Groups, operator algebras and approximation

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This thesis contains the results of research projects which were carried out during the past several years, and I would like to warmly thank all those who supported me during this period of work.

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Contents

Overview of the results

Two main objects of my research so far have been countable discrete groups and their operator algebras (C*-algebras and von Neumann algebras). Discrete groups are often succesfully studied using geometric and ergodic-theoretic methods, the corresponding areas of mathematics being called geometric resp. measured group theory. The worlds of groups, dynamical systems and operator algebras are deeply interconnected: studying spectral properties of groups or the corresponding dynamical systems amounts to passing to corresponding operator algebras, and often one can even pass back, understanding a discrete group or a dynamical system by looking at the operator algebra only. In this way, one studies objects through properties of their "relatives".

My main research interest lies exactly in understanding and exploiting the mutual connections between groups, groupoids and operator algebras. Amazingly, here one often profits from techniques from other areas of pure mathematics, notably algebraic topology (in connection to L^2-Betti numbers), differential and metric geometry (negatively curved groups), algebraic groups and Lie groups (which provide lattices as important examples of discrete groups), representation theory or even homological algebra. Given that, it's not surprising that the majority of my research projects came up through discussions with colleagues from neighbouring fields, and indeed this way of producing new results proved to be the most fruitful and fulfilling for me. It is safe to say that in the past several years I've been gradually learning several areas of mathematics (to name a few: (noncommutative) real algebraic geometry, coarse geometry, theory of semisimple algebraic groups over number fields and local fields) through completing research projects which borrow techniques from these.

This thesis has a cumulative form: each chapter is a research article, and therefore has its own abstract and bibliography. The majority of these publications have been peer-reviewed and published in various journals. Two chapters correspond to submitted articles currently under revision, and yet another two chapters correspond to articles in preparation; by the time this thesis arrives at the referees, they hopefully will be available in more complete form on the arXiv e-print archive.

What follows is a brief description of each chapter.

The first chapter contains the article [**AK15**] where together with David Kyed we investigated the version of first continuous L^2-cohomology of von Neumann algebras suggested by Andreas Thom as an interesting invariant potentially able to distinguish free group factors.

Unfortunately, as it turned out, this invariant vanishes – this is the result of article [**Ale14**] constituting the second chapter.

The third chapter contains the paper [**ANT16**] written together with Tim Netzer and Andreas Thom where we investigate quadratic modules, an important object in noncommutative real algebraic geometry, and show that semialgebraicity of free (= noncommutative) convex hulls of semialgebraic sets fail to be semialgebraic in general.

Chapter 4 contains the article [**AFS16**] where together with Martin Finn–Sell we investigated coarse geometry of spaces of graphs attached to a sofic approximation of a finitely generated group. These results and the corresponding questions lead to some further research work which is still in progress; I also have to mention that some questions which we posed at the end of this article have been meanwhile answered (in particular, by Tom Kaiser).

The fifth chapter corresponds to the article [**AFS18**] where together with Martin Finn–Sell we answered some questions on approximation properties of topological groupoids previously raised by Claire Anantharaman-Delaroche and Rufus Willett. It is a short paper mainly describing a very specific example of a topological groupoid with desired properties.

Chapter 6 contains the article [**AK19**] where together with David Kyed we made some progress on a question by Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam about uniqueness of C^*-norms on group rings.

Chapters 7 and 8 contain the articles [**AB18**] and [**AB19**] written together with Rahel Brugger on invariant random positive definite functions and some "strongly regular" subfactors of von Neumann algebras of lattices in higher rank Lie groups.

Finally, the last two chapters contain preliminary versions of some work-in-progress preprints. Chapter 9 contains the preliminary version of [**AFS**] which concerns representation theory of full groups of étale groupoids and its connection to semigroup theory; Chapter 10 contains a veru short outline of the preprint [**AC**] which describes maximal amenable subgroups of arithmetic groups and proves that they induce maximal amenable von Neumann subalgebras of their group von Neumann algebras.

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CHAPTER 1

Measure-continuous derivations on von Neumann algebras

Abstract

We prove that norm continuous derivations from a von Neumann algebra into the algebra of operators affiliated with its tensor square are automatically continuous for both the strong operator topology and the measure topology. Furthermore, we prove that the first continuous L^2-Betti number scales quadratically when passing to corner algebras and derive an upper bound given by Shen's generator invariant. This, in turn, yields vanishing of the first continuous L^2-Betti number for II_1 factors with property (T), for finitely generated factors with non-trivial fundamental group and for factors with property Gamma.

1. Introduction

The theory of L^2-Betti numbers has been generalized to a vast number of different contexts since the seminal work of Atiyah [**Ati76**]. One recent such generalization is due to Connes and Shlyakhtenko [**CS05**] who introduced L^2-Betti numbers for subalgebras of finite von Neumann algebras, with the main purpose being to obtain a suitable notion for arbitrary II_1-factors. Although their definitions are very natural, it has proven to be quite difficult to perform concrete calculations. The most advanced computational result so far is due to Thom [**Tho08**] who proved that the L^2-Betti numbers vanish for von Neumann algebras with diffuse center. Notably, the problem of computing a positive degree L^2-Betti number for a single II_1-factor has remained open for a decade at the time of writing! Due to this evident drawback, Thom [**Tho08**] introduced a continuous version of the first L^2-Betti number, which turns out to be much more manageable than its algebraic counterpart. The first continuous L^2-Betti number is defined as the von Neumann dimension of the first continuous Hochschild cohomology of the von Neumann algebra M with values in the algebra of operators affiliated with $M\overline{\otimes}M^{\mathrm{op}}$. The word 'continuous' here means that we restrict attention to those derivations that are continuous from the norm topology on M to the measure topology on the affiliated operators.

In this paper we continue the study of Thom's continuous version of the first L^2-Betti number and our first result (Theorem 3.1) shows that norm continuous derivations are automatically also continuous for both the strong operator topology and the measure topology.

This allows us to derive all previously known computational results concerning the first continuous L^2-Betti number, and furthermore to prove that it vanishes for II_1 factors with property (T) (Theorem 3.10). In Section 3.4 we prove that it scales quadratically when passing to corner algebras (Theorem 3.13) and is dominated by Shen's generator invariant (Corollary 3.17). Along the way, we give a new short cohomological proof of the fact that the (non-continuous) first L^2-Betti number vanishes for von Neumann algebras with diffuse center, and furthermore derive a number of new vanishing results regarding the first continuous L^2-Betti number, including the vanishing for II_1 factors with property Gamma and finitely generated factors with non-trivial fundamental group (Corollaries 3.17 & 3.18).

2. Preliminaries

In this section we briefly recapitulate the theory of non-commutative integration and the theory of L^2-Betti numbers for von Neumann algebras.

2.1. Non-commutative integration. Let us recall some facts from the theory of non-commutative integration, cf. [**Nel74**], [**Tak03**, IX.2]. Let N be a finite von Neumann algebra equipped with a normal, faithful, tracial state τ. Consider N in its representation on the GNS-space arising from τ, and let $\mathscr{N}$ be the algebra of (potentially) unbounded, closed, densely defined operators affiliated with N. We equip $\mathscr{N}$ with the *measure topology*, defined by the following two-parameter family of neighbourhoods of zero:

$$N(\varepsilon, \delta) = \{a \in \mathscr{N} \mid \exists p \in \mathrm{Proj}(N) : \|ap\| < \varepsilon,\ \tau(p^\perp) < \delta\}, \quad \varepsilon, \delta > 0.$$

With this topology, $\mathscr{N}$ is a complete [**Tak03**, Theorem IX.2.5] metrizable [**Rud73**, Theorem 1.24] topological vector space and the multiplication map

$$(a, b) \mapsto ab \colon \mathscr{N} \times \mathscr{N} \to \mathscr{N}$$

is uniformly continuous when restricted to products of bounded subsets [**Nel74**, Theorem 1]. Convergence with respect to the measure topology is also referred to as *convergence in measure*. We also introduce the notation

$$N(0, \delta) = \{a \in \mathscr{N} \mid \exists p \in \mathrm{Proj}(N) : ap = 0,\ \tau(p^\perp) < \delta\},$$

and

$$N(\varepsilon, 0) = \{a \in N \mid \|a\| < \varepsilon\} \subset \mathscr{N}.$$

Notice that $N(0, \delta)$ and $N(\varepsilon, 0)$ are *not* zero neighbourhoods in the measure topology, but merely G_δ sets However, the following additive and multiplicative properties continue to hold for all $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2 \geqslant 0$, cf. [**Nel74**, Theorem 1]:

$$N(\varepsilon_1, \delta_1) + N(\varepsilon_2, \delta_2) \subset N(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2), \tag{2.1}$$

$$N(\varepsilon_1, \delta_1) \cdot N(\varepsilon_2, \delta_2) \subset N(\varepsilon_1 \varepsilon_2, \delta_1 + \delta_2). \tag{2.2}$$

The noncommutative L^p-spaces $L^p(N,\tau)$ are naturally identified with subspaces of $\mathscr{N}$ [**Tak03**, Theorem IX.2.13]. We fix the notation $\xrightarrow{s}$ for strong convergence of elements in von Neumann algebras, $\xrightarrow{2}$ for the L^2-convergence and $\xrightarrow{m}$ for the convergence in measure of elements in $\mathscr{N}$. Clearly strong convergence implies convergence in 2-norm, and we remind the reader that for nets that are bounded in the operator norm the converse is also true — a fact we will use extensively in the sequel. As in the commutative case, the Chebyshev inequality can be used to establish the following fact.

LEMMA 2.1 ([**Nel74**, Theorem 5]). *For any $p \geqslant 1$ the inclusion $L^p(N,\tau) \subset \mathscr{N}$ is continuous; i.e. L^p-convergence implies convergence in measure.*

In [**CS05**] Connes and Shlyakhtenko introduced L^2-Betti numbers in the general setting of tracial $*$-algebras; if M is a finite von Neumann algebra and $\mathcal{A} \subset M$ is any weakly dense unital $*$-subalgebra its L^2-Betti numbers are defined as

$$\beta_p^{(2)}(\mathcal{A},\tau) = \dim_{M\bar{\otimes}M^{\mathrm{op}}} \mathrm{Tor}_p^{\mathcal{A}\odot\mathcal{A}^{\mathrm{op}}}(M\bar{\otimes}M^{\mathrm{op}},\mathcal{A}).$$

Here the dimension function $\dim_{M\bar{\otimes}M^{\mathrm{op}}}(-)$ is the extended von Neumann dimension due to Lück; cf. [**Lüc02**, Chapter 6]. This definition is inspired by the well-known correspondence between representations of groups and bimodules over finite von Neumann algebras, and it extends the classical theory by means of the formula $\beta_p^{(2)}(\Gamma) = \beta_p^{(2)}(\mathbb{C}\Gamma,\tau)$ whenever Γ is a discrete countable group. In [**Tho08**] it is shown that the L^2-Betti numbers also allow the following cohomological description:

$$\beta_p^{(2)}(\mathcal{A},\tau) = \dim_{M\bar{\otimes}M^{\mathrm{op}}} \mathrm{Ext}^p_{\mathcal{A}\odot\mathcal{A}}(\mathcal{A},\mathscr{U}),$$

where $\mathscr{U}$ denotes the algebra of operators affiliated with $M\bar{\otimes}M^{\mathrm{op}}$. It is a classical fact [**Lod98**, 1.5.8] that the Ext-groups above are isomorphic to the Hochschild cohomology groups of $\mathcal{A}$ with coefficients in $\mathscr{U}$, where the latter is considered as an $\mathcal{A}$-bimodule with respect to the actions

$$a\cdot\xi := (a\otimes 1^{\mathrm{op}})\xi \quad\text{and}\quad \xi\cdot b := (1\otimes b^{\mathrm{op}})\xi \quad \text{for } a,b\in\mathcal{A} \text{ and } \xi\in\mathscr{U}.$$

In particular, the first L^2-Betti number can be computed as the dimension of the right $M\bar{\otimes}M^{\mathrm{op}}$-module

$$H^1(\mathcal{A},\mathscr{U}) = \frac{\mathrm{Der}(\mathcal{A},\mathscr{U})}{\mathrm{Inn}(\mathcal{A},\mathscr{U})}.$$

Here $\mathrm{Der}(\mathcal{A},\mathscr{U})$ denotes the space of derivations from $\mathcal{A}$ to $\mathscr{U}$ and $\mathrm{Inn}(A,\mathscr{U})$ denotes the space of inner derivations. We recall that a linear map δ from $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{X}$ is called a *derivation* if it satisfies

$$\delta(ab) = a\cdot\delta(b) + \delta(a)\cdot b \quad \text{for all } a,b\in\mathcal{A},$$

and that a derivation is called *inner* if there exists a vector $\xi \in \mathcal{X}$ such that

$$\delta(a) = a \cdot \xi - \xi \cdot a \text{ for all } a \in \mathcal{A}.$$

When the bimodule in question is $\mathscr{U}$, with the bimodule structure defined above, the derivation property amounts to the following:

$$\delta(ab) = (a \otimes 1^{\mathrm{op}})\delta(b) + (1 \otimes b^{\mathrm{op}})\delta(a) \quad \text{for all } a, b \in \mathcal{A}.$$

Although the extended von Neumann dimension is generally not faithful, enlarging the coefficients from $M\overline{\otimes}M^{\mathrm{op}}$ to $\mathscr{U}$ has the effect that $\beta_1^{(2)}(\mathcal{A},\tau) = 0$ if and only if $H^1(\mathcal{A},\mathscr{U})$ vanishes [**Tho08**, Corollary 3.3 and Theorem 3.5]. In particular, in order to prove that $\beta_1^{(2)}(\mathcal{A},\tau) = 0$ one has to prove that every derivation from $\mathcal{A}$ into $\mathscr{U}$ is inner. These purely algebraically defined L^2-Betti numbers have turned out extremely difficult to compute in the case when $\mathcal{A}$ is M itself. Actually, the only computational result known in this direction (disregarding finite dimensional algebras) is that the they vanish for von Neumann algebras with diffuse center (see [**CS05**, Corollary 3.5] and [**Tho08**, Theorem 2.2]). In particular, for II_1-factors not a single computation of a positive degree L^2-Betti is known, and furthermore this seems out of reach with tools available at the moment. It is therefore natural to consider variations of the definitions above that take into account the topological nature of M, and in [**Tho08**] Thom suggests to consider a first cohomology group consisting of (equivalence classes of) those derivations $\delta\colon \mathcal{A} \to \mathscr{U}$ that are closable from the norm topology to the measure topology. Note that when $\mathcal{A}$ is norm closed these are exactly the derivations that are norm-measure topology continuous. We denote the space of closable derivations by $\mathrm{Der}_c(\mathcal{A},\mathscr{U})$, the continuous cohomology by $H_c^1(\mathcal{A},\mathscr{U})$ and by $\eta_1^{(2)}(\mathcal{A},\tau)$ the corresponding continuous L^2-Betti numbers; i.e.

$$\eta_1^{(2)}(\mathcal{A},\tau) = \dim_{M\overline{\otimes}M^{\mathrm{op}}} H_c^1(\mathcal{A},\mathscr{U}).$$

These continuous L^2-Betti numbers are much more manageable than their algebraic counterparts — for instance they are known [**Tho08**, Theorem 6.4] to vanish for von Neumann algebras that are non-prime and for those that contain a diffuse Cartan subalgebra.

Finally, let us fix a bit of notation. For the rest of this paper, we consider a finite von Neumann algebra M with separable predual M_*. We endow M with a fixed faithful, normal, tracial state τ and consider M in the GNS representation on the Hilbert space $\mathcal{H} = L^2(M,\tau)$. The trace τ induces a faithful, normal, tracial state on the von Neumann algebraic tensor product $M\overline{\otimes}M^{\mathrm{op}}$ of M with its opposite algebra; abusing notation slightly, we will still denote it by τ. We always consider $M\overline{\otimes}M^{\mathrm{op}}$ in the GNS representation on $L^2(M\overline{\otimes}M^{\mathrm{op}},\tau)$ and denote by $\mathscr{U}$ the algebra of closed, densely defined, unbounded operators affiliated with $M\overline{\otimes}M^{\mathrm{op}}$. More generally, if N is a finite von Neumann algebra endowed with a tracial state ρ we denote by $\mathscr{U}(N)$ the algebra of operators on $L^2(N,\rho)$ affiliated with N. We will use the symbol "$\overline{\otimes}$" to denote tensor products of von Neumann algebras and "$\odot$" to denote algebraic tensor products, and, unless explicitly stated otherwise, all subalgebras in M are implicitly assumed to contain the unit of M. Finally, when there is no source of confusion we will often suppress the notational reference to the trace τ, and simply write $L^2(M), \beta_p^{(2)}(M), \eta_1^{(2)}(M)$ etc.

3. Improving continuity

In this section we prove that a derivation which is continuous for the norm topology is automatically continuous for the strong operator topology and the measure topology as well. Intuitively, this statement is based on the fact that strong convergence is "almost uniform", which is known as the non-commutative Egorov theorem, cf. [**Tak02**, Theorem II.4.13]. The precise statement is as follows.

THEOREM 3.1.

(i) Let $A \subset M$ be a weakly dense C^*-algebra and let $\delta\colon A \to \mathscr{U}$ be a norm-measure continuous derivation. Then δ has a unique norm-measure continuous extension to M.

(ii) Let $\delta\colon M \to \mathscr{U}$ be a norm-measure continuous derivation. Then δ is also continuous from the strong operator topology to the measure topology.

(iii) Let $\delta\colon M \to \mathscr{U}$ be a norm-measure continuous derivation. Then δ is also continuous from the measure topology on M to the measure topology on $\mathscr{U}$; in particular, it has a unique measure-measure continuous extension to the algebra $\mathscr{M}$ of operators affiliated with M.

Note that the extension property in i) also follows from the rank-metric based arguments in [**Tho08**, Theorem 4.3 & Proposition 4.4], but the boundedness of the extension is not apparent from this.

PROOF. We first prove i). By Kaplansky's density theorem the unit ball $(A)_1$ is strongly dense in $(M)_1$, and for $a \in (M)_1$ we may therefore choose a sequence[1] $a_n \in (A)_1$ with $a_n \xrightarrow{s} a$. Let us first prove that $\delta(a_n)$ is Cauchy in measure. So we fix an $\varepsilon > 0$ and want to find an N_0 such that $\delta(a_n) - \delta(a_m) \in N(\varepsilon, \varepsilon)$ for $\min\{m, n\} > N_0$. We first make use of the norm-measure topology continuity of $\delta\colon A \to \mathscr{U}$ to find a $\gamma > 0$ such that $\|a\| \leqslant \gamma$ implies that $\delta(a) \in N(\varepsilon/3, \varepsilon/3)$. Consider now $\mathbb{N} \times \mathbb{N}$ with the ordering

$$(m, n) \geqslant (m', n') \text{ iff } m \geqslant m' \text{ and } n \geqslant n',$$

and the net of self-adjoint elements $b_{(m,n)} := (a_m - a_n)^*(a_m - a_n)$. Then for every $\xi \in \mathcal{H}$ we have $\|b_{(m,n)}\xi\| \leqslant 2\|(a_m - a_n)\xi\|$ and hence $b_{(m,n)} \xrightarrow{s} 0$. Let now $f\colon \mathbb{R} \to [0, 1]$ be a

[1]As the predual M_* is assumed separable, $(M)_1$ is a separable and metrizable space for the strong operator topology and we can therefore do with sequences rather than nets.

continuous function with $f(x) = 1$ for $x \leqslant \gamma^2/4$ and $f(x) = 0$ for $x \geqslant \gamma^2$ and consider the net $h_{(m,n)} := f(b_{(m,n)})$. It follows that $\|h_{(m,n)}\| \leqslant 1$ and

$$\frac{\gamma^2}{4}(1 - h_{(m,n)}) \leqslant b_{(m,n)}.$$

Since the C^*-algebra generated by $b_{(m,n)}$ is commutative and both $b_{(m,n)}$ and $1 - h_{(m,n)}$ are positive this implies

$$0 \leqslant (1 - h_{(m,n)})^*(1 - h_{(m,n)}) \leqslant \tfrac{16}{\gamma^4}\, b^*_{(m,n)} b_{(m,n)},$$

and thus

$$\|1 - h_{(m,n)}\|_2 \leqslant \tfrac{4}{\gamma^2}\|b_{(m,n)}\|_2 \to 0.$$

Hence $1 - h_{(m,n)} \xrightarrow{2} 0$, and the convergence therefore holds in measure as well. Also note that $\|b_{(m,n)} h_{(m,n)}\| \leqslant \gamma^2$ and hence

$$\|(a_m - a_n)h_{(m,n)}\| \leqslant \gamma. \tag{3.1}$$

We now use the derivation property to obtain:

$$\begin{aligned}\delta(a_m) - \delta(a_n) &= \delta((a_m - a_n)h_{(m,n)}) + \delta((a_m - a_n)(1 - h_{(m,n)}))\\ &= \delta((a_m - a_n)h_{(m,n)}) + (1 \otimes (1 - h_{(m,n)})^{\mathrm{op}})\delta(a_m - a_n)\\ &\quad - ((a_m - a_n) \otimes 1^{\mathrm{op}})\delta(h_{(m,n)}).\end{aligned}$$

By (3.1) and the choice of γ, the first summand is in $N(\varepsilon/3, \varepsilon/3)$. Let us now consider the second summand. The norm-measure topology boundedness of δ on A implies [**Rud73**, Theorem 1.32] that the set

$$\{\delta(a_m - a_n) \mid n, m \in \mathbb{N}\}$$

is bounded in $\mathscr{U}$. Hence it follows from the fact that $1 - h_{(m,n)} \xrightarrow{m} 0$ together with the uniform continuity of multiplication on bounded sets of $\mathscr{U}$, that there exists an N_1 such that $(1 \otimes (1 - h_{(m,n)})^{\mathrm{op}})\delta(a_m - a_n) \in N(\varepsilon/3, \varepsilon/3)$ for $\min\{m, n\} > N_1$. Lastly we consider the third term. Again by norm-boundedness of δ, the set $\{\delta(h_{(m,n)})\}_{n,m\in\mathbb{N}}$ is bounded in $\mathscr{U}$. As a_n is strongly convergent it also converges in 2-norm; hence $a_n \otimes 1^{\mathrm{op}}$ converges in 2-norm and is, in particular, a Cauchy sequence for the measure topology. Thus, there exists an N_2 such that

$$((a_m - a_n) \otimes 1^{\mathrm{op}})\delta(h_{(m,n)}) \in N(\varepsilon/3, \varepsilon/3)$$

for $\min\{n, m\} > N_2$. Taking $N_0 = \max\{N_1, N_2\}$ establishes that $\delta(a_n)$ is a Cauchy sequence in the measure topology. Appealing to the completeness of $\mathscr{U}$, we may now define $\delta(a) := \lim_n \delta(a_n)$; it is routine to check that this yields a well-defined derivation $\delta\colon M \to \mathscr{U}$. The continuity of the extension follows from its definition from which it is clear that the set $\delta((M)_1)$ is contained in the measure topology closure of the bounded set $\delta((A)_1)$ which is again bounded by [**Rud73**, Theorem 1.13]. Thus the extension maps norm bounded sets to measure topology bounded sets and is therefore continuous [**Rud73**, Theorem 1.32], and the proof of i) is complete.

Next we prove ii) and iii). If $\delta\colon M \to \mathscr{U}$ is a norm continuous derivation, then by repeating the above arguments for $m = \infty$, $a = a_\infty$, we get strong continuity of δ on bounded sets: if $a_n \xrightarrow{s} a$ and the sequence a_n is uniformly bounded, then $\delta(a_n) \xrightarrow{m} \delta(a)$. Since strong convergence implies L^2-convergence it also implies convergence in measure by Lemma 2.1, and hence it suffices to prove that δ is measure-measure continuous. To this end, by metrizability of the measure topology it suffices to take a sequence $a_n \in M$ such that $a_n \xrightarrow{m} 0$ and prove that $\delta(a_n) \xrightarrow{m} 0$. Since $a_n \xrightarrow{m} 0$ we get[2] a sequence of projections p_n such that $\|a_n p_n\| \to 0$ and $\tau(p_n) \to 1$ Thus, p_n is a norm bounded sequence that converges to 1 strongly and, by what was just proven, it follows that $\delta(p_n) \xrightarrow{m} \delta(1) = 0$. Now we use the derivation property of δ:

$$\delta(a_n) = \delta(a_n p_n) + (1 \otimes (1-p_n)^{\mathrm{op}})\delta(a_n) - (a_n \otimes 1^{\mathrm{op}})\delta(p_n). \tag{3.2}$$

As δ is norm-bounded, the first summand in (3.2) converges to 0 in measure. The second summand converges to 0 in measure because $1 - p_n \in N(0, \varepsilon_n)$ with $\varepsilon_n = 1 - \tau(p_n) \to 0$ and $\delta(a_n) \in N(\gamma_n, \varepsilon_n)$ for some $\gamma_n > 0$ [**Tak03**, Lemma IX.2.3]. The third summand converges to 0 in measure because $\delta(p_n) \xrightarrow{m} 0$ and multiplication is measure continuous. This finishes the proof. ☐

The next lemma shows that there is no hope for weaker continuity properties of derivations.

LEMMA 3.2. *Let M be a diffuse finite von Neumann algebra. The only derivation $\delta\colon M \to \mathscr{U}$ which is continuous from the ultraweak topology on M to the measure topology is the zero map.*

PROOF. Let $\delta\colon M \to \mathscr{U}$ be a derivation which is continuous from the ultraweak topology on M to the measure topology. Let $\{u_n\}_{n\in\mathbb{N}} \subset M$ be a sequence of unitaries weakly converging to zero and let $m \in M$ be given. Then $u_n m$ weakly converges to zero and we have

$$(u_n \otimes 1^{\mathrm{op}})\delta(m) = \delta(u_n m) - 1 \otimes m^{\mathrm{op}}\delta(u_n).$$

Because the ultraweak topology and the weak operator topology agree on bounded subsets of M, both summands on the right-hand side converge to zero in measure since δ is assumed continuous; hence $(u_n \otimes 1^{\mathrm{op}})\delta(m) \xrightarrow{m} 0$. Multiplying by the unitaries $(u_n^* \otimes 1^{\mathrm{op}})$, we infer $\delta(m) = 0$. ☐

[2] The existence of p_n can, for instance, be seen by noting that $\varepsilon_n := \inf\{\varepsilon > 0 \mid a_n \in N(\epsilon, \epsilon)\}$ must converge to zero if $a_n \xrightarrow{m} 0$.

REMARK 3.3. Bearing in mind the numerous automatic continuity results for derivations between operator algebras (see [**SS95**] and references therein), it is of course natural to ask if norm continuity of a derivation $\delta\colon M \to \mathscr{U}$ is also automatic. We were not able to prove this, and it seems to be a difficult question to answer. One reason being the absence of examples of finite von Neumann algebras (or C^*-algebras, for that matter) for which a non-inner derivation into the algebra $\mathscr{U}$ is known to exist. Moreover, the fact that we are considering the operators affiliated with $M\bar{\otimes}M^{\mathrm{op}}$ (as opposed to M itself) has to play a role if automatic norm continuity is to be proven, as there are examples of derivations from M into the operators affiliated with M which are not norm-measure topology continuous. This follows from [**BCS06**] where the authors exhibit a (commutative) finite von Neumann algebra M for which there exists a derivation δ: $\mathscr{U}(M) \to \mathscr{U}(M)$ which is not measure-measure continuous. If the restriction $\delta|_M$ were norm-measure continuous, then it is not difficult to see that the graph of the original derivation δ is closed (in the product of the measure topologies), and hence it cannot be discontinuous. If norm continuity turns out to be automatic, there is of course no difference between the ordinary and the continuous L^2-Betti numbers, and one might even take the standpoint that if there is no such automatic continuity, then continuity has to be imposed in order to get a satisfactory theory.

In this section we apply the above automatic continuity result to obtain information about the first continuous L^2-Betti number for von Neumann algebras. Some of the results presented are already known, or implicit in the literature, but since the proofs are knew and quite simple we hope they will shed new light on these results. The main new result in this section is Theorem 3.10 which shows that the first continuous L^2-Betti number vanishes for property (T) factors.

Recall that if $N \subset M$ is an inclusion of von Neumann algebras, then the normalizer of N in M is defined as the set of unitaries in M which normalize N:

$$\mathcal{N}_M(N) = \{u \in U(M) \,|\, u^*Nu = N\}.$$

The following lemma appears in [**Tho08**], but we include its short proof for the sake of completeness.

LEMMA 3.4 ([**Tho08**, Lemma 6.5]). *Let $\delta\colon M \to \mathscr{U}$ be a derivation which vanishes on a diffuse subalgebra $N \subset M$ and let $u \in \mathcal{N}_M(N)$. Then $\delta(u) = 0$.*

PROOF. Let $h \in N$ be a diffuse element. Since $\delta(u^*) = -(u^* \otimes u^{*\mathrm{op}})\delta(u)$ we get

$$\begin{aligned} 0 = \delta(uhu^*) &= (1 \otimes (hu^*)^{\mathrm{op}})\delta(u) + (uh \otimes 1^{\mathrm{op}})\delta(u^*) \\ &= (1 \otimes (hu^*)^{\mathrm{op}})\delta(u) - (uhu^* \otimes u^{*\,\mathrm{op}})\delta(u) \\ &= (u \otimes u^{*\mathrm{op}})(1 \otimes h^{\mathrm{op}} - h \otimes 1^{\mathrm{op}})(u^* \otimes 1)\delta(u). \end{aligned}$$

Since h is diffuse, $1 \otimes h^{\mathrm{op}} - h \otimes 1^{\mathrm{op}}$ is not a zero divisor in $\mathscr{U}$, and it follows that $\delta(u) = 0$. $\square$

The next lemma, which might be of independent interest, shows that if a von Neumann algebra only allows inner derivations into the algebra of operators affiliated with its double, then the same is true for the algebra of operators affiliated with the double of any ambient von Neumann algebra.

LEMMA 3.5. *Let $N \subset M$ be a sub-von Neumann algebra. Then the following statements are equivalent:*

(i) $\beta_1^{(2)}(N, \tau) = 0$,

(ii) every derivation $\delta\colon N \to \mathscr{U}$ is inner (where $\mathscr{U}$ is considered as an N-bimodule via the inclusion $N \subset M$).

PROOF. By [**Lod98**, 1.5.9] we have $H^1(N, \mathscr{U}) = \mathrm{Ext}^1_{N \odot N^{\mathrm{op}}}(N, \mathscr{U})$ and by [**Tho08**, Theorem 3.5] this right $\mathscr{U}$-module is isomorphic to

$$\mathrm{Hom}_{\mathscr{U}}\left(\mathrm{Tor}_1^{N \odot N^{\mathrm{op}}}(\mathscr{U}, N), \mathscr{U}\right).$$

Furthermore, by [**Tho08**, Corollary 3.3] this module vanishes exactly when

$$\dim_{M \overline{\otimes} M^{\mathrm{op}}} \mathrm{Tor}_1^{N \odot N^{\mathrm{op}}}(\mathscr{U}, N) = 0.$$

But since $\mathscr{U} \odot_{M \overline{\otimes} M^{\mathrm{op}}} -$ and $M \overline{\otimes} M^{\mathrm{op}} \odot_{N \overline{\otimes} N^{\mathrm{op}}} -$ are both flat and dimension preserving (see [**Rei01**, Proposition 2.1 and Theorem 3.11] and [**Lüc02**, Theorem 6.29]) we get

$$\begin{aligned}
\dim_{M \overline{\otimes} M^{\mathrm{op}}} \mathrm{Tor}_1^{N \odot N^{\mathrm{op}}}(\mathscr{U}, N) &= \dim_{M \overline{\otimes} M^{\mathrm{op}}} \mathrm{Tor}_1^{N \odot N^{\mathrm{op}}}(\mathscr{U} \odot_{M \overline{\otimes} M^{\mathrm{op}}} M \overline{\otimes} M^{\mathrm{op}}, N) \\
&= \dim_{M \overline{\otimes} M^{\mathrm{op}}} \mathscr{U} \odot_{M \overline{\otimes} M^{\mathrm{op}}} \mathrm{Tor}_1^{N \odot N^{\mathrm{op}}}(M \overline{\otimes} M^{\mathrm{op}}, N) \\
&= \dim_{M \overline{\otimes} M^{\mathrm{op}}} \mathrm{Tor}_1^{N \odot N^{\mathrm{op}}}(M \overline{\otimes} M^{\mathrm{op}}, N) \\
&= \dim_{M \overline{\otimes} M^{\mathrm{op}}} \mathrm{Tor}_1^{N \odot N^{\mathrm{op}}}(M \overline{\otimes} M^{\mathrm{op}} \odot_{N \overline{\otimes} N^{\mathrm{op}}} N \overline{\otimes} N^{\mathrm{op}}, N) \\
&= \dim_{M \overline{\otimes} M^{\mathrm{op}}} M \overline{\otimes} M^{\mathrm{op}} \odot_{N \overline{\otimes} N^{\mathrm{op}}} \mathrm{Tor}_1^{N \odot N^{\mathrm{op}}}(N \overline{\otimes} N^{\mathrm{op}}, N) \\
&= \dim_{N \overline{\otimes} N^{\mathrm{op}}} \mathrm{Tor}_1^{N \odot N^{\mathrm{op}}}(N \overline{\otimes} N^{\mathrm{op}}, N) \\
&= \beta_1^{(2)}(N, \tau).
\end{aligned}$$

$\square$

Since $H^1(M, \mathscr{U})$ is the algebraic dual of $H_1(M, \mathscr{U})$ [**Tho08**, Theorem 3.5] it follows from [**Tho08**, Corollary 3.3 & 3.4] that $\beta_1^{(2)}(M) = 0$ if and only if $H^1(M, \mathscr{U})$ is trivial. In the continuous case it is not so clear if the cohomology $H_c^1(M, \mathscr{U})$ is also a dual module, but as the following proposition shows, vanishing of $\eta_1^{(2)}(M)$ does actually imply innerness of all continuous derivations from M to $\mathscr{U}$. As we will see in the following subsections, $\eta_1^{(2)}(M)$ vanishes in a lot of special cases and these vanishing results can therefore be translated into automatic innerness of continuous derivations on M with values in $\mathscr{U}$. We point out that we do not know an example of a von Neumann algebra M for which $\eta_1^{(2)}(M) \neq 0$ and it cannot be excluded that a innerness of continuous derivations from M to $\mathscr{U}$ is automatic in general.

PROPOSITION 3.6. *We have $\eta_1^{(2)}(M) = 0$ if and only if $H_c^1(M, \mathscr{U}) = \{0\}$.*

The proof is a modification of an argument from [**TP11**].

PROOF. The "if" part of the statement is obvious so we only have to prove the converse. Assume therefore that $\eta_1^{(2)}(M) := \dim_{M\overline{\otimes}M^{\mathrm{op}}} H_c^1(M, \mathscr{U}) = 0$ and let $\delta \in \mathrm{Der}_c(M, \mathscr{U})$ be given; we need to prove that δ is inner. By Sauer's local criterion [**Sau05**, Theorem 2.4] we can find a partition $\{p_n\}_{n=1}^{\infty}$ of the unit in $M\overline{\otimes}M^{\mathrm{op}}$ such that $\delta(-)p_n \in \mathrm{Inn}(M, \mathscr{U})$ for every $n \in \mathbb{N}$. Thus, there exists $\xi_n \in \mathscr{U}$ such that

$$\delta(x)p_n = (x \otimes 1 - 1 \otimes x^{\mathrm{op}})\xi_n \quad \text{for all } x \in M,$$

and we may therefore furthermore assume that $\xi_n = \xi_n p_n$ for every $n \in \mathbb{N}$. We now claim that the series $\sum_{n=1}^{\infty} \xi_n$ converges in measure and that its limit implements δ. To prove the convergence it suffices to show that $S_k := \sum_{i=1}^{k} \xi_i$ is Cauchy in measure. For given $\varepsilon > 0$ and k sufficiently big we have $\sum_{i=k+1}^{\infty} \tau(p_i) < \varepsilon$, and therefore $q_k := \sum_{i=1}^{k} p_i$ satisfies $\tau(1-q) < \varepsilon$ and for every $l \geqslant k$ we have

$$(S_l - S_k)q = \sum_{i=k+1}^{l} \sum_{j=1}^{k} \xi_i p_j = \sum_{i=k+1}^{l} \sum_{j=1}^{k} \xi_i p_i p_j = 0,$$

since the p_n's are mutually orthogonal. Thus, S_k is Cauchy in the measure topology (even in the rank topology) and hence the limit $\xi := \lim_k^m S_k$ exists and this limit implements δ since

$$\begin{aligned}\delta(x) &= \lim_{k\to\infty}^{m} \delta(x)q_k = \lim_{k\to\infty}^{m} \sum_{i=1}^{k} \delta(x)p_i = \lim_{k\to\infty}^{m} \sum_{i=1}^{k} (x \otimes 1 - 1 \otimes x^{\mathrm{op}})\xi_i \\ &= \lim_{k\to\infty}^{m} (x \otimes 1 - 1 \otimes x^{\mathrm{op}})S_k = (x \otimes 1 - 1 \otimes x^{\mathrm{op}})\xi. \qquad \square\end{aligned}$$

Note that the above proof does not uses the continuity of the derivation δ at any point, so this also provides a proof of the fact that $\beta_1^{(2)}(M)$ vanishes iff $H^1(M, \mathscr{U}) = \{0\}$.

3.1. The case of diffuse center. Using methods from free probability, Connes and Shlyakhtenko proved in [**CS05**, Corollary 3.5] that $\beta_1^{(2)}(M, \tau) = 0$ when M has diffuse center, and using homological algebraic methods this was later generalized by Thom to higher L^2-Betti numbers in [**Tho08**, Theorem 2.2]. In this section we give a short cohomological proof of this result in degree one.

PROPOSITION 3.7 ([**CS05**, Corollary 3.5]). *If M has diffuse center then every derivation $\delta\colon M \to \mathscr{U}$ is norm-measure topology continuous and $\beta_1^{(2)}(M, \tau)$ is zero.*

PROOF. Since the center $Z(M)$ is diffuse we can choose an identification $Z(M) = L^\infty(\mathbb{T}) = L\mathbb{Z}$. Denote by h a diffuse, selfadjoint generator of $Z(M)$. To see that δ is bounded, it suffices to prove that its graph is closed [**Rud73**, Theorem 2.15]; let therefore $x_n \in M$ with $\|x_n\| \to 0$ and $\delta(x_n) \xrightarrow{m} \eta$. Then

$$\begin{aligned}0 = \delta([x_n, h]) &= (x_n \otimes 1^{\mathrm{op}})\delta(h) + (1 \otimes h^{\mathrm{op}})\delta(x_n) - (1 \otimes x_n^{\mathrm{op}})\delta(h) - (h \otimes 1^{\mathrm{op}})\delta(x_n)\\ &\xrightarrow{m} (1 \otimes h^{\mathrm{op}} - h \otimes 1)\eta,\end{aligned}$$

and since h is diffuse $(1 \otimes h^{\mathrm{op}} - h \otimes 1)$ is not a zero-divisor in $\mathscr{U}$; hence $\eta = 0$. We now claim that δ has to be inner on $Z(M) = L\mathbb{Z}$. If this were not the case, then, by Lemma 3.5, there exists a non-inner derivation $\delta' \colon L\mathbb{Z} \to \mathscr{U}(L\mathbb{Z}\bar{\otimes}L\mathbb{Z})$. By what was just proven, δ' is norm-measure continuous and therefore, by Theorem 3.1, also continuous for the strong operator topology. Because of this, the restriction of δ' to the complex group algebra $\mathbb{C}[\mathbb{Z}]$ has to be non-inner, contradicting the fact that $0 = \beta_1^{(2)}(\mathbb{Z}) = \beta_1^{(2)}(\mathbb{C}[\mathbb{Z}], \tau)$ (cf. [**CS05**, Proposition 2.3] and [**CG86**, Theorem 0.2]). Hence there exists $\xi \in \mathscr{U}$ such that δ agree with $\delta_\xi := [\cdot, \xi]$ on $Z(M)$. The difference $\delta - \delta_\xi$ therefore vanishes on $Z(M)$ and by Lemma 3.4 it has to vanish on every unitary in M. Since the unitaries span M linearly, we conclude that δ is globally inner. □

REMARK 3.8. By combining Lemma 3.4 and Proposition 3.7 with the automatic strong continuity from Theorem 3.1, one may at this point easily deduce the conclusion of [**Tho08**, Theorem 6.4]; namely that the first continuous L^2-Betti number vanishes for non-prime von Neumann algebras as well as for von Neumann algebras admitting a diffuse Cartan subalgebra. Since this will also follow from the more general vanishing results obtained in Section 3.4, we shall not elaborate further at this point.

3.2. Factors with property (T). If Γ is a countable discrete group with property (T) it is well known that its first L^2-Betti number vanishes. This observation dates back to the work of Gromov [**Gro93**], but the first complete proof was given by Bekka and Valette in [**BV97**]. Applying the recent techniques from[**TP11**, Theorem 2.2], this can now be deduced easily from the Delorme-Guichardet theorem (see e.g. [**BdlHV08**]), which characterizes property (T) of Γ in terms of vanishing of its first cohomology groups. The notion of property (T) for II_1-factors was introduced by Connes and Jones in [**CJ85**] and in [**Pet09a**] Peterson proved a version of the Delorme-Guichardet theorem in this context:

THEOREM 3.9 ([**Pet09a**, Theorem 0.1]). *Let M be a finite factor with separable predual. Then the following conditions are equivalent:*

(i) M has property (T)*;*

(ii) there exists a weakly dense $*$-subalgebra $M_0 \subset M$ which is countably generated as a vector space and such that every densely defined L^2-closable derivation from M into a Hilbert M-M-bimodule whose domain contains M_0 is inner.

In this section we apply Peterson's result to prove the following von Neumann algebraic version of the classical group theoretic result mentioned above.

THEOREM 3.10. *Let M be a II_1-factor with separable predual. If M has property* (T) *then* $\eta_1^{(2)}(M,\tau)=0$.

PROOF. Since M has property (T), we obtain from Theorem 3.9 a dense $*$-subalgebra $M_0 \subset M$ such that M_0 is countably generated as a vector space and such that any derivation from M_0 into a Hilbert M-bimodule H, which is closable as an unbounded operator from $L^2(M)$ to H, is inner. We first observe that

$$\dim_{M\bar{\otimes}M^{\mathrm{op}}} \mathrm{Der}_c(M,\mathscr{U}) = \dim_{M\bar{\otimes}M^{\mathrm{op}}}\{\delta \in \mathrm{Der}_c(M,\mathscr{U}) \mid \delta(M_0) \subset M\bar{\otimes}M^{\mathrm{op}}\}.$$

To see this, it suffices by Sauer's local criterion [**Sau05**, Theorem 2.4] to prove that for each continuous derivation $\delta\colon M \to \mathscr{U}$ and each $\varepsilon > 0$ there exists a projection $p \in M\bar{\otimes}M^{\mathrm{op}}$ such that $\tau(p^{\perp}) \leqslant \varepsilon$ and $\delta(-)p$ maps M_0 into $M\bar{\otimes}M^{\mathrm{op}}$. Choose a countable linear basis $(e_n)_{n=1}^{\infty}$ for M_0. Since each $\delta(e_n)$ is affiliated with $M\bar{\otimes}M^{\mathrm{op}}$ there exists a projection $p_n \in M\bar{\otimes}M^{\mathrm{op}}$ such that $\tau(p_n^{\perp}) \leqslant \frac{\varepsilon}{2^n}$ and such that $\delta(e_n)p_n \in M\bar{\otimes}M^{\mathrm{op}}$. The projection $p := \bigwedge_n p_n$ therefore satisfies the requirements. We now have to prove that

$$\dim_{M\bar{\otimes}M^{\mathrm{op}}}\{\delta \in \mathrm{Der}_c(M,\mathscr{U}) \mid \delta(M_0) \subset M\bar{\otimes}M^{\mathrm{op}}\} \leqslant 1,$$

and we will do so by proving that a continuous derivation $\delta\colon M \to \mathscr{U}$ for which $\delta(M_0) \subset M\bar{\otimes}M^{\mathrm{op}}$ has to be inner. We claim that it suffices to prove that δ is L^2-closable from M_0 to $L^2(M\bar{\otimes}M^{\mathrm{op}})$. Indeed, if this is the case, then by Peterson's result there exists a vector $\xi \in L^2(M\bar{\otimes}M^{\mathrm{op}})$ such that

$$\delta(a) = (a\otimes 1)\xi - (1\otimes a^{\mathrm{op}})\xi \quad \text{for all } a \in M_0.$$

Considering ξ as an operator in $\mathscr{U}$, we get that it implements δ on M_0 and hence by Theorem 3.1 it implements δ on all of M. Thus, our task is to show that $\delta\colon M_0 \to L^2(M\bar{\otimes}M^{\mathrm{op}})$ is L^2-closable. Let therefore $x_n \in M_0$ and assume that $x_n \xrightarrow{2} 0$ and $\delta(x_n) \xrightarrow{2} \eta$. Since convergence in 2-norm implies convergence in measure (Lemma 2.1) and since δ is continuous from the measure topology on M to the measure topology on $\mathscr{U}$ (Theorem 3.1) we obtain that $\delta(x_n) \xrightarrow{m} 0$ as well as $\delta(x_n) \xrightarrow{m} \eta$; hence $\eta = 0$ as the measure topology is Hausdorff. $\square$

3.3. Further remarks. In this section we collect a few observations concerning the first continuous L^2-Betti number. Some of them are easily derived from the results in [**Tho08**], but since they are also direct consequences of Theorem 3.1 we include them here for the sake of completeness.

PROPOSITION 3.11. *Let $A \subset M$ be a weakly dense $*$-subalgebra. Then*

$$\eta_1^{(2)}(M,\tau) \leqslant \eta_1^{(2)}(A,\tau) \leqslant \beta_1^{(2)}(A,\tau).$$

and if A is a C^*-algebra we have $\eta_1^{(2)}(M,\tau) = \eta_1^{(2)}(A,\tau)$.

Note that the inequalities in Proposition 3.11 are contained in [**Tho08**, Theorem 6.2], but that the equality when A is a C^*-algebra does not directly follow from this. Compare also with [**Tho08**, Theorem 4.6].

Proof. By Theorem 3.1 the map $H_c^1(M,\mathscr{U}) \longrightarrow H_c^1(A,\mathscr{U})$ induced by restriction is injective in general and an isomorphism when A is a C^*-algebra. That $\eta_1^{(2)}(A,\tau) \leqslant \beta_1^{(2)}(A,\tau)$ is clear, as we have an inclusion $\mathrm{Der}_c(A,\mathscr{U}) \subset \mathrm{Der}(A,\mathscr{U})$ for any algebra A. □

To illustrate the usefulness of the above result, we record the following consequences.

1) If Γ is a discrete countable group with $\beta_1^{(2)}(\Gamma) = 0$ then also $\eta_1^{(2)}(L\Gamma,\tau) = 0$. By [**CS05**, Proposition 2.3], $\beta_1^{(2)}(\Gamma) = \beta_1^{(2)}(\mathbb{C}\Gamma,\tau)$ and the claim therefore follows from Proposition 3.11. In particular, the first continuous L^2-Betti number of the hyperfinite factor R vanishes since $R \simeq L\Gamma$ for any amenable icc group Γ. The result about the hyperfinite factor can also be obtained directly from the definition of hyperfiniteness by realizing R as the von Neumann algebraic direct limit of matrix algebras, for which it is also known [**AK11**, Example 6.9] that the (non-continuous) L^2-Betti numbers of the corresponding *algebraic* direct limit vanishes.
2) It is well known [**BV97**] that when Γ is a discrete, countable group with property (T) then $\beta_1^{(2)}(\Gamma) = 0$ and hence also $\eta_1^{(2)}(L\Gamma,\tau) = 0$. Thus, in the case of factors arising from discrete groups, Theorem 3.10 can be deduced immediately.
3) For the von Neumann algebra $L^\infty(O_n^+)$ associated with the free orthogonal quantum group O_n^+ we have $\eta_1^{(2)}(L^\infty(O_n^+),\tau) = 0$. Denoting by $\mathrm{Pol}(O_n^+)$ the canonical dense Hopf $*$-algebra in $L^\infty(O_n^+)$, it is known that $\beta_1^{(2)}(\mathrm{Pol}(O_n^+),\tau) = 0$ (see [**Kye08**] for the case $n = 2$ and [**Ver09**] for the case $n \geqslant 3$) and hence, by Proposition 3.11, $\eta_1^{(2)}(L^\infty(O_n^+),\tau) = 0$.

Since $\eta_1^{(2)}(-)$ measures the dimension the space of continuous derivations it follows from the results already proven that this number is finite for von Neumann algebras that are finitely generated. Since we will use this repeatedly in the sequel, where a concrete upper bound will be of importance, we single this out by means of the following lemma.

Lemma 3.12. *If M is generated as a von Neumann algebra by n selfadjoint elements then $\eta_1^{(2)}(M,\tau) \leqslant n-1$.*

Proof. If M is generated by n selfadjoint elements $x_1,\dots,x_n$ then the complex subalgebra A generated by $\{1,x_1,\dots,x_n\}$ is a dense unital $*$-subalgebra in M and by Proposition 3.11 any continuous derivation $\delta\colon M \to \mathscr{U}$ is uniquely determined by its values on A. From the derivation property it follows that δ is already completely determined on its values on the generators $x_1,\dots,x_n$ and hence we get $\eta_1^{(2)}(M,\tau) \leqslant n-1$ as desired. □

3.4. The compression formula in continuous cohomology. Recall from [**CS05**, Theorem 2.4] that the algebraic L^2-Betti numbers scale quadratically when passing to corners;

more precisely if M is a finite factor and $p \in M$ is a non-zero projection then $\beta_n^{(2)}(pMp, \tau_p) = \tau(p)^{-2}\beta_n^{(2)}(M, \tau)$, where τ_p denotes the restriction of τ to the corner pMp rescaled with $\tau(p)^{-1}$. In this section we prove that the same holds true for the first continuous L^2-Betti number and, as a byproduct, provide a cohomological proof of the scaling formula for the first algebraic L^2-Betti number as well.

THEOREM 3.13. *Let M be a II_1-factor with trace-state τ and let $p \in M$ be a non-zero projection. Then $\eta_1^{(2)}(pMp, \tau_p) = \frac{1}{\tau(p)^2}\eta_1^{(2)}(M, \tau)$.*

PROOF. Denote $p \otimes p^{\mathrm{op}} \in M\overline{\otimes}M^{\mathrm{op}}$ by q and consider the right $qM\overline{\otimes}M^{\mathrm{op}}q$-linear map

$$\begin{aligned} \Phi_p \colon \mathrm{Der}(M, \mathscr{U}q) &\longrightarrow \mathrm{Der}(pMp, q\mathscr{U}q) \\ \delta &\longmapsto (p \otimes p^{\mathrm{op}}) \cdot \delta|_{pMp} \end{aligned}$$

Note that this map induces a map $\Phi_p \colon H^1(M, \mathscr{U}q) \to H^1(pMp, q\mathscr{U}q)$ on the ordinary cohomology as well as a map on the continuous cohomology $\Phi_p \colon H^1_c(M, \mathscr{U}q) \to H^1_c(pMp, \mathscr{U}q)$, since continuity of a derivation is preserved by construction, and an inner derivation implemented by $\xi \in \mathscr{U}q$ is mapped to the inner derivation on pMp implemented by $q\xi$. We also note that the restriction maps Φ_* are compatible with the order structure: if $r, s \in \mathrm{Proj}(M)$ and $r \leqslant s$ then $\Phi_r = \Phi_r \circ \Phi_s$. Our aim is to prove that the map Φ_p is an isomorphism of right $qM\overline{\otimes}M^{\mathrm{op}}q$-modules on the level of continuous 1-cohomology. Once this is established, the result follows from the general cut-down formula for the dimension function (see e.g. [**KPV12**, Lemma A.15]) since

$$\begin{aligned} \eta_1^{(2)}(pMp, \tau_p) &:= \dim_{qM\overline{\otimes}M^{\mathrm{op}}q} H^1_c\left(pMp, \mathscr{U}(pMp\overline{\otimes}(pMp)^{\mathrm{op}})\right) \\ &= \dim_{qM\overline{\otimes}M^{\mathrm{op}}q} H^1_c(pMp, q\mathscr{U}q) \\ &= \dim_{qM\overline{\otimes}M^{\mathrm{op}}q} H^1_c(M, \mathscr{U}q) \\ &= \dim_{qM\overline{\otimes}M^{\mathrm{op}}q} H^1_c(M, \mathscr{U})q \\ &= \frac{1}{(\tau \otimes \tau^{\mathrm{op}})(q)} \dim_{M\overline{\otimes}M^{\mathrm{op}}} H^1_c(M, \mathscr{U}) \\ &= \frac{1}{\tau(p)^2}\eta_1^{(2)}(M, \tau). \end{aligned}$$

By construction, Φ_p is right $q(M\overline{\otimes}M^{\mathrm{op}})q$-linear so we only have to provide the inverse to $\Phi_p \colon H^1(M, \mathscr{U}q) \to H^1(pMp, q\mathscr{U}q)$ and show that it maps $H^1_c(pMp, q\mathscr{U}q)$ to $H^1_c(M, \mathscr{U}q)$. To this end, choose n to be the smallest integer such that $n\tau(p) \geqslant 1$ and choose orthogonal projections $p_1, \dots, p_n \in M$ summing to 1_M such that $p_1, p_2, \dots, p_{n-1}$ are equivalent to p and p_n is equivalent to a subprojection f of p. We furthermore may, and will, assume that $p_1 = p$. This choice provides us with a $*$-isomorphism $s\mathbb{M}_n(pMp)s \cong M$ where $s \in \mathbb{M}_n(pMp) = \mathbb{M}_n(\mathbb{C}) \otimes pMp$ is the projection $\sum_{i=1}^{n-1} v_{ii} \otimes p + v_{nn} \otimes f$. Here, and in what follows, we denote by $\{v_{ij}\}_{i,j=1}^n$ the standard matrix units in $\mathbb{M}_n(\mathbb{C})$. In the sequel we will suppress this isomorphism and simply identify M and $s\mathbb{M}_n(pMp)s$.

Now we define two maps. The first one is "induction-to-matrices":

$$\operatorname{ind}_n\colon \operatorname{Der}(pMp, \mathscr{U}q) \to \operatorname{Der}\left(\mathbb{M}_n(pMp), \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})^{\mathrm{op}} \otimes q\mathscr{U}q\right)$$

given by

$$\operatorname{ind}_n(\delta)(x) = \sum_{i,j=1}^{n} (v_{i,1} \otimes v_{1,j}^{\mathrm{op}}) \otimes \delta((v_{1,i} \otimes p)x(v_{j,1} \otimes p)).$$

A direct computation verifies that $\operatorname{ind}_n(\delta)$ is indeed a derivation when $\mathbb{M}_n(\mathbb{C})\otimes\mathbb{M}_n(\mathbb{C})^{\mathrm{op}}\otimes q\mathscr{U}q$ is endowed with the natural $\mathbb{M}_n(\mathbb{C}) \otimes pMp$-bimodule structure given by

$$(a \otimes x) \otimes (b \otimes y)^{\mathrm{op}}.T := (a \otimes b^{\mathrm{op}} \otimes (x \otimes y^{\mathrm{op}}))\, T,\ \ a, b \in \mathbb{M}_n(\mathbb{C}), x, y \in pMp, T \in q\mathscr{U}q.$$

The map $\operatorname{ind}_n(\delta)$ descends to both cohomology and continuous cohomology since an inner derivation implemented by $\xi \in q\mathscr{U}q$ maps to the inner derivation implemented by $\sum_{i=k}^{n} v_{k1} \otimes v_{1k}^{\mathrm{op}} \otimes \xi$ and since $\operatorname{ind}_n(-)$ clearly maps continuous derivations to continuous derivations.

The second map is the compression map with respect to the projection s:

$$\begin{aligned}\Phi_s\colon \operatorname{Der}\left(\mathbb{M}_n(pMp), \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})^{\mathrm{op}} \otimes q\mathscr{U}q\right) &\longrightarrow \operatorname{Der}(M, \mathscr{U}q)\\ \delta &\longmapsto s \otimes s^{\mathrm{op}} \cdot \delta|_{s\mathbb{M}_n(pMp)s}\end{aligned}$$

Note that this restriction map indeed maps to $\operatorname{Der}(M, \mathscr{U}q)$: in the matrix picture p identifies with $v_{11} \otimes p$ which is a subprojection of s and hence

$$\begin{aligned}\mathscr{U}q &= \mathscr{U}\left(s \otimes s^{\mathrm{op}}(\mathbb{M}_n(\mathbb{C}) \otimes pMp)\overline{\otimes}(\mathbb{M}_n(\mathbb{C})^{\mathrm{op}} \otimes (pMp)^{\mathrm{op}})s \otimes s^{\mathrm{op}}\right)q\\ &= (s \otimes s^{\mathrm{op}})\mathscr{U}\left((\mathbb{M}_n(\mathbb{C}) \otimes pMp)\overline{\otimes}(\mathbb{M}_n(\mathbb{C})^{\mathrm{op}} \otimes (pMp)^{\mathrm{op}})\right)\\ &= (s \otimes s^{\mathrm{op}}).(\mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})^{\mathrm{op}} \otimes q\mathscr{U}q).\end{aligned}$$

We now claim that $\Phi_s \circ \operatorname{ind}_n$ is the inverse of Φ_p on the level of (continuous) cohomology. One composition can be easily computed using the order compatibility of the restriction maps: $\Phi_p \circ \Phi_s \circ \operatorname{ind}_n = \Phi_p \circ \operatorname{ind}_n$ which is the identity map even at the level of derivations. To see this, consider $x \in pMp$ and recall that in the matrix picture $p \in M$ identifies with the projection $v_{11} \otimes p$. Thus,

$$\begin{aligned}\Phi_p \circ \operatorname{ind}_n(\delta)(x) &= (v_{11} \otimes p) \otimes (v_{11} \otimes p)^{\mathrm{op}} \left(\sum_{i,j=1}^{n} v_{i1} \otimes v_{1j}^{\mathrm{op}} \otimes \delta((v_{i1} \otimes p)x(v_{j1} \otimes p))\right)\\ &= v_{11} \otimes v_{11}^{\mathrm{op}} \otimes \delta(x) = \delta(x).\end{aligned}$$

Next we have to compute $\Phi_s\circ\mathrm{ind}_n\circ\Phi_p$. We start with a derivation $\delta\in\mathrm{Der}(M,\mathscr{U}q)$. Consider the the following two systems of matrix units $\{v_{ij}\otimes(p-f)\}_{i,j=1}^{n-1}$ and $\{v_{ij}\otimes f\}_{i,j=1}^{n}$ in M and denote by A_0 the $*$-algebra they generate. This is a finite dimensional[3] C^*-algebra and hence $\beta_1^{(2)}(A_0,\tau|_{A_0})=0$ by [**CS05**, Proposition 2.9]. By Lemma 3.5, the restriction of δ to A_0 is therefore inner, so by subtracting an inner derivation we may assume that δ vanishes on A_0. Hence for all $a,b\in A_0$ and $x\in M$ we have $\delta(axb)=(a\otimes b^{\mathrm{op}})\delta(x)$. In particular, $\Phi_p(\delta)=\delta|_{pMp}$ since $p=v_{11}\otimes p\in A_0$. Thus, splitting the unit in M as $\sum_{i=1}^{n-1}v_{ii}\otimes p+v_{nn}\otimes f$ we have

$$\begin{aligned}
\delta(x)&=\sum_{i,j=1}^{n-1}\delta((v_{ii}\otimes p)x(v_{jj}\otimes p))+\sum_{i=1}^{n-1}\delta((v_{ii}\otimes p)x(v_{nn}\otimes f))+\\
&+\sum_{i=1}^{n-1}\delta((v_{nn}\otimes f)x(v_{ii}\otimes p))+\delta((v_{nn}\otimes f)x(v_{nn}\otimes f))\\
&=\sum_{i,j=1}^{n-1}\delta((v_{i1}\otimes p)(v_{1i}\otimes p)x(v_{j1}\otimes p)(v_{1j}\otimes p))+\\
&+\sum_{i=1}^{n-1}\delta((v_{i1}\otimes p)(v_{1i}\otimes p)x(v_{n1}\otimes p)(v_{1n}\otimes f))+\\
&+\sum_{i=1}^{n-1}\delta((v_{n1}\otimes f)(v_{1n}\otimes p)x(v_{i1}\otimes p)(v_{1i}\otimes p))+\\
&+\delta((v_{n1}\otimes f)(v_{1n}\otimes p)x(v_{n1}\otimes p)(v_{1n}\otimes f))\\
&=\sum_{i,j=1}^{n-1}(v_{i1}\otimes p)\otimes(v_{1j}\otimes p)^{\mathrm{op}}\delta((v_{1i}\otimes p)x(v_{j1}\otimes p))+\\
&+\sum_{i=1}^{n-1}(v_{i1}\otimes p)\otimes(v_{1n}\otimes f)\delta((v_{1i}\otimes p)x(v_{n1}\otimes p))+\\
&+\sum_{i=1}^{n-1}(v_{n1}\otimes f)\otimes(v_{1i}\otimes p)^{\mathrm{op}}\delta((v_{1n}\otimes f)x(v_{i1}\otimes p))+\\
&+(v_{n1}\otimes f)\otimes(v_{1n}\otimes f)\delta((v_{1n}\otimes p)x(v_{n1}\otimes p))\\
&=(s\otimes s^{\mathrm{op}}).\left(\sum_{i,j=1}^{n}(v_{i1}\otimes v_{1j}^{\mathrm{op}})\otimes\delta((v_{1i}\otimes p)x(v_{j1}\otimes p))\right)\\
&=(s\otimes s^{\mathrm{op}}).\mathrm{ind}_n(\delta|_{pMp})(x)
\end{aligned}$$

[3]The map $\mathbb{M}_{n-1}(\mathbb{C})\oplus\mathbb{M}_n(\mathbb{C})\ni(a,b)\mapsto a\otimes(p-f)+b\otimes f$ is surjective and an isomorphism in the generic case when $f\neq 0$ and $f\neq p$.

$$= (s \otimes s^{\mathrm{op}}).\mathrm{ind}_n(\Phi_p(\delta))(x)$$
$$= \Phi_s \circ \mathrm{ind}_n \circ \Phi_p(\delta)(x),$$

as desired. □

REMARK 3.14. More generally, for any $t > 0$ the t-th amplification of M is defined as $r\mathbb{M}_n(M)r$ where $n = \lfloor t \rfloor + 1$ and $r \in \mathbb{M}_n(M)$ is a projection with $(\mathrm{tr}_n \otimes \tau)(r) = t/n$. We note that the scaling formula holds true in this generality since applying it first to M considered as a corner in $\mathbb{M}_n(M)$ yields $\eta_1^{(2)}(M) = n^{-2}\eta_1^{(2)}(\mathbb{M}_n(M))$ and applying it once more with respect to the projection r therefore gives

$$\eta_1^{(2)}(M_t) = (\mathrm{tr}_n \otimes \tau)(r)^{-2}\eta_1^{(2)}(\mathbb{M}_n(M)) = \frac{n^2}{t^2}n^2\eta_1^{(2)}(M) = t^{-2}\eta_1^{(2)}(M).$$

The isomorphism provided in the proof of Theorem 3.4 is clearly also an isomorphism on the algebraic level so along the way we also proved the following special case of [**CS05**, Theorem 2.4].

PORISM 3.15. If M is a II_1-factor and $t > 0$ then $\beta_1^{(2)}(M_t) = t^{-2}\beta_1^{(2)}(M)$.

COROLLARY 3.16. *If M is a finitely generated II_1-factor with non-trivial fundamental group then $\eta_1^{(2)}(M, \tau) = 0$.*

PROOF. If M is generated by n elements $x_1, \dots, x_n$ then by extracting the real and imaginary part of these generators we get $2n$ selfadjoint generators $a_1, \dots, a_n, b_1, \dots, b_n$ and from Lemma 3.12 it follows that $\eta_1^{(2)}(M) \leqslant 2n < \infty$. Picking a non-trivial projection $p \in M$ such that $M \cong pMp$[4] we conclude from the scaling formula that

$$\eta_1^{(2)}(M, \tau) = \eta_1^{(2)}(pMp, \tau_p) = \frac{1}{\tau(p)^2}\eta_1^{(2)}(M, \tau),$$

and since we just argued that $\eta_1^{(2)}(M, \tau) < \infty$ this forces $\eta_1^{(2)}(M, \tau) = 0$. □

Recently, Shen [**She05**] introduced the *generator invariant* $\mathcal{G}(M)$ and proved that $\mathcal{G}(M) < \frac{1}{4}$ implies that M is singly generated. A further study of the generator invariant, as well as its hermitian analogue $\mathcal{G}_{\mathrm{sa}}(M)$, was undertaken in [**DSSW08**] where the authors, inter alia, prove a scaling formula for the invariant under the passage to corner algebras. This scaling formula implies that the class of II_1 factors with vanishing generator invariant[5] is stable under passing to corners. As a consequence, we obtain the following result showing, yet again, that the first continuous L^2-Betti number vanishes on a large class of II_1 factors.

COROLLARY 3.17. *For any II_1 factor M we have $\eta_1^{(2)}(M) \leqslant \mathcal{G}_{\mathrm{sa}}(M)$. In particular, factors that are either non-prime, admits a Cartan or has property Gamma has vanishing first continuous L^2-Betti number.*

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[4] Since M is a factor this isomorphism must intertwine the trace τ with τ_p.

[5] At the time of writing, no example of a II_1 factor with non-vanishing generator invariant is known.

The inequality in Corollary 3.17 can be deduced from the more general result [**DSSW08**, Corollary 5.12], but since the result there is stated without proof we find it worthwhile to include the short argument below. Note also that the vanishing of $\eta_1^{(2)}(-)$ in the non-prime and Cartan case was already proved by Thom in [**Tho08**] and that in the special case of a *group* von Neumann algebra, the result about property Gamma factors can be deduced from [**Pet09b**, Theorem 1.2] and the general inequality $\eta_1^{(2)}(L\Gamma) \leqslant \beta_1^{(2)}(\Gamma)$.

Proof. We first prove that if k is any integer then if $\mathcal{G}_{\rm sa}(M) < k$ then $\eta_1^{(2)}(M) \leqslant k$. To see this, just note that by [**DSSW08**, Theorems 3.1 & 5.5] we have that M is generated by $k+1$ selfadjoint elements and by Lemma 3.12 this implies that $\eta_1^{(2)}(M) \leqslant k$. The inequality is trivial when $\mathcal{G}_{\rm sa}(M) = \infty$, so assume that $\mathcal{G}_{\rm sa}(M) < \infty$ and put $t_n := \sqrt{\mathcal{G}_{\rm sa}(M) + \frac{1}{n}}$. Then by the scaling formula for sa [**DSSW08**, Corollary 5.6] we have $\mathcal{G}_{\rm sa}(M_{t_n}) < 1$ and hence by what was just proven also $\eta_1^{(2)}(M_{t_n}) \leqslant 1$. By Theorem 3.13 (see also Remark 3.14) we therefor have

$$\eta_1^2(M) = t_n^2 \eta(M_{t_n}) \leqslant t_n^2 \underset{n\to\infty}{\longrightarrow} \mathcal{G}_{\rm sa}(M).$$

That $\eta_1^{(2)}(M) = 0$ when M is non-prime, has a Cartan subalgebra or has property Gamma follows from the formula $\mathcal{G}_s a(M) = \frac{1}{2}\mathcal{G}_{\rm sa}(M)$ [**DSSW08**, Theorem 5.5] in conjunction with [**She05**, Section 6] where it is shown that $\mathcal{G}(M) = 0$ under the aforementioned hypotheses. $\square$

Corollary 3.18. *The first continuous L^2-Betti number vanishes on any class of singly generated* II_1*-factors that is stable under passing to corners.*

Proof. Let $\mathcal{C}$ be such a class of II_1 factors and note that $\eta_1^{(2)}(-)$ is bounded by 1 on $\mathcal{C}$. Let $M \in \mathcal{C}$ be given and choose a sequence of non-trivial projections $p_n \in M$ with $\tau(p_n) \to 0$. Since $\mathcal{C}$ is stable under passing to corners we obtain

$$1 \geqslant \eta_1^{(2)}(p_n M p_n, \tau_{p_n}) = \frac{1}{\tau(p_n)^2}\eta_1^{(2)}(M, \tau).$$

and hence

$$\eta_1^{(2)}(M, \tau) \leqslant \tau(p_n)^2 \underset{n\to\infty}{\longrightarrow} 0.$$

$\square$

Note that above corollary has the following curious consequence: If the notorious generator problem has a positive solution (i.e. every II_1 factor is singly generated) then the first continuous L^2-Betti number vanishes globally on the class of II_1 factors.

We end this section with a result regarding interpolated free group factors which shows that the first continuous L^2-betti number is "linear in the number of generators". This is another consequence of the scaling formula and the proof is verbatim the same as the corresponding proof regarding Shen's generator invariant given in [**DSSW08**]. We include it below for the sake of completeness.

PROPOSITION 3.19. *There exists* $a \in [0,1]$ *such that* $\eta_1^{(2)}(L\mathbb{F}_r) = a(1-r)$ *for every* $r \in]1,\infty[$. *In particular, if* $\eta_1^{(2)}(L\mathbb{F}_2) > 0$ *then the interpolated free group factors* $L\mathbb{F}_r$ *are pairwise non-isomorphic for* $r \in]1,\infty[$.

PROOF. Denote by $f\colon]0,\infty[\to \mathbb{R}$ the function $r \mapsto \eta_1^{(2)}(L\mathbb{F}_{r+1})$ and recall [**Dyk94**] that for $r > 1$ and $\lambda > 0$ the interpolated free group factors satisfy the scaling formula $L(\mathbb{F}_{1+\frac{r-1}{\gamma^2}}) = L(\mathbb{F}_r)_\gamma$. Hence

$$f\left(\frac{r-1}{\lambda^2}\right) = \eta_1^{(2)}\left(\mathbb{F}_{1+\frac{r-1}{\gamma^2}}\right) = \frac{1}{\gamma^2}\eta_1^{(2)}\left(L\mathbb{F}_r\right) = \frac{1}{\gamma^2}f(r-1).$$

The map f therefore satisfies $f(sr) = sf(r)$ for all $s, r > 0$ and thus $f(r) = rf(1) = r\eta_1^{(2)}(L\mathbb{F}_2)$; hence $a := \eta_1^{(2)}(L\mathbb{F}_2)$ does the job and since $\eta_1^{(2)}(L\mathbb{F}_2) \leqslant \beta_1^{(2)}(\mathbb{F}_2) = 1$ we have $a \in [0,1]$. The final statement concerning non-isomorphism follows trivially from this.

$\square$

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CHAPTER 2

First continuous L^2-cohomology of free group factors vanishes

Abstract

We prove that the first continuous L^2-cohomology of free group factors vanishes. This answers a question by Andreas Thom regarding continuity properties of free difference quotients and shows that one can not distinguish free group factors by means of first continuous L^2-Betti number.

1. Introduction

Introduced by topologists [**Ati76**], L^2-Betti numbers have been generalized to various contexts like groups, groupoids etc. Alain Connes and Dimitri Shlyakhtenko [**CS05**] introduced L^2-Betti numbers for subalgebras of finite von Neumann algebras, with the purpose to obtain a suitable notion for arbitrary II_1-factors and in the hope to get a nice homological invariant for them. Unfortunately, as of now there are only very few concrete calculations of them. The most advanced computational result so far is due to Andreas Thom [**Tho08**] who proved that the L^2-Betti numbers vanish for von Neumann algebras with diffuse center. To allow more computable examples, he also introduced a continuous version of the first L^2-Betti number [**Tho08**] which turns out to be much more manageable than its algebraic counterpart. The first continuous L^2-Betti number is defined as the von Neumann dimension of the first continuous Hochschild cohomology of the von Neumann algebra M with values in the algebra of operators affiliated with $M\overline{\otimes}M^{\mathrm{op}}$. The word 'continuous' here means that we restrict attention to derivations which are continuous from the norm topology on M to the measure topology on the affiliated operators.

So far only vanishing results were obtained about the first continuous L^2-Betti number: it has been shown to vanish for II_1-factors with Cartan subalgebras, non-prime II_1-factors [**Tho08**] as well as for II_1-factors with property (T), property Γ and finitely generated II_1 factors with nontrivial fundamental group [**AK13**]. The last result is due to a compression formula for the first continuous L^2-Betti number [**AK13**, Theorem 4.10].

The hope placed upon L^2-Betti numbers for group von Neumann algebras was to be able to connect them with L^2-Betti numbers of groups, thus obtaining a powerful invariant which would be able to distinguish free group factors, thus solving a long-standing problem in operator algebras. In fact, the attempt to do this can be formulated in a very concrete way using generators of the L^2-cohomology of the group ring $\mathbb{C}\mathbb{F}_n$ of the free group or some other subalgebras of $L\mathbb{F}_n$ generated by free elements. One possible choice of generators is to consider the so-called Voiculescu's free difference quotients [**Voi98**]. Andreas Thom posed a natural question in [**Tho08**], whether these derivations possess continuous extensions to operators from $L\mathbb{F}_n$ to $\mathscr{U}(L\mathbb{F}_n\overline{\otimes}L\mathbb{F}_n^{\mathrm{op}})$; a positive answer to this question would solve the free factor isomorphism problem.

In the present paper we answer this question in the negative; in fact, we show that the first continuous L^2-cohomology of free group factors vanishes; in particular, they can not be distinguished by this invariant. This also suggests that the invariant might be altogether trivial, i.e. that the first continuous L^2-cohomology might in fact vanish for all II_1-factors (while preparing the publication, we've been informed that Sorin Popa and Stefaan Vaes answered the above question affirmatively in [**PV14**]).

The result is established in several steps. First, we focus on the free group with three generators $\mathbb{F}_3$ and show that the canonical derivations which "derive in direction of a free generator" cannot be extended to the group von Neumann algebra. This is shown by analyzing their values on some specific elements for which the spectrum of the resulting operators can be calculated using free probability theory. To derive the vanishing of the whole continuous cohomology, we have to use certain automorphisms of the free group factors. Hereby we make use of certain weak mixing properties relative to a subalgebra; intuitively speaking, we are using the fact that there are enough automorphisms to move our derivations around; thus, the existence of one continuous non-inner derivation would automatically guarantee that all derivations of $\mathbb{C}\mathbb{F}_3$ are extendable, which yields a contradiction. Finally, we make use of the compression formula to extend the result from a single free group factor to all of them.

The author thanks Thomas Schick and Andreas Thom for helpful discussions and useful suggestions.

2. Preparatory results

In this section we set up the notation and briefly recapitulate the theory of non-commutative integration and the theory of L^2-Betti numbers for von Neumann algebras.

2.1. Notation. We consider finite von Neumann algebras M, N etc. with separable preduals. We always endow them with a fixed faithful normal tracial state (usually denoted by τ) and consider them in the corresponding GNS representation $L^2(N,\tau)$. If (N,τ) is a finite von Neumann algebra, then there is an induced a faithful normal tracial state on the von Neumann algebraic tensor product $N\overline{\otimes}N^{\mathrm{op}}$ of N with its opposite algebra; abusing notation slightly, we will still denote it by τ. We let $\mathscr{U}(N)$ be the algebra of closed densely defined operators on $L^2(N,\tau)$ affiliated with N. We equip $\mathscr{U}(N)$ with the *measure topology*, defined by the following two-parameter family of zero neighbourhoods:

$$N(\varepsilon,\delta)=\{a\in\mathscr{U}(N)\,|\,\exists p\in\mathrm{Proj}(N):\|ap\|<\varepsilon,\ \tau(p^{\perp})<\delta\},\quad \varepsilon,\delta>0.$$

With this topology, $\mathscr{U}(N)$ is a complete [**Tak03**, Theorem IX.2.5] metrizable [**Rud73**, Theorem 1.24] topological vector space and the multiplication map

$$(a,b) \mapsto ab\colon \mathscr{U}(N) \times \mathscr{U}(N) \to \mathscr{U}(N)$$

is uniformly continuous when restricted to products of bounded subsets [**Nel74**, Theorem 1]. Convergence with respect to the measure topology is also referred to as *convergence in measure* and denoted by $\xrightarrow{m}$. If $\xi \in \mathscr{U}(N)$ and $p \in N$ is its source projection, we denote $\operatorname{rk}\xi := \tau(p)$. Of course, we also have $\operatorname{rk}\xi = \tau(q)$, where q is the target projection of ξ.

Here and in the sequel $\odot$ denotes the algebric tensor product over $\mathbb{C}$. We freely identify M-M-bimodules with $M \odot M^{\mathrm{op}}$-modules. For $N = M\overline{\otimes}M^{\mathrm{op}}$ we equip $\mathscr{U}(M\overline{\otimes}M^{\mathrm{op}})$ with the M-M-bimodule structure

$$m \cdot \xi := (m \otimes 1^{\mathrm{op}})\xi \;\text{ and }\; \xi \cdot m := (1 \otimes m^{\mathrm{op}})\xi \;\text{ for } m, m \in M \text{ and } \xi \in \mathscr{U}.$$

All M-M-sub-bimodules of $\mathscr{U}(M\overline{\otimes}M^{\mathrm{op}})$ inherit this bimodule structure.

Let Γ be a discrete group with its natural left action λ on itself. Its von Neumann algebra $L\Gamma = \lambda(\Gamma)'' \subset \mathbb{B}(\ell^2(\Gamma))$ is equipped with the natural faithful normal tracial state $\tau(\cdot) := \langle \cdot\delta_e, \delta_e\rangle$. Notice that the GNS representation of $L\Gamma$ with respect to τ coincides with $\ell^2(\Gamma)$.

Let $\mathcal{A}$ be an algebra and $\mathcal{X}$ an $\mathcal{A}$-$\mathcal{A}$-bimodule. We recall that a linear map $\delta\colon \mathcal{A} \to \mathcal{X}$ is called a *derivation* if it satisfies

$$\delta(ab) = a \cdot \delta(b) + \delta(a) \cdot b \;\text{ for all } a, b \in \mathcal{A},$$

and that a derivation is called *inner* if there exists a vector $\xi \in \mathcal{X}$ such that

$$\delta(a) = a \cdot \xi - \xi \cdot a \text{ for all } a \in \mathcal{A}.$$

Consider a free group $\mathbb{F}$ generated by a set S. For $s \in S$, a derivation

$$\partial_s\colon \mathbb{C}\mathbb{F} \to L\mathbb{F}\overline{\otimes}L\mathbb{F}^{\mathrm{op}}$$

is defined uniquely by the properties

$$\partial_s(s) = s \otimes 1^{\mathrm{op}}, \quad \partial_s(s') = 0, \quad s \neq s' \in S.$$

In [**CS05**] Connes and Shlyakhtenko introduced L^2-Betti numbers in the general setting of tracial $*$-algebras; if M is a finite von Neumann algebra and $\mathcal{A} \subset M$ is any weakly dense unital $*$-subalgebra its L^2-Betti numbers are defined as

$$\beta_p^{(2)}(\mathcal{A}, \tau) = \dim_{M\overline{\otimes}M^{\mathrm{op}}} \operatorname{Tor}_p^{\mathcal{A}\odot\mathcal{A}^{\mathrm{op}}}(M\overline{\otimes}M^{\mathrm{op}}, \mathcal{A}).$$

Here the dimension function $\dim_{M\overline{\otimes}M^{\mathrm{op}}}(-)$ is Lück's extended von Neumann dimension [**Lüc02**, Chapter 6]. This definition is inspired by the well-known correspondence between representations of groups and bimodules over finite von Neumann algebras, and it extends the classical theory by means of the formula $\beta_p^{(2)}(\Gamma) = \beta_p^{(2)}(\mathbb{C}\Gamma, \tau)$ whenever Γ is a discrete countable group. In [**Tho08**] it is shown that the L^2-Betti numbers also allow the following cohomological description:

$$\beta_p^{(2)}(\mathcal{A}, \tau) = \dim_{M\overline{\otimes}M^{\mathrm{op}}} \mathrm{Ext}^p_{\mathcal{A}\odot\mathcal{A}}(\mathcal{A}, \mathscr{U}),$$

where $\mathscr{U} = \mathscr{U}(M\overline{\otimes}M^{\mathrm{op}})$ denotes the algebra of operators affiliated with $M\overline{\otimes}M^{\mathrm{op}}$. It is a classical fact [**Lod98**, 1.5.8] that the Ext-groups above are isomorphic to the Hochschild cohomology groups of $\mathcal{A}$ with coefficients in $\mathscr{U}$, where the latter is considered as an $\mathcal{A}$-bimodule with respect to the actions

$$a \cdot \xi := (a \otimes 1^{\mathrm{op}})\xi \quad \text{and} \quad \xi \cdot b := (1 \otimes b^{\mathrm{op}})\xi \quad \text{for } a, b \in \mathcal{A} \text{ and } \xi \in \mathscr{U}.$$

In particular, the first L^2-Betti number can be computed as the dimension of the right $M\overline{\otimes}M^{\mathrm{op}}$-module

$$H^1(\mathcal{A}, \mathscr{U}) = \frac{\mathrm{Der}(\mathcal{A}, \mathscr{U})}{\mathrm{Inn}(\mathcal{A}, \mathscr{U})}.$$

Here $\mathrm{Der}(\mathcal{A}, \mathscr{U})$ denotes the space of derivations from $\mathcal{A}$ to $\mathscr{U}$ and $\mathrm{Inn}(\mathcal{A}, \mathscr{U})$ denotes the space of inner derivations. These purely algebraically defined L^2-Betti numbers have turned out extremely difficult to compute in the case when $\mathcal{A}$ is M itself. Besides finite-dimensional algebras, the only computational result known in this direction vanishing for von Neumann algebras with diffuse centre (see [**CS05**, Corollary 3.5] and [**Tho08**, Theorem 2.2]). It is therefore natural to consider variations of the definitions above that take into account the topological nature of M. The continuous version of the first L^2-cohomology module was introduced by Andreas Thom in [**Tho08**], where one restricts attention to those derivations $\delta\colon \mathcal{A} \to \mathscr{U}$ which are closable from the norm topology to the measure topology. Note that when $\mathcal{A}$ is norm closed these are exactly the derivations that are norm–measure topology continuous by the closed graph theorem. We denote the space of closable derivations by $\mathrm{Der}_c(\mathcal{A}, \mathscr{U})$, the continuous cohomology by $H^1_c(\mathcal{A}, \mathscr{U})$ and by $\eta_1^{(2)}(\mathcal{A}, \tau)$ the corresponding continuous L^2-Betti numbers; i.e.

$$\eta_1^{(2)}(\mathcal{A}, \tau) = \dim_{M\overline{\otimes}M^{\mathrm{op}}} H^1_c(\mathcal{A}, \mathscr{U}).$$

Notice that by continuity of multiplication on $\mathscr{U}$, $\mathrm{Der}_c(\mathcal{A}, \mathscr{U})$ is naturally a right $\mathscr{U}$-module.

The first continuous L^2-Betti number satisfies the following compression formula analogous to [**CS05**, Theorem 2.4]:

THEOREM 2.1 ([**AK13**, Theorem 4.10]). *Let M be a II_1-factor with trace-state τ and let $p \in M$ be a non-zero projection. Then $\eta_1^{(2)}(pMp, \tau_p) = \frac{1}{\tau(p)^2}\eta_1^{(2)}(M, \tau)$.*

Although the extended von Neumann dimension is generally not faithful, enlarging the coefficients from $M\overline{\otimes}M^{\mathrm{op}}$ to $\mathscr{U}$ has the effect that $\beta_1^{(2)}(\mathcal{A}, \tau) = 0$ if and only if $H^1(\mathcal{A}, \mathscr{U})$ vanishes [**Tho08**, Corollary 3.3 and Theorem 3.5]. In particular, in order to prove that

$\beta_1^{(2)}(\mathcal{A},\tau)=0$ one has to prove that every derivation from $\mathcal{A}$ into $\mathscr{U}$ is inner. An analogous statement holds for continuous L^2-cohomology:

Proposition 2.2 ([**AK13**, Proposition 4.3]). *Let M be a finite von Neumann algebra. We have $\eta_1^{(2)}(M)=0$ if and only if $H_c^1(M,\mathscr{U})\cong 0$.*

2.2. Some properties of convergence in measure. Here we prove several lemmas which will help us to analyse convergence in measure of some particular operators. The following lemma tells us that we can analyse convergence in measure "locally".

Lemma 2.3. *Let $\{a_n\}_{n=1}^{\infty}\subset\mathscr{U}(N)$ be a sequence. Then $a_n\xrightarrow{m}0$ if and only if*

$$\forall\varepsilon>0\quad \tau\left(\chi_{[\varepsilon^2,+\infty)}(a_n^*a_n)\right)\to 0,\quad n\to\infty. \tag{2.1}$$

Proof. Suppose that (2.1) is satisfied. For every $\varepsilon>0$ set $p_n:=1-\chi_{[\varepsilon^2,+\infty)}(a^*a)$. We obtain

$$\|a_np_n\|^2=\|p_na_n^*a_np_n\|\leqslant\varepsilon^2$$

and

$$\tau(p_n)\to 1,\quad n\to\infty.$$

Thus, $a_n\xrightarrow{m}0$.

On the other hand, let $a_n\xrightarrow{m}0$ and suppose that (2.1) is false. Then, extracting a subsequence if needed, we may assume that

$$\tau\left(\chi_{[\varepsilon^2,+\infty)}(a_n^*a_n)\right)\geqslant\delta,\quad n\geqslant N_0.$$

Suppose now that there exists a sequence of projections $\{q_n\}_{n=1}^{\infty}$ such that

$$\|a_nq_n\|\to 0,\quad \tau(q_n)\to 1.$$

Then for sufficiently big n we get $\tau(q_n)>1-\delta/2$. Putting $r_n:=\chi_{[\varepsilon^2,+\infty)}(a_n^*a_n)$, we obtain

$$\|a_nq_n\|^2=\|q_na_n^*a_nq_n\|\geqslant\|r_n'a_n^*a_nr_n'\|\geqslant\varepsilon^2$$

for $r_n':=r_n\wedge q_n$, which satisfies $\tau(r_n')\geqslant\delta/2$, obtaining a contradiction. $\square$

Lemma 2.4. *Let $a_n\in\mathscr{U}(N)$ be a sequence of selfadjoint elements such that*

$$\forall\varepsilon>0\quad \tau\left(\chi_{[-\varepsilon,\varepsilon]}(a_n)\right)\to 0,\quad n\to\infty.$$

Then for every nonzero projection $p\in N$

$$a_np\not\to_m 0.$$

Proof. From the assumption of the lemma it immediately follows that

$$\forall\varepsilon>0\quad \tau\left(\chi_{[\varepsilon^2,+\infty)}(a_n^*a_n)\right)\to 1.$$

Setting $r_n:=\chi_{[\varepsilon^2,+\infty)}(a_n^*a_n)$, we get $\tau(r_n)\to 1,\quad n\to\infty$. Now, if $\tau(p)=\delta>0$, then there exists an N_0 such that $\tau(r_n)>1-\delta/2$, $n\geqslant N_0$. Thus for such n we obtain that the projections $q_n := r_n \wedge p$ satisfy $\tau(q_n) \geqslant \delta/2$. It follows that

$$q_n a_n^* a_n q_n \geqslant \varepsilon^2 q_n,$$

and hence by Lemma 2.3 $q_n a_n^* a_n q_n \not\to_m 0$. On the other hand, if $a_n p \xrightarrow{m} 0$, then by boundedness of q_n

$$q_n a_n^* a_n q_n = q_n p a_n^* a_n p q_n \xrightarrow{m} 0.$$

This proves the lemma. □

2.3. Continuity properties of derivations. We will be interested in continuity of certain derivations. To understand it, we recollect some useful notions and properties here.

Let $\mathcal{A} \subset M$ be a weakly dense $*$-subalgebra and $\delta \colon \mathcal{A} \to \mathscr{U}$ be a derivation. Let

$$P_\delta := \{p \in \operatorname{Proj}(M \overline{\otimes} M^{\mathrm{op}}) \mid \delta \cdot p \text{ is continuous}\}.$$

LEMMA 2.5. *P_δ is a complete sublattice of* $\operatorname{Proj}(M \overline{\otimes} M^{\mathrm{op}})$.

PROOF. If $\xi \in \mathscr{U}$, the right $\mathscr{U}$-submodule generated by $\delta \cdot \xi$ contains the derivation $\delta \cdot t(\xi)$, where $t(\xi)$ is the target projection of ξ. If now $p_1, p_2 \in P_\delta$, then $p_1 \wedge p_2 \in P_\delta$ for obvious reasons, and $p_1 \vee p_2 \in P_\delta$ because $\delta \cdot (p_1 + p_2)$ is continuous and $p_1 \vee p_2 = t(p_1 + p_2)$. If now $\{p_i\}_{i \in I}$ is an orthogonal family of projections with sum p, then

$$\sum_{i \in I} \tau(p_i) = \tau(p) \leqslant 1,$$

and therefore the series

$$\sum_{i \in I} \delta(x) \cdot p_i$$

converges uniformly in measure to $\delta(x) \cdot p$. As sums of uniformly convergent series of continuous maps are continuous, $\delta \cdot p$ is continuous. □

DEFINITION 2.6. We call the unique supremum of P_δ the *continuity projection* of δ.

Let $\delta \colon \mathcal{A} \to \mathscr{U}$ be a derivation and $\sigma \in \operatorname{Aut}(M)$ be an automorphism with $\sigma(\mathcal{A}) = \mathcal{A}$; by slight abuse of notation, we still denote by σ the induced automorphism $\sigma \otimes \sigma^{\mathrm{op}} \in \operatorname{Aut}(\mathscr{U})$. The map

$$\delta^\sigma := \sigma^{-1} \circ \delta \circ \sigma$$

is then a derivation $\mathcal{A} \to \mathscr{U}$. If δ is continuous, then δ^σ is obviously continuous as well. Notice that for $\xi \in \mathscr{U}$ we have

$$(\delta \cdot \xi)^\sigma = \delta^\sigma \cdot \sigma^{-1}(\xi).$$

The following observation is easy, but very useful.

LEMMA 2.7. *If $\Sigma \subset \operatorname{Aut}(M)$ is a subgroup of automorphisms of M leaving $\mathcal{A}$ invariant and $\delta^\sigma = \delta$ for $\sigma \in \Sigma$, then the continuity projection of δ is Σ-invariant.*

PROOF. Let p be the continuity projection of δ. Then $\delta \cdot p$ is continuous, and therefore

$$(\delta \cdot p)^\sigma = \delta^\sigma \cdot \sigma^{-1}(p).$$

is continuous as well. If for some $\sigma \in \Sigma$ we have $\sigma^{-1}(p) \neq p$, then $\sigma^{-1}(p) \vee p \in P_\delta$ has p as a proper subprojection, contradicting maximality. □

2.4. Some automorphisms of free group factors and their mixing properties. Let α be a trace-preserving automorphism of $(L\mathbb{Z}, \tau)$, where $\tau(x) = \langle x\delta_e, \delta_e\rangle$ is the canonical trace. It induces automorphisms of $L\mathbb{F}_3$ obtaining by decomposing $L\mathbb{F}_3$ as a free product with respect to the subalgebras $(L\mathbb{Z})_a$, $(L\mathbb{Z})_b$, $(L\mathbb{Z})_c$ generated by a, b and c using the free product decomposition $L\mathbb{F}_3 = (L\mathbb{Z})_a * (L\mathbb{Z})_b * (L\mathbb{Z})_c$. We denote

$$\mathrm{Aut}_a(L\mathbb{F}_3) = \{\alpha * \mathrm{id} \in \mathrm{Aut}(L\mathbb{F}_3 \cong (L\mathbb{Z})_a * L\mathbb{F}_2) \,|\, \alpha \in \mathrm{Aut}(L\mathbb{Z}, \tau)\},$$

$$\mathrm{Aut}_b(L\mathbb{F}_3) = \{\mathrm{id} * \beta * \mathrm{id} \in \mathrm{Aut}(L\mathbb{F}_3 \cong L\mathbb{Z} * (L\mathbb{Z})_b * L\mathbb{Z}) \,|\, \beta \in \mathrm{Aut}(L\mathbb{Z}, \tau)\}.$$

$$\mathrm{Aut}_c(L\mathbb{F}_3) = \{\mathrm{id} * \gamma \in \mathrm{Aut}(L\mathbb{F}_3 \cong L\mathbb{F}_2 * (L\mathbb{Z})_c) \,|\, \gamma \in \mathrm{Aut}(L\mathbb{Z}, \tau)\}.$$

We get actions of the groups $G_c = \mathrm{Aut}_c(L\mathbb{F}_3)$, $G_{bc} = \mathrm{Aut}_b(L\mathbb{F}_3) \times \mathrm{Aut}_c(L\mathbb{F}_3)$, and $G = \mathrm{Aut}_a(L\mathbb{F}_3) \times \mathrm{Aut}_b(L\mathbb{F}_3) \times \mathrm{Aut}_c(L\mathbb{F}_3)$ on $L\mathbb{F}_3$.

The following definition of the relative mixing property appeared in [**Pop07**, Definition 2.9]. We will use a strengthening of it.

DEFINITION 2.8. Let $N \subset M$ be a trace-preserving inclusion of finite von Neumann algebras and let $\sigma\colon G \to \mathrm{Aut}(M)$ be a trace-preserving action of a group G on M such that $\sigma_g(N) = N$ for all $g \in G$. The action σ is called weakly mixing relative to N if for every finite set $F \subset M \ominus N$ and for every $\varepsilon > 0$ there is a $g \in G$ such that

$$\|E_N(y^*\sigma_g(x))\|_2 < \varepsilon, \quad x, y \in F.$$

An action σ which is weakly mixing relative to $N = \mathbb{C}$ is called weakly mixing.

When N is pointwise fixed by the action, we get the following result resembling the classical equivalent characterisations of weakly mixing actions (cf. [**Vae07**, Proposition D.2]).

PROPOSITION 2.9. *Let $N \subset M$ be a trace-preserving inclusion of finite von Neumann algebras $\sigma\colon G \to \mathrm{Aut}(M)$ be a trace-preserving action such that $\sigma_g(n) = n$ for all $g \in G$ and $n \in N$. Then each of the following conditions implies the next one:*

(i) σ is weakly mixing relative to N;
(ii) for every $x_1, \ldots, x_n \in M \ominus N$ there exists a sequence $g_j \in G$ such that for every $y \in M$ $\|E_N(y^*\sigma_{g_j}(x_i))\|_2 \to 0$, $j \to \infty$, $i = 1, \ldots, n$.
(iii) every finite-dimensional invariant subspace of M is contained in N;
(iv) for every action ρ of G on a finite von Neumann algebra P

$$(M \overline{\otimes} P)^{\sigma \otimes \rho} = N \overline{\otimes} P^\rho.$$

PROOF. The proof of [**Vae07**, Prop. D.2] goes through with small modifications. i) $\Rightarrow$ ii) is immediate. To show ii) $\Rightarrow$ iii), take a finite-dimensional invariant subspace $V \subset M$. Consider the space $V' := (1-E_N)(V) \subset M \ominus N$; it is also finite-dimensional and G-invariant. But in view of ii) we have for every $x \in V'$ and $y \in M$ that

$$\tau(E_N(y^*\sigma_{g_j}(x))) = \langle \sigma_{g_j}(x), y \rangle \to 0, \quad j \to \infty.$$

It means that $\sigma_{g_j}(x) \to 0$ weakly as $j \to \infty$. As V' is finite-dimensional, this implies convergence to zero in norm, and as the G-action is norm-preserving, this means that $x = 0$. Thus, $V' = \{0\}$, thus V lies in N.

To show iii) $\Rightarrow$ iv), take $T \in (M \overline{\otimes} P)^{\sigma\otimes\rho} \subset L^2(M) \otimes L^2(P)$ and view it as a Hilbert–Schmidt operator $T\colon \overline{L^2(P)} \to L^2(M)$. Then the image of T is contained in M, and the operator TT^* is trace-class and commutes with the G-action. Taking its spectral projection, we obtain a finite-dimensional G-invariant subspace of M, which is necessarily contained in N by iii). Thus, the image of T lies in N, and therefore $T \in N \overline{\otimes} P^\rho$. $\square$

LEMMA 2.10. *Let Q and N be finite von Neumann algebras and $\sigma\colon G \to \mathrm{Aut}(Q)$ a trace-preserving weakly mixing action of a group G on Q. Then the action $\sigma * \mathrm{id}$ of G on the free product von Neumann algebra $M = Q * N$ is weakly mixing relative to N.*

PROOF. By a standard density argument it is enough to check the relative weak mixing condition on the algebraic free product $\mathcal{M} = Q *_{\mathrm{alg}} N \subset M$, which is the algebra generated by Q and N inside M. It is spanned by N and alternating products of elements from $N \ominus \mathbb{C}1$ und $Q \ominus \mathbb{C}1$; without loss of generality we may and will assume that the operator norms of all factors are bounded by 1. Moreover we have

$$E_N(n) = n, \quad n \in N$$

and

$$E_N(q_1 n_1 \cdots q_{k-1} n_{k-1} q_k) = 0$$

for $q_i \in Q \ominus \mathbb{C}1$, $n_i \in N \ominus \mathbb{C}1$. Thus, for $q_i, q_i' \in Q \ominus \mathbb{C}1$, $n_i, n_i' \in N \ominus \mathbb{C}1$ we get

$$E_N(q_1 n_1 \cdots q_k n_k n_\ell' q_\ell' \cdots n_1' q_1') = 0, \quad k \neq \ell,$$

$$E_N(q_1 n_1 \cdots q_k n_k n_k' q_k' \cdots n_1' q_1') = E_N(n_1 q_1 \cdots n_k q_k q_k' n_k' \cdots q_1' n_1') = \prod_{i=1}^k \tau(n_i n_i')\tau(q_i q_i').$$

Thus, $\mathcal{M} \ominus \mathcal{N}$ is spanned by the alternating products of elements from $N \ominus \mathbb{C}1$ and $Q \ominus \mathbb{C}1$, and it's enough to check the weak mixing property for such alternating products. Given a finite set F of them, let $F' \in Q \ominus \mathbb{C}1$ be all factors from $Q \ominus \mathbb{C}1$ occuring in the products. By the weak mixing property we find a $g \in G$ such that

$$|\tau(q\sigma_g(q'))| < \varepsilon < 1, \quad q, q' \in F'.$$

The above formulae for E_N then imply for two elements $x, y \in F$ that $E_N(y^*\sigma_g(x)) = 0$ unless

$$y^* = nq_1n_1 \cdots q_kn_k \text{ and } x = n'_kq'_k \cdots n'_1q'_1n',$$

where $q_i, q'_i \in Q \ominus \mathbb{C}1$, $n_i, n'_i \in N \ominus \mathbb{C}1$, $\|n_i\| \leqslant 1$, $\|n'_i\| \leqslant 1$, $\|q_i\| \leqslant 1$, $\|q'_i\| \leqslant 1$, $n, n' \in N$ with $\|n\| \leqslant 1$, $\|n'\| \leqslant 1$.

In this case we get

$$\|E_N(nq_1n_1 \cdots q_kn_k(\sigma_g * \mathrm{id})(n'_kq'_k \cdots n'_1q'_1n'))\|_2 \leqslant \prod_{i=1}^{k} |\tau(n_in'_i)\tau(q_i\sigma_g(q'_i))| < \varepsilon$$

which proves the statement. □

Using the existence of weakly mixing actions on an arbitrary finite von Neumann algebra (e.g. Bernoulli actions [**Pop06**, Sect. 2.4]) and Proposition 2.9, we obtain

Corollary 2.11. *Consider the actions of $G_c = \mathrm{Aut}_c(L\mathbb{F}_3)$, $G_{bc} = \mathrm{Aut}_b(L\mathbb{F}_3) \times \mathrm{Aut}_c(L\mathbb{F}_3)$, and $G = \mathrm{Aut}_a(L\mathbb{F}_3) \times \mathrm{Aut}_b(L\mathbb{F}_3) \times \mathrm{Aut}_c(L\mathbb{F}_3)$ on $L\mathbb{F}_3$ described above. Then*

(i) every G_{bc}-invariant element $L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}}$ is contained in $(L\mathbb{Z})_a \overline{\otimes} (L\mathbb{Z})_a^{\mathrm{op}}$;

(ii) every G_c-invariant element $L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}}$ is contained in $(L\mathbb{F}_2)_{ab} \overline{\otimes} (L\mathbb{F}_2)_{ab}^{\mathrm{op}}$.

3. Non-continuous derivations

Let $\mathbb{F}_3 = \langle a, b, c \rangle$ be a free group on three generators. We will naturally view $\mathbb{F}_2 = \langle b, c \rangle$ as a subgroup of $\mathbb{F}_3$. It's well-known that $\mathbb{F}_2 = \langle b, c \rangle$ contains a copy of $\mathbb{F}_\infty = \langle g_1, g_2, \dots \rangle$, and we well fix such a copy.

We recall that the derivation

$$\partial_a : \mathbb{C}[\mathbb{F}_3] \to \mathscr{U}(L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}})$$

is uniquely defined by the conditions

$$\partial_a(a) = a \otimes 1^{\mathrm{op}}, \quad \partial_a(b) = \partial_a(c) = 0.$$

It can obviously be extended to the algebra generated by a and $(L\mathbb{F}_2)_{bc}$, and we will still denote this extension by ∂_a.

Proposition 3.1. *Let $p \in L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}}$ be a nonzero projection. There is no norm-measure continuous extension of $\partial_a \cdot p$ to $L\mathbb{F}_3$.*

Proof. We will first construct a particular sequence $y_n \in \mathbb{C}[\mathbb{F}_3]$ such that $\|y_n\|_\infty \to 0$, $n \to \infty$, but $\partial_a(y_n) \not\to_m 0$. We set

$$x_n := g_1ag_2ag_3a \cdots ag_na \in \mathbb{C}[\mathbb{F}_3]$$

and consider

$$\partial_a(x_n) = \sum_{k=1}^{n} g_1a \cdots g_ka \otimes g_{k+1}a \cdots g_na$$

The elements

$$g_1a,\ g_1ag_2a, \ldots,\ g_1ag_2a\cdots g_na \in \mathbb{F}_3$$

are free. Indeed, the family $\{g_ia\}_{i=1}^n$ is free, because $\{g_i\}_{i=1}^n$ is a free family which is itself free from a.

Therefore the elements

$$(g_1a\cdots g_ka, g_{k+1}a\cdots g_na) \in \mathbb{F}_3 \times \mathbb{F}_3^{\mathrm{op}}, \quad k = \overline{1,n}$$

are free and hence form a freely independent family of unitaries $\{u_i\}_{i=1}^n \subset L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}}$. Consider the elements

$$h_n := \operatorname{Re}\partial_a(x_n) = \frac{1}{2}(\partial_a(x_n) + \partial_a(x_n)^*) = \frac{1}{2}\sum_{i=1}^n (u_i + u_i^*) \in L\mathbb{F}_3\overline{\otimes}L\mathbb{F}_3^{\mathrm{op}}.$$

The spectral density of this operator can be computed using free probability theory. Indeed, h_n is an instance of a scaled random walk operator on a free group, and therefore we can use the formula for the Cauchy transform for its spectral density from [**VDN92**, Example 3.4.5]:

$$G_{h_n}(\zeta) = \frac{n\zeta\sqrt{1-(2n-1)\zeta^{-2}} - (n-1)}{\zeta^2 - n^2}$$

Using the Stieltjes inversion formula, we get the spectral density of h_n:

$$d\mu_{h_n} = \begin{cases} \dfrac{n\sqrt{(2n-1)-x^2}}{n^2-x^2}dx, & x \in [-\sqrt{2n-1}, \sqrt{2n-1}], \\ 0 & \text{otherwise.} \end{cases}$$

We see that

$$\tau\left(\chi_{[-\varepsilon,\varepsilon]}\left(\frac{h_n}{\sqrt[4]{n}}\right)\right) = \int_{-\varepsilon\sqrt[4]{n}}^{\varepsilon\sqrt[4]{n}} \frac{n\sqrt{(2n-1)-x^2}\,dx}{n^2-x^2} \leqslant \frac{2\varepsilon\cdot n^{5/4}\sqrt{2n-1}}{n^2-\varepsilon^2\sqrt{n}} \to 0, \quad n\to\infty.$$

In view of Lemma 2.4, for every nonzero projection $p \in L\mathbb{F}_3\overline{\otimes}L\mathbb{F}_3^{\mathrm{op}}$ we get

$$\frac{h_n}{\sqrt[4]{n}}\cdot p \not\to_m 0, \quad n\to\infty.$$

In particular,

$$\partial_a\left(\frac{x_n}{\sqrt[4]{n}}\right) \not\to_m 0, \quad n \to \infty,$$

although $x_n/\sqrt[4]{n}$ converges to 0 in norm, because $\|x_n\|_\infty = 1$. Thus, $y_n := x_n/\sqrt[4]{n}$ satisfies the required properties, and the derivation ∂_a is not continuous.

Now we have to show that the continuity projection q of δ is equal to 0. The derivation ∂_a is invariant under the actions of the groups $\mathrm{Aut}_b(L\mathbb{F}_3)$ and $\mathrm{Aut}_c(L\mathbb{F}_3)$. By Lemma 2.7 we get that q is invariant under $\mathrm{Aut}_b(L\mathbb{F}_3)$ and $\mathrm{Aut}_c(L\mathbb{F}_3)$, hence by Corollary 2.11, i) it belongs to $(L\mathbb{Z})_a\overline{\otimes}(L\mathbb{Z})_a^{\mathrm{op}}$.

Now, for $\alpha \in \mathrm{Aut}_a(L\mathbb{F}_3)$ we infer that

$$\partial_a^\alpha(x) = \alpha^{-1}(\partial_a(\alpha(x)))\alpha^{-1}(q) = \partial_a \cdot (a^{-1} \otimes 1^{\mathrm{op}})\alpha^{-1}(\partial_a(\alpha(a)))\alpha^{-1}(q)$$

is continuous (the last equality follows by evaluating at a, b, c). The support projection of $(a^{-1} \otimes 1^{\mathrm{op}})\alpha^{-1}(\partial_a(\alpha(a)))\alpha^{-1}(q)$ is equal to $\alpha^{-1}(q)$, and by maximality of q it follows that $\alpha^{-1}(q) = q$. Therefore, $q = 0$ or $q = 1$, but the latter case is impossible because ∂_a is not continuous.

□

For obvious symmetry reasons, the statement of Proposition 3.1 is also true for the derivations

$$\partial_b,\ \partial_c \colon \mathbb{C}[\mathbb{F}_3] \to \mathscr{U}(L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}})$$

uniquely determined by the conditions

$$\partial_b(b) = b \otimes 1^{\mathrm{op}}, \quad \partial_b(a) = \partial_b(c) = 0$$

resp.

$$\partial_c(c) = c \otimes 1^{\mathrm{op}}, \quad \partial_c(a) = \partial_c(b) = 0.$$

Is is well-known that for every nonzero projection $p \in L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}}$ the derivations $\partial_a \cdot p$, $\partial_b \cdot p$, $\partial_c \cdot p$ are not inner. The derivations ∂_a, ∂_b, ∂_c freely generate the module $\mathrm{Der}(\mathbb{C}\mathbb{F}_3, \mathscr{U}(L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}}))$.

Theorem 3.2. *The first continuous L^2-cohomology of $L\mathbb{F}_3$ vanishes:*

$$H_c^1(L\mathbb{F}_3, \mathscr{U}(L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}})) \cong 0.$$

Proof. The restriction of δ to $\mathbb{C}\mathbb{F}_3$ is a derivation and therefore can be uniquely written as a combination of ∂_a, ∂_b, ∂_c:

$$\delta(x) = \partial_a(x) \cdot \xi_a'' + \partial_b(x) \cdot \xi_b'' + \partial_c(x) \cdot \xi_c'', \quad x \in \mathbb{C}\mathbb{F}_3$$

for some ξ_a'', ξ_b'', $\xi_c'' \in \mathscr{U}(L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}})$. As inner derivations are continuous, after subtracting an inner derivation we may assume that $\xi_c = 0$ and

$$\delta(x) = \partial_a(x) \cdot \xi_a' + \partial_b(x) \cdot \xi_b', \quad x \in \mathbb{C}\mathbb{F}_3.$$

The right $\mathscr{U}$-module generated by $\partial_a \cdot \xi_a$ contains $\partial_a \cdot p_a$, where p_a is the target projection of ξ_a. As the right $\mathscr{U}$-action preserves continuity, we may assume that $\xi_a = p_a$ and that δ has the form

$$\delta(x) = \partial_a(x) \cdot p_a + \partial_b(x) \cdot \xi_b, \quad x \in \mathbb{C}\mathbb{F}_3.$$

Multiplying from the right with $(1 - p_a)$ and using Proposition 3.1, we deduce $\xi_b(1 - p_a) = 0$. Thus, $\mathrm{rk}\,\xi_b \leqslant \mathrm{rk}\,p_a$; reasoning symmetrically, we infer $\mathrm{rk}\,\xi_b = \mathrm{rk}\,\xi_a = \mathrm{rk}\,p_a$. We also observe that for every $\delta \in \mathrm{Der}_c(\mathbb{C}\mathbb{F}_3, \mathscr{U}(L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}}))$, the elements p_a and ξ_b are uniquely determined.

Now, let

$$P_{\mathrm{cont}} = \{p \in \mathrm{Proj}(L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}}) \mid \exists \delta \in \mathrm{Der}_c(\mathbb{C}\mathbb{F}_3, \mathscr{U}(L\mathbb{F}_3 \overline{\otimes} L\mathbb{F}_3^{\mathrm{op}})) \colon p = p_a\}.$$

Analogously to Lemma 2.5, we are going to prove that P_{cont} is a complete lattice. Indeed, if $p_1, p_2 \in P_{\text{cont}}$, then $p_1 \wedge p_2 \in P_{\text{cont}}$ because $p_1 \cdot (p_1 \wedge p_2) = p_1 \wedge p_2$. Now, if δ_1 and δ_2 are derivations corresponding to p_1 resp. p_2, then $p_1 \vee p_2$, being the support projection of $p_1 + p_2$, corresponds to the derivation $\delta_1 + \delta_2$. Completeness of P_{cont} is proven as follows: if let $(p_i)_{i \in I}$ be an orthogonal family in P_{cont} with corresponding derivations δ_i with elements $\xi'_{b,i}$. By the observation above, $\text{rk}(\xi'_{b,i}) = \text{rk}(p_i)$. But then

$$\sum_{i \in I} \text{rk}(\xi'_{b,i}) = \sum_{i \in I} \text{rk}(p_i) \leqslant 1$$

and therefore as in Lemma 2.5 the series

$$\sum_{i \in I} \delta_i(x) = \sum_{i \in I} (\partial_a(x) \cdot p_i + \partial_b(x) \cdot \xi_{b,i})$$

converges uniformly in measure to a derivation δ having the supremum of p_i as the corresponding projection p_a.

Thus, P_{cont} is a complete lattice. Let p be its maximal element. In view of the equalities

$$\delta^\beta(x) = \partial_a(x) \cdot \beta^{-1}(p_a) + \beta^{-1}(\partial_b(\beta(x)))\beta^{-1}(\xi_b)$$

and

$$\delta^\gamma(x) = \partial_a(x) \cdot \gamma^{-1}(p_a) + \partial_b(x) \cdot \gamma^{-1}(\xi_b)$$

for $\beta \in \text{Aut}_b(L\mathbb{F}_3)$ and $\gamma \in \text{Aut}_c(L\mathbb{F}_3)$ we get by maximality that p is invariant under $\text{Aut}_b(L\mathbb{F}_3)$ and $\text{Aut}_c(L\mathbb{F}_3)$. Therefore by Corollary 2.11, i) we deduce that $p \in (L\mathbb{Z})_a \overline{\otimes} (L\mathbb{Z})_a^{\text{op}}$.

Now, for $\alpha \in \text{Aut}_a(L\mathbb{F}_3)$ we get that

$$\begin{aligned}\delta^\alpha(x) &= \alpha^{-1}(\partial_b(\alpha(x)))\alpha^{-1}(p) + \partial_b(x) \cdot \alpha^{-1}(\xi_b) \\ &= \partial_a \cdot (a^{-1} \otimes 1^{\text{op}})\alpha^{-1}(\partial_a(\alpha(x)))\alpha^{-1}(p) + \partial_b(x) \cdot \alpha^{-1}(\xi_b)\end{aligned}$$

is continuous. The support projection of $(a^{-1} \otimes 1^{\text{op}})\alpha^{-1}(\partial_a(\alpha(a)))\alpha^{-1}(p)$ is equal to $\alpha^{-1}(p)$, and by maximality of p it follows that $\alpha^{-1}(p) = p$. Therefore, $p = 0$ or $p = 1$.

If $p = 1$, we are given a continuous derivation of the form

$$\delta_0(x) = \partial_a(x) + \partial_b(x) \cdot \zeta_b, \quad x \in \mathbb{C}\mathbb{F}_3.$$

For $\gamma \in \text{Aut}_c(L\mathbb{F}_3)$ we obtain

$$\delta_0^\gamma(x) = \partial_a(x) + \partial_b(x) \cdot \gamma^{-1}(\zeta_b), \quad x \in \mathbb{C}\mathbb{F}_3.$$

As $\delta_0^\gamma - \delta_0$ is continuous, Proposition 3.1 implies that

$$\gamma^{-1}(\zeta_b) = \zeta_b, \quad \gamma \in \text{Aut}_c(L\mathbb{F}_3).$$

Thus using Corollary 2.11, ii), we get that $\zeta_b \in \mathscr{U}(L\mathbb{F}_2 \overline{\otimes} L\mathbb{F}_2^{\text{op}})$, where $L\mathbb{F}_2 \subset L\mathbb{F}_3$ is generated by a and b. Consider the restriction of the derivation δ to $L\mathbb{F}_2$. Subtracting an inner derivation and multiplying with a suitable element of $\mathscr{U}(L\mathbb{F}_2 \overline{\otimes} L\mathbb{F}_2^{\text{op}})$ from the right, we may assume that the derivation

$$\delta_1(x) = \partial_a(x) r_a, \quad x \in \mathbb{C}\mathbb{F}_2,$$

is continuous for some nonzero projection r_a. Thus, the continuity projection of $\partial_a : \mathbb{C}\mathbb{F}_2 \to \mathscr{U}(L\mathbb{F}_2 \overline{\otimes} L\mathbb{F}_2^{\mathrm{op}})$ is nonzero. Arguing as in Proposition 3.1, we deduce that the continuity projection of $\partial_a : \mathbb{C}\mathbb{F}_2 \to \mathscr{U}(L\mathbb{F}_2 \overline{\otimes} L\mathbb{F}_2^{\mathrm{op}})$ is equal to 1. For symmetry reasons, $\partial_b : \mathbb{C}\mathbb{F}_2 \to \mathscr{U}(L\mathbb{F}_2 \overline{\otimes} L\mathbb{F}_2^{\mathrm{op}})$ is continuous as well. Therefore the module $\mathrm{Der}_c(L\mathbb{F}_2, \mathscr{U}(L\mathbb{F}_2 \overline{\otimes} L\mathbb{F}_2^{\mathrm{op}}))$ of continuous derivations is two-dimensional, and hence the first continuous L^2-Betti number of $L\mathbb{F}_2$ is equal to 1:

$$\eta_1^{(2)}(L\mathbb{F}_2) = 1.$$

But then by the compression formula (Theorem 2.1) we obtain

$$\eta_1^{(2)}(L\mathbb{F}_3) = 2,$$

which contradicts the result of Proposition 3.1 that the submodule generated by ∂_a consists of discontinuous derivations. $\square$

As a corollary we get the following result.

THEOREM 3.3. *The first continuous L^2-cohomology of an interpolated free group factor $\mathbb{F}_r$, $1 < r < \infty$, vanishes:*

$$H_c^1(L\mathbb{F}_r, \mathscr{U}(L\mathbb{F}_r \overline{\otimes} L\mathbb{F}_r^{\mathrm{op}})) \cong 0.$$

PROOF. This follows immediately from the case $r = 3$ by the compression formula (Theorem 2.1) and Proposition 2.2. $\square$

In particular, the answer to the question of Andreas Thom in [**Tho08**] is negative: Voiculescu's free difference quotients don't have continuous extensions to $L\mathbb{F}_n$. Our result also allows to ask, whether $\eta_1^{(2)}(M) = 0$ for all II_1-factors M. While preparing this publication, we've been informed that Sorin Popa and Stefaan Vaes answered the above question affirmatively in [**PV14**] extending the key idea of Proposition 3.1 from free group factors to all finite von Neumann algebras.

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CHAPTER 3

Quadratic modules, C^*-algebras, and free convexity

Abstract

Given a quadratic module, we construct its universal C^*-algebra, and then use methods and notions from the theory of C^*-algebras to study the quadratic module. We define residually finite-dimensional quadratic modules, and characterize them in various ways, in particular via a Positivstellensatz. We give unified proofs for several existing strong Positivstellensätze, and prove some new ones. Our approach also leads naturally to interesting new examples in free convexity. We show that the usual notion of a free convex hull is not able to detect residual finite-dimensionality. We thus study a notion of free convexity which is coordinate-free. We characterize semialgebraicity of free convex hulls of semialgebraic sets, and show that they are not always semialgebraic, even at scalar level. This also shows that the membership problem for quadratic modules (a well-studied problem in Real Algebraic Geometry) has a negative answer in the non-commutative setup.

1. Introduction

Quadratic modules are well-studied objects in real algebra. They are generalizations of the cone of sums of squares, and play the role in Positivstellensätzen that ideals play in Nullstellensätzen. The commutative theory of quadratic modules is quite well-understood (see [**5**, **22**, **26**] or [**28**] for a survey). Interest in the non-commutative theory is much more recent (see [**29**] for a survey; more references to non-commutative Positivstellensätze can be found throughout this article). Also a quite new development with many recent results is free convexity (see for example [**8**, **13**, **15**, **16**]). Instead of looking at convex sets in $\mathbb{R}^d$ only, one considers sets of matrix-tuples of all sizes simultaneously. A suitable notion of convexity then relates the different matrix levels. All these notions are well-motivated by applications in such diverse areas as quantum physics, linear systems engineering, free probability and semidefinite optimization (see [**13**] for more information).

In this paper, our contribution is the following. In Section 2 we consider the C^*-algebra that one can canonically assign to an (archimedean) quadratic module (this was also done in [**7**]). With this construction, some of the most important methods from operator algebra pass to quadratic modules, as we explain. After assembling the necessary techniques, we define residually finite-dimensional (r.f.d.) quadratic modules in Section 3. This notion exists for C^*-algebras, and has interesting characterizations when formulated for quadratic modules. It corresponds to a Positivstellensatz with positivity at finite-dimensional representations. In this context we give alternative and uniform proofs for the strong Positivstellensätze from [**14**, **17**], and prove the same results for more classes of examples. In Section 4 we investigate free convexity, showing in particular that the coordinate-based approach is not always able to detect the property r.f.d. of a quadratic module. We thus suggest a coordinate-free approach towards free convexity. We characterize free convex hulls of semialgebraic sets to be semialgebraic at each matrix level. We produce examples showing that both matrix- and operator-convex hulls of free semialgebraic sets can fail to be semialgebraic, even at scalar level. This also shows that for a finitely generated quadratic module in a free algebra, the intersection with a finite-dimensional subspace can fail to be semialgebraic. So the membership problem has a negative answer in the non-commutative setup (see [**2**] for partial positive answers in the commutative case).

For further references on matrix- and operator convexity consult [**8**, **13**, **15**, **16**, **24**].

2. Universal C^*-algebras of quadratic modules

In this section we define the most important notions of the paper, and assemble important techniques for later use.

Definition 2.1. A quadratic module $(\mathcal{A}, \mathcal{Q})$ is a pair consisting of a unital complex $*$-algebra and a subset $\mathcal{Q} \subseteq \mathcal{A}^h$, where $\mathcal{A}^h$ denotes the $\mathbb{R}$-subspace of hermitian elements of $\mathcal{A}$, such that $1 \in \mathcal{Q}$ and

$$a, b \in \mathcal{Q}, c \in \mathcal{A} \quad \Rightarrow \quad c^*(a+b)c \in \mathcal{Q}.$$

Definition 2.2. The quadratic module $(\mathcal{A}, \mathcal{Q})$ is archimedean if $\ell - a^*a \in \mathcal{Q}$ for any $a \in \mathcal{A}$ and large enough ℓ, or equivalently $\ell - a \in \mathcal{Q}$ for any $a \in \mathcal{A}^h$ and large enough ℓ. It is also enough to require this for generators of $\mathcal{A}$ only (see [**7**] for technical details).

It's clear from the definitions that any quadratic module contains sums of squares

$$\Sigma^2\mathcal{A} = \left\{ \sum_{i=1}^n a_i^* a_i \,|\, n \in \mathbb{N},\, a_i \in \mathcal{A} \right\}.$$

In the sequel, if we don't specify a quadratic module in a $*$-algebra $\mathcal{A}$, we always assume that it comes with the smallest quadratic module $\Sigma^2\mathcal{A}$. Notice that for a C^*-algebra $\mathcal{A}$, $\Sigma^2\mathcal{A}$ is just the set of positive elements of $\mathcal{A}$ (for an introduction to C^*-algebras consult for example [**1**]).

Definition 2.3. Let $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{R})$ be two quadratic modules. Their tensor product

$$(\mathcal{A}, \mathcal{Q}) \otimes (\mathcal{B}, \mathcal{R})$$

is defined to be the smallest quadratic module in $\mathcal{A} \otimes \mathcal{B}$ containing the set $\{q \otimes r \mid q \in \mathcal{Q},\ r \in \mathcal{R}\}$.

EXAMPLE 2.4. (1) If $(\mathcal{B},\mathcal{R}) = \mathbb{M}_n(\mathbb{C})$, then

$$(\mathcal{A},\mathcal{Q}) \otimes \mathbb{M}_n(\mathbb{C}) = \left(\mathbb{M}_n(\mathcal{A}), \left\{ \sum_{\text{finite}} (a_i^* q a_j)_{ij} \mid a_1,\ldots,a_n \in \mathcal{A}, q \in \mathcal{Q} \right\}\right).$$

For sake of brevity we denote this quadratic module also by $\mathcal{Q} \otimes \mathbb{M}_n$. It is not hard to check that $\mathcal{Q} \otimes \mathbb{M}_n$ is archimedean, if $\mathcal{Q}$ was.

(2) Let $\mathbb{R} \subseteq \mathbf{R}$ be an extension of real closed fields. Then $\mathbb{C} \subseteq \mathbf{C} = \mathbf{R}[i]$ is an extension of algebraically closed fields, and $\mathbf{C}$ is a unital $*$-algebra over $\mathbb{C}$ with $\mathbf{C}^h = \mathbf{R}$. For $\mathcal{Q} = \Sigma^2\mathbf{C} = \mathbf{R}_+$ we find $\mathcal{Q} \otimes \mathbb{M}_n = \mathbb{M}_n(\mathbf{C})_+$, the set of positive semidefinite hermitian matrices over $\mathbf{C}$. Note that semidefiniteness has the same characterizations over $\mathbf{C}$ as over $\mathbb{C}$, for example by Tarski's Transfer Principle (see for example [**26**] for details on real closed fields and their model-theoretic properties).

DEFINITION 2.5. Let $(\mathcal{A},\mathcal{Q})$ and $(\mathcal{B},\mathcal{R})$ be two quadratic modules.
(i) A unital completely positive morphism (u.c.p. morphism)

$$\varrho\colon (\mathcal{A},\mathcal{Q}) \to (\mathcal{B},\mathcal{R})$$

is a unital $*$-linear map $\varrho\colon \mathcal{A} \to \mathcal{B}$ such that for every $n \in \mathbb{N}$,

$$(\varrho \otimes \mathrm{id})(\mathcal{Q} \otimes \mathbb{M}_n) \subseteq \mathcal{R} \otimes \mathbb{M}_n.$$

(ii) A homomorphism between quadratic modules

$$\pi\colon (\mathcal{A},\mathcal{Q}) \to (\mathcal{B},\mathcal{R})$$

is a unital $*$-homomorphism $\pi\colon \mathcal{A} \to \mathcal{B}$ such that $\pi(\mathcal{Q}) \subseteq \mathcal{R}$.
(iii) A representation of a quadratic module $(\mathcal{A},\mathcal{Q})$ is a homomorphism

$$\pi\colon (\mathcal{A},\mathcal{Q}) \to \mathbb{B}(\mathcal{H}),$$

where $\mathcal{H}$ is a Hilbert space.

Notice that any $*$-homomorphism between quadratic modules is obviously a u.c.p. morphism. Of course, a representation of $(\mathcal{A},\mathcal{Q})$ is just a $*$-representation $\pi\colon \mathcal{A} \to \mathbb{B}(\mathcal{H})$ fulfilling $\pi(q) \geqslant 0$ for all $q \in \mathcal{Q}$. We will see that representations of a quadratic module are in one-to-one correspondence with the representations of its universal C^*-algebra.

Although the collection of all representations of a quadratic module is in general not a set, this problem can be avoided by appropriately resticting the cardinality of the target Hilbert space, for instance, bounding it to the cardinality of the universal representation of the corresponding C^*-algebra. We will tacitly do this and denote by $\mathrm{Rep}(\mathcal{A},\mathcal{Q})$ the set of all representations of a quadratic module $(\mathcal{A},\mathcal{Q})$ and by $\mathrm{Rep}_{\mathrm{fd}}(\mathcal{A},\mathcal{Q})$ the subset of finite-dimensional representations.

DEFINITION 2.6. Let $(\mathcal{A}, \mathcal{Q})$ be an archimedean quadratic module. We equip the algebra $\mathcal{A}$ with the seminorm

$$\|a\|_{\mathcal{Q}} = \sup_{\pi \in \operatorname{Rep}(\mathcal{A},\mathcal{Q})} \|\pi(a)\|.$$

The supremum is finite, because

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| \leqslant \ell, \quad \pi \in \operatorname{Rep}(\mathcal{A}, \mathcal{Q}),$$

if $\ell - a^*a \in \mathcal{Q}$. In fact we have

$$\|a\|_{\mathcal{Q}} = \inf \left\{ \ell \mid \ell^2 - a^*a \in \mathcal{Q} \right\},$$

which follows by separation from the archimedean cone $\mathcal{Q}$, and the GNS construction.

As in [**7**], the (separated) completion of $\mathcal{A}$ with respect to $\|\cdot\|_{\mathcal{Q}}$ is denoted by $C^*(\mathcal{A}, \mathcal{Q})$ and called the universal C^*-algebra of $(\mathcal{A}, \mathcal{Q})$. We denote by $\iota: \mathcal{A} \to C^*(\mathcal{A}, \mathcal{Q})$ the canonical map with dense image.

The name "universal" comes from the following fact:

PROPOSITION 2.7. *Let $(\mathcal{A}, \mathcal{Q})$ be an archimedean quadratic module, $\iota: \mathcal{A} \to C^*(\mathcal{A}, \mathcal{Q})$ the canonical map, and $\mathcal{B}$ a C^*-algebra.*

(i) ι is a homomorphism of quadratic modules and respects the (semi)-norm.

(ii) There is a one-to-one correspondence between u.c.p morphisms $\varrho: (\mathcal{A}, \mathcal{Q}) \to \mathcal{B}$ and u.c.p. morphisms $\overline{\varrho}: C^*(\mathcal{A}, \mathcal{Q}) \to \mathcal{B}$. The correspondence is given by the formula $\varrho = \overline{\varrho} \circ \iota$. This correspondence maps homomorphisms to homomorphisms.

PROOF. (i) For any $\ell > \|q\|_{\mathcal{Q}}$ we have $\ell^2 - q^*q \in \mathcal{Q}$. Then

$$\ell - q = \frac{1}{2\ell}\big((\ell^2 - q^*q) + (\ell - q)^*(\ell - q)\big) \in \mathcal{Q}.$$

In particular $2\ell - q \in \mathcal{Q}$ and thus

$$\ell^2 - (\ell - q)^*(\ell - q) = \frac{1}{2\ell}\left((2\ell - q)^*q(2\ell - q) + q^*(2\ell - q)q\right) \in \mathcal{Q}.$$

This proves $\|\ell - q\|_{\mathcal{Q}} \leq \ell$, whenever $\ell > \|q\|_{\mathcal{Q}}$. The same then holds in $C^*(\mathcal{A}, \mathcal{Q})$, and this is well known to imply positivity in a C^*-algebra.

For (ii) first note that ι is u.c.p., and any u.c.p. map on $C^*(\mathcal{A}, \mathcal{Q})$ is continuous. Further, any u.c.p. map ϱ on $(\mathcal{A}, \mathcal{Q})$ factors through $C^*(\mathcal{A}, \mathcal{Q})$. Indeed if $\|a\|_{\mathcal{Q}} = 0$, then $\epsilon - a^*a \in \mathcal{Q}$ for all $\epsilon > 0$. Then

$$\begin{pmatrix} 1 & a \\ a^* & \epsilon \end{pmatrix} = \begin{pmatrix} 1 \\ a^* \end{pmatrix} \begin{pmatrix} 1 & a \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\epsilon - a^*a) \begin{pmatrix} 0 & 1 \end{pmatrix} \in \mathcal{Q} \otimes \mathbb{M}_2$$

and thus

$$\begin{pmatrix} 1 & \varrho(a) \\ \varrho(a)^* & 0 \end{pmatrix} \geqslant 0,$$

which implies $\varrho(a) = 0$. Finally, $(\iota\otimes\mathrm{id})(\mathcal{Q}\otimes\mathbb{M}_n)$ is dense in the positive elements of $C^*(\mathcal{A},\mathcal{Q})\otimes\mathbb{M}_n(\mathbb{C})$, so $\overline{\varrho}$ is u.c.p. $\square$

We now formulate some important techniques from operator algebra in the context of quadratic modules (see [**24**] for the corresponding results for C^*-algebras).

PROPOSITION 2.8 (Stinespring's Dilation Theorem). *Let $(\mathcal{A},\mathcal{Q})$ be an archimedean quadratic module and let*

$$\varrho\colon(\mathcal{A},\mathcal{Q})\to\mathbb{B}(\mathcal{H}_\varrho)$$

be a u.c.p. morphism. Then there is a representation $\pi\colon(\mathcal{A},\mathcal{Q})\to\mathbb{B}(\mathcal{H}_\pi)$ and an isometry $\gamma\colon\mathcal{H}_\varrho\hookrightarrow\mathcal{H}_\pi$ such that $\varrho = {}^\gamma\pi$, where

$${}^\gamma\pi(a) = (\gamma^*\circ\pi(a)\circ\gamma).$$

PROOF. For C^*-algebras, this is precisely the statement of Stinespring's Dilation Theorem ([**24**], Theorem 4.1). The version for quadratic modules is immediate from Proposition 2.7. $\square$

Now let $\mathcal{V}\subseteq\mathcal{A}$ be a unital $*$-subspace. For $n\in\mathbb{N}$ we equip the $*$-space $\mathcal{V}\otimes\mathrm{Mat}_n(\mathbb{C})$ with the convex cone

$$(\mathcal{Q}\otimes\mathbb{M}_n)_{|\mathcal{V}} := (\mathcal{Q}\otimes\mathbb{M}_n)\cap(\mathcal{V}\otimes\mathbb{M}_n(\mathbb{C}))\,.$$

A unital $*$-linear mapping $\varrho\colon\mathcal{V}\to\mathcal{B}(\mathcal{H})$ is again called u.c.p. if all mappings $\varrho\otimes\mathrm{id}$ map these cones to positive elements.

PROPOSITION 2.9 (Arveson's Extension Theorem). *Let $(\mathcal{A},\mathcal{Q})$ be an archimedean quadratic module and $\mathcal{V}\subseteq\mathcal{A}$ a unital $*$-subspace. Then any u.c.p. map $\varrho\colon\mathcal{V}\to\mathcal{B}(\mathcal{H})$ extends to a u.c.p. map $\varrho\colon\mathcal{A}\to\mathcal{B}(\mathcal{H})$.*

PROOF. Any u.c.p. map $\varrho\colon\mathcal{V}\to\mathbb{B}(\mathcal{H})$ factors through $\iota(\mathcal{V})$, by the same argument as in Proposition 2.7. We show that the resulting map $\overline{\varrho}\colon\iota(\mathcal{V})\to\mathbb{B}(\mathcal{H})$ is u.c.p., and the result then clearly follows from the standard version of Arveson's Extension Theorem [**24**, Theorem 7.5].

It is not hard to check that $C^*(\mathcal{A}\otimes\mathbb{M}_n(\mathbb{C}),\mathcal{Q}\otimes\mathbb{M}_n) = C^*(\mathcal{A},\mathcal{Q})\otimes\mathbb{M}_n(\mathbb{C})$ holds. Every $*$-linear functional $\varphi\colon\mathcal{A}\otimes\mathbb{M}_n(\mathbb{C})\to\mathbb{C}$ which is nonnegative on $\mathcal{Q}\otimes\mathbb{M}_n$ is automatically u.c.p., and thus extends to a u.c.p. functional on $C^*(\mathcal{A},\mathcal{Q})\otimes\mathbb{M}_n(\mathbb{C})$. So if $(\iota\otimes\mathrm{id})(M)\geqslant 0$ for some $M\in\mathcal{V}\otimes\mathbb{M}_n(\mathbb{C})$, then $M\in(\mathcal{Q}\otimes\mathbb{M}_n)^{\vee\vee}$, the double dual cone. Since $\mathcal{Q}\otimes\mathbb{M}_n$ is archimedean, this means $M+\epsilon\in\mathcal{Q}\otimes\mathbb{M}_n$, and this implies $(\varrho\otimes\mathrm{id})(M)\geqslant 0$. This proves that $\overline{\varrho}$ is u.c.p. $\square$

Given $\varrho\colon\mathcal{V}\to\mathbb{B}(\mathcal{H})$ $*$-linear, where $\mathcal{H}$ is of finite dimension n, we define a functional

$$\begin{aligned} c_\varrho\colon\mathcal{V}\otimes\mathbb{M}_n(\mathbb{C}) &\to\mathbb{C}\\ v\otimes M &\mapsto \mathrm{tr}(\varrho(v)M). \end{aligned}$$

Proposition 2.10 (Choi's Theorem). *Let $(\mathcal{A}, \mathcal{Q})$ be an archimedean quadratic module and $\mathcal{V} \subseteq \mathcal{A}$ a unital $*$-subspace. Let $\mathcal{H}$ be a Hilbert space of dimension $n < \infty$, and $\varrho\colon \mathcal{V} \to \mathbb{B}(\mathcal{H})$ unital $*$-linear. Then the following are equivalent:*

(i) ϱ *is u.c.p.*
(ii) $\varrho \otimes \mathrm{id}$ *maps* $(\mathcal{Q} \otimes \mathbb{M}_n)_{|\mathcal{V}}$ *to positive operators.*
(iii) c_ϱ *is nonnegative on* $(\mathcal{Q} \otimes \mathbb{M}_n)_{|\mathcal{V}}$.

Proof. Clear from the standard version of Choi's Theorem [**24**, Theorem 6.1] and the above considerations. □

We finally formulate the real closed separation theorem from [**23**] in the more general context of quadratic modules.

Definition 2.11. The quadratic module $(\mathcal{A}, \mathcal{Q})$ is called *tame*, if $\mathcal{Q} = \bigcup_{i \in I} \mathcal{Q}_i$, where

- $(I, \leq)$ is a directed poset
- each $\mathcal{Q}_i$ is a closed convex cone in a finite-dimensional subspace of $\mathcal{A}^h$
- $i \leq j \Rightarrow \mathcal{Q}_i \subseteq \mathcal{Q}_j$ for all $i, j \in I$
- for each finite-dimensional subspace $\mathcal{V} \subseteq \mathcal{A}$ and each $i \in I$ there exists $j \in I$ such that $\mathcal{V}^* \mathcal{Q}_i \mathcal{V} \subseteq \mathcal{Q}_j$.

Example 2.12. Assume $\mathcal{Q} \cap -\mathcal{Q} = \{0\}$ and $\mathcal{Q}$ admits a generating set $S \subseteq \mathcal{Q}$, such that

$$v^* q v = 0 \Rightarrow v = 0$$

holds for all $v \in \mathcal{A}, q \in S$. Then $(\mathcal{A}, \mathcal{Q})$ is tame. To see this, let

$$I = \{(\mathcal{V}, T) \mid \mathcal{V} \text{ finite-dimensional subspace of } \mathcal{A}, T \subseteq S \text{ finite}\}$$

be equipped with the obvious partial order. For $i = (\mathcal{V}, T) \in I$ we define

$$\mathcal{Q}_i = \left\{ \sum_{q \in T} \sum_j v_{qj}^* q v_{qj} \mid v_{qj} \in \mathcal{V} \right\}.$$

Using the arguments from [**25**, Proposition 2.6 and Lemma 2.7], closedness of $\mathcal{Q}_i$ follows if we show

$$\sum_{q \in T} \sum_j v_{qj}^* q v_{qj} = 0 \Rightarrow v_{qj} = 0 \quad \forall q, j.$$

But this is clear from our assumptions.

Theorem 2.13. *Let $(\mathcal{A}, \mathcal{Q})$ be a tame quadratic module, and $a \in \mathcal{A}^h \setminus \mathcal{Q}$. Then there exists an extension $\mathbb{R} \subseteq \mathbf{R}$ of real closed fields, a $\mathbf{C}$-vector space $\mathcal{H}$ with inner product, a $*$-homomorphism of $\mathbf{C}$-algebras*

$$\pi\colon \mathcal{A} \otimes_{\mathbb{C}} \mathbf{C} \to \mathbb{L}(\mathcal{H})$$

mapping $\mathcal{Q} \otimes \mathbf{R}_+$ *to positive operators, and* $\xi \in \mathcal{H}$ *with*

$$\langle \pi(a \otimes 1)\xi, \xi \rangle < 0.$$

Furthermore, ξ *can be assumed to be cyclic w.r.t.* π*, meaning each* $h \in \mathcal{H}$ *is of the form* $\pi(b)\xi$ *for some* $b \in \mathcal{A} \otimes \mathbf{C}$.

Proof. Since the argument is an adaption of the results in [**23**], we skip the technical details. For every $i \in I$ we find a linear functional $\varphi_i : \mathcal{A}^h \to \mathbb{R}$ with $\varphi_i \geq 0$ on $\mathcal{Q}_i$ and $\varphi(a) < 0$. Choose an ultrafilter ω on I containing all the the upper sets $\{i \in I \mid i \geq j\}$, and let $\mathbf{R} = \mathbb{R}^\omega$ be the ultrapower. Then the $\mathbb{R}$-linear map $\varphi : \mathcal{A}^h \to \mathbf{R}; b \mapsto (\varphi_i(b))_{i \in I}$ separates b from $\mathcal{Q}$ (using Łos's Theorem from model theory), and we can extend φ to a unital $*$-linear map $\varphi : \mathcal{A} \to \mathbf{C} = \mathbf{R}[i]$. Using the fourth property from the definition of a tame quadratic module (and Łos's Theorem again), one checks that φ is u.c.p. It follows that the $\mathbf{C}$-linear map $\varphi \otimes \mathrm{id} : \mathcal{A} \otimes_{\mathbb{C}} \mathbf{C} \to \mathbf{C}$ is nonnegative on $\mathcal{Q} \otimes \mathbf{R}_+$, and we perform the usual GNS construction, that works as well over $\mathbf{C}$. From this the result follows. □

3. Residually finite-dimensional quadratic modules

We now define the notion of a residually finite-dimensional quadratic module, and characterize it in several ways. First, the notion of the Fell topology for $*$-representations of C^*-algebras easily generalises to u.c.p. maps (see [**4**, Section F.2] for more details on the Fell topology).

Definition 3.1. Let $\mathcal{A}$ be a C^*-algebra and $\varrho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a u.c.p. map. A functional of positive type associated with ϱ is a functional

$$\varphi(a) = \langle \varrho(a)\xi, \xi \rangle \,,$$

where $\xi \in \mathcal{H}$ is a unit vector. We denote the set of such functionals by $\mathrm{Pos}(\varrho) \subset \mathcal{A}^*$.

Definition 3.2. Let $\mathcal{A}$ be a C^*-algebra and $\varrho : \mathcal{A} \to \mathbb{B}(\mathcal{H}_\varrho)$ be a u.c.p. map, let $F \subset \mathcal{A}$ and $\Phi \subset \mathrm{Pos}(\varrho)$ be finite. The neighborhood system

$$N(\varrho, F, \Phi, \varepsilon) = \left\{ \varrho' : \mathcal{A} \to \mathbb{B}(\mathcal{H}_{\varrho'}) \text{ u.c.p.} \mid \forall \varphi \in \Phi \; \exists \, \psi_i \in \mathrm{Pos}(\varrho') \; \forall a \in F : \left| \varphi(a) - \sum_{\text{finite}} \psi_i(a) \right| < \varepsilon \right\}$$

defines a topology on the set of u.c.p. maps $\mathcal{A} \to \mathcal{B}(\mathcal{H})$ called the Fell topology. We say that a u.c.p. map ϱ' is weakly contained in ϱ, denoted $\varrho' \prec \varrho$, if $\varrho' \in \overline{\{\varrho\}}$.

The Fell topology just defined clearly coincides with the usual Fell topology when restricted to the set of representations of $\mathcal{A}$. The Stinespring Dilation Theorem implies at once that any u.c.p. map is weakly contained in its Stinespring dilation.

DEFINITION 3.3. An archimedean quadratic module $(\mathcal{Q}, \mathcal{A})$ is residually finite-dimensional (r.f.d.) if its universal C^*-algebra $C^*(\mathcal{A}, \mathcal{Q})$ is residually finite dimensional, meaning that finite dimensional representations are dense in the set of all representations.

THEOREM 3.4. *For an archimedean quadratic module* $(\mathcal{A}, \mathcal{Q})$*, the following are equivalent:*

(i) $(\mathcal{A}, \mathcal{Q})$ *is r.f.d.*

(ii) For any $a \in \mathcal{A}^h$*, whenever* $\pi(a) \geqslant 0$ *for all* $\pi \in \mathrm{Rep}_{fd}(\mathcal{A}, \mathcal{Q})$*, then*

$$a + \epsilon \in \mathcal{Q} \quad \forall \epsilon > 0.$$

PROOF. (i) $\Rightarrow$ (ii): if $C^*(\mathcal{A}, \mathcal{Q})$ is r.f.d., $\pi(a) \geqslant 0$ for all $\pi \in \mathrm{Rep}_{fd}(\mathcal{A}, \mathcal{Q})$ is equivalent to $\pi(a) \geqslant 0$ for all $\pi \in \mathrm{Rep}(\mathcal{A}, \mathcal{Q})$. By the abstract Positivstellensatz [**29**], $a + \varepsilon \in \mathcal{Q}$ for all $\varepsilon > 0$ is equivalent to $\pi(a) \geqslant 0$ for all $\pi \in \mathrm{Rep}(\mathcal{A}, \mathcal{Q})$.

(ii) $\Rightarrow$ (i): $a + \varepsilon \in \mathcal{Q}$ for all $\varepsilon > 0$ implies $\iota(a) \geqslant 0$ in $C^*(\mathcal{A}, \mathcal{Q})$. If positivity in a C^*-algebra is detected by finite-dimensional representations, then the C^*-algebra is r.f.d. $\square$

REMARK 3.5. Besides the above Positivstellensatz, property r.f.d. is also interesting from a computational point of view. The (semi)-norm $\|a\|_{\mathcal{Q}}$ of an element $a \in \mathcal{A}$ can be approximated in the following way (see also [**11**] for more information): Upper bounds are obtained by computing numbers ℓ such that $\ell^2 - a^*a \in \mathcal{Q}$. This is a semidefinite program, if only finitely many generators of $\mathcal{Q}$ and sums of squares from a finite-dimensional subspace of $\mathcal{A}$ are used. Making these constraints less and less restrictive, the sequence of upper bounds converges to $\|a\|_{\mathcal{Q}}$ from above.

Now a sequence of lower bounds is obtained by computing $\sup\|\pi(a)\|$ over all representations π of some bounded dimension. In case of a finitely generated quadratic module in a finitely generated algebra, this is a semialgebraic decision problem, which is decidable. If $\mathcal{Q}$ is r.f.d., these lower bounds will also converge to $\|a\|_{\mathcal{Q}}$, with growing dimension.

LEMMA 3.6. *A* C^**-algebra* $\mathcal{A}$ *is r.f.d. if and only if the set of its finite-dimensional representations is dense in the set of u.c.p. maps* $\mathcal{A} \to \mathbb{B}(\mathcal{H})$*.*

PROOF. A C^*-algebra $\mathcal{A}$ is r.f.d. iff if the set of its finite-dimensional representations is dense in the set of all representations, which is in turn dense in the set of all u.c.p. maps by the remark above. $\square$

DEFINITION 3.7. A u.c.p. map $\varrho\colon \mathcal{A} \to \mathbb{B}(\mathcal{H})$ is finite-dimensional if $\mathcal{H}$ is finite-dimensional. A u.c.p. map $\varrho\colon \mathcal{A} \to \mathbb{B}(\mathcal{H})$ is strongly finite-dimensional if it possesses a finite-dimensional Stinespring dilation.

THEOREM 3.8. *For a unital* C^**-algebra* $\mathcal{A}$*, the following are equivalent:*

(i) $\mathcal{A}$ *is r.f.d.*

(ii) *For every finite-dimensional u.c.p. map $\varrho\colon \mathcal{A} \to \mathbb{B}(\mathcal{H})$, every finite $F \subset \mathcal{A}$ and every $\varepsilon > 0$ there exists a strongly finite-dimensional u.c.p. map $\tilde{\varrho}\colon \mathcal{A} \to \mathbb{B}(\mathcal{H})$ such that*

$$\|\varrho(a) - \tilde{\varrho}(a)\| \leq \varepsilon, \quad a \in F.$$

(iii) *Every state $\omega\colon \mathcal{A} \to \mathbb{C}$ is a weak*-limit of states associated to finite-dimensional representations.*

Proof. The equivalence of (i) and (iii) is well-known, and (ii) obviously implies (iii). The proof that (i) implies (ii) is essentially the argument from [**19**, Proposition 2.2]. □

Example 3.9. If $\mathcal{A}$ is commutative (with trivial involution), then every archimedean quadratic module in $\mathcal{A}$ is r.f.d. This follows for example via Theorem 3.4 from the commutative archimedean Positivstellensatz [**22**, Theorem 5.4.4], or can be deduced via functional calculus for commuting families of operators.

Example 3.10. Let Γ be a group, and $\mathcal{A} = \mathbb{C}\Gamma$ the group algebra, equipped with $\mathcal{Q} = \Sigma^2\mathcal{A}^2$. Then r.f.d. for $\mathcal{Q}$ is a well-studied property in group theory. There are groups which are r.f.d., for example free groups $\mathbb{F}_m$ [**6**] or surface groups [**21**], and there are groups which are not r.f.d., like $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ (they have Kazhdan's Property (T) and thus each finite-dimensional representation is an isolated point in the Fell topology, see [**4**]).

Example 3.11. We consider the class of examples from [**14**]. Let $\mathcal{A} = \mathbb{C}\langle z_1, \ldots, z_n\rangle$ be the free algebra with $z_i^* = z_i$. Fix Hermitian matrices $M_1, \ldots, M_n \in \mathbb{M}_s(\mathbb{C})^h$ and let

$$\mathcal{L} = I_s + M_1 z_1 + \cdots + M_n z_n$$

be the associated linear matrix pencil. Then

$$\mathcal{Q} = \left\{ \sum_j p_j^* p_j + q_j^* \mathcal{L} q_j \mid p_j, q_j \in \mathcal{A}^s \right\}$$

is a quadratic module in $\mathcal{A}$ which is r.f.d. This can be deduced from the Positivstellensatz in [**14**], but we prove it directly, and in fact re-prove this Positivstellensatz with our method. A representation $\pi \in \mathrm{Rep}(\mathcal{A}, \mathcal{Q})$ is just a tuple $\underline{T} = (T_1, \ldots, T_n)$ of self-adjoint operators on a Hilbert space $\mathcal{H}$, fulfilling

$$\mathcal{L}(\underline{T}) = I_s \otimes \mathrm{id}_{\mathcal{H}} + M_1 \otimes T_1 + \cdots + M_n \otimes T_n \geqslant 0$$

(to see this, use that π can be assumed to admit a cyclic vector). For any isometry $\gamma\colon \tilde{\mathcal{H}} \to \mathcal{H}$ we set $\gamma^* \underline{T} \gamma = (\gamma^* T_1 \gamma, \ldots, \gamma^* T_n \gamma)$ and find

$$\mathcal{L}(\gamma^* \underline{T} \gamma) = (I_s \otimes \gamma)^* \mathcal{L}(\underline{T})(I_s \otimes \gamma) \geqslant 0.$$

So the tuple $\gamma^* \underline{T} \gamma$ gives rise to a representation $\tilde{\pi}$ of $\mathcal{Q}$ again. The usual compression trick [**23**, Theorem 6.1] shows that for finite-dimensional subspaces $\mathcal{V} \subseteq \mathcal{A}$, $\mathcal{H}_0 \subseteq \mathcal{H}$, there is some isometry $\gamma\colon \mathcal{H}_1 \to \mathcal{H}$ from a finite-dimensional space $\mathcal{H}_1 \supseteq \mathcal{H}_0$, such that

$$p(\gamma^* \underline{T} \gamma) \equiv \gamma^* p(\underline{T}) \gamma \tag{3.1}$$

on $\mathcal{H}_0$, for all $p \in \mathcal{V}$. Thus $\tilde{\pi} \in \operatorname{Rep}(\mathcal{A}, \mathcal{Q})$ is close to π in the Fell topology. This shows that $(\mathcal{A}, \mathcal{Q})$ is r.f.d., even in a very strong sense.

We strengthen the argument to prove the strong Positivstellensatz from [**14**]. First check that $\mathcal{Q}$ is tame, using Example 2.12. Then for $a \in \mathcal{A}^h \setminus \mathcal{Q}$, use the real closed separation from Theorem 2.13, and obtain $\pi(z_1), \ldots, \pi(z_n) \in \mathbb{L}(\mathcal{H})$ with $I_s \otimes \mathrm{id}_{\mathcal{H}} + M_1 \otimes \pi(z_1) + \cdots + M_n \otimes \pi(z_n) \geqslant 0$ (again use that π admits a cyclic vector). Now apply the same compression trick as before to π, and obtain a finite-dimensional representation over $\mathbf{C}$ in which a is not positive. Using Tarski's Transfer Principle, such a representation also exists over $\mathbb{C}$. We have thus shown: If $a \in \mathcal{A}^h$ is nonnegative on $\operatorname{Rep}_{fd}(\mathcal{A}, \mathcal{Q})$, then $a \in \mathcal{Q}$.

Example 3.12. This is the example from [**17**]. Let $\mathcal{A} = \mathbb{C}\langle z_1, z_1^*, \ldots, z_n, z_n^* \rangle$ and

$$\mathcal{Z} = \left\{ (M_1, \ldots, M_n) \mid M_i \text{ matrices}, \sum_i M_i^* M_i = I \right\}.$$

Then

$$\mathcal{Q} = \Sigma^2 \mathcal{A} + \left\{ p \in \mathcal{A}^h \mid p \equiv 0 \text{ on } \mathcal{Z} \right\}$$

is r.f.d. This can be deduced from the Positivstellensatz in [**17**], which we again re-prove it with our above separation method. Let $\pi \in \operatorname{Rep}(\mathcal{A}, \mathcal{Q})$ be given. With $T_i = \pi(z_i) \in \mathbb{B}(\mathcal{H})$ we have $\sum_i T_i^* T_i = \mathrm{id}_{\mathcal{H}}$. Let $\mathcal{V} \subseteq \mathcal{A}, \mathcal{H}_0 \subseteq \mathcal{H}$ be finite-dimensional subspaces. Without loss of generality assume that $\mathcal{V} = \mathcal{A}_d$, the space of all polynomials of degree at most $d \geq 2$. Inductively define

$$\mathcal{H}_{i+1} = \operatorname{span}\{p(\underline{T})h \mid p \in \mathcal{V}, h \in \mathcal{H}_i\}$$

for $i = 0, 1$. Let $\gamma\colon \mathcal{H}_2 \hookrightarrow \mathcal{H}$ be the embedding and consider the compressed operators $M_i := \gamma^* T_i \gamma \in \mathbb{B}(\mathcal{H}_2)$. We have $p(\underline{M}) \equiv \gamma^* p(\underline{T}) \gamma$ on $\mathcal{H}_1$ for all $p \in \mathcal{V}$, and thus

$$M\colon \mathcal{H}_1 \to \mathcal{H}_2^n; h \mapsto (M_1 h, \ldots, M_n h)$$

is an isometry. So we can extend to an isometry

$$\tilde{M} = (\tilde{M}_1, \ldots, \tilde{M}_n)\colon \mathcal{H}_2 \to \mathcal{H}_2^n,$$

and thus obtain a finite-dimensional representation $\tilde{\pi}$ of $(\mathcal{A}, \mathcal{Q})$ on $\mathcal{H}_2$. Now one checks that $p(\underline{\tilde{M}}) \equiv \gamma^* p(\underline{T}) \gamma$ on $\mathcal{H}_0$, and so $\tilde{\pi}$ is close to π in the Fell topology. So $(\mathcal{A}, \mathcal{Q})$ is r.f.d. in a strong sense.

Now we do the same over a real closed field $\mathbf{R}$. This time, we first pass to

$$\mathcal{B} = \mathcal{A} / \{p \in \mathcal{A} \mid p \equiv 0 \text{ on } \mathcal{Z}\},$$

where $\Sigma^2\mathcal{B}$ is tame, as is easily checked (using Example 2.12). We separate by a real-closed representation and lift it to $\mathcal{A}$. Then we do the compression as described above, and transfer to $\mathbb{C}$ in the end. We have shown: If $a \in \mathcal{A}^h$ is nonnegative on $\mathrm{Rep}_{fd}(\mathcal{A}, \mathcal{Q})$, then $a \in \mathcal{Q}$.

Example 3.13. Essentially the same methods can be used to show that the following quadratic modules are r.f.d., and even fulfill the strong Positivstellensatz

$$a \in \mathcal{A}^h, a \geqslant 0 \text{ on } \mathrm{Rep}_{fd}(\mathcal{A}, \mathcal{Q}) \;\Rightarrow a \in \mathcal{Q}.$$

(i) $\mathcal{A} = \mathbb{C}\langle z_1, z_1^*, \dots, z_n, z_n^*\rangle$ with $\mathcal{Q}$ generated by either

$$1 - \sum_{i=1}^n z_i^* z_i \quad \text{or} \quad 1 - z_i^* z_i \text{ for all } i.$$

Here we can separate and compress without any further adjustments.

(ii) $\mathcal{A} = \mathbb{C}\langle z_{ij}, z_{ij}^* \mid 1 \le i, j \le n\rangle$ and $\mathcal{Q} = \Sigma^2\mathcal{A} + \left\{p \in \mathcal{A}^h \mid p \equiv 0 \text{ on } \mathcal{Z}\right\}$, where

$$\mathcal{Z} = \left\{(M_{11}, M_{12}, \dots, M_{nn}) \mid M_{ij} \text{ matrices}, \left(M_{ij}\right)_{ij} \text{ unitary}\right\}.$$

After separating and compressing, we invoke Choi's matrix-trick from [**6**] and a suitable permutation of rows and columns.

(iii) $\mathcal{A} = \mathbb{C}\langle z_{ij}, z_{ij}^* \mid 1 \le i, j \le n\rangle$ and $\mathcal{Q}$ generated (as in Example 3.11) by the quadratic matrix polynomial

$$\mathcal{P} = I_n - (z_{ij})_{ij}^*(z_{ij})_{ij} \in \mathbb{M}_n(\mathcal{A})^h.$$

This is even simpler as (ii).

Example 3.14. Let $\mathcal{A} = \mathbb{C}\langle u, u^*, v, v^*\rangle/(z^*z = zz^* = v^*v = vv^* = 1)$ and let $\mathcal{Q}$ be generated by

$$\epsilon^2 - (uv - vu)^*(uv - vu)$$

for some $\epsilon > 0$ (this is called the *soft torus*). It is shown in [**9**] that $(\mathcal{A}, \mathcal{Q})$ is rfd.

Example 3.15. Let $\Gamma = \mathbb{F}_2 \times \mathbb{F}_2$ and $\mathcal{A} = \mathbb{C}\Gamma$. Then $\mathcal{A}$ is rfd if and only if Connes' Embedding Conjecture is true (see [**20**]).

Example 3.16. Let $\mathcal{A} = \mathbb{C}\langle z, z^*\rangle/(zz^* - 1)$ be the Toeplitz algebra. Then $\mathcal{Q}$ is not rfd. Finite dimensional representations correspond to unitary matrices, but the left-shift on $\ell^2(\mathbb{N})$ yields a representation that cannot be approximated by finite-dimensional representations, since it is not unitary.

4. Free convexity

Let us briefly introduce the main concepts of free convexity, as in [**8, 13, 15, 16**]. For some $n \geq 1$ we consider subsets $C_s \subseteq \mathrm{Her}_s(\mathbb{C})^n$ of n-tuples of hermitian matrices of size s, for all

$s \geq 1$. The whole collection $C = \bigcup_{s\geq 1} C_s$ is called matrix-convex, if it is closed under block-diagonal sums and compressions via isometries. That means, whenever $\underline{A} = (A_1, \ldots, A_n) \in C_s, \underline{B} = (B_1, \ldots, B_n) \in C_r$, and $V \in \mathbb{M}_{s,r}(\mathbb{C})$ with $V^*V = 1$, then

$$\underline{A} \oplus \underline{B} = \left(\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} \right) \in C_{s+r}$$

and

$$V^* \underline{A} V = (V^* A_1 V, \ldots, V^* A_n V) \in C_r.$$

This easily implies that each C_s is convex in the real vectorspace $\mathrm{Her}_s(\mathbb{C})^n$, but matrix convexity of C is a stronger assumption in general.

For any set $C = \bigcup_s C_s$, its matrix-convex hull $\mathrm{mconv}(C)$ is the smallest matrix-convex superset of C. In case that C is already closed under block-diagonal sums, it is easy to see that we only need to add compressions to obtain the matrix convex hull:

$$\mathrm{mconv}(C)_s = \{V^* \underline{A} V \mid r \geq s, \underline{A} \in C_r, V \in \mathbb{M}_{r,s}(\mathbb{C}), V^*V = 1\}. \tag{4.1}$$

Now assume $\mathcal{A} = \mathbb{C}\langle z_1, \ldots, z_n \rangle$ with $z_i^* = z_i$ and $p \in \mathcal{A}^h$. Define

$$C(p)_s = \{(A_1, \ldots, A_n) \in \mathrm{Her}_s(\mathbb{C})^n \mid p(A_1, \ldots, A_n) \geqslant 0\}$$

and $C(p) = \bigcup_s C(p)_s$, a so-called (basic closed) free semialgebraic set (finite intersections of such sets are also called basic closed). Understanding the matrix convex hullof such sets is one of the main issues in the above mentioned papers.

Note that one can also use operators instead of matrices to define free convex hulls. For a Hilbert space $\mathcal{H}$ define

$$C(p)_{\mathcal{H}} = \Big\{(T_1, \ldots, T_n) \mid T_i \in \mathbb{B}(\mathcal{H})^h, p(T_1, \ldots, T_n) \geqslant 0\Big\}$$

and the operator convex hull $\mathrm{oconv}(C(p))$ as

$$\mathrm{oconv}(C(p))_s = \{V^* \underline{T} V \mid \mathcal{H} \text{ Hilbert space }, \underline{T} \in C(p)_{\mathcal{H}}, V\colon \mathbb{C}^s \to \mathcal{H} \text{ isometry}\}.$$

Now, interesting (and previously open) questions are:

- Can r in (4.1) be bounded in terms of s (and maybe other data)?
- Is $\mathrm{mconv}(C(p))$ and/or $\mathrm{oconv}(C(p))$ semialgebraic in any (free) sense?
- Is at least each $\mathrm{mconv}(C(p))_s$ and/or $\mathrm{oconv}(C(p))_s$ semialgebraic in the usual sense?

We will answer these questions to the negative below. But let us first define a broader and coordinate-free notion of free convexity. Example 4.3 will show that this might be useful. Note that all of the following concepts coincide for an archimedean quadratic module $(\mathcal{A}, \mathcal{Q})$ and its universal C^*-algebra $C^*(\mathcal{A}, \mathcal{Q})$.

Definition 4.1. Let $\mathcal{A}$ be a C^*-algebra. The convex hull of $\mathrm{Rep}(\mathcal{A})$ is defined as

$$\mathrm{conv}\,\mathrm{Rep}(\mathcal{A}) = \{\varrho\colon \mathcal{A} \to \mathbb{B}(\mathcal{H}) \mid \mathcal{H} \text{ Hilbert space}, \varrho \text{ u.c.p.}\}.$$

The convex hull of $\operatorname{Rep}_{fd}(\mathcal{A})$ is defined as

$$\operatorname{conv}\operatorname{Rep}_{fd}(\mathcal{A}) = \{\varrho\colon \mathcal{A} \to \mathbb{B}(\mathcal{H}) \mid \varrho \text{ strongly finite dimensional u.c.p.}\}.$$

Remark 4.2. (i) Convex hulls are the smallest supersets closed under compressions via isometries. This immediately follows from Stinespring's Dilation Theorem. Also note that the sets $\operatorname{Rep}(\mathcal{A}), \operatorname{Rep}_{fd}(\mathcal{A})$ and their convex hulls are closed under finite direct sums.

(ii) In case $\mathcal{A} = \mathbb{C}\langle z_1, \ldots, z_n\rangle$, we obtain the old notions of free convex hulls when we restrict the u.c.p. maps to the space $\mathcal{V} = \operatorname{span}\{z_1, \ldots, z_n\}$.

Example 4.3. Let $\mathcal{A} = \mathbb{C}\langle z, z^*\rangle/(zz^* - 1)$ be the Toeplitz algebra with $\mathcal{Q} = \Sigma^2\mathcal{A}$, and $\mathcal{V} = \operatorname{span}\{z, z^*\}$. For any finite-dimensional $\varrho \in \operatorname{conv}\operatorname{Rep}(\mathcal{A}, \mathcal{Q})$ there is a strongly finite dimensional $\tilde{\varrho} \in \operatorname{conv}\operatorname{Rep}_{fd}(\mathcal{A}, \mathcal{Q})$ with $\varrho \equiv \tilde{\varrho}$ on $\mathcal{V}$. In fact $\varrho(z) = \gamma^*\pi(z)\gamma$ for some $\pi \in \operatorname{Rep}(\mathcal{A}, \mathcal{Q})$ and some isometry γ. So $\varrho(z)$ is a finite-dimensional contraction, and thus admits a finite-dimensional unitary dilation.

On the other hand, $\mathcal{Q}$ is not rfd, as shown in Example 3.16, so for other subspaces $\mathcal{V}$ we don't even get a good approximation by strongly finite-dimensional morphisms (by Theorem 3.8). This suggests that restricting all the maps to a generating subspace $\mathcal{V}$ of $\mathcal{A}$, as done in free convexity throughout so far, is not always a good idea. This approach will for example not be able to detect the property rfd. This is why we proposed the above notion of free convex hulls.

Given a set $\mathcal{T}$ of mappings defined on $\mathcal{A}$, and a subset $\mathcal{V} \subseteq \mathcal{A}$, we call

$$\mathcal{T}_{|\mathcal{V}} = \left\{\varrho_{|\mathcal{V}} \mid \varrho \in \mathcal{T}\right\}$$

the projection of $\mathcal{T}$ to $\mathcal{V}$. Note that in a finite-dimensional real vectorspace, there is a notion of *semialgebraic set*, which is independent of the choice of a basis.

Theorem 4.4. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{V} \subseteq \mathcal{A}$ a finite-dimensional unital $*$-subspace and $\mathcal{H}$ a Hilbert space with* $\dim(\mathcal{H}) = n < \infty$. *Then the following are equivalent:*

(i) The projection of $\{\varrho\colon \mathcal{A} \to \mathbb{B}(\mathcal{H}) \mid \varrho \in \operatorname{conv}\operatorname{Rep}(\mathcal{A})\}$ to $\mathcal{V}$ is semialgebraic.

(ii) $(\Sigma^2\mathcal{A} \otimes \mathbb{M}_n)_{|\mathcal{V}}$ is semialgebraic.

Proof. A unital $*$-linear mapping $\varrho\colon \mathcal{V} \to \mathbb{B}(\mathcal{H})$ is in the projection from (i), if and only if it is u.c.p, by Arvesons's Extension Theorem. By Choi's Theorem, this is equivalent to c_ϱ being nonnegative on $(\Sigma^2\mathcal{A} \otimes \mathbb{M}_n)_{|\mathcal{V}}$. So the projection from (i) is just the dual of the closed set $(\Sigma^2\mathcal{A} \otimes \mathbb{M}_n)_{|\mathcal{V}}$. This proves the claim. $\square$

Example 4.5. Let $\mathcal{A}$ be a commutative C^*-algebra. An element from $\mathcal{V} \otimes \mathbb{M}_n(\mathbb{C})$ is positive if and only if it is positive under each representation of the form $\pi \otimes \mathrm{id}$, where $\pi \in \operatorname{Rep}(\mathcal{A})$. Since $\mathcal{A}$ is rfd we can restrict to $\pi \in \operatorname{Rep}_{fd}(\mathcal{A})$, and since $\mathcal{A}$ is commutative, even to one-dimensional representations.

If $\mathcal{A}$ is the universal C^*-algebra of a finitely generated quadratic module in a finitely generated commutative algebra, then the set of one-dimensional representations is semialgebraic. Thus $(\Sigma^2\mathcal{A} \otimes \mathbb{M}_n)_{|\mathcal{V}}$ and the corresponding projection of $\operatorname{conv}\operatorname{Rep}(\mathcal{A})$ to $\mathcal{V}$ are always semialgebraic.

EXAMPLE 4.6. Let $\mathcal{A} = \mathbb{C}\mathbb{F}_m$ be the group algebra of the free group. Again $(\Sigma^2\mathcal{A}\otimes\mathbb{M}_n)_{|\mathcal{V}}$ is always semialgebraic. As before, an element from $\mathcal{V}\otimes\mathbb{M}_n(\mathbb{C})$ is positive if and only if it is positive at each representation $\pi\otimes\mathrm{id}$, where $\pi\in\operatorname{Rep}(\mathcal{A})$. Choi's proof that $\mathcal{A}$ is rfd shows that we can even restrict to representations π of some fixed dimension, depending only on $\mathcal{V}$ and n. So $(\Sigma^2\mathcal{A}\otimes\mathbb{M}_n)_{|\mathcal{V}}$ can be defined by a formula in the language of ordered rings and is thus semialgebraic. The same reasoning applies to all quadratic modules from Examples 3.11, 3.12 and 3.13.

In general, the projections of both $\operatorname{conv}\operatorname{Rep}(\mathcal{A})$ and $\operatorname{conv}\operatorname{Rep}_{fd}(\mathcal{A})$ are not semialgebraic, answering the above questions.

THEOREM 4.7. *There exists a finitely generated quadratic module $\mathcal{Q}$ in the free algebra $\mathcal{A} = \mathbb{C}\langle z_1,\dots,z_n\rangle$, such that already the projection of*

$$\left\{\rho\colon\mathcal{A}\to\mathbb{C}\mid\rho\in\operatorname{conv}\operatorname{Rep}_{fd}(\mathcal{A},\mathcal{Q})\right\}$$

to $\mathcal{V}=\operatorname{span}\{z_1,\dots,z_n\}$ is not semialgebraic.

PROOF. Instead of the free algebra, we work with the group algebra $\mathcal{A} = \mathbb{C}\Gamma$ of the discrete Heisenberg group $\Gamma = \langle a,b,c \mid c = aba^{-1}b^{-1}, ca = ac, cb = bc\rangle$ and $\mathcal{Q} = \Sigma^2\mathcal{A}$. By lifting the relations as pairs of inequalities to the free algebra, one obtains an example in the free algebra.

Each irreducible n-dimensional representation of $\mathcal{A}$ maps c to an n-th root of unity. This is true since c lies in the center of $\mathcal{A}$, and as a commutator has determinant one. Any n-th root of unity is attained through a representations, by [**10**]. Let $\tilde{\mathcal{V}} = \operatorname{span}\{(c+c^*)/2, (c-c^*)/(2i)\}\subseteq\mathcal{A}$. Then the projection of $\left\{\varrho\colon\mathcal{A}\to\mathbb{C}\mid\varrho\in\operatorname{conv}\operatorname{Rep}_{fd}(\mathcal{A})\right\}$ to $\tilde{\mathcal{V}}$ is

$$\operatorname{conv}\left\{(x,y)\in\mathbb{R}^2\mid x+iy \text{ roof of unity}\right\},$$

which is not semialgebraic. $\square$

REMARK 4.8. The example also shows that there is no bound on r in (4.1).

THEOREM 4.9. *There exists a finitely generated quadratic module $\mathcal{Q}$ in the free algebra $\mathcal{A} = \mathbb{C}\langle z_1,\dots,z_n\rangle$, such that already the projection of*

$$\{\rho\colon\mathcal{A}\to\mathbb{C}\mid\rho\in\operatorname{conv}\operatorname{Rep}(\mathcal{A},\mathcal{Q})\}$$

to $\mathcal{V}=\operatorname{span}\{z_1,\dots,z_n\}$ is not semialgebraic.

PROOF. Again we work in the group algebra $\mathbb{C}\Gamma$ of the discrete Heisenberg group $\Gamma = \langle a,b,c \mid c = aba^{-1}b^{-1}, ca = ac, cb = bc\rangle$. This time let $\tilde{\mathcal{V}} = \operatorname{span}\{a+a^*+b+b^*, (c+$

$c^*)/2$, $(c - c^*)/(2i)\}$. The classification of irreducible representations of Γ is well-known, the spectral properties of the above operators in these representations are extensively studied in [**3**], and we will use these results. The irreducible representations of Γ are parametrised by the circle $\{e^{i\theta} \mid \theta \in [0, 2\pi]\}$, and in every such irreducible representation we have $\pi_\theta(c) = e^{i\theta}$, $\pi_\theta(ab) = e^{i\theta}\pi_\theta(ba)$, so in an irreducible representation a and b generate a noncommutative torus with parameter θ. We denote $H_\theta = \pi_\theta(a + a^* + b + b^*)$. Using the automorphism of the noncommutative torus which maps the generators to the negatives of them, it is not hard to see that the spectrum of H_θ is symmetric.

Now a u.c.p. map $\varrho\colon \mathcal{A} \to \mathbb{C}$ is just a state on $C^*(\Gamma)$, and it's a well-known general fact that the states on a C^*-algebra form a closed convex set whose extremal points are the pure states coming from irreducible representations. Thus, the projection of $\{\varrho\colon \mathcal{A} \to \mathbb{C} \mid \varrho \in \operatorname{conv} \operatorname{Rep}(\mathcal{A})\}$ to $\tilde{\mathcal{V}}$ is the closed convex hull

$$C = \overline{\operatorname{conv}} \left\{ (\pm\|H_\theta\|, \cos\theta, \sin\theta) \in \mathbb{R}^3 \mid \theta \in [0, 2\pi] \right\}.$$

The function $\theta \mapsto \|H_\theta\|$ describes the boundary of the "Hofstadter butterfly" [**18**] (see Figure 1 for a picture), and is known to be non-differentiable at the points where $\theta/(2\pi)$ is rational [**12, 27**]. So if C were a semialgebraic set, its intersection with the cylinder $Z = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 = 1\}$ would also be semialgebraic, and thus the functions $\theta \mapsto \pm\|H_\theta\|$ whose graphs form the (relative) boundary of the set $Z \cap C$ would be piecewise smooth, which yields a contradiction. Therefore C is not semialgebraic. $\square$

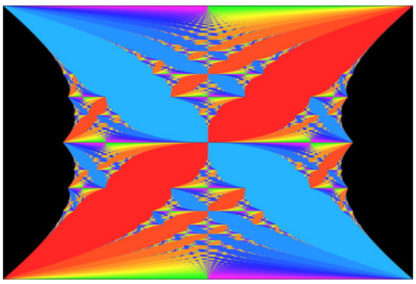

FIGURE 1. The Hofstadter butterfly
`https://commons.wikimedia.org/wiki/File%3AHofstadter's_butterfly.png`

The membership problem from real algebraic geometry is the following: Given a finitely generated quadratic module $(\mathcal{A}, \mathcal{Q})$ and a finite-dimensional $\mathbb{R}$-subspace $\mathcal{V} \subseteq \mathcal{A}^h$, is $\mathcal{Q} \cap \mathcal{V}$ a semialgebraic set? This is known to be true in certain cases, but an open question in general [**2**].

COROLLARY 4.10. *There is a finitely generated quadratic module in the free algebra* $\mathbb{C}\langle z_1, \ldots, z_n\rangle$, *for which the membership problem has a negative answer.*

Proof. This follows from Theorem 4.9 combined with Theorem 4.4. In fact, if $\mathcal{Q} \cap \mathcal{V}$ is semialgebraic, then so is its closure $\overline{\mathcal{Q} \cap \mathcal{V}}$, and this is equivalent to condition (ii) for the universal C^*-algebra $C^*(\mathcal{A}, \mathcal{Q})$ in Theorem 4.4. $\square$

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CHAPTER 4

Sofic boundaries of groups and coarse geometry of sofic approximations

Abstract

Sofic groups generalise both residually finite and amenable groups, and the concept is central to many important results and conjectures in measured group theory. We introduce a topological notion of a sofic boundary attached to a given sofic approximation of a finitely generated group and use it to prove that coarse properties of the approximation (property A, asymptotic coarse embeddability into Hilbert space, geometric property (T)) imply corresponding analytic properties of the group (amenability, a-T-menability and property (T)), thus generalising ideas and results present in the literature for residually finite groups and their box spaces. Moreover, we generalise coarse rigidity results for box spaces due to Kajal Das, proving that coarsely equivalent sofic approximations of two groups give rise to a uniform measure equivalence between those groups. Along the way, we bring to light a coarse geometric view point on ultralimits of a sequence of finite graphs first exposed by Ján Špakula and Rufus Willett, as well as proving some bridging results concerning measure structures on topological groupoid Morita equivalences that will be of interest to groupoid specialists.

1. Introduction

Finite approximation of infinite objects is a fundamental tool in the modern mathematician's toolkit, and it has been used to great effect in the authors' favourite areas of mathematics: in the realm of operator algebras the notions of nuclearity, exactness and quasidiagonality for C^*-algebras [**SWW15**, **TWW15**, **BK97**], and the corresponding notion of hyperfiniteness for von Neumann algebras [**MvN43**] have given rise to the classification programs of C^*-algebras [**Ell76**, **Kir99**] and von Neumann algebras [**Con76**]. Their natural group theoretic counterpart is amenability.

The aforementioned types of approximation are quite strong and therefore restrictive: they correspond to the "amenable world" of groups and operator algebras. While interesting and beautiful in its own right, it does not encompass many natural and important examples in group theory and operator algebras – say, the free groups and operator algebraic objects related to them. However, one would like to extend the idea of finitary approximation as well beyond amenability. In the realm of operator algebras, such an approximation was suggested by Alain Connes in [**Con76**] and lead to the famous *Connes Embedding Conjecture*. By the remarkable work of Eberhard Kirchberg [**Kir93**] it was shown to be equivalent to the so-called QWEP conjecture for C^*-algebras.

What one sees by studying the above is a relaxation of *algebra homomorphisms* to maps that are *approximately* homomorphisms. This suggests a more general notion of finite approximation should exist for groups when we allow for a *metric* on the finite set on which we attempt to approximate. This leads to the definition of a *sofic* group.

To make sense of what an "approximate" map to a finite group is, one chooses finite symmetric groups as targets and equips them with the normalised Hamming distance. A group Γ is sofic if it is possible to find approximations of arbitrary finite subsets of Γ in symmetric groups $\mathrm{Sym}(X)$ that are approximately injective and approximately multiplicative with respect to this distance. A countable collection $\mathcal{X}$ of finite sets X_i that witness stronger and stronger approximations for an exhaustion of the group Γ is a *sofic approximation* of Γ. Examples of sofic groups include amenable groups and residually finite discrete groups. Sofic groups were introduced by Mikhail Gromov [**Gro99**] in his work on Gottschalk's surjunctivity conjecture, and expanded on (and named by) Benjamin Weiss in [**Wei00**]. Since then they have played a fundamental role in research in dynamical systems.

The purpose of this paper is to introduce a general technique for studying sofic approximations of groups from the coarse geometric point of view and to give a mechanism for transferring topological (in this context, coarse geometric) properties from the approximation back to the group. The vessel we use to complete this journey is coarse geometric in nature and was initially introduced by George Skandalis, Jean-Louis Tu and Guoliang Yu in [**STY02**], where a *topological groupoid* was constructed to emulate the role of a group in certain aspects of the Baum–Connes conjecture for metric spaces. The second author of this paper studied this groupoid and certain of its *reductions* in [**FSW14**] and [**FS14**] in the context of *box spaces* associated to residually finite discrete groups.

A *box space* associated to a residually finite discrete group Γ and a chain of subgroups $\{N_i\}_i$ is a metric space, denoted $\square\Gamma$, constructed from the Cayley graphs of the finite quotients Γ/N_i. This is a particular example of a sofic approximation of a residually finite group.

Box spaces can be a powerful tool, both to differentiate between coarse properties (as in [**AGŠ12**]) and to provide a finite dimensional test for analytic properties of the group Γ. Notably, the following correspondences between coarse geometric properties of the box space and analytic properties of the group are known:

- $\square\Gamma$ has Property A if and only if Γ is amenable [**Roe03**, Proposition 11.39];
- $\square\Gamma$ has an asymptotic coarse embedding (or a fibred coarse embedding) into Hilbert space if and only if Γ is a-T-menable [**Wil15**, **FS14**, **CWY13**, **CWW13**];
- $\square\Gamma$ has geometric property (T) if and only if Γ has property (T) [**WY14**].

The method presented in [**FS14**] for producing these results was to associate to any given box space $\square\Gamma$ a topological boundary that admits a free Γ-action – this boundary action is a particular component of the coarse groupoid of Skandalis–Tu–Yu. The main idea in this paper is to generalise this procedure to a sofic approximation of a sofic group, but in this setting the counting measures on each "box" will play a fundamental role. More precisely, we associate to a given sofic approximation a topological groupoid that we call the *sofic coarse boundary groupoid*. The base space of this groupoid – the *sofic boundary* – is constructed from the "box space" of graphs coming from the sofic approximation. It carries a natural invariant measure coming from the counting measure on the graphs and has a nice closed saturated subset Z of full measure – the *core* of the sofic boundary – restricted to which, the sofic coarse boundary groupoid turns out to be a crossed product by an action of Γ as in the traditional box space case. This allows us to prove:

THEOREM 1.1. *Let Γ be a sofic group, $\mathcal{X}$ a sofic approximation of Γ, and X be the space of graphs constructed from $\mathcal{X}$. Then:*

(i) If X has property A then Γ is amenable (Theorem 4.5);
(ii) If X admits an asymptotic coarse embedding into Hilbert space, then Γ is a-T-menable (Theorem 4.12);
(iii) If X has boundary geometric property (T) then G has property (T) (Theorem 4.25).

At this point, it is natural to ask about the converse statements. There appears to be little hope of establishing them in full generality, the main technical reason being that the core of the sofic boundary is a proper subset of it, and there is no control of what happens on the complement. We explain this issue in more detail in the final section of the paper.

However, if the sofic approximation is coming from the group being *locally embeddable into a finite group* (or briefly an LEF group), the core is the entire boundary, which allows us to recover the converse to the above statements, thus reproving the known results about LEF groups from the literature [**MS13**, **MOSS15**].

Transitioning from coarse invariants (that are *topological* invariants of a groupoid) to measurable invariants, we begin to investigate the question: to what extent a sofic approximation is a "coarse invariant" of the sofic group? To this end, we were able to prove the following:

THEOREM 1.2. *(Theorem 5.13) Let Γ, Λ be sofic groups with sofic approximations $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let $X_{\mathcal{X}}$ and $X_{\mathcal{Y}}$ be their associated spaces of graphs. If $X_{\mathcal{X}}$ and $X_{\mathcal{Y}}$ are coarsely equivalent, then Γ and Λ are quasi-isometric and uniformly measure equivalent.*

This theorem generalises part of the work in [**KV15**], and the main result of [**Das15**] to the case that Γ and Λ are sofic, as opposed to residually finite, and the technique is completely different – we construct a *Morita equivalence* bispace for the sofic coarse boundary groupoids. This bispace looks very much like the topological coupling introduced by Gromov in his dynamic classification of quasi-isometries between groups. Given appropriate measures on the groupoids, we construct a measure on the bispace, which turns the topological Morita equivalence into a measurable one – and this allows us to deduce the uniform measure equivalence combining the topological and measure-theoretic properties of sofic coarse boundary groupoids. As was pointed out in [**Das15**], by combining a result of Damien Gaboriau [**Gab02**, Theorem 6.3] with Theorem 5.13 we are able to conclude facts concerning the rigidity of ℓ^2-Betti numbers of sofic groups with coarsely equivalent approximations:

COROLLARY 1.3. *If Γ and Λ are finitely generated sofic groups with coarsely equivalent sofic approximations, then their ℓ^2-Betti numbers are proportional.*

The downside of the topological groupoid we construct to settle the above questions is that the unit space is not second countable, therefore not metrizable (and thus not a *standard* as a probability space). We remedy this situation by providing a recipe for constructing many different second countable versions of the groupoid using ideas from [**STY02**, **Exe08**]. The following should be considered as a topological result in line with the standartisation theorem for measurable actions proved by Alessandro Carderi in [**Car15**, Theorem A].

THEOREM 1.4. *Let Γ be a sofic group, $\mathcal{X}$ a sofic approximation of Γ, X the associated total space of the family of graphs attached to $\mathcal{X}$ and $Z \subset X$ the core of a sofic approximation. Then there exists a second countable étale, locally compact, Hausdorff topological groupoid $\mathcal{G}$ with following properties:*

- *(i) the base space $\mathcal{G}^{(0)} =: \widehat{X}$ is a compactification of X (in particular, it's a quotient of βX through a quotient map $p\colon \beta X \to \widehat{X}$),*
- *(ii) $p(Z) \subset \partial\widehat{X}$ is invariant and satisfies $\mathcal{G}|_{p(Z)} \cong p(Z) \rtimes \Gamma$. As a consequence, we have an almost everywhere isomorphism*

$$(\mathcal{G}|_{\partial\widehat{X}}, \nu_{p_*\mu}) \to (\widehat{X}, p_*\mu) \rtimes \Gamma.$$

As an example of this process, we construct the minimal topological groupoid introduced in [**AN12**] for a residually finite discrete group and a corresponding Farber chain of finite index subgroups.

The paper is organised as follows. In Section 2 we recapitulate the necessary definitions and results both from the theory of sofic group approximations and groupoids arising from coarse geometry. Section 3 introduces our main player, the sofic coarse boundary groupoid associated with a fixed sofic approximation of a group and studies its properties; in particular, we introduce the core of a sofic approximation as the closure of the "good set" in the approximating graphs. Section 4 is devoted to the proof of the main Theorem 1.1 and its converse in the case of an LEF group. Finally, in Section 5 we prove that coarse equivalence of two sofic approximations implies quasi-isometry and uniform measure equivalence of groups (Theorem 5.13). In the last section we discuss some related open questions that might be of interest for further investigation.

2. Preliminaries

In this section we introduce the necessary definitions, facts and references for coarse groupoids and sofic groups.

2.1. Groupoids from coarse geometry. We recapitulate some particular examples of groupoids that appear later in the paper. For a basic introduction to étale groupoids we recommend [**Exe08**], for their representation theory [**SW12**] and finite approximation properties [**ADR00**]. We also suggest the collected references of [**STY02**], [**Roe03**] and [**SW16**] for the notion of coarse groupoid and its properties.

Example 2.1. Let X be a topological Γ-space. Then the *transformation groupoid* associated to this action is given by the data $X \times G \rightrightarrows X$ with $s(x,g) = x$ and $r(x,g) = g.x$. We denote this by $X \rtimes \Gamma$. A basis $\{U_i\}$ for the topology of X lifts to a basis for the topology of $X \rtimes \Gamma$, given by sets $[U_i, g] := \{(u,g) \mid u \in U_i\}$.

Example 2.2. We move now to examples of groupoids coming from uniformly discrete metric spaces of bounded geometry. We define a groupoid which captures the coarse information associated to X. Consider the collection $\mathcal{S}$ of the R-neighbourhoods of the diagonal in $X \times X$; that is, for every $R > 0$ the set

$$E_R = \{(x,y) \in X \times X \mid d(x,y) \leqslant R\}$$

Let $\mathcal{E}$ be the coarse structure generated by $\mathcal{S}$ as in [**Roe03**]; it is called the *metric coarse structure* on X. If X is a uniformly discrete metric space of bounded geometry, then this coarse structure is uniformly locally finite, proper and weakly connected – thus of the type studied by Skandalis, Tu and Yu in [**STY02**].

We now define the coarse groupoid following the approach of [**SW16**, Appendix C]. Let βA denote the Stone-Čech compactification of a set A. Set $G(X) := \bigcup_{R>0} \overline{E_R}$, where the closure $\overline{E}_R$ takes place in $\beta X \times \beta X$ and $G(X)$ has the weak topology coming from the union – with this topology $G(X)$ is a locally compact, Hausdorff topological space, which becomes a groupoid with the pair groupoid operations from $\beta X \times \beta X$. Another possible approach (for instance that adopted originally in [**STY02**] or in [**Roe03**]) is to consider graphs of partial translations on X and form a *groupoid of germs* from this data [**Exe08**]. Each approach has value, depending on the particular situation.

One advantage of working with groupoids is that they come with many possible *reductions*.

Definition 2.3. A subset of $C \subseteq G^{(0)}$ is said to be *saturated* if for every element $\gamma \in G$ with $s(\gamma) \in C$ we have $r(\gamma) \in C$. For such a subset we can form a subgroupoid of G, denoted by G_C which has unit space C and $G_C^{(2)} = \{(\gamma, \gamma') \in G^{(2)} \mid s(\gamma), r(\gamma) = s(\gamma'), r(\gamma') \in C\}$. The groupoid G_C is called the *reduction of G to C*.

REMARK 2.4. For a uniformly discrete metric space X of bounded geometry there are natural reductions of $G(X)$ that are interesting to consider. It is easy to see that the set X is an open saturated subset of βX and in particular this means that the Stone-Čech boundary $\partial\beta X$ is saturated. We remark additionally that the groupoid $G(X)|_X$ is the pair groupoid $X \times X$ (as the coarse structure is weakly connected).

DEFINITION 2.5. The *boundary groupoid* $\partial G(X)$ associated to X is the groupoid reduction $G(X)|_{\partial\beta X}$.

2.2. Box spaces as an example. Let $\mathcal{X} = \{X_i\}_i$ be a family of finite connected graphs of uniformly bounded vertex degree.

DEFINITION 2.6. The *space of graphs* associated to $\mathcal{X}$ is the set $X := \bigsqcup_i X_i$, equipped with any metric d that satisfies:

(i) $d|_{X_i}$ is the metric coming from the edges of the graph X_i;
(ii) $d(X_i, X_j) \to \infty$ as $i + j \to \infty$.

We remark that any two metrics that satisfy i) and ii) are coarsely equivalent, and thus we need not be more specific about the rates of divergence.

Natural examples of graph families, and thus spaces of graphs, come from finitely generated residually finite discrete groups. Let $\Gamma = \langle S \rangle$ be finitely generated and residually finite. Then, for any chain (i.e. a nested family of finite index subgroups with trivial intersection) $\mathcal{H} = \{H_i\}_i$ we can consider the Schreier coset graphs:

$$X_i := \mathrm{Cay}(\Gamma/H_i, S).$$

REMARK 2.7. We note that there are various conditions in the literature that one could reasonably put into such a chain of finite index subgroups, for instance asking for each to be normal subgroups, or more generally to separate points from the entire conjugacy class of the subgroup H_i (which is called *semi-conjugacy separating* in [**FSW16**] and appears first in [**SWZ14**]), or to ask that the family is *Farber* (that is, for any $g \in \Gamma$, $n_i(g) = o(n_i)$, where n_i is the number of conjugates of H_i in Γ and $n_i(g)$ is the number of conjugates of H_i containing g [**Far98**, **AN12**]).

For simplicity, suppose the chain consists of normal subgroups. Then the space of graphs associated to $\mathcal{X} = \{X_i\}_i$ is called the *box space* of Γ with respect to $\mathcal{H}$, and denoted by $\square_{\mathcal{H}}\Gamma$.

This construction and the many results concerning it in the literature drive the coarse geometric aspect of this paper. We will focus on the coarse groupoid (and its boundary), to get a better feeling for it in a simpler case than will appear later on.

DEFINITION 2.8. Let $\mathcal{S}$ be a family of subsets in $X \times X$. The family $\mathcal{S}$ *generates* $\mathcal{E}$ *at infinity* if for every $R > 0$ there are finitely many sets $S_1, \ldots, S_n \in \mathcal{S}$ and a finite subset

$F \subset X \times X$ such that

$$E_R \subseteq \left(\bigcup_{k=1}^{n} S_k\right) \cup F.$$

REMARK 2.9. The above definition is equivalent to asking that $\overline{E_R} \setminus E_R \subseteq \bigcup_{k=1}^{n} \overline{S_k} \setminus S_k$, where the closure is taken in $\beta X \times \beta X$.

If Γ is a discrete group acting on X, let $E_g := \{(x, x.g) \mid x \in X\}$ be the g-diagonal in X. We say that the action of Γ generates the metric at infinity if the set $\{E_g \mid g \in \Gamma\}$ satisfies Definition 2.8.

PROPOSITION 2.10 ([**FSW14**, Proposition 2.5]). *Let X be a uniformly discrete bounded geometry metric space and let Γ be a finitely generated discrete group. If Γ acts on X so that the induced action on βX is free on $\partial\beta X$ and the action generates the metric coarse structure at infinity, then $\partial G(X) \cong \partial\beta X \rtimes \Gamma$.* □

The following example is the basic model we will build on in Section 3 for sofic groups.

EXAMPLE 2.11. Let $X = \square_{\mathcal{H}}\Gamma$ be the box space of a residually finite group Γ with normal chain $\mathcal{H}$. Then, considering the metric d from Definition 2.6 we see that the sets E_R decompose as

$$E_R = \bigsqcup_i E_{R,i} \sqcup F_R,$$

where $E_{R,i}$ is the R-neighbourhood of the diagonal in X_i and $F_R = \{(x,y) \mid x \in X_i, y \in X_j, i \neq j,\ d(x,y) \leqslant R\}$. This observation allows us to reduce to considering the set $E_{R,\infty} = \bigsqcup_i E_{R,i} \subset E_R$, as these sets have the same Stone-Čech boundary.

As the group Γ is residually finite, each of the $E_{R,i}$ decomposes as $\bigsqcup_{|g|\leqslant R} E_{g,i}$ when i is sufficiently large – in particular, $\partial\beta E_{R,\infty} = \bigsqcup_{|g|\leqslant R} \partial\beta E_g$, and so the group, acting by translations, generates the metric coarse structure at infinity. This action is free at the boundary by residual finiteness of Γ: for each $g \in \Gamma$ the orbit graph for the action of g on $\square\Gamma$ has degree at most 2, and thus is at most 3-coloured by Brookes' theorem. The Stone-Čech boundaries of each colour set are then permuted by the element g and have empty intersection. Thus Proposition 2.10 implies that $\partial G(X) \cong \partial\beta X \rtimes \Gamma$.

2.3. A formal definition of soficity.

Let us give a formal definition of a sofic group:

DEFINITION 2.12 (see [**Pes08**, Theorem 3.5]). A group Γ is *sofic* if for every finite subset $F \subset \Gamma$ and every $\varepsilon > 0$ there exists a finite set X, a map $\sigma\colon \Gamma \to \mathrm{Sym}(X)$ and a subset $Y \subset X$ with $|Y| \geqslant (1-\varepsilon)|X|$ such that

$$\sigma(g)\sigma(h)(y) = \sigma(gh)(y), \quad g, h \in F,\ y \in Y$$

and

$$\sigma(g)(y) \neq y, \quad g \in F \setminus \{e\},\ y \in Y.$$

The map σ is said be an (F,ε)-*injective almost action* on the set X if the condition above holds.

We note that if Γ is sofic, then by fixing a nested sequence of sets F_i that exhaust the group, choosing a sequence $\varepsilon_i \to 0$, and letting X_i be a set with an (F_i,ε_i)-injective almost actions of Γ, we obtain a sequence of sets together with almost actions of Γ; such a sequence called a *sofic approximation of* Γ.

We remark that soficity generalises both being residually finite and being amenable for a group Γ. We refer the reader to the book [**CSC12**] for more details of the permanence properties of sofic groups, and we also note that there is, at time of writing, no group that is known to be non-sofic.

In the remaining part of this section, we will give a more geometric definition of soficity which will allow us to apply coarse geometric methods.

2.4. Ultralimits and local convergence of graphs.

Definition 2.13. Let $\mathcal{X} = \{X_i\}_i$ be a countable family of finite graphs of bounded degree, X be the space of graphs attached to $\mathcal{X}$ and let $\omega \in \partial\beta\mathbb{N}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $\underline{x}$ be a sequence of points in X, and let $S(\underline{x})$ be the set of all $\underline{y} = (y_n)_n$ such that $\sup_n(d(x_n, y_n)) < \infty$. We define a (pseudo-)metric on $S(\underline{x})$ by

$$d_\omega(\underline{y}, \underline{z}) = \lim_\omega d(y_n, z_n)$$

and the *ultralimit along* ω, denoted $X(\omega, \underline{x})$, to be the canonical quotient metric space obtained from $(S(\underline{x}), d_\omega)$ by identifying all pairs of points at distance 0.

This notion of ultralimit has a natural description in terms of the coarse boundary groupoid $\mathcal{G} := \partial G(X)$ from the previous section. Let $\eta = \lim_\omega x_n$ be the point in the Stone-Čech boundary that corresponds to $\underline{x}$ and $\omega \in \partial\beta\mathbb{N}$.

Proposition 2.14. *Let $\mathcal{G}_\eta$ be the source fibre of $\mathcal{G}$ at $\eta \in \partial\beta X$. Equip $\mathcal{G}_\eta$ with the metric*

$$d_\eta((\eta_1, \eta), (\eta_2, \eta)) = \inf\{R > 0 \mid (\eta_1, \eta_2) \in \overline{E_R}\}.$$

Then the map $f : X(\omega, \underline{x}) \to \mathcal{G}_\eta$ given by $[(y_n)] \mapsto (\lim_\omega y_n, \eta)$ is a basepoint preserving isometry.

Proof. For any points $[(y_n)], [(z_n)] \in X(\omega, \underline{x})$, we have

$$\begin{aligned} d_\omega([(y_n)], [(z_n)]) &= \inf\{R > 0 \mid \omega(\{n \in \mathbb{N} \mid d(y_n, z_n) \leqslant R\}) = 1\} \\ &= \inf\{R > 0 \mid \omega(\{n \in \mathbb{N} \mid (y_n, z_n) \in E_R\}) = 1\} \\ &= \inf\{R > 0 \mid \lim_\omega (y_n, z_n) \in \overline{E_R}\} \\ &= \inf\{R > 0 \mid (\lim_\omega y_n, \lim_\omega z_n) \in \overline{E_R}\} \\ &= d_\eta(\lim_\omega y_n, \lim_\omega z_n). \end{aligned}$$

Hence f is isometric and maps into $\mathcal{G}_\eta$. It remains to prove that f is surjective.

Let $(\eta', \eta) \in \mathcal{G}_\eta$. Using the view on $G(X)$ in terms of germs of partial translations as in [**STY02**, Proposition 3.2] or [**Roe03**, Chapter 10], we obtain a partial translation $t : A \to B$ between subsets $A, B \subset X$ such that $\eta \in \overline{A} \subset \beta X$, $\eta' \in \overline{B} \subset \beta X$ and with $\bar{t}(\eta) = \eta'$.As $\eta = \lim_\omega(x_n)$, we have that the set $E = \{n \in \mathbb{N} \mid x_n \in A\}$ has ω-measure 1, and therefore we can define another sequence with terms:

$$y_n := \begin{cases} x_n \text{ if } n \notin E \\ t(x_n) \text{ if } n \in E. \end{cases}$$

As η' is the unique point in the closure of the graph of t satisfying $(\eta', \eta) \in \overline{\operatorname{graph}(t)}$, we have that

$$(\eta', \eta) = \lim_\omega(t(x_n), x_n) = \lim_\omega(y_n, x_n),$$

and thus $\eta' = \lim_\omega y_n$. $\square$

We remark that for a fixed ultrafilter $\eta \in \partial\beta X$ one can always find a sequence $\underline{x}$ tending to infinity and an ultrafilter $\omega \in \partial\beta\mathbb{N}$ such that $\eta = \lim_\omega \underline{x}$. There will in general be many such choices, but the above proposition ensures that they will give isometric fibres.

Ideally, we would like to remove the dependence on the base point from this process. The suggested method (say of [**BS01**] or [**AL07**]) is to make this choice *uniformly at random*, and to do this we need a measure on $\partial\beta X$.

Given the sequence of counting measures μ_i on each $X_i \in \mathcal{X}$ and fixing an ultrafilter $\omega \in \partial\beta\mathbb{N}$, we can obtain a measure μ on the Stone-Čech boundary of X corresponding to the state

$$\mu(f) = \lim_\omega \frac{1}{|X_i|} \sum_{x \in X_i} f(x), \quad f \in C(\beta X). \tag{2.1}$$

Note that $\mu(X) = 0$, whence $\mu(\partial\beta X) = 1$. Armed with this measure on $\partial\beta X$, we can now formulate a notion of graph convergence:

Definition 2.15. A sequence of graphs $\mathcal{X}$ of bounded degree is said to *Benjamini–Schramm converge* to a graph Y if the set

$$\{x = \lim_\omega x_n \in \partial\beta X \mid X(\omega, \underline{x}) \cong (Y, y) \text{ for some } y \in Y\}$$

of ultralimits that are isomorphic as pointed graphs to Y has μ-measure 1.

A first remark concerning this definition is that the basepoint in Y does not matter if Y is vertex transitive. The second remark we make is that this definition can also be made using *labelled* graphs.

Let S be a finite set of labels. Suppose also that each X_i admits an S-edge labelling. Then any ultralimit of the sequence $X(\omega, \underline{x})$ also admits an S-labelling. In this case, we can ask that Y admits a labelling and that the base point preserving isometries occurring in the definition can be taken as isometries of labelled graphs.

REMARK 2.16. The traditional formulation of Benjamini–Schramm convergence (found for instance in [**BS01**]) uses converging probabilities of isometry types of balls. It is equivalent to this more topological formulation by realising an ultralimit $X(\omega, \underline{x})$ as a union of balls around $\underline{x}$ and studying how these can be obtained from the sequence $\mathcal{X}$ using ω. This works equally well in labelled and non-labelled settings.

REMARK 2.17. This Benjamini–Schramm convergence should be thought of as an "almost everywhere" (in terms of the normalised counting measure) version of the convergence in the space of marked graphs – if a sequence of bounded degree finite graphs converges there to a fixed graph, then it Benjamini–Schramm converges to that graph – in fact, the set of measure 1 will be the entire boundary in that case.

The following definition is central to the paper:

DEFINITION 2.18. Let Γ be a finitely generated group with a finite generating set S. Γ is *sofic* if there exists a sequence $\mathcal{X}$ of bounded degree, finite S-labelled graphs such that $\mathcal{X}$ Benjamini–Schramm converges to $(\mathrm{Cay}(G,S), e_G)$.

It is equivalent to Definition 2.12 by an argument present in [**Pes08**, Theorem 5.1], which constructs the (S-labelled) graph structure on the sets X_i appearing in Definition 2.12 by connecting each $x \in X_i$ with $\sigma_i(s)$ by an edge labelled with $s \in S$; we will always equip X_i coming from a sofic approximation with this graph structure and (slightly abusing notation) also call the resulting sequence $\mathcal{X}$ a sofic approximation of Γ. The following lemma asserts that we can assume these graphs to be connected, which we will always do.

LEMMA 2.19. *Let $\Gamma = \langle S \rangle$ be a finitely generated sofic group and let $\mathcal{X}' = \{X_i', \sigma_i'\}_i$ be a sofic approximation; equip X_i' with the graph structure described above. For each i there is a connected component $X_i \subset X_i'$ and maps $\sigma_i \colon \Gamma \to \mathrm{Sym}(X_i)$ coinciding with σ_i' on the generating set S such that $\mathcal{X} = \{X_i, \sigma_i\}_i$ is a sofic approximation with X_i. In particular, the graph structure coming from $\mathcal{X}$ makes X_i connected.*

PROOF. Let $X_{i,j}'$, $j = 1, \dots, n_i$ be the connected components of X_i' and let $Y_i' \subseteq X_i'$ be the subsets from Definition 2.12. Increasing i if needed, we may assume without loss of generality that $S \subset F_i$. Observe that

$$|Y_i'| = \sum_{j=1}^{n_i} |Y_i' \cap X_{i,j}'| \geqslant (1-\varepsilon)|X_i'| = (1-\varepsilon) \sum_{j=1}^{n_i} |X_{i,j}'|.$$

This implies that there is at least one connected component $X_{i,j}'$ such that $|Y_i' \cap X_{i,j}'| \geqslant (1-\varepsilon)|X_{i,j}'|$; we denote it by X_i and set $Y_i^{(0)} := Y_i' \cap X_i$.

Observe that by definition of the graph structure and by preceding construction:

- the connected components $X'_{i,j}$ are invariant under $\sigma'_i(S)$;
- $|Y_i^{(0)}| \geqslant (1-\varepsilon)|X_i|$.

For $g \in F_i$, we set $Y_{i,g} \coloneqq \{x \in X_i \mid \sigma'_i(g)(x) \in X_i\}$. We define $\sigma_i(g) \in \mathrm{Sym}(X_i)$ for $g \in F_i$ by (arbitrarily) extending the partial bijection $\sigma'_i(g)\colon Y_{i,g} \to X_i$ to a permutation $\sigma_i(g) \in \mathrm{Sym}(X_i)$ and we set $\sigma_i(g) = \mathrm{id}_{X_i}$ for $g \notin F_i$. The above properties guarantee that $\mathcal{X} = \{X_i, \sigma_i\}_i$ is the desired sofic approximation:

- as $\sigma_i(g)$ coincides with $\sigma'_i(g)$ on the points which remain in X_i under the latter permutation, the set

$$Y_i \coloneqq \{x \in X_i \mid \forall g, h \in F_i \; \sigma_i(g)\sigma_i(h)(x) = \sigma_i(gh)(x) \text{ and } \forall g \in F_i \setminus \{e\} \; \sigma_i(g)(x) \neq x\}$$

  contains $Y_i^{(0)}$ and therefore satisfies $|Y_i| \geqslant (1-\varepsilon)|X_i|$;
- $\sigma_i(s) = \sigma'_i(s)$ for all $s \in S$, and therefore the graph structure associated with σ_i is the same as the one coming from σ'_i.

This finishes the proof. □

3. The sofic coarse boundary groupoid

Let $\Gamma = \langle S \rangle$ be a finitely generated sofic group and $\mathcal{X}$ be a sofic approximation of Γ. The main idea of this paper is that the space of graphs X associated with $\mathcal{X}$ can be thought of as a box space for sofic group. In this section we will analyse the boundary groupoid attached with X, defined in the previous section. We will also explain how this analysis connects with the sofic core of the sofic approximation. We remark that being finitely generated by S gives rise to a natural quotient map $\pi_\Gamma : F_S \to \Gamma$, where F_S is the free group on the letters S.

Definition 3.1. Let $\mathcal{G}$ be the coarse boundary groupoid associated with the space of graphs X of a sofic approximation $\mathcal{X} = \{X_i, \sigma_i\}_i$ as defined in the previous section. $\mathcal{G}$ is called the *sofic coarse boundary groupoid* associated with the sofic approximation $\mathcal{X}$. Its base space $\partial\beta X$ is called the *sofic boundary* of $\mathcal{X}$.

Remark 3.2. For a sofic group Γ with a sofic approximation $\mathcal{X}$ and the attached space of graphs X, for $\mu_{\mathcal{X}}$-almost all $\omega \in \beta X$, the range fibre $r^{-1}(\omega)$ is isometric to $\mathrm{Cay}(\Gamma, S)$, as $\mathcal{X}$ is a sofic approximation. Let δ_ω be the Dirac mass at ω and let $\mathrm{Ind}(\delta_\omega)$ be the induced representation of $G(X)$ associated with the measure δ_ω as in [**SW12**]. Then $C^*(G(X), \delta_\omega)$, obtained through the the representation $\mathrm{Ind}(\delta_\omega)$ of $G(X)$ on $L^2(r^{-1}(\omega), \lambda^\omega)$, is a subalgebra of $C^*_u(\Gamma)$ [**SW16**, Appendix C].

As $\mathcal{G}$ is a locally compact étale groupoid, it can be considered as a Borel groupoid using the natural Borel σ-algebra obtained from the open subsets of $\mathcal{G}$. Our goal in this section is to relate $\mathcal{G}$ to an action Γ, both measurably and topologically. To do this, we introduce an action of F_S on $\partial\beta X$. Note that each X_i is an S-labelled finite graph, with labelled edges constructed using the permutations $\sigma_i(s)$. This defines an action of F_S on X_i. We then extend this action continuously to the Stone-Čech boundary, obtaining an F_S-action denoted τ. We remark that when the graphs are regular, it is precisely the action defined in [**FSW14**, Lemma 3.26]. The action τ is in general not free, but is still connected with the groupoid $\mathcal{G}$.

Definition 3.3. A τ-diagonal on the boundary is a set of the form:

$$A_P := \{(\omega, \tau(P)(\omega)) \mid \omega \in \partial\beta X\}$$

for each $P \in F_S$.

Proposition 3.4. *$\mathcal{G}$ is isomorphic to the orbit equivalence relation $\mathcal{R}_\tau$ of the action $\tau : F_S \to \mathrm{Homeo}(\partial\beta X)$, where this equivalence relation is given the weak topology generated by the clopen sets $\{A_P\}_{P\in F_S}$.*

Proof. We check that, for each $n \in \mathbb{N}$, the sets ∂E_n and $\bigcup_{|P|\leq n} A_P$ are equal. We first observe that if $\gamma \in \partial E_n$ then there is a net of pairs $((x_\lambda, y_\lambda))_\lambda$ with limit γ, and $d(x_\lambda, y_\lambda) \leqslant n$ on a convergent subnet.

However, as the distance here is the natural edge metric on a graph, to be at distance of at most n means that x_λ and y_λ are connected by an S-labelled path of length of most n. From this we conclude that the F_S-action by the concatenation of the labels will map x_λ to y_λ.

To see the reverse inclusion, we observe that anything belonging to at least one of the A_P's must be a limit of a net of pairs of the form $(x_\lambda, \tau(P)(x_\lambda))$. Therefore this net consists of pairs whose distances are bounded precisely by the length of P, which was supposed less than n. $\square$

We now return to Γ. For each $g \in \Gamma$, the map $\sigma(g)$ defined by performing $\sigma_i(g)$ in each graph X_i defines a bijection of X to itself. Extending these maps continuously gives us a collection of homeomorphisms $\sigma(g)$ on βX. We remark that this gives a map $\Gamma \to \mathrm{Homeo}(\partial\beta X)$, which is in general *not* a homomorphism of groups, but it is quite close to a homomorphism when we make use of the fact that the soficity of Γ is being witnessed by $\mathcal{X}$.

Let $Y \subset X$ be the the disjoint union of each Y_i coming from Definition 2.12. As the sets Y_i^c are at most μ_i-measure ε_i (and tending to 0) we have that $\mu(\overline{Y}) = 1$, where μ is the probability measure on $\partial\beta X$ defined in (2.1). For any element $\omega \in \partial Y$, the maps $\sigma(g)\sigma(h)$ and $\sigma(gh)$ coincide, and thus the map σ is a homomorphism of groups after throwing out a set of measure 0 in $\partial\beta X$. In particular, this is an example of a "near action" of Γ in the sense of [**GTW05**].

This is not yet useful topologically, but we can still make the following definition:

Definition 3.5. The σ-diagonals in $\partial\beta X \times \partial\beta X$ are sets of the form:

$$E_g := \{(x, \sigma(g)x) \mid x \in \partial\beta X\},$$

for $g \in \Gamma$.

Now we relate the equivalence relation $\mathcal{R}_\tau$ to the Γ-near action on $\partial\beta X$ by finding an F_S-invariant subset of $\partial\beta X$ on which the free group action really agrees with the Γ-near action.

DEFINITION 3.6. The set

$$Z := \bigcap_{g\in\Gamma} \sigma(g)(\partial Y)$$

is called the *core* of the sofic boundary $\partial\beta X$. It depends on the choice of the subsets $Y_i \subset X_i$ satisfying the conditions of Definition 2.12.

As ∂Y is clopen and the maps $\sigma(g)$ are all homeomorphisms, the core Z is a closed subset of $\partial\beta X$ that is invariant under the maps $\sigma(g)$. Using de Morgan's law, it's clear that $\mu(Z) = 1$; in particular the core is not empty.

For $K \subset \partial\beta X \times \partial\beta X$, we denote by K^Z the restriction $K \cap (Z \times Z)$.

LEMMA 3.7. *We have the following compatibility between the action of F_S and the action of Γ on Z:*

(i) For $g \neq h \in \Gamma$, we have that $\partial E_g^Z \cap \partial E_h^Z = \varnothing$.
(ii) $\mathrm{Stab}_{F_S}(Z) = \ker(\pi_\Gamma : F_S \to \Gamma)$;
(iii) If $\pi_\Gamma(P) = \pi_\Gamma(Q)$ then $A_P^Z = A_Q^Z$.

PROOF. For i), let $(\omega, \sigma(g)(\omega)) = (\omega, \sigma(h)(\omega)) \in \partial E_g^Z \cap \partial E_h^Z$. Thus, $\omega = \sigma(g)^{-1}\sigma(h)(\omega)$. As $Z \subset \partial Y$, we have that $\omega = \sigma(g^{-1}h)(\omega)$, however this can only happen if $g^{-1}h = e$.

The proofs of the remaining points follow directly from a key observation that comes from the definition of Z: if $w = a_{s_1}\cdots a_{s_n} \in F_S$, then $\tau(w)(\omega) = \sigma(s_1)\cdots\sigma(s_n)(\omega) = \sigma(\pi_\Gamma(w))(\omega)$ for every $\omega \in Z$. ii) and iii) are now deduced by elementary calculations using this observation. $\square$

We conclude that the set Z is a closed subset which is invariant under the equivalence relation $\mathcal{R}_\tau$, and thus under $\mathcal{G}$. In fact, combining with the arguments in the proof of Proposition 3.4, we can observe:

LEMMA 3.8. *There is a homeomorphism $\partial E_n^Z = \bigsqcup_{|g|\leqslant n} \partial E_g^Z$, given explicitly by the map*

$$\Theta : \partial E_n^Z \to \bigsqcup_{|g|\leqslant n} \partial E_g^Z,$$

$$\gamma \mapsto (s(\gamma), \pi_\Gamma(P)(s(\gamma))).$$

$\square$

The main result of this section is the following:

THEOREM 3.9. *The reduction groupoid $\mathcal{G}|_Z$ and the transformation groupoid $Z \rtimes \Gamma$ are topologically isomorphic.*

PROOF. The technique of the proof is similar to that of Proposition 3.4. As $\mathcal{G}|_Z = \bigcup_n \partial E_n^Z$, and $Z \rtimes \Gamma$ is the disjoint union $\bigsqcup_{g\in\Gamma} \partial E_g^Z$, we obtain a map $\Theta : \mathcal{G}|_Z \to Z \rtimes \Gamma$ using the (obviously compatible) map from Lemma 3.8. It remains to see that it is both a homeomorphism and a homomorphism of groupoids.

We observe that:

(i) both groupoids have a basis of topology given by clopen slices [**Exe10**, Proposition 4.1];
(ii) as $\mathcal{G}$ has the weak topology, it is sufficient to consider slices contained in $\overline{E_n}$, i.e we can consider slices $U \subset \partial E_n^Z$ when working with $\mathcal{G}|_Z$;
(iii) slices of the form $(U, g) := \{(\omega, \sigma(g)\omega) \mid \omega \in U\}$ for some clopen $U \subset Z$ generate the topology of $Z \rtimes \Gamma$.

Given a slice $U \subset \mathcal{G}|_Z$ contained in some ∂E_n^Z, we can see that $\Theta(U)$, by Lemma 3.7 iv), is contained within a finite disjoint union of clopen sets ∂E_g^Z. This means, in particular, that $\Theta(U) = \bigsqcup_g (U_g, g)$, which are open and disjoint. A similar argument proves that the map Θ is continuous.

To complete the proof we must show that the map is a homomorphism. This, however, follows from Lemma 3.7 ii) and the fact the map $\pi_\Gamma : F_S \to \Gamma$ is a group homomorphism. □

Recall that the measure μ is naturally extended to a Borel measure $\nu := \mu \circ \lambda$ on $\mathcal{G}|_Z$, defined by:

$$\int_{\gamma\in\mathcal{G}} f d\nu = \int_{x\in\partial\beta X} \left(\sum_{s(\gamma)=x} f(\gamma) \right) d\mu(x)$$

for every Borel measurable function f on $\mathcal{G}|_Z$.

COROLLARY 3.10. *The measure $\nu = \mu \circ \lambda$ is invariant for $\mathcal{G}|_Z$ (and thus for $\mathcal{G}$).*

PROOF. We compute:

$$\int_{\gamma\in\mathcal{G}|_Z} f d\nu = \sum_{g\in\Gamma} \int_{\gamma\in\partial E_g^Z} f d\nu.$$

We now analyse the last integral under the map $\gamma \mapsto \gamma^{-1}$, where it transforms to:

$$\int_{\gamma^{-1}\in\partial E_g^Z} f d\nu = \int_{x\in Z} \sum_{\substack{s(\gamma^{-1})=x \\ \gamma^{-1}\in\partial E_g^Z}} f(\gamma^{-1}) d\mu(x).$$

The conditions on the integrand here are equivalent to the statement that $\gamma \in \partial E_{g^{-1}}^Z$ and that $s(\gamma) = \sigma(g)(x)$. As μ and Z are both invariant under $\sigma(g)$, performing a change of variables $x \mapsto \sigma(g)^{-1}(x)$ we see that this last integral is equal to:

$$\int_{x\in Z} \sum_{\substack{s(\gamma)=x \\ \gamma\in\partial E_{g^{-1}}^Z}} f(\gamma) d\mu(x) = \int_{\gamma\in\partial E_{g^{-1}}^Z} f d\nu.$$

However, as we are summing over the group Γ, this completes the proof. $\square$

Thus $(\mathcal{G}|_Z, \nu)$ is a measured groupoid and the topological isomorphism of Theorem 3.9 gives us an isomorphism of measured groupoids $(\mathcal{G}|_Z, \nu) \cong (Z, \mu) \rtimes \Gamma$. Thus, if we extend the action of Γ on $\partial\beta X$ by letting every element of Γ act by the identity on the complement of Z, we obtain an almost everywhere isomorphism[1] as in [**Ram82**] for $\mathcal{G}$ and $\partial\beta X \rtimes \Gamma$:

THEOREM 3.11. *The measured groupoids* $(\mathcal{G}, \nu)$ *and* $(\partial\beta X, \mu) \rtimes \Gamma$ *(where each element of* Γ *is defined to act by the identity transformation on the complement of* Z*) are almost everywhere isomorphic as Borel measured groupoids.*

PROOF. The map defined in the proof of Theorem 3.9 is a well defined groupoid homomorphism of topological groupoids, but the set of elements in $\mathcal{G}$ for which this map is not well defined have measure 0; this is precisely the definition of an almost everywhere isomorphism: just map the elements $\gamma = (\omega, \omega') \in \mathcal{G}|_{Z^c}$ to any pair $(\omega, \tau(P_\gamma))$ and notice that the homomorphism rule will hold almost everywhere for the appropriate measure on $\mathcal{G}$. $\square$

REMARK 3.12. In the purely measurable setting, given a sofic approximation $\mathcal{X}$ and an ultrafilter $\omega \in \partial\beta\mathbb{N}$, one can naturally define the ultraproduct measure space

$$\prod_{i\to\omega}(X_i, \mu_i)$$

which will carry a natural Γ-action: viewing the sofic approximation σ as an embedding of Γ into the ultraproduct of permutation groups,

$$\sigma\colon \Gamma \hookrightarrow \prod_{i\to\omega} \mathrm{Sym}(X_i),$$

one uses natural embeddings $\mathrm{Sym}(X_i) \hookrightarrow \mathbb{M}_{|X_i|}(\mathbb{C})$ as permutation matrices to obtain a unitary representation

$$\sigma\colon \Gamma \hookrightarrow \mathcal{U}\left(\prod_{i\to\omega}(M_{|X_i|}(\mathbb{C}), \mathrm{tr}_i)\right),$$

where tr_i denotes the normalized trace. As permutation matrices normalize the subalgebra of diagonal matrices $A_i \subset M_{|X_i|}(\mathbb{C})$, we obtain a natural action of Γ on the ultraproduct

$$\Gamma \curvearrowright \prod_{i\to\omega}(A_i, \mathrm{tr}_i),$$

and this latter ultraproduct is by construction isomorphic to

$$\prod_{i\to\omega}(A_i, \mathrm{tr}_i) \cong \prod_{i\to\omega}(\ell^\infty(X_i), \mu_i) \cong L^\infty\left(\prod_{i\to\omega}(X_i, \mu_i)\right)$$

[1]This is just an isomorphism in parts of the measured groupoid literature, cf. [**DKP14**].

On the other hand, by definition of the ultraproduct

$$\prod_{i\to\omega}(\ell^\infty(X_i),\mu_i)\cong \ell^\infty(X)/\{f\in\ell^\infty(X)\,|\,\lim_{i\to\omega}\mu_i(f^*f)=0\}\cong L^\infty(\partial\beta X,\mu).$$

Therefore measure theoretically our construction yields nothing but the ultraproduct measure space naturally associated with the sofic approximation.

Remark 3.13. The results in this section should be thought of as an "almost everywhere" version of Example 2.11, where the set Z should be considered as the appropriate boundary set to attach to the space of graphs X of a sofic approximation $\mathcal{X}$.

4. From sofic approximations to analytic properties of the group

In this section we prove the results announced in Theorem 1.1, and we recall the necessary definitions (or references) of the coarse geometric and analytic properties that we need to keep this paper approximately self contained.

4.1. Amenability. Let X be a uniformly discrete metric space of bounded geometry. We begin with a few definitions concerning X:

Definition 4.1. X is *amenable* if for every $R>0,\varepsilon>0$ there exists a finite set $F\subset X$ such that

$$\frac{|\partial_R F|}{|F|}<\varepsilon,$$

where $\partial_R F$ is the R-boundary of F, that is the set of points in the R-neighbourhood of F that do not themselves belong to F.

Equivalent to this metric definition is a functional one:

Definition 4.2. X is (R,ε)-amenable if there exists a norm one probability measure ϕ on X such that:

$$\sum_{(x,y)\in E_R}|\phi(x)-\phi(y)|\leqslant\varepsilon.$$

A space X is *amenable* if it is (R,ε)-amenable for every $R>0$, $\varepsilon>0$ [**BW92**].

This leads nicely to a functional definition of *property A*, a coarse notion of amenability introduced by Yu in [**Yu00**], which is heavily studied in the literature. For a comprehensive survey on what is known about property A, see [**Wil09**].

Definition 4.3. X has *Property A* if for every $R>0,\varepsilon>0$, there exists an $S>0$ and a function $\eta\colon X\to\mathrm{Prob}(X)$, written $x\mapsto\eta_x$ with the following properties:

(i) each η_x is supported in a ball of radius at most S around x;
(ii) for any pair $(x,y)\in E_R$, we have: $\|\eta_x-\eta_y\|\leqslant\varepsilon$.

Condition ii) for η is known as being (R,ε)-variation.

For *families* of metric spaces, we can study *uniform* properties of the family. In this context, a family $\mathcal{X} = \{X_\alpha\}_\alpha$ has property A uniformly if, for every $R > 0, \varepsilon > 0$ and there is an $S > 0$ independent of α such that X_α satisfies conditions in the definition of property A for parameters R, ε, S.

EXAMPLE 4.4. For families of metric spaces, we know the following:

(i) Any sequence of finite graphs $\{X_i\}_i$ with degree bounded below by 3, above uniformly and girth tending to ∞, does not have property A uniformly, where girth is the length of the shortest simple cycle [**Wil11**];

(ii) Any box space of any residually finite amenable group has property A (in fact, this characterises amenability for a residually finite group) [**Roe03**, Chapter 11].

Here is the amenability part of the main result of this paper.

THEOREM 4.5. *Let Γ be a sofic group and let $\mathcal{X}$ be the a sofic approximation of Γ. If $\mathcal{X}$ has property A uniformly, then Γ is amenable.*

PROOF. Suppose the space of graphs X associated with $\mathcal{X}$ has property A. Then the full coarse groupoid – and thus $\mathcal{G}$, which is a closed reduction – is topologically amenable as a groupoid [**STY02**]. Applying this closed reduction fact again, $Z \rtimes \Gamma$ is therefore topologically amenable – but since Z has a Γ-invariant probability measure, this can happen if and only if Γ is amenable [**AD02**, Example 2.7.(3)]. $\square$

4.2. Amenable limits. As a basic application of the ideas from Section 2.4, we also give an answer to the following natural question: given a graph sequence with property A, can one use the measure μ to tell "how many" ultralimits are amenable as metric spaces?

Let A_{amen} denote the set of ultralimits of a graph sequence $\mathcal{X}$ that are amenable as metric spaces.

PROPOSITION 4.6.

(i) If $\mathcal{X} = \{X_i\}_i$ is a family of finite graphs with bounded degree that has property A uniformly, then there exists an ultralimit $X(\omega, \underline{x})$ that is (R, ε)-amenable;

(ii) If $\mathcal{X}$ has property A and Benjamini–Schramm converges to a graph X, then $\mu(A_{\text{amen}}) \in \{0, 1\}$;

(iii) For every $q \in \mathbb{Q} \cap [0, 1]$ there is a sequence of finite graphs $\mathcal{X}$ of bounded degree that have $\mu(A_{\text{amen}}) = q$.

PROOF. For i): as $\mathcal{X}$ has property A uniformly, for each $R, \varepsilon > 0$ we can find an $S > 0$ (independent of i) and a function, for each i:

$$\eta \colon X_i \to \mathrm{Prob}(X_i),$$

satisfying:

- each η_x is supported in a ball of radius at most S around x;
- for any pair $(x, y) \in E_R$, we have: $\|\eta_x - \eta_y\| \leqslant \varepsilon/N_R$,

where N_R is the uniform upper bound on the cardinality of a ball of radius R in X_i.

We now unpack the latter point (and using $\|\eta_x\| = 1$) into:

$$\sum_{z\in X_i} |\eta_x(z) - \eta_y(z)| \leqslant \frac{\varepsilon}{N_R} \sum_{z\in X_i} |\eta_x(z)|.$$

Fixing $x \in X_i$ and summing over the ball of radius R around x gives:

$$\sum_{z\in X_i} \sum_{y\in B_R(x)} |\eta_x(z) - \eta_y(z)| \leqslant \varepsilon \sum_{z\in X_i} |\eta_x(z)|.$$

Now summing over all possible $x \in X_i$, we obtain

$$\sum_{z\in X_i} \sum_{(x,y)\in E_R} |\eta_x(z) - \eta_y(z)| \leqslant \varepsilon \sum_{z\in X_i} \sum_{x\in X_i} |\eta_x(z)|.$$

It follows from this that there must be some $z_i \in X_i$ such that:

$$\sum_{(x,y)\in E_R} |\eta_x(z) - \eta_y(z)| \leqslant \varepsilon \sum_{x\in X_i} |\eta_x(z)|.$$

This lets us define $\phi : X_i \to [0, 1]$ by $\phi(x) = \eta_x(z_i)$, and then by the above we deduce:

$$\sum_{(x,y)\in E_R} |\phi(x) - \phi(y)| \leqslant \varepsilon\|\phi\|_1.$$

and as η_x is supported in a ball of radius S for each x, ϕ also is supported in a ball of radius S.

Repeating this for each X_i and renormalising, we see that for every $R > 0, \varepsilon > 0$ there exists $S > 0$ such that for every $i \in \mathbb{N}$ there is an $z_i \in X_i$ and a function $\phi_i : X_i \to [0, 1]$ supported in the ball of radius S around z_i such that:

$$\sum_{(x,y)\in E_R} |\phi_i(x) - \phi_i(y)| \leqslant \varepsilon.$$

Now take $\underline{z} = (z_i)_i$ and fix any nonprincipal ultrafilter $\omega \in \partial\beta\mathbb{N}$. We claim that the ultralimit $X(\omega, \underline{z})$ is (R, ε)-amenable. Indeed, if we let $B = B_{R+S}(\underline{z})$ in $X(\omega, \underline{z})$, then the set:

$$E = \{i \in \mathbb{N} \mid B_{R+S}(x_i) \text{ is isometric to } B\}$$

has ω-measure 1.

Now, for each $i \in E$ we can use a fixed isometry to transplant ϕ_i onto the set B. We note that these new transplanted functions also satisfy:

$$\sum_{(x,y)\in E_R^{X(\omega,\underline{z})}} |\phi_i(x) - \phi_i(y)| \leqslant \varepsilon.$$

As B is bounded, we can now take the ultralimit $\phi = \lim_\omega \phi_i$, which now clearly satisfies:

$$\sum_{(x,y)\in E_R^{X(\omega,\underline{z})}} |\phi(x) - \phi(y)| \leqslant \varepsilon.$$

For ii), observe that a graph family $\mathcal{X}$ converges to a graph X locally implies that μ-almost all $X(\omega, \underline{x})$ are isometric to X, that is we can find a base point $x \in X$ and a basepoint preserving isometry $X(\omega, \underline{x}) \to (X, x)$ for almost all admissible sequences $\underline{x}$.

Running the proof of i) sequentially for the sequence $(R_n, \varepsilon_n) = (n, \frac{1}{n})$, we construct a family of ultralimits denoted by Y_n. Now, either Y_n is isometric to X for arbitrarily large n, or it isn't – and the first case gives us that X is amenable (as it's (R, ε)-amenable for all $R, \varepsilon > 0$). To complete the proof, notice that because of the local convergence, the second case happens for a set of possible admissible sequences of μ-measure 0.

For iii): fix $q = \frac{a}{b} \in \mathbb{Q}$. Consider the graph family $\mathcal{X} = \{X_i\}_i$ with

$$X_i = \bigsqcup_{k=1}^{a} Y_i \sqcup \bigsqcup_{k=a+1}^{b} Z_i,$$

where Y_i is a cycle of length at i and Z_i is a family of bounded degree graphs with all vertices of degree at least three and girth at least i. Let X be the space of graphs attached with $\mathcal{X}$, and let Y and Z be the spaces of graphs attached with the sequences $\mathcal{Y} = \{Y_i\}_i$, $\mathcal{Z} = \{Z_i\}_i$ respectively. Then the boundary $\partial\beta X$, by definition, splits into $\bigsqcup_{k=1}^{a} \partial\beta Y \sqcup \bigsqcup_{k=a+1}^{b} \partial\beta Z$, and thus $\mu\,(\bigsqcup_{k=1}^{a} \partial\beta Y) = q$. So for the first part of the claim, it is enough to see that $A_{\text{amen}} = \bigsqcup_{k=1}^{a} \partial\beta Y$. This is clear, however, as any ultralimit of the sequence Z_i is an infinite tree with all vertices of degree at least three, which is certainly not amenable (this proves $A_{\text{amen}} \subset \bigsqcup_{k=1}^{a} \partial\beta Y$). For the other inclusion, notice that any ultralimit attached the sequence $\mathcal{Y}$ is a copy of the integer bi-infinite ray – this is certainly amenable as a metric space (using the Følner argument for the integers). □

4.3. a-T-menability. The following is a compression of definitions taken from [**Tu99**] and [**AD13**].

DEFINITION 4.7. Let G be a groupoid.

- A (real) conditionally negative definite function on G is a function $\psi\colon G \to \mathbb{R}$ such that:
  (i) $\psi(x) = 0$ for every $x \in G^{(0)}$;
  (ii) $\psi(g) = \psi(g^{-1})$ for every $g \in G$;
  (iii) For every $x \in G^{(0)}$, every $g_1, ..., g_n \in G^x$, and all real numbers $\lambda_1, ..., \lambda_n$ with $\sum_{i=1}^n \lambda_i = 0$ we have:

$$\sum_{i,j} \lambda_i \lambda_j \psi(g_i^{-1} g_j) \leqslant 0$$

- A locally compact, Hausdorff groupoid G is *a-T-menable* if there exists a proper, continuous, conditionally negative definite function $\psi\colon G \to \mathbb{R}$. This definition applies to groups: a group Γ is a-T-menable if is satisfies ii).
- A Borel groupoid (G, ν) is *a-T-menable* if there exists a proper, Borel, conditionally negative definite function $G \to \mathbb{R}$. In this context, properness means that $\nu(\{g \in G \mid \psi(g) \leqslant c\}) < \infty$ for every $C > 0$.

If G is locally compact, Hausdorff, topologically a-T-menable groupoid, then the associated Borel groupoid (G, ν_μ) is a-T-menable in the sense of iii) for any quasi-invariant measure μ on $G^{(0)}$. It's also transparent that topological a-T-menability passes to closed subgroupoids.

Related to this are various notions of a coarse embedding for a metric space X.

Definition 4.8. A metric space X *coarsely embeds* into Hilbert space H if there exist maps $f : X \to H$, and non-decreasing $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}$ such that:

(i) for every $x, y \in X$, $\rho_1(d(x,y)) \leqslant \|f(x) - f(y)\| \leqslant \rho_2(d(x,y))$;
(ii) for each i, we have $\lim_{r\to\infty} \rho_i(r) = +\infty$.

The connection with groupoids here is that a result of [**STY02**], which states that X coarsely embeds into Hilbert space if and only if $G(X)$ is topologically a-T-menable. In [**Wil15**], Willett introduced a property sufficient for the a-T-menability of the boundary groupoid associated with a sequence of bounded degree graphs:

Definition 4.9. Let $\mathcal{X} = \{X_i\}_i$ be a sequence of finite graphs of bounded degree. Then the sequence $\mathcal{X}$ *asymptotically (coarsely) embeds into Hilbert space* if there exist non-decreasing control functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}$ and symmetric, normalised kernels:

$$K_i : X_i \times X_i \to \mathbb{R},$$

and a sequence of non-negative real numbers $(R_i)_i$ tending to infinity satisfying:

(i) for all i, and all $x, y \in X_i$:

$$\rho_1(d(x,y)) \leqslant K_i(x,y) \leqslant \rho_2(d(x,y));$$

(ii) for any i and any subset $\{x_1, ..., x_n\} \subset X_i$ of diameter at most R_i, and any collection of real numbers $\lambda_1, ..., \lambda_n$ with $\sum_i \lambda_i = 0$ we have:

$$\sum_{i,j} \lambda_i \lambda_j K_i(x_i, x_j) \leqslant 0.$$

The key point here is the parameter family $(R_i)_i$. If this sequence grows faster than the sequence of diameters, then the family $\mathcal{X}$ is coarsely embeddable into Hilbert space (uniformly in i). However, this might grow slower than the diameter as is the case when the space X *fibred coarsely embeds into Hilbert space* but does not coarsely embed into Hilbert space. The following is [**Wil15**, Lemma 5.3], which is proved using the techniques of [**FS14**]:

PROPOSITION 4.10. *If $\mathcal{X}$ is an asymptotically coarsely embeddable family of finite graphs of bounded degree, then the boundary groupoid $\mathcal{G}$ of the associated space of graphs X is topologically a-T-menable.* □

Let $\mathcal{G}$ be the coarse boundary groupoid of the graphs obtained from the sofic approximation and $Z \subset \partial\beta X$ be a core of the sofic boundary.

PROPOSITION 4.11. *If $\mathcal{G}|_Z$ is a-T-menable, then Γ is a-T-menable.*

PROOF. As $\mathcal{G}|_Z \cong Z \rtimes \Gamma$ and carries an invariant measure, in view of [**BG13**, Corollary 5.11] it is enough to prove that the action of Γ on Z is a-T-menable in the sense of [**BG13**, Definition 5.5]; this, however, immediately follows from a-T-menability of $\mathcal{G}|_Z \cong Z \rtimes \Gamma$.

□

THEOREM 4.12. *If Γ is a sofic group admitting a sofic approximation $\mathcal{X}$ that asymptotically embeds into Hilbert space. Then Γ is a-T-menable.*

PROOF. As X asymptotically coarsely embeds into Hilbert space, the groupoid $\mathcal{G}$ is topologically a-T-menable. As $\mathcal{G}|_Z$ is closed, it also topologically a-T-menable. The result now follows from Proposition 4.11. □

4.4. Property (T).

DEFINITION 4.13. A finitely generated discrete group $\Gamma = \langle S \rangle$ has *property (T)* if for any unitary representation $\pi : \Gamma \to \mathcal{U}(H)$ that has *almost invariant vectors* has an invariant vector. Here, a vector $v \in H$ is ε-invariant If

$$\max_{s\in S} \|\pi(s)v - v\| \leqslant \varepsilon,$$

and π has *almost invariant vectors* if for every $\varepsilon > 0$ there is a ε-invariant vector.

Given a uniformly discrete metric space X of bounded geometry, there is a way to associate a C^*-algebra to X that bridges operator algebraic properties with coarse geometric properties. Let $\ell^2(X)$ be the complex Hilbert space spanned by Dirac functions δ_x for each point $x \in X$. Any bounded linear operator $T \in \mathbb{B}(\ell^2(X))$ can be uniquely represented as a matrix $(T_{x,y})$ indexed by $X \times X$ where the entries are defined by $T_{x,y} = \langle T\delta_x, \delta_y \rangle$.

For $T \in \mathbb{B}(\ell^2(X))$ we can define the propagation of T by the formula:

$$\text{Propagation}(T) := \sup\{d(x, y) \mid T_{x,y} \neq 0\}.$$

DEFINITION 4.14. The $*$-subalgebra of $\mathcal{B}(\ell^2 X)$ consisting of operators with finite propagation is denoted $\mathbb{C}[X]$. The closure of $\mathbb{C}[X]$ in the operator norm of $\ell^2(X)$ is called the *uniform Roe algebra of* X and is denoted by $C^*_u(X)$.

A representation of $\mathbb{C}[X]$ is a $*$-homomorphism $\pi : \mathbb{C}[X] \to \mathbb{B}(H)$, where H is some Hilbert space. Each injective representation π gives rise to a completion $C^*_\pi(X) := \overline{\pi(\mathbb{C}[X])} \subset \mathbb{B}(H)$. In this context we think of $C^*_u(X)$ as the *regular* completion.

Using this observation, it is possible to show that a *maximal* C^*-norm makes sense and this leads to:

DEFINITION 4.15. The maximal Roe algebra $C^*_{\max}(X)$ is the completion of $\mathbb{C}[X]$ in the norm

$$\|T\| := \sup\{\|\pi(T)\| | \ \pi \text{ a cyclic representation of } \mathbb{C}[X]\}.$$

DEFINITION 4.16. Let X be a coarse space with uniformly locally finite coarse structure $\mathcal{E}$, and let $E \in \mathcal{E}$ be an entourage. Then the *E-Laplacian*, denoted by Δ^E, is the element of $\mathbb{C}[X]$ with matrix entries defined by:

$$\Delta^E_{x,y} = \begin{cases} -1, & (x,y) \in (E \cup E^{-1}) \setminus \operatorname{diag}(E), \\ \left|\{z \in X \mid (x,z) \in (E \cup E^{-1}) \setminus \operatorname{diag}(E)\right|, & x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $E \subset \operatorname{diag}(X)$ then $\Delta^E = 0$.

EXAMPLE 4.17.

(i) If X is a connected graph of bounded degree, then the set E_1, that is all pairs of points of distance 1 (i.e the edges of the graph) generates the metric. In particular, Δ^{E_1} is the unnormalised graph Laplacian of X;

(ii) If Γ is a finitely generated group, and then we can refine this above example to get the Laplacian:

$$\Delta^{E_1} = 1 - \sum_{s \in S} [s],$$

where $[s]$ is the formal element in the group ring $\mathbb{C}\Gamma$ given by $s \in S$, and S (symmetrically) generates Γ – this *group Laplacian* will be denoted by Δ_Γ.

This latter example connects with property (T) via a result of Valette [**Val84**, Theorem 3.2], which states that $\Gamma = \langle S \rangle$ has property (T) if and only if 0 is isolated in the spectrum of the operator Δ_Γ in the maximal group C^*-algebra $C^*(\Gamma)$.

Before moving onto the main result of this section, we point out that we can identify the algebraic Roe algebra, up to $*$-isomorphism, with the groupoid convolution algebra $C_c(G(X))$ [**Roe03**, Section 10.4], [**SW16**, Appendix C]. In this way, groupoid reductions such as restricting to the boundary $\partial\beta X$ give rise to representations of $\mathbb{C}[X]$.

DEFINITION 4.18. A representation of $\mathbb{C}[X]$ (or equivalently $C_c(G(X))$) is a *boundary representation* whenever the ideal

$$I_X = \{T \in \mathbb{C}[X] \mid T_{x,y} \neq 0 \text{ for only finitely many } x, y \in X\}$$

is contained in the kernel.

Note that in groupoid terms, I_X is precisely the ideal $C_c(X \times X)$ in $C_c(G(X))$. Thus, a representation of $C_c(G(X)$ is a boundary representation if and only if it factors through $C_c(\partial G(X))$.

DEFINITION 4.19. The *boundary completion* $C^*_\partial(X)$ of $\mathbb{C}[X]$ is its separated completion in the seminorm

$$\|T\|_\partial \coloneqq \sup\{\|\pi(T)\|| \ \pi \text{ a boundary representation of } \mathbb{C}[X]\}$$

We can now state the relevant form of the definition of geometric property (T), using [**WY14**, Proposition 5.2]:

DEFINITION 4.20. A space X has *geometric property (T)* (resp. *geometric property (T) for boundary representations*) if there exists[2] an entourage $E \in \mathcal{E}$ and a $c > 0$ such that $\mathrm{Spec}_{\max}(\Delta^E)$ (resp. $\mathrm{Spec}_\partial(\Delta^E)$) is contained in $\{0\} \cup [c, \infty)$. Here $\mathrm{Spec}_{\max}$ denotes the spectrum in $C^*_{\max}(X)$ and Spec_∂ denotes the spectrum in $C^*_\partial(X)$.

We note that the presence of the invariant measure μ on $\partial\beta X$ allows us to use the following well known C^*-algebraic fact:

LEMMA 4.21 ([**WY14**, Section 7]). *Let $\Gamma \curvearrowright X$ be an action of Γ on a compact Hausdorff space. Then $C^*_{\max}(\Gamma) \to C(X) \rtimes_{\max} \Gamma$ is injective if and only if X has an invariant measure.* □

COROLLARY 4.22. *Let Γ be a sofic group and Z be a core of its sofic approximation. Then the natural map $C^*_{\max}(\Gamma) \to C(Z) \rtimes_{\max} \Gamma$ is injective.*

DEFINITION 4.23. We call any representation π of $C_c(G(X))$ that factors through $C_c(\mathcal{G}|_Z)$ *sofic with respect to Z* or a *Z-representation.* The *sofic completion* $C^*_s(X)$ of $\mathbb{C}[X]$ is its completion in the norm

$$\|T\|_s \coloneqq \sup\{\|\pi(T)\|| \ \pi \text{ a } Z\text{-representation of } \mathbb{C}[X]\}$$

Note that $C^*_s(X) \cong C^*_{\max}(\mathcal{G}|_Z)$.

DEFINITION 4.24. $\mathcal{X}$ has *geometric property (T) for Z-representations* if there exists $E \in \mathcal{E}$ and a $c > 0$ such that $\mathrm{Spec}_s(\Delta^E) \subset \{0\} \sqcup [c, \infty)$, where Spec_s is the spectrum in $C^*_s(X)$.

THEOREM 4.25. *Let Γ be a sofic group, $\mathcal{X}$ a sofic approximation and X the corresponding space of graphs. Then Γ has property (T) if and only if X has geometric property (T) for Z-representations for any sofic core $Z \subset \partial\beta X$.*

[2]This is equivalent to "for every" entourage, as the referenced proposition explains.

Proof. The proof is follows that of [**WY14**, Theorem 7.1], making use of the fact that the operator $\Delta_\Gamma = \sum_{s\in S} 1-[s] \in \mathbb{C}\Gamma$ maps to the operator $\Delta^Z = \sum_{s\in S} 1-\sigma(s)$ in $C(Z)\rtimes_{\mathrm{alg}}\Gamma$, and thus it satisfies:

$$\mathrm{Spec}_{\max}(\Delta_\Gamma) = \mathrm{Spec}_{\max}(\Delta^Z).$$

The result now follows from [**Val84**, Theorem 3.2], which shows that property (T) is equivalent to a spectral gap for Δ_Γ. ☐

Corollary 4.26. *If $\mathcal{X}$ has either geometric property (T) or geometric property (T) for boundary representations, then Γ has property (T).* ☐

4.5. Locally embeddable into finite groups and some examples. A group that is locally embeddable into finite groups has a $\varepsilon = 0$ sofic approximation $\mathcal{X}$, which we call an *LEF approximation*. The set Z in this case is the entire boundary $\partial\beta X$. From this we can observe that it is possible to prove the converse of some of the results from the previous section. This reproves essentially all of the results from [**MS13**] and [**MOSS15**]. The arguments are straightforward after unpacking all of the definitions using groupoids.

Theorem 4.27. *Let Γ be LEF, let $\mathcal{X}$ be a LEF approximation and let X be the space of graphs constructed as in section 2.4. Then:*

(i) Γ is amenable if and only if X has property A;
(ii) Γ has property (T) if and only if X has geometric property (T).

Proof. It clearly suffices to prove the converses.

For i): as $\partial G(X)$ is topologically amenable, it has weak containment and a nuclear reduced groupoid C^*-algebra by [**BO08**, Corollary 5.6.17]. Additionally, the sequence

$$0 \to \mathcal{K}(\ell^2(X)) \to C^*_u(X) \to C^*_r(\partial G(X)) \to 0,$$

is exact because of weak containment. It follows that $C^*_u(X)$ is nuclear, which is a well known characterisation of property A [**STY02**].

To show ii), we have to exhibit spectral gap of its Laplacian $\Delta \in C^*_{\max}(X)$.

By [**Sha00**, Theorem 6.7], every marked group $\Gamma = \langle S \mid R\rangle$ with property (T) is a (marked) quotient of a finitely presented group $\Gamma_0 = \langle S \mid R_0\rangle$ with property (T); without loss of generality we may assume $R_0 \subset R$. Given an LEF approximation of Γ, we thus obtain homomorphisms $\varphi_i : \mathbb{F}_S \to X_i$ into some finite groups X_i which (together with their S-labellings) consitute the LEF approximation. As R_0 is finite, after finitely many steps (say, for $i \geqslant N_0$) the maps φ_i descend to homomorphisms $\overline{\varphi_i}\colon \Gamma_0 \to X_i$. Thus, putting $X' = \bigsqcup_{i\geqslant N_0} X_i$, we get a homomorphism $\varphi\colon C^*(\Gamma_0) \to C^*_{\max}(X')$ sending the group Laplacian Δ_{Γ_0} to the Laplacian Δ on X'. As Γ_0 has property (T), Δ_{Γ_0} has spectral gap and therefore so does the Laplacian Δ on X'; in other words, this space has geometric property (T). As the Laplacian on a finite graph always has spectral gap, adding back the graphs X_i for $i < N_0$ retains spectral gap for the Laplacian. Thus, X has geometric property (T), as claimed.

□

We remark that there are many interesting groups that are not residually finite, but are LEF – chief amongst these are topological full groups of Cantor minimal systems, introduced by Giordano, Putman, and Skau [**GPS99**], proved to be LEF by Grigorchuk and Medynets [**GM14**], amenable by Juschenko–Monod [**JM13**] and have a simple commutator subgroup by Matui [**Mat06**].

5. Coarse equivalence, quasi-isometry and uniform measure equivalence

In this section we prove that coarsely equivalent sofic approximations give rise to a uniform measure equivalence between groups, using Morita equivalence of groupoids as a tool. We first recall some definitions concerning the various notions of equivalence for groupoids that appear in the literature.

Definition 5.1. (A linking groupoid) Let G be a groupoid and let T be a set with a map $f : T \to G^{(0)}$. Then the set

$$G[T] := \left\{(t, t', g) \in (T \times T) \times G \mid g \in G_{f(t')}^{f(t)}\right\}$$

is a groupoid with the obvious operations. If G is a locally compact Hausdorff groupoid, T is a locally compact Hausdorff topological space and the map f is continuous, then $G[T]$ is a locally compact Hausdorff topological groupoid.

For any sets X, Y, T with maps $f : X \to T$, $g : Y \to T$ we denote the pullback by $X \times_{f,g} Y$, or $X \times_T Y$ if there is no ambiguity.

Definition 5.2. (A groupoid action) Let G be a groupoid and let M be a set. M is a (right) G-space if there exists

- a map $p : M \to G^{(0)}$ (called the anchor map) and
- a map $M \times_{p,r} G \to M$ denoted by $(z, g) \mapsto zg$ (called the action map)

with the following properties:

- $p(zg) = s(g)$ for all $(z, g) \in M \times_{p,r} G$;
- $z(gh) = (zg)h$ whenever $p(z) = r(g)$ and $s(g) = r(h)$;
- $zp(z) = z$ for every $z \in M$.

This allows us to define a natural "crossed product" groupoid $M \rtimes G$ with base space M, which consists of the elements $(z, z', g) \in (M \times M) \times G$ that satisfy $z = z'g$. Note that since $M \rtimes G \to M \times G$ given by $(z, z', g) \mapsto (z, g)$ is injective, we can also consider $M \rtimes G$ as a subset of $M \times G$, which we will do. We can also define a left G-space similarly using the source map instead of the range map: we denote the groupoid constructed from a left action by $G \ltimes M$.

Every groupoid G naturally acts on its base space $G^{(0)}$ using $\mathrm{id}\colon G^{(0)} \to G^{(0)}$ as the anchor map and the multiplication as the action map. From the algebraic structure of the groupoid it easily follows that the orbit relation on $G^{(0)}$ defined by $x \sim y$ iff $x = y \cdot g$ for some $g \in G$ is an equivalence relation. The corresponding quotient is denoted by $G^{(0)}/G$. For a set M with a G-action the quotient space by the action is defined through $M/G \coloneqq M/(M \rtimes G)$.

So far we have mentioned nothing concerning the topological structure of the action and the crossed product space in case G and M have topologies. This can be adjusted by putting sufficient continuity and openness conditions on the maps above, which is discussed at length in [**Tu04**, Section 2]. The main result of these considerations which we will need is the following:

PROPOSITION 5.3. [**Tu04**, Proposition 2.29] *Let G_1 and G_2 be two topological groupoids, let s_i, r_i be the open source and range maps of G_i. Then the following are equivalent:*

(i) there exists a set T with $f_i : T \to G_i^{(0)}$ open surjective maps such that $G_1[T] \cong G_2[T]$;
(ii) there exists a space M with two continuous maps $\rho : M \to G_1^{(0)}$, $\sigma : M \to G_2^{(0)}$ such that ρ is the anchor map for a left action of G_1 on M, σ is the anchor map of a right action of G_2 on M such that these actions commute, are free and the action of G_2 is ρ-proper, the action of G_1 is σ-proper and:

$$M/G_2 \to G_1^{(0)} \text{ and } G_1\backslash M \to G_2^{(0)}$$

are homeomorphisms.

Two topological groupoids that satisfy either of the two equivalent conditions above will be called Morita equivalent.

REMARK 5.4. The main point to raise here is that the space M in the proof of i) $\Rightarrow$ ii) is constructed as follows [**Tu04**, Proposition 2.29]: take M_1 to be the space $G_1 \times_{s,f_1} T$, and M_2 to be the space $T \times_{f_2,r} G_2$. These are then combined over the $G_i[T]$-action on the right on M_1 and the left on M_2 to the space $M \coloneqq M_1 \times_{G_1[T]} M_2$, which amounts of dividing the space $M_1 \times_T M_2$ by the relation generated by $(z, z') \sim (zg, g^{-1}z)$, where $g \in G_1[T]$. The space M then admits a bispace structure which implements ii).

REMARK 5.5. The notion of Morita equivalence can also be defined for measured groupoids in a similar manner, replacing topological conditions by measurable ones, and we will make use of it later. We refer the reader to [**Lan01**] and references therein for discussion of definitions Morita equivalence for various categories of groupoids and operator algebras and connections between them.

EXAMPLE 5.6. A coarse equivalence f produces a "coarse correspondence", as in [**STY02**], between $G(X)$ and $G(Y)$. This is a groupoid $G(X \sqcup Y)$ constructed from a "linking" coarse structure, defined using the coarse structure $\mathcal{E}(f) \coloneqq \mathcal{E}^X_{\text{met}} \sqcup \mathcal{E}^Y_{\text{met}} \sqcup \mathcal{E}^{XY} \sqcup \mathcal{E}^{YX}$, where the sets in $\mathcal{E}^{XY}$ are precisely those of the form $F \times f(F)$, similarly defining those in $\mathcal{E}^{YX}$ using the coarse inverse of f. This coarse structure is uniformly locally finite if E^X_{met} and E^Y_{met} are [**STY02**, Proposition 2.3].

This coarse correspondence allows us to construct a topological space $T = \beta X \sqcup \beta Y$ that implements a topological Morita equivalence between $G(X)$ and $G(Y)$ in the sense of Proposition 5.3. The proof of this is a part of the content of a remark from the beginning of Section 3.4 of [**STY02**].

LEMMA 5.7. *If X and Y are coarsely equivalent by a pair of maps $f : X \to Y$ and $k : Y \to X$, then $G(X)[T] \cong G(Y)[T]$ for $T = \beta X \sqcup \beta Y$ and maps $p_X : T \to \beta X$ (resp. $p_Y : T \to \beta Y$) given by*

$$p_X(\omega) = \begin{cases} \omega \text{ if } \omega \in \beta X \\ \overline{f}(\omega) \text{ if } \omega \in \beta Y \end{cases}.$$

and a similar definition for p_Y. □

The space M whose construction was outlined in Remark 5.4 is a quotient of

$$M := G(X) \times_{s,p_X} T \times_{p_Y,r} G(Y)/\sim \tag{5.1}$$

where $\sim$ implements the identification of points in T who are joined by continuous extensions of the coarse maps $f : X \to Y$ and $k : Y \to X$. We also remark, that as the sets X and Y are invariant in their respective coarse groupoids, these bispaces restrict to bispaces over the boundary groupoids $\partial G(X)$ resp. $\partial G(Y)$.

LEMMA 5.8. *Let Γ and Λ be sofic groups with $\mathcal{X}$, and $\mathcal{Y}$ sofic approximations of Γ and Λ respectively, and suppose that $f : X_{\mathcal{X}} \to X_{\mathcal{Y}}$ is a coarse equivalence of the associated spaces of graphs. Then the set $\widetilde{f}(Z_{\mathcal{X}}) \cap Z_{\mathcal{Y}}$ has positive measure in $\partial\beta X_{\mathcal{Y}}$.*

PROOF. By [**KV15**, Lemma 1] we can assume that $f(X_i) \subset Y_i$, and that $f|_{X_i}$ is a (C, C)-quasi-isometry (for some constant $C > 0$). As f is a coarse equivalence, there is a constant $n > 0$ such that $X_{\mathcal{Y}} = N_n(f(X_{\mathcal{X}}))$, where N_n is the n-neighbourhood of $f(X_{\mathcal{X}})$ in $X_{\mathcal{Y}}$. We also observe that $N_m(A) = N_1(N_{m-1}(A))$ for all subsets $A \subseteq X_{\mathcal{Y}}$ and all $m \in \mathbb{N}$. It follows by induction that, for all i:

$$|N_i(f(X_i))| \leqslant |S_\Lambda|^i |f(X_i)|,$$

where S_Λ is the finite generating set of Λ. This shows that $\widetilde{f}(Z_{\mathcal{X}})$ has measure at least $\frac{1}{|S_\Lambda|^n}|Y_i|$ in $\partial\beta Y$ as the image preserves unions and the measure $Z_{\mathcal{X}}^c$ is 0. This completes the proof, since $Z_{\mathcal{Y}}$ has $\mu_{\mathcal{Y}}$-measure 1. □

In fact, we can say more using the observation that $\widetilde{f}(Z_{\mathcal{X}}) \cap Z_{\mathcal{Y}} \neq \varnothing$: it allows us to construct a quasi-isometry using the transplanting technique of [**KV15**, Proposition 3]:

PROPOSITION 5.9. *Let Γ and Λ be sofic groups, with sofic approximations $\mathcal{X}$ and $\mathcal{Y}$ respectively. If the spaces of graphs $X_{\mathcal{X}}$ and $X_{\mathcal{Y}}$ attached with $\mathcal{X}$ and $\mathcal{Y}$ are coarsely equivalent, then Γ and Λ are quasi-isometric.*

PROOF. The proof of this fact is precisely the proof of [**KV15**, Proposition 3], except that instead of using convergence of marked groups (i.e ultralimits of groups using the identity as base point), we use ultralimits along a base point sequence $(x_i)_i$, such that $\eta = \lim_\omega x_i$ satisfies: $f(\eta) \in \widetilde{f}(Z_{\mathfrak{X}}) \cap Z_{\mathfrak{Y}}$. □

Finally, we consider the analogous notion of measure equivalence, as was considered in [**Das15**] for box spaces of residually finite discrete groups.

DEFINITION 5.10 ([**Gro93**, **Sha04**, **Das15**]). Two groups Γ and Λ are *measure equivalent* if there exists a essentially free Borel measure (Γ, Λ)-space (X, μ) such that there are finite volume fundamental domains $X_\Gamma \subset X \supset X_\Lambda$ for the actions. A measure equivalence is *uniform* if additionally, for every $g \in \Gamma$ (resp. $h \in \Lambda$) there exists a finite subset $S_g \subset \Lambda$ (resp. $T_h \subset \Gamma$) such that

$$gX_\Lambda \subset X_\Lambda S_g \text{ and } X_\Gamma h \subset T_h X_\Gamma$$

Our aim is to prove that if Γ and Λ are sofic groups with coarsely equivalent approximations, then Γ and Λ are uniformly measure equivalent. To accomplish this, we need to take the topological Morita equivalence M of $G(X_{\mathfrak{X}})$ and $G(X_{\mathfrak{Y}})$ provided by a coarse equivalence $f : X_{\mathfrak{X}} \to X_{\mathfrak{Y}}$, and turn it into a Morita equivalence between measured groupoids. To do this, we have to analyse the correspondence between invariant measures and measures on a quotient by a free and proper action for étale groupoids:

PROPOSITION 5.11. *Let G and H be étale groupoids and let X be a free and proper G-H-space. Then there is a one-to-one correspondence between G-invariant Radon measures ρ on X and Radon measures μ on $G\backslash X$. Moreover, this correspondence is additive and H-equivariant.*

PROOF. Each G-invariant Radon measure ρ on X defines a Radon measure $\mu = {}_G\rho$ on $G\backslash X$ using the pushforward of ρ over a subset $U \subset X$ such that the quotient map is one-to-one on U. This construction is H-equivariant as the H-action commutes with the G-action on X.

To go back, we use the construction from [**SW12**, Section 3]: let X be a free and proper left G-space. Then $G\backslash X$ is a locally compact Hausdorff space, and for each $x \in X$, the map $\gamma \mapsto \gamma \cdot x$ is a homeomorphism of $G_{r(x)}$ onto the orbit $G \cdot x$. We define a Radon measure $\rho^{G\cdot x}$ on X with support $G \cdot x$ by

$$\rho^{G\cdot x}(f) := \int_G f(\gamma^{-1}(x)) d\lambda^{r(x)}(\gamma)$$

Our definition is independent of our choice of x in its orbit by left-invariance of the Haar system λ. Additionally, the map

$$[x] \mapsto \rho^{[x]}(f)$$

is continuous on $G\backslash X$. Given a finite Radon measure μ on $G\backslash X$, we define a Radon measure ρ_μ on X by

$$\rho_\mu(f) = \int_{G\backslash X} \int_X f(y) d\rho^{[x]}(y) d\mu([x])$$

The measure ρ is G-invariant by construction, as $\rho^{[x]}$ is invariant and supported on a G-orbit. On the other hand, as the actions of G and H on X commute and because the measures $\rho^{[x]}$ are defined by integrating over the orbit, they are H-equivariant: for all $\chi \in H$ we have $\rho^{[x]\cdot\chi} = \chi_*\rho^{[x]}$.

It's routine to check that these constructions are additive, inverse to each other and therefore define a one-to-one correspondence as claimed. $\square$

In the situation of the above proposition we say that μ is the quotient measure corresponding to ρ and write $\mu = {}_{G}\underline{\rho}$ and that ρ is the measure induced by μ through the action of G and write $\rho = {}^{G}\overline{\mu}$; we use corresponding notations for right actions.

Corollary 5.12. *Let G and H be étale groupoids with invariant measures μ and η on $G^{(0)}$ and $H^{(0)}$ respectively and let M be a Morita equivalence between them. If $\underline{{}^{G}\overline{\mu}}_{H}$ on $H^{(0)}$ is absolutely continuous with respect to η and ${}_{G}\underline{\overline{\eta}^{H}}$ on $G^{(0)}$ is absolutely continuous with respect to μ, then (G, ν_μ) and (H, ν_η) are Morita equivalent as measured groupoids in the sense of* [**Lan01**].

Proof. The absolute continuity assumptions imply that the measure

$$\rho := {}^{G}\overline{\mu} + \overline{\eta}^{H}$$

descends to measures ${}_{G}\underline{\rho}$ and $\underline{\rho}_{H}$ which are equivalent to μ resp. η. Thus, (M, ρ) is a Morita equivalence between the measured groupoids (G, ν_μ) and (H, ν_η). $\square$

Using this we can prove:

Theorem 5.13. *Let Γ and Λ be sofic groups with approximations $\mathcal{X}$ and $\mathcal{Y}$ respectively. If the associated spaces of graphs $X_{\mathcal{X}}$ and $X_{\mathcal{Y}}$ are coarsely equivalent, then the groups Γ and Λ are uniformly measure equivalent.*

Proof. In order to appeal to Corollary 5.12, we have to show that the limits μ and η of counting measures on the base spaces of $\mathcal{G}_\Gamma$ and $\mathcal{G}_\Lambda$ satisfy the absolute continuity assumption. To check this, recall the construction of the space M following (5.1):

$$M := \partial G(X) \times_{s,p_X} T \times_{p_Y,r} \partial G(Y)/\!\sim$$

where $\sim$ implements the identification of points in $T = \partial\beta X \sqcup \partial\beta Y$ who are joined by continuous extensions of the coarse maps $f : X \to Y$ and $k : Y \to X$. It follows that the measure $\underline{{}^{G}\overline{\mu}}_{H}$ is equal to the pushforward $\overline{f}_*\mu$ of the measure μ under the coarse equivalence map f, and similarly, ${}_{G}\underline{\overline{\eta}^{H}}$ is equal to the pushforward $k_*\eta$ under the coarse inverse. As coarse maps have uniformly finite fibres, the absolute continuity follows. Thus, Corollary 5.12 yields a measurable Morita equivalence (M, ρ) between $(\mathcal{G}_\Gamma, \nu_\mu)$ and $(\mathcal{G}_\Lambda, \nu_\eta)$.

To show that Γ and Λ are uniformly measure equivalent, we fix fundamental domains $X_\Gamma, X_\Lambda \subset M$ with compact closures for the $\mathcal{G}_\Gamma$ and $\mathcal{G}_\Lambda$-actions respectively and let $\{U_g\}_{g\in\Gamma}$

and $\{U_h\}_{h\in\Lambda}$ be covers of $\mathcal{G}_\Gamma$ and $\mathcal{G}_\Lambda$ by compact open slices, each of which restricts to a slice of the form $[Z_\mathcal{X}, g]$ on $\mathcal{G}|_{Z_\mathcal{X}} \cong Z_\mathcal{X} \rtimes \Gamma$ and $[Z_\mathcal{Y}, h]$ on $\mathcal{G}|_{Z_\mathcal{Y}} \cong Z_\mathcal{Y} \rtimes \Lambda$.

Fix $h \in \Lambda$. The set $\{U_g X_\Lambda\}_{g\in\Gamma}$ is an open cover of M, thus in particular it covers $X_\Gamma U_h$, which is a compact subset of M as the right $\mathcal{G}_\Lambda$ action is proper and U_h is a compact open slice of $\mathcal{G}_\Lambda$. Now, compactness of $X_\Gamma U_h$ allows us to extract a finite subcover $\{U_g X_\Gamma\}_{g\in T_h}$ for some finite set $T_h \subset \Gamma$.

To finish the proof, we remark that the the almost everywhere isomorphisms $\mathcal{G}_\Gamma \cong \partial\beta X \rtimes \Gamma$ and $\mathcal{G}_\Gamma \cong \partial\beta X \rtimes \Lambda$ constructed in Theorem 3.11 give rise to actions of Γ and Λ on M with (measurable) fundamental domains X_Γ and X_Λ such that gX_Γ and $X_\Gamma h$ coincide with $U_g X_\Gamma$ and $X_\Gamma U_h$ up to null sets. Thus, the set T_h satisfies the condition in the Definition 5.10, and symmetrisation of the argument for the $\mathcal{G}_\Lambda$-action provides for evey $g \in G$ a finite set S_g with the necessary properties. This finishes the proof. $\square$

Appealing to [**Gab02**, Theorem 6.3], we now obtain:

COROLLARY 5.14. *If Γ and Λ are finitely generated sofic groups with coarsely equivalent sofic approximations, then their ℓ^2-Betti numbers are proportional.*

This Corollary has immediate applications to distinguishing families of finite graphs up to coarse equivalence. In particular, it allows us to see that box spaces of products of free groups with different number of factors are not coarsely equivalent [**Gab02**, Corollaire 0.3]) as they have ℓ^2-Betti numbers that are not proportional – we remark that this is considered directly in the work of Das [**Das15**], and we draw attention to it again due to recent interest in this question [**KV15**, **Del16**].

5.1. Remarks about bilipschitz equivalence. Let Γ and Λ be sofic groups with approximations $\mathcal{X}$ and $\mathcal{Y}$ respectively. If $X_\mathcal{X}$ and $X_\mathcal{Y}$ are bilipschitz equivalent via a map f, then they are certainly coarsely equivalent and so the results of the previous section apply. However, as in the remark that precedes [**Sha04**, Definition 2.1.4], we can say quite a bit more concerning the relationship between Γ and Λ in this instance.

Notably, the following basic observations can be used to simplify and improve on the results from Section 5:

(i) Bilipschitz equivalences are bijections, so the pushforward $f_*\mu_\mathcal{X}$ agrees with $\mu_\mathcal{Y}$. This means that Lemma 5.8 is a triviality, as $\tilde{f}(Z_\mathcal{X})$ is $\mu_\mathcal{Y}$-measure 1. We also remark that any bijection from $X_\mathcal{X}$ to $X_\mathcal{Y}$ will also give a homeomorphism between $\partial\beta X_\mathcal{X}$ and $\partial\beta X_\mathcal{Y}$;
(ii) Let μ and η be measures on $G^{(0)}$ and $H^{(0)}$ respectively (as in Corollary 5.12). Then applying i), but this time in the construction of the bimodule measure ρ induced from μ, we see that actually $\underline{{}^G\overline{\mu}_H} = \eta$. As a consequence,
   (a) we do not need to use the sum of $\rho := {}^G\overline{\mu} + \overline{\eta}^H$ in the proof of Corollary 5.12;

(b) there is a common fundamental domain in a topological and measurable sense (as a consequence of the homeomorphism between $\partial\beta X_{\mathcal{X}}$ and $\partial\beta X_{\mathcal{Y}}$).

Additionally, one can improve Proposition 5.9.

PROPOSITION 5.15. *Let Γ and Λ be sofic groups, with sofic approximations $\mathcal{X}$ and $\mathcal{Y}$ respectively. If the spaces of graphs $X_{\mathcal{X}}$ and $X_{\mathcal{Y}}$ attached with $\mathcal{X}$ and $\mathcal{Y}$ are bilipschitz equivalent, then Γ and Λ are bilipschitz equivalent.* □

This has additional consequences due to results by Medynets–Thom–Sauer [**MTS15**, Theorem 3.2]:

COROLLARY 5.16. *Let Γ and Λ be sofic groups, with sofic approximations $\mathcal{X}$ and $\mathcal{Y}$ respectively. If the spaces of graphs $X_{\mathcal{X}}$ and $X_{\mathcal{Y}}$ attached with $\mathcal{X}$ and $\mathcal{Y}$ are bilipschitz equivalent, then there exist minimal, continuous orbit equivalent actions of Γ and Λ on some Cantor set C.* □

6. A standardisation of the base space

This section is dedicated to the proof of the following theorem:

THEOREM 6.1. *Let Γ be a sofic group, $\mathcal{X}$ be a sofic approximation of Γ and X the associated total space of the family of graphs attached to $\mathcal{X}$. Then there exists a second countable étale, locally compact, Hausdorff topological groupoid $\mathcal{G}$ with following properties:*

(i) the base space $\mathcal{G}^{(0)} =: \widehat{X}$ is a compactification of X (in particular, it's a quotient of βX through a quotient map $p\colon \beta X \to \widehat{X}$),

(ii) $p(Z) \subset \partial\widehat{X}$ is invariant and satisfies $\mathcal{G}|_{p(Z)} \cong p(Z) \rtimes \Gamma$. As a consequence, we have an almost everywhere isomorphism

$$(\mathcal{G}|_{\partial\widehat{X}}, \nu_{p_*\mu}) \to (\widehat{X}, p_*\mu) \rtimes \Gamma.$$

Morally, this means that although the space $(\partial\beta X, \mu)$ is not a standard probability space, we can use $\widehat{X}$ to make arguments *as if* we were actually in $\partial\beta X$, whilst actually working in a standard Borel probability space.

EXAMPLE 6.2. Let Γ be a residually finite, finitely generated discrete group, let $\mathcal{X}$ be a sofic approximation made up of finite quotients of Γ and let X be the space of graphs associated to $\mathcal{X}$. Then by considering the Boolean algebra B generated by $\mathrm{Cofin}(X) \cup \mathrm{Fin}(X) \cup \{\mathrm{Sh}(eN_i)\}_i$, where

$$\mathrm{Sh}(eN_i) := \bigcup_{j \geqslant i} \{x \in X_j \mid \pi_{i,j}(x) = e_i N_i\}$$

is the *shadow* of eN_i in X, we obtain a second countable, locally compact, Hausdorff étale groupoid $\mathcal{G}_B$, which is homeomorphic to $X_B \rtimes \Gamma$, and $X_B \cong \widehat{\Gamma}_{\mathcal{X}}$ is the profinite completion associated with the family of finite quotients $\mathcal{X}$. This dynamical system was introduced in

[**AN12**], where it was shown to be minimal (and in this case, as subgroups in question are normal, it's also free). A similar construction using the shadows of the identity would give us the boundary ∂T as defined in [**AN12**] when the chain is *Farber*. This example shows that one can choose the appropriate Boolean algebra depending on the goals in question.

The ideas used in the proof stem from the work of Skandalis–Tu–Yu [**STY02**], where one pushes the failure of second countability of $G(X)$ purely into the unit space: this allows one to make use of the groupoid equivariant KK-theory of Pierre-Yves Le Gall [**LG01**] to describe the coarse Baum–Connes conjecture attached to X.

Proof of Theorem 6.1. We first recall the outcome of [**STY02**, Lemma 3.3], which states that any countable generating set $\mathcal{A}$ of the metric coarse structure on a space X gives rise to a second countable, étale, locally compact Hausdorff groupoid $G_{\mathcal{A}}$, such that the coarse groupoid $G(X)$ is homeomorphic to the transformation groupoid $\beta X \rtimes G_{\mathcal{A}}$[3]. We construct $\mathcal{A}$ in what follows.

In light of Section 3, we can construct generators using those given by the labelling, i. e. by considering the entourages E_P, where P is a word in the free group on the alphabet S. These clearly generate the metric for the space X (as a total space of the family $\mathcal{X}$). This family doesn't give us a good unit space however, as each of the elements we are using here are bijections (thus the base space $X_{\mathcal{A}}$ of $G_{\mathcal{A}}$ would end up being a point).

To remedy this, we consider the set $\mathcal{B}$ of all countable Boolean subalgebras of $\mathbf{2}^X$ that contain the set Y and some infinite set that is not cofinite. Note that if the approximation $\mathcal{X}$ is a LEF approximation (i.e $\varepsilon = 0$ for i), then $Y = X$ and subsequently, this is all countable Boolean subalgebras with at least one infinite, not cofinite set.

Fix $B \in \mathcal{B}$. By taking the inverse semigroup generated by B and the transformations $\tau(w)$ for $w \in F_S$, we get a countable pseudogroup. Let $\mathcal{A}$ be this set of partial transformations $\tau(g)|_A$, where $A \in \mathcal{B}$, extended continuously to βX. Applying [**STY02**, Lemma 3.3], we obtain a second countable étale groupoid $G_{\mathcal{A}}$. Its base space $X_{\mathcal{A}}$ is a quotient of βX and we denote the quotient map by $p\colon \beta X \to X_{\mathcal{A}}$. Pushing forward the measure μ along the map $p\colon \beta X \to X_{\mathcal{A}}$, and using the Urysohn metrization theorem, we obtain that $(X_{\mathcal{A}}, p_*\mu)$ has the structure of a standard Borel probability measure space. Since μ is supported inside the boundary $\partial\beta X$, it follows that $p_*\mu(\partial X_{\mathcal{A}}) = 1$, and we again obtain a boundary type groupoid $\mathcal{G}_{\mathcal{A}} := G_{\mathcal{A}}|_{\partial X_{\mathcal{A}}}$, and as the set Y from Section 3 belongs to the Boolean algebra generating $\mathcal{G}_{\mathcal{A}}$, we can see that the F_S-action factors through Γ up to null sets. This allows us to run the arguments of Section 3 again to obtain an almost everywhere isomorphism of groupoids (through a μ-inessential reduction). This finishes the proof. $\square$

[3]Skandalis-Tu-Yu give a "by hand" proof of this result: a slightly more modern approach to it would be to make use of the fact that a pseudogroup in this context gives us a inverse monoid, and then construct from that, using well known techniques of [**Exe08**], a groupoid with all the desired properties.

7. Concluding remarks and further questions

We finish the paper with a few questions and comments on the surrounding literature, concerning primarily the interactions between the geometric and probabilistic points of view on sofic groups and graphs. Throughout, let Γ be a sofic group, $\mathcal{X}$ a sofic approximation and $\mathcal{G}|_Z$ be the sofic coarse groupoid restricted to the sofic core.

The statement of our main result immediately suggests a question about the converse:

QUESTION 7.1. To which extent do the converse statements to the one of Theorem 1.1 hold?

Because soficity gives only a measure-theoretic control of actions on the sofic boundary, we do not expect the converse to hold true in full generality. On the other hand, as amenability, a-T-menability and property (T) of discrete groups are visible at the level of measure-preserving actions, it is natural to expect that they will be visible at the sofic boundary; it is natural to expect some form of probabilistic manifestation of coarse-geometric properties there.

DEFINITION 7.2 ([**Ele07**, **Sch08**]). A family of finite graphs $\mathcal{Y} = \{Y_i\}_i$ of bounded degree is a *hyperfinite family* if for every $\varepsilon > 0$ and for each $i \in \mathbb{N}$ there exists a decomposition of Y_i into $K_{\epsilon,i}$ finite sets $U_{i,j}$ such that

(i) each $U_{i,j}$ is uniformly bounded;
(ii) the size of each set $E(U_{i,j}, U_{i,j'})$ is at most $\varepsilon|Y_i|$ whenever $j \neq j'$, where $E(U_{i,j}, U_{i,j'})$ is the set of edges between $U_{i,j}$ and $U_{i,j'}$.

A combination of Theorem 4.5 with [**Sch08**, Theorem 1.1] shows that property A for a sofic approxiation implies hyperfiniteness of that approximation.

QUESTION 7.3. Does hyperfiniteness of a sofic approximation imply property A for that approximation?

Here the fact that we ask this for a sofic approximation is important, as the implication does not hold for a general Benjamini–Schramm convergent sequence of graphs[4].

One approach to this question would be to use a property equivalent to property A called the *metric sparsification property* [**CTWY08**], which was shown to be equivalent in [**Sak14**] to a graph family being weighted hyperfinite (as defined in by Elek and Timár in [**EAT11**]). However, there is a subtlety here – the measure on the groupoid $\mathcal{G}$ only deals with counting measures on the graphs, whereas the weighted notion of hyperfiniteness from [**EAT11**] is dealing with limits of arbitrary measures on the graph family $\mathcal{X}$.

Another recent development in [**Kun16**] classified measurably those approximations coming from groups with property (T). The natural analogue of hyperfiniteness in this setting is the following:

[4]This was communicated by Gabor Elek, in a personal communication.

THEOREM 7.4 ([**Kun16**, Theorem 1]). *Let Γ be a property (T) group and let $\mathcal{X} = \{X_i\}_i$ be a family of bounded degree graphs that Benjamini-Schramm converge to the Cayley graph of Γ. Then there is a $c > 0$ and a family of regular graphs $\mathcal{Y} = \{Y_i\}_i$ such that:*

(i) $V(X_i) = V(Y_i)$ *for every i;*
(ii) $\lim_{i\to\infty} \frac{|E(X_i)\triangle E(Y_i)|}{|V(X_i)|} = 0$;
(iii) Each Y_i is a vertex disjoint union of c-expanders.

In other words, the graphs Y_i are obtained by "rewiring" X_i in an asymptotically negligible manner.

QUESTION 7.5. Can a sofic approximation of a property (T) group be "asymptotically rewired" to have some form of geometric property (T)?

As Theorem 4.25 implies that geometric (T), boundary geometric (T) or geometric (T) for sofic representations imply the conclusion of [**Kun16**, Theorem 1], the above is asking about a strengthening of the latter.

As the results of [**Kun16**] are statements about the ergodic decomposition of the measure, and these specific questions motivate the following:

QUESTION 7.6 (Ergodic decomposition). What properties do the subgroupoids of $\mathcal{G}|_Z$ that correspond to the ergodic components have?

Notice that this question connects very nicely to older results, notably [**Ele07**] and [**Sch08**].

On a related note, there are many measurable notions from the literature, such as cost [**Ele07**], entropy and mean dimension [**DKP14**] that all apply to *measured* groupoids – the topological groupoid defined in Section 3 can also be considered in this setting, and after passing through the standardisation process of Section 6 we obtain groupoids that allow for these notions to be applied. This mirrors the work of Carderi [**Car15**], as remarked earlier.

Our standardisation process produces topological groupoids, but is far from giving a unique space – the difference being that we use countable Boolean subalgebras of $\mathbf{2}^{\mathcal{X}}$, as opposed to countable Borel σ-algebras – and these each give potentially give rise to very different metrisable dynamical systems. On the other hand, properties such as amenability and property (T) will pass to these systems without any loss. This naturally leads to the following question:

QUESTION 7.7. What is the interaction between coarse properties of $\mathcal{X}$ and the measurable properties of its various standardisations? More concretely, can we show that for these systems, the invariants such as entropy (or mean dimension) do not depend on the choice of countable Boolean subalgebra? Is there an "clopen" analogue of the main results in [**Car15**]?

Finally, we end this section with a remark about a specific sofic group that itself does not have property A.

EXAMPLE 7.8 (Non-exact groups that are sofic). It is known, by a construction proposed in [**AO14**] and completed in [**Osa14**] that there are groups that are a-T-menable, but do not have property A. A natural observation is that any such Γ is direct limit of hyperbolic, CAT(0)-cubulable groups Γ_m – and as hyperbolic CAT(0)-cubical groups are residually finite [**Ago13**], Γ will be LEF (see [**CSC12**] for a proof of this, in the more general sofic setting).

In this situation, any LEF sequence will mostly likely be asymptotically coarsely embeddable, but it will not satisfy a notion of "asymptotic property A" that will be introduced in [**Pil16**], which is some form of groupoid exactness that appears to fail in the general setting – this is related to doing coarse geometry on groupoids with metrizable range fibres as in [**TWY16**] or [**AD16**].

QUESTION 7.9. What can we say concerning the asymptotic geometry of the sofic approximations of the above monster groups? Can we use embeddings of sofic approximations to construct new exotic monster groups with strange properties?

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CHAPTER 5

Non-amenable principal groupoids with weak containment

Abstract

We construct examples of principal groupoids that have weak containment but are not amenable, thus answering questions by Claire Anantharaman-Delaroche and Rufus Willett.

1. Amenability and weak containment

Through its many guises, amenability of a group has become a focal concept within both group theory and operator algebras. By a classical result of Andrzej Hulanicki ([**Hul64**]), the amenability of a discrete group is equivalent to the property that all unitary representations of the group are weakly contained in the left regular representation – we refer to this property as weak containment.

Recently, there has been interest in how far Hulanicki's classical result can be generalised, and in particular it has been shown by Rufus Willett in [**Wil15**] to fail for groupoids that are bundles of groups. The purpose of this note is to address Question 4.1 from [**AD16**] (that was also raised in Remark 3.6 of [**Wil15**]), namely we give an example of a *principal* groupoid that has weak containment, but is not amenable.

For a information about étale groupoids, their C^*-algebras and representations, we suggest [**BO08**, Chapter 5]. For more general information concerning locally compact groupoids, we refer to [**Ren80**] and [**ADR00**] and references therein.

1.1. Preliminaries. Throughout the text, $\mathcal{G}$ will be an étale Hausdorff topological groupoid, and for any subset of the unit space $U \subset \mathcal{G}^{(0)}$ we will denote by $\mathcal{G}|_U$ the *restriction* of $\mathcal{G}$ to U, i.e the subgroupoid of $\mathcal{G}$ consisting of all the elements with both source and range in U. We remark that this groupoid is open (resp. closed) if U is open (resp. closed) in $\mathcal{G}^{(0)}$.

Definition 1.1. $\mathcal{G}$ has *weak containment* if the left regular representation $\lambda : C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$ is a $*$-isomorphism.

From [**ADR00**, Theorem 6.1.4] it is known that all measurewise amenable groupoids have weak containment. We recall the general strategy used in [**Wil15**] to construct a non-amenable groupoid with weak containment.

Definition 1.2. Let Γ be a residually finite finitely generated discrete group and let $\mathcal{N} := \{N_i\}_i$ be a family of nested, finite index normal subgroups of Γ with trivial intersection.

Let π_i be the quotient map $\Gamma \to \Gamma/N_i$. The HLS[1] groupoid $\mathcal{G}$ associated to Γ and $\mathcal{N}$ is:

$$\mathcal{G} := \bigsqcup_{i \in \mathbb{N}^+} \{i\} \times X_i$$

where

$$X_i = \begin{cases} \Gamma/N_i \text{ if } i \in \mathbb{N} \\ \Gamma \text{ if } i = \infty \end{cases}$$

equipped with the topology generated by the following sets:

- the singletons $\{(i, g)\}$;
- the tails: $\{(i, \pi_i(g)) \mid i \in \mathbb{N}^+, i > N\}$ for every fixed $g \in \Gamma$ and $N \in \mathbb{N}$.

One can check that equipped with this topology and the obvious partially defined product and inverse $\mathcal{G}$ becomes an étale, locally compact Hausdorff groupoid with unit space $\mathbb{N}^+$. Moreover, it is amenable if and only if Γ is amenable.

Considering the open invariant set $U := \mathbb{N} \subset \mathcal{G}^{(0)}$, we obtain a commuting diagram with exact rows consisting of C^*-algebras associated with the restriction groupoids $\mathcal{G}|_U$ and $\mathcal{G}|_{U^c}$:

$$\begin{tikzcd}
0 \arrow[r] & C^*(\mathcal{G}|_U) \arrow[r] \arrow[d] & C^*(\mathcal{G}) \arrow[r] \arrow[d] & C^*(\mathcal{G}|_{U^c}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^*_r(\mathcal{G}|_U) \arrow[r] & C^*_r(\mathcal{G}) \arrow[r] \arrow[dr] & Q_U \arrow[r] \arrow[d] & 0 \\
& & & C^*_r(\mathcal{G}|_{U^c}) \arrow[r] & 0
\end{tikzcd}$$

where Q_U is the quotient by the ideal $C^*_r(\mathcal{G}|_U)$. The groupoid $\mathcal{G}|_U$ is amenable and therefore has weak containment, and so to deduce weak containment for $\mathcal{G}$ it is enough to show that the map $C^*(\mathcal{G}|_{U^c}) \to Q_U$ is isometric. In [**Wil15**], it is then proved that $C^*(\mathcal{G}|_{U^c}) \cong C^*(\Gamma)$, $C^*_r(\mathcal{G}|_{U^c}) \cong C^*_r(\Gamma)$, and that vertical arrows come from canonical maps between these; therefore weak containment is automatic if Γ is amenable. In the non-amenable case, weak containment is deduced from the property FD of Lubotzky–Shalom ([**LS04**]):

DEFINITION 1.3. Let Γ be a countable discrete group. Γ has *property FD* if finite dimensional representations are dense in the unitary dual of Γ; a family of finite quotients $\mathcal{X} := \{\Gamma/N_\kappa\}_\kappa$ is an *FD family* if the set of representations of Γ which factor through the quotient maps $\{\pi_\kappa : \Gamma \to \Gamma/N_\kappa\}_\kappa$ is dense in the unitary dual of Γ.

This is then used to deduce the key result in [**Wil15**]: if $\mathcal{X}$ is an FD family, then the C^*-algebra Q_U in the corresponding HLS groupoid $\mathcal{G}$ is isomorphic to the maximal group C^*-algebra $C^*(\Gamma)$ through the vertical map $C^*(\Gamma) \cong C^*(\mathcal{G}|_{U^c}) \to Q_U$ in the above diagram.

[1]After Nigel Higson, Vincent Lafforgue and George Skandalis who first considered this construction for a related purpose in [**HLS02**].

However, non-amenability of Γ implies that the groupoid $\mathcal{G}$ is non-amenable, and this finishes the construction.

2. Constructing examples

Let Γ be a non-amenable residually finite group with a countable nested (FD) family $\mathcal{X}$ and let $\mathcal{G}$ be the HLS groupoid from the previous section. We are going to consider a transformation groupoid constructed from $\mathcal{G}$ and the set of finite quotients $\mathcal{X}$. Let $X := \bigsqcup_i X_i$. We begin by constructing the unit space for this groupoid as a second countable compactification of X.

For $g \in X_i$, consider the *shadow of* g *in* X:

$$\mathrm{Sh}(g) := \bigcup_{j \geqslant i} \{x \in X_j \mid \pi_{i,j}(x) = g\},$$

where $\pi_{i,j} \colon \Gamma/N_j \to \Gamma/N_i$ is the canonical quotient map.

Let B be the $\mathcal{G}$-invariant C^*-subalgebra of $\ell^\infty(X)$ generated by $\{\delta_x\}_{x\in X}$ and the sets of projections $\{\mathbf{1}_{\mathrm{Sh}(g))}\}_{g\in X_i}$ for all $i \in \mathbb{N}$. We will consider the spectrum of B, which we denote by $\widehat{X}$. As B is $\mathcal{G}$-invariant, $\widehat{X}$ carries a natural $\mathcal{G}$-action, and we consider the transformation groupoid $G := \widehat{X} \rtimes \mathcal{G}$.

We remark that $G^{(0)}$ contains a obvious open invariant subset $X \subset G^{(0)}$ corresponding to the ideal generated by δ_x, $x \in X$, and let $\partial X \subset G^{(0)}$ be the closed (compact) complement. The following lemma describes it as a Γ-space.

Lemma 2.1.

(i) The space ∂X is Γ-equivariantly homeomorphic to $\widehat{\Gamma}_{\mathcal{X}}$, the profinite completion of Γ with respect to the family $\mathcal{X}$.

(ii) The algebra $A := C(\partial X)$ is a direct limit of finite-dimensional Γ-C^*-algebras A_i, such that the action on A_i factors through Γ/N_i.

Proof. The inclusion $\partial X \subset \widehat{X}$ gives rise to a restriction homomorphism $r \colon B = C(\widehat{X}) \to C(\partial X) = A$ which obviously contains all elements δ_x, $x \in X$, in its kernel. Thus, $A = C(\partial X)$ is generated by images of the projections $p_{i,g} := \mathbf{1}_{\mathrm{Sh}(g)}$, $i \in \mathbb{N}$, $g \in X_i$.

Consider the finite-dimensional C^*-algebras A_i generated by the projections $p_{i,g}$, $g \in X_i$. Notice that the action of Γ on A_i obviously factors through Γ/N_i, as it is isomorphic to the natural left action of Γ on $\mathbb{C}[\Gamma/N_i]$. Moreover, there are natural Γ-equivariant injective homomorphisms

$$\rho_{i,j} \colon A_i \to A_j,$$
$$\rho_{i,j}(p_{i,g}) = \sum_{\pi_{i,j}(g')=g} p_{j,g'}$$

corresponding to (Γ-equivariant) projections $\Gamma/N_j \twoheadrightarrow \Gamma/N_i$.

Furthermore, the element $p_{i,g} - \rho_{i,i+1}(p_{i,g})$, considered as an element of $C(\widehat{X})$, equals δ_g, and therefore the kernel of the restriction map r is generated by such differences. As a consequence, we get a Γ-equivariant isomorphism $A \cong \varinjlim A_i$, whence the boundary ∂X is the inverse limit of the corresponding projective system of Γ-spaces. By the remark above, this projective system of spaces is naturally identified with the projective system $\{\Gamma/N_i\}_{i\in\mathbb{N}}$, equipped with the natural left Γ-action. This finishes the proof.

□

REMARK 2.2. As the C^*-algebras A_i are finite-dimensional, they have natural regular representations $\lambda_i\colon A_i \to \mathbb{B}(A_i)$, where A_i carries the Hilbert space structure obtained from the natural trace $\tau_i\colon p_{i,g} \mapsto 1$ as well as a unitary representation $\alpha_i\colon \Gamma \to \mathcal{U}(A_i)$ given by the Γ-action. Let ϕ_i be the bijection that sends $p_{i,g}$ to $\pi_i(g)$. This induces an isomorphism ϕ_i between $A_i \rtimes \Gamma/N_i$ and the full matrix algebra $\mathbb{M}_{|X_i|}$.

The consequence of Lemma 2.1 is that we can identify the boundary piece of G as

$$G|_{\partial X} \cong \widehat{\Gamma}_{\mathfrak{X}} \rtimes \Gamma,$$

where the latter groupoid is the transformation groupoid with the natural free action. It follows that G is a principal groupoid as the action on X is obviously free: $G|_X \cong \bigsqcup_{i\in\mathbb{N}}(X_i \rtimes \Gamma/N_i)$ by construction.

Attached with this decomposition of $\widehat{X}$ into X and ∂X we obtain a commuting diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C^*(G|_X) \arrow[r] \arrow[d] & C^*(G) \arrow[r] \arrow[d] & C^*(G|_{\partial X}) \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & C_r^*(G|_X) \arrow[r] & C_r^*(G) \arrow[r] \arrow[dr] & Q_X \arrow[r] \arrow[d] & 0 \\
& & & C_r^*(G|_{\partial X}) \arrow[r] & 0
\end{tikzcd}$$

LEMMA 2.3.

(i) If the map $q\colon C^*(G|_{\partial X}) \to Q_X$ in the above diagram is an isomorphism, then G has weak containment;

(ii) If the map $Q_X \to C_r^*(G|_{\partial X})$ in the above diagram is not an isomorphism, then G is non-amenable.

PROOF. As $G|_X \cong \bigsqcup_i X_i \rtimes \Gamma/N_i$ is a disjoint union of pair groupoids with the obvious discrete topology, it is amenable and therefore has weak containment. i) now follows from the above diagram by the five lemma. ii) follows as amenability passes to restrictions to closed invariant subsets. □

As a final preliminary before proving our result, we describe an ambient setting for Q_X and $C_r^*(G)$.

Lemma 2.4. *There is a natural isometric embedding*

$$C_r^*(G) \hookrightarrow \prod_{i \in \mathbb{N}} \mathbb{M}_{|X_i|},$$

which induces an isometric embedding

$$\iota\colon Q_X \hookrightarrow \frac{\prod_j \mathbb{M}_{|X_j|}}{\bigoplus_j \mathbb{M}_{|X_j|}}.$$

Proof. Since X is dense in $\widehat{X}$, we can use [**KS04**, Corollary 2.4 a)] to see that the norm of an element $f \in C_r^*(G)$ is equal to $\sup_{x \in X} \|\lambda_x(f)\|$, where λ_x is the left regular representation on $s^{-1}(x)$ (which is equal to X_i if $x \in X_i$). Thus we get a natural embedding

$$C_r^*(G) \hookrightarrow \prod_{i \in \mathbb{N}} \mathbb{M}_{|X_i|},$$

where $\mathbb{M}_{|X_i|}$ is the full matrix algebra over X_i (viewed as bounded operators on $\ell^2(X_i)$). As $G|_X$ is a union of pair groupoids, we get $C_r^*(G|_X) \cong \bigoplus_j \mathbb{M}_{|X_j|}$, which implies that Q_X is isometrically embedded into $\frac{\prod_j \mathbb{M}_{|X_j|}}{\bigoplus_j \mathbb{M}_{|X_j|}}$.

$\square$

Our goal now is connect the maximal crossed product of A by Γ with Q_X using that Γ has property (FD).

Proposition 2.5. *The maximal crossed product $A_i \rtimes \Gamma$ embeds into $\prod_{j \geqslant i} A_j \rtimes \Gamma/N_j$ by the natural maps $\rho_{i,j} \rtimes \pi_j$.*

Proof. The claim is equivalent to the statement that every representation of $A_i \rtimes \Gamma$ is weakly contained in a representation factoring through $A_j \rtimes \Gamma/N_j$. To this end, consider an arbitrary representation $\sigma\colon A_i \rtimes \Gamma \to \mathbb{B}(\mathcal{H})$ and an element

$$x = \sum_{g \in \Gamma/N_i} p_{i,g} f_g \in A_i \rtimes_{\mathrm{alg}} \Gamma,$$

where $f_g \in \mathbb{C}[\Gamma]$ and let $\xi, \eta \in \mathcal{H}$ be arbitrary vectors. We have

$$\langle x\xi, \eta \rangle = \sum_{g \in \Gamma/N_i} \langle f_g \xi, p_{i,g} \eta \rangle$$

By property (FD) of Γ for every $\varepsilon > 0$ we get a $j \geqslant i$, representation $\sigma'\colon \Gamma \twoheadrightarrow \Gamma/N_j \to \mathcal{U}(\mathcal{H}')$ and vectors $\xi_1', \ldots, \xi_N'$, $\eta_1', \ldots, \eta_N' \in \mathcal{H}'$ such that

$$\left| \langle x\xi, \eta \rangle - \sum_{g \in \Gamma/N_i} \sum_{\ell=1}^{N} \langle \sigma'(f_g)\xi_\ell', \eta_\ell' \rangle \right| < \varepsilon.$$

Consider now the Hilbert space $\mathcal{H}'' := \mathcal{H}' \otimes A_j$ and the representation $\sigma'' := \sigma' \otimes \alpha_j\colon \Gamma \to \mathcal{U}(\mathcal{H}'')$ (which factors through Γ/N_j) as well as the representation $m_{i,j} := \mathrm{id}_{\mathcal{H}'} \otimes (\lambda_j \circ \rho_{i,j})\colon B_i \to$

$\mathbb{B}(\mathcal{H}'')$. It's easy to see that these give a covariant pair and that for every $h \in \Gamma/N_j$ we have an equality of matrix coefficients

$$\langle \sigma'(f_g)\xi'_\ell, \eta'_\ell \rangle = \left\langle \sigma''(f_g)(\xi'_\ell \otimes p_{j,h}), \eta'_\ell \otimes \sum_{g' \in \Gamma/N_j} p_{j,g'} \right\rangle$$

Therefore any matrix coefficient of any representation of $A_i \rtimes \Gamma$ is approximated by a matrix coefficient of a representation factoring through $A_j \rtimes \Gamma/N_j$ for a suitable j, which ends the proof. □

We now can prove the following:

Proposition 2.6. *The maximal crossed product $A \rtimes \Gamma$ is isomorphic to Q_X through the canonical quotient map $q\colon A \rtimes \Gamma \to Q_X$.*

Proof. In view of Lemma 2.4, it is enough to prove that the composition

$$\iota \circ q\colon A \rtimes \Gamma \to \frac{\prod_j \mathbb{M}_{|X_j|}}{\bigoplus_j \mathbb{M}_{|X_j|}}$$

is isometric. By Lemma 2.1 and the continuity of the maximal crossed product functor, for this it is enough to prove that the map $\iota \circ q$ is isometric on $A_i \rtimes \Gamma$.

To this end, take an arbitrary element of the algebraic crossed product $A_i \rtimes_{\mathrm{alg}} \Gamma$

$$z = \sum_{g \in \Gamma/N_i} p_{i,g} f_g,$$

where $f_g \in \mathbb{C}[\Gamma]$, and observe that it lifts to $C^*(G)$ as the family of elements

$$(z_j)_j = \left(\sum_{g \in \Gamma/N_i} p_{i,g}|_{X_j} \pi_j(f_g) \right)_j \in C(X_j) \rtimes \Gamma/N_j, \quad j \geqslant i.$$

Using the isomorphisms $\phi_j\colon A_j \rtimes \Gamma/N_j \to \mathbb{M}_{|X_j|}$ defined in Remark 2.2, we now see that the image of z under the composition $\iota \circ q$ coincides with $(\phi_j \circ (\rho_{i,j} \rtimes \pi_j))(z) \in \prod_{j \geqslant i} \mathbb{M}_{|X_j|}$, because $\rho_{i,j}(p_{i,g})(x) = p_{i,g}(x)$ for all $x \in X_j$.

Therefore the map $\iota \circ q\colon A \rtimes \Gamma \to \frac{\prod_j \mathbb{M}_{|X_j|}}{\bigoplus_j \mathbb{M}_{|X_j|}}$ coincides with the map $A \rtimes \Gamma \to \frac{\prod_j \mathbb{M}_{|X_j|}}{\bigoplus_j \mathbb{M}_{|X_j|}}$ induced by $\phi_j \circ (\rho_{i,j} \rtimes \pi_j)$. The latter is isometric by the previous proposition, and therefore we are done.

□

Theorem 2.7. *Let Γ be a non-amenable residually finite group with a countable nested (FD) family $\mathcal{X}$. Then the groupoid G constructed above is principal and non-amenable, but has weak containment.*

Proof. In view of Lemma 2.3 and Proposition 2.6, it remains to prove that the map $Q_X \to C^*_r(G|_{\partial X})$ is not an isomorphism. We remark that ∂X has a Γ-invariant probability measure obtained by taking the weak* limit of the normalised counting measures on each X_i. By [**WY14**, Lemma 7.1, Remark 7.1], we get that Q_X contains $C^*(\Gamma)$ as a $*$-subalgebra, which maps onto $C^*_r(\Gamma)$ under the quotient map $Q_X \to C^*_r(G|_{\partial X})$. Hence the map $Q_X \to C^*_r(G|_{\partial X})$ is not an isomorphism. This finishes the proof. $\square$

We remark that [**LS04**, Theorems 2.2 and 2.8] give a wealth of examples of Γ that satisfy the conditions above: notably free groups and surface groups (also cyclic extensions of these groups).

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CHAPTER 6

Uniqueness questions for C^*-norms on group rings

Abstract

We provide a large class of discrete amenable groups for which the complex group ring has several C^*-completions, thus providing partial evidence towards a positive answer to a question raised by Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam.

1. Introduction

The interplay between group theory and operator algebras dates back to the seminal papers by Murray and von Neumann [**MvN36**] and by choosing different completions of a discrete countable group Γ one obtains interesting analytic objects; for instance the Banach algebra $\ell^1(\Gamma)$, the full and reduced C^*-algebras $\mathrm{C}^*(\Gamma)$ and $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$, and the group von Neumann algebra $L\Gamma$. In general there are many norms on, say, $\ell^1(\Gamma)$ such that the completion with respect to this norm gives a C^*-algebra, and the question of when the C^*-completion is unique (in which case Γ is said to be C^**-unique*) has been studied by various authors [**LN04**, **Boi84**, **Bar83**]. A C^*-unique discrete group is evidently amenable and it is, to the best of the authors' knowledge, an open question whether the converse is true, although it is known to be false in the more general context of locally compact groups [**LN04**]. More recently, the paper [**GMRr16**] put emphasis on the question of when the complex group algebra $\mathbb{C}\Gamma$ has a unique C^*-completion. As is easily seen [**GMRr16**, Proposition 6.7], if Γ is locally finite (i.e. if every finitely generated subgroup is finite) then $\mathbb{C}\Gamma$ has a unique C^*-completion, and [**GMRr16**, Question 6.8] asks if the converse is true. The present paper provides partial evidence towards a positive answer to this, in that we prove that for the following classes of non-locally finite groups have several C^*-completions.

Theorem A (see Proposition 2.4 and Corollary 3.7). *The class of countable groups Γ for which $\mathbb{C}\Gamma$ does not have a unique C^*-norm includes the following:*

(i) Infinite groups of polynomial growth.
(ii) Torsion free, elementary amenable groups with a non-trivial, finite conjugacy class.
(iii) Groups with a central element of infinite order.

The key to the proof of of (i) and (ii) is the so-called strong Atiyah conjecture (see Section 3.1) which predicts a concrete restriction on the von Neumann dimension of kernels of elements in the complex group algebra under the left regular representation — notably these are predicted to be either zero or one if the group in question is torsion free.

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2. Basic results on C^*-uniqueness

In what follows, all discrete groups are implicitly assumed to be at most countable. We will use several operator algebras associated to a discrete group Γ: the maximal C^*-algebra $\mathrm{C}^*(\Gamma)$, the reduced C^*-algebra $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ and the von Neumann algebra $L\Gamma$. For more information on these, we refer to [**BO08**, §2.5]. We recall that $L\Gamma = (\lambda(\mathbb{C}\Gamma))'' \subset \mathbb{B}(\ell^2\Gamma)$ is generated by the left regular representation $\lambda\colon \mathbb{C}\Gamma \to \mathcal{B}(\ell^2\Gamma)$ and carries a canonical, faithful, normal trace given by $\tau(x) = \langle x\delta_e\rangle\, \delta_e$. In what follows, tr will denote the normalized trace on $\mathbb{M}_n(\mathbb{C})$ while Tr will denote the non-normalized trace.

We begin by formally introducing the notion of C^*-uniqueness. In order to avoid a notational conflict with the already existing notions studied in [**LN04**, **Boi84**], we emphasize that we are investigating the uniqueness of C^*-norms on the complex group algebra in contrast to the ℓ^1-algebra.

Definition 2.1. Let Γ be a discrete group. $\mathbb{C}\Gamma$ is said to be:

(i) C^*-unique if it carries a unique C^*-norm;
(ii) $\mathrm{C}^*_{\mathrm{r}}$-unique if no C^*-norm on $\mathbb{C}\Gamma$ is properly majorised by the reduced C^*-norm.

Γ is said to be algebraically C^*- (respectively $\mathrm{C}^*_{\mathrm{r}}$-)unique if $\mathbb{C}\Gamma$ is C^*- (respectively $\mathrm{C}^*_{\mathrm{r}}$-)unique.

Amenable groups are characterized by the property that the maximal and reduced C^*-algebras coincide, and thus a nonamenable group is never algebraically C^*-unique; on the other hand, for amenable groups the above notions coincide. Note also that the class of C^*-simple groups, which has recently received a lot of attention [**BKKO17**, **LB17**], falls within the class of algebraically $\mathrm{C}^*_{\mathrm{r}}$-unique groups. As already mentioned in the introduction, algebraic C^*-uniqueness appeared in the recent paper [**GMRr16**] in which the authors observed that locally finite groups have this property and asked if this characterizes the class of locally finite groups. Below we prove a few basic permanence results regarding algebraic C^*-uniqueness, but before doing so we give an alternative characterization, which is straightforward algebraic adaptation of the similar result for ℓ^1-algebras [**Bar83**, Proposition 2.4].

Lemma 2.2. *Let Γ be a discrete group. Then $\mathbb{C}\Gamma$ is C^*-unique (respectively $\mathrm{C}^*_{\mathrm{r}}$-unique) if and only if every nontrivial closed, two-sided ideal in $\mathrm{C}^*(\Gamma)$ (respectively $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$) intersects $\mathbb{C}\Gamma$ non-trivially.*

Proof. We give the proof for the statement about algebraic C*-uniqueness; the other case is obtained by replacing $C^*(\Gamma)$ by $C^*_r(\Gamma)$ throughout the proof. Assume that there is a non-trivial ideal $J \trianglelefteq C^*(\Gamma)$ intersecting $\mathbb{C}\Gamma$ trivially and denote by $q\colon C^*(\Gamma) \to C^*(\Gamma)/J$ the quotient map. Composing q with the inclusion $\mathbb{C}\Gamma \hookrightarrow C^*(\Gamma)$ yields a faithful representation of $\mathbb{C}\Gamma$ and it defines a C*-norm on it that is properly majorised by the maximal norm by non-triviality of J. Conversely, if there is a C*-norm on $\mathbb{C}\Gamma$ which is properly majorised by the norm coming from $C^*(\Gamma)$, then $C^*(\Gamma)$ surjects onto the corresponding quotient, and the kernel of this surjection is a non-trivial ideal intersecting $\mathbb{C}\Gamma$ trivially. $\square$

Corollary 2.3. *Let Γ and Λ be discrete groups. If $\mathbb{C}(\Gamma \times \Lambda)$ is C*-unique (respectively C^*_r-unique), then so are $\mathbb{C}\Gamma$ and $\mathbb{C}\Lambda$.*

Proof. Let $J \trianglelefteq C^*_r(\Gamma)$ be a non-trivial ideal intersecting $\mathbb{C}\Gamma$ trivially. Then $J \otimes_{\max} C^*_r(\Lambda) \trianglelefteq C^*(\Gamma) \otimes_{\max} C^*(\Lambda) = C^*(\Gamma \times \Lambda)$ is a non-trivial ideal intersecting $\mathbb{C}(\Gamma \times \Lambda) = \mathbb{C}\Gamma \otimes_{\mathrm{alg}} \mathbb{C}\Lambda$ trivially. The same proof with C^* replaced by C^*_r and $\otimes_{\max}$ replaced by $\otimes_{\min}$ works for the reduced case. $\square$

Proposition 2.4. *If Γ is a discrete group with a central element of infinite order then $\mathbb{C}\Gamma$ is not C^*_r-unique.*

Proof. Denote by Z the subgroup in Γ generated by a central element of infinite order. Then $C^*_r(Z) \cong C(S^1)$ and $LZ \cong L^\infty(S^1)$ via the Fourier transform and we denote by $p \in LZ$ the projection corresponding to the characteristic function of the upper half circle $\{e^{i\theta} \mid \theta \in [0,\pi]\}$. Define $\pi := \lambda_\Gamma p$; i.e. the left regular representation of Γ restricted to the invariant subspace $p\ell^2(\Gamma)$. Choosing a non-zero function $f \in C(S^1)$ supported in the *lower* half circle we obtain a non-zero element $x \in C^*_r(Z) \subset C^*_r(\Gamma)$ with $xp = 0$ and hence the norm on $C^*_r(\Gamma)$ induced by π is not the one induced by λ_Γ. We now only need to see that π is faithful on $\mathbb{C}\Gamma$. To this end, consider the trace-preserving conditional expectation $E\colon L\Gamma \to LZ$ [**BO08**, Lemma 1.5.11] and assume that $a \in \mathbb{C}\Gamma$ is in the kernel of π. Then a^*a is also in the kernel of π and since E is an LZ-bimodule map [**BO08**, Proposition 1.5.7] we get

$$0 = E(\lambda_\Gamma(a^*a)p) = E(\lambda_\Gamma(a^*a))p.$$

However, $E(\mathbb{C}\Gamma) \subset \mathbb{C}Z \simeq \mathrm{Pol}(z,\bar{z}) \subset C(S^1)$ and therefore $E(\lambda_\Gamma(a^*a)) = 0$ and since E is trace-preserving and the trace on $L\Gamma$ is faithful we conclude that a^*a, and hence a, is zero. $\square$

Corollary 2.5. *An abelian group is algebraically* C**-unique if and only if it is locally finite (i.e. pure torsion).*

Remark 2.6. The result in Corollary 2.5 was also observed, independently and with different proofs, by Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam (unpublished).

Remark 2.7. The class of locally finite groups has many stability properties — for instance it is closed under subgroups, quotients and extensions and, moreover, being virtually locally finite is the same as being locally finite. However, verifying these properties for the class of C*-unique groups seems to be a much bigger challenge.

3. The strong Atiyah conjecture and $\mathrm{C}^*_{\mathrm{r}}$-uniqueness

3.1. The strong Atiyah conjecture. The key to our main result is the so-called strong Atiyah conjecture which is briefly described in the following. A good general reference is [**Lüc02**, Chapter 10] where all of the results below can be found, and to which we also refer for the original references. Let Γ be a discrete group and denote by $\frac{1}{|\mathrm{FIN}(\Gamma)|}\mathbb{Z}$ the additive subgroup in $\mathbb{Q}$ generated by the set

$$\left\{ \frac{1}{|\Lambda|} \;\middle|\; \Lambda \leq \Gamma \text{ a finite subgroup} \right\}.$$

Given a matrix $A \in \mathbb{M}_n(\mathbb{C}\Gamma)$ we denote by $L_A \in L\Gamma \otimes \mathbb{M}_n(\mathbb{C}) \subset \mathcal{B}(\ell^2(\Gamma)^n)$ the bounded operator given by left multiplication with A (via the left regular representation of Γ). The strong Atiyah conjecture then predicts that

$$\dim_{L\Gamma} \ker(L_A) := (\tau \otimes \mathrm{Tr})(P_{\ker L_A}) \in \frac{1}{|\mathrm{FIN}(\Gamma)|}\mathbb{Z}.$$

Here $\dim_{L\Gamma}(-)$ denotes the von Neumann dimension of the (right) Hilbert $L\Gamma$-module $\ker(L_A)$ defined as the non-normalized trace of the kernel projection $P_{\ker L_A}$; see [**Lüc02**] for details on this. It should be noted that the strong Atiyah conjecture is false in general [**Lüc02**, Theorem 10.23], but is known to hold for all groups which have a bound on the order of finite subgroups and belong to Linnell's class $\mathcal{C}$ [**Lüc02**, Theorem 10.19], the latter being the smallest class of groups which contain all free groups, is closed under directed unions and extensions by elementary amenable groups (i.e., if $\Lambda \trianglelefteq \Gamma$, $\Lambda \in \mathcal{C}$ and Γ/Λ is elementary amenable, then $\Gamma \in \mathcal{C}$). The above discussion motivates the following notion.

Definition 3.1. Let Γ be a countable group. The *torsion multiplier* of Γ is defined as

$$\theta(\Gamma) = \frac{1}{\mathrm{lcm}\{|H| |\; H \leqslant \Gamma \text{ finite}\}} \in [0, 1].$$

In this definition, and in what follows, we use the convention that the least common multiple (lcm) of an infinite set of natural numbers is infinity and that $\frac{1}{\infty} = 0$. Note that if Γ has an upper bound on the set of finite subgroups, then

$$\frac{1}{|\mathrm{FIN(G)}|}\mathbb{Z} = \{n\theta(\Gamma) \mid n \in \mathbb{Z}\},$$

and $\frac{1}{|\mathrm{FIN(G)}|}\mathbb{Z}$ has 0 as an accumulation point otherwise. In view of this, the strong Atiyah conjecture for a group Γ with $\theta(\Gamma) > 0$ implies that the possible kernel dimensions are properly quantized in the sense that they can only take values in the discrete set $\{n\theta(\Gamma) \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Theorem A (i) and (ii) will follow directly from our main technical result, Theorem 3.6 below. The key idea in the proof is to play the aforementioned "quantization" of the kernel dimensions against an abundance of central projections in $L\Gamma$ with small traces which provide representations of $C^*_r(\Gamma)$ with non-trivial kernels. To quantify this, we need the following definition.

DEFINITION 3.2. The *central granularity* of Γ is defined as

$$\sigma(\Gamma) = \inf\{\tau(p) \mid p \in \mathrm{Proj}(Z(L\Gamma)),\, p \neq 0\} \in [0, 1].$$

We note that $\sigma(\Gamma) < 1$ if and only if $Z(L\Gamma)$ is nontrivial which is equivalent Γ not being icc[1]. The next proposition computes the central granularity of Γ in group-theoretic terms. Recall that the *FC-centre* Γ_{fc} is the normal subgroup of Γ consisting of all elements with finite conjugacy classes.

PROPOSITION 3.3. *Let* $\Gamma_{\mathrm{fc}} \trianglelefteq \Gamma$ *be the FC-centre of* Γ*. Then*

$$\sigma(\Gamma) = \frac{1}{|\Gamma_{\mathrm{fc}}|},$$

where the right-hand side is interpreted as 0 *if* $|\Gamma_{\mathrm{fc}}| = \infty$*.*

PROOF. Γ_{fc} is an increasing union of a sequence of finitely generated normal subgroups $\Lambda_n \trianglelefteq \Gamma$; to see this, note that Γ_{fc} is clearly an increasing union of a sequence of finitely generated subgroups Λ'_n, and defining Λ_n to be generated by the Γ-conjugacy classes of a finite system of generators for Λ'_n yields the desired sequence of finitely generated subgroups which are normal in Γ. We now have two cases to consider:

(i) all Λ_n are finite (equivalently, Γ_{fc} is a torsion group),
(ii) Λ_n is infinite for some n.

In case (i), setting $p_n := \frac{1}{|\Lambda_n|}\sum_{g\in\Lambda_n} g$, we get a projection $p_n \in L\Gamma_{\mathrm{fc}}$ with $\tau(p_n) = \frac{1}{|\Lambda_n|}$; moreover, p_n is central in $L\Gamma$ since Λ_n is normal in Γ. This proves that $\sigma(\Gamma) = 0$ if Γ_{fc} is an infinite torsion group (in this case $|\Lambda_n| \to \infty$). If Γ_{fc} is finite, then the sequence stabilizes, and therefore we get a central projection p in $L\Gamma$ with trace $\frac{1}{|\Gamma_{\mathrm{fc}}|}$. The centre of $L\Gamma$ consists of elements whose associated Fourier series in $\ell^2(\Gamma) = L^2(L\Gamma, \tau)$ are supported only on Γ_{fc} and are constant along conjugacy classes, and is therefore contained in the centre of $L\Gamma_{\mathrm{fc}}$; hence we get $\frac{1}{|\Gamma_{\mathrm{fc}}|} \geqslant \sigma(\Gamma) \geqslant \sigma(\Gamma_{\mathrm{fc}})$. But we also have $L\Gamma_{\mathrm{fc}} = \mathbb{C}\Gamma_{\mathrm{fc}}$ which by representation theory of finite groups is isomorphic to a direct sum of matrix algebras $\bigoplus_\pi \mathbb{M}_{d_\pi}(\mathbb{C})$ with the trace given by $\bigoplus_\pi \frac{d_\pi^2}{|\Gamma_{\mathrm{fc}}|}\,\mathrm{tr}$; thus, the minimal central projection has trace $\frac{1}{|\Gamma_{\mathrm{fc}}|} = \sigma(\Gamma_{\mathrm{fc}})$; this proves the claim.

In case (ii) we fix an $n \in \mathbb{N}$ such that $\Lambda_n =: \Lambda$ is infinite and note that since Λ is generated by a finite number of elements with finite conjugacy classes, its centralizer $C_\Gamma(\Lambda)$ is of finite index in Γ. We now claim that $L\Lambda$ has a diffuse von Neumann subalgebra and thus projections of arbitrarily small trace. This can be seen as follows: if $L\Lambda$ has a direct summand of type II_1, it is clear because such von Neumann algebras are diffuse. Otherwise $L\Lambda$ is of type I,

[1]recall that a group is *icc* if all non-trivial conjugacy classes are infinite

but then Λ is virtually abelian [**Lüc97**, Lemma 3.3], and hence, being infinite by assumption and finitely generated by construction, contains a copy of $\mathbb{Z}$ which generates a diffuse von Neumann algebra $L\mathbb{Z} \cong L^\infty(S^1)$. In view of the above, for an arbitrary $\varepsilon > 0$ there is a projection $p \in L\Lambda \subset L\Gamma_{\mathrm{fc}}$ of trace $\tau(p) < \frac{\varepsilon}{[\Gamma:C_\Gamma(\Lambda)]}$. Now let

$$q := \bigvee_{g\in\Gamma} {}^g p,$$

where ${}^g p := gpg^{-1}$. Then q is a central projection in $L\Gamma$. Moreover, p is invariant under the centralizer $C_\Gamma(\Lambda)$ and upon choosing coset representatives $g_1, \ldots, g_{[\Gamma:C_\Gamma(\Lambda)]}$ for $\Gamma/C_\Gamma(\Lambda)$ we obtain that

$$q = \bigvee_{i=1}^{[\Gamma:C_\Gamma(\Lambda)]} {}^{g_i} p$$

and hence $\tau(q) \leqslant [\Gamma : C_\Gamma(\Lambda)] \cdot \tau(p) < \varepsilon$. Thus $\sigma(\Gamma) = 0$. □

LEMMA 3.4. *Let Γ be a discrete non-icc group. For every $\varepsilon > 0$ there exists a nonzero projection $p \in Z(L\Gamma)$ with $\tau(p) < \sigma(\Gamma) + \varepsilon$ and a nonzero, central element $x \in \mathrm{C}^*_{\mathrm{r}}(\Gamma)$ with $xp^\perp = 0$.*

PROOF. Since Γ is non-icc, $Z(L\Gamma) \neq \mathbb{C}1$ so $\sigma(\Gamma) < 1$. Let $\varepsilon > 0$ be given and assume, without loss of generality, that $\sigma(\Gamma) + \varepsilon < 1$. One has $Z(L\Gamma) = Z(\mathrm{C}^*_{\mathrm{r}}(\Gamma))'' = Z(\mathbb{C}\Gamma)''$, as can bee seen for instance by using Kaplansky's density theorem together with the center valued trace, and noting that $Z(\mathbb{C}\Gamma)$ consists of the elements whose coefficients are constant along conjugacy classes. By Gelfand duality, $Z(\mathrm{C}^*_{\mathrm{r}}(\Gamma))$ is isomorphic to the C^*-algebra $C(Z)$ of continuous functions on its Gelfand spectrum Z, which is a compact Hausdorff space; it is metrizable because $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ is separable. The canonical trace τ thus gives a regular Borel probability measure μ on Z [**Rud66**, Theorem 2.14] and an isomorphism $Z(L\Gamma) = Z(\mathrm{C}^*_{\mathrm{r}}(\Gamma))'' \cong L^\infty(Z, \mu)$ compatible with the natural inclusions. Projections in $Z(L\Gamma)$ correspond via this isomorphism to measurable subsets of Z (up to null sets), and we therefore obtain a measurable subset $A \subset Z$ such that $0 < \mu(A) < \sigma(\Gamma) + \varepsilon/2$. By regularity of μ, there exists $U \supseteq A$ open such that

$$0 < \mu(A) \leqslant \mu(U) < \sigma(\Gamma) + \varepsilon < 1.$$

Now, there is a non-zero element $x \in C(Z)$ vanishing on the compact set $K := Z \setminus U$ (for instance, the distance function to K); letting p be the projection corresponding to U finishes the proof. □

The following lemma gives a concrete description of the decomposition of the left regular representation of a discrete group Γ over the cosets of a finite index normal subgroup Λ.

LEMMA 3.5. *Let $\Lambda \trianglelefteqslant \Gamma$ be a normal subgroup of finite index. For every choice of coset representatives $g_1, \ldots, g_{[\Gamma:\Lambda]} \in \Gamma$ there exists a trace-preserving inclusion of von Neumann algebras $\pi\colon (L\Gamma, \tau) \hookrightarrow (\mathbb{M}_{[\Gamma:\Lambda]}(L\Lambda), \tau \otimes \mathrm{tr})$ which restricts to corresponding inclusions at the* *level of reduced* C^**-algebras and complex group rings, and which for* $x \in L\Lambda$ *is given by*

$$\pi(x) = \operatorname{diag}({}^{g_1}x, {}^{g_2}x, \dots, {}^{g_{[\Gamma:\Lambda]}}x), \tag{3.1}$$

where ${}^g x = gxg^{-1}$ *is the conjugation action of* $g \in \Gamma$ *on* $L\Lambda$.

Proof. Choose coset representatives $g_1, g_2, \dots, g_{[\Gamma:\Lambda]}$ of Γ/Λ and consider the isomorphisms of Hilbert spaces

$$\ell^2(\Gamma) \cong \bigoplus_{i=1}^{[\Gamma:\Lambda]} \ell^2(g_i^{-1}\Lambda) \cong \bigoplus_{i=1}^{[\Gamma:\Lambda]} \ell^2(\Lambda).$$

These induce a $*$-isomorphism $\pi\colon \mathbb{B}(\ell^2\Gamma) \xrightarrow{\cong} \mathbb{M}_n(\mathcal{B}(\ell^2(\Lambda)))$. It is routine to check that π restricts to a trace-preserving inclusion of $\mathbb{C}\Gamma$ into $\mathbb{M}_{[\Gamma:\Lambda]}(\mathbb{C}\Lambda)$ which automatically implies the corresponding results for the reduced C^*-algebras and von Neumann algebras. Finally, for $h \in \Lambda$ we have

$$\pi(h) = \operatorname{diag}(\lambda(h), \dots, \lambda(h)) \in \bigoplus_{i=1}^{[\Gamma:\Lambda]} \mathbb{B}(\ell^2(g_i^{-1}\Lambda)),$$

and thus formula (3.1) follows in view of the identity

$$hg_i^{-1}h' = g_i^{-1}({}^{g_i}h)h', \quad h, h' \in \Lambda. \qquad \square$$

Theorem 3.6. *Let* Λ *be a discrete group satisfying the strong Atiyah conjecture and let* $\Lambda \trianglelefteq \Gamma$ *be a finite index inclusion of* Λ *into a group* Γ *as a normal subgroup. If* $[\Gamma : \Lambda]^2 \cdot \sigma(\Lambda) < \theta(\Lambda)$ *then* $\mathbb{C}\Gamma$ *is not* $\mathrm{C}^*_{\mathrm{r}}$*-unique.*

Proof. The assumption $[\Gamma : \Lambda]^2 \cdot \sigma(\Lambda) < \theta(\Lambda)$ forces Λ to be non-icc and applying Lemma 3.4 we get a projection $p \in Z(L\Lambda)$ with $\tau(p) < \frac{\theta(\Lambda)}{[\Gamma:\Lambda]^2}$ and a non-zero central element $x \in \mathrm{C}^*_{\mathrm{r}}(\Lambda)$ with $xp^\perp = 0$. We are going to construct a representation of $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ which is injective on $\mathbb{C}\Gamma$ but with x in the kernel. To this end, consider a set of coset representatives $g_1, \dots, g_{[\Gamma:\Lambda]}$ for Γ/Λ and the $*$-homomorphism $\pi\colon L\Gamma \to \mathbb{M}_{[\Gamma:\Lambda]}(L\Lambda)$ provided by Lemma 3.5. From this we obtain a central projection $q := \bigvee_{i=1}^{[\Gamma:\Lambda]} {}^{g_i}p \in Z(L\Lambda)$, and cutting π with the complement of $\tilde{q} := \operatorname{diag}(q, \dots, q) \in Z(\mathbb{M}_{[\Gamma:\Lambda]}(L\Lambda))$, we get a representation

$$\pi_q\colon \mathrm{C}^*_{\mathrm{r}}(\Gamma) \to \mathbb{B}\left(\ell^2(\Lambda)^{[\Gamma:\Lambda]}\tilde{q}^\perp\right),$$

$$a \mapsto \pi(a)\tilde{q}^\perp.$$

As $q^\perp = \bigwedge_{i=1}^{[\Gamma:\Lambda]} {}^{g_i}(p^\perp)$ and $xp^\perp = 0$, it follows that $x \in \ker \pi_q$ in view of (3.1). Let $a \in \mathbb{C}\Gamma \cap \ker \pi_q$. This means that $\pi(a)\tilde{q}^\perp = 0$, and thus the kernel projection r of $\pi(a)$ satisfies $r \geqslant \tilde{q}^\perp$. Therefore

$$(\tau \otimes \mathrm{Tr})(r) \geqslant (\tau \otimes \mathrm{Tr})(\tilde{q}^\perp) \geq [\Gamma : \Lambda](1 - [\Gamma : \Lambda]\tau(p)) > [\Gamma : \Lambda] - \theta(\Lambda).$$

On the other hand, the assumption $[\Gamma : \Lambda]^2 \cdot \sigma(\Lambda) < \theta(\Lambda)$ forces an upper bound on the order of finite subgroups in Λ, i.e. $\theta(\Lambda) > 0$, and since Λ is furthermore assumed to satisfy the strong Atiyah conjecture we obtain (using the notation of Section 3.1) that

$$\dim_{L\Lambda}(\ker(L_A)) = (\tau \otimes \mathrm{Tr})(P_{\ker L_A}) \in \{n\theta(\Lambda) \mid n \in \mathbb{Z}\}$$

for any matrix $A \in \mathbb{M}_{[\Gamma:\Lambda]}(\mathbb{C}\Lambda)$. Thus $(\tau \otimes \mathrm{Tr})(r) \leqslant [\Gamma : \Lambda] - \theta(\Lambda)$ unless $\pi(a) = 0$. This proves that π_q is injective on $\mathbb{C}\Gamma$ and hence completes the proof. □

As a corollary, we deduce that some important families of groups are not $\mathrm{C}^*_{\mathrm{r}}$-unique. In particular this includes the groups mentioned in Theorem A (i) and (ii), and together with Proposition 2.4 this completes the proof of Theorem A.

Corollary 3.7. *All groups in following classes are not* $\mathrm{C}^*_{\mathrm{r}}$*-unique:*

(i) Torsion free, non-icc groups satisfying the strong Atiyah conjecture; in particular all elementary amenable, non-icc, torsion free groups.

(ii) Virtually polycyclic groups with infinite FC-centre; in particular, all infinite groups of polynomial growth.

Proof. To see (i), note that the existence of a non-trivial finite conjugacy class implies the existence of a non-trivial central element in $\mathbb{C}\Gamma$ (namely the sum of the elements in the finite conjugacy class) and hence a non-trivial projection in $Z(L\Gamma)$; thus $\sigma(\Gamma) < 1$. Moreover, since Γ is torsion free, $\theta(\Gamma) = 1$ and since Γ is assumed to satisfy the strong Atiyah conjecture it follows $\mathrm{C}^*_{\mathrm{r}}$-unique by Theorem 3.6. The last statement in (i) follows directly from this since the elementary amenable groups are contained in Linnell's class $\mathcal{C}$ (see Section 3.1) for which the strong Atiyah conjecture is known to hold in the presence of a bound on the order of finite subgroups [**Lüc02**, Theorem 10.19].
To see (ii), let $\Lambda \trianglelefteq \Gamma$ be a normal finite index polycyclic subgroup of Γ. As Γ has infinite FC-centre, so does Λ and the FC-centre of Λ is moreover finitely generated by polycyclicity. A classical result by Hirsch [**Hir46**, Theorem 3.21] implies that the orders of finite subgroups of Λ are bounded; thus $\theta(\Lambda) > 0$. On the other hand, $\sigma(\Lambda) = 0$ by Proposition 3.3 (ii). Moreover, polycyclic groups, being elementary amenable, satisfy the strong Atiyah conjecture. Thus, $\mathbb{C}\Gamma$ follows non-$\mathrm{C}^*_{\mathrm{r}}$ unique by Theorem 3.6.

Finally, the claim about infinite groups of polynomial growth follows by first observing that by Gromov's theorem [**Gro81**] these are exactly finitely generated virtually nilpotent groups. As finitely generated nilpotent groups are polycyclic, the claim follows once we argue that virtually nilpotent groups automatically have infinite FC-centre. To see this, recall that a finitely generated virtually nilpotent group Γ contains a finite index torsion free nilpotent normal subgroup Λ (by polycyclicity and [**Hir46**, Theorem 3.21]). Now it follows that the centre of Λ is infinite, and therefore so is the FC-centre Λ_{fc}; but as $\Lambda \trianglelefteq \Gamma$ is a finite index inclusion, $\Lambda_{\mathrm{fc}} \subseteq \Gamma_{\mathrm{fc}}$. Thus, Γ_{fc} is infinite. □

Addendum

During the meeting "C^*-algebras" in 2019 in Oberwolfach it was pointed out by Narutaka Ozawa that the original question of Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam [**GMRr16**, Question 6.8] actually has negative answer: the group ring of the lamplighter group $\Gamma = \mathbb{Z}/2 \wr \mathbb{Z}$ has unique C*-norm. We provide his argument here for the sake of completeness: setting $H = \bigoplus_{\mathbb{Z}} \mathbb{Z}/2$, we see that $C^*(\Gamma) \cong C^*(H) \rtimes \mathbb{Z}$ is a crossed product of the Bernoulli action which is topologically free, so by the Archbold–Spielberg theorem [**AS94**, Theorem 1] every ideal I in $C^*(\Gamma)$ intersects $C^*(H)$ nontrivially. But the group H is locally finite, so by C*-uniqueness I intersects its group ring nontrivially.

In view of this result, it would be of interest to exactly characterise amenable groups Γ which have a unique C*-norm on $\mathbb{C}\Gamma$. In particular, it would be interesting to know whether there is a torsion-free amenable group with a unique C*-norm on its complex group ring.

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CHAPTER 7

Invariant random positive definite functions

Abstract

We give the definition of an invariant random positive definite function on a discrete group, generalizing both the notion of an invariant random subgroup and a character. We use von Neumann algebras to show that all invariant random positive definite functions on groups with infinite conjugacy classes which integrate to the regular character are constant.

1. Introduction

In the last years there has been a lot of progress about invariant random subgroups (IRSes), which shifted the attention in the study of ergodic group actions from their orbit equivalence relations to their stabilizers [**AGV14**], [**AGN17**], [**Gel18**], [**ABB+11**],[**ABB+17**]. IRSes are a tool to study actions, but also behave similarly to normal subgroups.

We define a generalization of invariant random subgroups, which we call invariant random positive definite functions. An invariant random positive definite function (i.r.p.d.f.) is a measurable Γ-equivariant map

$$\varphi\colon \Omega \to \mathrm{PD}(\Gamma),$$

where (Ω, μ) is a standard probabilitiy space with a measure preserving Γ-action, and $\mathrm{PD}(\Gamma)$ are the normalized positive definite functions ϕ on Γ with Γ-action given by $(g.\phi)(h) = \phi(g^{-1}hg)$ for $\phi \in \mathrm{PD}(\Gamma)$ and $g, h \in \Gamma$. This specializes to the definition of an IRS if we demand each $\varphi(\omega)$ to be the characteristic function of the stibilizer subgroup of ω.

The definition of an i.r.p.d.f. is also closely related to the notion of a character on Γ, i.e. a conjugation invariant normalized positive definite function. Indeed, if φ is an i.r.p.d.f.,

$$\mathbb{E}[\varphi] := \int_\Omega \varphi(\omega)\, d\omega$$

is a character.

A construction of Anatoly Vershik shows that in the case of $\Gamma = S_\infty$ every extremal character, except for the regular, the trivial and the alternating character, is of this form for a non-constant i.r.p.d.f. φ [**VK81**]. Some of these i.r.p.d.f.'s are IRSes, some are "twisted IRSes" arising from cocyles of the action.

Our main result is the following theorem. We call this phenomenon "disintegration rigidity" of the regular character $\delta_e \in \mathrm{Ch}(\Gamma)$.

THEOREM 1.1 (Theorem 5.1). *Let Γ be a group where every nontrivial conjugacy class is infinite and let $\varphi\colon \Omega \to \mathrm{PD}(\Gamma)$ be an i.r.p.d.f. on Γ with $\mathbb{E}[\varphi] = \delta_e$. Then $\varphi(\omega) = \delta_e$ for almost every $\omega \in \Omega$.*

Γ having infinite conjugacy classes is equivalent to $\delta_e \in \mathrm{Ch}(\Gamma)$ being an extremal character, hence the theorem states disintegration rigidity of δ_e in all cases where it has a chance to be disintegration rigid.

The main step in the proof of this theorem is to translate a given ergodic i.r.p.d.f. φ with $\mathbb{E}[\varphi] = \delta_e$ into a random variable $f\colon \Omega \to L^1(L\Gamma)$ which fulfills the invariance condition $f(\gamma.\omega) = \pi(\gamma^{-1})f(\omega)\pi(\gamma)$. We then show that such a function must be constantly 1, using that the conjugation action of Γ on $L\Gamma$ is weakly mixing. Then φ also must be constant. This method might also apply to other characters than the regular one.

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2. Preliminaries

As invariant random positive definite functions generalize both characters and invariant random subgroups we first collect some information about these.

2.1. Characters on discrete groups. Let Γ be a discrete, countable group.

DEFINITION 2.1. A function $\phi\colon \Gamma \to \mathbb{C}$ is called *positive definite* if for all $g_1, \ldots, g_n \in \Gamma$ the matrix $[\phi(g_j^{-1}g_i)] \in \mathbb{M}_n(\mathbb{C})$ is positive or, equivalently, if ϕ induces a state on $\mathbb{C}\Gamma$.

DEFINITION 2.2. A *character* $\tau \in \mathrm{Ch}(\Gamma)$ is a conjugation-invariant positive definite function on Γ normalized by $\tau(e) = 1$. A character is called *extremal* if it is not a non-trivial convex combination of two different characters.

The characters of a given group Γ form a Choquet simplex, i.e. every character can be uniquely decomposed as a convex combination of extremal ones [**Tho64b**].

If ϕ is a positive definite function, then $\langle g, h\rangle = \phi(h^*g)$ for $g, h \in \Gamma$ extends to a prescalar product on $\mathbb{C}\Gamma$. Let H be the separated completion and denote the image of δ_g in H again by δ_g. Then $\pi(g)\colon \delta_h \mapsto \delta_{gh}$ extends uniquely to a unitary operator $\pi(g) \in U(H)$. We get a unitary representation $\pi\colon \Gamma \to U(H)$ such that $\delta_e \in H$ is cyclic and

$$\phi(g) = \langle \pi(g)\delta_e, \delta_e \rangle$$

for all $g \in \Gamma$. The triple (H, π, δ_e) is unique with these properties up to a unitary. This is called the *GNS construction* of ϕ. Sometimes we will also call the von Neumann algebra $\pi(\Gamma)'' \subset B(H)$ the GNS construction of ϕ.

If $\phi = \tau$ is a character, its GNS construction is a finite von Neumann algebra with trace extending the character. We denote this trace again by τ and get $L^2(\pi(\Gamma)'', \tau) = H$. In this case we also have a unitary right representation

$$\rho\colon \Gamma \to U(H), \quad \rho(g)\colon \delta_h \mapsto \delta_{hg^{-1}}.$$

Restricted to $\pi(\Gamma)'' \subset L^2(\pi(\Gamma)'', \tau)$, the maps $\pi(g)$ and $\rho(g)$ correspond to $x \mapsto \pi(g)x$ and $x \mapsto x\pi(g^{-1})$ when x is viewed as an operator $x \in B(H)$. In particular,

$$\Gamma \to \mathrm{Aut}(\pi(\Gamma)''), \quad g \mapsto (x \mapsto \pi(g)x\pi(g^{-1})),$$

is a trace-preserving action.

In the case of the regular character δ_e we get the group von Neumann algebra $L\Gamma$ as GNS construction.

By [**Tho64b**], a character is extremal if and only if its von Neumann algebra $\pi(\Gamma)'' \subset B(H)$ is a factor.

Definition 2.3. The type of a character is the type I of its von Neumann algebra (e.g. I, II_1 etc.).

Since the GNS construction of a character is finite, an extremal character can only be of type I_n or II_1.

2.2. Invariant random subgroups. The name "invariant random subgroup" is due to [**AGV14**]. However, the concept is much older and was, for example, studied by Vershik in the 80s and by Stuck-Zimmer in the 90s.

Definition 2.4. An invariant random subgroup (IRS) is a map given by

$$\varphi\colon \Omega \to \mathrm{Sub}(\Gamma), \quad \omega \mapsto \mathrm{Stab}(\omega) = \{\gamma \in \Gamma \mid \gamma.\omega = \omega\},$$

for a measure preserving action $\Gamma \curvearrowright (\Omega, \mu)$ on a standard probability space.

In fact, invariant random subgroups were originally defined as conjugation invariant measures on $\mathrm{Sub}(\Gamma)$. One can show that this is equivalent to the above definition [**AGV14**, Proposition 13]. We use this formulation because it will fit with our definition of invariant random positive definite functions and makes our notation easier.

If $\varphi: \Omega \to \mathrm{PD}(\Gamma)$ is an IRS,

$$\mathbb{E}[\varphi]: \gamma \mapsto \mu(\{\omega \mid \gamma.\omega = \omega\})$$

is a character.

EXAMPLE 2.5. Let $\Omega = \{1, \dots, n\}$, let μ be the normalized counting measure and $\Gamma = S_n$ the symmetric group. Then $\varphi(i) = \{\sigma \mid \sigma(i) = i\}$ for $i \in \Omega$ is an IRS where $\mathbb{E}[\varphi] = \mathrm{tr}$ is the normalized trace on matrices. The trace tr is not an extremal character on S_n.

The following two theorems show that for $\Gamma = S_\infty$, many characters arise in this way.

THEOREM 2.6 ([**Tho64a**]). *Every extremal character on S_∞ is of the form*

$$\tau_{\alpha,\beta}(g) = \prod_{k \geq 2} s_k^{r_k(g)},$$

where $r_k(g)$ is the number of cycles of length k in g, $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ and $\beta = (\beta_n)_{n\in\mathbb{N}}$ are sequences with $\alpha_n \geq \alpha_{n+1} \geq 0$ and $\beta_n \geq \beta_{n+1} \geq 0$ for all $n \in \mathbb{N}$ and such that

$$\sum_{n\in\mathbb{N}} \alpha_n + \sum_{n\in\mathbb{N}} \beta_n \leq 1$$

and the s_k are given by

$$s_k := \sum_{n\in\mathbb{N}} \alpha_n^k + (-1)^{k+1} \sum_{n\in\mathbb{N}} \beta_n^k.$$

All such $\tau_{\alpha,\beta}$ are extremal characters and $\tau_{\alpha,\beta} = \tau_{\alpha',\beta'}$ implies $\alpha = \alpha'$ and $\beta = \beta'$.

All extremal characters on S_∞ exept for the trivial character and the alternating character are of type II.

REMARK 2.7. In the theorem the trivial character belongs to $\alpha = (1, 0, 0, \dots)$ and $\beta = 0$, the alternating character belongs to $\alpha = 0$ and $\beta = (1, 0, 0, \dots)$ and the regular character belongs to $\alpha = \beta = 0$.

THEOREM 2.8 ([**VK81**]). *Using the notation of Theorem 2.6, assume $\beta = 0$, let*

$$\delta = 1 - \sum_{n\in\mathbb{N}} \alpha_n$$

and let $\Omega = \mathbb{N} \sqcup [0, \delta]$ with probability measure μ which is $(\alpha_n)_{n\in\mathbb{N}}$ on $\mathbb{N}$ and the Lebesgue measure on $[0, \delta]$. Let $\Omega = \prod_1^\infty \Omega$ with measure $m_{\alpha,0} = \prod_1^\infty \mu$ and let S_∞ act on $(\Omega, m_{\alpha,0})$ by permutation of the coordinates.

Then $\tau_{\alpha,0} = \mathbb{E}[\varphi]$ for this IRS φ.

3. Invariant random positive definite functions

DEFINITION 3.1. Let Γ be a discrete group. An invariant random positive definite function (i.r.p.d.f.) is a measurable Γ-equivariant map

$$\varphi: \Omega \to \mathrm{PD}(\Gamma),$$

where (Ω, μ) is a standard probabilitiy space with a measure preserving Γ-action and $\mathrm{PD}(\Gamma)$ are the positive definite functions ϕ on Γ with $\phi(e) = 1$ and Γ-action given by $(g.\phi)(h) = \phi(g^{-1}hg)$ for $\phi \in \mathrm{PD}(\Gamma)$.

We often write φ_ω for $\varphi(\omega)$.

Definition 3.2. An i.r.p.d.f. φ is called *ergodic* if the action $\Gamma \curvearrowright (\Omega, \mu)$ is ergodic.

We say φ is *extremal* if $\varphi = c\varphi_1 + (1-c)\varphi_2$ for i.r.p.d.f.'s $\varphi_i \colon \Omega \to \mathrm{PD}(\Gamma)$ and $c \in (0,1)$ implies that $\varphi_1 = \varphi_2 = \varphi$.

When viewing the i.r.p.d.f.'s with given $\Gamma \curvearrowright \Omega$ as Γ-equivariant positive definite functions $\varphi \colon \Gamma \to L^\infty(\Omega, \mu)$, they form a compact convex subset of $\ell^\infty(\Gamma, L^\infty(\Omega, \mu))$ with the topology of pointwise weak* convergence. By the Kreĭn–Milman Theorem, the space of these functions is then equal to the closed convex hull of its extremal points. Hence as for characters, every i.r.p.d.f. is the convex integral of extremal i.r.p.d.f.'s.

Example 3.3. Invariant random subgroups are i.r.p.d.f.'s because the subgroups $\mathrm{Sub}(\Gamma)$ of Γ are canonically embedded in $\mathrm{PD}(\Gamma)$ by taking the characteristic function and the stabilizers of an action fulfill the invariance condition in Definition 3.1.

As for invariant random subgroups, if $\varphi \colon \Omega \to \mathrm{PD}(\Gamma)$ is an i.r.p.d.f.,

$$\mathbb{E}[\varphi] = \int_\Omega \varphi_\omega \, d\mu(\omega)$$

is a character.

Question 3.4. Does ergodicity and extremality of φ imply that $\mathbb{E}[\varphi]$ is extremal as a character?

A positive answer to this question would mean that it every i.r.p.d.f. can be decomposed into i.r.p.d.f.'s with an extremal character as expectation.

Example 3.5. Let (S, λ) be the unit sphere in $\mathbb{C}^n$ with Lebesgue measure and let Γ be a discrete subgroup of the unitary group $U(n)$ acting on S in the natural way. Then

$$\varphi \colon S \to \mathrm{PD}(\Gamma), \quad \varphi_\xi(\gamma) = \langle \gamma.\xi, \xi \rangle \quad \forall \xi \in S, \gamma \in \Gamma$$

is an i.r.p.d.f. for which $\mathbb{E}[\varphi] = \mathrm{tr}$ is the normalized trace on matrices, which is an extremal character on Γ iff Γ generates $\mathbb{M}_n(\mathbb{C})$ as an algebra. For such Γ, φ is an extremal i.r.p.d.f..

Example 3.6. Let (S^1, λ) be the circle with Lebesgue measure and trivial action of $\mathbb{Z}$. Then

$$\varphi \colon S^1 \to \mathrm{PD}(\mathbb{Z}), \quad \varphi_z(n) = z^n$$

is an i.r.p.d.f. with $\mathbb{E}[\varphi] = \delta_e$. Here δ_e is not extremal and φ_z is an extremal character for every $z \in S^1$. In this way every decomposition of a non-extremal character into extremal ones gives an i.r.p.d.f. with trivial action.

EXAMPLE 3.7. Let G be a compact group with Haar measure μ and $\Gamma < G$. Let Γ act on G by left multiplication. Let $\pi\colon G \to U(H)$ be a unitary representation and $\xi \in H$ a unit vector. Then

$$\varphi^{\xi}\colon (G,\mu) \to \mathrm{PD}(\Gamma), \quad \varphi^{\xi}_g(h) = \langle \pi(hg)\xi, \pi(g)\xi\rangle$$

is an i.r.p.d.f.. If $\pi\colon G \to U(\mathbb{C}^n)$ is irreducible and Γ is dense, then $\mathbb{E}[\varphi^{\xi}](\gamma) = \mathrm{tr}(\pi(\gamma))$, which is an extremal character on Γ, and φ^{ξ} is ergodic and extremal.

Example 3.7 shows that, in contrast to the situation for characters, the decomposition of an i.r.p.d.f. into extremal i.r.p.d.f.'s is not unique: Take an irreducible representation $\pi\colon G \to U(\mathbb{C}^n)$, an orthonormal basis (ξ_i) of $\mathbb{C}^n$ and $\Gamma < G$ dense. Then

$$\sum_{i=1}^{n} \frac{1}{n}\varphi^{\xi_i} \equiv \mathrm{tr}\circ\pi.$$

For different bases we get different φ^{ξ_i}'s, so this gives different convex decompositions of the constant i.r.p.d.f. $\mathrm{tr}\circ\pi$ into extremal i.r.p.d.f.'s..

THEOREM 3.8 ([**VK81**], Theorem 3). *In the notation of Theorem* 2.6*, let*

$$\delta = 1 - \sum_{n\in\mathbb{N}} \alpha_n - \sum_{n\in\mathbb{N}} \beta_n,$$

$\mathbb{N}_+ = \mathbb{N}_- = \mathbb{N}$ *and* $\mathcal{Q} = \mathbb{N}_+ \sqcup \mathbb{N}_- \sqcup [0,\delta]$ *with the probability measure* μ *which is* $(\alpha_n)_{n\in\mathbb{N}}$ *on* $\mathbb{N}_+$*,* $(\beta_n)_{n\in\mathbb{N}}$ *on* $\mathbb{N}_-$ *and the Lebesgue measure on* $[0,\delta]$*. Then let* $\Omega = \prod_1^\infty \mathcal{Q}$ *with the measure* $m_{\alpha,\beta} = \prod_1^\infty \mu$ *and let* S_∞ *act on* $(\Omega, m_{\alpha,\beta})$ *by permutation of the coordinates.*

For $g \in S_\infty$ *and* $\omega \in \Omega$ *define* $\mathrm{sgn}(g,\omega)$ *to be* 1 *if*

$$\prod_{(i,j)\colon \omega_i,\omega_j\in\mathbb{N}_-,\, i<j} (g(j)-g(i))$$

is positive and -1 *otherwise. This fulfills the cocycle identity*

$$\mathrm{sgn}(gh,\omega) = \mathrm{sgn}(h,\omega)\,\mathrm{sgn}(g,h.\omega). \tag{3.1}$$

Let

$$\varphi_\omega(g) = \begin{cases} \mathrm{sgn}(g,\omega) & \text{if } g.\omega = \omega,\\ 0 & \text{if } g.\omega \neq \omega.\end{cases}$$

Then $\tau_{\alpha,\beta} = \mathbb{E}[\varphi]$*.*

The following theorem proves that the above φ is an i.r.p.d.f.. If β is non-trivial, then φ is not an IRS.

THEOREM 3.9. *Let* $\Gamma \curvearrowright (\Omega,\mu)$ *be a p.m.p. action and* $c\colon \Gamma\times\Omega \to S^1$ *a cocycle as in* (3.1)*. Then*

$$\varphi_\omega(g) = \begin{cases} c(g,\omega) & \text{if } g.\omega = \omega,\\ 0 & \text{if } g.\omega \neq \omega,\end{cases}$$

is an i.r.p.d.f..

If c is not constantly 1, φ is not an IRS because it takes values outside $\{0, 1\}$.

PROOF. To show that φ is invariant we need that $c(g, h\omega) = c(h^{-1}gh, \omega)$ if $h^{-1}gh.\omega = \omega$. By the cocycle identity we have

$$1 = c(1, \omega) = c(h^{-1}h, h^{-1}gh.\omega) = c(h, \omega)c(h^{-1}, gh.\omega)$$

and hence

$$c(h^{-1}gh, \omega) = c(h, \omega)c(h^{-1}g, h.\omega) = c(h, \omega)c(g, h.\omega)c(h^{-1}, gh.\omega) = c(g, h.\omega).$$

Now we show that φ_ω is positive definite for a.e. $\omega \in \Omega$. Let $\mathcal{R} \subset \Omega \times \Omega$ be the orbit equivalence relation of $\Gamma \curvearrowright (\Omega, \mu)$, equipped with the measure $\mu_\mathcal{R}$ which is μ on Ω and the counting measure in each fiber, i.e., for $A \subset \mathcal{R}$ measurable

$$\mu_\mathcal{R}(A) := \int_\Omega |\{(x, y) \in A\}| dx.$$

Then $\pi \colon \Gamma \to U(L^2(\mathcal{R}))$, given by

$$(\pi(g)\xi)(x, y) = c(g, x)\xi(g.x, y)$$

is a unitary representation and for every $X \subset \Omega$ we find a vector $\xi_X = \chi_{\{(x,x)|x \in X\}} \in L^2(\mathcal{R})$ such that

$$\int_X \varphi_\omega(g) = \langle \pi(g)\xi_X, \xi_X \rangle .$$

Hence for every $a \in \mathbb{C}\Gamma$ we have

$$\int_X \varphi_\omega(a^*a) \geq 0$$

for all $X \subset \Omega$ and hence $\varphi_\omega(a^*a) \geq 0$ almost everywhere. □

Up to now, all our examples of i.r.p.d.f.'s which integrate to a type II character are of this form. In particular, they are supported on an IRS in the sense that $\varphi_\omega(\gamma) = 0$ if $\gamma.\omega \neq \omega$. This leads to the following questions.

QUESTION 3.10. Is every i.r.p.d.f. φ such that $\mathbb{E}[\varphi]$ is of type II_1 supported on an IRS ?

QUESTION 3.11. Is every i.r.p.d.f. which is supported on an IRS as in Theorem 3.9 ?

4. Connections to von Neumann algebras

In this section we translate i.r.p.d.f.'s into the language of von Neumann algebras in order to be able to use von Neumann methods to study them in the next section. For the relevant theory of von Neumann algebras see [**Bla06**],[**ADP**],[**Hou**].

Fix a discrete group Γ, a character $\tau \in \mathrm{Ch}(\Gamma)$ and an ergodic, measure preserving action $\alpha \colon \Gamma \curvearrowright (\Omega, \mu)$ on a standard probability space. Let $A := L^\infty(\Omega, \mu)$ and write again α for the corresponding action on A. Let $\pi \colon \Gamma \to U(H)$ be the GNS representation of τ.

LEMMA 4.1. *Let φ be an i.r.p.d.f. with $\mathbb{E}[\varphi] = \tau$ and for each $\omega \in \Omega$ let $(\pi_\omega, H_\omega, \xi_\omega)$ be the GNS construction of φ_ω. Let*

$$H_\varphi := \int_\Omega^\oplus H_\omega \, d\mu(\omega)$$

be the direct integral of Hilbert spaces, $\xi = (\xi_\omega)_{\omega \in \Omega} \in H_\varphi$ and

$$\pi_\varphi = \int_\Omega^\oplus \pi_\omega \, d\mu(\omega) \colon \Gamma \to B(H_\varphi)$$

the direct integral of representations. Then $\pi_\varphi(\Gamma)'' \cong \pi(\Gamma)''$ with isomorphism taking $\pi_\varphi(\gamma)$ to $\pi(\gamma)$ for all $\gamma \in \Gamma$.

PROOF. Let $p \in B(H_\varphi)$ be the orthogonal projection onto the cyclic representation of ξ. Then $p \in \pi_\varphi(\Gamma)'$. As $\mathbb{E}[\varphi] = \tau$ we have

$$\langle \pi_\varphi(\gamma)\xi, \xi \rangle = \int_\Omega \langle \pi_\omega(\gamma)\xi_\omega, \xi_\omega \rangle \, d\mu(\omega) = \int_\Omega \varphi_\omega(\gamma) \, d\mu(\omega) = \tau(\gamma)$$

for all $\gamma \in \Gamma$. So $(p(H_\varphi), \pi_\varphi, \xi)$ is a GNS triple for τ and therefore by uniqueness of the GNS construction

$$\pi(\Gamma)'' \cong (p\, \pi_\varphi(\Gamma)\, p)'' = p(\pi_\varphi(\Gamma))''$$

with isomorphism taking $\pi(\gamma)$ to $p\pi_\varphi(\gamma)$ for all $\gamma \in \Gamma$. Now we show that

$$\Phi \colon (\pi_\varphi(\Gamma))'' \to p(\pi_\varphi(\Gamma))'', \ x \mapsto px,$$

is an isomorphism. It is clearly a surjective homomorphism. For injectivity let $x \in (\pi_\varphi(\Gamma))''$ with $\Phi(x^*x) = px^*xp = 0$. Then for all $a \in \mathbb{C}\Gamma$ we have

$$0 = \langle x^*x\pi_\varphi(a)\xi, \pi_\varphi(a)\xi \rangle = \int_\Omega \langle (x^*x)_\omega \pi_\omega(a)\xi_\omega, \pi_\omega(a)\xi_\omega \rangle \, d\mu(\omega)$$

and therefore $\langle (x^*x)_\omega \pi_\omega(a)\xi_\omega, \pi_\omega(a)\xi_\omega \rangle = 0$ for almost all $\omega \in \Omega$. But $\pi_\omega(\mathbb{C}\Gamma)\xi_\omega$ is dense in H_ω, so $(x^*x)_\omega = 0$ for almost all ω. Hence $x = 0$ and Φ is injective.

Composing the two isomorphisms we get $\pi(\Gamma)'' \cong p\pi_\varphi(\Gamma)'' \cong \pi_\varphi(\Gamma)''$ with isomorphism mapping $\pi(\gamma)$ to $\pi_\varphi(\gamma)$. □

LEMMA 4.2. *Let $M := (A \cup \pi_\varphi(\Gamma))'' = \int_\Omega^\oplus \pi_\omega(\Gamma)'' d\mu(\omega)$. Then M is a finite von Neumann algebra.*

PROOF. Let $u \in M$ be such that $u^*u = 1$. By the Kaplansky Density Theorem we find a sequence of finite sums

$$t_n = \sum_i p_{n,i} x_{n,i}$$

converging to u in the strong* topology such that $\|t_n\| \leq 1$ for all n, $p_{n,i} \in A$ are mutually orthogonal projections for fixed n and $x_{n,i} \in \pi_\varphi(\Gamma)''$. We then have $t_n^* t_n \xrightarrow{s^*} 1$ since the strong* topology is jointly continuous on bounded sets. Hence $|t_n| \xrightarrow{s^*} 1$ by [**Tak02**, Lemma II.4.6].

Letting

$$f(t) := \begin{cases} 1-2t, & 0 \leqslant t \leqslant 1/2, \\ 0, & 1/2 \leqslant t \leqslant 1, \end{cases}$$

we obtain (again by [**Tak02**, Lemma II.4.6]) $f(|t_n|) \xrightarrow{s^*} 0$, and therefore $|t_n|+f(|t_n|) \xrightarrow{s^*} 1$. However, as $1/2 \leqslant t + f(t) \leqslant 1$ on $[0,1]$, we also have $1/2 \leqslant |t_n|+f(|t_n|) \leqslant 1$.

Let $t_n = u_n|t_n|$ be the polar decomposition of t_n. Then we have

$$u_n(|t_n|+f(|t_n|)) \xrightarrow{s^*} u$$

because $f(|t_n|) \xrightarrow{s^*} 0$. On the other hand, $|t_n|+f(|t_n|)$ is invertible with the inverse bounded by 2 and $(|t_n|+f(|t_n|))^{-1} \xrightarrow{s^*} 1$ again by [**Tak02**, Lemma II.4.6]. Therefore,

$$u_n = u_n(|t_n|+f(|t_n|))(|t_n|+f(|t_n|))^{-1} \xrightarrow{s^*} u. \tag{4.1}$$

Let $x_{n,i} = v_{n,i}|x_{n,i}|$ be the polar decomposition of $x_{n,i}$. Then

$$u_n = \sum_i p_{n,i} v_{n,i} \tag{4.2}$$

because using that A commutes with $\pi_\varphi(\Gamma)''$ and that the $p_{n,i}$ are mutually orthogonal we get that

$$|t_n| = \sum_i p_{n,i}|x_{n,i}|,$$

and hence

$$\left(\sum_i p_{n,i} v_{n,i}\right)|t_n| = \sum_i p_{n,i} v_{n,i}|x_{n,i}| = \sum_i p_{n,i} x_{n,i} = t_n.$$

Now (4.2) and (4.1) imply that

$$u_n^* u_n = \sum_i p_{n,i} v_{n,i}^* v_{n,i} \xrightarrow{s^*} 1,$$

and therefore

$$\sum_i p_{n,i} \xrightarrow{s^*} 1. \tag{4.3}$$

Since $\pi_\varphi(\Gamma)'' \cong \pi(\Gamma)''$ is finite, there exist partial isometries $w_{n,i} \in \pi_\varphi(\Gamma)''$ such that $u_{n,i} = v_{n,i} + w_{n,i}$ are unitaries. Let $q_{n,i} := w_{n,i}^* w_{n,i}$ be the source projections of the $w_{n,i}$. Then

$$\sum_i p_{n,i} q_{n,i} = \sum_i p_{n,i}(1 - v_{n,i}^* v_{n,i}) \leq 1 - \sum_i p_{n,i} v_{n,i}^* v_{n,i} = 1 - u_n^* u_n \xrightarrow{s^*} 0,$$

and therefore

$$\sum_i p_{n,i} w_{n,i} = \left(\sum_i p_{n,i} w_{n,i}\right)\left(\sum_i p_{n,i} q_{n,i}\right) \xrightarrow{s^*} 0.$$

Thus by (4.2)

$$\sum_i p_{n,i}u_{n,i} = u_n + \sum_i p_{n,i}w_{n,i} \xrightarrow{s^*} u,$$

and therefore, since the $u_{n,i}$ are unitaries,

$$\sum_i p_{n,i} = \sum_i p_{n,i}u_{n,i}u_{n,i}^* \xrightarrow{s^*} uu^*.$$

Hence $uu^* = 1$ by (4.3), which means that M is finite. □

LEMMA 4.3. *If τ is extremal, we have $M \cong A\overline{\otimes}\pi_\varphi(\Gamma)''$ with isomorphism taking $xa \in M$ to $a \otimes x \in A\overline{\otimes}\pi_\varphi(\Gamma)''$ for all $a \in A$ and $x \in \pi_\varphi(\Gamma)''$.*

PROOF. Since M is finite by the previous lemma, there exists a normal faithful conditional expectation $E\colon M \to \pi_\varphi(\Gamma)''$. Since $\pi_\varphi(\Gamma)''$ and A commute and E is $\pi_\varphi(\Gamma)''$-linear,

$$E(a) = E(\pi_\varphi(\gamma)a\pi_\varphi(\gamma^{-1})) = \pi_\varphi(\gamma)E(a)\pi_\varphi(\gamma^{-1})$$

for all $\gamma \in \Gamma$ and $a \in A$. Thus, $E(A)$ is contained in the center of $\pi_\varphi(\Gamma)'' \cong \pi(\Gamma)''$, which is equal to $\mathbb{C}$ since τ is extremal. Now the claim follows from [**Str81**, Theorem 9.12]. □

On M resp. $L^1(M)$ we define a Γ-action θ by

$$\theta_\gamma(a \otimes m) = \alpha_\gamma(a) \otimes \pi(\gamma)m\pi(\gamma^{-1}).$$

By M^θ resp. $L^1(M)^\theta$ we denote the elements that are invariant under θ.

PROPOSITION 4.4. *Given an ergodic action and an extremal character $\tau \in \mathrm{Ch}(\Gamma)$ there is a one-to-one correspondence between i.r.p.d.f.'s $\varphi\colon \Omega \to \mathrm{PD}(\Gamma)$ with $\mathbb{E}[\varphi] = \tau$ and positive selfadjoint elements $f \in L^1(M)^\theta$ with $\int_\Omega f_\omega \, d\mu(\omega) = 1$ such that*

$$\varphi_\omega(\gamma) = \tau(\pi(\gamma)f_\omega).$$

PROOF. By Lemma 4.1 and Lemma 4.3, we have $\pi(\Gamma)'' \cong \pi_\omega(\Gamma)''$ for a.e. $\omega \in \Omega$ with the canonical isomorphism sending $\pi(\gamma)$ to $\pi_\omega(\gamma)$ for each $\gamma \in \Gamma$. As $\varphi_\omega(\gamma) = \langle \pi_\omega(\gamma)\xi_\omega, \xi_\omega\rangle$, we can extend it to

$$\varphi_\omega\colon \pi_\omega(\Gamma)'' \to \mathbb{C},\ x \mapsto \langle x\xi_\omega, \xi_\omega\rangle\,,$$

which is a positive normal functional on $\pi_\omega(\Gamma)''$ and therefore on $\pi(\Gamma)''$. So by [**Tak03**, Lemma IX.2.12] there exists a unique positive element $f_\omega \in L^1(\pi(\Gamma)'', \tau)$ such that $\varphi_\omega(x) = \tau(xf_\omega)$ for all $x \in \pi(\Gamma)''$. Let $f\colon \Omega \to L^1(\pi(\Gamma)'')$, $\omega \mapsto f_\omega$. To see that f is θ-invariant, we calculate

$$\tau(\pi(\gamma)f_{\alpha_{\gamma'}(\omega)}) = \varphi_{\alpha_{\gamma'}(\omega)}(\gamma) = \varphi({\gamma'}^{-1}\gamma\gamma') = \tau(\pi(\gamma)\pi(\gamma')f_\omega\pi({\gamma'}^{-1})),$$

so $\alpha_{\gamma'}^{-1}(f)_\omega = f_{\alpha_{\gamma'}(\omega)} = \pi(\gamma')f_\omega\pi({\gamma'}^{-1})$ for all $\gamma' \in \Gamma$ by uniqueness of f, hence $\theta(f) = f$. It follows that $\|f_\omega\|_1$ is Γ-invariant and hence constant, so $f \in L^1(M)^\theta$. We have for all $\gamma \in \Gamma$

$$\tau\left(\pi(\gamma)\int f_\omega \, d\mu(\omega)\right) = \int \tau\left(\pi(\gamma)f_\omega\right)\, d\mu(\omega) = \int \varphi_\omega(\gamma)\, d\mu(\omega) = \tau(\pi(\gamma)),$$

hence $\int f_\omega \, d\mu(\omega) = 1$. By [**Lüc02**, Lemma 8.3 (3)], f is a selfadjoint operator.

Conversely it is easy to check that such an f defines an i.r.p.d.f. φ with $\mathbb{E}(\varphi) = \tau$ by $\varphi_\omega(\gamma) = \tau(\pi(\gamma) f_\omega)$. $\square$

Remark 4.5.

(i) If φ is as in Example 3.5 with Γ big enough so that φ is extremal, we have $f \colon S \to \mathbb{M}_n(\mathbb{C})$ with f_ξ the orthogonal projection on $\operatorname{span}(\xi)$.
(ii) Similarly, if Γ in Example 3.7 is dense and π irreducible, we find $f \colon G \to \mathbb{M}_n(\mathbb{C})$ where f_g is the orthogonal projection on $\operatorname{span}(\pi(g)\xi)$.
(iii) The i.r.p.d.f. in Example 3.6 is not of the form as in Proposition 4.4. Hence the ergodicity and extremality assumptions are necessary (or at least one of them is).

Lemma 4.6. *In fact, for $f \in L^1(M)^\theta$ as in Proposition 4.4 the condition that $\int_\Omega f_\omega \, d\mu(\omega) = 1$ is equivalent to $\tau_M(f) = 1$, where $\tau_M = \int_\Omega \otimes \tau$ is the trace on M.*

Proof. Let f be constructed from φ as above. Then

$$\tau_M(f) = \int_\Omega \tau(f_\omega) \, d\mu(\omega) = \int_\Omega \varphi_\omega(e) \, d\mu(\omega) = \int_\Omega 1 \, d\mu(\omega) = 1.$$

For the other direction let first $p \in M^\theta$ be a projection. Then

$$\tau(\gamma) = \tau\left(\pi(\gamma) \int p_\omega \, d\mu(\omega)\right) + \tau\left(\pi(\gamma) \int (1-p)_\omega \, d\mu(\omega)\right)$$

is a convex decomposition into two characters. So by extremality of τ,

$$\int p_\omega \, d\mu(\omega) = \tau_M(p) \cdot 1.$$

Now let $f \in L^1(M)^\theta$ be positive selfadjoint with $\tau_M(f) = 1$. Then it follows from the above and the spectral theorem for f that $\int f_\omega \, d\mu(\omega) = \tau_M(f) \cdot 1 = 1$. $\square$

Lemma 4.7. *For τ extremal and α ergodic the extremal i.r.p.d.f.'s φ given α and $\mathbb{E}[\varphi] = \tau$ correspond to minimal projections in M^θ. M^θ is a direct sum of matrix algebras.*

Proof. Let $\varphi \colon \Omega \to \mathrm{PD}(\Gamma)$ be an extremal i.r.p.d.f. and $f \in L^1(M)^\theta$ as in Proposition 4.4 such that $\tau(f_\omega \pi(\gamma)) = \varphi_\omega(\gamma)$ for a.e. $\omega \in \Omega$ and all $\gamma \in \Gamma$. Assume that f is not a scalar multiple of a projection. Then there is a $c \in \mathbb{R}^+$ such that

$$f^{<c} := \chi([0,c))f \quad \text{and} \quad f^{\geq c} := \chi([c,\infty))f,$$

are both nonzero with $\chi(I)$ denoting the spectral projection on I. These are again positive elements in M^θ hence $\tau_M(f^{<c})^{-1} f^{<c}$ and $\tau_M(f^{\geq c})^{-1} f^{\geq c}$ define two different i.r.p.d.f.'s $\varphi^{<c}$ and $\varphi^{\geq c}$ such that

$$\varphi = \tau_M(f^{<c})\varphi^{<c} + \tau_M(f^{\geq c})\varphi^{\geq c}$$

contradicting the extremality of φ. So $f = \tau_M(p)^{-1}p$ for some projection $p \in M^\theta$. If p is not minimal in M^θ, say $q < p$ and $q \in M^\theta$, then again q and $p - q$ define two i.r.p.d.f.'s such that a convex combination gives φ, which contradicts extremality.

Conversely every minimal projection $p \in M^\theta$ gives an extremal i.r.p.d.f. φ because if there was a decomposition $\varphi = c\varphi_1 + (1-c)\varphi_2$ for some $0 < c < 1$ and different i.r.p.d.f.'s φ_i, this would give different positive elements $f_1, f_2 \in M^\theta$ such that $\tau_M(p)^{-1}p = cf_1 + (1-c)f_2$, which is not possible for a minimal projection p.

Since the set of i.r.p.d.f.'s is the closed convex hull of its extremal points, every positive trace 1 element of M^θ is a convex integral of minimal projections. This means M^θ is generated by its minimal projections, hence it is of type I with no diffuse part, i.e., $\mathcal{Z}(M^\theta) = L^\infty(X, \mu)$ such that every point in X has positive mass. Since it is also finite, it follows that M^θ is a (maybe infinite) direct sum of matrix algebras. $\square$

REMARK 4.8. Let $\tau \in \mathrm{Ch}(\Gamma)$ be an extremal character, $\alpha\colon \Gamma \curvearrowright \Omega$ an ergodic action and θ corresponding to α and τ as in Proposition 4.4. Then, for i.r.p.d.f.'s associated to α, we have the following observations.

(i) As M^θ is a direct sum of matrix algebras every i.r.p.d.f. φ with $\mathbb{E}[\varphi] = \tau$ is a convex combination of countably many extremal ones.
(ii) $M^\theta = \mathbb{C}$ iff the constant i.r.p.d.f. τ is the only one with $\mathbb{E}[\varphi] = \tau$. It is also equivalent to the constant τ being an extremal i.r.p.d.f.. If this is true for all α, τ is disintegration rigid.
(iii) M^θ is abelian iff the decomposition of i.r.p.d.f.'s with $\mathbb{E}[\varphi] = \tau$ into extremal ones is unique.
(iv) M^θ is finite-dimensional iff every i.r.p.d.f. is a finite convex sum of extremal ones.

5. Disintegration rigidity of the regular character on i.c.c. groups

In this section we show the following theorem.

THEOREM 5.1. *Let Γ be a group with infinite conjugacy classes. Let $\varphi\colon \Omega \to \mathrm{PD}(\Gamma)$ be an i.r.p.d.f. on Γ with $\mathbb{E}[\varphi] = \delta_e$. Then $\varphi(\omega) = \delta_e$ for almost every $\omega \in \Omega$.*

DEFINITION 5.2. If the conclusion of the theorem holds, we say (Γ, δ_e) is *disintegration rigid.*

REMARK 5.3. Theorem 2.6, Remark 2.7 and Theorem 3.8 show that the regular character, the trivial character and the alternating character are the only disintegration rigid characters on S_∞. Indeed, if $S_\infty \curvearrowright (\Omega, m_{\alpha,\beta})$ is the action from Theorem 3.8 such that $\tau_{\alpha,\beta}$ is none of these three characters, we have $0 < \alpha_1 < 1$ or $0 < \beta_1 < 1$. Assume w.l.o.g. that $0 < \alpha_1 < 1$.

Then for every nontrivial $g \in S_\infty$ and $j \in \operatorname{supp}(g) = \{j |\, g(j) \neq j\}$

$$\begin{aligned}0 &< m_{\alpha,\beta}\left(\{\omega \in \Omega |\, \omega_i = 1 \in \mathbb{N}_+ \ \forall i \in \operatorname{supp}(g)\}\right)\\ &\leq m_{\alpha,\beta}\left(\{\omega \in \Omega |\, g.\omega = \omega\}\right)\\ &\leq 1 - m_{\alpha,\beta}\left(\{\omega \in \Omega |\, \omega_j = 1 \in \mathbb{N}_+,\ g.\omega_j \neq 1 \in \mathbb{N}_+\}\right) < 1.\end{aligned}$$

Hence the φ in Theorem 2.8 is non-constant with $\mathbb{E}[\varphi] = \tau_{\alpha,\beta}$.

The trivial and the alternating character are clearly disintegration rigid because every positive definite function takes values in the unit disk, and thus, if an i.r.p.d.f. intergrates to a character which takes values only on the boundary of the unit disk, the i.r.p.d.f. has to be constant.

Definition 5.4. A trace-preserving action on a finite von Neumann algebra $\Gamma \to \operatorname{Aut}(M)$ is called *weakly mixing* if $\mathbb{C} \cdot 1$ is the only finite-dimensional, Γ-invariant subspace in M.

The following lemma might be known to experts but we give a proof for the sake of completeness.

Lemma 5.5. *Let Γ be an i.c.c. group. Then the conjugation action on $L\Gamma$ is weakly mixing.*

Proof. Let $\Gamma = \{\gamma_j |\, j \in \mathbb{N}\}$ be an enumeration of Γ. Assume $H \subset L\Gamma \subset \ell^2(\Gamma)$ is an Γ-invariant, finite-dimensional subspace and let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis of H such that $\xi_1 \notin \mathbb{C}\delta_e$. Then for every $\varepsilon > 0$ there is a $K \in \mathbb{N}$ such that

$$\left\| \xi_j - \sum_{i=1}^{K} \langle \xi_j, \delta_{\gamma_i} \rangle\, \delta_{\gamma_i} \right\| < \varepsilon \quad \text{for all } j = 1, \ldots, n. \tag{5.1}$$

Let $F = \{\gamma_1, \ldots, \gamma_K\}$. Then by [**CSU16**, Proposition 3.4] there exists a $\gamma \in \Gamma$ such that

$$\gamma F \gamma^{-1} \cap F \subset \{e\}. \tag{5.2}$$

Let $H_F := \operatorname{span}(F)$ and P_F the orthogonal projection on H_F.

As $\{\gamma\xi_1\gamma^{-1}, \ldots, \gamma\xi_n\gamma^{-1}\}$ is again an orthonormal basis of H we have $c_j \in \mathbb{C}$ with $\sum_{j=1}^n |c_j|^2 = 1$ such that

$$\xi_1 = \sum_{j=1}^{n} c_j \gamma \xi_j \gamma^{-1} = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{K} \langle \xi_j, \delta_{\gamma_i} \rangle\, \delta_{\gamma\gamma_i\gamma^{-1}} + \sum_{i=K+1}^{\infty} \langle \xi_j, \delta_{\gamma_i} \rangle\, \delta_{\gamma\gamma_i\gamma^{-1}} \right).$$

We have $\sum_{i=1}^{K} \langle \xi_j, \delta_{\gamma_i} \rangle \delta_{\gamma\gamma_i\gamma^{-1}} \in H_F^{\perp} + \mathbb{C}\delta_e$ because of (5.2), which together with (5.1) implies

$$\begin{aligned} \|P_F(\xi_1)\| &\leq |\langle \xi_1, \delta_e \rangle| + \left\| P_F \left(\sum_{j=1}^{n} c_j \sum_{i=K+1}^{\infty} \langle \xi_j, \delta_{\gamma_i} \rangle \delta_{\gamma\gamma_i\gamma-1} \right) \right\| \\ &\leq |\langle \xi_1, \delta_e \rangle| + \varepsilon \sum_{j=1}^{n} |c_j| \\ &\leq |\langle \xi_1, \delta_e \rangle| + n\varepsilon. \end{aligned}$$

Since $\|P_F(\xi_1)\| > 1 - \varepsilon$ by (5.1), we get a contradiction when choosing $\varepsilon < n^{-1}(1 - |\langle \xi_1, \delta_e \rangle|)$. □

Definition 5.6. We call an extremal character *conjugation weakly mixing* if the conjugation action on its GNS construction is weakly mixing.

Question 5.7. Which other characters are conjugation weakly mixing?

The following statement contains Theorem 5.1 as a special case.

Theorem 5.8. *Let τ be a conjugation weakly mixing character on Γ. Then (Γ, τ) is disintegration rigid.*

Proof. We first assume that α is ergodic. An action on a finite von Neumann algebra $\sigma\colon \Gamma \curvearrowright N$ is weakly mixing if and only if for every action $\alpha\colon \Gamma \curvearrowright A$ on a finite von Neumann algebra one has $(A \overline{\otimes} N)^{(\alpha\otimes\sigma)} = A^{\alpha} \otimes 1$ [**Vae07**, Proposition D.2]. So if we take $A = L^{\infty}(\Omega)$ as in Section 4 and $N = \pi(\Gamma)''$, Lemma 5.5 implies that

$$M^{\theta} = (A \overline{\otimes} N)^{(\alpha\otimes\mathrm{conj}(\pi))} = A^{\alpha} = \mathbb{C}.$$

τ is extremal because if the conjugation action is weakly mixing, it must be ergodic, hence the GNS construction is a factor. Hence by Proposition 4.4 every i.r.p.d.f. φ with $\mathbb{E}[\varphi] = \tau$ is given by an element in M^{θ}, which proves the statement in the ergodic case.

The general case follows by ergodic decomposition: Let φ be an i.r.p.d.f. with $\mathbb{E}[\varphi] = \tau$. Then the restriction to the ergodic components are ergodic i.r.p.d.f.'s. The expectation values of these ergodic i.r.p.d.f.'s integrate to τ and are therefore by extremality μ-almost surely equal to τ. Hence we can apply the statement to them and get that they are equal to τ ν-almost surely, which implies that φ is equal τ μ-almost surely. □

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CHAPTER 8

A rigidity result for normalized subfactors

Abstract

We show a rigidity result for subfactors that are normalized by a representation of a lattice Γ in a higher rank simple Lie group with trivial center into a finite factor. This implies that every subfactor of $L\Gamma$ which is normalized by the natural copy of Γ is trivial or of finite index.

1. Introduction

It seems like a natural generalization of Margulis' Normal Subgroup Theorem to ask wheter every regular subfactor of the group von Neumann algebra of a lattice in a higher-rank simple Lie group with trivial center is trivial or of finite index.

In this article we make a small step into the direction of answering this question by looking at the special case where a subfactor $N \subset L\Gamma$ is actually normalized by a unitary representation $\pi\colon \Gamma \to U(L\Gamma)$ such that N and $\pi(\Gamma)$ generate M. We use methods developed by Jesse Peterson for the proof of his character rigidity theorem to prove the following theorem.

THEOREM 1.1 (Theorem 4.7). *Let Γ be a lattice in a simple real Lie group G which has trivial center and real rank at least 2. Let M be a finite factor, $N \subset M$ a subfactor and $\pi\colon \Gamma \to \mathcal{N}_M(N)$ a unitary representation of Γ into the normalizer of N such that the action $\Gamma \curvearrowright M$ given by $\alpha_\gamma(x) = \pi(\gamma)x\pi(\gamma^{-1})$ is ergodic and $M = (N \cup \pi(\Gamma))''$. Then $M \neq N \rtimes \Gamma$ or $[M : N] < \infty$.*

Peterson's proof is inspired by Margulis' proof in the sense that the proof of the normal subgroup theorem is based on the fact that an amenable discrete group with property (T) is finite, whereas the proof of character rigidity is based on the fact that an amenable factor with property (T) is finite-dimensional.

We adjust Peterson's proof to the situation of subfactors described above by putting coefficients in N into it. Then we use that if an inclusion $N \subset M$ is both coamenable and corigid and the relative commutant is finite-dimensional, the inclusion is of finite index.

Another approach to the result above would be to study the character $\gamma \mapsto \|E_N(\pi(\gamma))\|_2^2$. If this happens to be extremal, the above theorem follows directly from character rigidity [**Pet16**, Theorem C]: if the GNS construction of this character generates a finite dimensional von Neumann algebra, then the index is finite; if it is the regular character, then M is the crossed product $N \rtimes \Gamma$. However, as we observe in the last section, no direct reduction to the extremal case seems possible: the above character happens to be extremal if and only if it is the regular character.

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2. Preliminaries

Let M for the rest of this text be a finite factor with trace τ, $N \subset M$ a subfactor and let Γ be a discrete group. With $\mathcal{N}_M(N) := \{u \in U(M) | \, uNu^* = N\}$ we denote the normalizer of N inside M. $[M : N] := \dim_N(L^2M)$ is the index of $N \subset M$. We denote by J the antilinear, bounded operator on $L^2(M)$ that extends $x \mapsto x^*$ for $x \in M$.

2.1. Coamenability and Corigidity of inclusions of von Neumann algebras. Amenability and property (T) of M in Peterson's proof will be replaced by coamenability and corigidity of the inclusion $N \subset M$.

Definition 2.1 ([**Pop86**, 3.2.3 (ii)]). Let M be a factor and N a von Neumann subalgebra. Then $N \subset M$ is *coamenable* if there exists a conditional expectation $E: B(L^2M) \cap (JNJ)' \to M$.

Definition 2.2 ([**Pop06**]). Let M be a factor and N a von Neumann subalgebra. Then $N \subset M$ is called *corigid* if every M-bimodule H with N-central norm one vectors $\xi_n \in H$ such that $\|x\xi_n - \xi_n x\| \to 0$ for all $x \in M$ contains a non-zero M-central vector.

Note that in [**Pop86**] coamenability is called amenability and corigidity is called rigidity.

Theorem 2.3 ([**Pop86**, 4.1.8 (iv)]). *If an inclusion $N \subset M$ is coamenable and corigid and $N' \cap M$ is finite dimensional, then the inclusion is of finite index.*

2.2. Actions on von Neumann algebras.

Definition 2.4. An action $\sigma: \Gamma \to \mathrm{Aut}(M)$ of a group on a von Neumann algebra is *ergodic* if the fixed point algebra is $\mathbb{C}$.

Let us recall the definition und some properties of induced actions on von Neumann algebras.

Definition 2.5. Let $\Gamma \subset G$ be a closed subgroup of a locally compact group and $\theta: \Gamma \to \mathrm{Aut}(M)$ a continuous action. Pick a Borel section $s: G/\Gamma \to G$ and let $\chi: G \times G/\Gamma \to \Gamma$ be the cocycle given by $\chi(g, x) = s(gx)^{-1} g s(x)$.

Then the *induced action* $\tilde{\theta}$ of G on $L^\infty(G/\Gamma)\overline{\otimes} M$, which we view as bounded functions from G/Γ to M, is given by

$$\tilde{\theta}_g(f)(x) := \theta_{\chi(g,g^{-1}x)} f(g^{-1}x),$$

for $g \in G$, $f \in L^\infty(G/\Gamma)\overline{\otimes} M$ and $x \in G/\Gamma$.

REMARK 2.6. Let R be the G-action on $L^\infty(G)$ given by right multiplication. Then

$$\Psi\colon L^\infty(G/\Gamma)\overline{\otimes}M \to (L^\infty(G)\overline{\otimes}M)^{(R\otimes\theta)(\Gamma)}, \quad \Psi(f)(g) = \theta_{s(\Gamma)\chi(g,g\Gamma)}(f(g\Gamma))$$

is an isomorphism and

$$\Psi(\tilde{\theta}_g(f)) = L \otimes \mathrm{id}(g)\Psi(f),$$

where L is the G-action on $L^\infty(G)$ given by left multiplication.

LEMMA 2.7. $(L^\infty(G/\Gamma)\overline{\otimes}M)^{\tilde{\theta}(G)} \cong 1 \otimes M^{\theta(\Gamma)}$. *In particular, $\tilde{\theta}$ is ergodic iff θ is.*

PROOF. By Remark 2.6,

$$(L^\infty(G/\Gamma)\overline{\otimes}M)^{\tilde{\theta}(G)} \cong (L^\infty(G)\overline{\otimes}M)^{(R\otimes\theta)(\Gamma)\cup(L\otimes\mathrm{id})(G)} = 1 \otimes M^{\theta(\Gamma)},$$

hence $(L^\infty(G/\Gamma)\overline{\otimes}M)^{\tilde{\theta}(G)} = \mathbb{C}$ if and only if $M^{\theta(\Gamma)} = \mathbb{C}$. $\square$

3. A question about regular subfactors of the von Neumann algebra of lattices in higher-rank groups

We want to study possible analogues of Margulis' Normal Subgroup Theorem [**Mar91**, Theorem IX.5.3] in the setting of subfactors.

THEOREM 3.1 (Margulis' Normal Subgroup Theorem). *Let Γ be an irreducible lattice in a higher-rank simple Lie group G with trivial center. Then every normal subgroup of Γ is trivial or of finite index.*

A typical example of such a group is $\mathrm{PSL}(n,\mathbb{Z}) \subset \mathrm{PSL}(n,\mathbb{R})$ for $n \geq 3$. Margulis' Theorem was generalized by J. Peterson in [**Pet14**] as follows.

THEOREM 3.2 (Peterson). *Let G be a property (T) semi-simple Lie group with trivial center, no compact factors, and real rank at leasdt 2, and let $\Gamma < G$ be an irreducible lattice in G. Then for every unitary representation π of Γ such that $\pi(\Gamma)''$ is a finite factor π extends to an isomorphism $L\Gamma \to \pi(\Gamma)''$ or $\pi(\Gamma)''$ is finite-dimensional.*

It should be possible to do everything in this article with the same assumptions on Γ as in the above theorem. We restrict ourselves to the simple real case to avoid some technicalities.

When replacing groups by factors the analogue of a normal subgroup is a regular subfactor.

DEFINITION 3.3. An inclusion of von Neumann algebras $N \subset M$ is *regular* if the normalizer of N generates M, i.e., $\mathcal{N}_M(N)'' = M$.

QUESTION 3.4. Is it true that if Γ is as above and $N \subset L\Gamma$ a regular subfactor then $N = \mathbb{C}$ or $[L\Gamma : N] < \infty$?

This question has probably been asked before, but we couldn't find it in the literature.

In the following we restrict our attention to the situation where the image of Γ is not only in $\mathcal{N}_M(N)''$, but even in $\mathcal{N}_M(N)$ in order to make the question accessible to Peterson's methods from the proof of character rigidity. We allow subfactors in von Neumann algebras a bit more general than $L\Gamma$.

ASSUMPTION. For the rest of this article let Γ be a lattice in a simple real Lie group G which has trivial center and real rank at least 2. Let M be a finite factor with trace τ, $N \subset M$ a subfactor and $\pi\colon \Gamma \to \mathcal{N}_M(N)$ a representation of Γ into the normalizer of N such that the action $\Gamma \curvearrowright M$ given by $\alpha_\gamma(x) = \pi(\gamma)x\pi(\gamma^{-1})$ is ergodic and $M = (N \cup \pi(\Gamma))''$.

EXAMPLE 3.5. $M = L\Gamma$ with $\lambda : \Gamma \to L\Gamma$ the left regular representation and $N \subset L\Gamma$ a subfactor which is normalized by $\lambda(\Gamma)$ is as in the assumption. M is a factor and the conjugation action is ergodic because Γ is i.c.c..

4. Peterson machine with coefficients

In this section we adjust the proof of Peterson's character rigidity theorem in [**Pet14**] and [**Pet16**] to the situation described above by putting coefficients in N into it. Setting $N = \mathbb{C}$ gives back the proof of character rigidity.

We will need a bunch of subgroups, which we first define in the case of $G = \mathrm{SL}(n, \mathbb{R})$.

EXAMPLE 4.1. For $G = \mathrm{SL}(n, \mathbb{R})$ let P be the subgroup of upper triangular matrices and V the subgroup of upper triangular matrices with 1 on the diagonal. Fix numbers $0 = j_0 < j_1 < j_2 < \cdots < j_k = n$. We define now subgroups consisting each of all block matrices in $\mathrm{SL}(n, \mathbb{R})$ of a certain structure:

$$P_0 := \left\{ \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ 0 & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk} \end{pmatrix} \right\}, \quad V_0 := \left\{ \begin{pmatrix} \mathbb{1} & A_{12} & \dots & A_{1k} \\ 0 & \mathbb{1} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{1} \end{pmatrix} \right\},$$

$$R_0 := \left\{ \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk} \end{pmatrix} \right\}, \quad L_0 := \left\{ \begin{pmatrix} V_{11} & 0 & \dots & 0 \\ 0 & V_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V_{kk} \end{pmatrix} \right\},$$

where A_{il} are arbitrary matrices of size $(j_i - j_{i-1}) \times (j_l - j_{l-1})$, V_{ii} are upper triangular matrices with 1 on the diagonal and $\mathbb{1}$ is an identity matrix of fitting size.

For each of these subgroups we denote by $\overline{P}, \overline{V}$, ect. the corresponding transposed subgroup.

DEFINITION 4.2. For general G, let S be an $\mathbb{R}$-split maximal torus, P a minimal parabolic subgroup containing S and $V < P$ its unipotent radical. Let $\overline{P}$ be the opposite parabolic and $\overline{V}$ its unipotent radical. Let P_0 be another parabolic subgroup s.t. $P < P_0 \lneq G$. Let V_0 be the unipotent radical of P_0 and $\overline{P_0}, \overline{V_0}$ the corresponding opposite subgroups. Let R_0 be the reductive component of P_0 containing S so that $P_0 = R_0 \rtimes V_0$ and $\overline{L_0} = R_0 \cap \overline{V}$. Then $\overline{V} = \overline{V_0} \rtimes \overline{L_0}$. See [**Mar91**, I.1.2] for the definitions.

We have the following commuting diagram:

$$\begin{array}{ccc} \overline{V} = \overline{V_0} \rtimes \overline{L_0} & \xrightarrow{(v,l)\mapsto v} & \overline{V_0} \\ {\scriptstyle v\mapsto vP}\downarrow & & \downarrow{\scriptstyle v\mapsto vP_0} \\ G/P & \xrightarrow{gP\mapsto gP_0} & G/P_0. \end{array}$$

LEMMA 4.3. *The vertical maps map measures in the class of the Haar measure to G-quasiinvariant measures. They are measure isomorphisms when equipping the quotient spaces with these measures.*

PROOF. Let $\mu_{\overline{V}}$ be a left Haar measure on $\overline{V}$ and let $\lambda \in \mathcal{M}(G/P)$ be the image of $\mu_{\overline{V}}$. This defines by [**Bou04**, Proposition VII.2.1.4] a measure $\lambda^{\#}$ on G given by

$$\int_G f d\lambda^{\#} = \int_{G/P} \int_P f(gp) d\mu_P(p) d\lambda(gP),$$

where μ_P is a left Haar measure on P. By [**Mar91**, Lemma IV.2.2], the map $(\overline{V} \times P, \mu_{\overline{V}} \otimes \mu_P) \to (G, \mu_G)$, $(v,p) \mapsto vp^{-1}$, is a homeomorphism onto the image and a measure isomorphism, μ_G being a suitably normalized left Haar measure on G. This implies that $\lambda^{\#} = (1_{\overline{V}} \otimes \Delta_P) \cdot \mu_G$, where Δ_P is the modular function on P. It follows by [**Bou04**, Lemma VII.2.5.4] that λ is G-quasiinvariant. Now the same follows for the images of measures that are strongly equivalent to Haar measure and analogously for such measures on $\overline{V_0}$.

The vertical maps are then measure isomorphisms because they map the measures to each other and are injective. □

Let ν and ρ be probability measures on $\overline{V_0}$ resp. $\overline{L_0}$ in the class of the Haar measure; the image of ν on G/P_0 is still denoted by ν. We equip $\overline{V} = \overline{V_0} \rtimes \overline{L_0}$ with the product measure $\nu \times \rho$.

Let G act on $\overline{V}$ and $\overline{V_0}$ in the way that makes the above diagram G-equivariant. This transforms the action of $\overline{V}$ on G/P to left multiplication on $\overline{V}$ and the action of R_0 on G/P to the action induced by conjugation on $\overline{V}$.

Let σ be the corresponding action of Γ on $L^\infty(G/P)$ and σ^0 the corresponding Koopman representation on $L^2(G/P)$. Let

$$P_1 := 1 \otimes P_{\hat{1}} \in L^\infty(G/P) \overline{\otimes} B(L^2M),$$

where $P_{\hat{1}}$ is the orthogonal projection on $\mathbb{C}\hat{1} \subset L^2M$ with M as in the assumption. Let

$$\mathcal{B} := (L^\infty(G/P) \overline{\otimes} B(L^2M)) \cap \{\sigma_\gamma \otimes (J\pi(\gamma)J) | \gamma \in \Gamma\}' \cap (1 \otimes JNJ)'.$$

LEMMA 4.4. *There exists a conditional expectation*

$$E\colon (L^\infty(G/P) \overline{\otimes} B(L^2M)) \cap (1 \otimes JNJ)' \to \mathcal{B}.$$

Proof. Let $H = L^2(M)$ and let $\theta\colon \Gamma \to \mathrm{Aut}(B(H))$ be conjugation with $J\pi(\cdot)J$. Let $\tilde{\theta}$ be the induced action of G on $L^\infty(G/\Gamma)\overline{\otimes}B(H)$ as in Definition 2.5 with a section $s\colon G/\Gamma \to G$ and $\chi\colon G\times G/\Gamma \to \Gamma$ given by $\chi(g,x) = s(gx)^{-1}gs(x)$. $\tilde{\theta}$ is also well-defined on $L^\infty(G/\Gamma)\overline{\otimes}B(H)\cap (1\otimes JNJ)'$, which we view as bounded functions from G/Γ to $B(H)\cap(JNJ)'$. To see this let $f \in L^\infty(G/\Gamma)\overline{\otimes}(B(H)\cap(JNJ)')$, $x\in G/\Gamma$, $n\in N$, $g\in G$, $\gamma = \chi(g, g^{-1}x)$ and calculate

$$\begin{aligned}
\tilde{\theta}_g(f)(x)JnJ &= J\pi(\gamma)Jf(g^{-1}x)J\pi(\gamma^{-1})JJnJ\\
&= J\pi(\gamma)Jf(g^{-1}x)J\alpha_{\gamma^{-1}}(n)JJ\pi(\gamma^{-1})J\\
&= J\pi(\gamma)JJ\alpha_{\gamma^{-1}}(n)Jf(g^{-1}x)J\pi(\gamma^{-1})J\\
&= JnJJ\pi(\gamma)Jf(g^{-1}x)J\pi(\gamma^{-1})J\\
&= JnJ\tilde{\theta}_g(f)(x).
\end{aligned}$$

P is amenable [**Mar91**, IV.4.4], hence [**Pet16**, Theorem 7.4] gives that there is a conditional expectation

$$\begin{aligned}
E\colon L^\infty(G/\Gamma)\otimes(B(H)\cap(JNJ)') &\to (L^\infty(G/\Gamma)\otimes(B(H)\cap(JNJ)'))^{\tilde{\theta}(P)}\\
&= (L^\infty(G/\Gamma)\otimes B(H))^{\tilde{\theta}(P)}\cap(1\otimes JNJ)'.
\end{aligned}$$

But

$$\begin{aligned}
(L^\infty(G/\Gamma)\overline{\otimes}B(H))^{\tilde{\theta}(P)} &\cong (L^\infty(G)\overline{\otimes}B(H))^{(L\otimes\mathrm{id}(P))\times(R\otimes\theta(\Gamma))}\\
&\cong (L^\infty(G)\overline{\otimes}B(H))^{(R\otimes\mathrm{id}(P))\times(L\otimes\theta(\Gamma))}\\
&\cong (L^\infty(G/P)\overline{\otimes}B(H))^{\sigma\otimes\theta(\Gamma)}.
\end{aligned}$$

Here the first isomorphism is the map Ψ given in Remark 2.6 and the second isomorphism is $f\mapsto (g\mapsto f(g^{-1}))$. Thus

$$(L^\infty(G/\Gamma)\overline{\otimes}B(H))^{\tilde{\theta}(P)}\cap(JNJ)' = (L^\infty(G/P)\overline{\otimes}B(H))^{\sigma\otimes\theta(\Gamma)}\cap(1\otimes JNJ)' = \mathcal{B}.$$

$\square$

The following lemma is [**Pet14**, Lemma 4.4], only with different $\mathcal{B}$, which does not change the proof. We give it anyway in order to provide more details.

Lemma 4.5. *Let*

$$x = x^* \in \mathcal{B}\subset L^\infty(G/P)\overline{\otimes}B(L^2M) = L^\infty(\overline{V_0})\overline{\otimes}L^\infty(\overline{L_0})\overline{\otimes}B(L^2M)$$

and view it as a function from $\overline{V_0}$ *to* $L^\infty(\overline{L_0})\overline{\otimes}B(L^2M)$. *Let*

$$x_0\in L^\infty(\overline{L_0})\overline{\otimes}B(L^2M)$$

be in the SOT-essential range of x. *Then there exists a* $y = y^*\in\mathcal{B}$ *such that* $yP_1\in L^\infty(\overline{L_0})\overline{\otimes}B(L^2M)$ *and* $P_1yP_1 = P_1x_0P_1$.

PROOF. That x_0 is in the SOT-essential range of x means that there are subsets $E_j \subset \overline{V_0}$ of positive measure such that for all $\eta \in L^2(M)$, $\xi_L \in L^2(\overline{L_0})$ and $\epsilon > 0$ there exists an N with

$$\int_{\overline{L_0}} \|(x(v,l) - x_0(l))\eta\|^2 |\xi_L(l)|^2 \, d\rho(l) < \epsilon \tag{4.1}$$

for all $j > N$ and all $v \in E_j$. By the proof of [**Pet14**, Lemma 4.3] there are $\gamma_j \in \Gamma$ and $h_j \in \overline{V_0} \rtimes \mathcal{Z}(R_0)$ such that $\gamma_j h_j^{-1} \to e$ and $\nu(h_j E_j) \to 1$. We first show that these can be chosen in a way that $\sigma_{\gamma_j}(x) \to x_0$ in SOT. Take a countable SOT-basis of neighborhoods of zero in the unit ball of $L^\infty(\overline{V_0}) \overline{\otimes} L^\infty(\overline{L_0}) \overline{\otimes} B(L^2 M)$, denoted by $\{U_j\}_{j \in \mathbb{N}}$, such that $U_j \searrow \{0\}$. As the action σ is strongly continuous, there are numbers $k(j) \in \mathbb{N}$ and neighborhoods $e \in O_j \subset G$ such that $O_j \subset O_i$ if $j > i$ and

$$\sigma_g(x_0 + U_{k(j)}) \subset x_0 + U_j \quad \forall g \in O_j. \tag{4.2}$$

We can choose the γ_j in [**Pet14**, Lemma 4.3] in a way that $\gamma_j h_j^{-1} \in O_j$ for all j. We will show now first that $\sigma_{\gamma_j}(x) \to x_0$ if $\sigma_{h_j}(x) \to x_0$ and then that $\sigma_{h_j}(x) \to x_0$, all in SOT.

So assume that $\sigma_{h_j}(x) \to x_0$, hence $\forall j \, \exists N : \sigma_{h_i}(x) \in x_0 + U_{k(j)}$ for all $i > N$. Then by (4.2) $\sigma_{\gamma_i}(x_0) = \sigma_{\gamma_i h_i^{-1}} \sigma_{h_i}(x_0) \in x_0 + U_j$ for all $i > \max\{N, j\}$ because $\gamma_i h_i^{-1} \in O_j$. So then $\sigma_{\gamma_j}(x) \to x_0$.

To show that $\sigma_{h_j}(x) \to x_0$ let $\eta \in L^2 M, \xi_L \in L^2(\overline{L_0}), \xi_V \in L^2(\overline{V_0})$. Then, as the $h_j \in \overline{V_0} \rtimes \mathcal{Z}(R_0)$ act trivially on $\overline{L_0}$ and using (4.1),

$$\begin{aligned}
&\|1_{h_j E_j}(\sigma_{h_j}(x) - x_0)(\xi_V \otimes \xi_L \otimes \eta)\|^2 \\
&= \int_{h_j E_j \times \overline{L_0}} \|(\sigma_{h_j}(x)(v,l) - x_0(l))\eta\|^2 |\xi_V(v)\xi_L(l)|^2 \, d\nu(v) d\rho(l) \\
&= \int_{h_j E_j \times \overline{L_0}} \|(x(h_j^{-1}v, l) - x_0(l))\eta\|^2 |\xi_V(v)\xi_L(l)|^2 \, d\nu(v) d\rho(l) \\
&= \int_{E_j \times \overline{L_0}} \|(x(v,l) - x_0(l))\eta\|^2 |\xi_V(h_j v)\xi_L(l)|^2 \, d((h_j^{-1})_* \nu)(v) d\rho(l) \\
&= \int_{E_j} \left(\int_{\overline{L_0}} \|(x(v,l) - x_0(l))\eta\|^2 |\xi_L(l)|^2 dl \right) |\xi_V(h_j v)|^2 \, d((h_j^{-1})_* \nu)(v) \\
&< \int_{h_j E_j} \epsilon |\xi_V(v)|^2 \, d\nu(v) \le \epsilon \|\xi_V\|^2.
\end{aligned}$$

So $1_{h_j E_j}(\sigma_{h_j}(x) - x_0) \to 0$ in SOT, and since $\nu(h_j E_j) \to 1$, also $\sigma_{h_j}(x) - x_0 \to 0$ in SOT.

Let y be a WOT cluster point of the set $\{\pi(\gamma_j) x \pi(\gamma_j^{-1})\}$. Then $y \in \mathcal{B}$ because $x \in \mathcal{B}$ and conjugation with JNJ and $J\pi(\Gamma)J$ commutes with conjugation with $\pi(\Gamma)$. Also yP_1 is a WOT cluster point of

$$\begin{aligned}\{\pi(\gamma_j)x\pi(\gamma_j^{-1})P_1\} =&\{\pi(\gamma_j)(J\pi(\gamma_j)J)(J\pi(\gamma_j^{-1})J)x(J\pi(\gamma_j)J)P_1\}\\ =&\{\pi(\gamma_j)(J\pi(\gamma_j)J)\sigma_{\gamma_j}(x)P_1\}\end{aligned}$$

Since $\sigma_{\gamma_j}(x) \to x_0$ in SOT, yP_1 must then also be a WOT cluster point of

$$\{\pi(\gamma_j)(J\pi(\gamma_j)J)x_0P_1\} \subset L^\infty(\overline{L_0})\overline{\otimes}B(L^2M),$$

so $yP_1 \in L^\infty(\overline{L_0})\overline{\otimes}B(L^2M)$. P_1yP_1 is a WOT cluster point of

$$\{P_1\pi(\gamma_j)x\pi(\gamma_j^{-1})P_1\} = \{P_1(J\pi(\gamma_j)J)x(J\pi(\gamma_j^{-1})J)P_1\} = \{P_1\sigma_\gamma(x)P_1\}.$$

So again since $\sigma_{\gamma_j}(x) \to x_0$, $P_1yP_1 = P_1x_0P_1$. □

PROPOSITION 4.6. *If M is not isomorphic to $N \rtimes \Gamma$ with isomorphism extending π, then $\mathcal{B} = M$, hence $N \subset M$ is coamenable.*

PROOF. Assume that M is not isomorphic to $N \rtimes \Gamma$ with isomorphism extending π. Then there is a $\gamma_0 \in \Gamma \setminus \{e\}$ and an $n \in N$ such that $c_0 := \tau(\pi(\gamma_0)n) \neq 0$.

Let x, x_0 and y be as in the above lemma. We want to show that x is a constant function. Let $\theta\colon \Gamma \to \mathrm{Aut}(M)$, different as in the proof of Lemma 4.4, be conjugation by $\pi(\cdot)$ and the induced action $\tilde{\theta}\colon G \curvearrowright L^\infty(G/\Gamma)\overline{\otimes}M$ as in Definition 2.5.

θ is ergodic by assumption, hence $\tilde{\theta}$ is ergodic by Lemma 2.7. It is still ergodic when restricted to $\overline{V_0} \rtimes \mathcal{Z}(R_0)$ because $\overline{V_0} \rtimes \mathcal{Z}(R_0)$ is not compact and hence every $\overline{V_0} \rtimes \mathcal{Z}(R_0)$-invariant vector must also be G-invariant by the Howe-Moore property of G [**HM79**, Theorem 5.2]. Now [**Pet14**, Lemma 3.2] gives us that for every neighborhood $e \in O \subset G$ and $\Gamma_O = \Gamma \cap O(\overline{V_0} \rtimes \mathcal{Z}(R_0))$ we have

$$\tau(\pi(\gamma_0)n) = J\tau(\pi(\gamma_0)n)J \in \overline{\mathrm{conv}}^{SOT}\{J\pi(\gamma^{-1})\pi(\gamma_0)n\pi(\gamma)J \,|\, \gamma \in \Gamma_O\}. \tag{4.3}$$

We want to show now that $[\sigma^0_{\gamma_0} \otimes c_0, P_1x_0P_1]$ is zero. By Lemma 4.5 $[\sigma^0_{\gamma_0} \otimes c_0, P_1x_0P_1] = [\sigma^0_{\gamma_0} \otimes c_0, P_1yP_1]$. The approximation (4.3) of $\tau(\pi(\gamma_0)n)$ gives for every O an approximation

$$[\sigma^0_{\gamma_0} \otimes c_0, P_1yP_1] \overset{SOT}{\sim} \sum_{i=1}^k c_iP_1\,[\sigma^0_{\gamma_0} \otimes J\pi(\gamma_i^{-1}\gamma_0)n\pi(\gamma_i)J, y]\,P_1$$

with $\gamma_i \in \Gamma_O$ and $\sum_{i=1}^k c_i = 1$.

Now write $\gamma_i = g_ih_i$ where $g_i \in O$ and $h_i \in \overline{V_0} \rtimes \mathcal{Z}(R_0)$. We have $\sigma_{h_i}(yP_1) = yP_1$ and $\sigma_{h_i}(P_1y) = P_1y$ since $yP_1, P_1y \in L^\infty(\overline{L_0})\overline{\otimes}B(L^2M)$ and taking O small enough we get $\sigma_{\gamma_i}(y)P_1 = \sigma_{\gamma_i}(yP_1) \sim yP_1$ and $P_1\sigma_{h_i}(y) = \sigma_{h_i}(P_1y) \sim P_1y$ in SOT. Then

$$\begin{aligned}&[\sigma^0_{\gamma_0} \otimes c_0, P_1x_0P_1]\\ \overset{\text{WOT}}{\sim}&\sum_{i=1}^k c_iP_1\,[\sigma^0_{\gamma_0} \otimes J\pi(\gamma_i^{-1}\gamma_0)n\pi(\gamma_i)J, \sigma_{\gamma_i}(y)]\,P_1\end{aligned}$$

$$
\begin{aligned}
&=\sum_{i=1}^{k} c_i P_1 \left[\sigma^0_{\gamma_0} \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, \sigma_{\gamma_i}(y)\right] (J\alpha_{\gamma_i^{-1}}(n)J)P_1 \\
&=\sum_{i=1}^{k} c_i P_1 \left[\sigma^0_{\gamma_0} \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, (\sigma^0_{\gamma_i} \otimes 1)y(\sigma^0_{\gamma_i^{-1}} \otimes 1)\right] (J\alpha_{\gamma_i^{-1}}(n)J)P_1 \\
&=\sum_{i=1}^{k} c_i P_1 (\sigma^0_{\gamma_i} \otimes 1) \left[\sigma^0_{\gamma_i^{-1}\gamma_0\gamma_i} \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, y\right](\sigma^0_{\gamma_i^{-1}} \otimes 1) (J\alpha_{\gamma_i^{-1}}(n)J)P_1 \\
&=0.
\end{aligned}
$$

In the second step we used that y and hence also $\sigma_{\gamma_i}(y)$ commutes with JNJ. In the last step we used that $[\sigma^0_{\gamma_i^{-1}\gamma_0\gamma_i} \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, y] = 0$ because $y \in \mathcal{B}$. So we found $[\sigma^0_{\gamma_0} \otimes c_0, P_1 x_0 P_1] = 0$ and hence $\sigma_{\gamma_0}(P_1 x_0 P_1) = P_1 x_0 P_1$. Since $\tau(\pi(\gamma_0)n) = \tau(\pi(\gamma)\pi(\gamma_0)n\pi(\gamma^{-1})) = \tau(\pi(\gamma\gamma_0\gamma^{-1})\alpha_\gamma(n))$, we get $\sigma_\gamma(P_1 x_0 P_1) = P_1 x_0 P_1$ for all $\gamma \in \langle\langle\gamma_0\rangle\rangle$ in the normal closure of γ_0. By [**Pet16**, Theorem 10.10] the action of $\langle\langle\gamma_0\rangle\rangle$ on $L^\infty(G/P)$ is ergodic, so $P_1 x_0 P_1 \in \mathbb{C}1 \otimes P_{\hat{1}}$. Since x_0 was arbitrary in the range of x, we conclude that $P_1 x P_1 \in L^\infty(\overline{V_0}) \otimes P_{\hat{1}}$ and hence $P_1 \mathcal{B} P_1 \subset L^\infty(\overline{V_0}) \otimes P_{\hat{1}}$. This means $\mathcal{B} \subset L^\infty(\overline{V_0}) \overline{\otimes} B(L^2 M)$ because if $x \in \mathcal{B}$ and $a, b \in M$, we have

$$\left\langle x\hat{a}, \hat{b}\right\rangle = \left\langle (b^* x a)\hat{1}, \hat{1}\right\rangle = \left\langle (P_1 b^* x a P_1)\hat{1}, \hat{1}\right\rangle \in L^\infty(\overline{V_0})$$

since $b^* x a \in \mathcal{B}$. But $\overline{V_0} = G/P_0$ and G is generated by the P_0's [**Mar91**, Proposition I.1.2.2], so we get

$$\mathcal{B} = B(L^2 M) \cap (J\pi(\Gamma)J \cup JNJ)' = M.$$

Now $N \subset M$ is coamenable by Lemma 4.4. □

THEOREM 4.7. *Let Γ be a lattice in a simple real Lie group G which has trivial center and real rank at least 2. Let M be a finite factor, $N \subset M$ a subfactor and $\pi\colon \Gamma \to \mathcal{N}_M(N)$ a representation of Γ into the normalizer of N such that the action $\Gamma \curvearrowright M$ given by $\alpha_\gamma(x) = \pi(\gamma)x\pi(\gamma^{-1})$ is ergodic and $M = (N \cup \pi(\Gamma))''$.*

Then M is isomorphic to $N \rtimes \Gamma$ with isomorphism extending π or $[M : N] < \infty$.

PROOF. If π does not extend to an isomorphism $M \cong N \rtimes \Gamma$, the inclusion $N \subset M$ is coamenable by Proposition 4.6. Then by the proof of [**BMO19**, Lemma 2.1] there is a nonzero projection $q \in N' \cap M$ such that $q(N' \cap M)q$ is completely atomic. Hence the center of $N' \cap M$ is atomic since the conjugation action of Γ on it is ergodic and the existence of q implies that it is not diffuse. Since the action is also trace preserving, it is finite. $N' \cap M$ must be of type I since it contains a minimal projection and the action is ergodic, hence it is finite dimensional.

$N \subset M$ is also corigid because Γ has property (T) [**Pop86**, 4.1.7 (ii)]. So the inclusion is of finite index by Theorem 2.3. □

In the case where $M = L\Gamma$ and π is the left regular representation we have $M \cong N \rtimes \Gamma$ with isomorphism extending π if and only if $N = \mathbb{C}$. Hence we get

COROLLARY 4.8. *Let Γ be a lattice in a simple real Lie group G which has trivial center and real rank at least 2. Let $N \subset L\Gamma$ a subfactor which is normalized by the natural copy of Γ in $L\Gamma$. Then $N = \mathbb{C}$ or $[L\Gamma : N] < \infty$.*

Corollary 4.8 can also be obtained without Theorem 4.7 from [**CD19**, Theorem 3.15 1)] and [**Bru18**, Section 4.3]. In [**CD19**] similar results are proven for groups with positive ℓ^2-Betti numbers and acylindrically hyperbolic groups. S. Das brought to our attention that S. Popa observed that in the above situation the index is an integer as is always true for regular subfactors with finite index, see also [**CS06**, Theorem 4.5].

5. On deducing the theorem from character rigidity

It seems natural to try to deduce Theorem 4.7 from character rigidity [**Pet16**, Theorem C] by applying it to the character $\gamma \mapsto \|E_N(\pi(\gamma))\|_2^2$. The following connection is easy to deduce:

LEMMA 5.1. *Let $N \subset M$ and π be as above and let $\varphi(\gamma) = \|E_N(\pi(\gamma))\|_2^2$. If the GNS construction of this character generates a finite dimensional von Neumann algebra, then $[M : N] < \infty$. If it is the regular character, then $M = N \rtimes \Gamma$.*

PROOF. If φ is the regular character,

$$E_N(\pi(\gamma)) = \begin{cases} 0 & \gamma \neq e \\ 1 & \gamma = e \end{cases},$$

hence $M = N \rtimes \Gamma$.

We now describe the GNS construction of φ. Let $e_N \in B(L^2M)$ be the orthogonal projection onto N and $\langle M, N\rangle := \{\sum_{k=1}^n x_k e_N y_k | \, n \in \mathbb{N}, x_k, y_k \in M\}'' \subset B(L^2M)$ the basic construction with semifinite trace given by $Tr(xe_N y) = \tau(xy)$. From this we get a Hilbert space $\mathcal{H} := L^2(\langle M, N\rangle, Tr)$ as the completion of the finite elements of $\langle M, N\rangle$ with the norm $\|x\|_2 = Tr(x^*x)^{\frac{1}{2}}$. Define a unitary representation $\theta\colon \Gamma \to U(\mathcal{H})$ by

$$\theta_\gamma(xe_N y) = \pi(\gamma)xe_N y\pi(\gamma)^*.$$

Then, using $e_N x e_N = E_N(x)e_N$ for all $x \in M$, we get

$$\begin{aligned}\langle\theta_\gamma(e_N), e_N\rangle = \langle\pi(\gamma)e_N\pi(\gamma)^*, e_N\rangle &= Tr(e_N\pi(\gamma)e_N\pi(\gamma)^*e_N) = Tr(E_N(\pi(\gamma))e_N\pi(\gamma)^*e_N) \\ &= Tr(E_N(\pi(\gamma))e_N E_N(\pi(\gamma)^*)) = \tau_M(E_N(\pi(\gamma))E_N(\pi(\gamma))^*) = \tau(\gamma),\end{aligned}$$

hence $\mathcal{H}_\varphi := \overline{\operatorname{span}}\{\sigma_\gamma(e_N) | \gamma \in \Gamma\} \subset \mathcal{H}, e_N$, and σ form a GNS triple for φ.

Assume $\sigma(\Gamma)'' \subset B(\mathcal{H}_\varphi)$ is finite dimensional. Say it is generated as a vector space by $\sigma_{\gamma_1}, \dots, \sigma_{\gamma_n}$. Then for all $x, y \in N$

$$\pi(\gamma)xe_Ny\pi(\gamma') = \pi(\gamma)e_Nxy\pi(\gamma)^*\pi(\gamma\gamma') = \theta_\gamma(e_Nxy)\pi(\gamma\gamma') = \sum_{i=1}^n c_i\pi(\gamma_i)e_Nxy\pi(\gamma_i)^*\pi(\gamma\gamma')$$

for some $c_i \in \mathbb{C}$. So since M is generated by N and $\pi(\Gamma)$, $\langle M, N\rangle$ is generated over M by $\pi(\gamma_1)e_N, \dots, \pi(\gamma_n)e_N$. Hence $[M : N] = [\langle M, N\rangle : M] < \infty$. □

In particular, if we knew that φ was extremal or could reduce the situation to the extremal case, the theorem would follow. However, things are more complicated.

Definition 5.2. A trace-preserving action on a finite von Neumann algebra $\sigma\colon \Gamma \curvearrowright M$ that leaves a von Neumann subalgebra $N \subset M$ invariant is called *weakly mixing relative to* N if for any finite set $F \subset M$ with $E_N(x) = 0$ for all $x \in F$ there exist $\gamma_n \in \Gamma$ such that for all $\eta, \eta' \in F$, $\|E_N(\eta^*\sigma_{\gamma_n}(\eta'))\|_2 \to 0$ for $n \to \infty$.

Lemma 5.3. *The following are equivalent.*

(i) φ *is extremal.*
(ii) $\sigma_\gamma(x) = \pi(\gamma)x\pi(\gamma)^*$ *is weakly mixing relative to* N.
(iii) $M = N \rtimes \Gamma$.

Proof. φ is extremal if and only if $\theta(\Gamma)'' \subset B(L^2(\langle M, N\rangle, Tr))$ is a factor. We have a dense inclusion $\theta(\Gamma)'' \subset L^2(\theta(\Gamma)'') \cong \mathcal{H}_\varphi$ sending $\theta(\gamma) \in \theta(\Gamma)''$ to $\pi(\gamma)e_N\pi(\gamma)^* \in \mathcal{H}_\varphi$ such that the conjugation action on $\theta(\Gamma)''$ translates to the action σ^N on $\mathcal{H}_\varphi$ given by

$$\sigma^N_\gamma(xe_Ny) = \pi(\gamma)x\pi(\gamma)^*e_N\pi(\gamma)y\pi(\gamma)^*.$$

So φ is extremal iff every σ^N-invariant vector in $\mathcal{H}_\varphi$ is trivial. Since $N^\sigma = \mathbb{C}$ this is equivalent to

$$(\mathcal{H}_\varphi)^{\sigma^N} \subset L^2(Ne_N). \tag{5.1}$$

Condition (5.1) is equivalent to σ being weakly mixing because it is implied by *(iii)* in [**Pop07**, Lemma 2.10] and implies *(i)* with the same proof as for *(iii)* $\Rightarrow$ *(i)*. Hence i) and ii) are equivalent.

If φ is extremal, then $M = N \rtimes \Gamma$ or $[M : N]$ by character rigidity and Lemma 5.1. In the second case, since N is a factor, $L^2(M)$ has a finite orthogonal basis $\eta_1, \dots, \eta_k$ over N with $\|\eta\|_2 = \sum_{i=1}^k \|E_N(\eta_i^*\eta)\|_2$ for and all $\eta \in L^2(M)$ (see [**ADP**, 8.4-8.6]). Then for all $\gamma \in \Gamma$,

$$\sum_{i=1}^k \|E_N(\eta_i^*\sigma_\gamma(\eta)\|_2 = \|\sigma_\gamma(\eta)\|_2 = \|\eta\|_2,$$

hence $\|E_N(\eta_i^*\sigma_{\gamma_n}(\eta))\|_2$ cannot go to zero for some γ_n and all $i = 1, \dots, k$. □

EXAMPLE 5.4. If $\Lambda \triangleleft \Gamma$ is a normal subgroup, $\pi\colon \Gamma \to L\Gamma = M$ the left regular representation and $N = L\Lambda$, then φ is the regular character on Γ/Λ which is not extremal if the index is finite. The decomposition into extremal characters corresponds to the decomposition of the left regular representation of Γ/Λ into irreducible representations, which doesn't seem to be nicely reflected in the situation of $\Gamma \curvearrowright N \subset M$.

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CHAPTER 9

On representation theory of topological full groups of étale groupoids

1. Inverse semigroups, groupoids and their C^*-algebras

1.1. A brief introduction to semigroup theory. We consider the general text by Lawson [**Law98**] for a complete introduction to inverse semigroup theory, and recall the basic notions with references below. Recall that a *semigroup* is a set S, together with an associative binary operation. If additionally it has a unit element, then we say it is a *monoid*.

DEFINITION 1.1. Let S be a semigroup. We say S is *inverse* if there exists a unary operation $* : S \to S$ satisfying the following identities:

(i) $(s^*)^* = s$.
(ii) $ss^*s = s$ and $s^*ss^* = s^*$ for all $s \in S$.
(iii) $ef = fe$ for all idempotents $e, f \in S$.

An element $e \in S$ satisfying $e^2 = e$ is called an *idempotent*, and the set of idempotents is denoted $E(S)$ - in an inverse semigroup this is a (commutative) subsemigroup. Inverse semigroups also carry a natural partial order, one induced from the subsemigroup of idempotents.

DEFINITION 1.2. Two elements s and t in an inverse semigroup S (with zero) are *compatible* if st and st are elements of $E(S)$, and *orthogonal* if $s^*t = st^* = 0$. A finite set of elements $F \subset S$ is a compatible (resp. orthogonal) set if each pair of elements $s, t \in F$ are compatible (resp. orthogonal).

A fundamental example of such an object is the *symmetric inverse monoid* on any set X, denoted by $I(X)$. This is defined equipping the collection of all partial bijections of X to itself equipped the composition defined on common intersections. The Wagner–Preston theorem [**Law98**, Section 1.5, Theorem 1], the analogue of Cayley's theorem for inverse semigroups, says that every abstract inverse semigroup can be realised as a subsemigroup of $I(X)$ for some set X. The natural order, $s \leq t$ then says that t extends s on a larger domain, and the compatibility condition says that s and t agree on the intersection of their domain, and that their inverses agree on the intersection of their ranges.

In this situation the function, $s \vee t$, defined by doing both s and t simultaneously on the union of their domains, is well defined and belongs to $I(X)$. A subsemigroup of $I(X)$ is *distributive* if, given any compatible subset $F \subset S$, the join over F: $\bigvee_{s \in F} s$ also belongs to S.

1.2. Groupoids by example. For groupoids in the context in which we are studying them, a useful reference is [**Exe08**]. *Topological groupoids* are elementary models for 'non-commutative spaces' and appear throughout noncommutative geometry, index theory and operator algebras as a primary source of examples.

DEFINITION 1.3. A *groupoid* is a set $\mathcal{G}$ equipped with the following information:

(i) A subset $\mathcal{G}^{(0)}$ consisting of the objects of $\mathcal{G}$, denote the inclusion map by $i : \mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$.
(ii) Two maps, r and $s : \mathcal{G} \to \mathcal{G}^{(0)}$ such that $r \circ i = s \circ i = Id$.
(iii) An involution map ${}^{-1} : \mathcal{G} \to \mathcal{G}$ such that $s(g) = r(g^{-1})$.
(iv) A partial product $\mathcal{G}^{(2)} \to \mathcal{G}$ denoted $(g, h) \mapsto gh$, with $\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} | s(g) = r(h)\} \subseteq \mathcal{G} \times \mathcal{G}$ being the set of pairs it is possible to compose.

Moreover we ask the following:

- The product is associative where it is defined in the sense that for any pairs: $(g, h), (h, k) \in \mathcal{G}^{(2)}$ we have $(gh)k$ and $g(hk)$ are defined and equal.
- For all $g \in \mathcal{G}$ we have $r(g)g = gs(g) = g$.

A groupoid is *principal* if $(r, s) : \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is injective. A groupoid $\mathcal{G}$ is a *topological groupoid* if both $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are topological spaces, and the maps $r, s, {}^{-1}$ and the composition are all continuous. A Hausdorff, locally compact topological groupoid $\mathcal{G}$ is *proper* if (r, s) is a proper map and *étale* or *r-discrete* if the map r is a local homeomorphism. When $\mathcal{G}$ is étale, s and the product are also local homeomorphisms, and $\mathcal{G}^{(0)}$ is an open subset of $\mathcal{G}$ [**Exe08**, Section 3].

DEFINITION 1.4. A *bisection* in $\mathcal{G}$ is a subset $U \subset \mathcal{G}$ such that the range (or source) map $U \mapsto r(U)$ is a bijection. The set of open bisections $\mathcal{G}^o$ forms a basis for the topology, in the case $\mathcal{G}$ is étale. We say that an étale groupoid is *ample* if the set of compact open bisections $\mathcal{G}^a$ is a basis for the topology of $\mathcal{G}$ (this follows more along the lines of the definition given in Paterson [**Pat99**] or Renault [**Ren80**].

Convention. From now on, all groupoids considered in this paper will be assumed ample and with compact base space $\mathcal{G}^{(0)}$.

EXAMPLE 1.5. Let G be a discrete group and let X be a compact, Hausdorff topological G-space. Then the product $G \times X$ can be equipped with a groupoid structure that encodes the action of G on X, as follows, with product given by $(g, x)(h, y) = (gh, y)$ whenever $x = hy$, inverse $(g, x)^{-1} = (g^{-1}, gx)$ and source and range maps $s(g, x) = x$, $r(g, x) = gx$. We can topologise this by considering the sets $(g, U) = \{(g, x) \mid x \in U\}$, where U is a open subset of X. This topological groupoid is denoted by $X \rtimes G$ and is called the *action groupoid*.

In the above example stabilisers may occur. To produce a principal groupoid, one uses the groupoid of germs construction.

DEFINITION 1.6. Let f, g be partial homeomorphisms of X and let $x \in X$ be in the domain of both f and g. Then f and g have the same germ at x, denoted $f \sim_x g$, if there is some neighbourhood U of x on which f and g agree.

Thus we can define, for any étale groupoid $\mathcal{G}$ a corresponding groupoid of germs Germs($\mathcal{G}$) by considering the semigroup of ample bisections $\mathcal{G}^a$ and letting Germs($\mathcal{G}$) be the set of equivalence classes of germs of bisections. This inherits the product, inverse, range and source maps from the semigroup $\mathcal{G}^a$, and is a surjective groupoid image of $\mathcal{G}$, by mapping an element $\gamma \in \mathcal{G}$ to the germ of any bisection containing γ at $s(\gamma)$. By the remark above, this map is surjective (as every element is contained in some bisection) but not injective in general.

Germs($\mathcal{G}$) can be given a topology in the following way: for a (cl)open set $U \in \mathcal{G}^{(0)} = \text{Germs}(\mathcal{G})^{(0)}$, and an element in $A \in \mathcal{G}^a$ we can consider the sets $O_A = \{[A_x] \mid x \in s(A)\}$, and note that by declaring these sets be clopen when appropriate shows immediately that the map $\mathcal{G} \to \text{Germs}(\mathcal{G})$ defined above is open.

DEFINITION 1.7. An ample étale groupoid $\mathcal{G}$ is

- a *groupoid of germs* or *effective*, if for every non-identity $g \in \mathcal{G}$ and every bisection $A \in \mathcal{G}^a$ containing g, there is an element $h \in A$ such that $s(h) \neq r(h)$,
- *essentially principal* or *topologically principal* if the set of points with trivial isotropy is dense in the unit space of $\mathcal{G}$,
- *minimal* if the only non-empty closed invariant subset of $\mathcal{G}^{(0)}$ is $\mathcal{G}^{(0)}$.

The following is a standard result (see [**Nek17**, Proposition 2.1], [**LM15**], or [**BCFS14**, Lemma 3.1]):

PROPOSITION 1.8. *Let $\mathcal{G}$ be a topologically principal, Hausdorff ample étale groupoid, then it is a groupoid of germs. If $\mathcal{G}$ is second countable, Hausdorff groupoid of germs, then $\mathcal{G}$ is topologically principal.* $\square$

1.3. Boolean inverse monoids and full groups. In this section we give specifics concerning the semigroup structure on bisections, and the reconstruction techniques that make noncommutative Stone duality work.

The set of ample bisections $\mathcal{G}^a$ is a *distibutive inverse semigroup* under composition of bisections – if A and B are bisections, then the set product AB is also a bisection, as is A^{-1} and $A \cup B$ whenever A and B are compatible bisections. Note that the idempotent elements in $\mathcal{G}^a$ are clopen subsets of $\mathcal{G}^{(0)}$.

If $\mathcal{G}^{(0)}$ is compact, then $E(\mathcal{G}^a)$ is a Boolean algebra, and $\mathcal{G}^a$ is a *Boolean inverse monoid* [**LL13**], that is an distributive inverse monoid with a Boolean algebra of idempotents. These have been studied in the context of Stone dualities in the noncommutative setting [**LL13**]

DEFINITION 1.9. Let $\mathcal{G}$ be a second countable ample étale groupoid with compact base space. Then the subgroup of bijections of $\mathcal{G}^a$, denoted $F(\mathcal{G})$, is called the topological full group of $\mathcal{G}$.

The relationship of the full group to the original groupoid is described below.

LEMMA 1.10. *Let $F(\mathcal{G})$ be the full group of an ample topologically principal, Hausdorff étale groupoid $\mathcal{G}$ with compact base space X. Then* $\mathrm{Germs}(F(\mathcal{G}) \curvearrowright X)$ *is a closed subgroupoid of $\mathcal{G}$.*

PROOF. The subgroupoid $\mathcal{G}^{(0)} \rtimes F(\mathcal{G})$ is closed in $\mathcal{G}^{(0)} \rtimes \mathcal{G}^a$ (when $\mathcal{G}^a$ is given the discrete topology), and because the germ topology coincides with the quotient topology when $\mathcal{G}$ is Hausdorff, and this quotient map is in fact closed. □

We can improve this precisely when the full group completely describes the local structure of $\mathcal{G}^a$ in the following sense.

DEFINITION 1.11. $\mathcal{G}^a$ is *piecewise factorisable* if for every $A \in \mathcal{G}^a$ there are orthogonal idempotents $e_1, ..., e_n \in E(\mathcal{G}^a)$ and group elements $g_1, ..., g_n \in F(\mathcal{G})$ such that $A = \vee_{i=1}^n e_i g_i$.

Using [**Law15**, Theorem 2.2], we can strengthen Lemma 1.10 to the following statement:

PROPOSITION 1.12. *Let $F(\mathcal{G})$ be the full group of an ample essentially principal, Hausdorff étale groupoid $\mathcal{G}$ with compact base space X such that $\mathcal{G}^a$ is piecewise factorisable. Then* $\mathrm{Germs}(F(\mathcal{G}) \curvearrowright X)$ *is homeomorphic to $\mathcal{G}$.* □

For instance, this will occur for topologically free actions of discrete groups, because the original group will belong to the full group as a subgroup in that case.

2. Representations of inverse semigroups, groupoids and their full groups

In this section we recall the construction of a groupoid C^*-algebras. Following this, we illustrate that the groupoid representation associated to a full group (or certain subgroups of a full group) appears naturally as a reduction of the left regular representation of the associated Boolean inverse monoid of clopen bisections. This allows us to also formulate a clean connection between the "rigid" stabilisers of clopens and the groupoid representation being considered.

2.1. Groupoids associated to inverse semigroups. For any inverse semigroup S, it is possible to manufacture a variety of ample étale groupoids from S that satisfy different purposes. We first recall some definitions, following the notation from [**Exe08**] and [**Sta16**].

DEFINITION 2.1. A *filter* in $E(S)$ is a non-empty subset $\eta \subset E(S)$ such that:

(i) $0 \notin \eta$,
(ii) if $e, f \in \eta$, then $ef \in \eta$ and,

(iii) $e \in \eta$ and $e \leq f$, then $f \in \eta$.

The set of filters is denoted $\widehat{E}(S)$, and can be viewed as a subspace of $\mathbf{2}^{E(S)}$. For finite sets $X, Y \subset E(S)$, let

$$U(X,Y) = \{\eta \in \widehat{E}(S) \mid X \subset \eta, Y \cap \eta = \emptyset\}.$$

The sets of this form are clopen and generate the topology on $\widehat{E}(S)$, as X and Y are varied over all finite subsets of $E(S)$. With this topology, the space $\widehat{E}(S)$ is called the spectrum of $E(S)$.

Recall that an *ultrafilter* is a filter that is not properly contained in any other filter. The set of ultrafilters is denoted by $\widehat{E}_\infty(S)$, and as a subspace of $\widehat{E}(S)$ this *may not be closed.* Let $\widehat{E}_{tight}(S)$ denote the closure of $\widehat{E}_\infty(S)$ in the topology of $\widehat{E}(S)$ – when $E(S)$ is a Boolean algebra, we know immediately that $\widehat{E}_{tight}(S) = \widehat{E}_\infty(S)$ by Stone duality.

The second definition we recall is that of an inverse semigroup action:

DEFINITION 2.2. An *action* of an inverse semigroup S on a locally compact space X is a semigroup homomorphism $\alpha : S \to I(X)$ such that:

(i) $\alpha(s)$ is continuous for each $s \in S$,
(ii) the domain of $\alpha(s)$ is open for each $s \in S$, and
(iii) the union of the domains of the $\alpha(s)$, $s \in S$ is equal to X.

If α is an action of S on X, then as with group actions, we will write $\alpha : S \curvearrowright X$. The above implies that $\alpha(s)^{-1} = \alpha(s^*)$, and that each $\alpha(s)$ is a partial homemorphism of X. For each $e \in E(S)$, the map $\alpha(e)$ is the identity on some open subset D_e^α, and one easily sees that the domain of $\alpha(s)$ is $D_{s^*s}^\alpha$, and the range is $D_{ss^*}^\alpha$, that is:

$$\alpha(s) : D_{s^*s}^\alpha \to D_{ss^*}^\alpha.$$

We can now introduce the two groupoids we will associate to any inverse semigroup S, and recall some facts concerning them from the literature that we will need in the sequel.

EXAMPLE 2.3. There is a natural action of S on $\widehat{E}(S)$, and the *universal groupoid* of S, denoted $G(S)$, is the groupoid of germs of this action. This was introduced by Paterson in [**Pat99**], and appears in Exel [**Exe08**] and has alternative descriptions, for instance found inLawson–Lenz [**LL13**].

EXAMPLE 2.4. In addition to the universal groupoid, there is the *tight reduction* $G_{tight}(S)$, which is the reduction of $G(S)$ to the invariant subspace $\widehat{E}_{tight}(S)$. This was introduced by Exel [**Exe08**], and various of its properties have been studied in [**Exe09**] and [**Exe10**] – an alternative description of this groupoid would be to consider the natural action of S on $\widehat{E}_{tight}(S)$ and then take the groupoid of germs of this (following for instance the ideas of Exel [**Exe10**]).

In [**Exe10**], Exel proved a reconstruction theorem, which is useful in this context, which states that if one begins with an ample étale groupoid $\mathcal{G}$, then $G_{tight}(\mathcal{G}^a) \cong \mathcal{G}$ via the procedure above applied to $S = \mathcal{G}^a$. More generally, this is an example of a *noncommutative Stone duality* [**Law15**], and has been studied in much more generality for classes of inverse semigroups by Lawson–Lenz [**LL13**] – it has even been improved to a categorical notion. For an inverse semigroup, we remark that the above procedure performed in a loop provides:

$$G_{tight}(G_{tight}(S)^a) \cong G_{tight}(S)$$

for any inverse semigroup S.

2.2. Inverse semigroup C^*-algebras. In this section we study from the perspective of the inverse monoid associated representations of the group of invertible elements, with the examples from the previous section in mind. First we recall the construction, and then discuss the notions of induction and restriction in this context.

Let S be an inverse monoid. Following [**Pat99**, Section 2.1], we define the monoid algebra $\mathbb{C}S$ by extending the multiplication on S linearly and the inversion antilinearly:

$$\left(\sum_{s\in S} a_s s\right)\left(\sum_{t\in S} a_t t\right) = \sum_{s,t\in S} a_s b_t \cdot st,$$

$$\left(\sum_{s\in S} a_s s\right)^* = \sum_{s\in S} \overline{a_s} s^*.$$

This makes $\mathbb{C}S$ a $*$-algebra, and every element $s \in S$ is a partial isometry in $\mathbb{C}S$. Therefore, we can define the *universal C^*-algebra of S* by completing $\mathbb{C}S$ with respect to the norm

$$\|a\| = \sup\|\pi(a)\|,$$

where the supremum is taken over all $*$-representations of $\mathbb{C}S$. Similar to the case of discrete groups, $*$-representation of S are in one-to-one correspondence with $*$-representations of $C^*(S) := \overline{\mathbb{C}S}^{\|\cdot\|}$.

We denote by $\ell^2 S$ the Hilbert space with basis S and let

$$\lambda \colon S \to \mathbb{B}(\ell^2 S)$$

$$\lambda(s)\delta_t := \begin{cases} \delta_{st}, & tt^* \leqslant s^*s, \\ 0, & \text{otherwise} \end{cases}$$

be the left regular representation of S; it extends to an injective representation of $\mathbb{C}S$ (in fact, by a Theorem of Wordingham [**Wor82**], it extends to a injective representation of $\ell^1 S$). Observe that if $G \subset S$ is a subgroup of S that shares the unit of S, then $\lambda(G)$ consists of unitaries. The *reduced C^*-algebra of S* is then defined as $C^*_r(S) := \overline{\lambda(\mathbb{C}S)} \subset \mathbb{B}(\ell^2 S)$.

If S has a zero element $o \in S$, then $I_o := \mathbb{C} \cdot o$ is an ideal in $\mathbb{C}S$ and we have

$$\lambda(o)\delta_s = 0, \quad s \neq o, \quad \lambda(s)\delta_o = \delta_o, \quad s \in S.$$

We remark that o gives rise to a central projection in any $*$-representation of S and so if we consider the algebras:

$$C^*_o(S) = C^*(S)/I_o,$$
$$C^*_{r,o}(S) = C^*_r(S)/\lambda(I_o)$$

we obtain a decomposition $C^*_-(S) \cong C^*_{-,o}(S) \oplus I_o$, where $-$ is a place holder for maximal or reduced respectively. We call $C^*_o(S)$ and $C^*_{r,o}(S)$ the *truncated* universal and reduced C^*-algebras of S.

Notice that in discussing properties like nuclearity or exactness it does not matter whether to work with the truncated or full version of inverse monoid C^*-algebras. It is also easy to see that $*$-representations of $C^*_o(S)$ correspond to $*$-representations of S where o acts as the zero operator.

2.3. Groupoid C^*-algebras and some representation theory. Let $\mathcal{G}$ be a locally compact étale groupoid. Then the algebra of continuous, compactly supported functions

$$C_c(\mathcal{G}) = \{f : \mathcal{G} \to \mathbb{C} \mid \operatorname{supp} f \text{ compact}\}$$

is a $*$-algebra under convolution and pointwise conjugation. It admits a maximal norm [**Pat99**, **Ren80**] satisfying the usual universal property. Denote the completion in this maximal norm by $C^*\mathcal{G}$ when there is no ambiguity. In addition to this maximal norm, there also a *reduced norm*, defined as follows:

$$\|f\|_r := \sup_{x \in \mathcal{G}^{(0)}} \|\lambda_x(f)\|,$$

where $\lambda_x : C_c(\mathcal{G}) \to \mathfrak{B}(\ell^2(\mathcal{G}x))$ is the natural left regular representation of $\mathcal{G}$ on the orbit of x given by... Denote the completion of $C_c(\mathcal{G})$ in this norm by $C^*_r(\mathcal{G})$, the reduced C^*-algebra of $\mathcal{G}$. Note that by the universal property of $C^*(\mathcal{G})$, there is a surjective quotient homomorphism $C^*\mathcal{G} \to C^*_r(\mathcal{G})$.

Given a closed subset $C \subset \mathcal{G}^{(0)}$, we define the restriction groupoid $\mathcal{G}|_C$ by $s^{-1}(C) \cap r^{-1}(C)$, and note that if C is invariant under $\mathcal{G}$ then this is a closed subgroupoid of $\mathcal{G}$, and that there is a natural restriction map from $C_c(\mathcal{G})$ to $C_c(\mathcal{G}|_C)$, which extends continuously to both maximal and regular representations.

Applying this construction to the groupoids attached to an inverse semigroup S, we obtain a natural surjective homomorphism $C^*_-(G(S)) \to C^*_-(G_{tight}(S))$ (where $-$ is a placeholder for maximal or reduced), because $\widehat{E}_{tight}$ is a closed invariant subset of $\widehat{E}$, for every inverse semigroup S [**Exe08**].

Finally, we can relate this construction directly to $C^*_-(S)$ defined in the previous section. By the work of Paterson [**Pat99**], Exel [**Exe08**] and Khoskham–Skandalis [**KS02**] , we have the following isomorphism:

$$C^*_-(S) \cong C^*_-(G(S)).$$

More is known concerning the connection between representations of S and those of $G(S)$: in [**Exe08**], Exel showed that representations of S by partial isometries are in one-to-one correspondence with representations of $C_c(G(S))$, and that tight representations of S are in one-to-one correspondence with representations of $C_c(G_{tight}(S))$, i.e those that factor through the canonical quotient map $C_c(G(S)) \to C_c(G_{tight}(S))$. We will make use of this fact in the next section.

2.4. Groupoid and Koopman representations of $F(\mathcal{G})$**.** We begin this section with a construction fundamental to later aspects of this paper.

Let $\mathcal{G}^a$ be the Boolean inverse semigroup attached to a ample étale groupoid $\mathcal{G}$. Then by appealing to Exel's reconstruction theorem [**Exe10**], or one of the various non-commutative Stone dualities [**LL13**], the groupoid $G_{tight}(\mathcal{G}^a)$ is topologically isomorphic to $\mathcal{G}$, and so the restriction of functions coupled with the identification of groupoid and semigroup C^*-algebras described in the previous section induces a quotient homomorphism $\pi : \mathbb{C}\mathcal{G}^a \to C_c(\mathcal{G})$, which maps a compact open bisection U to its characteristic function $1_U \in C_c(\mathcal{G})$.

Definition 2.5. The canonical representation $\pi : F(\mathcal{G}) \to U(C_c(\mathcal{G}))$ obtained by restricting the construction above is called the *canonical groupoid representation of* $F(\mathcal{G})$.

A representation $\pi\colon F(\mathcal{G}) \to U(\mathcal{H})$ is a *groupoid representation of* $F(\mathcal{G})$ if it factors through the canonical groupoid representation π. If $\theta\colon C_c(\mathcal{G}) \to \mathbb{B}(\mathcal{H})$ is any representation of $\mathcal{G}$, we denote the corresponding groupoid representation of $F(\mathcal{G})$ by $\pi_\theta := \theta \circ \pi$.

Remark 2.6. This definition of a groupoid representation is nothing but a representation of $F(\mathcal{G})$ that comes from a tight completion of Boolean inverse monoid $\mathcal{G}^a$. This follows from [**Exe08**, Theorem 13.3] and the identifications from the previous section.

As remarked above, Lemma 1.12 shows that $F(\mathcal{G})$ has the same orbits as $\mathcal{G}$ when acting on X. This makes the quasi-regular representations:

$$\rho_x : F(\mathcal{G}) \to \mathfrak{B}(\ell^2(\mathcal{G}_x)$$

important in determining the structure of the groupoid representation associated to the regular representation of $\mathcal{G}$.

These sorts of representation were studied independently by Birget [**Bir04**] and Nekrasheyvch [**Nek04**] in a particular example (which will will study in detail in the next section) and appear in [**DG17**] in the context of weakly branch groups.

Another key example of a representation is the *Koopman representation* of $F(\mathcal{G})$. Let μ be a $F(\mathcal{G})$-quasi-invariant measure on $\mathcal{G}^{(0)}$ and let $\kappa\colon F(\mathcal{G}) \to U(L^2(\mathcal{G}^{(0)}, \mu))$ be the representation defined in the following way:

$$\kappa(g)(f)(x) = \sqrt{\frac{d\mu(g^{-1}(x)}{d\mu(x)}} f(g^{-1}x), \quad f \in L^2(\mathcal{G}^{(0)}, \mu)$$

This representation is extended to compact open bisections $U \in \mathcal{G}^a$, acting now by partial isometries rather than unitaries by the similar formula:

$$\kappa(U)(f)(x) = \sqrt{\frac{d\mu(U^*(x)}{d\mu(x)}} f(U^*x).$$

LEMMA 2.7. *The Koopman representation $\kappa : \mathcal{G}^a \to \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu)$ defined above is a tight representation of $\mathcal{G}^a$, thus induces a groupoid representation of $F(\mathcal{G})$ in the sense of Definition 2.5.*

PROOF. As $E(\mathcal{G}^a)$ is a Boolean algebra by [**Exe08**, Proposition 11.9] it suffices to check that the representation induces a Boolean algebra homomorphism. This is true, since the idempotents are mapped precisely to characteristic functions of clopens via this representation. $\square$

2.5. Representations associated to subgroups of inverse semigroups. For an inverse monoid S there is no relationship between the reduced C^*-algebra of a inverse submonoid T of S and $C^*_r(S)$ in general. What we show in the following Proposition is that if T is a group, then some connection exists.

PROPOSITION 2.8. *Let S be an inverse monoid and $U \subset S$ be a subgroup of S with the same identity element. Then $C^*_r(U)$ is a subquotient of $C^*_r(S)$.*

PROOF. Let $\lambda\colon \mathbb{C}S \to \mathbb{B}(\ell^2 S)$ denote the left regular representation of S and let

$$A = \overline{\lambda(\mathbb{C}U)} \subset C^*_r(S) \subset \mathbb{B}(\ell^2 S).$$

Consider the orthogonal projection $p\colon \ell^2 S \to \ell^2 U$ and observe that for every $g \in U$

$$p\lambda(g)p = \lambda_g\colon \ell^2 U \to \ell^2 U,$$

where λ_g denotes the left regular representation of U applied to $g \in U$. Moreover, for each $g \in U$, we know that $\ell^2 U \subset \ell^2 S$ is an invariant subspace for the unitary $\lambda(g)$, hence p commutes with $\lambda(g)$ for all $g \in U$ and the map

$$\pi\colon A \to C^*_r(U),$$

$$a \mapsto pap,$$

is a $*$-homomorphism of C^*-algebras; it is surjective because it has dense range. Thus, $C^*_r(U)$ is a quotient of A. $\square$

2.6. Induction of representations. Denote by $\operatorname{Res}_S^U \pi$ the representation obtained by restricting π to a group representation of the unit subgroup U.

PROPOSITION 2.9. *Let S be an inverse monoid, and U the group of units of S. For every unitary representation $\pi : U \to \mathbb{B}(H)$ there is a representation $\operatorname{Ind}_U^S : S \to \mathbb{B}(H')$ such that π is contained in $\operatorname{Res}_S^U \operatorname{Ind}_U^S \pi$.*

PROOF. We construct a representation as follows. Observe that U acts on S by bijections, and thus the orbit space S/U is well defined, let $p : S \to S/U$ be the map that assigns each $s \in S$ it's U-orbit sU. We note that unlike the group case, these might not give bijective copies of U – denote by U_s the stabiliser subgroup for each $s \in S$. Now consider the space of functions:

$$\mathcal{F} := \{f : S \to H \mid f(su) = \pi(u)^{-1} f(s) \text{ and } p(\operatorname{supp} f) \text{ is finite}\}.$$

As with group induction, our goal is to construct an inner product on this space. Let $f, f' \in \mathcal{F}$, and define:

$$\langle f, f' \rangle := \sum_{x \in S/U} \langle f(x), f'(x) \rangle_H.$$

This is well defined, since the map

$$x \mapsto \langle f(x), f'(x) \rangle$$

is constant on the right U-orbits. Take H' to the Hilbert space completion of $\mathcal{F}$ using this inner product, and define the map $\operatorname{Ind}_U^S \pi(s)$ using the left action of S on itself:

$$\operatorname{Ind}_U^S \pi(s) f(x) := f(s^* x).$$

This is clearly a homomorphism of S, and each $\operatorname{Ind}_U^S \pi(s)$ is a partial isometry as it defines a linear map between the subspaces: H'_{s^*s} and H'_{ss^*}, where H'_e is the closure in H' of all the functions containing the orbit of e in their support.

The final claim concerning the restriction $\operatorname{Res}_S^U \operatorname{Ind}_U^S \pi : U \to U(H')$ follows by restricting the representation to the U-invariant subspace H'_{1_S}, this has a free U-action, in particular this gives rise to a multiple of original representation π.

□

In the case of the representation λ_S we can describe the weak equivalence class of $\pi = \operatorname{Res}_S^U \lambda_S$.

PROPOSITION 2.10. *$\pi = \operatorname{Res}_S^U \lambda_S$ is weakly equivalent to $\bigoplus_{e \in E(S)/U} \lambda_{U/U_e}$, where λ_{U/U_e} is the quasiregular representation of U on $\ell^2(U/U_e)$.*

PROOF. Splitting S into right U-orbits we decompose $\ell^2(S)$ into $\oplus_{s \in S/U} \ell^2(sU)$. Letting U_s be the stabiliser of sU under the left action, we obtain that the restriction of λ_S decomposes as $\bigoplus_{s \in S/U} \lambda_{U/U_s}$, however, we note that since $U_s = U_{ss^*}$, that this summand is weakly equivalent to the one in the claim. □

3. Comparing some natural representations of the full group

The goal of this section is to relate these representations of $F(\mathcal{G})$ following the ideas of [**DG17**] and [**DG15**] where the corresponding theory was developed in the weakly branch case. The main result is the following:

THEOREM 3.1. *Let $\mathcal{G}$ be an ample étale groupoid of germs and let $F(\mathcal{G})$ be its topological full group. Let λ be the left regular representation of $\mathcal{G}$. Then for the representations π_λ, π_κ and ρ_x defined above we have:*

(i) If $\mathcal{G}$ is minimal then
- *(a) $\pi_\lambda \sim \rho_x$ for every $x \in X$.*
- *(b) ρ_x is irreducible for each $x \in X$.*
- *(c) For $x, y \in X$, with x and y in distinct $\mathcal{G}$-orbits, ρ_x and ρ_y are not unitarily equivalent.*

(ii) If $\mathcal{G}$ is topologically amenable, then $\pi_\kappa \prec \pi_\lambda$.

(iii) If $\mathcal{G}$ is minimal and topologically amenable, then $\pi_\kappa \sim \pi_\lambda$.

PROOF OF (I) A), (II) AND (III). For (i) a), we observe that $\|\pi(g)\|_r$ is the reduced groupoid norm of the element $g \in C_c(\mathcal{G})$, and thus it bounds above the norm $\|\lambda_x(\pi(g))\|$ for each x. Note that the representation $\lambda_x \circ \pi$ agrees with the quasi-regular representation ρ_x of $F(\mathcal{G})$. By a result of Khoshkam and Skandalis [**KS02**, Corollary 2.4], it is sufficient to compute the supremum over a dense subset of the unit space, such as any single $\mathcal{G}$-orbit. But by Lemma 1.12, these are precisely the $F(\mathcal{G})$-orbits.

For (ii), we observe that as $\mathcal{G}$ is amenable, the reduced C^*-algebra $C^*_r(\mathcal{G})$ is isomorphic to the maximal C^*-algebra $C^*(\mathcal{G})$, and so it satisfies the universal property of the maximal completion. By Lemma 2.7, the representation κ is a groupoid representation of $F(\mathcal{G})$, thus it is obtained through some completion of $C_c(\mathcal{G})$ [**Exe08**, Theorem 13.3]. Then $\|\kappa(g)\| = \|\pi(g)\|_{\mathbb{B}(L^2(X,\mu))} \leqslant \|\pi(g)\|_\lambda = \|\pi_\lambda(g)\|$, where the middle inequality follows from the fact that the reduced norm and maximal norm on $C_c(\mathcal{G})$ agree. Thus $\pi_\kappa \prec \pi_\lambda$.

(iii) follows from (ii) and the fact that $C^*_r(\mathcal{G})$ is a simple C^*-algebra when $\mathcal{G}$ is minimal. $\square$

To show (i) b) and c) we use Mackey's criterion for irreducibility and disjointness of quasi-regular representations in terms of commensurators of subgroups and the notion of quasi-conjugacy.

DEFINITION 3.2. Recall that $H, K < G$ are *commensurable* if $H \cap K$ has finite index in H and K and are *quasi-conjugate* if $H^g := gHg^{-1}$ is commensurate with K for some $g \in G$. the commensurator is defined, for $H \leqslant G$, as:

$$\mathrm{Comm}_H(G) := \{g \in G \mid H \cap H^g \text{ has finite index in } H \text{ and } H^g\}$$

Let us recall Mackey's criterion:

THEOREM 3.3 ([**Mac76**, Corollary 7]). *Let G be a countable discrete group, H, K be subgroups of G.*

(i) The quasi-regular representation $\rho_{G/H}$ is irreducible if and only if $\mathrm{Comm}_G(H) = H$.
(ii) The quasi-regular representations $\rho_{G/H}$ and $\rho_{G/K}$ are unitarily equivalent if and only if H and K are quasi-conjugate in G.

□

DEFINITION 3.4. Let $U \subseteq \mathcal{G}^{(0)}$ be clopen. The rigid stabiliser $F(\mathcal{G})_(U)$ of U is the subgroup of $F(\mathcal{G})$ consisting of all elements which act trivially on $\mathcal{G}^{(0)} \setminus U$.

The following technical lemma will be used in the sequel.

LEMMA 3.5. *Let $\mathcal{G}$ be a minimal groupoid of germs. Then*

(i) each rigid stabiliser $F(\mathcal{G})_{(U)}$ is isomorphic to the full group $F(\mathcal{G}|_U)$ of the restriction of $\mathcal{G}$ to U.
(ii) for every pair of non-empty clopen subsets $U, V \neq \mathcal{G}^{(0)}$ and every point $x \in V$ there exists a $g \in F(\mathcal{G})$ such that $x \in g(U) \cap V$ (in particular, $g(U) \cap V \neq \varnothing$).

PROOF. To show (i), we observe that for every non-empty clopen $U \subset \mathcal{G}^{(0)}$, the restriction groupoid $\mathcal{G}|_U$ is also a groupoid of germs, and the rigid stabilisers $F(\mathcal{G})_{(U)}$ are precisely the groups $F(\mathcal{G}|_U)$ extended trivially on the complement of U. By [**BG00**, Lemma 3.1], these contain arbitrarily long finite orbits, thus $F(\mathcal{G})_{(U)}$ is infinite.

For (ii): Notice that due to the minimality of the groupoid, for every non-empty clopen U, the union $\bigcup_{g \in F(\mathcal{G})} g(U)$ is equal to $\mathcal{G}^{(0)}$, thus one of them contains x. This proves (ii). □

The following Lemmas are modifications of the work of Bartholdi–Grigorchuk [**BG00**] and Dudko–Grigochuk [**DG15**] into the setting of full groups.

LEMMA 3.6. *Let $\mathcal{G}$ be a minimal groupoid of germs, $\Gamma = F(\mathcal{G})$ its topological full group, $x \in \mathcal{G}^{(0)}$ and $S_x = \mathrm{Stab}(x) < \Gamma$. Then $\mathrm{Comm}_\Gamma(S_x) = S_x$.*

PROOF. Let $g \in \Gamma \setminus S_x$. We will show the subgroup $S_x^g \cap S_x$ has infinite index in S_x^g by showing that the orbit $S_x^g \cdot x$ is infinite.

First we observe that as the base space is Hausdorff and totally disconnected, there is a clopen neighbourhood U of x such that $g(U) \cap U = \varnothing$. Now $(S_x^g)_{(U)} = F(\mathcal{G})_{(U)}$ since $g(U) \subset \mathcal{G}^{(0)} \setminus U$ by assumption, wherefore any element of $F(\mathcal{G})$ that fixes $\mathcal{G}^{(0)} \setminus U$ pointwise must fix $g(U)$ pointwise, thus belonging to $S_{gx} = S_x^g$. Hence we have $(S_x \cap S_x^g)_{(U)} = S_x \cap F(\mathcal{G})_{(U)} = (S_x)_{(U)}$.

The orbit of $F(\mathcal{G})_{(U)} \cdot x$ is infinite by Lemma 3.5 (i) and minimality of the resctriction groupoid. As moreover $F(\mathcal{G})_{(U)} \leqslant S_x^g$, the orbit $S_x^g \cdot x$ also follows infinite, which concludes the proof. □

LEMMA 3.7. *If $x, y \in X$ belong to distinct $\mathcal{G}$-orbits, then S_x and S_y are not quasi-conjugate in G.*

PROOF. Our goal is to show that $S_x y$ and $S_y x$ are infinite, which is equivalent to quasi-conjugacy after replacing x by gx for some appropriate g. This follows from the definition 3.2.

Let $\{W_i\}_{i\in\mathbb{N}}$ be a nested family of neighbourhoods of x with $\cap_i W_i = \{x\}$. Then, pick an i large enough such that $y \notin W_i$, and find a clopen neighbourhood $V \ni y$ such that $V \cap W_i = \emptyset$ (which can be done because X is regular).

Now we will make repeated use of Lemma 3.5 (ii) to complete the proof. Let $h \in F(\mathcal{G})_{W_i}$ be some element that is not trivial, and then for each $j \geq i$, find a k_j such that $k_j(\mathrm{supp}(h)) \cap W_j$ contains x, and is thus not empty. Let V_j denote $k_j^{-1}(k_j(\mathrm{supp}(h) \cap W_j)$.

Set, for each j, $Z_j = W_{j+1} \setminus W_j$, which is clopen as all the W_j are. Then again, apply Lemma 3.5 (ii) to obtain a z_j such that $z_j(Z_j) \cap h(V_j)$ is not empty and contains $hk_j^{-1}x$. Then the elements $z_j^{-1}hk_j^{-1}$ does not stabilise x (so it is not trivial), belongs to $F(\mathcal{G})_{W_i}$ (and thus S_y, since $y \notin W_i$) and maps x into Z_j.

Note that for each $j, l \in \mathbb{N}$, the sets Z_j and Z_l have empty intersection, so the points $z_j^{-1}hk_j^{-1}x$ are all distinct, and this provides infinitely many distinct points in the orbit $S_y x$. By symmetry, we can construct an infinite subset of the orbit $S_x y$. □

Now, parts (i) b), c) of Theorem 3.1 follow from Mackey's criterion directly.

4. Amenability and C^*-simplicity of $F(\mathcal{G})$

4.1. Invariant means and amenability.

DEFINITION 4.1. Let S be a Boolean inverse semigroup (such as $\mathcal{G}^a$). Then S has an *invariant mean* if there is a function $\mu : E(S) \to [0, \infty)$ such that:

(i) for any $s \in S$, we have that $\mu(s^*s) = \mu(ss^*)$,
(ii) if e and f are orthogonal idempotents, then $\mu(e \vee f) = \mu(e) + \mu(f)$.

A mean μ is *normalised* at $e \in E(S)$ if $\mu(e) = 1$. By convention, if we don't mention any normalisation then we suppose $\mu(1) = 1$, and *faithful* if $\mu(e) = 0$ implies $e = 0$.

The existence of invariant means on Boolean inverse monoids is investigated by Kudryavtseva–Lawson–Lenz–Resende in [**KLLR16**], and we will go into detail into the ideas in that work in the next section. Additionally in the context of an ample groupoid $\mathcal{G}$, Starling [**Sta16**] addressed the existence of invariant means on $\mathcal{G}^a$ in the context of both invariant measures on the base space of $\mathcal{G}$ and traces on the reduced C^*-algebra $C_r^*(\mathcal{G})$.

THEOREM 4.2. *Suppose $\mathcal{G}$ is an ample Hausdorff étale groupoid, then the following sets are in bijection:*

- $IM(\mathcal{G}) := \{\mu \in \mathrm{Prob}(G^{(0)}) \mid \mu(s(U)) = \mu(r(U))\ \forall U \in \mathcal{G}^a\}$

- $M(\mathcal{G}^a) := \{m \mid m \textit{ a mean on } E(\mathcal{G}^a)\}$.
- $T(C_r^*(\mathcal{G})) = \{\textit{traces on } C_r^*(\mathcal{G})\}$.

However, asking that the groupoid $\mathcal{G}$ preserves a measure on its own base space $\mathcal{G}^{(0)}$ is not the only possible notion of amenability one could consider – the other is *topological amenability* [**ADR00**]. We now relate the existence of an invariant mean on $\mathcal{G}^a$ to the amenability of $F(\mathcal{G})$ in the minimal case, partly addressing Remark 2.25 in [**KLLR16**].

THEOREM 4.3. *Let $\mathcal{G}$ be a minimal ample Hausdorff étale groupoid of germs. Then $F(\mathcal{G})$ is amenable if and only if the following three conditions are satisfied:*

(i) $\mathcal{G}^a$ admits an invariant mean,
(ii) $\mathcal{G}$ is topologically amenable and
(iii) the stabiliser $F(\mathcal{G})_U$ is amenable for each clopen subset $U \subset \mathcal{G}^{(0)}$.

PROOF. Suppose $F(\mathcal{G})$ is amenable, then it is immediate that (iii) holds, since subgroups of amenable groups are amenable. To conclude (ii), we appeal to minimality: as $\mathcal{G}$ minimal, we can apply Lemma 1.12 to conclude that $\mathcal{G} = Germs(F(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)})$ is a groupoid quotient of $F(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$. Since $F(\mathcal{G})$ is amenable, the transformation groupoid $F(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is topologically amenable. By applying Proposition 5.1.2 from [**ADR00**], which states that topological amenability is groupoid extension closed, we can conclude that $\mathcal{G}$ is topologically amenable. Finally, i) is Proposition 2.24 in [**KLLR16**] (which was the original motivation for considering this problem).

We now show the other direction by supposing conditions (i), (ii) and (iii). We will appeal to representation theory to conclude the result. We analyse groupoid representations of $F(\mathcal{G})$: the first step is to use group to inverse semigroup induction to understand the natural groupoid representation in $C_r^*(\mathcal{G})$, and the second step involves studying the Koopman representation of $F(\mathcal{G})$. We combine these using Theorem 3.1 using assumption ii).

Let $\pi := \mathrm{Res}^{\mathcal{G}^a}_{F(\mathcal{G})} \lambda_{\mathcal{G}^a}$ be the representation of $F(\mathcal{G})$ obtained by completing $\mathbb{C}F(\mathcal{G})$ in $C_r^*(\mathcal{G}^a)$. By Proposition 2.10 $\tilde{\pi}$ is weakly equivalent to $\oplus_U \lambda_{F(\mathcal{G})/F(\mathcal{G}_U)}$. However, since each $F(\mathcal{G})_U$ is amenable, each representation $\lambda_{F(\mathcal{G})/F(\mathcal{G}_U)}$ is weakly contained in the left regular representation $\lambda_{F(\mathcal{G})}$, so we know that π is weakly equivalent to $\lambda_{F(\mathcal{G})}$.

Consider the groupoid representation π_λ of $F(\mathcal{G})$ in $C_r^*(\mathcal{G})$. This representation π_λ is weakly contained in π, since the algebraic map:

$$\mathbb{C}\mathcal{G}^a \to C_c(\mathcal{G})$$

extends surjectively to reduced completions, so for any $g \in F(\mathcal{G})$, there is an inequality of norms $\|\pi_\lambda(g)\| \leq \|\pi(g)\|$. Appealing to a result of Dixmier (here referenced as Theorem F.4.4 from Bekka–de la Harpe–Valette [**BdlHV08**]) this is equivalent to weak containment of π_λ in π.

The second step is to analyse the Koopman representation of $F(\mathcal{G})$, which is also a groupoid representation by Lemma 2.7. As $\mathcal{G}^a$ admits an invariant mean, $\mathcal{G}$ admits an invariant measure by Corollary 4.12 in [**Sta16**]. Note that this measure is automatically $F(\mathcal{G})$-invariant, and so by Theorem 5.7 in [**Bek16**], the representation π_κ has almost invariant vectors; this means exactly that $1_{F(\mathcal{G})}$ is weakly contained in π_κ.

Since $\mathcal{G}$ is topologically amenable, we can apply part (iii) of Theorem 3.1 to obtain the following chain of weak containments and equivalences:

$$1_{F(\mathcal{G})} \prec \pi_\kappa \sim \pi_\lambda \prec \pi \sim \lambda_{F(\mathcal{G})}.$$

Thus $F(\mathcal{G})$ is amenable (since weakly containment of the trivial representation in the left regular representation is a characterisation). $\square$

We remark that it is possible for $F(\mathcal{G})$ to be non-amenable, whilst $\mathcal{G}^a$ admits an invariant mean – due to results of Starling [**Sta16**] and Kerr–Nowak [**KN12**], it is the case for any residually finite action of a free group (or more generally, an action of a non-amenable group preserving a probability measure on the Cantor set).

This also connects with the work of Haagerup–Olesen [**HKO16**] part of which is concerned with C^*-simplicty of Thompson's group T and the amenability of F.

LEMMA 4.4. *Let $\mathcal{G}$ be a minimal ample étale groupoid with Cantor set base space X. Then the representation $\pi : G \to C_c(\mathcal{G})$ does not extend faithfully to $\mathbb{C}G$ for any subgroup of $F(\mathcal{G})$ with non-trivial rigid stabilisers. Thus it is not injective for any faithful groupoid representation of G.*

PROOF. Note that by minimality, for every clopen subset $U \subset X$, the subgroup $F(\mathcal{G}|_U)$ is infinite - so choose a partition of X into U and U^c, and then consider $g_1 \in F(\mathcal{G}|_U)$ and $g_2 \in F(\mathcal{G}|_{U^c})$. These elements are faithfully represented in $C_c(\mathcal{G})$ via the construction at the beginning of section 2.4, and satisfy the formula:

$$1 + g_1 g_2 = g_1 + g_2.$$

Thus, the element $a = g_1g_2 - g_1 - g_2 + 1 = 0$ in any faithful groupoid representation of $F(\mathcal{G})$ extended linearly to $\mathbb{C}F(\mathcal{G})$. $\square$

THEOREM 4.5. *Let $\mathcal{G}$ be a minimal ample étale groupoid of germs with Cantor set base space X. If the stabiliser $F(\mathcal{G})_U$ is amenable for every clopen subset U, then the group $F(\mathcal{G})$ is not C^*-simple.*

PROOF. This uses part of the argument for Theorem 4.5, but we repeat it here. Let $\pi := \mathrm{Res}^{\mathcal{G}^a}_{F(\mathcal{G})} \lambda_{\mathcal{G}^a}$ be the representation of $F(\mathcal{G})$ obtained by completing $\mathbb{C}F(\mathcal{G})$ in $C^*_r(\mathcal{G}^a)$. By Proposition 2.10 $\tilde{\pi}$ is weakly equivalent to $\oplus_U \lambda_{F(\mathcal{G})/F(\mathcal{G}_U)}$. However, since each $F(\mathcal{G})_U$ is amenable, each representation $\lambda_{F(\mathcal{G})/F(\mathcal{G}_U)}$ is weakly contained in the left regular representation $\lambda_{F(\mathcal{G})}$ (by induction), so we know that the groupoid representation π_λ is weakly contained in $\lambda_{F(\mathcal{G})}$. However, the representation π_λ has an algebraic kernel by Lemma 4.4, so $C^*_{\pi_\lambda}(F(\mathcal{G})$ is a proper quotient of $C^*_r(F(\mathcal{G}))$, which completes the proof. $\square$

This theorem can be regarded as a conceptualisation of the ideas of Haagerup–Oleson [**HKO16**], which appears also in the work of Nekrasheyvch [**Nek04**] and Birget [**Bir04**].

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CHAPTER 10

Maximal amenable subgroups of arithmetic groups

1. Preliminaries

The main goal of this work is to study maximal amenable subgroups of S-arithmetic groups. Already from the definition of an S-arithmetic group it's clear that here it is indispensable to study groups up to commensurability, because an arithmetic group is defined up to commensurability. Passing to a commensurable group, while preserving amenability, can interfere with maximality: a finite group $\mathbb{Z}/6$ is maximal amenable in $\mathbf{SL}_2(\mathbb{Z})$. Therefore, we'll first study maximal amenable commensurability classes, whose representatives we'll call *commensurably maximal amenable subgroups.*

We primarily follow the notation and conventions of [**Mar91**], occasionally using some results on algebraic groups from other sources. In particular, we freely use results on

- algebraic groups, tori and parabolic subgroups [**Mar91**, 0.20–0.29]
- local and global fields [**Mar91**, 0.31–0.34]
- algebraic groups over local and global fields and their arithmetic subgroups [**Mar91**, I.1–I.3]

Let $\mathbf{G}$ be a semisimple algebraic group defined over a global field K of characteristic zero, $\mathscr{R}$ the set of (equivalence classes of) its nonequivalent valuations, $\mathscr{R}_\infty \subset \mathscr{R}$ the set of Archimedean valuations; let $\mathscr{T} \subset \mathscr{R}$ be the subset of valuations for which $\mathbf{G}$ is anisotropic. Fix a subset $S \subset \mathscr{R}$ containing $\mathscr{R}_\infty \setminus \mathscr{T}$ and let $K(S)$ be the ring of S-integral elements of K.For a K-subgroup $\mathbf{H} < \mathbf{GL}_n$, we set $\mathbf{H}(K(S)) \coloneqq \mathbf{H} \cap \mathbf{GL}_n(K(S))$.

In view of [**Mar91**, Lemma I.3.1.1] if we change the K-embedding $\mathbf{H} < \mathbf{GL}_n$, the group $\mathbf{H}(K(S))$ changes to a commensurable one. In view of this the following notion does not depend on the choice of the K-embedding $\mathbf{H} < \mathbf{GL}_n$:

Definition 1.1. A subgroup of $\mathbf{H}$ is called S-arithmetic if it is commensurable to $\mathbf{H}(K(S))$.

We'll often need to neglect a passage to a finite index sub/overgroup, so the statements as stated now have to be taken up to commensurability; we'll use $\prec$ and $\simeq$ for inclusions up to finite index resp. commensurability.

2. Classification of commensurably maximal amenable subgroups

Throughout this section, we'll fix a reductive K-subgroup $\mathbf{G} < \mathbf{GL}_n$ and set $\Gamma \coloneqq \mathbf{G}(K(S))$.

LEMMA 2.1. *Let $\Lambda_0 < \Gamma$ be amenable. Then, there is a finite index subgroup $\Lambda < \Lambda_0$ such that*

(i) $\Lambda_u = \{g \in \Lambda \mid g \text{ unipotent}\}$ *is a normal subgroup of* Λ *and*
(ii) $[\Lambda, \Lambda] \leqslant \Lambda_u$*; in particular, if* $\Lambda_u = 1$*, then* Λ *is abelian.*

PROOF. In view of the Tits alternative [**Tit72**], we can assume Λ to be solvable. Therefore it is contained in a Borel subgroup $\mathbf{B} < \mathbf{G}$ [**Bor12**, Corollary 11.17 (ii)]. Then $\Lambda_u = \{g \in \Lambda \mid g \text{ unipotent}\}$ is a normal subgroup of Λ. The second statement follows because $[\mathbf{B}, \mathbf{B}]$ is unipotent by the Lie–Kolchin theorem [**Bor12**, Corollary 10.5].

□

PROPOSITION 2.2. *Let $\Lambda < \Gamma$ be commensurably maximal amenable. The following properties are equivalent:*

(i) Λ *contains a unipotent element,*
(ii) *there is a proper parabolic subgroup $\mathbf{Q} < \mathbf{G}$ defined over $\mathbb{Q}$ such that $\Lambda \prec \mathbf{Q}(K(S))$ and $\Lambda_u \simeq \mathbf{U}(K(S))$.*

PROOF. Let's prove that (ii) implies (i). Suppose that $\Lambda \subseteq \mathbf{Q}(K(S))$ but $\Lambda_u = 1$. Consider the unipotent radical $\mathbf{R} \coloneqq R_u(\mathbf{Q})$; it is defined over K. Then the group generated by Λ and $\mathbf{R}(K(S))$ is still amenable, but contains Λ with infinite index, contradicting maximality. This finishes the proof of (ii) $\Rightarrow$ (i).

To show that (i) implies (ii), suppose that $\Lambda_u \neq 1$. Consider $\mathbf{U} = \overline{\Lambda_u}$, the Zariski closure of Λ_u; it is a unipotent K-subgroup of $\mathbf{G}$ by a result of Margulis [**Zim84**, Proposition 3.1.18]. Consider its normaliser $\mathbf{Q} \coloneqq N_{\mathbf{G}}(\mathbf{U})$, it's also defined over K. As Λ normalizes Λ_u and is contained in $\Gamma = \mathbf{G}(K(S))$, $\Lambda < \mathbf{Q}(K(S))$.

We now claim that $\mathbf{Q}$ is parabolic. To this end, we have to show that $\mathbf{U}$ is equal to the unipotent radical $\mathbf{R} \coloneqq R_u(\mathbf{Q})$ of $\mathbf{Q}$ [**Hum75**, Corollary 30.3.B]. By (the proof of) Borel–Tits theorem [**MT11**, Theorem 17.10], $\mathbf{U} \leqslant \mathbf{R}$. If now $\mathbf{U} \neq \mathbf{R}$, then the group $\Lambda_1 \coloneqq \mathbf{R}(K(S)) < \Gamma$ is nilpotent and contains Λ_u as a subgroup of infinite index. Passing to a subgroup of finite index of Λ if necessary, we may assume that Λ normalises Λ_1. Then the group $\Lambda' \coloneqq \langle \Lambda, \Lambda_1 \rangle < \Gamma$ is a quotient of the amenable subgroup $\Lambda \ltimes \Lambda_1$ and therefore amenable; however, it contains Λ with infinite index, contradicting maximality of the latter. Thus $\mathbf{U} = \mathscr{R}_u(\mathbf{Q})$ and therefore $\mathbf{Q}$ is parabolic. Now, $\mathbf{U}(K(S))$ is normalised by $\mathbf{Q}(K(S))$, so by maximality $\Lambda_u \simeq \mathbf{U}(K(S))$. This finishes the proof of (i) $\Rightarrow$ (ii). □

Let $\Lambda < \mathbf{Q}(K(S))$ be commensurably maximal amenable; by the previous proposition we can assume that $\Lambda_u = \mathbf{U}(K(S))$. If $\mathbf{Q} = \mathbf{L} \ltimes \mathbf{U}$ is the Levi decomposition of $\mathbf{Q}$, then $\mathbf{Q}(K(S))$ has finite index in $\mathbf{U}(K(S)) \rtimes \mathbf{L}(K(S))$ by [**Mar91**, Lemma I.3.1.1 (iv)]. Therefore Λ is commensurable to $\mathbf{U}(K(S)) \rtimes \Lambda'$, where $\Lambda' \leqslant \mathbf{L}(K(S))$ is commensurably maximal amenable because Λ is. By construction, Λ' does not contain unipotent elements. Therefore the previous proposition reduces the study of commensurably maximal amenable subgroups to the abelian case.

PROPOSITION 2.3. *Let $\Lambda < \Gamma$ be abelian and commensurably maximal amenable such that $\mathbf{T} = \overline{\Lambda}$ is connected and let $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T}) = C_{\mathbf{G}}(\Lambda)$ be the centraliser of Λ. Then*

(i) $\operatorname{rk}_K \mathbf{L} = 0$*,*

(ii) for every $v \in S$, $\operatorname{rk}_{K_v} \mathbf{L} = \operatorname{rk}_{K_v} \mathbf{T}$.

PROOF. $\mathbf{T}$ is a K-torus in $\mathbf{G}$ by [**Zim84**, Proposition 3.1.8], and thus $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T})$ is a Levi subgroup of $\mathbf{G}$ defined over K [**MT11**, Proposition 12.10].

If $\operatorname{rk}_K \mathbf{L} > 0$, then $\mathbf{L}$ contains a non-trivial K-torus $\mathbf{S}$. Now, [**Mar91**, Proposition I.1.1.3] provides the existence of a unipotent K-subgroup $\mathbf{U} < \mathbf{G}$ which is normalised by $C_{\mathbf{G}}(\mathbf{S})$. But then $\Lambda < C_{\mathbf{G}}(\mathbf{S})$ is contained in the Levi subgroup of the parabolic subgroup which normalises $\mathbf{U}(K(S))$; hence Λ and $\mathbf{U}(K(S))$ generate an amenable subgroup in which Λ has infinite index, contradicting maximality. This shows that $\operatorname{rk}_K \mathbf{L} = 0$, proving (i).

As $\Lambda < \Gamma$ is commensurably maximal amenable, the inclusion $\Lambda < \mathbf{L}(K(S))$ has to be of finite index. Decomposing $\mathbf{L} \simeq \mathbf{T} \times \mathbf{L}'$, we therefore obtain that $\mathbf{L}'(K(S))$ must be finite. As it is a lattice in L'_S [**Mar91**, Theorem I.3.2.5], the latter must be compact which implies (ii). □

Collecting the above considerations together, we now have

THEOREM 2.4. *Let $\mathbf{G} < \mathbf{GL}_n$ be a reductive algebraic group defined over a global field K of characteristic zero, $S \subset \mathscr{R}$ a subset of valuations of K, $\Gamma := \mathbf{G}(K(S))$. A subgroup $\Lambda < \Gamma$ is commensurably maximal amenable if and only if there exists a K-parabolic subgroup $\mathbf{Q} \leqslant \mathbf{G}$ such that Λ is commensurable to*

$$\Lambda_{\mathbf{T} \ltimes \mathbf{U}} = (\mathbf{T} \ltimes \mathbf{U})(K(S)) \simeq \mathbf{T}(K(S)) \ltimes \mathbf{U}(K(S)),$$

where $\mathbf{U} = R_u(\mathbf{Q})$ is the unipotent radical of $\mathbf{Q}$ and $\mathbf{T} < \mathbf{L}'$ is a K-anisotropic maximal torus in the Levi subgroup $\mathbf{L}$ of $\mathbf{Q}$. Moreover, for every $v \in S$, $\operatorname{rk}_{K_v} \mathbf{L} = \operatorname{rk}_{K_v} \mathbf{T}$.

3. Toral subgroups

From the previous considerations we see that every commensurably maximal amenable subgroup decomposes into the unipotent part and a part associated to a maximal torus in a reductive group. This motivates the following definition.

DEFINITION 3.1. A subgroup $\Lambda < \Gamma$ is called *toral* if it is commensurable to $\mathbf{T}(K(S))$, where $\mathbf{T}$ is a K-anisotropic maximal torus in $\mathbf{G}$.

LEMMA 3.2. *Let $\Lambda < \Gamma$ be a toral maximal amenable subgroup with corresponding torus $\mathbf{T}$. Then $\mathbf{T}(K(S))$ is a normal subgroup of Λ. Moreover, the quotient $\Lambda/\mathbf{T}(K(S))$ embeds into the K-points of the Weyl group $W(\mathbf{T}, \mathbf{G})(K) = N_{\mathbf{G}}(\mathbf{T})(K)/C_{\mathbf{G}}(\mathbf{T})(K)$ of $\mathbf{T}$.*

Proof. Let $\Lambda_0 \coloneqq \Lambda \cap \mathbf{T}(K(S))$; by assumption, it has finite index in both Λ and $\mathbf{T}(K(S))$. Consider $g \in \Gamma$. We claim that if $g \notin N_{\mathbf{G}}(\mathbf{T})$, then the group Γ_0 generated by g and Λ_0 is not amenable. Supposing otherwise, we first observe that since Λ is toral, it is commensurably maximal amenable, so no finite index subgroup of Λ can normalize a unipotent element and therefore Γ_0 has no unipotent elements. Hence Γ_0 is virtually abelian wherefore its Zariski closure has $\mathbf{T}$ as the connected component of identity; the lemma follows. $\square$

Lemma 3.3. *Let $v \in S$, $k = K_v$ and $\Lambda < \Gamma$ be a toral maximal amenable subgroup with corresponding torus $\mathbf{T}$. Let $\mathbf{T}_s$ be the k-split part of $\mathbf{T}$, $\mathbf{M} = C_{\mathbf{G}}(\mathbf{T}_s)$ and let $\mathbf{P}$ be the corresponding k-parabolic subgroup of $\mathbf{G}$. Consider the action $\mathbf{G}(k) \curvearrowright \mathbf{G}(k)/\mathbf{P}(k)$.*

Then $\operatorname{Fix}_{\mathbf{T}(k)}(\mathbf{G}(k)/\mathbf{P}(k)) = W(\mathbf{T}_s, \mathbf{G})(k)[\mathbf{P}(k)]$.

Proof. Take an element $g \in \mathbf{G}(k)$ satisfying $g^{-1}\mathbf{T}(k)g \subset \mathbf{P}(k)$. Then there exists a unipotent element $u \in R_u(\mathbf{P})(k)$ such that $\mathbf{T}' \coloneqq u^{-1}g^{-1}\mathbf{T}gu \leqslant \mathbf{M}$; thus, $\mathbf{T}'$ commutes with $\mathbf{T}_s$. Consider now $\mathbf{L} \coloneqq C_{\mathbf{G}}(\mathbf{T}) \leqslant \mathbf{M}$ and $\mathbf{L}' \coloneqq u^{-1}g^{-1}\mathbf{L}gu = C_{\mathbf{G}}(\mathbf{T}')$; in particular, $\mathbf{T}_s \leqslant \mathbf{L}'$. Now, by Proposition 2.3, $\operatorname{rk}_k \mathbf{L} = \operatorname{rk}_k \mathbf{T} = \dim \mathbf{T}_s = \operatorname{rk}_k \mathbf{T}' = \operatorname{rk}_k \mathbf{L}'$. Thus, $\mathbf{T}_s$ is a maximal k-split torus of $\mathbf{L}'$ which by construction and rank properties of $\mathbf{L}'$ is unique and central; thus, $\mathbf{T}'_s = \mathbf{T}_s$. Now, $gu \in N_{\mathbf{G}}(\mathbf{T}_s)(k)$; as $u \in \mathbf{P}(k)$, the claim follows.

$\square$

We recall the definition of singular subgroups from [**BC15**].

Definition 3.4. Let Γ be a discrete group. An amenable subgroup $\Lambda < \Gamma$ is called singular if there exists a compact Γ-space X such that for every Λ-invariant probability measure μ on X and for every $g \in \Gamma \setminus \Lambda$ we have $g_*\mu \perp \mu$.

Note that singularity is a strenghtening of maximal amenability. The main application of singularity is the following theorem [**BC15**, Theorem A]:

Theorem 3.5. *Let Γ be a countable discrete group and $\Lambda < \Gamma$ be an amenable singular subgroup. Then for any trace preserving action $\Gamma \curvearrowright (Q, \tau)$ on a finite amenable von Neumann algebra, $Q \rtimes \Lambda$ is maximal amenable inside $Q \rtimes \Gamma$.*

Proposition 3.6. *A toral maximal amenable subgroup Λ is singular.*

Proof. Let
$$X = \prod_{v \in S} \mathbf{G}(K_v)/\mathbf{P}_v(K_v),$$
where $\mathbf{P}_v$ is the parabolic subgroup whose Levi subgroup is the centraliser $C_{\mathbf{G}}(\mathbf{T}_{v,s})$ of the K_v-split part of $\mathbf{T}$ as in Lemma 3.3. We will show that Λ is singular in Γ with respect to X.

First we observe that the action of Γ on X is *algebraic* and in particular *smooth* and thus every probability measure preserved by Λ is supported on a finite orbit. Also observe that as $\mathbf{T}$ is connected, every finite subset of X invariant by $\mathbf{T}(K(S))$ is actually pointwise invariant.

Let $\gamma \in \Gamma$ and assume that $\gamma \operatorname{Fix}_{\mathbf{T}}(X) \cap \operatorname{Fix}_{\mathbf{T}}(X)$ is not empty. Without loss of generality, we can assume that $\prod_{v \in S} \gamma \mathbf{P}_v(K_v) \in \operatorname{Fix}_{\mathbf{T}}(X)$, so $\gamma^{-1}\mathbf{T}(K_v)\gamma \subset \mathbf{P}_v(K_v)$ for all $v \in S$. But by the Lemma 3.3, the K_v-parabolic in which the torus $\gamma^{-1}\mathbf{T}\gamma$ lies is uniquely determined by it up to the Weyl group of $\mathbf{T}_{v,s}$. Therefore $\gamma^{-1}\mathbf{T}_{v,s}\gamma = \mathbf{T}_{v,s}$ for all $v \in S$, so $\gamma \in N_{\mathbf{G}}(\mathbf{T})(K(S))$; by Lemma 3.2, $\gamma \in \Lambda$.

□

THEOREM 3.7. *A commensurably maximal amenable subgroup* $\Lambda \simeq (\mathbf{T} \ltimes \mathbf{U})(K(S))$ *whose toral part* $\mathbf{T}(K(S))$ *is maximal amenable in* $\mathbf{L}(K(S))$, *where* $\mathbf{L}$ *is the corresponding Levi subgroup, is singular.*

SKETCH OF PROOF. Use the K_v-parabolic subgroup $\mathbf{P}$ whose central torus is generated by $Z(\mathbf{L})$ and $\mathbf{T}_v^{(s)}$ and prove that the action of $(\mathbf{T} \rtimes \mathbf{U})(K_v)$ on $\mathbf{G}(K_v)/\mathbf{P}(K_v)$ has only "Weyl many" fixed points. This is achieved by reducing to the toral case as follows. Take $g \in \mathbf{G}(K_v)$ such that $g(\mathbf{T} \ltimes \mathbf{U})(K_v)g^{-1} \subset \mathbf{P}(K_v)$. First, observing that $\mathbf{P} \leqslant \mathbf{Q}$, we get the following: if $\mathbf{U} \leqslant g\mathbf{P}g^{-1} \leqslant g\mathbf{Q}g^{-1}$, then $\mathbf{U} \leqslant g\mathbf{Q}g^{-1} \cap \mathbf{Q}$. Now, [**BT65**, Corollaire 4.5] ensures that $g\mathbf{Q}g^{-1} = \mathbf{Q}$ and as $\mathbf{Q}$ is self-normalizing, $g \in \mathbf{Q}$; in particular, $g^{-1}\mathbf{U}g = \mathbf{U}$, and hence $\mathbf{U} < \mathbf{P}$. The rest follows as in Lemma 3.3. □

Combined with Theorem 3.5, this provides a vast class of new examples of maximal amenable inclusions of von Neumann algebras.

4. Constructing toral subgroups

In this section we provide a very explicit method for completely describing toral maximal amenable subgrops of $\mathrm{SL}(n,\mathbb{Z})$, because the $\mathbb{Q}$-anisotropic tori there have a very nice description: every such torus corresponds to a number field E with $[E:\mathbb{Q}] = n$.

PROPOSITION 4.1. *Let* E *be a number field of degree* n *over* $\mathbb{Q}$. *Take an integral basis of* E *over* $\mathbb{Q}$ *and consider the corresponding multiplication representation* $\mu\colon E \to \mathbb{M}_n(\mathbb{Q})$. *Let* $E_1^\times \coloneqq \{\alpha \in E \mid N(\alpha) = 1\}$ *be the subset of norm* 1 *elements. Then* $\mu(E_1^\times)$ *is the set of* $\mathbb{Q}$*-points of a* $\mathbb{Q}$*-anisotropic torus* $\mathbf{T}$ *in* $\mathbf{SL}_n$, *and every* $\mathbb{Q}$*-anisotropic torus in* $\mathbf{SL}_n$ *arises in this way.*

Moreover, in this case $\mathbf{T}(\mathbb{Z}) = \mu(\mathcal{O}_1^\times)$, *where* $\mathcal{O}$ *is the ring of integers of* E, *and the image of the integral points of the normaliser* $N_{\mathbf{SL}_n}(\mathbf{T})(\mathbb{Z})$ *in the Weyl group is isomorphic to the automorphism group* $\operatorname{Aut}(E/\mathbb{Q})$.

PROOF. The first assertion is well-known, see, for instance [**PR13**, Section 9]. Now, every automorphism $\sigma \in \operatorname{Aut}(E/\mathbb{Q})$ is represented by an integral matrix in the integral basis of E and therefore correponds to an element of $N_{\mathbf{SL}_n}(\mathbf{T})(\mathbb{Z})$. On the other hand, every element in the normaliser clearly induces an automorphism of E by conjugating the image of the embedding $\mu\colon E \hookrightarrow \mathbb{M}_n(\mathbb{Q})$; the assertion follows. □

EXAMPLE 4.2. Taking the number field $K = \mathbb{Q}[x]/\langle x^3 + x - 1\rangle$, we obtain a maximal amenable subgroup of $\mathrm{SL}(3,\mathbb{Z})$ isomorphic to $\mathbb{Z}$ generated by

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

There are no integral elements in the Weyl group because $\mathrm{Aut}(E/\mathbb{Q}) = \{1\}$.the Galois group is isomorphic to S_3.

EXAMPLE 4.3. Consider the torus in $\mathrm{SL}(n)$ associated to the number field $K = \mathbb{Q}[x]/\langle x^4 - 8x^3 + 20x^2 - 16x + 1\rangle$. Its integer points are generated by the central element $c = \mathrm{diag}(-1,-1,-1,-1)$ and

$$g_1 = \begin{pmatrix} 0 & -1 & -2 & -4 \\ 8 & 16 & 31 & 62 \\ -6 & -12 & -24 & -49 \\ 1 & 2 & 4 & 8 \end{pmatrix}, g_2 = \begin{pmatrix} -1 & -1 & -2 & -5 \\ 9 & 15 & 31 & 78 \\ -6 & -11 & -25 & -69 \\ 1 & 2 & 5 & 15 \end{pmatrix}, g_3 = \begin{pmatrix} -4 & -1 & -1 & -1 \\ 13 & 12 & 15 & 15 \\ -7 & -7 & -8 & -5 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Here the Galois group $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, and there is an integral element

$$w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -16 & -1 & -4 & -16 \\ 16 & 0 & 1 & 8 \\ -4 & 0 & 0 & -1 \end{pmatrix}$$

which is nontrivial in the Weyl group.

The subject of our forthcoming research which is currently in progress is the explicit algorithmic construction of maximal amenable subgroups in all classical arithmetic groups (using the computer algebra system MAGMA to desribe generators of toral subgroups and their unipotent complements; in fact, both examples above were constructed using computer algebra systems).

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Erklärung

Ich versichere, dass ich die Habilitationsschrift mit dem Titel

„Groups, operator algebras and approximation“

selbständig und ohne andere als die darin angegebenen Hilfsmittel angefertigt habe. Die wörtlich oder inhaltlich übernommenen Stellen wurden als solche gekennzeichnet.

Die eigene Mitarbeit bei den gemeinschaftlichen Arbeiten – neben allgemeiner Mitausarbeitung der Ideen und Mitgestaltung der Beweise – erstreckt sich insbesondere auf:

- Kapitel 1: Beweise von: Theorem 3.1, Proposition 3.6, Theorem 3.9 und 3.10, Berechnung für die Kompressionsformel im Abschnitt 3.4;
- Kapitel 2: vollständige Eigenarbeit;
- Kapitel 3: Proposition 2.7, Satz 3.8 (ii) sowie Satz 4.9;
- Kapitel 4: Abschnitt 3, Sätze aus Abschnitten 4.3 und 4.4 sowie im Abschnitt 5: Lemma 5.8, Proposition 5.11 und Satz 5.14.
- Kapitel 5: Lemma 2.1, Lemma 2.4, Proposition 2.5.
- Kapitel 6: Proposition 3.3, Lemma 3.5, Ausarbeitung von Satz 3.6 und Korollar 3.7;
- Kapitel 7: Lemma 4.1, Lemma 4.2, Lemma 4.3;
- Kapitel 8: Proposition 4.6, Abschnitt 5;
- Kapitel 9: Abschnitt 2;
- Kapitel 10: Proposition 2.3, Proposition 3.4, Abschnitt 4.

Dresden, den 5. Dezember 2019

(Dr. Vadim Alekseev)

Erklärung

Ich habe bislang an keiner anderen Universität einen Habilitationsgesuch eingereicht.

Dresden, den 5. Dezember 2019

(Dr. Vadim Alekseev)