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On a partially simple ribbon fusion of links

by

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Abstract

In recent papers [2, 3], Tsukamoto and the authors defined a transformation of links, called a simple ribbon fusion. In this paper, we define another transformation called a partially simple ribbon fusion and study its several properties as well as the difference between the two transformations. By definition, a simple ribbon fusion consists of finitely many elementary simple ribbon fusions. We investigate the relation between a partially simple ribbon fusion and an elementary simple ribbon fusion.

keywords; Simple ribbon fusion

1 Introduction.

All links are assumed to be ordered and oriented, and they will be considered up to ambient isotopy in the oriented 3-sphere S^3.

In [2, 3], Tsukamoto and the authors define a transformation called a simple ribbon fusion, which is a generalization of a simple ribbon move (cf. [4]), and study its several properties. A link L is called the link which can be obtained from a link ℓ by a *simple ribbon fusion* if there are disjoint unions of non-singular disks $\mathcal{D} = \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^m$ and bands $\mathcal{B} = \mathcal{B}^1 \cup \cdots \cup \mathcal{B}^m$ such that $L = (\ell \cup \partial(\mathcal{D} \cup \mathcal{B})) - \mathrm{int}(\mathcal{B} \cap \ell)$ and that they satisfy the following, where $\mathcal{D}^k = D_1^k \cup \cdots \cup D_{m_k}^k$ and $\mathcal{B}^k = B_1^k \cup \cdots \cup B_{m_k}^k$.

(1) $\ell \cap \mathcal{D} = \emptyset$.

(2) For each k and i, $B_i^k \cap \ell = \partial B_i^k \cap \ell = \{\text{a single arc}\}$ and $B_i^k \cap \partial\mathcal{D} = \partial B_i^k \cap \partial D_i^k = \{\text{a single arc}\}$.

(3) For each k and i, $B_i^k \cap \mathrm{int}\,\mathcal{D} = B_i^k \cap \mathrm{int}\,D_{i+1}^k = \mathcal{B} \cap \mathrm{int}\,D_{i+1}^k = \{\text{an arc of ribbon type}\}$, where we consider the lower index modulo m_k.

When $m = 1$, we call the simple ribbon fusion an *elementary simple ribbon fusion* [2].

In this paper, we introduce another transformation called a partially simple ribbon fusion and investigate the difference of an elementary simple ribbon fusion and a partially simple ribbon fusion. We also study some properties of a partially simple ribbon fusion. A link L is called the link which can be obtained from a link ℓ by a *partially simple ribbon fusion* if there are disjoint unions of non-singular disks $\mathcal{D} = \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^m$ and bands $\mathcal{B} = \mathcal{B}^1 \cup \cdots \cup \mathcal{B}^m$ such that $L = (\ell \cup \partial(\mathcal{D} \cup \mathcal{B})) - \mathrm{int}(\mathcal{B} \cap \ell)$ and that they satisfy the following, where $\mathcal{D}^k = D_1^k \cup \cdots \cup D_{m_k}^k$ and $\mathcal{B}^k = B_1^k \cup \cdots \cup B_{m_k}^k$.

(1) The link $L_k = (\ell \cup \partial(\mathcal{D}^k \cup \mathcal{B}^k)) - \mathrm{int}(\mathcal{B}^k \cap \ell)$ can be obtained from ℓ by a simple ribbon fusion with respect to $\mathcal{D}^k \cup \mathcal{B}^k$ for each k.

(2) $\mathcal{B}^k \cap \mathcal{D}^l = \emptyset$ for each k, l $(1 \leq k < l \leq m)$.

We note that if the condition (2) is replaced with the condition that $\mathcal{B}^k \cap \mathcal{D}^l = \emptyset$ for each k, l $(k \neq l)$, then L is obtained from ℓ by a simple ribbon fusion. Hence if L can be obtained from ℓ by a simple ribbon fusion, then L can be obtained from ℓ by a partially simple ribbon fusion. However, we show that the converse does not hold.

Theorem 1. *There is a pair of links ℓ and L such that L can be obtained from ℓ by a partially simple ribbon fusion but L can not be obtained from ℓ by a simple ribbon fusion.*

We reveal a relation between a partially simple ribbon fusion and an elementary simple ribbon fusion as follows.

Theorem 2. *A link L can be obtained from a link ℓ by a partially simple ribbon fusion if and only if there is a sequence $L_0(=\ell), L_1, \ldots, L_m(=L)$ of links such that L_k can be obtained from L_{k-1} by an elementary simple ribbon fusion for $k = 1, \ldots, m$.*

In [1], Goldberg introduced the *disconnectivity number* of a link L, denoted by $\nu(L)$, which is the maximal number of connected components of all the Seifert surfaces for L. For each integer r $(1 \leq r \leq \nu(L))$, the *r-th genus* of L, denoted by $g_r(L)$, is the minimal number of genera of all the Seifert surfaces for L with r connected components.

As an extension of Theorem 1.1 in [2], Theorem 2 implies the following.

Corollary 3. *Let L be a link obtained from a link ℓ by a partially simple ribbon fusion. Then we have that $\nu(L) \leq \nu(\ell)$ and that $g_r(L) \geq g_r(\ell)$ for each integer r $(1 \leq r \leq \nu(L))$. Moreover, if $\nu(L) = \nu(\ell)(= p)$ and $g_p(L) = g_p(\ell)$, then L is ambient isotopic to ℓ.*

2 Proof of Theorems.

Let L be a link obtained from a link ℓ by a simple ribbon fusion with respect to $\mathcal{D} = \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^m$ and $\mathcal{B} = \mathcal{B}^1 \cup \cdots \cup \mathcal{B}^m$. We say that $D_i^k \cup B_i^k$ $(\subset \mathcal{D} \cup \mathcal{B})$ is *trivial*, if there is a non-singular disk Δ_i^k with $\partial \Delta_i^k = \partial D_i^k$ such that $\mathrm{int}\, \Delta_i^k \cap (L \cup \mathcal{B}) = \emptyset$. A simple ribbon fusion is said to be *irreducible* if $D_i^k \cup B_i^k$ is not trivial for any i, k.

Lemma 4. *Let L be a non-prime and non-split link. If L is obtained from a link ℓ by a simple ribbon fusion with respect to $\mathcal{D} \cup \mathcal{B}$, then there is no non-trivial decomposition sphere Σ of L with $\Sigma \cap \ell = \emptyset$.*

Proof. By definition, if $D_i^k \cup B_i^k$ $(\subset \mathcal{D} \cup \mathcal{B})$ is trivial for some k and i, then L is ambient isotopic to the link $(L - \partial(\mathcal{D}^k \cup \mathcal{B}^k)) \cup (\mathcal{B}^k \cap \ell)$. This implies that L can be obtained from ℓ by a simple ribbon fusion with respect to $(\mathcal{D} - \mathcal{D}^k) \cup (\mathcal{B} - \mathcal{B}^k)$. Thus we may assume that a simple ribbon fusion is irreducible.

Suppose that there is a non-trivial decomposition sphere Σ of L with $\Sigma \cap \ell = \emptyset$. Since $\Sigma \cap \ell = \emptyset$, we can deform Σ by isotopy so that $\Sigma \cap \mathcal{B} = \emptyset$. Then there is a disk D_i^k of $\mathcal{D}$ such that $\Sigma \cap L = \Sigma \cap (\partial D_i^k - \partial B_i^k)$ which consists of two points. Therefore $\Gamma(= \Sigma \cap \mathcal{D})$ consists of a simple arc, say γ, proper on D_i^k and some simple loops, where we note that $\gamma \cap \mathcal{B} = \emptyset$.

Suppose that Γ contains a simple loop c. Let $D_i^k(c)$ be the disk on D_i^k with $\partial D_i^k(c) = c$. First we consider the case where $D_i^k(c)$ does not contain $\alpha_i^k = \mathrm{int}\, D_i^k \cap \mathcal{B}$. Then we obtain two 2-spheres one of which is a non-trivial decomposition sphere Σ' of L with $\Sigma' \cap \ell = \emptyset$ by attaching $D_i^k(c)$ to Σ, namely we replace a neighborhood of c on Σ with two parallel copies of $D_i^k(c)$. By applying the above transformation at an innermost loop on $D_i^k(c)$ in turn as illustrated in Figure 1, we can take a non-trivial decomposition sphere, denoted by Σ again, of L with $\Sigma \cap \ell = \emptyset$ such that Γ does not contain such a loop c.

Next we consider the case where $D_i^k(c)$ contains α_i^k. We may assume that c is innermost on Σ with respect to γ, namely for the disk, denoted by Σ_c on Σ bounded by c, $\mathrm{int}\, \Sigma_c \cap \mathcal{D} = \emptyset$.

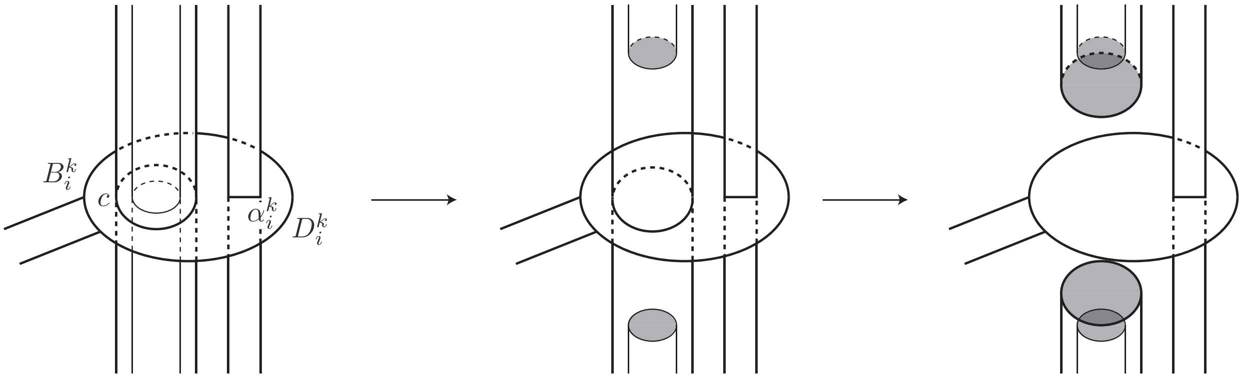

Figure 1:

Then $E = (D_i^k - D_i^k(c)) \cup \Sigma_c$ is a non-singular disk such that $\operatorname{int} E \cap (L \cup \mathcal{B}) = \emptyset$ and thus $D_i^k \cup B_i^k$ is trivial, which contradicts to the irreducibility of the simple ribbon fusion. Hence we obtain that $\Gamma = \gamma$.

Since γ is proper on D_i^k and $\Sigma \cap \mathcal{B} = \emptyset$, we have $D_i^k - \gamma$ consists of two disks, say D_{i0}^k and D_{i1}^k, where $\partial D_{i1}^k \cap \partial B_i^k \neq \emptyset$. First we consider the case where α_i^k is contained in D_{i1}^k. Then Σ decomposes L into two links such that one of which contains ∂D_{i0}^k as a component. This contradicts to that L is non-split or that Σ is a non-trivial decomposition sphere of L.

Next we consider the case where α_i^k is contained in D_{i0}^k. We consider a simple loop κ intersecting each α_i^k at a point on $\mathcal{D}^k \cup \mathcal{B}^k$, which is one component of an attendant link with respect to $\mathcal{D} \cup \mathcal{B}$ (see, [2, 3]). Since $\Sigma \cap (\mathcal{B} \cup \mathcal{D}) = \gamma$, we have that $\Sigma \cap \kappa = \gamma \cap \kappa$ which is a point. However, since κ is a loop, $\Sigma \cap \kappa$ consists of even points, which is a contradiction. □

Proof of Theorem 1. Let L be the link as illustrated in Figure 2. Then L can be obtained from the split link ℓ consisting of the trivial knot and the right-handed trefoil knot by a partially simple ribbon fusion with respect to $(B_1 \cup B_2 \cup B_3) \cup (D_1 \cup D_2 \cup D_3)$. We denote by K_1 and K_2 the components of L, and by $K_1 \circ K_2$ the split link consisting of K_1 and K_2.

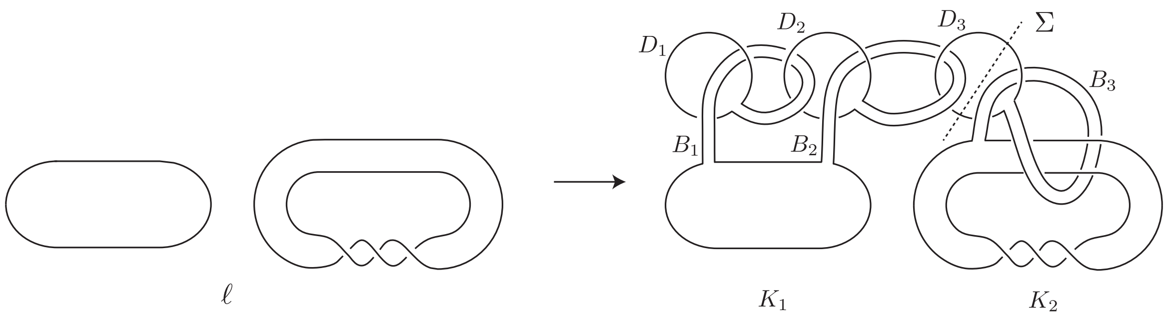

Figure 2:

First we show that L is non-split. We have that $\operatorname{span} V(L) = 18$ and $\operatorname{span} V(K_1 \circ K_2) = 16$, where $\operatorname{span} V(X)$ is the difference between the maximum degree and the minimum degree of the

Jones polynomial of X. This implies that L is not ambient isotopic to $K_1 \circ K_2$, namely L is non-split.

Next we show that L is non-prime. Let Σ be the decomposition sphere of L which satisfies that $\Sigma \cap \ell = \emptyset$ as illustrated in Figure 2. Since span $V(K_1) = 6$ and span $V(K_2) = 9$, namely K_1 and K_2 are non-trivial, L is non-prime and thus Σ is non-trivial. Hence L can not be obtained from ℓ by a simple ribbon fusion by Lemma 4. $\square$

To prove Theorem 2, we give the following lemma.

Lemma 5. [2, Lemma 4.7] *Let L be a link obtained from a link ℓ by a simple ribbon fusion. Then there is a sequence $L_0(=\ell), L_1, \ldots, L_m(=L)$ of links such that L_k can be obtained from L_{k-1} by an elementary simple ribbon fusion for $k = 1, \ldots, m$.*

Proof of Theorem 2. Since a partially simple ribbon fusion consists of finitely many simple ribbon fusions, we obtain the necessity by Lemma 5.

Conversely, suppose that there is a sequence $L_0(=\ell), L_1, \ldots, L_m(=L)$ of links such that L_k can be obtained from L_{k-1} by an elementary simple ribbon fusion with respect to $\mathcal{D}^k \cup \mathcal{B}^k$ for $k = 1, \ldots, m$. Let $\mathcal{D} = \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^m$ and $\mathcal{B} = \mathcal{B}^1 \cup \cdots \cup \mathcal{B}^m$. To prove that $L(= L_m)$ can be obtained from $\ell(= L_0)$ by a partially simple ribbon fusion, it is sufficient to do that we can deform $\mathcal{D} \cup \mathcal{B}$ by isotopy so that it satisfies the following claims.

(1) For each k and i, $B_i^k \cap \ell = \partial B_i^k \cap \ell = \{\text{a single arc}\}$.

(2) $\mathcal{B}$ is a disjoint union of bands.

(3) For each k, $(\mathcal{B}^1 \cup \cdots \cup \mathcal{B}^{k-1}) \cap \mathcal{D}^k = \emptyset$.

(4) $\mathcal{D}$ is a disjoint union of disks.

(1) Suppose that $B_i^k \cap \ell = \emptyset$ and $B_q^p \cap \ell = \partial B_q^p \cap \ell = \{\text{a single arc}\}$ for each $p < k$ and q. We deform B_i^k along $\partial((\mathcal{B}^1 \cup \cdots \cup \mathcal{B}^{k-1}) \cup (\mathcal{D}^1 \cup \cdots \cup \mathcal{D}^{k-1}))$ by isotopy so that $B_i^k \cap \ell = \partial B_i^k \cap \ell = \{\text{a single arc}\}$ as illustrated in Figure 3. By repeating the deformation, we obtain that $B_i^k \cap \ell = \partial B_i^k \cap \ell = \{\text{a single arc}\}$ for each k and i.

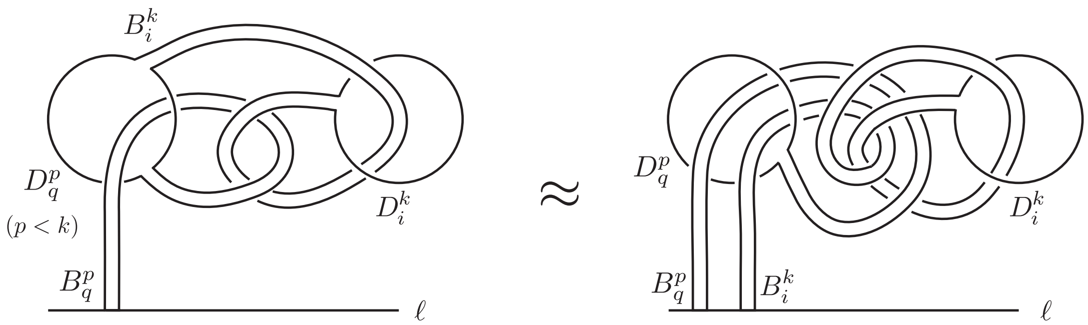

Figure 3:

(2) Suppose that $\mathcal{B}^p \cap \mathcal{B}^k \neq \emptyset$ for $p < k$. By thinning $\mathcal{B}^k$ enough, we may assume that $\mathcal{B}^p \cap \mathcal{B}^k$ consists of arcs in $\operatorname{int} \mathcal{B}^p$. There are two bands B_q^p of $\mathcal{B}^p$ and B_i^k of $\mathcal{B}^k$ such that $B_i^k \cap B_q^p \neq \emptyset$. We deform B_i^k along B_q^p by isotopy so that $B_i^k \cap B_q^p = \emptyset$ as illustrated in Figure 4. By repeating the deformation, we obtain that $\mathcal{B}$ is a disjoint union of bands.

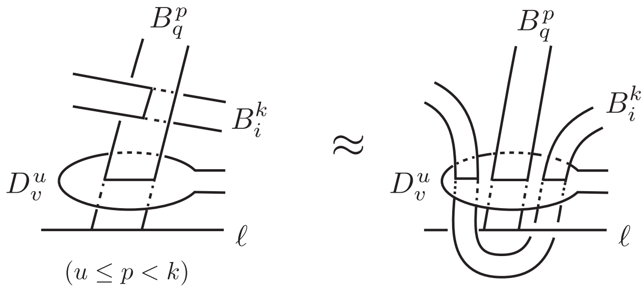

Figure 4:

(3) Suppose that $\mathcal{B}^p \cap \mathcal{D}^k \neq \emptyset$ for $p < k$. Then there is a band B_q^p of $\mathcal{B}^p$ such that $B_q^p \cap \mathcal{D}^k \neq \emptyset$. Since $L_{k-1} \cap \mathcal{D}^k = \emptyset$, we may assume that $B_q^p \cap \mathcal{D}^k$ consists of arcs in B_q^p each of which connects ∂D_q^p and ℓ, where we note that $\#((D_q^p \cap \mathcal{D}^k) \cap \alpha_{j-1}) = \#(B_q^p \cap \mathcal{D}^k)$. On the other hand, since any loop of $D_q^p \cap \mathcal{D}^k$ bounds a disk in $\mathcal{D}^k$, there is no loop γ of $D_q^p \cap \mathcal{D}^k$ with $\operatorname{lk}(\gamma, \alpha_q^p) = \pm 1$. Then there exists an arc of $D_q^p \cap \mathcal{D}^k$ such that its subarc bounds a disk δ on D_q^p with a proper subarc of α_q^p as illustrated in Figure 5. Then we may assume that $\delta \cap (D_q^p \cap \mathcal{D}^k) = \emptyset$.

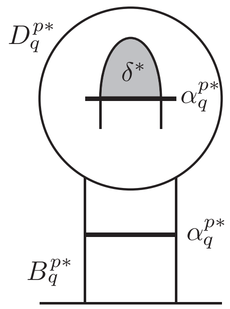

Figure 5: Pre-images of $\mathcal{D}^k \cap D_q^p$ and δ

If $\delta \cap (D_q^p \cap \mathcal{B}^k) \neq \emptyset$, that is, there exists an arc β of $D_q^p \cap \mathcal{B}^k$ which is contained in δ, then we deform $\mathcal{D}^k \cup \mathcal{B}^k$ along δ by isotopy as illustrated in Figure 6. We note that if $\delta \cap (D_q^p \cap \mathcal{B}^k) = \emptyset$, then we deform $\mathcal{D}^k$ only. By repeating the deformation, we obtain that $(\mathcal{B}^1 \cup \cdots \cup \mathcal{B}^{k-1}) \cap \mathcal{D}^k = \emptyset$ for each k.

(4) Suppose that $(\mathcal{D}^1 \cup \cdots \cup \mathcal{D}^{k-1}) \cap \mathcal{D}^k \neq \emptyset$ for some k. Since $\mathcal{D}^k \cap L_{k-1} = \emptyset$ and $(\mathcal{B}^1 \cup \cdots \cup \mathcal{B}^{k-1}) \cap \mathcal{D}^k = \emptyset$, we have that $(\mathcal{D}^1 \cup \cdots \cup \mathcal{D}^{k-1}) \cap \mathcal{D}^k$ consists of a disjoint union of simple loops. Let γ be a loop of $(\mathcal{D}^1 \cup \cdots \cup \mathcal{D}^{k-1}) \cap \mathcal{D}^k$ which is innermost on $\mathcal{D}^k$ and δ the disk on $\mathcal{D}^k$ with $\partial\delta = \gamma$. Let σ be a disk on D_q^p of $\mathcal{D}^p$ with $\partial\sigma = \gamma$ for $p < k$. Since γ is innermost on $\mathcal{D}^k$, we have that $\operatorname{int} \delta \cap \mathcal{D}^p = \emptyset$. Let $\gamma^+ = \partial N(\gamma : D_q^p - \sigma) - \gamma$ and δ^+ a disk parallel to δ

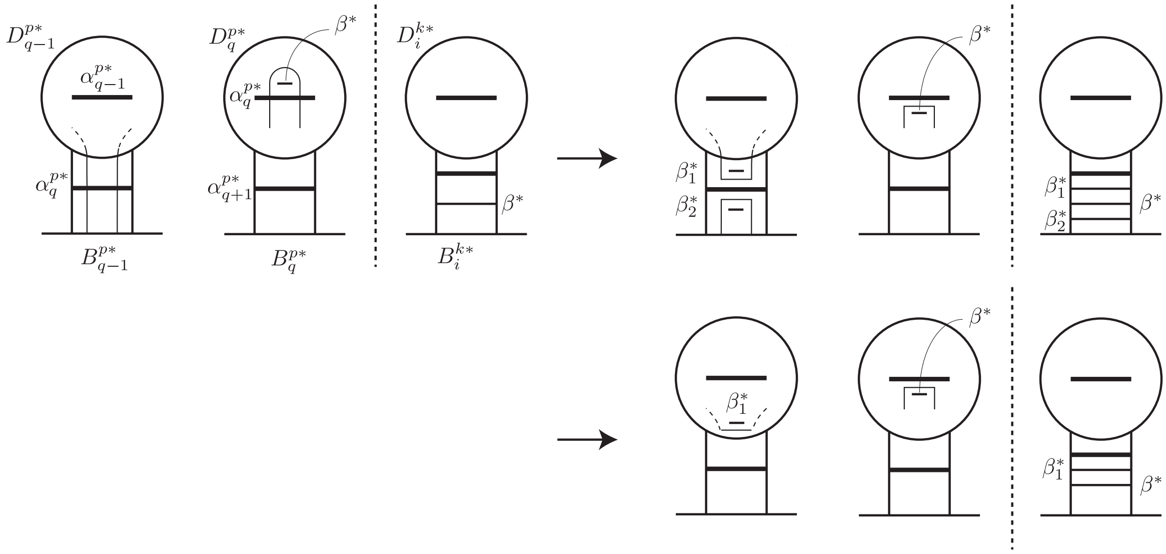

Figure 6:

with $\partial\delta^+ = \gamma^+$. We deform D^p_q into $D^{p+}_q = (D^p_q - N(\sigma : D^p_q)) \cup \delta^+$ by isotopy as illustrated in Figure 7. By repeating the deformation, we obtain that $\mathcal{D}$ is a disjoint union of disks.

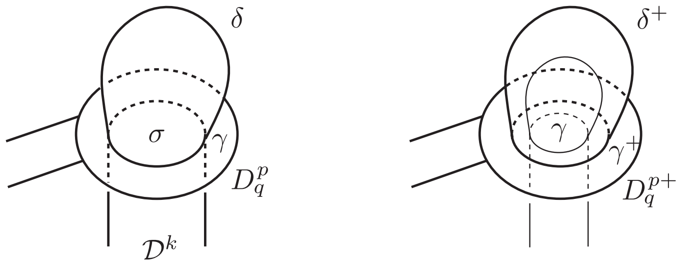

Figure 7:

Therefore we obtain the sufficiency. □

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