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On inverse limit sequence

Ralph Bennett
University of Tennessee

To the Graduate Council:

I am submitting herewith a thesis written by Ralph Bennett entitled "On inverse limit sequence." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts, with a major in Mathematics.

William S. Mahavier, Major Professor

We have read this thesis and recommend its acceptance:

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

August 11, 1962

To the Graduate Council:

I am submitting herewith a thesis written by Ralph Bennett entitled "On Inverse Limit Sequences." I recommend that it be accepted for nine quarter hours of credit in partial fulfillment of the requirements for the degree of Master of Arts, with a major in Mathematics.

William S. Mahavier
Major Professor

We have read this thesis and recommend its acceptance:

John W. Neuberger

E. Cohen

Accepted for the Council:

Hilton A. Smith
Dean of the Graduate School

ON INVERSE LIMIT SEQUENCES

A Thesis

Presented to

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

by

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CHAPTER I

INTRODUCTION

The following arose out of an unsuccessful attempt to answer the question "is there a map of the unit interval onto itself whose inverse limit is hereditarily indecomposable?" This question naturally leads to the broader problem of determining what sort of continua may be obtained by taking the inverse limit of a single map on the unit interval. A very limited number of answers to this problem will be found in Chapter IV, chiefly dealing with how to obtain indecomposable continua. Chapter V gives some examples to show why Chapter IV contains very little in the way of theorems characterizing the inverse limits by means of reasonable properties of the map. Some examples are also given of continua which may be obtained.

A complete answer is given in Chapter III to the question of what may be obtained as the inverse limit of a sequence of functions on the unit interval. The answer is complete since it is that every compact chainable continuum may be so obtained, and only such continua may be obtained. The question of which compact chainable continuum one will get with a given sequence of maps is not answered.

The study of inverse limits has developed in two principal directions. The first direction is abstract homology theory, which is the source of the concept. This direction will not be considered. The second direction is apparently an outgrowth of the first. It consists of giving examples of unusual continua conveniently generated as inverse limits and the study of the properties used in generating the examples.

One of the classical sets of examples of indecomposable continua, the solenoids, are very nicely given as inverse limits. R. D. Anderson and Gustave Choquet [1]* have given an example of a compact continuum contained in the plane no two of whose non-degenerate subcontinua are homeomorphic. The construction was by means of inverse limits.

In connection with the general study of inverse limits, M. K. Fort, Jr., and Jack Segal [7] have given a necessary and sufficient condition that an inverse limit be locally connected. J. R. Isbell [9] has shown that an inverse limit on compact subsets of E^n can be embedded in E^{2n} . Eilenberg and Steenrod [6; Chapter X] stated a theorem which apparently asserts that each compact space is an inverse limit on a sequence of compact triangulable spaces.

The results listed above are those which are most closely connected with the problems considered here. There seems to have been little systematic study of inverse limits for their own sake. Accessible information is of a very fragmentary nature.

Some elements of the style and notation to be followed should be mentioned, perhaps as a warning. Basic topological theorems and definitione will be used without author reference and sometimes without specific assertion of the particular item being employed. All topological spaces considered are assumed to be metric and of diameter 1, if not otherwise specified. Much of the considerations to follow depend on the concept

*The numbers which appear in brackets correspond to the numbers in the Bibliography at the end of this paper.

and properties of a product space. For a metric space sequence $(X_1, d_1), (X_2, d_2), \ldots$, the product πX_i will always be metrized by the function D defined by

$$D(x, y) = \Sigma_{k=1}^{\infty} d_k(x_k, y_k)\, 2^{-k}$$

where x and y are points in πX_i and x_k and y_k always denote the coordinates of x and y in X_k. This metric is equivalent to the product topology. A concise discussion of product spaces and metric spaces can be found in Kelley [10]. The symbol "iff" is used as an abbreviation for the phrase "if and only if", generally in definitions. The closure of a set A will be denoted A^-. The word "continuum" means only a closed and connected set and if compactness is wanted it will be stated in the phrase "compact continuum." The reader should not assume that since some proofs are given in great detail that all are so given, nor should he assume that all the proofs are only sketched upon reading some excessively concise remarks indicative of the outlines of a demonstration.

CHAPTER II

BASIC PROPERTIES

A sequence $(X_1, f_1), (X_2, f_2), \ldots$ will be called an inverse limit sequence iff each X_i is a topological space and each f_i is a continuous transformation from X_{i+1} into X_i. If P is a property of topological spaces (maps), an inverse limit sequence will be said to have or satisfy property P iff each space (map) in the sequence has property P. The word map, if used, will mean continuous transformation. The inverse limit of an inverse limit sequence $(X_1, f_1), (X_2, f_2), \ldots$ is the set of all sequences $(x_1, x_2, \ldots)$ so that for each positive integer i, x_i is in X_i and $f_i(x_{i+1})$ is x_i. The inverse limit is denoted $\lim(X_i, f_i)$ or more concisely as $\lim f_i$, and is always considered as a subspace of the product space of the X_i. If f is a map from a space X into itself the inverse limit of f is the inverse limit of the sequence $(X_1, f_1), (X_2, f_2), \ldots$ where for each positive integer i, X_i is X and f_i is f, and is denoted $\lim f$.

Strictly speaking, one should distinguish between a topological space, and the set of points of the topological space. This will not be done. For a sequence of spaces, the projection map from the product space onto the k-th coordinate space is always denoted as P_k. If i and k are positive integers and i is less than k, f_{ik} will denote the map from X_k into X_i defined by $f_{ik} = f_i f_{i+1} \ldots f_{k-1}$. The identity map from X_i onto X_i is denoted f_{ii}. A somewhat confusing property of this notation is that $f_{i,i+1}$ is f_i.

In the theorems of this chapter (X_i, f_i) always denotes an inverse limit sequence.

Theorem 1. *If* A *is a subset of* $\lim f_i$, *and* k *and* m *are positive integers with* m *greater than* k , *then* $P_k(A)$ *is* $f_{km}[P_m(A)]$.

Proof. If x is in $P_k(A)$, there is a point y in A whose k-th coordinate is x . The m-th coordinate of y is necessarily a point z so that $f_{km}(z)$ is x . That is, $P_k(A)$ is a subset of $f_{km}(P_k(A))$. Conversely, if z is in $P_m(A)$, there is a point y in A so that the m-th coordinate of y is z . By definition, the k-th coordinate of y is $f_{km}(z)$, and so $f_{km}(P_m(A))$ is a subset of $P_k(A)$.

Theorem 2. *If* $\lim f_i$ *exists it is a closed subset of the product space of the* X_i .

Proof. Suppose y is in $(\lim f_i)^-$ but not in $\lim f_i$. Since y is not in $\lim f_i$ there is a positive integer n so that $f_n(P_{n+1}(y))$ is not $P_n(y)$. There are disjoint open subsets U and V of X_n containing $P_n(y)$ and $f_n(P_{n+1}(y))$ respectively. Since f_n is continuous, there is an open subset W of X_{n+1} containing $P_{n+1}(y)$ so that $f_n(W)$ is contained in V . Let Z denote the open subset $P_n^{-1}(U) \cap P_{n+1}^{-1}(W)$ of πX_i . Since Z contains y , Z must contain a point of $\lim f_i$. But $P_{n+1}(Z \cap \lim f_i)$ is contained in W and $f_n(P_{n+1}(Z \cap \lim f_i))$ $= P_n(Z \cap \lim f_i)$ is contained in V . But also $P_n(Z \cap \lim f_i)$ is contained in U , contradicting the disjointness of U and V .

Theorem 3. *If for each* n , f_n *is onto, then* $P_n(\lim f_i)$ *is* X_n *for each* n . *Conversely, if for each* n , $P_n(\lim f_i)$ is X_n , *then each* f_n *is onto.* (The proof is obvious.)

<u>Theorem</u> 4. <u>If</u> y <u>is a point of</u> $\lim f_i$ <u>and</u> U <u>is an open subset of</u> $\lim f_i$ <u>containing</u> y , <u>then there is a positive integer</u> p , <u>so that for each integer</u> n <u>greater than</u> p <u>there is an open subset</u> V <u>of</u> X_n <u>containing</u> $P_n(y)$ <u>so that if</u> z <u>is in</u> $\lim f_i$ <u>and</u> $P_n(z)$ <u>is in</u> V , <u>then</u> z <u>is in</u> U . (<u>That is,</u> $P_n^{-1}(V) \cap \lim f_i$ <u>is an open subset of</u> U <u>containing</u> y .)

<u>Proof</u>. By the definition of the product topology, there is a positive integer p and a collection $U_1, \ldots, U_p$ of open sets so that for each i , $U_i \subset X_i$ and so that y is in $[\lim f_i \cap ((\pi_{i=1}^{p} U_i) \times \pi_{i=p+1}^{\infty} X_i]$ $\subset U$. Pick n to be any integer greater than p . Since f_{in} is continuous for $i = 1, \ldots, n$ and $f_{in}(P_n(y)) = P_i(y)$, for each i from 1 to p there is an open subset V_i of X_n containing $P_n(y)$ so that $f_{in}(V_i) \subset U_i$. Define $V = \bigcap_{i=1}^{p} V_i$. Since projections are continuous $P_n^{-1}(V)$ is an open subset of the product space. Suppose $z \in \lim f_i$ and $P_n(z)$ is in V . For each i from 1 to p , $P_i(z) = f_{in}(P_n(z))$ is in U_i , and so z is in U . Therefore $P_n^{-1}(V) \cap \lim f_i$ is an open subset of U containing y .

<u>Theorem</u> 5. <u>If</u> (X_i, f_i) <u>is a compact inverse limit sequence, the inverse limit exists and is compact. If the sequence is also connected, the inverse limit is connected.</u>

<u>Proof</u>. For each positive integer n , denote as F_n the transformation from $\pi_{i=n}^{\infty} X_i$ into $\pi_{i=1}^{\infty} X_i$ defined by $F_n(x) = (f_{1n}(x_n),$ $f_{2n}(x_n), \ldots, f_{nn}(x_n), x_{n+1}, \ldots)$ where x is $(x_n, x_{n+1}, \ldots)$. Suppose U is an open subset of $\pi_{i=1}^{\infty} X_i$ containing $F_n(x)$. There is a

positive integer $p > n$ and open subsets $U_1, \dots U_p$ of $X_1, \dots, X_p$ respectively so that $F_n(x) \varepsilon (\pi_{i=1}^p U_i) \times \pi_{i=p+1}^{\infty} X_i \subset U$. For each i from 1 to n there is an open subset V_i of X_n containing x_n so that $f_{in}(V_i) \subset U_i$. Define $V = \bigcap_{i=1}^{n} V_i$, and $W = V \times (\pi_{i=n+1}^p U_i) \times (\pi_{i=p+1}^{\infty} X_i)$. Clearly W is an open subset of $\pi_{i=n}^{\infty} X_i$ containing x so that $F_n(W) \subset U$. Therefore each F_n is continuous.

For each n denote $M_n = F_n(\pi_{i=n}^{\infty} X_i)$. Since products of compact spaces are compact and the F_n are continuous, each M_n is compact. If x is in the domain of F_{n+1}, $x = (x_{n+1}, x_{n+2}, \dots)$ and $x' = (f_n(x_{n+1}), x_{n+1}, \dots,)$, then $F_n(x') = F_{n+1}(x)$. So for each n, $M_{n+1} \subset M_n$. Therefore $\bigcap_{n=1}^{\infty} M_n$ exists and is compact.

Suppose $x \varepsilon \bigcap_{n=1}^{\infty} M_n$. For any positive integer k, since x is in M_{k+1}, $P_k(x) = f_k(P_{k+1}(x))$, and x is in $\lim f_i$. Conversely, if $x = (x_1, x_2, \dots)$ is in $\lim f_i$, then for any positive integer k, $x = F_k(x')$ where $x' = (x_k, x_{k+1}, \dots)$, and so x is in $\bigcap_{n=1}^{\infty} M_n$. So $\lim f_i$ is $\bigcap_{n=1}^{\infty} M_n$ and exists and is compact.

If each X_i is connected, then $\pi_{i=n}^{\infty} X_i$ is connected and M_n is connected since the image of a connected space under a continuous transformation. Then $\bigcap_{n=1}^{\infty} M_n = \lim f_i$ is connected.

It should be noted that Theorems 1, 2, 3 and 5 have been proved by Eilenberg and Steenrod [6] and also by Hocking and Young [8], and are noted in Capel [5]. An example given in the latter can be slightly altered to show the use of Theorems 3 and 5. Denote as X the non-negative integers with the usual topology and as f the homeomorphism from X into X given by $f(x) = x + 1$. For any positive integer x,

$f^{-1}(x) = x - 1$. Note that a point z in an inverse limit always has the property that $P_{n+1}(z)$ is in $f_n^{-1}(P_n(z))$. Suppose x is in the inverse limit of f, $x = (x_1, x_2, \dots)$. The (x_1+1)-st coordinate of x is $f^{-x_1}(x_1) = 0$. But 0 has no preimage under f . So f has no inverse limit.

An easy application of Theorem 4 gives the following.

Theorem 6. *If K is a closed subset of $\lim f_i$, and for each n , $P_n(K) = X_n$, then $K = \lim f_i$.*

Proof. Suppose y is in $\lim f_i$ and U is an open set containing y . There is a positive integer n and an open subset V of X_n so that if z is in $\lim f_i$ and $P_n(z)$ is in V , then z is in U . Since $P_n(K) = X_n$, there is a point z of K so that $P_n(z)$ is in V , and so there is a point of K in U . So y is a limit point of K , and since K is closed, y is in K . Therefore K is $\lim f_i$.

The following theorem says approximately that the same inverse limit is obtained if a subsequence of the inverse limit sequence is taken. This may be considered as a variation of Lemma 2-84 of [8] or of Theorem 2.11 of [5].

Theorem 7. *Suppose (X_i, f_i) is an inverse limit sequence, $\lim (X_i, f_i)$ exists and $n_1, n_2, \dots$ is an increasing sequence of positive integers. Then $\lim (Y_i, g_i)$ exists and is homeomorphic to $\lim (X_i, f_i)$ where for each i , $Y_i = X_{n_i}$ and $g_i = f_{n_i, n_{i+1}}$.*

Proof. Denote as F the transformation from $\lim (X_i, f_i)$ defined by $F(x) = (x_{n_1}, x_{n_2}, \dots)$ where $x = (x_1, x_2, \dots)$. Clearly F

is into $\lim(Y_i, g_i)$, and $\lim(Y_i, g_i)$ exists. If $F(x) = F(y)$, then for each positive integer i , $P_{n_i}(x) = P_{n_i}(y)$. If k is a positive integer there is an integer i so that $n_i > k$, and then $P_k(x)$ $= f_{k,n_i}(P_{n_i}(x)) = f_{k,n_i}(P_{n_i}(y)) = P_k(y)$, and $x = y$. So F is 1-1 . Suppose $x' = (x_1', x_2', \ldots)$ is in $\lim(Y_i, g_i)$. If $x = (f_{1n_1}(x_1'), f_{2n_1}(x_1'), \ldots, x_1', f_{n_1+1,n_2}(x_2'), \ldots)$ clearly x is in $\lim(X_i, f_i)$ and $F(x) = x'$. So F is onto. Suppose U is open in $\lim(Y_i, g_i)$ and $F(x)$ is in U . There is a positive integer $k > 1$ and an open subset V of Y_k containing $P_k(F(x))$ so that a point z of $\lim(Y_i, g_i)$ is in U if $P_k(z)$ is in V . Define $W = [(\pi_k^{n_k-1} X_i) \times V \times (\pi_{i=n_k+1}^{\infty} X_i)]$. Obviously $P_k[F(W \cap \lim(X_i, f_i))] \subset V$ and x is in $W \cap \lim(X_i, f_i)$. So F is continuous. Suppose U is open in $\lim(X_i, f_i)$ and x is in U . By Theorem 4 there is a positive integer p so that for any integer n greater than p there is an open subset V of X_n containing $P_n(x)$ so that the open subset $P_n^{-1}[V] \cap \lim(X_i, f_i)$ contains x and is contained in U . Choose n to be any n_k greater than p and V a set with the foregoing properties. Clearly $F(P_{n_k}^{-1}[V] \cap \lim(X_i, f_i))$ contains $F(x)$ and is open in $\lim(Y_i, g_i)$ since it is $P_k^{-1}[V] \cap \lim(Y_i, g_i)$. So F is 1-1 continuous and open, and is a homeomorphism.

There are several corollaries to this theorem. Corollary 9 yields an amusing example in Chapter V, that for any compact continuous curve T (Peano space) there is a map f from T onto T so that the inverse limit of f is a chainable indecomposable continuum. Since any such continuum can be embedded in the plane and T may be of high dimension, this is not the expected thing.

Corollary 8. *Suppose (X_i, f_i) is an inverse limit sequence, $\lim(X_i, f_i)$ exists and n is a positive integer. Then $\lim(Y_i, g_i)$ exists and is homeomorphic to $\lim (X_i, f_i)$ where $Y_i = X_{(n-1)+i}$,* $g_i = f_{(n-1)+i}$ for each i.

Corollary 9. *Suppose X and Y are topological spaces, f is a map of Y into X, and the inverse limit of fg exists. Then the inverse limit of gf exists and is homeomorphic to the inverse limit of fg.*

Proofs. Corollary 8 is obtained by considering the increasing sequence $n, n + 1, n + 2, \ldots$ and applying Theorem 7 directly. Corollary 9 is obtained by considering (X_i, f_i) where $X_i = X$ for i odd, $X_i = Y$ for i even, $f_i = g$ for i odd and $f_i = f$ for i even. Taking the sequence $1, 3, 5, \ldots$, Theorem 7 gives $\lim (X_i, f_i)$ is homeomorphic to $\lim gf$. Taking the sequence $2, 4, 6, \ldots$ Theorem 7 gives $\lim (X_i, f_i)$ is homeomorphic to $\lim fg$. So $\lim fg$ is homeomorphic to $\lim gf$.

For a map f of X into X, f^1 will denote f, and for n a positive integer, f^{n+1} denotes ff^n.

Corollary 10. *If f is a map from X into X, $\lim f$ exists and n is a positive integer, then $\lim f^n$ exists and is homeomorphic to $\lim f$.*

Corollary 10 is a simplified version of Corollary 11.

Corollary 11. *Suppose* f *is a map from* X *into* X , *lim* f *exists and* $n_1, n_2, \ldots$ *is a sequence of positive integers, (not necessarily increasing.) Then* $\lim (X_i, f_i)$ *exists and is homeomorphic to* $\lim f$ *where* $X_i = X$ *for each* i *and* $f_i = f^{n_i}$.

Proof. Consider the sequence $1, n_1 + 1, n_1 + n_2 + 1, \ldots$ and $\lim f$. Clearly Theorem 7 applies and gives the desired result.

Recall that a collection of open sets $U_1, \ldots, U_n$ in a metric space is called a chain if U_k intersects U_m if and only if $|k - m| \leq 1$. The maximum diameter of the open sets making up a chain (if such exists) is called the mesh of the chain. A set M in a metric space is said to be chainable iff for each positive real number d , there is a chain of mesh less than d so that every point of M is contained in some member of the chain. (Such a chain is said to cover M.)

One may refer to a paper by R. H. Bing, [3], and a paper by Lida K. Barrett [2] for some interesting results on chainable continua, and (especially) for references to other results. The following shows that inverse limits on the unit interval are all chainable. This at least prohibits some pathological continua occurring, but not sufficiently many so as to make the problem uninteresting.

Theorem 12. *If* (X_i, f_i) *is an onto inverse limit sequence so that each* X_i *is compact and chainable, then* $\lim (X_i, f_i)$ *is chainable.*

Proof. The principal fact used in the proof is that a continuous transformation from a compact metric space into a metric space is uniformly continuous. Note that if $U_1, U_2, \ldots, U_k$ is a chain covering X_n, $P_n^{-1}[U_i] \cap \lim (X_i, f_i) = U_i'$, $i = 1, \ldots, k$ is a collection of open sets covering $\lim (X_i, f_i)$ and U_i' intersects U_j' if and only if $P_n[U_i']$ intersects $P_n[U_j']$, hence if and only if U_i intersects U_j. But then $U_1', \ldots, U_k'$ is a chain (of open subsets of $\lim (X_i, f_i)$ covering $\lim (X_i, f_i)$.

Suppose ε is a positive number. There is a positive integer k so that 2^{-k} is less than $(\varepsilon/2)$. For each positive integer i not greater than k, since f_{ik} is uniformly continuous, there is a positive number δ_i so that if x and y are in X_k and $d_k(x, y) < \delta_i$, then $d_i(f_{ik}(x), f_{ik}(y)) < 2^{-k}$. There is a positive number δ less than the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Since X_k is chainable there is a chain of mesh less than δ covering X_k, say $U_1, \ldots, U_n$. Denote each set of the form $P_k^{-1}(U_i) \cap \lim f_i$ as V_i. Then $V_1, \ldots, V_n$ is a chain covering $\lim f_i$. Suppose x and y are in some V_i. Since $d_k(P_k(x), P_k(y)) < \delta$, we have

$$D(x,y) = \Sigma_{i=1}^{\infty} d_i(P_i(x),P_i(y))2^{-i} \leq \Sigma_{i=1}^{k} 2^{-k}2^{-i} + \Sigma_{i=k+1}^{\infty} 2^{-i} < 2^{-k} + 2^{-k} < \varepsilon .$$

The assumption that all the X_i have diameter not greater than 1 was used in the inequality $d_i(P_i(x),P_i(y))\ 2^{-i} \leq 2^{-i}$ for i greater than k. So $V_1, \ldots, V_n$ is a chain of mesh less that ε covering $\lim f_i$.

The following obvious theorem and its corollary give that every compact space is the inverse limit of some inverse limit sequence. This is not very useful since the space used is the space to be duplicated, and we can obtain no information about the space to be duplicated by looking at the spaces in the inverse limit sequence. In fact, one can easily show that if $(X_1, f_1), (X_2, f_2), \ldots$ is an inverse limit sequence and each f_i is an onto homeomorphism, then $\lim f_i$ is topologically equivalent to X_1 .

Theorem 13. *If each f_i is 1-1 and $\lim f_i$ exists, then for each positive integer n, the projection P_n maps $\lim f_i$ 1-1 into X_n.*

Corollary 14. *If $(X_1, f_1), (X_2, f_2), \ldots$ is a 1-1 onto compact inverse limit sequence, then $\lim f_i$ is topologically equivalent to X_1.*

Proof. One only needs recall that a continuous 1-1 map on a compact space is a homeomorphism, and therefore the continuous 1-1 onto transformation P_1 from $\lim f_i$ onto X_1 is a homeomorphism. Properly the homeomorphism is $P_1 | \lim f_i$.

CHAPTER III

AN EQUIVALENCE THEOREM

The constructive proof of the principal theorem of this chapter is broken down into several lemmas. The first may seem to be out of place. However, it gives conditions under which one compact chainable continuum is homeomorphic to another, and the conditions of the first lemma will be used to guide the construction. This may make the constructive process clearer than it would have been, had the construction been made without and it were shown at the end that the construction is adequate.

In general a chain will be given a name which is a capital letter, possibly with subscripts, and its links will be denoted by the same letter not capitalized, and with subscripts. The notation the chain $C = c_1, \ldots, c_n$ will mean $c_1, \ldots, c_n$ are distinct links of the chain, and that they are in order. That is, c_i intersects c_j if and only if $|i - j| < 2$.

A chain C will be said to be δ-spread iff if x and y are points of non-adjacent links of C , then the distance from x to y is not less than δ . A chain C will be said to be spread if there is some positive number δ so that C is δ-spread.

If each of $C_1, C_2, \ldots$ and $D_1, D_2, \ldots$ is a sequence of chains, $C_1, C_2, \ldots$ will be said to be similar to $D_1, D_2, \ldots$ iff each is a decreasing sequence of chains so that for each positive integer n C_n and

D_n have the same number of links and if each of j and k is a positive integer so that C_{n+1} contains at least j links and C_n contains at least k links, and $c_{n+1,j} \subset c_{n,k}$, then there is a positive integer m so that $d_{n+1,j} \subset d_{n,m}$ and $|m - k| < 2$. A chain E is said to have mesh less than the number t iff each link of E is of diameter less than t. A sequence $E_1, E_2, \ldots$ of chains is said to be decreasing iff there is a sequence $t_1, t_2, \ldots$ of real numbers with limit 0 so that for each positive integer n, E_n has mesh less than t_n and the closure of each link of E_{n+1} is contained in some link of E_n.

Lemma 1. *Suppose each of* M *and* N *is a compact chainable continuum and each of* $C_1, C_2, \ldots$ *and* $D_1, D_2, \ldots$ *is a decreasing sequence of chains so that*

1. $C_1, C_2, \ldots$ *is similar to* $D_1, D_2, \ldots$.
2. *For each positive integer* n,
   - A. *Each point of* M *is in some link of* C_k *and each point of* N *is in some link of* D_n.
   - B. *There is a positive number* δ_n *so that* D_n *is* δ_n-*spread and* D_{n+1} *has mesh less than* $\delta_n/4$.
   - C. *Each link of* C_n *contains some point of* M *and each link of* D_n *contains some point of* N.

Then M *is homeomorphic to* N.

Proof. First a function is defined from M into N. Suppose p is a point of M, and n is a positive integer. Define K_{np} by agreeing that q is in K_{np} iff q is in N and there are positive integers

j and s so that each of j and s is not greater than the number of links in C_n , p is in c_{nj} , q is in d_{ns} and $|s - j| < 3$. The set K_{np} consists of the points of N in approximately the same position in D_n as p is in in C_n . Certainly K_{np} always exists by condition 2.C. It will be shown that $\bar{K}_{n+1,p}$ is always contained in K_{np} , and that $K_{n,p}$ is of diameter at most 7 times the mesh of D_n . Since $\bar{K}_{np}$ is a closed subset of N , then $\bar{K}_{np}$ is compact, and we will have that $\bar{K}_{1p}$, $\bar{K}_{2p}$, ... is a nested sequence of compact sets whose diameters have limit 0 and therefore $\bigcap_{n=1}^{\infty}\bar{K}_{np}$ which is $\bigcap_{n=1}^{\infty} K_{np}$ exists and is a single point. The function f from M into N is defined by $f(p) = \bigcap_{n=1}^{\infty}K_{np}$.

To prove the assertions about K_{np} , note that if p is in c_{nj} then p is not contained in a link of C_n not adjacent to c_{nj} . So if p is in c_{nj} , and p is in $c_{nj'}$, and s is a positive integer not greater than the number of links of C_n and $|j' - s| < 3$, it is true that $|j - s| \leq 3$ since $|j - j'| \leq 1$. So K_{np} is contained in the union of d_{nj} and the three links to each side of d_{nj} , a set of diameter at most 7 times the mesh of D_n . Restated, K_{np} is contained in the union of at most 7 consecutive links of D_n no more than three of which are on the same side of d_{nj} . Next suppose that p is in $c_{n+1,k}$. There is a positive integer j so that $c_{n+1,k}$ is contained in c_{nj} . Since the C-sequence and the D-sequence are similar, there is a positive integer k' so that $d_{n+1,j}$ is contained in $d_{n,k'}$, and $|k - k'| \leq 1$. Because p is in c_{nk} , if $|r - k| < 3$, we have the intersection of d_{nr} and N is contained in K_{np} . But if d_{nr} is a link of D_n adjacent to $d_{nk'}$, then $|r - k| \leq |r - k'| + |k' - k| < 3$ and so K_{np} contains the intersection of N and links of D_n either adjacent to $d_{nk'}$ or $d_{nk'}$. Since $K_{n+1,p}$ contains at most the intersection of N and $d_{n+1,j}$ and of three links to each side of $d_{n+1,j}$, and the mesh of D_{n+1} is less than $1/4$ the width of the gap between non-adjacent links of D_n , then four consecutive links of D_{n+1} , one of which is $d_{n+1,j}$ have the property that their union does not intersect a link of D_n not adjacent to $d_{n,k'}$ and so $K_{n+1,p}$ does not intersect a link of D_n not adjacent to $d_{n,k'}$. Therefore K_{np} contains $\bar{K}_{n+1,p}$.

It remains to be shown that f is 1-1 , onto and continuous. Since M is compact, this will be sufficient to know that f is a homeomorphism of M onto N .

Suppose that p and q are different points of M . Choose an integer r so that C_r is of mesh less than $1/15$ the distance from p to q . There are integers j and k so that p is in c_{rj} and q is in c_{rk} . Necessarily $|j - k| > 15$. But K_{rp} is contained in the union of at most 7 links of D_r , one of which is d_{rj} and K_{rq} is contained in the union of 7 consecutive links of D_r one of which is d_{rk} . Since $|j - k| > 15$, these two sets of 7 links contain no adjacent links and so K_{rp} and K_{rq} are disjoint. Since $f(p)$ is in K_{rp} and $f(q)$ is in K_{rq} , $f(p)$ is not $f(q)$, and f is 1-1 .

Suppose U is open and $f(p)$ is in U . There is a positive number ε so that the ε-sphere about $f(p)$ is contained in U . There is a positive integer r so that D_r has mesh less than $\varepsilon/7$. Choose j to be any positive integer so that p is in c_{rj}. If q is in c_{rj} then K_{rq} is contained in the union of 7 consecutive links of D_r, one of which is d_{rj}, and this set contains $f(q)$ and is of diameter less than seven times a number greater than the mesh of D_r. This set also contains $f(p)$ and so $f(p)$ is a distance less that $7(\varepsilon/7)$ from $f(q)$, or $f(q)$ is in U. So $f(M \cap c_{rj})$ is contained in U and f is continuous.

Since f is continuous and 1-1 on the compact set M, $f(M)$ is compact and therefore closed. Suppose f is not onto. Then there is a point z in N not in $f(M)$. Since $f(M)$ is closed there is a positive number ε so that the sphere about z of radius ε does not intersect $f(M)$. Again choose a positive integer r so that D_r has mesh less than $\varepsilon/7$. Let j be an integer so that z is in d_{rj}. There is a point p of M in c_{rj} by condition 2C. But as above in showing that f is continuous, $f(c_{rj} \cap M)$ is contained in the ε-sphere about z, and in particular $f(p)$ is within ε of z, contrary to assumption. Therefore f is onto.

This completes the proof of the first lemma.

The second lemma is a restatement of Lemma 1 of [2], and will not be proved here.

<u>Lemma</u> 2. <u>If</u> M <u>is a compact chainable continuum, there is a decreasing sequence of chains</u> $C_1, C_2, \ldots$ <u>so that each</u> C_n <u>is spread, and so that for each positive integer</u> n, <u>each point of</u> M <u>is contained in some link of</u> C_n <u>and each link of</u> C_n <u>contains a point of</u> M <u>and a link of</u> C_{n+1}.

The next Lemma provides the constructive apparatus.

Lemma 3. Suppose E and F are chains containing m and n links respectively so that each link of F contains some link of E and each link of E is contained in some link of F. Suppose further that n is greater than 3, δ and t are positive numbers and Q is an integer greater than 1 so that F is δ-spread, E has mesh less than δ/Q and $t/\delta > 1/(Q-1)$, and V is a spread chain of n connected open subsets of [0,1] which covers [0,1] so that V has mesh less than $2/n$. Then there is a spread chain U of m connected open subsets of [0,1] which covers [0, 1] of mesh less than $2/m$ and a continuous function g from [0, 1] into [0,1] so that the closure of the g-image of each link of U is contained in some link of V and is of length less than t, each link of V contains the closure of the image of some link of U and so that if $g(u_j)$ is contained in V_k then there is an integer p so that e_j is contained in f_p and $|p - k| < 2$.

I. The chain U is constructed by choosing a number w less than $1/(6m)$ and defining u_1 to be $[0, (1/m) + w)$, u_2 to be $((1/m) - w, (2/m) + w)$, ... and u_m to be $((m-1/m) - w, 1]$. Clearly U is a chain of m connected open subsets of [0,1] covering [0,1], U has mesh less than $2/m$ and is $(1/3m)$-spread. There is also a positive number x so that V is x-spread and x is less than $t/2$.

II. Let k_1 denote a positive integer so that e_1 is contained in f_{k_1}. Denote as j_1 the smallest positive integer so that e_{j_1} is not contained in f_{k_1}. Since F is spread more than the diameter of e_{j_1} and e_{j_1} intersects f_{k_1} in a point of e_{j_1-1}, there is only one integer k_2 so that e_{j_1} is contained in f_{k_2} and k_2 is either $k_1 + 1$ or $k_1 - 1$. Denote $k_2 - k_1$ as i_1, and

$k_1 - i_1$ as k_2' . Two cases will be considered:

A. Either $k_2' < 1$ or $k_2' > n$ or for $1 \leq j \leq j_1$, e_j is not contained in $f_{k_2'}$.

B. There is an integer j_1' with $1 \leq j_1' < j_1$ so that e_{j_1}' is contained in $f_{k_2'}$.

The second case is that E runs over into the link of F on the other side of f_{k_1} from f_{k_2} before running inside f_{k_2} , and leaving f_{k_1} . The first case is that the second does not occur. Denote the endpoint of v_{k_1} in v_{k_2} as b_1 and the endpoint of v_{k_1} not in v_{k_2} as a_1 . Denote the endpoint of v_{k_2} in v_{k_1} as a_1' and the endpoint of $v_{k_2'}$ in v_{k_1} as b_1' , if $v_{k_2'}$ is defined. Either $a_1 < b_1' < a_1' < b_1$ or $a_1 > b_1' > a_1' > b_1$. In either case, $b_1' - a_1' > x$. Choose a number x_1 less than x , $|a_1 - b_1'|/2$ and $|a_1' - b_1|/2$ but greater than 0 .

In case A define g_1 to be the function from $[0, (j_1/m) + w]$ onto $\frac{a_1' + b_1}{2}$. The following are true of g_1 : (1) g_1 is continuous, (2) For $1 \leq j \leq j_1$, $(g_1(u_j))^-$ is contained in $v_{k_1} \cap v_{k_2}$ and is of diameter 0 . (3) For each link f_k of F containing a link e_j of E with $1 \leq j \leq j_1$, v_k contains the closure of the image of some link $u_{j'}$ of U with $1 \leq j' \leq j_1$. (4) If $g_1(u_j)$ is contained in v_k for $1 \leq j \leq j_1$, then there is an integer p so that $e_j \subset f_p$ and $|p - k| < 2$. (The integer k is either k_1 or k_2 , $p = k_1$ for $1 \leq j < j_1$, and $p = k_2$ for $j = j_1$ will do.)

Suppose in case B that $a_1 < b_1' < a_1' < b_1$. Then define g_1 from $[0, (j_1 - 1/m) + w]$ into v_{k_1} by $g_1(0) = b_1' - x_1$, $g_1((1/m) + w) = b_1' - (x_1/2)$,

$g_1((^{j_1-1}/m) - w) = a_1' + (^{x_1}/2)$ and $g_1((^{j_1}/m) + w) = a_1' + x_1$, and g_1 is linear on the intervals $[0, (^1/m) + w]$, $[(^1/m) + w, (^{j_1-1}/m) - w]$ and $[(^{j_1-1}/m) - w, (^{j_1}/m) + w]$. If $a_1 > b_1' > a_1' > b_1$, define $g_1(0) = b_1' + x_1$, $g_1((^1/m) + w) = b_1' + (^{x_1}/2)$, $g_1((^{j_1-1}/m) - w) = a_1' - (^{x_1}/2)$ and $g_1((^{j_1}/m) + w) = a_1' - x_1$, with the corresponding linearity on the intervals. Note that (1) g_1 is continuous. (2) $g_1(u_1)^- = [b_1' - x_1, b_1' - (^{x_1}/2)]$ or $[b_1' + (^{x_1}/2), b_1' + x_1]$ is contained in $v_{k_2}' \cap v_{k_1}$ and is of diameter less than $^t/2$ and $g_1(u_{j_1})^- = [a_1' + (^{x_1}/2), a_1' + x_1]$ or $[a_1' - x_1, a_1' - (^{x_1}/2)]$ is contained in $v_{k_1} \cap v_{k_2}$ and is of diameter less than $^t/2$, and for $1 \leq j \leq j_1$, $g_1(u_j)^-$ is contained in v_{k_1}. Since g_1 is linear on the interval $[(^1/m) + w, (^{j_1-1}/m) - w]$ there is a number s_1 so that if r is the length of a subinterval, then its g_1-image is of length $s_1 r$. Now $[(^1/m) + w, (^{j_1-1}/m) - w]$ has length $(^{j_1-2}/m) - 2w$ and its image has length less than 1. So the inequality $((^{j_1-2}/m) - 2w)s_1 < 1$ holds.

Since F is δ-spread, the mesh of E is less than $^\delta/Q$ and e_{j_1} and $e_{j_1'}$ are in non-adjacent links of F, $Q < j_1 - j_1' \leq j_1 - 1$. Then $1 < Q \leq j_1 - 2$ and so $^1/(j_1 - 2) \leq {}^1/Q < 1$. Applying this to the inequality above and using $2w < {}^1/3m$, and $^1/Q < {}^t/6$ one sees that

$$^1/Q > s_1\left(\frac{j_1 - 2}{mQ} - \frac{2w}{Q}\right) \geq s_1\left(\frac{1}{m} - 2w\right) > s_1\left(\frac{1}{m} - \frac{1}{3m}\right) > (^{s_1}/3)(^2/m).$$

So $^t/2 > {}^3/Q > s_1(^2/m)$. But U has mesh less than $^2/m$, and so the length of the image of u_j for $1 < j < j_1$ is less than $^t/2 + {}^t/2 = t$. (The link u_j may be split up into a piece in $[0, (^1/m) + w]$ or $[(^{j_1-1}/m) - w, (^{j_1}/m) + w]$ and a piece in $[(^1/m) + w, (^{j_1-1}/m) - w]$.)
(3) If a link f_k of F contains a link e_j of E with $1 \leq j \leq j_1$,

then v_k contains the closure of the image of some link $u_{j'}$ of U with $1 \leq j' \leq j_1$, since k is either k_1, k_2, or k_2'. (4) If $g_1(u_j)$ is contained in v_k for $1 \leq j \leq j_1$, then there is an integer p so that $e_j \subset f_p$ and $|p - k| < 2$. (Again, k is k_1 or k_2 or k_2' and $1 \leq j < j_1$, then $e_j \subset f_{k_1}$ and $p = k_1$ gives $|p - k| < 2$. If j is j_1, then k is k_2 or k_1, $e_{j_1} \subset f_{k_2}$ and $p = k_2$ gives $|p - k| < 2$.)

It probably should have been noted earlier that j_1 exists since $n > 3$. In fact $j_1 < m$. It could easily happen that k_1 is 1 or m, in which case, Case A applies since no e_j is contained in $f_{k_2'}$ because no such link of F exists. No use was made of $v_{k_2'}$ in constructing g_1 in part A, and no difficulties arise. However, in the next part, III, the existence of a j_2 is more worrisome. In either Case A or Case B a function g_1 has been constructed satisfying the conditions of the conclusion for g on the first j_1 links of U.

III. The next thing to do is to consider as many as possible consecutive links of E in f_{k_2} starting with e_{j_1}. There are five cases to be considered, to be from two major cases.

A. For $j_1 \leq j \leq m$, e_j is contained in f_{k_2}.

B. There is an integer j so that $j_1 < j$ and e_j is not contained in f_{k_2}.

Choose j_2 to be the least such integer j. Denote as k_3 the integer so that e_{j_2} is contained in f_{k_3}. As in II, there is only one such integer and it is either $k_2 + 1$ or $k_2 - 1$. Denote $k_3 - k_2$ as i_2, and $k_2 - i_2$ as k_3'. Three subcases of B may be considered.

B_1. We have k_3 is k_1 and no link e_j of E, $j_1 \leq j \leq j_2$, is contained in $f_{k_3'}$.

B_2. We have k_3 is k_1 and for some integer j_2' with $j_1 \leq j_2' < j_2$, $e_{j_2'}$ is contained in $f_{k_3'}$.

B_3. We have k_3 is not k_1.

Two subcases of Case A will be considered. One cannot actually occur in this second step of the construction, but its inclusion makes further steps exactly the same as the second. Recall that k_2 is $k_1 + i_1$, and $|i_1| = 1$. The number $k_1 + 2i_1$ which we denote k_2'' is so that $|k_2'' - k_2| = 1$. If $f_{k_2''}$ is defined, then it is the link of F adjacent to f_{k_2} but not f_{k_1}. Two things could conceivably happen.

<u>Case</u> A_1. No e_j for $j_1 \leq j \leq m$, is contained in $f_{k_2''}$ (or $f_{k_2''}$ is not defined.)

<u>Case</u> A_2. For some integer j_2'' with $j_1 < j_2'' \leq m$, $e_{j_2''}$ is contained in $f_{k_2''}$.

In Case A_1, define g_2 to be the map from $[0, 1]$ into v_{k_1} so that g_2 is g_1 on $[0, (j_1/m) + w]$ and so that g_2 takes $g_1((j_1/m) + w, 1]$ onto $g_1((j_1/m) + w)$. Clearly (1) g_2 is continuous. (2) For $j_1 + 1 < j \leq m$, $g_2(u_j)^-$ is contained in $v_{k_1} \cap v_{k_2}$ and is of diameter 0. $g_2(u_{j_1+1})^-$ is $g_1([(j_1/m)-w, (j_1/m) + w]) \cup g_2([(j_1/m) + w, (j_1+1/m)+ w])$, which is of diameter less than $t/2 + 0$, and $g_2(u_{j+1})^-$ is contained in $v_{k_1} \cap v_{k_2}$. (3) For each link f_k of F containing a link e_j of E with $j_1 \leq j \leq m$, v_k contains the closure of the image of u_j. (Since k is either k_1 or k_2.) (4) If $g_2(u_j)$ is contained in v_k for $j_1 \leq j \leq m$, there is an integer p so that e_j is contained in f_p and

$|p - k| < 2$. We can invariably choose p to be k_2 .

In Case A_2, denote the endpoint of v_{k_2} in $v_{k_2''}$ as b_1'' and the endpoint of $v_{k_2''}$ in v_{k_2} as a_1''. Either $a_1' < b_1 < a_1'' < b_1''$ or $a_1' > b_1 > a_1'' > b_1''$. Somewhat confusingly, the endpoints of v_{k_2} are a_1' and b_1'' . Choose a positive number x_2' to that x_2' is less than $t/2$ and less than $|a_1'' - b_1''|/2$. Define the map g_2 from $[0, 1]$ into $[0, 1]$ by g_2 is g_1 on $[0, (j_1/m) + w]$, $g_2((m-1/m) - w)$ is $a_1'' + (x_2'/2)$ and $g_2(1)$ is $a_1'' + x_2'$ if $a_1'' < b_1''$, $g_2((m - 1/m) - w)$ is $a_1'' - (x_2'/2)$ and $g_2(1)$ is $a_1'' - x_2''$ if $a_1'' > b_1''$, and g_2 is linear on $[(j_1/m) + w , (m-1/m) - w]$ and on $[(m-1/m) - w, 1]$. Since e_{j_1-1} is contained in f_{k_1} and $e_{j_2''}$ is contained in $f_{k_2''}$, a link of F not adjacent to f_{k_1} , there are at least Q links of E between e_{j_1-1} and $e_{j_2''}$. So $Q \leq j_2'' - (j_1 - 1) - 1 = j_2'' - j_1 \leq m - j_1$. Since g_2 is linear on $[(j_1/m) + w, (m - 1/m) - w]$ there is a number s_2 so that if r is the length of a subinterval of $[(j_1/m) + w, (m - 1/m - w]$, then its g_2-image is of length $s_2 r$. Since the whole interval is of length $((m - j_1 - 1)/m) - 2w$ and its image has length less than 1 , $s_2(((m - j_1 - 1)/m) - 2w) < 1$. Since $Q - 1 \leq m - j_1 - 1$, and $t/6 > 1/(Q - 1) \geq 1/(m - j_1 - 1)$, then $t/6 > s_2((1/m) - 2w/(m - j_1 - 1)) \geq s_2((1/m) - 2w) > s_2(2/(3m))$, and so $t/2 > s_2(2/m)$. We have (1) g_2 is continuous. (2) For $j_1 + 1 < j < m - 1$, $g_2(u_j)^-$ is of diameter less than $t/2$ and is contained in v_{k_2} . The set $g_2(u_m)^-$ is of diameter $x_2'/2$, hence of diameter less than $t/2$, and is contained in $v_{k_2''}$. The set $g_2(u_{j_1+1})^-$ is $(g_1[(j_1/m) - w, (j_1/m) + w] \cup g_2[(j_1/m) + w, ((j_1+1)/m + w])$ and is of diameter less than the sum of the length of $g_1(u_{j_1})^-$ and

$s_2(1/m)$, which is less than t . Similarly $g_2(u_{m-1})^-$ is of diameter less than t , and both $g_2(u_{j_1+1})^-$ and $g_2(u_{m-1})^-$ are contained in v_{k_2}. (3) For each link f_k of F so that f_k contains a link e_j of E with $j_1 \leq j \leq m$, v_k contains the closure of the g_2-image of a link $u_{j'}$ of U with $j_1 - 1 \leq j' \leq m$, since k is either k_1, k_2 or k_2'' . (4) If $g_2(u_j)$ is contained in v_k for $j_1 \leq j \leq m$, then k is either k_1, k_2 or k_2'' and $p = k_2$ gives both that e_j is contained in f_p and $|p - k| < 2$.

How to continue the construction, and finish it, when the chain E ends in a link of F has now been shown. In both cases A_1 and A_2 a function g_2 was constructed which fulfilled all the conditions of the conclusion. The only part of the conclusion which might be in doubt is that each link of V contains the image of some link of U . Any link v_k of V contains the image of some link of U since f_k contains some e_j . Either $1 \leq j \leq j_1$ or $j_1 \leq j \leq m$. But one of the observations numbered (4) after the construction of g_1 and g_2 applies, and v_k contains the closure of the image of some link of U .

Case B_1 is that k_1 is k_3 and no link of E is contained in $f_{k_2'}$ for $j_1 \leq j \leq j_2$. To duplicate this behavior it is sufficient to map all the links u_j of U with $j_1 \leq j \leq j_2$ into $v_{k_1} \cap v_{k_2}$. Take g_2 to be g_1 on $[0, (j_1/m) + w]$ and so that g_2 maps all of $[(j_1/m) + w , (j_2/m) + w]$ onto $g_1((j_1/m) + w)$. The required four properties hold for g_2 . The statements of these properties can be obtained by replacing m with j_2 in the statement of the resulting properties in Case A_1 . The proofs are almost exactly identical.

Case B_2 is that k_3 is k_1 and for some integer j_2' so that $j_1 < j_2' < j_2$, $e_{j_2'}$ is contained in $f_{k_3'}$. To duplicate this, we shall make the images run over into $v_{k_3'}$ and back into $v_{k_1} \cap v_{k_2}$, which is $v_{k_3} \cap v_{k_2}$. Define a_2 and b_2 to be the endpoints of v_{k_2} in $v_{k_3'}$ and v_{k_3} respectively, a_2' to be the endpoint of v_{k_3} in v_{k_2} and b_2' to be the endpoint of $v_{k_3'}$ in v_{k_2}. Either $a_2 < b_2' < a_2' < b_2$ or $a_2 > b_2' > a_2' > b_2$. Let x_2 be a positive number less than half the overlap of each of v_{k_2} and v_{k_3} and v_{k_3} and $v_{k_3'}$, and also less than $^t/2$. Recall that g_1 is so that $g_1((^{j}1/m) + w)$ is in $v_{k_1} \cap v_{k_2}$ which is $v_{k_2} \cap v_{k_3}$. We choose g_2 to be the function defined on $[0, (^{j}2/m) + w]$ so that g_2 is g_1 on $[0, (^{j}1/m) + w]$, $g_2(((j_2'-1)/m) + w)$ is $b_2' - x_2$ and $g_2((^{j}2'/m) + w)$ is $b_2' - x_2$ if $a_2 < b_2'$, $g_2(((j_2' - 1)/m) + w)$ is $b_2' + x_2$ and $g_2((j_2'/m) + w)$ is $b_2' + x_2$ if $a_2 > b_2'$, $g_2((j_2/m) + w)$ is $(a_2' + b_2)/2$, and g_2 is linear on the intervals $[(j_1/m) + w, ((j_2' - 1)/m + w]$, $[((j_2' - 1)/m) + w, (j_2'/m) + w]$ and $[j_2'/m) + w, (j_2(m) + w]$. Since e_{j_1-1} and e_{j_2} are contained in f_{k_1} and $e_{j_2'}$ is contained in $f_{k_3'}$, $j_2' - j_1 + 1 < Q$ and $j_2 - j_2' < Q$. Similar to the above one can show (1) g_2 is continuous. (2) For $j_1 \leq j \leq j_2$ $g_2(u_j)^-$ is contained in v_{k_2} and is of diameter less than t. Moreover, $g_2(u_{j_2'})^-$ is contained in $v_{k_3}' \cap v_{k_2}$ and $g_2((j_2/m) + w)$ is in $v_{k_2} \cap v_{k_3}$. (3) for each link f_k of F containing a link e_j of E with $j_1 \leq j \leq j_2$, there is an integer j' so that $j_1 - 1 \leq j' \leq j_2$ and $g_2(u_{j'})^-$ is contained in v_k. (Again k is either k_3, k_2 or k_3' and one may choose j' to be $j_1 - 1$, $j_1 - 1$ and j_2' respectively.)

(4) If $g_2(u_j)^-$ is contained in v_k for $j_1 \leq j \leq j_2$, then there is an

integer p so that e_j is contained in f_p and $|p - k| < 2$. We have k is either k_3, k_2 or k_3' , $p = k_2$ will do if $j_1 \leq j < j_2$, and $p = k_1 = k_3$ will do if j is j_2 .

Case B_3 is that k_3 is not k_1 , in which case k_3' is k_1 and will be ignored. The map g_2 from $[0, (j_2/m) + w]$ is defined so that g_2 is g_1 on $[0, (j_1/m) + w]$, $g_2(((j_2 - 1)/m) - w]$ is $a_2' + (x_2/2)$ and $g_2((j_2/m) + w)$ is $a_2' + x_2$ if $a_2' < b_2$, $g_2(((j_2 - 1)/m) + w)$ is $a_2' - x_2$ and $g_2((j_2/m) + w)$ is $a_2' - x_2$ if $a_2' > b_2$, and g_2 is linear on $[(j_1/m) + w, ((j_2-1)/m) + w]$, and on $[((j_2-1)/m) + w, (j_2/m)+ w]$. Again $j_2 - j_1 > Q$ and the following hold: (1) g_2 is continuous. (2) For $j_1 \leq j \leq j_2$, $g_2(u_j)^-$ is contained in v_{k_2} and is of length less than t , and $g_2(u_{j_1})^-$ is contained in $v_{k_2} \cap v_{k_3}$. (3) For each link f_k of F containing a link e_j of E with $j_1 \leq j \leq j_2$, there is an integer j' with $j_1 \leq j' \leq j_2$ so that $g_2(u_{j'})^-$ is contained in v_k . (4) If $g_2(u_j)^-$ is contained in v_k for $j_1 \leq j \leq j_2$, then there is an integer p so that e_j is contained in f_p and $|p - k| < 2$.

IV. Clearly the above process can be continued, if necessary. Step III constructed a function g_2 so that g_2 fulfills all the requirements for the function g of the conclusion on at least the first two links of U . If g_2 does not have domain $[0,1]$, the next constructive step would involve first seeing if all the links e_j of E with $j_2 \leq j \leq m$ are contained in f_{k_3} . If so, precisely the same procedure as in Case A_1 or Case A_2 of III could be followed. If not, then a precise analogue of Case B_1 , Case B_2 or Case B_3 would hold. A function g_3 could then be constructed so that g_3 is an extension of g_2 , g_3 is continuous, and g_3 fulfills the properties of g for at least one more link of U than g_2 does.

This completes the demonstration of Lemma 3.

The first three lemmas may be described in a heuristic fashion. The proof of the theorem of this chapter involves imitating a sequence of chains which characterize a chainable continuum with a sequence of continuous functions on [0,1]. The imitation will be faithful enough so that the inverse limit of the sequence of functions is homeomorphic to the original chainable continuum. The first lemma gives a standard of accuracy for the imitation to follow. The second lemma asserts there is a sequence of chains of a nice enough nature "characterizing" the continuum which we will see are comfortably imitable. The third lemma says the imitation may be made so as to satisfy many conditions if the chains satisfy a few conditions.

The following two lemmas continue the process. The fourth merely gives that the imitation, still to be put together out of the pieces provided by the lemmas, satisfies one more of the criteria of adequacy of the first lemma. The fifth lemma is really inessential, but says the imitation to be constructed may be chosen to be of a more pleasing nature.

Lemma 4. *Suppose* $f_1, f_2, \ldots$ *is an inverse limit sequence on* [0,1] *and* $U_1, U_2, \ldots$ *is a sequence of chains of open subsets of* [0, 1] *each covering* [0, 1] *so that for all positive integers* n *if* $u_{n,j}$ *is a link of* U_n, *then there is a link* $u_{n+1,k}$ *of* U_{n+1} *so that* $f_n(u_{n+1,k})^-$ *is contained in* $u_{n,j}$. *Then for each positive integer* n, *if* $u_{n,j}$ *is a link of* U_n, *there is a point* p *in* $\lim f_i$ *so that* $P_n(p)$ *is in* $u_{n,j}$.

Proof. Recall that in the proof of Theorem 5 of Chapter I, when given a compact inverse limit sequence (X_i, f_i) then a sequence of continuous functions $F_1, F_2, \ldots$ was constructed. Each F_i was defined from $\pi_{k=i}^{\infty} X_k$ into $\pi_{k=1}^{\infty} X_k$ by $F_i(x) = (f_{1,i}(x_i), f_{2,i}(x_i), \ldots, f_{i,i}(x_i), x_{i+1}, \ldots)$ where $(x_i, x_{i+1}, \ldots)$ was a point of $\pi_{k=i}^{\infty} X_k$. The functions had the properties that the range of each F_i contained the range of F_{i+1}, and that the intersection of the ranges of all the F_i existed and was $\lim f_i$.

In this case each X_i is $[0,1]$. Suppose u_{nj} is a link of U_n. By hypothesis there is a sequence $u_{n+1,j_1}, u_{n+2,j_2}, \ldots$ of links of $U_{n+1}, U_{n+2}, \ldots$ so that $f_n(u_{n+1,j_1})^-$ is contained in u_{nj} and for each positive integer i, $f_{n+i}(u_{n+i+1,j_{i+1}})^-$ is contained in u_{n+i,j_i}. For each positive integer i, $f_{n+i}(u_{n+i+1,j_{i+1}})^-$ is compact and so

$f_{n+i}(u_{n+i+1,j_{i+1}})^-$ is compact and so $f_{n+i}(u_{n+i+1,j_{i+1}})^- \times \pi_{k=n+i+1}^{\infty}(X_k)$ is compact and its image under F_{n+i} is therefore compact. Define D_i to be $f_{n+i}(u_{n+i+1,j_{i+1}})^- \times \pi_{k=n+i+1}^{\infty} X_k$ for each positive integer i. Clearly for each i, $F_{n+i}(D_i)$ contains $F_{n+i+1}(D_{i+1})$ and $P_n(F_{n+i}(D_i))$ is contained in u_{nj}. Since each $F_{n+i}(D_i)$ is compact $\bigcap_{k=1}^{\infty} F_{n+k}(D_k)$ exists, is compact, is a subset of $\lim f_i$ and $P_n(\bigcap_{k=1}^{\infty} F_{n+k}(D_k))$ is contained in u_{nj}, and the lemma is proved.

Lemma 5. *Suppose $f_1, f_2, \ldots$ is a sequence of continuous functions from $[0, 1]$ into $[0, 1]$ and $\lim f_i$ is non-degenerate. Then there is a sequence $g_1, g_2, \ldots$ of continuous functions from $[0, 1]$ onto $[0, 1]$ so that $\lim f_i$ is homeomorphic to $\lim g_i$.*

Proof. For each positive integer n define a_n and b_n to be the left and right endpoints of the possibly degenerate interval, $\bigcap_{i=1}^{\infty} f_{n,n+i}[0,1]$. For each positive integer i, $f_{n,n+i}$ is continuous and so $f_{n,n+i}[0,1]$ is a compact subcontinuum of $[0,1]$. Moreover since $f_{n,n+i+1}[0,1]$ is $f_{n,n+i}(f_{n+i}[0,1])$, $f_{n,n+i+1}[0,1]$ is contained in $f_{n,n+i}[0,1]$ and the intersection of a nested sequence of compact continua is a compact continuum, $\bigcap_{i=1}^{\infty} f_{n,n+i}[0,1]$ exists and is a subcontinuum of $[0,1]$. So it is proper to call $\bigcap_{i=1}^{\infty} f_{n,n+i}[0,1]$ by the name $[a_n, b_n]$ if we allow degenerate intervals.

It is also true that $f_n(\bigcap_{i=1}^{\infty} f_{n+1,n+1+i}[0,1])$ is contained in $\bigcap_{i=1}^{\infty} f_n(f_{n+1,n+i+1}[0,1])$ which is $\bigcap_{i=2}^{\infty} f_{n,n+i}[0,1]$ and is contained in $[a_n,b_n]$, since $f_{n,n+1}[0,1]$ contains $f_{n,n+2}[0,1]$. For each positive integer n, $f_n[a_{n+1}, b_{n+1}]$ is contained in $[a_n, b_n]$. Suppose p is in $[a_n, b_n]$ but not in $f_n[a_{n+1}, b_{n+1}]$. Now $f_n^{-1}(p)$ is closed and so compact, and does not intersect $[a_{n+1}, b_{n+1}]$. Since for each i, $f_{n+1,n+1+i}[0,1]$ is compact, it must be there is a positive integer j so that $f_{n+1,n+1+j}[0,1]$ does not intersect $f_n^{-1}(p)$ or $\bigcap_{i=1}^{\infty}(f_{n+1,n+i+1}[0,1] \cap f_n^{-1}(p)$ would exist and be contained in $f_n^{-1}(p) \cap [a_{n+1}, b_{n+1}]$ which does not exist. But then $f_n f_{n+1,n+1+j}[0,1]$ does not contain p and does contain $[a_n, b_n]$ since it is $f_{n,n+1+j}[0,1]$. A contradiction has been produced, and so we know for each n, $f_n[a_{n+1},b_{n+1}]$ is $[a_n,b_n]$. Suppose p is in $\lim f_i$. For all positive integers n and i, $f_{n,n+i}(P_{n+i}(p))$ is $P_n(p)$ and so $P_n(p)$ is in $f_{n,n+i}[0,1]$. That is, $P_n(p)$ is in $[a_n,b_n]$. The obvious thing to do is cut each f_n down to $[a_n, b_n]$.

For each positive integer define f_n^* to be $f_n|[a_n,b_n]$, where "$|$" denotes restriction. Clearly $\lim(f_n^*,[a_n,b_n])$ is $\lim f_n$, since if p is in $\lim f_n$, then $P_n(p)$ is in $[a_n,b_n]$ for each n and $f_n^*(P_{n+1}(p))=f_n(P_{n+1}(p))= P_n(p)$ and p is in $\lim(f_n^*, [a_n,b_n])$. The containment in the other direction is even more obvious.

Since $\lim f_n$ is non-degenerate, there is an integer r so that $[a_r, b_r]$ is non-degenerate and necessarily for $k > r$, $[a_k, b_k]$ is non-degenerate. For each positive integer k not less than r define T_k by $T_k(x) = (x - a_k)/(b_k - a_k)$. Each T_k is a linear homeomorphism from $[a_k, b_k]$ onto $[0, 1]$. For each integer k not less than r define g_k to be $T_k f_k^* T_{k+1}^{-1}$. The inverse limit of the sequence $g_r, g_{r+1} \ldots$ is by Theorem 7 of Chapter I, the inverse limit of the sequence T_r, f_r^*, $T_{r+1}^{-1}, T_{r+1}, f_{r+1}^*, T_{r+2}^{-1}, T_{r+2}, f_{r+2}^*$, etc. But this is the inverse limit of $f_r^*, f_{r+1}^*, f_{r+2}^*, \ldots$, which is the inverse limit of $f_1, f_2, \ldots$. Each g_k is from $[0, 1]$ onto $[0,1]$, since $g_k[0,1] = T_k f_k^*[a_{k+1}, b_{k+1}] = T_k[a_k,b_k]$ $= [0,1]$ and so $\lim f_i$ is $\lim g_i$ and the lemma is proved.

Theorem. A compact continuum M is chainable if and only if homeomorphic to the inverse limit of a sequence of continuous functions from $[0, 1]$ into $[0, 1]$. Moreover, M is a non-degenerate compact chainable continuum iff there is a sequence of continuous functions from $[0,1]$ onto $[0,1]$ whose inverse limit is homeomorphic to M.

Proof. The "moreover" is a direct and simple consequence of Lemma 5 and the first part of the theorem. Lemma 5 and Theorems 5 and 12 of Chapter II give immediately that each inverse limit of a sequence of maps on $[0,1]$ is a chainable compact continuum, with the obvious observation that a point is chainable.

It remains to be shown that if M is a compact chainable continuum, then M is homeomorphic to the inverse limit of a sequence of functions on [0,1]. If M is degenerate, then M is homeomorphic to the inverse limit of the function g defined by $g(x) = 0$ for x in [0,1]. The inverse limit of g is the point $(0,0,0, \ldots)$. Suppose M is non-degenerate. By Lemma 2 there is a decreasing sequence $C_1, C_2, \ldots$ of spread chains so that for each positive integer n, each point of M is contained in some link of C_n, and each link of C_n contains some link of C_{n+1} and a point of M.

Since M is non-degenerate, there is a positive integer n_1 so that C_{n_1} contains more than four links. Let r_1 denote the number of links in C_{n_1}, δ_1 a positive numbers so that C_{n_1} is δ_1-spread, t_1 denote the number $1/2$ and Q_1 denote any integer greater than 4 so that ${}^{t_1}/16$ is greater that ${}^{1}/(Q_1 - 1)$. Since $C_1, C_2, \ldots$ is decreasing there is an integer n_2 greater than n_1 so that C_{n_2} has mesh less than ${}^{\delta_1}/Q_1$. Let U_1 be any spread chain of r_1 connected open subsets of [0,1] covering [0,1] which is of mesh less than ${}^{2}/r_1$. Let r_2 denote the number of links in C_{n_2}. By Lemma 3 there is a spread chain U_2 of r_2 connected open subsets of [0,1] covering [0,1] which has mesh less than ${}^{2}/r_2$ and there is a continuous function g_1 from [0,1] into [0,1] so that the closure of the g_1-image of each link of U_2 is contained in some link of U_1 and is of length less than t_1, each link of U_1 contains the closure of the g_1-image of some link of U_2 and so that if $g_1(u_{2j})$ is contained in u_{1k}, then there is an integer p so that $c_{n_2,j}$ is contained in $c_{n_1,p}$ and $|p - k| < 2$.

Suppose this process has been continued for $m - 1$ steps in the manner described below. An increasing sequence $n_1, n_2, \ldots n_m$ of positive integers and a sequence of integers $r_1, r_2, \ldots, r_m$ each greater than 4 exists so that for $1 \leq i \leq m$, C_{n_i} contains r_i links and r_i is greater than 2^{i+1}. A sequence $\delta_1, \delta_2, \ldots, \delta_m$ of positive numbers exists so that for $1 \leq j \leq m$, C_{n_j} is δ_j-spread and for $1 \leq j \leq m - 1$, $C_{n_{j+1}}$ is of mesh less than $\delta_j/4$. A sequence $U_1, \ldots, U_m$ of spread chains of connected open subsets of $[0,1]$ covering $[0,1]$ exists so that for $1 \leq j \leq m$, U_j contains r_j links and is of mesh less than $2/r_j$. A sequence $g_1, g_2, \ldots, g_{m-1}$ of continuous functions from $[0,1]$ into $[0,1]$ exists so that for $1 \leq j \leq m - 1$, the closure of the g_j-image of each link of U_{j+1} is contained in some link of U_j and each link of U_j contains the closure of the g_j-image of some link of U_{j+1}. Moreover, for any two points x and y in a link of U_j, $|g_{ij}(x) - g_{ij}(y)| < 2^{-j}$ where $i < j \leq m$ and g_{ij} denotes $g_i \, g_{i+1} \, \cdots \, g_{j-1}$. Note that U_j is of mesh less than 2^{-j} for $1 \leq j \leq m$ since of mesh less than $2/r_j$ and $r_j \geq 2^{j+1}$.

Now g_{im} is continuous for $i < m$ and so uniformly continuous. Clearly there is a positive number t_m so that if $|x - y| < t_m$, then $|g_{im}(x) - g_{im}(y)| < 2^{-m-1}$, and so that t_m is less than 2^{-m-1}. Choose an integer Q_m greater than 1 so that $t_m/6 > 1/(Q_m - 1)$. Choose an integer n_{m+1} greater than n_m so that the mesh of $C_{n_{m+1}}$ is less than δ_m/Q_m and is less $\delta_m/4$, and $C_{n_{m+1}}$ has more than 2^{m+2} links. Denote the number of links of $C_{n_{m+1}}$ as r_{m+1}. By Lemma 3 there is a continuous function g_m from $[0,1]$ into $[0,1]$ and a spread chain U_{m+1} of r_{m+1} connected open subsets of $[0,1]$ covering $[0,1]$ so that U_{m+1} has mesh less than 2^{-m-1} (which is more than 2 times the reciprocal of the number of links in $C_{n_{m+1}}$), the closure of the g_m-image of each link of U_{m+1} is contained in some link of U_m and is of length less than t_m, each link of U_m contains the g_m-image of some link of U_{m+1} and so that if $g_m(u_{m+1,j})$ is contained in $u_{m,k}$ then there is a positive integer p so that $c_{n_{m+1},j}$ is contained in $c_{n_m,p}$ and $|k - p| < 2$.

By induction the above process can be continued for all positive integers. Let D_i denote C_{n_i} for each positive integer i . Therefore there exists a decreasing sequence $D_1, D_2, \ldots$ of spread chains so that for each positive integer i , each point of M is in some link of D_i and each link of D_i contains a point of M . There is a sequence $\delta_1, \delta_2, \ldots$ of positive numbers so that for each positive integer i , D_i is δ_i spread and D_{i+1} has mesh less than $\delta_i/4$. There is a sequence $U_1, U_2, \ldots$ of spread chains of connected open subsets of $[0,1]$ covering $[0,1]$ so that each U_i has the same number of links as D_i . There is a sequence $g_1, g_2, \ldots$ of continuous functions from $[0,1]$ into $[0,1]$ so that each U_i has mesh less than 2^{-i} so that for any two points x and y of a link of U_i , $|g_{ji}(x) - g_{ji}(y)| < 2^{-i}$ for all j less than i and so that the closure of the g_i-image of each link of U_{i+1} is contained in some link of U_i , each link of U_i contains the closure of the g_i-image of some link of U_{i+1} and so that if $g_i(u_{i+1,j})$ is contained in $u_{i,k}$ there is a positive integer p so that $d_{i+1,j}$ is contained in $d_{i,p}$ and $|p - k| < 2$.

Let T denote the inverse limit of $g_1, g_2, \ldots$. Let V_1 denote the chain of open subsets of T covering T whose links are $(u_{11} \times \pi_{i=2}^{\infty} [0,1]) \cap T$, $(u_{12} \times \pi_{i=2}^{\infty}[0,1]) \cap T$, etc. For integers j greater than 1 let V_j denote the chain of open subsets of T covering T whose links are $(\pi_{i=1}^{j-1} [0,1] \times u_{j1} \times \pi_{i=j+1}[0,1]) \cap T$,

$(\pi_{i=1}^{j-1}[0,1] \times u_{j2} \times \pi_{i=j+1}^{\infty}[0,1]) \cap T$, etc. Each of the links of each V_j exists and contains a point of T by Lemma 4. Suppose v_{jk} is a link of V_j . Let x and y denote any two points of v_{jk} . Since x and y are in T , and $P_j(x)$ and $P_j(y)$ are in u_{jk} , the distance from x to y is

$$\Sigma_{i=1}^{\infty}|P_i(x) - P_i(y)|2^{-i} \leq \Sigma_{i=1}^{j-1}|g_{ij}(P_j(x)) - g_{ij}(P_j(y))|2^{-i}$$

$$+ |P_j(x) - P_j(y)| + \Sigma_{i=j+1}^{\infty}2^{-i}$$

$$\leq 2^{-j}\Sigma_{i=1}^{j-1} 2^{-i} + 2^{-j} + 2^{-j} < 3(2^{-j}) .$$

For any link $v_{j+1,i}$ of V_{j+1} , there is a link $u_{j,k}$ of U_j so that $g_j(u_{j+1,i})^{-}$ is contained in u_{jk} . Since $v_{j+1,i} = g_j P_{j+1}(v_{j+1,i})$, and since $u_{j+1,i} \supset P_{j+1}(v_{j+1,i})$ we have $\bar{v}_{j+1,i} \subset v_{jk}$. So V_1, V_2 is a decreasing sequence of chains. Obviously, by the construction, $V_1, V_2, \ldots$ is similar to $D_1, D_2, \ldots$ and by Lemma 1, T is homeomorphic to M .

This completes the proof of the theorem.

CHAPTER IV

SOME THEOREMS ON DECOMPOSABILITY

This chapter is principally concerned with a few theorems that say when the inverse limit of a single function on [0,1] is indecomposable. A continuum is said to be indecomposable iff it is not the union of two proper subcontinua. It is well-known that a non-degenerate compact continuum is indecomposable iff it contains three points between each pair of which it is irreducible. That is a non-degenerate compact continuum M is indecomposable iff it contains three points a, b, and c so that no proper subcontinuum of M contains more than one of a, b, and c.

The first theorem gives that continuous functions from [0,1] into [0,1] of a particularly simple type have very decomposable inverse limits, which are in fact arcs. An arc may be defined as a non-degenerate compact metrizable continuum with at most two non-cut points. A standard theorem is that a compact continuum is an arc iff it is homeomorphic to the unit interval (with the usual topology.) A much stronger form of the first theorem may be found in Capel [5].

Theorem 1. *If f is a monotone continuous function from [0,1] onto [0,1], then lim f is an arc.*

Proof. It will be assumed that f is non-decreasing. If f were non-increasing, then f^2 is non-decreasing and the inverse limit of f is homeomorphic to the inverse limit of a non-decreasing function by

Corollary 10 of Chapter II.

Since f is onto, its inverse limit is non-degenerate. Since f is onto and non-decreasing, $f(0) = 0$ and $f(1) = 1$. So $(0,0, \ldots)$ and $(1,1,\ldots)$ are points of the inverse limit of f. Suppose p is a point of the inverse limit of f so that for some positive integer n, $P_n(p)$ is not 0 and is not 1. For each integer m greater than n, $P_m(p)$ is neither 0 nor 1. Define A_m to be $(\lim f) \cap P_m^{-1}([0,P_m(p)))$ and B_m to be $(\lim f) \cap P_m^{-1}((P_m(p),1])$ for each integer m greater than n. Define A to be $\bigcup_{m=n+1}^{\infty} A_m$ and B to be $\bigcup_{m=n+1}^{\infty} B_m$. Each of A and B is an open subset of $\lim f$. Suppose q is in $\lim f$ and q is not p. For some positive integer j, $P_j(p)$ is not $P_j(q)$, and j can be chosen larger than n. If $P_j(p) < P_j(q)$, then q is in A_j. If $P_j(p) < P_j(q)$, then q is in B_j. So $\lim f - p$ is $A \cup B$. Suppose q is a point in A and q is a point in B. For some integer j greater than n, $P_j(q) > P_j(p)$, and for some integer k greater than n, $P_k(q) < P_k(p)$. If j is greater than k, then $f(P_j(p)) \leq f(P_j(q))$ and $P_{j-1}(p) \leq P_{j-1}(q)$, $fP_{j-1}(p) \leq fP_{j-1}(q)$ and $P_{j-2}(p) \leq P_{j-2}(q)$, etc., since f is non-decreasing. But then also $P_k(p) \leq P_k(q)$ contrary to assumption. A similar argument suffices if j is less than k. Therefore A and B do not intersect, and p separates $\lim f$.

It has been shown that each point of $\lim f$ other than $(0,0,\ldots)$ and $(1,1, \ldots)$ separates $\lim f$, and $\lim f$ is an arc.

A simple example shows that this onto condition of the first theorem cannot easily be dispensed with, and requiring f simply to have a non-degenerate range is not sufficient. Let f be the function from $[0,1]$ into $[0,1)$ defined by $f(x) = (x/2) + (1/4)$. A simple computation shows $f^{-n}(x) = 2^n x - 2^{n-1} + 1/2$, and the only number with enough preimages in $[0,1]$ is $1/2$. That is, the inverse limit of f is $(1/2, 1/2, 1/2, \ldots)$.

The first theorem may be considered as a slight strengthening of the fact that an onto homeomorphism has an inverse limit homeomorphic to its domain, since an onto monotone map is "almost" a homeomorphism. The above example also shows that into homeomorphisms do not share this property.

The second theorem gives a simple condition which implies the inverse limit is indecomposable. An example in Chapter V shows this is not a necessary and sufficient condition.

<u>Theorem</u> 2. <u>Suppose</u> g <u>is a continuous function from</u> $[0, 1]$ <u>onto</u> $[0, 1]$ <u>and there are numbers</u> a, b, <u>and</u> c <u>so that</u> $a < b < c$ <u>and either</u> $g(a) = g(c) = 1$ <u>and</u> $g(b) = 0$ <u>or</u> $g(a) = g(c) = 0$ <u>and</u> $g(b) = 1$. <u>Then the inverse limit of</u> g <u>is indecomposable</u>.

<u>Proof</u>. Suppose that $g(a) = g(c) = 0$ and $g(b) = 1$. Since g is continuous there are numbers d and e with $a \leq d < b < e < c$ so that $g(d) = d$ and $g(e) = e$. Since $d < b$ and $g(b) = 1$, there is a number h so that $d < h < b$ and $g(h) = b$, and since $b < e < c$ and $g(b) = 1$, there is a number k so that $b \leq k < e$ and $g(k) = c$. Now $g^2(h) = 1$ and $g^2(k) = 0$ and $h < k$ so there is a number f so that $h < f < k$ and $g^2(f) = f$. We have now that $g^2[d,f]$ contains $[d,f]$ and $g^2(h)$, which is 1 . So $g^2[d,f]$ contains $[d, 1]$. Also $g^2 g^2[d,f]$ contains

$[d,1]$ and g^2k which is 0, and so $g^4[d,f] = [0,1]$. Similarly $g^4[f,e]$ and $g^4[d,e]$ are $[0,1]$.

The inverse limit of g^4 is homeomorphic to the inverse limit of g. Also g^4 leaves the three numbers d, e and f fixed and so $(d,d,d, \ldots)$, $(e,e,e, \ldots)$ and $(f,f,f, \ldots)$ are points of $\lim g^4$. Suppose T is a subcontinuum of $\lim g^4$ containing $(d,d,d, \ldots)$ and $(e,e,e, \ldots)$. Since T is connected and for each positive integer n, P_{n+1} is continuous, $P_{n+1}(T)$ contains $[d,e]$. But $g^4P_{n+1}(T) = P_n(T)$ by Theorem 1 of Chapter II, so for each positive integer n, $P_n(T) = [0,1]$. By Theorem 6 of Chapter II, T is $\lim g^4$, and so $\lim g^4$ is irreducible from $(d,d,d, \ldots)$ to $(e,e,e,\ldots)$. Similarly $\lim g^4$ is irreducible from $(d,d,d, \ldots)$ to $(f,f,f, \ldots)$ and from $(e,e,e, \ldots)$ to $(f,f,f, \ldots)$ and $\lim g^4$ is indecomposable.

Suppose instead that $g(a) = g(c) = 1$ and $g(b) = 0$. Then since g is continuous, there are numbers r, s and t and u, v and w so that $a \leq r < s < t \leq b \leq u < v < w \leq c$ so that $g(r) = g(w) = c$, $g(s) = g(v) = b$ and $g(t) = g(u) = a$. Then $s < u < v$ and $g^2(s) = g^2(v) = 0$ and $g^2(u) = 1$ and by the above argument the inverse limit of g^2 is indecomposable. But since the inverse limit of g^2 is homeomorphic to the inverse limit of g, the inverse limit of g is indecomposable, and the theorem is proved.

Actually Theorem 2 is a special case of Theorem 3, but its proof is enough different to make it interesting in itself. The proof of Theorem 3 depends on some concepts and theorems of which the author first read in a paper of Lida K. Barrett [2]. The essential concepts used here are of a defining sequence and of a chain looping in another. If M is a chainable continuum, a sequence $C_1, C_2, \ldots$ of chains is said to be a defining sequence for M iff $C_1, C_2, \ldots$ is a decreasing sequence of spread chains so that for each positive integer n, each point of M is contained in some link of C_n and each link of C_n contains some point of M and a link of C_{n+1}. A chain C is said to loop in a chain D iff each link of C is contained in some link of D and there is a subchain C' of C so that either the first and last links of C' are contained in the first link of D and some link of C' is contained in the last link of D or the first and last links of C' are contained in the last link of D and some link of C' is contained in the first link of D. Theorem 2 of [2] may be restated as follows: *If M is an indecomposable compact continuum and $C_1, C_2, \ldots$ is a defining sequence for M, then for each positive integer n there is an integer i greater than n so that C_i loops in C_n. Conversely if there is a defining sequence $C_1, C_2, \ldots$ for the compact chainable continuum M so that for each positive integer n there is an integer i greater than n so that C_i loops in C_n, then M is indecomposable.*

Analogously, if f is a function from $[0,1]$ into $[0,1]$ and ε is a positive number, f is said to ε-loop iff there are numbers a, b and c with $a < b < c$ so that either $|f(a) - 1| < \varepsilon$, $|f(c) - 1| < \varepsilon$ and $|f(b) - 0| < \varepsilon$ or $|f(a) - 0| < \varepsilon$, $|f(c) - 0| < \varepsilon$ and $|f(b) - 1| < \varepsilon$. So a function ε-loops if it follows the behavior of the function of Theorem 2 to within ε. An analogue to the theorem quoted above is immediate, and the proof is only sketched.

Theorem 3. If f is a continuous function from $[0,1]$ onto $[0,1]$, the inverse limit of f is indecomposable if and only if for each positive *number* ε *there is a positive integer* n *so that* f^n ε-*loops*.

Proof. Suppose $\lim f$ is indecomposable, and ε is a positive number. By methods similar to those used before, a defining sequence of chains can be produced for $\lim f$ by taking a sequence of chains U_1, U_2, ... each covering $[0,1]$, and this can be arranged so that U_1 has mesh less than ε. Now since for some integer $n+1$ the chain resulting from U_{n+1} loops in the chain resulting from U_1, we have integers i, j, and k with $i < j < k$ so that either $f^n(u_{n+1,i}) \cup f^n(u_{n+1,k}) \subset u_{1,1}$ and $f^n(u_{n+1,j})$ is contained in the last link of U_1, or $f^n(u_{n+1,i}) \cup f^n(u_{n+1,k})$ is contained in the last link of U_1 and $f^n(u_{n+1,j}) \subset u_{1,1}$. Choosing points a, b and c so that a is in $u_{n+1,i}$, b is in $u_{n+1,j}$ and c is in $u_{n+1,k}$ it is clear that f^n ε-loops.

Conversely if for each positive number ε, there is a positive integer n so that f^n ε-loops we can construct a defining sequence V_1, V_2, ... for $\lim f$ so that for each positive integer n, V_{n+1} loops in V_n, in an obvious way.

Morton Brown [3] has proved a theorem similar to this, which shows that a property much like ε-looping is necessary and sufficient for an inverse limit sequence to have an hereditarily indecomposable inverse limit. His theorem is concerned with much more general inverse limit sequences, although the condition on the maps is necessarily somewhat more restrictive.

The following gives another condition under which an inverse limit is decomposable, and it is a simple corollary to Theorem 3 that the inverse limit of f is decomposable.

Theorem 4. *Suppose* f *is a continuous function from* $[0,1]$ *onto* $[0,1]$ *and there is a number* b *so that* $0 < b < 1$ *and* $f(b) = b$,

$f[0,b] = [0,b]$ *and* $f[b,1] = [b,1]$. *Then the inverse limit of* f *is decomposable,* $\lim f$ *is the union of* $\lim(f|[0,b])$ *and* $\lim(f|[b,1])$ *and these proper subcontinua of* $\lim f$ *intersect in only one point.*

Proof. Since $f|[0,b]$ is a continuous function from $[0,b]$ onto $[0,b]$, $\lim(f|[0,b])$ is a compact continuum and similarly $\lim(f|[b,1])$ is a compact continuum. Suppose p is in $\lim f$. Either $P_n(p) = b$ for all n, or for some n, $P_n(p)$ is in $[0,b)$ or for some n, $P_n(p)$ is in $(b,1]$. If for some n $P_n(P)$ is in $[0, b)$, then for m less than n, $P_m(p) = f^{n-m}(P_n(p)) = (f|[0,b])^{n-m}(P_n(P))$ and $P_m(p)$ is in $[0,b]$. Clearly also it cannot be that for some m greater than n, $P_m(p)$ is in $[b,1]$ or $P_n(p)$ would have to be in $[b,1]$, and so p is in $\lim(f|[0,b])$. If for some n, $P_n(p)$ is in $(b,1]$, we see similarly that p is in $\lim(f|[b,1])$. So $\lim(f|[0,b]) \cup \lim(f|[b,1])$. No point with first coordinate 0 is in $\lim(f|[b,1])$ and no point with first coordinate 1 is in $\lim(f|[0,b])$. So $\lim(f|[0,b])$ and $\lim(f|[b,1])$ are proper subcontinua of $\lim f$ whose union is $\lim f$, $\lim f$ is decomposable, and clearly $\lim(f|[0,b]) \cap \lim(f|[b,1])$ is $(b,b,b, \ldots)$.

The next theorem is used mainly in some of the examples of Chapter V. A topological ray is a continuum homeomorphic to the non-negative real numbers with the usual topology. A non-degenerate connected set is a topological ray iff separable, locally compact and it has one non-cut point so that each other point separates it into two connected sets.

Theorem 5. Suppose $f_1, f_2, \ldots$ *is a sequence of continuous functions from* $[0,1]$ *into* $[0,1]$ *and the sequence* $g_1, g_2, \ldots$ *of continuous functions from* $[0,1]$ *onto* $[0,1]$ *is so that for each positive integer* n, $g_n(0) = 0$, $g_n(1/4) = 1$, $g_n(1/2) = f_n(0)/2 + 1/2$, g_n *is linear on*

$[0, 1/4]$ *and* $[1/4, 1/2]$ *and for* x *in* $[1/2, 1]$, $g_n(x) = (1/2)f_n(2x-1) + 1/2$. *Then* $\lim g_i$ *is the union of a topological ray* R *and a continuum* M *so that* R *and* M *are disjoint, each point of* M *is a limit point of* R *and* M *is homeomorphic to* $\lim f_i$.

Proof. Let R denote the set of points of $\lim g_i$ with some coordinate less than $1/2$. Let M denote the set of points of $\lim g_i$ with all coordinates greater than or equal to $1/2$. Clearly $R \cup M$ is $\lim g_i$ and R and M are disjoint. Define T from $[0,1]$ onto $[1/2,1]$ by $T(x) = x/2 + 1/2$. Clearly T^{-1} is 1-1 from $[1/2,1]$ onto $[0,1]$ and is so that $T^{-1}(x)$ is always $2x - 1$. Then $g_n|[1/2,1]$ is Tf_nT^{-1}, and the inverse limit of $g_1|[1/2,1]$, $g_2|[1/2,1]$, ... is homeomorphic to $\lim f_n$ by Theorem 7 of Chapter I. (The explicit proof is similar to the proof of Lemma 5 of Chapter III.) But this continuum is M and so M is homeomorphic to $\lim f_i$.

Now we define a relation $<$ on R. We shall say that $p < q$ iff for some positive integer n, $P_n(p) < P_n(q) < 1/2$, where here $<$ is the normal order on the real numbers. Observe that if p is in R and $P_n(p) < 1/2$, then $P_{n+1}(p) = (1/4)P_n(p)$, $P_{n+2}(p) = (1/16)P_n(p)$, etc. For all integers m greater than n, $P_m(p)$ is $(1/4)^{m-n}P_n(p)$, and a coordinate of a point of R which is in $[0,1/2]$ completely determines the point. So if p and q are in R there is a positive integer n so that both $P_n(p)$ and $P_n(q)$ are in $[0,1/2)$, and $<$ is defined for all pairs p and q in R. Clearly also $<$ is transitive and anti-reflexive.

The next thing to show is that if $p < q$ for p and q in R, then $[p,q]$ is an arc, where $[p,q]$ denotes the set of all points x of

R so that $p \leq x \leq q$. Choose a positive integer n so that both $P_n(p)$ and $P_n(q)$ are less than $1/2$. Clearly $[p,q]$ consists of the points r of $\lim g_i$ so that $P_n(p) \leq P_n(r) \leq P_n(q)$ and the function S from $[P_n(p), P_n(q)]$, defined by $S(x)$ is the unique point of $\lim g_i$ whose n-th coordinate is x, is 1-1 and continuous and onto $[p,q]$, and so $[p,q]$ is an arc.

Therefore R is arcwise connected and connected. Since R is an open subset of $\lim g_i$, R is separable and locally compact. It is also true that $R - (0,0,0, \ldots)$ is arcwise connected by the preceding argument and so $(0,0, \ldots)$ is a non-cut point of R . For p in R not $(0,0,0, \ldots)$, for q and r less than p , there is an arc of points less than p containing q and r and so the set of points less than p is connected. Similarly the set of points of R greater than p is connected. Choosing an integer n so that $P_n(p) < 1/2$, we have $R - p$ is $(R \cap P_n^{-1}[0,P_n(p))) \cup (R \cap P_n^{-1}(P_n(p),1])$ which are disjoint open subsets of R and so p separates R . Therefore R is a topological ray.

Suppose x is a point of M , and U is an open subset of $\lim g_i$ containing x . By Theorem 4 of Chapter II there is a positive integer n and an open subset V of $[0,1]$ so that $P_n^{-1}(V) \cap \lim g_i$ contains x and is contained in U . By the way g_n was constructed there is a number s in $[0,1/4]$ so that $g_n(s) = P_n(x)$. The unique point y of R so that $P_{n+1}(y) = s$ has the property that $P_n(y) = P_n(x)$ and so y is in U , since $P_n(y)$ is in V . So R is dense in $\lim g_i$.

This completes the proof of Theorem 5.

A theorem of R. H. Bing [3] is that each compact chainable continuum can be embedded in the plane. There is a hereditarily indecomposable compact non-degenerate chainable continuum which by the theorem of Chapter III can be represented as the inverse limit of a sequence of continuous functions from [0,1] onto [0,1] , f_1, f_2, ... By the preceding theorem there is a chainable continuum which is the union of such an hereditarily indecomposable continuum and a ray dense in it. Embedding this in the plane, we obtain a very wild topological ray, one whose set of limit points not in itself is an hereditarily indecomposable continuum.

From some of the preceding comments, the following is clear.

Corollary 6. *If M is a chainable compact continuum, there is a compact chainable continuum which is the union of a topological ray R and a continuum M' homeomorphic to M so that M' and R are disjoint and R is dense in M'.*

CHAPTER V

SOME EXAMPLES

The first example shows that the converse of Theorem 1 of Chapter IV does not hold.

Example 1. There is a continuous function from $[0,1]$ onto $[0,1]$ which is not monotone and whose inverse limit is an arc.

Define g on $[0,1]$ by $g(0) = 0$ $g(1/2) = 1/4$, $g(7/12) = 1/2$ $g(8/12) = 1/4$, $g(3/4) = 1/2$ $g(1) = 1$ and g is linear on the intervals $[0, 1/2]$, $[1/2, 7/12]$, $[7/12, 8/12]$, $[8/12, 3/4]$ and on $[3/4,1]$. The function g is continuous and not monotone. Note that for x in $(1/2,1]$, $g^{-1}(x)$ is degenerate and is in $(3/4,1]$. If a point of $\lim g$ has a coordinate in $(1/2,1]$, then all further coordinates are in $(3/4,1]$ and they are a monotone non-decreasing sequence. Suppose x is $1/2$. Then $g^{-1}(x)$ consists of $7/12$ and $3/4$, and the preimages of $7/12$ and $3/4$ are in $(3/4,1]$. So if a point of $\lim g$ has n-th coordinate $1/2$ the $(n + 2)$-cordinate is in $(3/4,1]$ and all further coordinates form a monotone increasing sequence. Suppose x is in $(1/4,1/2)$. There are three g-preimages of x, all in $(1/2,1]$, and so the g-preimages of the g-preimages are in $(3/4,1]$. The g-preimages of $1/4$ are $1/2$ and $7/12$, and we have seen that the g-preimages of the g-preimages of $1/2$ and $7/12$ are in $(3/4,1]$. On $[0, 1/2)$, $g(x) = x/2$ and for x in $[0, 1/4)$, $g^{-1}(x) = 2x$. For x in $(0, 1/4)$ there is a positive integer m so that $1/2 > 2^m \geq 1/4$, $g^{-m}(x)$ is in $[1/4,1/2)$. But then if a point of lim g has a coordinate in $(0, 1/4)$ it eventually has a coordinate in $[1/4, 1/2)$ and we have just seen that if a point of lim g has a coordinate in $[1/4, 1/2]$ it has a coordinate in $(3/4, 1]$. In summary, it has been shown that every point other than $(0, 0, 0, \ldots)$ has a coordinate in $(3/4, 1]$. Moreover since g^{-1} is 1-1 from $(3/4,1]$ into $(3/4,1]$, for each positive integer n and number x in $(3/4,1]$, there is only one point p of lim g so that $P_n(p) = x$. If p is a point of lim g, and p is not $(0, 0, \ldots)$ or $(1, 1, \ldots)$ then there is a positive integer n so that $P_n(p)$ is in $(3/4, 1)$ and lim g - p $= (\lim g \cap P_n^{-1}[0, P_n(p))) \cup (\lim g \cap P_n^{-1}(P_n(p),1])$ which are separated sets. Therefore lim g is a compact continuum in a metric space with at most two non-cut points and is an arc.

The second example shows that there is a function f from $[0,1]$ onto $[0,1]$ whose inverse limit is indecomposable and so that there are no numbers a, b and c with $a < b < c$ so that either $f(a) = f(c) = 0$ and $f(b) = 1$ or $f(b) = 0$ and $f(a) = f(c) = 1$. That is, the converse of Theorem 2 of Chapter IV does not hold.

<u>Example</u> 2. There is a continuous function g from $[0, 1]$ onto $[0, 1]$ whose inverse limit is indecomposable so that $g^{-1}(0) = 0$ and $g^{-1}(1) = 1$. Define g on $[0,1]$ by $g(0) = 0$, $g(1/3) = 1/4$, $g = (4/9) = 3/4$, $g(5/9) = 1/4$, $g(2/3) = 3/4$ and $g(1) = 1$, and g is linear on $[0, 1/3]$, $[1/3, 4/9]$, $[4/9, 5/9]$, $[5/9, 2/3]$ and $[2/3,1]$. Clearly g is continuous and $g^{-1}(0) = 0$, $g^{-1}(1) = 1$. A little thought will show that for all x in $[0,1]$, $g(1-x) = 1 - g(x)$. On

$[0, 1/3]$, $g(x) = 3/4\, x$, and so for any positive integer n , $g^n(x) = (3/4)^n x$ for x in $[0, 1/3]$. If ε is a positive number there is a positive integer n so that $(3/4)^n < \varepsilon$. Since $g(1 - x) = 1 - g(x)$, then $g^2(1 - x) = g(1 - g(x)) = 1 - g^2(x)$ and clearly for any positive integer n , $g^n(1 - x) = 1 - g^n(x)$. Now $g(5/9) = 1/4$ and so $g^{n+1}(5/9) = (3/4)^n(1/4) < \varepsilon$. Also $g(4/9) = 3/4$, and $g^{n+1}(4/9) = g^n(3/4) = g^n(1 - 1/4) = 1 - (3/4)^n(1/4)$. Moreover $g^{n+1}(0) = 0$. So we have $|g^{n+1}(0) - 0| = 0 < \varepsilon$, $|g^{n+1}(4/9) - 1| = |(3/4)^n(1/4)| < \varepsilon$ and $|g^{n+1}(5/9) - 0| = |(3/4)^n(1/4)| < \varepsilon$, and g^{n+1} ε-loops. Since for each positive number ε there is a positive integer n so that g^{n+1} ε-loops, by Theorem 3 of Chapter IV, the inverse limit of g is indecomposable.

It is by no means necessary that a function with the properties stated in Example 2 need have such an obvious up and down character. In fact, there is a continuous g from $[0, 1]$ onto $[0, 1]$ whose inverse limit is indecomposable so that for each x in $[0, 1]$ $g(x) \leq x$. This is neither surprising enough nor instructive enough to warrant its inclusion.

The next is an example of a continuum which can be obtained as the inverse limit of a single function.

<u>Example</u> 3. A continuous function from $[0,1]$ onto $[0,1]$ whose inverse limit is a topological "Sin(1/x)" curve.

For each n let δ_n be the identity map of $[0,1]$ onto $[0,1]$. Using the procedure of Theorem 5 of Chapter IV we have that the inverse limit of $g_1, g_2, \ldots$ where for each n $g_n(0) = 0$, $g_n(1/4) = 1$, $g_n(1/2) = 1/2$, $g_n(1) = 1$ and g_n is linear on $[0, 1/4]$, $[1/4, 1/2]$ and

[1/2,1] is the union of a topological ray and an arc disjoint from the ray so that the ray is dense in the arc. Each g_n is g_1 and so this is the inverse limit of a single function. A "Sin(1/x)" curve is also the union of an arc and a ray of this form. Unfortunately this is not enough information to know that the inverse limit of g_1 is a "Sin(1/x)" curve. The actual verification would be very tedious and will not be done here.

If f is a monotone non-decreasing function from [0, 1] into [0,1], then $f, f^2, f^3, \ldots$ converge pointwise to a continuous function, whose graph is an arc. If f is onto, then the inverse limit of f is an arc. Consider the function g of Example 1. The graphs of $g, g^2, g^3, \ldots$ converge to the set of all points (x,y) with x and y in [0,1] and either $y = 0$ or $x = 1$. This set is also an arc, and the inverse limit of g is an arc. Consider the function g_1 of Example 3. The graphs of $g_1, g_1^2, g_1^3, \ldots$ have a sequential limiting set which is a polygonal "Sin(1/x)" curve in the upper half of the unit square with an arc trailing off along the lower left-hand edge of the unit square. These and some other things of the same sort make it seem that there must be a definite connection between convergence of the graphs of a function and its inverse limit. No specific conjecture is offered.

The next example is an illustration of the case with which some examples of continua with specified properties may be obtained as inverse limits and also of a continuum which can be obtained as the inverse limit of a single function. It is essentially just an iteration of the procedure of Theorem 5 of Chapter IV.

Example 4. A continuum with only two topologically different non-degenerate subcontinua, one of which is an arc, and which is the union of a ray dense in the continuum and a homeomorphic image of the continuum disjoint from the ray.

Define a map f from $[0, 1/2]$ onto $[0,1]$ by $f(0) = 0$, $f(1/4) = 1$, $f(1/2) = 1/2$ and f is linear on $[0, 1/4]$ and $[1/4,1/2]$. Define the map T from $[0,1]$ onto $[1/2,1]$ by $T(x) = (x/2) + (1/2)$. The map T is 1-1 and maps $[0, 1/2]$ onto $[1/2, 3/4]$. Define g by $g = f$ on $[0, 1/2]$, $g = TfT^{-1}$ on $[1/2,3/4]$, $g = T^2fT^{-2} = TgT^{-1}$ on $[3/4,7/8]$, and in general, g is T^nfT^{-n} on $[1 - 2^{-n}, 1 - 2^{-n-1}]$, and $g(1) = 1$. Essentially, the graph of g is formed by taking the graph of f, squeezing down to half its height and width, shifting up 1/2 and over onto $[1/2, 3/4]$, then squeezing f down to 1/4 its height and width, shifting up 3/4 and over onto $[3/4, 7/8]$, etc. We have that T^{-1} is given by $T^{-1}(x) = 2x - 1$, and so $T^{-1}(1 - 2^{-n}) = 2 - 2^{-n+1} - 1 = 1 - 2^{-n+1}$ for all positive integers n. Moreover $T^{-n}(1 - 2^{-n}) = 1 - 2^0 = 0$, $T^{-n+1}(1-2^{-n}) = 1 - 2^{-1} = 1/2$ and $T^nf(0) = T^n(0) = T^{n-1}(1/2) = T^{n-1}f(1/2)$ and so g is well defined on points of the form $1 - 2^{-n}$. A little reflection shows $T^n[0,1] = [1 - 2^{-n},1]$ and so it is always true that $g[1\cdot 2^{-n}, 1 - 2^{-n-1}]$ is contained in $[1-2^{-n},1]$. Since g is piecewise linear on $[0,1)$ and by the preceding remark the limit of g at 1 (from the left) is 1, g is continuous.

Note now that TgT^{-1} on $[1/2, 1]$ is g on $[1/2,1]$, and so $\lim g$ is homeomorphic to $\lim(g\,|[1/2,1]$, precisely as in Theorem 5 of Chapter IV. So $\lim g$ is homeomorphic to the union of a topological ray

R and a continuum M so that R and M are disjoint, R is dense in M and M is homeomorphic to lim g . Next, recall that the only non-degenerate compact subcontinuum of a ray is an arc. So if H is a compact subcontinuum of lim g and H is contained in a ray in lim g , H is an arc. Since lim g is compact, each subcontinuum of lim g is compact and so if H is a non-degenerate subcontinuum of lim g and H is not an arc, H is not contained in a ray in lim g.

Note that T and g commute. If x is a point in $[1 - 2^{-n}, 1 - 2^{-n-1}]$, then $T(x)$ is in $[1 - 2^{-n-1}, 1 - 2^{-n-2}]$. On $[1 - 2^{-n}, 1 - 2^{-n-1}]$, g is $T^n f T^{-n}$ and on $[1 - 2^{-n-1}, 1 - 2^{-n-2}]$, g is $T^{n+1} f T^{-n-1}$. So $gT(x) = T^{n+1} f T^{-n-1} T(x) = T^{n+1} f T^{-n}(x) = Tg(x)$. Clearly the map h from lim g into lim g defined by $h(p) = (TP_1(p), TP_2(p), \ldots)$ is a homeomorphism which leaves $(1,1,1,\ldots)$ fixed. As in the proof of Theorem 5 of Chapter IV, the set R of all points of limit g with some coordinates less than 1/2 is a ray, and the rest is homeomorphic to lim g , as remarked before. Suppose H is a non-degenerate proper sub-continuum of lim g which is not an arc. Either H intersects R or H does not. If H intersects R , H is clearly homeomorphic to lim g since H is not contained in R and therefore H is only lim g less a half-open arc beginning the ray R , and a slight displacement along R gives H is homeomorphic to lim g . Suppose H does not intersect R . Then there is a first integer n so that $h^{-n}(H)$ intersects R . Since H is not an arc, $h^{-n}(H)$ is not contained in R and so $h^{-n}(H)$ is homeomorphic to lim g . Since h is a homeomorphism, H is homeomorphic to lim g .

The next example shows that things are not always what they might seem.

Example 5. If M is a non-degenerate compact continuous curve, there is a continuous function g from M onto M whose inverse limit is a chainable indecomposable continuum.

Since M is normal and connected there is a continuous function f from M onto $[0,1]$. Let a be any point so that $f(a) = 0$ and b any point so that $f(b) = 1$. By a standard theorem there is a continuous function T_2 from $[1/6, 2/6]$ onto M and a continuous function T_5 from $[4/6, 5/6]$ onto M. Now either $T_2(1/6)$ is a or there is an arc α_1 from a to $T_2(1/6)$ contained in M. In either case there is a continuous function T_1 from $[0, 1/6]$ into M so that $T_1(0) = a$ and $T_1(1/6) = T_2(1/6)$. Similarly there are continuous functions T_3, T_4 and T_6 from $[2/6, 3/6]$, $[3/6, 4/6]$, and $[5/6, 1]$, respectively, into M so that $T_3(2/6) = T_2(2/6)$, $T_3(3/6) = b$, $T_4(3/6) = b$, $T_4(4/6) = T_5(4/6)$, $T_6(5/6) = T_5(5/6)$ and $T_6(1) = a$. Define T to be the continuous function from $[0,1]$ onto M so that $T|[(n-1/6, (n/6)]$ is T_n for $n=1,2,3,4,5,6$. Now define g to be Tf. The function g is continuous from M onto M. By Corollary 9 of Chapter II, the inverse limit of g is homeomorphic to the inverse limit of fT. Now fT is continuous, $fT(0) = f(a) = 0$, $fT(1/2) = f(b) = 1$ and $fT(1) = f(a) = 0$ and the inverse limit of fT is a chainable indecomposable continuum by Theorem 2 of Chapter IV.

So although one gets chainable continua out of inverse limits on chainable compact continua, it is by no means necessary to take chainable continua to get a chainable inverse limit.

On the very unlikely chance that one now believes that inverse limits always are chainable, examples of non-chainable indecomposable inverse limit continua will now be given. It is, of course, necessary to know that an n - od is not chainable for any positive integer n greater than 2 . A continuum M will be said to be an n - od iff M contains a point p and a sequence of arcs $A_1, A_2, \ldots, A_n$ so that M is $\cup_{i=1}^{n} A_i$ and $A_i \cap A_j$ is p for i not j .

Example 6. For each integer n greater than 2 there is a continuous function from an n -od onto itself whose inverse limit is indecomposable and contains an n-od.

Suppose n is greater than 2 and M is an n-od. Denote the "arms" of M as $A_1, A_2, \ldots, A_n$ and the point at which these arcs meet denote as p . Suppose f is a continuous transformation from M onto M so that f leaves p fixed and for each integer m less than n , $f(A_m) = A_m \cup A_{m+1}$, and $f(A_n) = A_n \cup A_1$. Then $f^2(A_1) = A_1 \cup A_2 \cup A_3$, $f^3(A_1) = A_1 \cup A_2 \cup A_3 \cup A_4$, etc., and $f^n(A_1) = M$. Suppose M_1 and M_2 are proper subcontinua of lim f so that $M_1 \cup M_2$ = lim f . For each positive integer k , $P_k(M_1) \cup P_k(M_2)$ is M , since f is onto. It must be that for one of M_1 and M_2 , for each positive integer k , $P_k(M_i)$ contains one of the arcs A_j . But then for each positive integer k , $P_k(M_i)$ is $f^n P_{k+n}(M_i)$ which is M , and so M_i is lim f . So lim f is indecomposable.

Clearly f might have been chosen so that there are subarcs B_1, $B_2, \ldots, B_n$ of $A_1, A_2, \ldots, A_n$ respectively, each containing p so that f is 1-1 from each B_i onto A_i . It is not difficult to see that f restricted to the set $\cup_{i=1}^{n} B_i$ is an into homeomorphism, and that $\lim f$ contains a homeomorphic image of the n-od $\cup_{i=1}^{n} B_i$.

The following conjecture is offered principally because it was very difficult to find a counter-example. It is conjectured that any piece-wise linear map of the unit square onto itself has a decomposable inverse limit. An unsuccessful search for a nice map of the unit square onto itself with an indecomposable inverse limit led to Example 5. Another problem suggested by some of the preceding work is to find some topologically different indecomposable inverse limits of single functions from [0, 1] into [0, 1] .

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