}$$

$$z_0 = \frac{\pi}{2} - \phi + 2m\pi(\text{minimum}), \;\; z_0 = \frac{3\pi}{2} - \phi + 2m\pi \;(\text{maximum}), \quad m=0, \pm 1, \ldots \tag{44}$$

In conclusion, the optimal choices for the time delay and the feedback gains suppresses the two-period quasi-periodic motion, eliminates the frequency splitting and reduces the peak amplitude.

The strategy control can be summarized in the following two steps:

(i) if the system response is simply periodic, the response amplitude can be reduced and the saddle-center bifurcation removed by correct choices of the feedback gains as illustrated in the above-mentioned case iii). Moreover, the response amplitude can be reduced by the choice (44) for the time delay, that is not dependent on the feedback gains;

(ii) if a two-period quasi-periodic motion exists, the only way to avoid it and suppress the frequency splitting is to choose the feedback gains and the time delay in such a way as to satisfy condition (42).

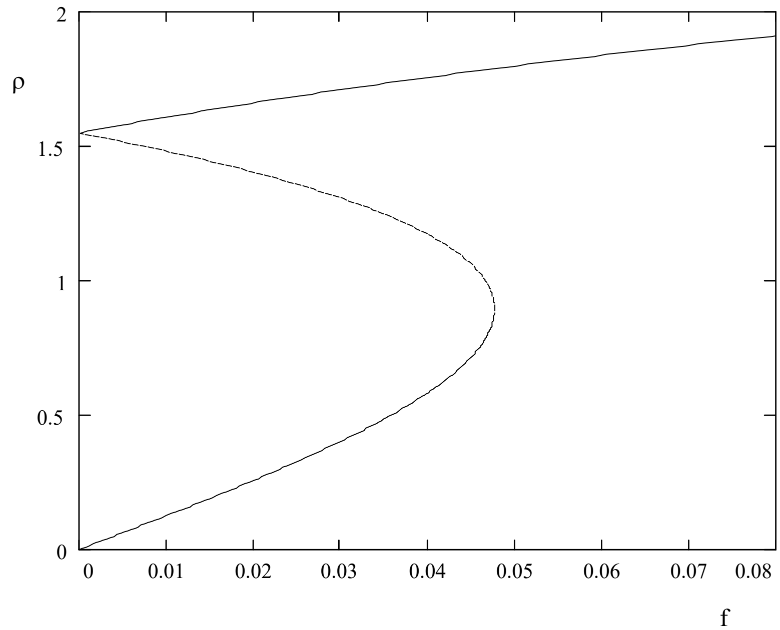

FIGURE 1: External excitation (f)-response (ρ) curve. Solid lines stand for stable solutions, dashed lines for unstable solutions. ($A=B=1,\ C=0.1,\ \sigma=0.04$)

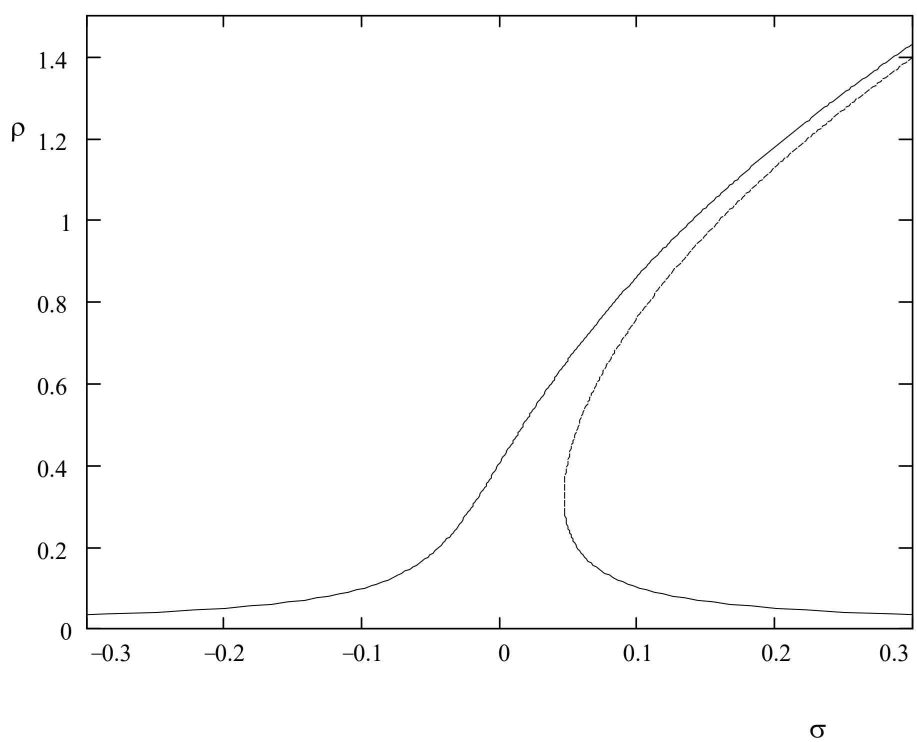

FIGURE 2: Frequency (σ)-response (ρ) curve. Solid lines stand for stable solutions, dashed lines for unstable solutions. ($A=B=1,\ C=0.3,\ f_0=0.02$)

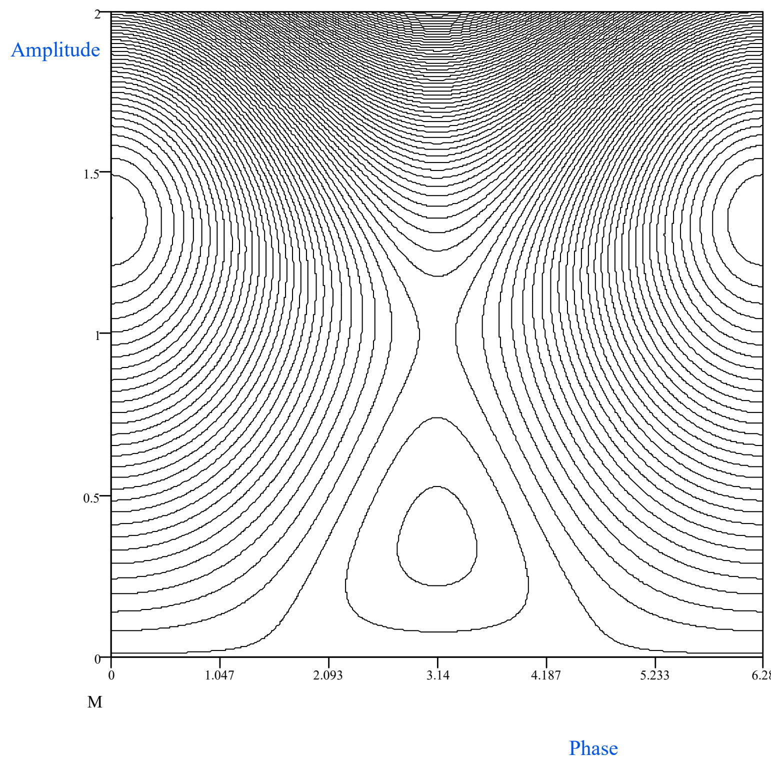

FIGURE 3: Phase space representation of solutions for the system of equations (19-20) in the case ii), $f_0 < 4/3\sqrt{3}$. Note the presence of two elliptic points and one saddle point.

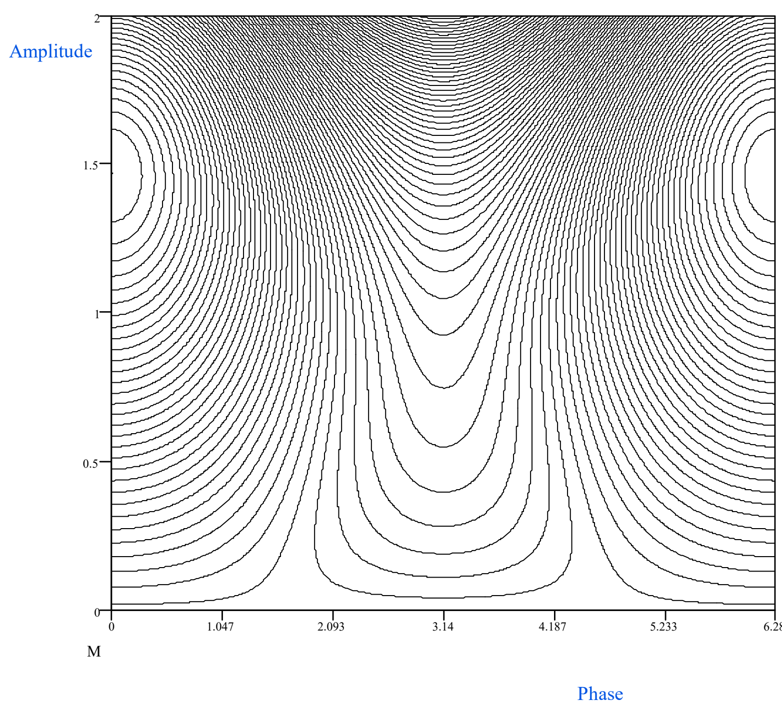

FIGURE 4: Phase space representation of solutions for the system of equations (19-20) in the case ii), $f_0 > 4/3\sqrt{3}$. The smaller elliptic point and the saddle point have disappeared.

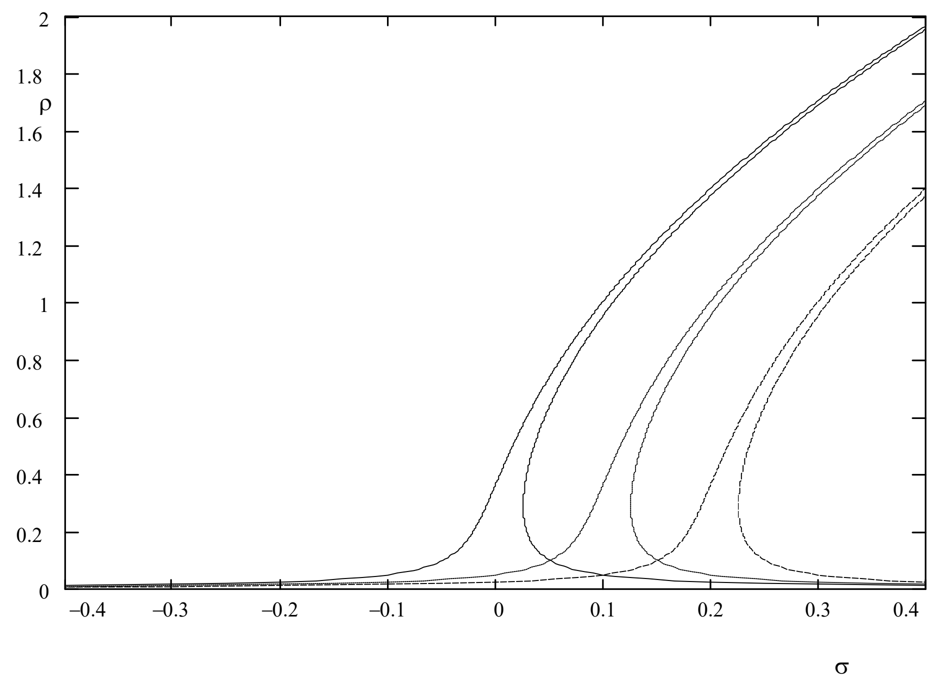

FIGURE 5: Frequency (σ)-response(ρ) curves for the uncontrolled system (solid line) and the linear controlled system ($E_1=0.2$, $E_2=0$-dot line and $E_1=0.4$, $E_2=0$-dash line). ($A=B=1$, $C=0.25$, $f_0=0.01$)

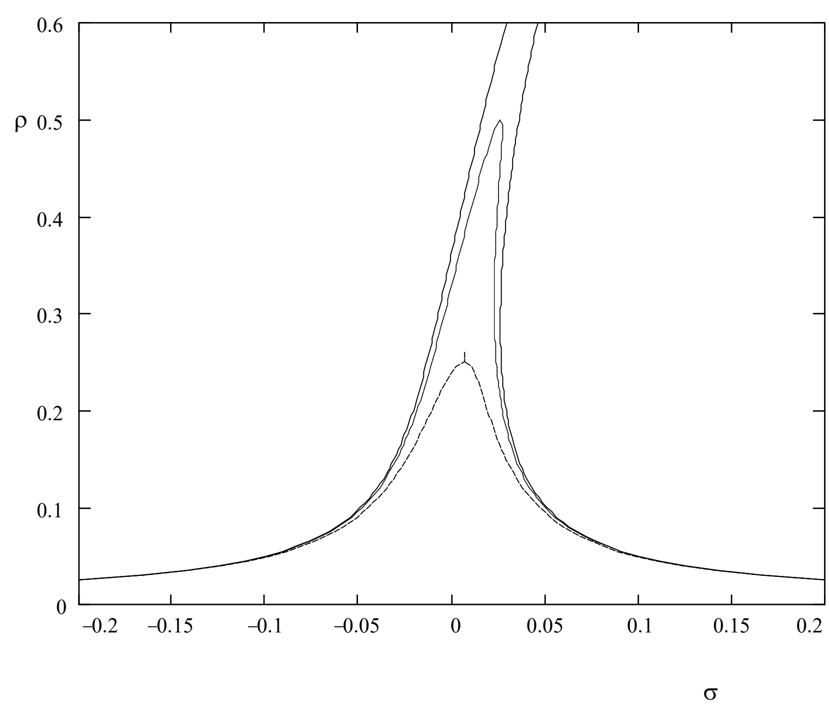

FIGURE 6: Frequency (σ)-response(ρ) curves for the uncontrolled system (solid line) and the nonlinear controlled system ($A=B=1$, $C=0.25$, $f_0=0.01$).

(E_1*=0,* E_2*=0.02*-dot line and E_1*=0,* E_2*=0.04*-dash line)

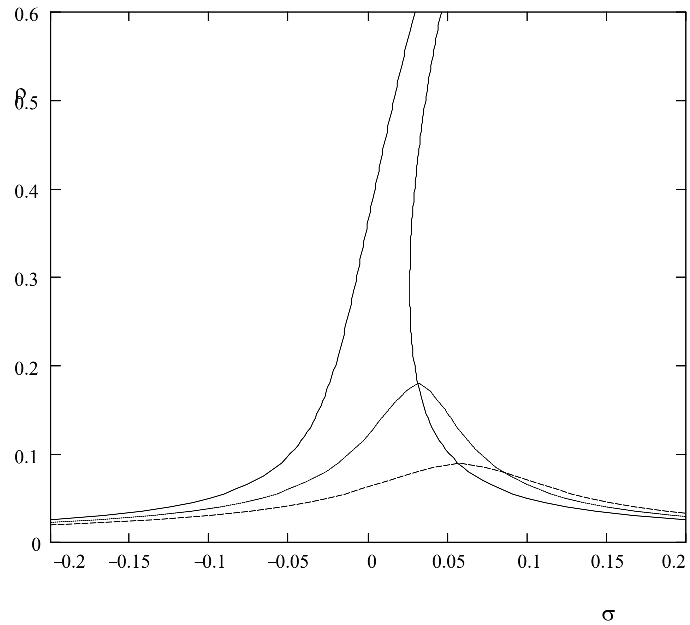

FIGURE 7: Frequency (σ)-response(ρ) curves for the uncontrolled system (solid line) and the linear and nonlinear controlled system (*A=B=1, C=0.25,* f_0*=0.01*).
(E_1*=*E_2*=0.057*-dot line and E_1*=*E_2*=0.113*-dash line)

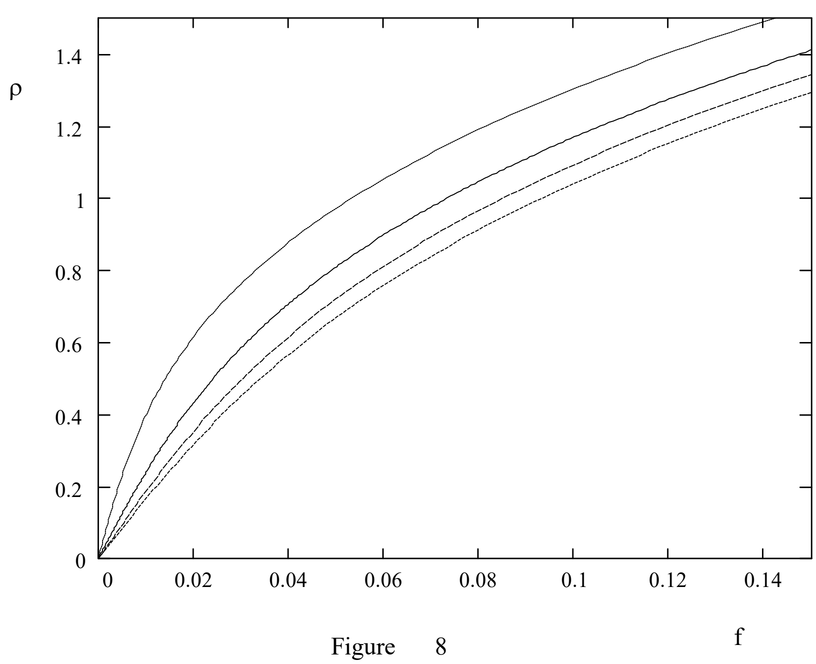

FIGURE 8: External excitation(f)-response(ρ) curve for the uncontrolled system (curve *A,* solid line), the controlled system without time delay (z_0*=0*, curve *B,* dash line), and those with time delays corresponding to the minimum ($z_0=\pi/3$, curve *C,* dash-dot line) and the maximum ($z_0=4\pi/3$, curve *D,* dot line) value for the response. (*A=B=1, C=0.1,* σ*=-0.02,* f_0*=0.01,* E_1*=0.01,* E_2*=0.017,* $\varphi = \pi/6$)

4. Conclusion

We considered the forced KP equation with delay feedback linear control terms nd found an approximate solution with a perturbation method. Deriving a system of nonlinear model equations describing the modulation of the amplitude and of the phase of the oscillation, we were able to understand the dependence of the equilibrium points (periodic solutions of the original system) on the external force and the existence of bifurcations and stable periodic solutions for appropriate parameters values. A saddle-center bifurcation and jump phenomena (discontinuous transitions between two solutions) characterize the uncontrolled system. Using energy considerations we find a frequency splitting triggered by the external excitation and the existence and characteristics of closed orbits of the slow flow equations which correspond to a two-period quasi-periodic modulated motion for the forced KP equation.

A correct choice of the feedback gains and time delay can successfully accomplish a vibration control, shift or remove the saddle-center bifurcation, suppress the quasi-periodic motion and reduce the amplitude peak of the resonant response.

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