12-1-2006

Linear Conditions Imposed on Flag Varieties

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ABSTRACT. We study subvarieties of the flag variety called Hessenberg varieties, defined by certain linear conditions. These subvarieties arise naturally in applications including geometric representation theory, number theory, and numerical analysis. We describe completely the homology of Hessenberg varieties over $\mathrm{GL}_n(\mathbb{C})$ and show that they have no odd-dimensional homology. We provide an explicit geometric construction which partitions each Hessenberg variety into pieces homeomorphic to affine space. We characterize these affine pieces by fillings of Young tableaux and show that the dimension of the affine piece can be computed by combinatorial rules generalizing the Eulerian numbers. We give an equivalent formulation of this result in terms of roots. We conclude with a section on open questions.

1. Introduction

The full flag variety over $\mathrm{GL}_n(\mathbb{C})$ is the collection of nested complex vector spaces $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n$ where V_i is i-dimensional. Given a linear operator X on $\mathbb{C}^n$, the set of flags that are stabilized by X—that is, flags $V_1 \subsetneq \cdots \subsetneq V_n$ such that $XV_i \subseteq V_i$ for each i—is an important subvariety of the full flag variety called the Springer-Grothendieck fiber. Geometric representation theorists use this subvariety to construct the irreducible representations of the symmetric group ([CG, section 3.6] has background and references).

More generally, fix any nondecreasing function $h : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ such that $h(i) \geq i$ for each i, and consider the flags

$$\mathcal{H}(X, h) = \{\text{flags } V_1 \subseteq \cdots \subseteq V_n \text{ such that } XV_i \subseteq V_{h(i)} \text{ for each } i\}.$$

The subvariety $\mathcal{H}(X, h)$ is called a Hessenberg variety, and the map h is a Hessenberg function.

For example, consider the set of flags with $XV_i \subseteq V_{i+1}$ whenever i is less than n. This parametrizes the bases that put the operator X into Hessenberg form, a form used in a common algorithm to compute eigenvalues (see [dMS] for more about the QR algorithm). The natural generalization presented here was defined in [dMPS].

Our main theorem explicitly partitions each Hessenberg variety into affine spaces satisfying weak closure rules. This decomposition is a paving and is the intersection of $\mathcal{H}(X, h)$ with a special Bruhat decomposition of the flag variety. Pavings give the homology of $\mathcal{H}(X, h)$, and hence a combinatorial description of its Betti numbers. We conclude that Hessenberg varieties have no odd-dimensional homology.

For notational convenience, we give the main result here in the case when $X = N$ is nilpotent. Theorems 6.1 and 7.1 have the result for general X in two different forms. If the nilpotent operator N has Jordan blocks of size $d_1 \geq \ldots \geq d_k$, then associate to it the Young diagram λ_N with row lengths $d_1 \geq \ldots \geq d_k$. Our Young

2000 *Mathematics Subject Classification.* Primary 14M15, 14F25, 05E10.

diagrams are left-aligned and top-aligned. For example, Figure 1 shows a nilpotent with Jordan blocks of dimension 3 and 1 and the associated Young diagram.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow$$

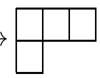


FIGURE 1. The Young diagram corresponding to one nilpotent operator

The cells of the paving are indexed by Young tableaux that are filled with the numbers from 1 to n without repetition. Each tableau defines a permutation w of n letters for which $w^{-1}(k)$ is the number of boxes to the left of or below the box filled by k (including the box itself).

Theorem 1.1. *Fix a nilpotent N. The Hessenberg variety $\mathcal{H}(N,h)$ is paved by affines. Each nonempty cell corresponds to a unique filling of λ_N in which $\boxed{k \mid j}$ occurs only if $k \leq h(j)$. This correspondance is a bijection. The dimension of a nonempty cell is the number of pairs i,k such that*

1. *i is below or anywhere to the left of k (see Figure 2),*
2. *$k < i$, and*
3. *if there is a box immediately to the right of k that is filled by j then $i \leq h(j)$.*

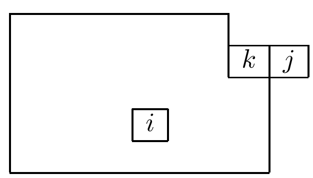


FIGURE 2. Configuration of triples

This result extends N. Spaltenstein's description of the Springer fibers' components, the case when $h(i) = i$ [Sp]. In particular, it can be used to give a new proof that the rank of each irreducible representation of the symmetric group is the number of standard fillings of its Young tableau. It also partially extends the work of F. de Mari, C. Procesi, and M. Shayman paving Hessenberg varieties by affines when X is regular semisimple [dMPS], and of C. de Concini, G. Lusztig, and C. Procesi paving Springer fibers by affines [dCLP]. Our methods are different from theirs though similar in spirit to Spaltenstein's or to those in [KnM]. B. Kostant used a different Bruhat decomposition to pave one Hessenberg variety when X is regular nilpotent, giving a geometric construction of the quantum cohomology of the flag variety [K]. According to personal communications [C] and announcements [BC, Theorem 3], D. Peterson has other uncirculated results studying Hessenberg varieties when X is regular nilpotent. Our methods do not use torus actions, as there is no obvious torus action for general X. Rather than using one-dimensional deformations as in [V] or restricting to intersections with codimension-one Schubert varieties as in [So], our approach makes fewer deformations of higher dimension in each Schubert cell.

Our proof begins by describing $\mathcal{H}(X,h)$ in terms of matrices g for which $g^{-1}Xg$ is zero in fixed coordinates, and then reducing to the case when $g=u$ is upper-triangular. The entries of the matrices $u^{-1}Xu$ need not be linear nor affine functions of the entries of u. However, the entries of the i^{th} row of $u^{-1}Xu$ are affine functions of the i^{th} row of u. For instance, when X is nilpotent with a single Jordan block its conjugate by an upper-triangular u is

$$u^{-1}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} u = \begin{pmatrix} 0 & 1 & u_{23}-u_{12} & u_{24}-u_{12}(u_{34}-u_{23})-u_{13} \\ 0 & 0 & 1 & u_{34}-u_{23} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Section 2 has the necessary background on the Bruhat decomposition and pavings. In Section 3 we partition the upper-triangular matrices into subgroups called *rows* and show that conjugation by a row is an affine transformation of the row's entries.

In our example, the functions of u_{1j} in the first row have the same rank regardless of the other u_{ij}. This is true if X is in *highest form*, defined for any linear operator in Section 4. Section 5 has the paper's key lemma. That lemma is one step in the main theorem of Section 6, which proves that each cell of a Bruhat decomposition intersects $\mathcal{H}(X,H)$ in an iterated tower of affine fiber bundles. The main theorem is described using tableaux in Section 7 and using roots in Section 8. Section 9 has open questions and conjectures about Hessenberg varieties, including whether they are pure dimensional and how many components they have.

This work was partially supported by an NDSE graduate fellowship and was part of the author's doctoral dissertation. I thank Emina Alibegović, Jared Anderson, Henry Cohn, William Fulton, Gil Kalai, David Kazhdan, Robert Lazarsfeld, David Nadler, Arun Ram, Eric Sommers, and the anonymous referee for valuable comments. I am especially grateful to my advisor, Robert MacPherson.

2. Pavings and the Bruhat decomposition

In this section we describe a classical partition of the flag variety called the Bruhat decomposition. We also precisely define pavings, the special partitions of a variety used in this paper, sometimes called cellular decompositions.

Definition 2.1. A paving of the variety $\mathcal{X}$ is an ordered partition $\mathcal{X}=\coprod_{i=0}^{\infty}\mathcal{X}_i$ so that each finite union $\coprod_{i=0}^{j}\mathcal{X}_i$ is Zariski-closed in $\mathcal{X}$. If in addition each $\mathcal{X}_i$ is homeomorphic to affine space $\mathbb{R}^{d_i}$ then $\coprod_{i=0}^{\infty}\mathcal{X}_i$ is a paving by affines.

Our pavings have a finite number of pieces. We call the $\mathcal{X}_i$ cells. Figure 3 shows three spheres glued successively at a point like a string of beads. It is paved by four affine cells: the marked point and each S^2 without its leftmost point. The closure of a cell need not cover the cells it intersects, as it must in a CW-decomposition.

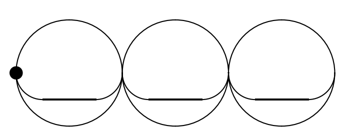


Figure 3. A Space Paved by Four Cells

Pavings by affines determine Betti numbers [F, 19.1.11]:

Proposition 2.2. *Let $\mathcal{X} = \coprod \mathcal{X}_i$ be a paving by a finite number of affines $\mathcal{X}_i$ with each $\mathcal{X}_i$ homeomorphic to $\mathbb{C}^{d_i}$. Then the nonzero cohomology groups of $\mathcal{X}$ are*

$$H^k(\mathcal{X}) = \bigoplus_{i \text{ such that } 2d_i = k} \mathbb{Z}.$$

The full flag variety has a well-known paving by affines called the Bruhat decomposition. Recall that the flag $V_1 \subseteq \cdots \subseteq V_n$ is determined by any matrix g whose first i column vectors generate the i^{th} vector space V_i. The flag corresponding to g is denoted $[g]$.

The next definition parametrizes the cells of this paving [H, section 28.4].

Definition 2.3. Let w be a permutation matrix. The group U_w of upper-triangular matrices associated to w is defined as $U_w = \{u : u \in U, w^{-1}uw \text{ is lower-triangular}\}$.

We now state a classical result in the language of this paper. Write e_i for the basis vector of $\mathbb{C}^n$ which has one in the i^{th} position and zero otherwise. The permutation matrix w corresponds to the permutation of $\{1, 2, \ldots, n\}$ given by $e_i w = e_{w(i)}$.

Proposition 2.4. *The flag variety is paved by affines $\coprod_{w \in S_n} C_w$. The Schubert cell C_w is the set of flags $[U_w w]$, which is homeomorphic to $U_w w$ and has dimension $|\{(i,j) : 1 \leq i < j \leq n, w(i) > w(j)\}|$.*

Proof. The Schubert cells are described in [H, section 28.3]. The U_w parametrize the cells by [H, section 28.4]. The cells form a paving by [BL, section 2.10]. □

The matrix description of the flag variety gives a different formulation of the definition of Hessenberg varieties.

Definition 2.5. The Hessenberg space H associated to h is the linear subspace of matrices X whose $(i,j)^{th}$ entry $X_{ij} = 0$ if $i > h(j)$.

Section 8 has an intrinsic definition of Hessenberg spaces from [dMPS]. The next proposition relates the linear subspace H to the function h. Its proof is immediate from $w^{-1}E_{jk}w = E_{w(j),w(k)}$, where E_{jk} is the matrix basis unit with 1 in its (j,k) entry and zero everywhere else.

Proposition 2.6. *The matrix basis unit $E_{jk} \in wHw^{-1}$ if and only if $w(j) \leq h(w(k))$.*

An alternate definition of Hessenberg varieties first given in [dMPS] is

$$\mathcal{H}(X, H) = \{\text{flags } [g] : g^{-1}Xg \in H\} = \mathcal{H}(X, h).$$

Conjugation by $g \in GL_n(\mathbb{C})$ is a homeomorphism of Hessenberg varieties in two ways.

Proposition 2.7. *Fix X and H and $g_0 \in GL_n(\mathbb{C})$. The Hessenberg variety $\mathcal{H}(g_0^{-1}Xg_0, H)$ is homeomorphic to $\mathcal{H}(X, H)$.*

Proof. Using associativity gives $\mathcal{H}(g_0^{-1}Xg_0, H) = g_0^{-1}\mathcal{H}(X, H)$. Multiplication is an automorphism of flags so this is homeomorphic to $\mathcal{H}(X, H)$. □

Proposition 2.8. *Fix a matrix X, a Hessenberg space H, and $g_0 \in GL_n(\mathbb{C})$. The Hessenberg variety $\mathcal{H}(g_0^{-1}Xg_0, g_0^{-1}Hg_0)$ is homeomorphic to $\mathcal{H}(X, H)$.*

Proof. By definition, $\mathcal{H}(g_0^{-1}Xg_0, g_0^{-1}Hg_0) = \{\text{flags } [g_0^{-1}gg_0] : g^{-1}Xg \in H\}$. Conjugation is an automorphism of flags so this is homeomorphic to $\mathcal{H}(X, H)$. □

These show that the topology and geometry of an arbitrary Hessenberg variety $\mathcal{H}(X,H)$ are the same as when X, H, and the underlying basis are in fixed relative position. In what follows, we assume that X and H are in fixed conjugacy classes without further comment.

3. Rows of upper-triangular matrices

This section describes a decomposition of the upper-triangular invertible matrices into subgroups called rows and shows how rows act on arbitrary matrices. A similar partition is used implicitly in [Ste, section 2.C] and in [CP, section 3].

Unless otherwise stated all matrices are $n \times n$ with complex coefficients. We use X to denote an arbitrary matrix, N to denote a nilpotent upper-triangular matrix, and S to denote a diagonal matrix. Write U for the group of upper-triangular matrices with ones on the diagonal. Let X_{jk} be the $(j,k)^{th}$ entry of the matrix X.

Definition 3.1. The i^{th} row U_i is the subgroup $U_i = \{u \in U : u_{jk} = 0 \text{ if } j \neq i, k\}$.

We distinguish the rows U_i from the Schubert cell subgroups U_w by subscripts: i, j, k always denote an integer, while w always denotes a permutation matrix. Note that $U_i \cap U_j$ is the identity if $i \neq j$. The rows generate all of U because each row is a product of one-parameter subgroups, as in [H, Proposition 28.1].

Proposition 3.2. *The group U factors uniquely as $U = U_{n-1}U_{n-2}\cdots U_1$.*

This result together with Proposition 2.4 shows that representatives for each Schubert cell factor uniquely as $(U_w \cap U_{n-1})(U_w \cap U_{n-2})\cdots(U_w \cap U_1)w$.

We use rows because of their group structure, given next. Its proof is immediate.

Proposition 3.3. *U_i is naturally isomorphic to the additive group $\mathbb{C}^{n-i}$. If u and v are elements of U_i then $(uv)_{ik} = u_{ik} + v_{ik}$ for each $k > i$. In particular, the entries of the inverse u^{-1} are given by $\left(u^{-1}\right)_{ik} = -u_{ik}$ for each $k > i$.*

The group U_i acts on a matrix X by left-multiplication, right-multiplication, or conjugation. In each case most of the rows of X are preserved, as the following makes precise.

Proposition 3.4. *Fix u in U_i.*

1. *$(uX)_{jk} = X_{jk}$ except possibly when $j = i$.*
2. *$(Xu)_{jk} = X_{jk}$ except possibly in rows j for which X_{ji} is nonzero.*
3. *If X is upper-triangular then $\left(u^{-1}Xu\right)_{jk} = X_{jk}$ except possibly when $j \leq i$.*

Proof. The first two parts restate matrix multiplication.

Since X is upper triangular the product $(Xu)_{jk} = X_{jk}$ except perhaps in a row j with $j \leq i$ by Part 2. By Part 1 the product $\left(u^{-1}Xu\right)_{jk} = (Xu)_{jk}$ except perhaps when $j = i$. Thus $\left(u^{-1}Xu\right)_{jk} = X_{jk}$ whenever $j > i$. □

Denote the i^{th} row vector of X by $X_{i\bullet}$. Let $X = S + N$ be upper-triangular. The next result shows that the i^{th} row of $u^{-1}Xu$ is the image under an affine transformation of the i^{th} row of u, namely the translation of a linear map on $u_{i\bullet}$.

Proposition 3.5. *The map $u_{i\bullet} \mapsto u^{-1}(S+N)u_{i_\bullet}$ is an affine transformation of the entries of $u_{i\bullet}$. Explicitly,*

$$\left(u^{-1}(S+N)u\right)_{i\bullet} = S_{ii}u_{i\bullet} + \left(u^{-1}\right)_{i\bullet}(S+N).$$

Proof. We prove this by comparing the k^{th} entry of each vector. Note that

$$\begin{aligned}\left(u^{-1}(S+N)u\right)_{ik} &= \sum_{j=1}^{n} \left(u^{-1}\right)_{ij} \left((S+N)u\right)_{jk} \\ &= \sum_{j=1}^{n} \left(u^{-1}\right)_{ij} (S_{ji}+N_{ji})u_{ik} + \sum_{j=1}^{n} \left(u^{-1}\right)_{ij} (S_{jk}+N_{jk})u_{kk}.\end{aligned}$$

The first sum simplifies to $(u^{-1})_{ii}(S_{ii}+N_{ii})u_{ik}$ because if $i>j$ then $(u^{-1})_{ij}=0$ and if $i<j$ both S_{ji} and N_{ji} vanish. Since $N_{ii}=0$ and $(u^{-1})_{ii}=1$ this is $S_{ii}u_{ik}$.

The second sum is the k^{th} entry of $(u^{-1})_{i\bullet}(S+N)$ by definition. $\square$

4. Highest forms of linear operators

This section introduces one of the main tools of our proof: the *highest form* for linear operators. We first define the highest form of a nilpotent matrix and then reduce the general case to a sum of nilpotents. We begin with some linear algebra.

Definition 4.1. Fix a matrix X. The entry X_{ik} is a pivot of X if X_{ik} is nonzero and if all entries below and to its left vanish, that is $X_{ij}=0$ if $j<k$ and $X_{jk}=0$ if $j>i$.

Given i, define r_i to be the row of $X_{r_i,i}$ if the entry is a pivot and zero if not.

Definition 4.2. Fix an upper-triangular nilpotent matrix N. Then N is in highest form if the pivots form a nondecreasing sequence, namely $r_1 \leq r_2 \leq \cdots \leq r_n$.

By definition $r_i = r_j$ only if both are zero, so only initial columns of a matrix in highest form can be zero. Columns with pivots are linearly independent, so when N is in highest form its first dim $(\ker N)$ columns are zero.

To construct a highest form for N fill the Young diagram λ_N constructed in the Introduction with 1 to n starting at the bottom of the leftmost column, incrementing by one while moving up, then moving to the lowest box of the next column and repeating. The highest form for N is the matrix with $N_{ij}=1$ if i fills the box to the left of j and $N_{ij}=0$ otherwise, as in Figure 4.

$$\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \longleftrightarrow \begin{array}{|c|c|c|} \hline 3&5&6\\ \hline 2&4 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \longleftrightarrow \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}$$

Figure 4. Jordan canonical form, the Young diagram, and the highest form

The main property of the highest form is that conjugation by U preserves it.

Proposition 4.3. *If N is nilpotent and in highest form and $u \in U$ then $u^{-1}Nu$ is in highest form. The entry $N_{r_j,j}$ is a pivot if and only if $(u^{-1}Nu)_{r_j,j}$ is. If so, $N_{r_j,j} = (u^{-1}Nu)_{r_j,j}$.*

Proof. The entry $(Nu)_{jk}$ is the sum of N_{jk} and multiples of $N_{j1}, \ldots, N_{jk-1}$. This means $(Nu)_{jk} = N_{jk}$ for each column up to and including the first nonzero column in the j^{th} row of N. Similarly $(u^{-1}Nu)_{jk} = (Nu)_{jk}$ for each row after and including the last nonzero row in the k^{th} column of Nu. So the pivots of $u^{-1}Nu$ are in the same entries with the same values as in Nu, which are in the same entries with the same values as in N. □

We now describe highest form for an arbitrary upper-triangular matrix $S+N$, where S is diagonal and N is nilpotent. If c is an eigenvalue of S then let E_c be its eigenspace. Recall that S induces a decomposition of the total vector space $\mathbb{C}^n = \bigoplus_{\text{eigenvalues } c \text{ of } S} E_c$.

Inclusion and then projection gives a map from the semigroup $\mathrm{End}(\mathbb{C}^n)$ to $\mathrm{End}(E_c)$. For instance, the image of $S+N$ under this map is the composition

$$E_c \hookrightarrow \mathbb{C}^n \xrightarrow{S+N} \mathbb{C}^n \longrightarrow \left(\mathbb{C}^n / \bigoplus_{c' \neq c} E_{c'} \right) \cong E_c.$$

The matrix for $(S+N)_c$ is given by the dim $E_c \times$ dim E_c minor of $S+N$ obtained by removing the j^{th} row and j^{th} column if $S_{jj} \neq c$. This is shown in Figure 5.

$$S+N = \begin{pmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad \mapsto \qquad (S+N)_1 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

FIGURE 5. An example of $S+N$ and $(S+N)_1$

Note that N_c is the strictly upper-triangular part of $(S+N)_c$.

Definition 4.4. $S+N$ is in highest form if the following hold:

(1) $S+N$ is upper triangular;
(2) if $S_{ii} = S_{jj}$ then $S_{ii} = S_{kk}$ for each k between i and j; and
(3) N_c is in highest form for each eigenvalue c of S.

The diagonal blocks of a matrix in highest form are in highest form. However, highest form matrices need not be block diagonal in general. Condition 2 is designed so the map $Y \mapsto Y_c$ is a morphism of semigroups, as in the next lemma. Again e_i is the standard basis vector in $\mathbb{C}^n$.

Lemma 4.5. *$(XY)_c = X_cY_c$ for all upper-triangular matrices X and Y if and only if there are i and j so that E_c is the span of the basis vectors $e_i, e_{i+1}, e_{i+2}, \ldots, e_{i+j}$.*

Proof. The coefficient of e_k in $XYe_{i'}$ is $\sum_{j=k}^{i'} x_{kj}y_{ji'}$. If E_c satisfies the hypothesis then for each e_k spanning E_c the entries x_{kj} and $y_{ji'}$ are in X_c and Y_c respectively as long as j is between k and i'. Consequently $(XY)_c = X_cY_c$.

Conversely, suppose e_i, e_{i+k}, and e_{i+j} are vectors with $0 < k < j$ and with e_i, e_{i+j} in E_c while e_{i+k} is not. If X is a matrix nonzero only in entry $X_{i,i+j}$ and Y is nonzero only in entry $Y_{i+j,i+k}$ then $X_c = Y_c = 0$ but $(XY)_c$ is nonzero. □

To construct a matrix in highest form which is conjugate to $S+N$, write $S+N$ in Jordan canonical form $\sum(S_i+N_i)$ with blocks S_i+N_i corresponding to distinct eigenvalues c_i. If N_i' is highest form for N_i then the matrix $\sum(S_i+N_i')$ is in highest form, called the permuted Jordan form of $S+N$.

The next proof extends Proposition 4.3 to general linear operators.

Proposition 4.6. *If $S+N$ is in highest form and u is in U then $u^{-1}(S+N)u$ is in highest form. The (r_k,k) entry of N_c is a pivot if and only if the (r_k,k) entry of $(u^{-1}Nu)_c$ is a pivot. In this case the two entries are equal.*

Proof. Note that $u^{-1}(S+N)u$ is upper-triangular if $S+N$ is. Direct computation shows $u^{-1}(S+N)u = S+(u^{-1}Nu+u^{-1}Su-S) = S+N'$ for some nilpotent N'.

Fix an eigenvalue c of S. By Lemma 4.5 we know $(u^{-1}Nu)_c = u_c^{-1}N_cu_c$. Proposition 4.3 applies since N_c is in highest form and u_c is upper-triangular with ones on the diagonal. $\square$

5. Paving Hessenberg varieties by affines

In this section we prove that if X is in highest form, each row of each Schubert cell is in $\mathcal{H}(X,h)$ if and only if certain affine conditions hold. This is the key step in the paper.

Recall that $X_{i\bullet}$ is the i^{th} row of X, that $X_{\bullet j}$ is the j^{th} column, and H is the Hessenberg space given by h in Definition 2.5. The next lemma identifies $\{u \in U_i : \left(u^{-1}Nu\right)_{i\bullet} \in \left(wHw^{-1}\right)_{i\bullet}\} \cap U_w$ as the solution to an affine system of equations and finds its rank.

Lemma 5.1. *Fix a permutation w, a row U_i, a Hessenberg space H, and N in highest form. If the pivots of N are in nonzero entries of wHw^{-1} then the set $\{u \in U_i : \left(u^{-1}Nu\right)_{i\bullet} \in \left(wHw^{-1}\right)_{i\bullet}\} \cap U_w$ is homeomorphic to $\mathbb{C}^d$ for*

$$d = |\{k : k > i, w(i) > w(k), h(w(j)) \geq w(i) \text{ if } N_{kj} \text{ is a pivot in } N\}|.$$

The inequality $h(w(j)) \geq w(i)$ does not apply if the k^{th} row of N has no pivot.

Proof. The i^{th} row of $u^{-1}Nu$ is $(u^{-1})_{i\bullet}N$ by Proposition 3.5. Examining the condition $(u^{-1})_{i\bullet}N \in \left(wHw^{-1}\right)_{i\bullet}$ for each column gives the system of equations

$$(u^{-1})_{i\bullet}N_{\bullet j} = 0 \qquad \text{for } j \text{ such that } w(i) > h(w(j)).$$

Each equation in this system has the form

$$(1, -u_{i,i+1}, \ldots, -u_{i,n}) \cdot (N_{i,j}, \ldots, N_{n,j})^t = 0$$

for j satisfying $w(i) > h(w(j))$. Adding the constraint that $u \in U_w$ gives the following affine system of equations in the free entries u_{ik}:

$$(5.1) \qquad (u_{ik_1}, u_{ik_2}, \ldots, u_{ik_{d_i}}) \begin{pmatrix} N_{k_1 j} \\ N_{k_2 j} \\ \vdots \\ N_{k_{d_i} j} \end{pmatrix} = N_{ij} \qquad \begin{array}{l} \text{for } j \text{ with } w(i) > h(w(j)) \\ \text{and } k_l \text{ with } w(i) > w(k_l). \end{array}$$

The linear system of equations $\mathbf{x}M = \mathbf{v}$ has a solution if and only if the rank of the coefficient matrix M equals that of the extended matrix $\binom{\mathbf{v}}{M}$. To prove this here, we show that if either N_{ij} or one of the $N_{k_l j}$ is nonzero then in fact one of the $N_{k_l j}$ is a pivot in N.

Indeed, if N_{ij} or $N_{k_l j}$ is nonzero then N has a pivot N_{kj} in some row $k \geq i$. The pivots of N are in wHw^{-1} by hypothesis. This means that $w(k) \leq h(w(j))$ by

Proposition 2.6. In addition $w(i) > h(w(j))$ by hypothesis on j. Hence $w(i) > w(k)$ and so N_{kj} is one of the entries of the column vector of Equation (5.1).

The dimension of the solution space is the number of free entries in $U_i \cap U_w$ less the number of pivots of N in the coefficient matrix of Equation (5.1). The set $\{k : k > i, w(i) > w(k)\}$ indexes the free entries while $\{k : k > i, w(i) > w(k), N_{kj}$ is a pivot and $w(i) > h(w(j))\}$ indexes the rank of the coefficient matrix. This proves the claim. $\square$

This extends to general linear operators in much the same way.

Lemma 5.2. *Fix a permutation w, a row U_i, a Hessenberg space H, and $S+N$ in highest form. If the pivots of each submatrix N_c are in wHw^{-1} then the set $\{u \in U_i : \left(u^{-1}(S+N)u\right)_{i\bullet} \in \left(wHw^{-1}\right)_{i\bullet}\} \cap U_w$ is homeomorphic to $\mathbb{C}^d$ for*

$$\begin{aligned} d &= |\{k : k > i, w(i) > w(k), \\ &\qquad h(w(j)) \geq w(i) \text{ if } N_{kj} \text{ is a pivot in } N_{S_{ii}}, S_{kk} = S_{ii}\}| \\ &+ |\{k : k > i, h(w(k)) \geq w(i) > w(k), S_{kk} \neq S_{ii}\}|. \end{aligned}$$

Proof. The i^{th} row of $u^{-1}(S+N)u$ is $S_{ii}u_{i\bullet} + (u^{-1})_{i\bullet}(S+N)$ by Proposition 3.5. The condition that this be in $\left(wHw^{-1}\right)_{i\bullet}$ gives the system of equations

$$S_{ii}u_{ij} + (u^{-1})_{i\bullet}(S+N)_{\bullet j} = 0 \qquad \text{for } j \text{ such that } w(i) > h(w(j)).$$

Each equation in this system is of the form

$$S_{ii}u_{ij} + (1, -u_{i,i+1}, \cdots, -u_{i,n}) \cdot (N_{ij}, \cdots, N_{j-1,j}, S_{jj}, 0, \ldots, 0)^t = 0$$

for j such that $w(i) > h(w(j))$. Adding the condition that $u \in U_w$ gives the system

$$(u_{ik_1}, u_{ik_2}, \ldots, u_{ik_{d_i}}) \begin{pmatrix} N_{k_1 j} \\ N_{k_2 j} \\ \vdots \\ S_{jj} - S_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = N_{ij} \qquad \begin{array}{l} \text{for } j \text{ such that } w(i) > h(w(j)) \\ \text{and } k_l \text{ such that } w(i) > w(k_l). \end{array}$$

As in the previous lemma, we show that the rank of the coefficient matrix is unchanged if the vector of solutions (N_{ij}) is inserted as the top row.

We study the cases when $S_{ii} = S_{jj}$ and when $S_{ii} \neq S_{jj}$ separately. Let c_i be the cardinality $|\{S_{jj} : j > i, S_{jj} = S_{ii}\}|$ so $S_{jj} - S_{ii}$ is zero exactly when j is at most $i + c_i$. The columns with $j > i + c_i$ have a pivot in position (j, j) regardless of N_{ij}. For each such j we know $w(i) > w(j)$ since $h(w(j)) \geq w(j)$.

The first c_i columns and rows of this system satisfy $S_{jj} - S_{ii} = 0$ and so form the system of Equation (5.1). Its pivots are computed in Lemma 5.1. Each is a pivot in the original system because the $(k_l, j)^{th}$ entry is zero when k_l is greater than j.

The rank of the entire matrix is therefore

$$\begin{aligned} &|\{k : k > i, w(i) > w(k), w(i) > h(w(j)) \text{ and } N_{kj} \text{ is a pivot in } N_{S_{ii}}, S_{ii} = S_{jj}\}| \\ &\quad + |\{k : k > i, w(i) > w(k), w(i) > h(w(k)), S_{ii} \neq S_{kk}\}|. \end{aligned}$$

Since the dimension of $U_i \cap U_w$ is $|\{k : k > i, w(i) > w(k)\}|$ the claim follows. $\square$

6. The Main Theorems

We now demonstrate that requiring each row of a flag in $\mathcal{H}(X,h)$ to satisfy the Hessenberg conditions gives the structure of an iterated tower of affine fiber bundles on each Bruhat cell in $\mathcal{H}(X,h)$. This constructs a paving by affines on the Hessenberg variety. We use the Hessenberg space H determined by h as in Definition 2.5, as well as the description of the Schubert cells in Proposition 2.4.

Theorem 6.1. *Fix a Hessenberg space H and a basis for which $S+N$ is in highest form and in permuted Jordan form. Let $\{C_w\}$ be the Schubert cells.*

The intersections $C_w \cap \mathcal{H}(S+N,H)$ form a paving by affines of $\mathcal{H}(S+N,H)$. The cell $C_w \cap \mathcal{H}(S+N,H)$ is nonempty if and only if N is in wHw^{-1}. If nonempty, the cell $C_w \cap \mathcal{H}(S+N,H)$ is homeomorphic to $\mathbb{C}^d$ for

$$\begin{aligned} d &= |\{(i,k) : k > i, w(i) > w(k), \\ &\qquad h(w(j)) \geq w(i) \text{ if } N_{kj} \text{ is a pivot in } N_{S_{ii}}, S_{kk} = S_{ii}\}| \\ &+ |\{(i,k) : k > i, h(w(k)) \geq w(i) > w(k), S_{kk} \neq S_{ii}\}|. \end{aligned}$$

Proof. The Schubert cells $\{C_w\}$ form a paving of the full flag variety. The Hessenberg variety $\mathcal{H}(S+N,H)$ is a closed subvariety of the flag variety so the intersections $C_w \cap \mathcal{H}(S+N,H)$ pave the Hessenberg variety.

We now identify the nonempty cells. If N is in wHw^{-1} then the flag $[w]$ is in $\mathcal{H}(S+N,H)$. Conversely, if the flag $[uw]$ is in $\mathcal{H}(S+N,H)$ then $w^{-1}u^{-1}(S+N)uw \in H$. This implies that the pivots of each submatrix $(u^{-1}Nu)_{S_{ii}}$ are in wHw^{-1}. Since N is in highest form, its pivots are in the same positions as those of $u^{-1}Nu$ by Proposition 4.3. Each pivot of N is a pivot of some $N_{S_{ii}}$ because $S+N$ is in permuted Jordan form. The pivots of each $N_{S_{ii}}$ are in wHw^{-1} if and only if those of $(u^{-1}Nu)_{S_{ii}}$ are. The only nonzero entries of N are pivots so N is in wHw^{-1}.

Next, suppose $C_w \cap \mathcal{H}(S+N,H)$ is nonempty. Define

$$\begin{aligned} Z_i = \big\{u \in \big(U_{n-1}U_{n-2}\cdots U_i\big) \cap U_w : \\ \big(u^{-1}(S+N)u\big)_{j\bullet} \in \big(wHw^{-1}\big)_{j\bullet} \text{ for all } j > i\Big\}. \end{aligned}$$

For instance, $Z_{n-1} = U_{n-1} \cap U_w$ since wHw^{-1} always contains the span of E_{nn}. Also, observe that Z_1 is homeomorphic to $C_w \cap \mathcal{H}(S+N,H)$ under the map which sends $u \mapsto uw$. We will show that Z_1 is affine and compute its dimension.

To do this, we factor each element in Z_i uniquely as $u'u$ for $u' \in U_{n-1}\cdots U_{i+1}$ and $u \in U_i$ by Proposition 3.2. Conjugation by U_i only affects the first i rows of an upper triangular matrix by Proposition 3.4, so $u^{-1}({u'}^{-1}(S+N)u')u$ agrees with ${u'}^{-1}(S+N)u'$ in rows $i+1$ and higher. Thus, this factorization satisfies the additional conditions that $u' \in Z_{i+1}$ and that $u \in U_i \cap U_w$ has $\Big(u^{-1}({u'}^{-1}(S+N)u')u\Big)_{i\bullet} \in \big(wHw^{-1}\big)_{i\bullet}$. This gives a well-defined map $\pi_i : Z_i \to Z_{i+1}$ sending $u'u$ to u'.

We now show that $\pi_i : Z_i \to Z_{i+1}$ is an affine fiber bundle and compute its rank. For each element $u' \in Z_{i+1}$, the operator ${u'}^{-1}(S+N)u'$ is in highest form and has its pivots in the same position as $S+N$. Consequently, the hypotheses of Lemma 5.2 hold. Lemma 5.2 states that for each $u' \in Z_{i+1}$, the preimage $\pi_i^{-1}(u') \subseteq Z_i$ is affine of dimension

$$\begin{aligned} d_i &= |\{k : k > i, w(i) > w(k), \\ &\qquad h(w(j)) \geq w(i) \text{ if } N_{kj} \text{ is a pivot in } N_{S_{ii}}, S_{kk} = S_{ii}\}| \\ &+ |\{k : k > i, h(w(k)) \geq w(i) > w(k), S_{kk} \neq S_{ii}\}|. \end{aligned}$$

The fiber $\pi_i^{-1}(u')$ is the set of solutions $\mathbf{x}_{u'}$ to the affine system $\mathbf{x}_{u'}M_{u'} = \mathbf{v}_{u'}$, where $M_{u'}$ and $\mathbf{v}_{u'}$ vary continuously (by conjugation) in u'. In other words $\pi_i : Z_i \longrightarrow Z_{i+1}$ is a fiber bundle.

We produce a bundle homeomorphism from $\pi_i : Z_i \longrightarrow Z_{i+1}$ to the trivial bundle of rank d_i over Z_{i+1}. Let I be the set of indices used to define d_i in Lemma 5.2. For each $u' \in Z_{i+1}$, Lemma 5.2 shows that the (i,k) entry of the matrices in $\pi_i^{-1}(u')$ is free whenever $k \in I$. The map sending $u'u \mapsto (u', (u_{ik})_{k\in I})$ has a continuous inverse given by the system $\mathbf{x}_{u'}M_{u'} = \mathbf{v}_{u'}$ and so is the desired bundle homeomorphism. Given this bundle map, if Z_{i+1} is homeomorphic to affine space then Z_i is homeomorphic to affine space of dimension dim $Z_{i+1} + d_i$.

Finally, consider the sequence $Z_1 \xrightarrow{\pi_1} Z_2 \xrightarrow{\pi_2} Z_3 \cdots \xrightarrow{\pi_{n-2}} Z_{n-1}$. Each map π_i is an affine fiber bundle of rank d_i. We know $Z_{n-1} = U_{n-1} \cap U_w$ is affine and write its dimension as $d_{n-1} = |\{k : k > n-1, w(n-1) > w(k)\}|$ to stress the analogy to the other d_i. Inducting on i, we may assume the base space of $Z_i \xrightarrow{\pi_i} Z_{i+1}$ is homeomorphic to affine space, and so its total space Z_i is homeomorphic to affine space of dimension dim $Z_{i+1} + d_i$. By induction Z_1 is homeomorphic to $\mathbb{C}^d$ with $d = d_1 + \cdots + d_{n-1}$. $\square$

This along with Proposition 2.2 leads to an immediate corollary when the base field is $\mathbb{C}$.

Corollary 6.2. *Hessenberg varieties have no odd-dimensional cohomology.*

The main theorem is much simpler if the operator is nilpotent or semisimple.

Corollary 6.3. *Fix a Hessenberg space H. Let N be a nilpotent matrix in highest form and in permuted Jordan form. Let $\{C_w\}$ be the Schubert cells.*

The intersections $C_w \cap \mathcal{H}(N,H)$ form a paving by affines of $\mathcal{H}(N,H)$. The cell $C_w \cap \mathcal{H}(N,H)$ is nonempty if and only if N is in wHw^{-1}. If nonempty, the cell $C_w \cap \mathcal{H}(N,H)$ is homeomorphic to $\mathbb{C}^d$ for

$$d = |\{(i,k) : k > i, w(i) > w(k), h(w(j)) \geq w(i) \text{ if } N_{kj} \text{ is nonzero}\}|.$$

The proof of this is immediate, as is that of the next corollary.

Corollary 6.4. *Fix a Hessenberg space H. Let S be a diagonal matrix in highest form and let $\{C_w\}$ be the Schubert cells of the flag variety. The intersections $C_w \cap \mathcal{H}(S,H)$ form a paving by affines of $\mathcal{H}(S,H)$. The cell $C_w \cap \mathcal{H}(S,H)$ is homeomorphic to $\mathbb{C}^d$ for*

$$d = |\{(i,k) : k > i, w(i) > w(k), h(w(k)) \geq w(i) \text{ if } S_{kk} \neq S_{ii}\}|.$$

In particular, the intersection of each Schubert cell with $\mathcal{H}(S,H)$ is nonempty!

Corollary 6.5. *If S is diagonal then the Euler characteristic $\chi(\mathcal{H}(S,h))$ is $n!$ for every Hessenberg function h.*

Proof. Since $w^{-1}Sw$ is diagonal for each permutation, every Schubert cell C_w intersects $\mathcal{H}(S,h)$ in a nonempty affine cell $\mathbb{C}^{d_w}$. Since the cohomology is only even-dimensional, the Euler characteristic of $\mathcal{H}(S,h)$ is the total number of cells. $\square$

7. Tableaux Interpretations

We describe the main theorems combinatorially using Young diagrams.

To each linear operator X we associate a multitableau λ_X as follows. If $\sum(S_i + N_i)$ is a Jordan canonical form for X then λ_X is the collection of tableaux λ_{N_i} associated to N_i as in the Introduction. We assume tableaux are ordered vertically by size as shown in Figure 6. Note that λ_X is independent of the numerical eigenvalues of S_i. When X is nilpotent this definition reduces to that of Figure 1.

$$\begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&2&1&0\\ 0&0&0&0&0&2&0\\ 0&0&0&0&0&0&2 \end{pmatrix} \longleftrightarrow \begin{array}{c} \begin{array}{|c|c|c|}\hline 5&6&7\\ \hline 4 \\ \cline{1-1} \end{array} \\ \\ \begin{array}{|c|c|}\hline 2&3\\ \hline 1 \\ \cline{1-1} \end{array} \end{array} \longleftrightarrow \begin{pmatrix} 2&0&0&0&0&0&0\\ 0&2&1&0&0&0&0\\ 0&0&2&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

FIGURE 6. The Jordan form, multitableau with base filling, and highest form of a general linear operator

The base filling of λ_X is that for which each λ_{N_i} is filled according to the rules in Figure 4 except that the lowest number in λ_{N_i} is one more than the highest in $\lambda_{N_{i-1}}$. Figure 6 demonstrates this. The box containing i in this filling of λ_X is called the i^{th} box.

We associate each filling of the multitableau λ_X to a unique permutation w according to the convention that the i^{th} box contains $w(i)$. For instance, the i^{th} box of the base filling contains i.

Theorem 7.1. *Fix any linear operator X and Hessenberg function h. The Hessenberg variety $\mathcal{H}(X,h)$ is paved by affines. The nonempty cells are naturally in bijection with the fillings of λ_X which contain the configuration* $\begin{array}{|c|c|}\hline k & j \\ \hline\end{array}$ *only if $k \leq h(j)$. The dimension of a nonempty cell is the sum of:*

1. *the number of pairs i,k in the corresponding filling of λ_X such that*
   - *i and k are in the same tableau,*
   - *the box filled by i is to the left of or directly below the box filled by k,*
   - *$k < i$, and*
   - *if j fills the box immediately to the right of k then $i \leq h(j)$.*
2. *the number of pairs i,k in λ_X such that*
   - *i and k are in different tableaux,*
   - *the box filled with i is below k, and*
   - *$k < i \leq h(k)$.*

The first condition is illustrated in Figure 2 and the second in Corollary 7.2.

Proof. Write i' for the index of the box containing i, respectively j' and k'. This means that $w(i') = i$ so $i > k$ if and only if $w(i') > w(k')$.

The i'^{th} box is in the same tableau as the k'^{th} box if and only if $S_{i'i'} = S_{k'k'}$.

Box i' sits left of or directly below box k' if and only if $k' > i'$ by the labelling convention.

The nilpotent part of a permuted Jordan form is the sum of $E_{k'j'}$ over (k', j') such that box j' sits to the right of box k'. X is in wHw^{-1} exactly when each of these summands is and each $E_{k'j'}$ is in wHw^{-1} exactly when $k = w(k') \leq h(w(j')) = h(j)$ by Proposition 2.6. $\square$

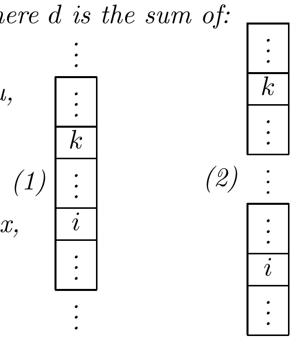
We prove Theorem 1.1, paving nilpotent Hessenberg varieties using tableaux.

Proof. If N is nilpotent its multitableau consists of exactly one tableau. Condition 2 of Theorem 7.1 never applies so Condition 1 gives the dimension. $\square$

The following interprets the main theorem for semisimple operators.

Corollary 7.2. *Fix a Hessenberg space H. Let S be a diagonal matrix and λ_S its associated multitableau. The Schubert cell C_w intersects the Hessenberg variety $\mathcal{H}(S, H)$ in a space homeomorphic to $\mathbb{C}^d$ where d is the sum of:*

1. *the number of pairs i,k such that*
   - *i and k are in the same tableau,*
   - *i is below k, and*
   - *$k < i$.*
2. *the number of pairs i,k such that*
   - *i and k are in different tableaux,*
   - *i is below k, and*
   - *$k < i \leq h(k)$.*



Proof. The nilpotent associated to each eigenspace is the zero matrix so each Young diagram is a single column. This implies that every Schubert cell intersects the Hessenberg variety and that the first condition of Theorem 7.1 simplifies as given. $\square$

8. Root system interpretation

The main theorem can also be expressed in terms of roots. For general background on Lie algebras, the reader is referred to [H2].

Recall that the Lie algebra of $GL_n(\mathbb{C})$ is $\mathfrak{gl}_n(\mathbb{C})$, which we think of as $n \times n$ matrices over $\mathbb{C}$. Fix the Borel subalgebra $\mathfrak{b}$ of upper-triangular matrices in $\mathfrak{gl}_n(\mathbb{C})$.

The standard embedding of $\mathfrak{gl}_n(\mathbb{C})$ into the space of matrices associates the matrix E_{ij} with $i < j$ to the root vector E_α where $\alpha = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$. The root α can also be regarded as the linear functional on diagonal matrices with $\alpha(S) = S_{jj} - S_{ii}$.

The set of positive roots Φ^+ are the roots α for which E_α is upper-triangular. The set of negative roots Φ^- are the roots $-\alpha$ for α in Φ^+. They correspond to the lower-triangular matrices by the map which sends E_{ji} to $-\alpha$ if $E_\alpha = E_{ij}$. The action of the permutation w on the set of roots is defined by $w^{-1}\alpha = \beta$ if $w^{-1}E_\alpha w = E_\beta$.

With this notation a Hessenberg space H can be defined intrinsically as a vector subspace of $\mathfrak{gl}_n(\mathbb{C})$ which contains $\mathfrak{b}$ and which is closed under Lie bracket with $\mathfrak{b}$ as in [dMPS]. We write Φ_H to denote the roots whose root spaces span H.

The definition of highest form operators can be extended to root spaces by the standard embedding. If $S + N$ is in highest form we denote by Φ_{S+N} the set of roots corresponding to the pivots of N_c over all eigenvalues c of S.

Theorem 8.1. *Fix a Hessenberg space H. Fix $\mathfrak{b}$ with respect to which $S + N$ is in highest form and permuted Jordan form. The intersection $C_w \cap \mathcal{H}(S + N, H)$ is* *nonempty if and only if $w^{-1}\Phi_{S+N}$ is in Φ_H. If so $C_w \cap \mathcal{H}(S+N,H)$ is homeomorphic to $\mathbb{C}^d$ for*

$$\begin{aligned} d &= |\{\alpha \in \Phi^+ : \alpha(S) = 0, w^{-1}\alpha \in \Phi^-, w^{-1}(\alpha+\beta) \in \Phi_H \text{ for some } \beta \in \Phi_{S+N}\}| \\ &+ |\{\alpha \in \Phi^+ : \alpha(S) \neq 0, w^{-1}\alpha \in \Phi_H, w^{-1}\alpha \in \Phi^-\}|. \end{aligned}$$

Proof. Write N in terms of root vectors as $\sum_{\beta \in \Phi_{S+N}} E_\beta$.

The pivot E_β is in wHw^{-1} if and only if $w^{-1}\beta \in \Phi_H$ by Proposition 2.6.

If $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{k-1}$ then $S_{ii} = S_{kk}$ if and only if $\alpha(S) = 0$, which describes two of the conditions in the theorem.

The condition $h(w(k)) \geq w(i)$ is equivalent to $w^{-1}\alpha \in \Phi_H$ by Proposition 2.6.

The root α satisfies $k > i$ and $w(i) > w(k)$ if and only if $\alpha \in \Phi^+$ and $w^{-1}\alpha \in \Phi^-$ according to the characterization of the Bruhat decomposition in Proposition 2.4.

The condition that N_{kj} be a pivot in $N_{S_{ii}}$ indicates that $\beta = \alpha_k + \alpha_{k+1} + \cdots + \alpha_{j-1}$ is a root in Φ_{S+N}. The root $\alpha + \beta$ corresponds to E_{ij}. This means that the condition $w^{-1}(\alpha+\beta) \in \Phi_H$ is equivalent to $w^{-1}E_{ij}w \in H$, which in turn is just $w(i) \leq h(w(j))$. □

The theorem also simplifies when the operator is either nilpotent or semisimple.

9. Open Questions

Many questions about Hessenberg varieties remain, some of which are described here.

9.1. **Geometric properties.** One of the most fundamental unanswered questions about the geometry of Hessenberg varieties is:

Question 1. *Is every Hessenberg variety pure dimensional?*

In every known example, the answer to this is yes. This also raises the following.

Question 2. *What is the dimension of the Hessenberg variety $\mathcal{H}(X,H)$?*

The answer is known for various examples, including the Springer fibers (where it is $\sum_{i=1}^k (i-1)d_i$ if the Jordan blocks have size $d_1, \ldots, d_k$ [Sp]) and regular nilpotent Hessenberg varieties (namely $\sum_{i=1}^n (h(i)-i)$ [ST]). It is unknown in general.

The answer to the next question is known for the Springer fiber, where it is the dimension of the corresponding irreducible representation of the symmetric group ([Sp], [CG, 3.6.2]).

Question 3. *How many components does $\mathcal{H}(X,H)$ have?*

This paper has discussed Hessenberg varieties over $GL_n(\mathbb{C})$. Hessenberg varieties are defined for general complex linear algebraic groups (see [dMPS]), and the same questions can be posed in the general setting.

Question 4. *How many of these results hold for general G?*

9.2. **Closure relations.** Given a Schubert cell C_w, classical results show that the cell C_x lies in its closure if and only if w is a product of simple transpositions $w = s_1 \cdots s_k$ and $x = s_{i_1} \cdots s_{i_{k'}}$ with $1 \leq i_1 < \cdots < i_{k'} \leq k$ (see [BL, section 2.7]). The closure $\overline{C_w}$ is a Schubert variety, whose geometry and associated combinatorics has been extensively studied [BL].

The cells of a general Hessenberg variety are intersections with Schubert cells. However, the closure relations of these intersections are not in general restrictions of the closure relations of the full Schubert cells.

Question 5. *What are the closure relations for cells in a Hessenberg variety? For which x does $C_x \cap \mathcal{H}(X, H)$ intersect the closure of $C_w \cap \mathcal{H}(X, H)$?*

The answer to an apparently simpler question is also unknown.

Question 6. *If $C_w \cap \mathcal{H}(X, H)$ is nonempty, for which permutations x does the flag given by x lie in the closure of $C_w \cap \mathcal{H}(X, H)$?*

9.3. **Betti numbers.** The previous results established that the odd-dimensional Betti numbers for Hessenberg varieties are zero. They also provide an algorithm to generate tables of the even-dimensional Betti numbers, which are available at

`http://www.math.lsa.umich.edu/~tymoczko`

The even-dimensional Betti numbers for Hessenberg varieties $\mathcal{H}(N, H)$ have closed formulae when N is a regular nilpotent operator, i.e., N consists of a single Jordan block. These Betti numbers are both symmetric (namely $b_i = b_{k-i+1}$ for each i) and unimodal (namely $b_1 \leq b_2 \leq b_3 \cdots \leq b_{\lceil k/2 \rceil}$) by [ST]. Yet most of these varieties are singular.

The even-dimensional Betti numbers for general Hessenberg varieties need not be symmetric. Robert MacPherson conjectured the following, which is true in all known cases. It is the combinatorial description of the hard Lefschetz property and has been studied in other contexts [Sta].

Question 7. *For any Hessenberg variety $\mathcal{H}(X, H)$ the even-dimensional Betti numbers are unimodal and satisfy $b_i \leq b_{k-i+1}$ for all i between 1 and $k/2$.*

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