Generalized minimum 0-extension problem and discrete convexity

Martin Dvorak Vladimir Kolmogorov
Institute of Science and Technology Austria
{martin.dvorak,vnk}@ist.ac.at

Abstract

Given a fixed finite metric space (V, μ), the *minimum* 0*-extension problem*, denoted as `0-Ext`$[\mu]$, is equivalent to the following optimization problem: minimize function of the form $\min_{x \in V^n} \sum_i f_i(x_i) + \sum_{ij} c_{ij}\mu(x_i, x_j)$ where c_{ij}, c_{vi} are given nonnegative costs and $f_i : V \to \mathbb{R}$ are functions given by $f_i(x_i) = \sum_{v \in V} c_{vi}\mu(x_i, v)$. The computational complexity of `0-Ext`$[\mu]$ has been recently established by Karzanov and by Hirai: if metric μ is *orientable modular* then `0-Ext`$[\mu]$ can be solved in polynomial time, otherwise `0-Ext`$[\mu]$ is NP-hard. To prove the tractability part, Hirai developed a theory of discrete convex functions on orientable modular graphs generalizing several known classes of functions in discrete convex analysis, such as $L^\natural$-convex functions.

We consider a more general version of the problem in which unary functions $f_i(x_i)$ can additionally have terms of the form $c_{uv;i}\mu(x_i, \{u, v\})$ for $\{u, v\} \in F$, where set $F \subseteq \binom{V}{2}$ is fixed. We extend the complexity classification above by providing an explicit condition on (μ, F) for the problem to be tractable. In order to prove the tractability part, we generalize Hirai's theory and define a larger class of discrete convex functions. It covers, in particular, another well-known class of functions, namely submodular functions on an integer lattice.

Finally, we improve the complexity of Hirai's algorithm for solving `0-Ext`$[\mu]$ on orientable modular graphs.

1 Introduction

Consider a metric space (V, μ) where V is a finite set and μ is a nonnegative function $V \times V \to \mathbb{R}$ satisfying the axioms of a metric: $\mu(x, y) = 0 \Leftrightarrow x = y$, $\mu(x, y) = \mu(y, x)$, $\mu(x, y) + \mu(y, z) \geq \mu(x, z)$ for all $x, y, z \in V$. We study optimization problems of the following form:

$$\min_{x \in V^n} f(x), \qquad f(x) = \sum_{i \in [n]} f_i(x_i) + \sum_{1 \leq i < j \leq n} c_{ij}\mu(x_i, x_j) \tag{1}$$

where weights c_{ij} are nonnegative. If unary terms $f_i : V \to \mathbb{R}$ are allowed to be arbitrary nonnegative functions then this is a well-studied *Metric Labeling Problem* [21]. Another important special case is when the unary terms are given by

$$f_i(x_i) = \sum_{v \in V} c_{vi}\mu(x_i, v) \tag{2}$$

with nonnegative weights c_{vi}. This is a classical facility location problem, known as *multifacility location problem* [32]. It can be interpreted as follows: we are going to locate n new facilities in V, where the facilities communicate with each other and communicate with existing facilities in V. The cost of the communication is propositional to the distance. The goal is to find a location of minimum total communication cost. The multifacility location problem is also equivalent to the *minimum* 0*-extension problem* formulated by Karzanov [19]. We denote `0-Ext`$[\mu]$ to be class of problems of the form (1),(2).

Optimization problems of the form above have applications in computer vision and related clustering problems in machine learning [21, 10, 4, 13]. $\texttt{0-Ext}[\mu]$ also includes a number of basic combinatorial optimization problems. For example, the multiway cut problem on k vertices can be obtained by setting (V, μ) to be the uniform metric on $|V| = k$ elements; it can be solved in polynomial time (via a maximum flow algorithm) if $k = 2$, and is NP-hard for $k \geq 3$.

We explore a generalization of $\texttt{0-Ext}[\mu]$ in which the unary terms are given by

$$f_i(x_i) = \sum_{v \in V} c_{vi}\mu(x_i, v) + \sum_{U \in F} c_{Ui}\mu(x_i, U) \tag{3}$$

Here F is a fixed set of subsets of V, c_{vi}, c_{Ui} are nonnegative weights, and $\mu(x_i, U) = \min_{v \in U} \mu(x_i, v)$. We refer to this generalization as $\texttt{0-Ext}[\mu, F]$. In the facility location interpretation, allowing terms of the form $c_{Ui}\mu(x_i, U)$ means that the i-th facility can be "served" by any of the facilities in U, and it can choose to communicate with the closest facility to minimize the communication cost.

Note that $\texttt{0-Ext}[\mu] = \texttt{0-Ext}[\mu, \varnothing]$. Furthermore, $\texttt{0-Ext}[\mu, 2^V]$, where $2^V = \{U \mid U \subseteq V\}$ is the set of all subsets of V, is the *restricted Metric Labeling Problem* [8], which is equivalent to the Metric Labeling Problem [6, Section 5.2].

1.1 Complexity classifications

The computational complexity of $\texttt{0-Ext}[\mu]$ has been established in [19, 14]. The tractability criterion is based on the properties of graph $H_\mu = (V, E, w)$ defined as the minimal undirected weighted graph whose path metric equals μ. Clearly, we have

$$E = \left\{ xy \in \binom{V}{2} \;\middle|\; \forall z \in V - \{x, y\} : \; \mu(x, y) < \mu(x, z) + \mu(z, y) \right\}$$

and w is the restriction of μ to E. For brevity, we usually denote the elements of $\binom{V}{2}$ as xy instead of $\{x, y\}$.

In order to state the classification of $\texttt{0-Ext}[\mu]$, we need to introduce a few definitions.

Orientable modular graphs Let us fix metric μ. For nodes $x, y \in V$ let $I(x, y) = I_\mu(x, y)$ be the *metric interval of* x, y, i.e. the set of points $z \in V$ satisfying $\mu(x, z) + \mu(z, y) = \mu(x, y)$. Metric μ is called *modular* if for every triplet $x, y, z \in V$ the intersection $I(x, y) \cap I(y, z) \cap I(x, z)$ is non-empty. (Points in this intersection are called *medians* of x, y, z.) We say that graph H is *modular* if $H = H_\mu$ for a modular metric μ.

Let o be an edge-orientation of Γ with the relation $\to_o$ on $V \times V$. This orientation is called *admissible for* Γ if, for every 4-cycle x_1, x_2, x_3, x_4, condition $x_1 \to_o x_2$ implies $x_4 \to_o x_3$. Γ is called *orientable* if it has an admissible orientation.

Theorem 1 ([19])**.** *If H_μ is not orientable or not modular then* $\texttt{0-Ext}[\mu]$ *is NP-hard.*

Theorem 2 ([14])**.** *If H_μ is orientable modular then* $\texttt{0-Ext}[\mu]$ *can be solved in polynomial time.* [1]

Our results We extend the classification above to problems $\texttt{0-Ext}[\mu, F]$ in which all subsets $U \in F$ have cardinality 2, i.e. $F \subseteq \binom{V}{2}$. To formulate the tractability criterion, we need to introduce some definitions. Let o be an orientation of (H, F), i.e. each edge of H is assigned an orientation, and each element of F is assigned an orientation. We say that o is *admissible for* (H, F) if it is admissible for H and, for every $\{x, y\} \in F$ with $x \to_o y$, the following holds: if P is a shortest x-y path in H then all edges of P are oriented according to o. We say that H is *F-orientable modular* if it is orientable modular and (H, F) admits an admissible orientation o. We can now formulate the main result of this paper.

[1] In this result μ is implicitly assumed to be rational-valued, since μ is treated as part of the input. The same remark applies to later results on VCSPs.

Theorem 3. *If H_μ is F-orientable modular then* $\texttt{0-Ext}[\mu, F]$ *can be solved in polynomial time. Otherwise* $\texttt{0-Ext}[\mu, F]$ *is NP-hard.*

To prove the tractability part, we define *L-convex functions on extended modular complexes*, and show that they can be minimized in polynomial time. This generalizes *L-convex functions on modular complexes* introduced by Hirai in [14, 15].

1.2 Discrete convex analysis

As Hirai remarks, the approach in [14, 15] had been inspired by discrete convex analysis developed in particular in [12, 26, 29, 31, 27] and [11, Chapter VII]. This is a theory of convex functions on integer lattice $\mathbb{Z}^n$, with the goal of providing a unified framework for polynomially solvable combinatorial optimization problems including network flows, matroids, and submodular functions. Hirai's work extends this theory to more general graph structures, in particular to orientable modular graphs, and provides a unified framework for polynomially solvable minimum 0-extension problems and related multiflow problems.

We develop a yet another generalization. To illustrate the relation to previous work, consider two fundamental classes of functions on the integer lattice $V = [k] = \{1, 2, \ldots, k\}$: *submodular functions* and *$L^\natural$-convex functions*. These are functions $f : [k]^n \to \overline{\mathbb{R}}$ satisfying conditions (4) and (5), respectively: [2]

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y) \qquad \forall x, y \in [k]^n \tag{4}$$

$$f(x) + f(y) \geq f(\lceil \tfrac{1}{2}(x+y) \rceil) + f(\lfloor \tfrac{1}{2}(x+y) \rfloor) \qquad \forall x, y \in [k]^n \tag{5}$$

where all operations are applied componentwise.

If, for example, $f(x) = \sum_i f_i(x_i) + \sum_{ij} f_{ij}(x_j - x_i)$, then f is submodular if all functions f_{ij} are convex, and f is $L^\natural$-convex if all functions f_i and f_{ij} are convex. The class of submodular functions on $[k]$ is strictly larger than the class of $L^\natural$-functions. However, $L^\natural$-functions possess additional properties that allow more efficient minimization algorithms, such as the Steepest Descent Algorithm [28, 22, 30].

The theory developed in [14, 15] covers $L^\natural$-convex functions and several other function classes, such as bisubmodular functions, *k-submodular functions* [17], *skew-bisubmodular functions* [18], and *strongly-tree submodular functions* [23]. However, it excludes submodular functions on $[k]$ for $k \geq 3$, which is a fundamental class of functions in discrete convex analysis. This paper fills this gap by introducing a unified framework that includes all classes of functions mentioned above.

Algorithms for solving $\texttt{0-Ext}[\mu]$ and $\texttt{0-Ext}[\mu, F]$ The tractability of $\texttt{0-Ext}[\mu]$ for orientable modular μ was proven in [15] as follows. Given an instance $f : V^n \to \overline{\mathbb{R}}$, Hirai defines a different instance $f^*_\times : (V^*)^n \to \overline{\mathbb{R}}$ with the same minimum, where $|V^*| = O(|V|^2)$. Function $f^*_\times$ is then minimized using the *Steepest Descent Algorithm* (SDA). This is an iterative technique that at each step computes a minimizer of $f^*_\times$ in a certain local neighborhood of the current iterate (by solving a linear programming relaxation). [15] shows that SDA terminates after at most $O(|V^*|)$ steps. We refer to this technique as the *SDA* approach.*

This approach does not seem to be applicable to our generalization $\texttt{0-Ext}[\mu, F]$. We present an alternative algorithm (for tractable classes of $\texttt{0-Ext}[\mu, F]$) that minimizes function f directly via a version of SDA that we call *zigzag SDA*. We prove that it terminates after $O(|V|^2)$ steps.

Comparison of SDA* and zigzag SDA approaches for tractable classes of $\texttt{0-Ext}[\mu]$ In both approaches the bound $O(|V|^2)$ on the number of steps can actually be refined, depending on the structure of graph H_μ and on the chosen orientation (see Sections 2 and 3). If this structure is

[2] We use a common notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ where ∞ is an infinity element treated as follows: $\infty \cdot 0 = 0$, $x < \infty$ ($x \in \mathbb{R}$), $\infty + x = \infty$ ($x \in \overline{\mathbb{R}}$), $x \cdot \infty = \infty$ ($a \in \mathbb{R} : a > 0$). We also denote $\mathbb{R}_+, \mathbb{Q}_+, \mathbb{Z}_+$ to be the sets of nonnegative elements of $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$, respectively.

taken into account, then the bounds for SDA* and zigzag SDA are not easy to compare. However, the size of local neighborhoods (and accordingly the size of the LP that needs to be solved at each step) is never bigger in the latter approach: for each local neighborhood in zigzag SDA there is a local neighborhood in SDA* of the same or bigger size. Furthermore, for certain μ's the size of this neighborhood can be $\Theta(|V^*|) = \Theta(|V|^2)$ in SDA*, but it is always $O(|V|)$ in zigzag SDA. Thus, zigzag SDA has better complexity compared to SDA*, if stated in terms of $|V|$ and n (considering that the complexity of Linear Programming is superlinear in the size of the LP).

Orthogonal generalizations of the minimum 0-extension problem In [15] Hirai defined *L-extendable functions on swm-graphs*, which generalizes L-convex functions on orientable modular graphs. Minimizing L-extendable functions on swm-graphs is an NP-hard problem (unless the graph is orientable modular); however, these functions admit a *discrete relaxation* on an orientable modular graph that can be minimized in polynomial time and yields a partial optimal solution for the original function. The theory of swm-graphs has been developed, in particular, by Chalopin, Chepoi, Hirai and Osajda [5]. In [16] Hirai and Mizutani considered minimum 0-extension problem for *directed* metrics, and provided some partial results (including a dichotomy for directed metrics of a star graph).

1.3 Valued Constraint Satisfaction Problems (VCSPs)

Results of this paper can be naturally stated in the framework of *Valued Constraint Satisfaction Problems* (VCSPs). This framework is defined below.

Let us fix a finite set D called a *domain.* A *cost function over D of arity n* is a function of the form $f : D^n \to \overline{\mathbb{R}}$. It is called finite-valued if $f(x) < \infty$ for all $x \in D^n$. We denote $\texttt{dom}\, f = \{x \in D^n \mid f(x) < \infty\}$. A *(VCSP) language over D* is a (possibly infinite) set Φ of cost functions over D. Language Φ is called finite-valued if all functions $f \in \Phi$ are finite-valued.

A *VCSP instance* $\mathcal{I}$ is a function $D^n \to \overline{\mathbb{R}}$ given by

$$f_{\mathcal{I}}(x) \;=\; \sum_{t\in T} f_t(x_{v(t,1)}, \ldots, x_{v(t,n_t)}) \tag{6}$$

It is specified by a finite set of variables $[n]$, finite set of terms T, cost functions $f_t : D^{n_t} \to \overline{\mathbb{R}}$ of arity n_t and indices $v(t,k) \in [n]$ for $t \in T, k = 1, \ldots, n_t$. A solution to $\mathcal{I}$ is a labeling $x \in [n]^V$ that minimizes $f_{\mathcal{I}}(x)$. Instance $\mathcal{I}$ is called a *Φ-instance* if all terms f_t belong to Φ. The set of all Φ-instances is denoted as VCSP(Φ). Language Φ with finite $|\Phi|$ is called *tractable* if instances in VCSP(Φ) can be solved in polynomial time, and *NP-hard* if VCSP(Φ) is NP-hard. If $|\Phi|$ is infinite then Φ is called tractable if every finite $\Phi' \subseteq \Phi$ is tractable, and NP-hard if there exists finite $\Phi' \subseteq \Phi$ which is NP-hard.

A key algorithmic tool in the VCSP theory is the *Basic Linear Programming* (BLP) relaxation of instance $\mathcal{I}$. We refer to [24] for the description of this relaxation. We say that BLP *solves instance* $\mathcal{I}$ if this relaxation is tight, i.e. its optimal value equals $\min_{x\in D^n} f_{\mathcal{I}}(x)$.

The following results are known; we refer to Section 6 for the definition of a "binary symmetric fractional polymorphism".

Theorem 4 ([24])**.** *Let Φ be a finite-valued language. Then BLP solves Φ if and only if Φ admits a binary symmetric fractional polymorphism. If the condition holds, then an optimal solution of an Φ-instance can be computed in polynomial time.*

Theorem 5 ([33])**.** *If a finite-valued language Φ does not satisfy the condition in Theorem 4 then Φ is NP-hard.*

Application to the minimum 0-extension problem Consider again a metric space (V, μ) and subset $F \subseteq \binom{V}{2}$. For a set $U \subseteq V$ let $\delta_U : V \to \{0, \infty\}$ be the indicator function of set U, with $\delta_U(v) = 0$ iff $v \in U$. For brevity, we write $\delta_{\{u_1,\ldots,u_k\}}$ as $\delta_{u_1\ldots u_k}$. Clearly, the minimum 0-extension problems introduced earlier can be equivalently defined by the following languages over domain $D = V$:

$$
\begin{aligned}
\texttt{0-Ext}[\mu] &= \{\mu\} \cup \{\delta_u : u \in V\} \\
\texttt{0-Ext}[\mu, F] &= \{\mu\} \cup \{\delta_u : u \in V\} \cup \{\delta_U : U \in F\}
\end{aligned}
$$

Note that the existence of the dichotomy given in Theorem 3 follows from Theorems 4 and 5 (but not the specific criterion for tractability).[3]

The rest of the paper is organized as follows. Section 2 reviews Hirai's theory and defines *L-convex functions on modular complexes* and the SDA algorithm for minimizing them. Section 3 generalizes this to *L-convex functions on extended modular complexes* and presents zigzag SDA algorithm. Both sections use the notion of *submodular functions on valuated modular semilattices*, which is formally defined in Section 4. All proofs missing in Section 3 are given in Sections 5 and 6.1; this completes the proof of the tractability direction of Theorem 3. The NP-hardness direction of Theorem 3 is proven in Section 6.2.

2 Background on orientable modular graphs

Notation for graphs If H is a simple undirected graph and o its edge orientation, then the pair (H, o) can be viewed as a simple directed graph. We will usually denote this graph as $\Gamma = (V_\Gamma, E_\Gamma, w_\Gamma)$. Γ is called *oriented modular*, or a *modular complex*, if it is admissible orientation of an orientable modular graph [14].

We let $\to_\Gamma$ be the edge relation of Γ, i.e. condition $u \to_\Gamma v$ means that there is an edge from u to v in Γ. When Γ is clear from the context, we may omit subscript Γ and write $V, E, w, \to$, etc. A *path* $(u_0, u_1, \ldots, u_k)$ in a directed graph Γ is defined as a path in the undirected version of Γ, i.e. for each i we must have either $u_i \to u_{i+1}$ or $u_{i+1} \to u_i$. An x-y path in Γ is a path from $x \in V$ to $y \in V$. With some abuse of notation we sometimes view Γ as a set of its nodes, and write e.g. $v \in \Gamma$ to mean $v \in V$.

Orbits For an undirected graph $H = (V, E, w)$ edges $e, e' \in E$ are called *projective* if there is a sequence of edges $(e_0, e_1, \ldots, e_m)$ with $(e_0, e_m) = (e, e')$ such that e_i, e_{i+1} are vertex-disjoint and belong to a common 4-cycle of H. Clearly, projectivity is an equivalence relation on E. An equivalence class of this relation in called an *orbit* [14]. Edge weights $w : E \to \mathbb{R}_{>0}$ are called *orbit-invariant* if $w(e) = w(e')$ for any pair of edges e, e' in the same orbit (equivalently, for vertex-disjoint edges e, e' belonging to a common 4-cycle).

Theorem 6 ([1, 20])**.** *Consider undirected graph $H = H_\mu = (V, E, w)$.*

(a) *If μ is modular then w is orbit-invariant, and path P is shortest in H if and only if it is shortest in $(V, E, 1)$.*

(b) *The following conditions are equivalent: (i) H is (orientable) modular; (ii) w is orbit-invariant and $(V, E, 1)$ is (orientable) modular.*

Metric spaces For a weighted directed or undirected graph $G = (V, E, w)$ let μ_G and d_G (or simply μ and d, when G is clear) be its path metrics w.r.t. edge lengths w and 1, respectively. If graph G is directed then edge orientations are again ignored.

For a metric space (V, μ), subset $U \subseteq V$ is called *convex* if $I(x, y) \subseteq U$ for every $x, y \in U$. Note, if G is an orientable (or oriented) modular graph then the definitions of the metric interval $I(x, y)$ and of convex sets coincide for metric spaces (V_G, μ_G) and (V_G, d_G) (by Theorem 6).

[3] Theorems 3 and 4 are not directly applicable to $\texttt{0-Ext}[\mu]$ since $\texttt{0-Ext}[\mu]$ is not finite-valued. However, we can get a finite-valued language by replacing functions $\delta_u : V \to \{0, \infty\}$ with functions $\mu_u : V \to \mathbb{R}$ defined via $\mu_u(x) = \mu(x, u)$. It is not difficult to show that such transformation does not affect the complexity of $\texttt{0-Ext}[\mu]$. A similar remark applies to $\texttt{0-Ext}[\mu, F]$.

Posets A modular complex Γ known to be an acyclic graph [14, Lemma 2.3], and thus induces a partial order $\preceq$ on V. Partially ordered sets (posets) play a key role in the study of oriented modular graphs. Below we describe basic facts about posets and terminology that we use, mainly following [14, 15].

Consider poset $\mathcal{L}$ with relation $\preceq$. For elements p, q with $p \preceq q$, the *interval* $[p, q]$ is the set $\{x \in \mathcal{L} \mid p \preceq x \preceq q\}$. A *chain from p to q of length k* is a sequence $u_0 \prec u_1 \prec \ldots \prec u_k$ with $(u_0, u_k) = (p, q)$, where notation $a \prec b$ means that $a \preceq b$ and $a \neq b$. The length $r[p, q]$ of interval $[p, q]$ is defined as the maximum length of a chain from p to q. If $\mathcal{L}$ has the lowest element (denoted as 0) then the *rank* $r(a)$ of element $a \in \mathcal{L}$ is defined by $r(a) = r[0, a]$, and elements of rank 1 are called *atoms*.

Element q *covers* p if $p \prec q$ and there is no element u with $p \prec u \prec q$. The *Hasse diagram of* $\mathcal{L}$ is a directed graph on $\mathcal{L}$ with the set of edges $\{p \rightarrow q \mid q \text{ covers } p\}$, and the *covering graph of* $\mathcal{L}$ is the corresponding undirected graph.

A pair $x, y \in \mathcal{L}$ is said to be *upper-bounded* (resp. *lower-bounded*) if x, y have a common upper bound (resp. common lower bound). The lowest common upper bound, if exists, is denoted by $x \vee y$ (the "join" of x, y), and the greatest common upper bound, if exists, is denoted by $x \wedge y$ (the "meet" of x, y). $\mathcal{L}$ is called a *(meet-)semilattice* if every pair $x, y \in \mathcal{L}$ has a meet, and it is called a *lattice* if every pair $x, y \in \mathcal{L}$ has both a join and a meet. If $\mathcal{L}$ is a semilattice and $x, y \in \mathcal{L}$ are upper-bounded then $x \vee y$ is known to exist.

A (positive) *valuation* of a semilattice $\mathcal{L}$ is a function $v : \mathcal{L} \rightarrow \mathbb{R}$ satisfying

$$v(q) - v(p) > 0 \qquad \forall p, q \in \mathcal{L} : \ p \prec q \tag{7a}$$

$$v(p) + v(q) = v(p \wedge q) + v(p \vee q) \qquad \forall p, q \in \mathcal{L} : \ p, q \text{ upper-bounded} \tag{7b}$$

In particular, if $\mathcal{L}$ is a lattice then (7b) should hold for all $p, q \in \mathcal{L}$. A semilattice with valuation v will be called a *valuated semilattice*. We will view the Hasse diagram of a valuated semilattice $\mathcal{L}$ (and the corresponding covering graph) as weighted graphs, where the weight of $p \rightarrow q$ is given by $v(q) - v(p) > 0$.

A lattice $\mathcal{L}$ is called *modular* if for every $x, y, z \in \mathcal{L}$ with $x \preceq z$ there holds $x \vee (y \wedge z) = (x \vee y) \wedge z$. A semilattice $\mathcal{L}$ is called *modular* [2] if for every $p \in \mathcal{L}$ poset $(\{x \in \mathcal{L} \mid x \preceq p\}, \preceq)$ is a modular lattice, and for every $x, y, z \in \mathcal{L}$ the join $x \vee y \vee z$ exists provided that $x \vee y$, $y \vee z$, $z \vee x$ exist. It is known that a lattice $\mathcal{L}$ is modular if and only if its rank function $r(\cdot)$ is a valuation [3, Chapter III, Corollary 1]. Furthermore, a (semi)lattice is modular if and only if its covering graph is modular, see [34, Proposition 6.2.1]; [2, Theorem 5.4]. The Hasse diagram of a (valuated) modular semilattice is known to be oriented modular [14, page 13].

Boolean pairs From now on we fix a modular complex $\Gamma = (V, E, w)$. Graph B is called a *cube graph* if it is isomorphic to the Hasse diagram of the Boolean lattice $\{0, 1\}^k$ for some $k \geq 0$. A pair of vertices (p, q) of Γ is called a *Boolean pair* if Γ contains cube graph B as a subgraph so that p and q are respectively the source and the sink of B. Let $\sqsubseteq$ be the following relation on V: $p \sqsubseteq q$ iff (p, q) is a Boolean pair in Γ. We have $p \sqsubseteq p$ for any $p \in V$, and condition $p \sqsubseteq q$ implies that $p \preceq q$. Graph Γ is called *well-oriented* if the opposite implication holds, i.e. if relations $\preceq$ and $\sqsubseteq$ are the same. Note that relation $\sqsubseteq$ is not necessarily transitive. We write $p \sqsubset q$ to mean $p \sqsubseteq q$ and $p \neq q$. Let $\Gamma^{\sqsubset}$ be the graph with nodes V and edges $\{p \rightarrow q \mid p, q \in V, p \sqsubset q\}$. Clearly, Γ is a subgraph of $\Gamma^{\sqsubset}$ (ignoring edge weights).

For a vertex $p \in V$ define the following subsets of V:

$$\mathcal{L}_p^{\uparrow}(\Gamma) = \{q \in V \mid p \preceq q\} \qquad \mathcal{L}_p^{\downarrow}(\Gamma) = \{q \in V \mid q \preceq p\} \tag{8a}$$

$$\mathcal{L}_p^{+}(\Gamma) = \{q \in V \mid p \sqsubseteq q\} \qquad \mathcal{L}_p^{-}(\Gamma) = \{q \in V \mid q \sqsubseteq p\} \qquad \mathcal{L}_p^{\pm}(\Gamma) = \mathcal{L}_p^{+}(\Gamma) \cup \mathcal{L}_p^{-}(\Gamma) \tag{8b}$$

When Γ is clear from the context, we will omit it for brevity (and if Γ is not clear, we may write $\preceq_\Gamma$ and $\sqsubseteq_\Gamma$ instead of $\preceq$ and $\sqsubseteq$). We view $\mathcal{L}_p^{\uparrow}, \mathcal{L}_p^{+}$ as posets with relation $\preceq$, and $\mathcal{L}_p^{\downarrow}, \mathcal{L}_p^{-}$ as posets with the reverse of relation $\preceq$. Note, if Γ is well-oriented then $\mathcal{L}_p^+ = \mathcal{L}_p^\uparrow$ and $\mathcal{L}_p^- = \mathcal{L}_p^\downarrow$ for every node p of Γ.

Lemma 7 ([14, Proposition 4.1, Theorem 4.2, Lemma 4.14])**.** *Let Γ be a modular complex.*

(a) *If elements a, b are upper-bounded then $a \vee b$ exists. Similarly, if a, b are lower-bounded then $a \wedge b$ exists.*

(b) *Consider elements p, q with $p \preceq q$. Then $\mathcal{L}_p^\uparrow$, $\mathcal{L}_p^\downarrow$, $\mathcal{L}_p^+$, $\mathcal{L}_p^-$ are modular semilattices, and $[p, q] = \mathcal{L}_p^\uparrow \cap \mathcal{L}_p^\downarrow$ is a modular lattice. Furthermore, these (semi)lattices are convex in Γ, and function $v(\cdot)$ defined via $v(a) = \mu_\Gamma(p, a)$ is a valid valuation of these (semi)lattices.*

We will always view $\mathcal{L}_p^\uparrow$, $\mathcal{L}_p^\downarrow$, $\mathcal{L}_p^+$, $\mathcal{L}_p^-$ as **valuated** semilattices, where the valuation is defined as in the lemma.

L-convex functions Next, we review the notion of *an L-convex function on a modular complex* Γ introduced by Hirai in [14, 15]. The definition involves the following steps.

- First, Hirai defines the notion of a *submodular function on a valuated modular semilattice* $\mathcal{L}$. These are functions $f : \mathcal{L} \to \overline{\mathbb{R}}$ satisfying certain linear inequalities. Formulating these inequalities is rather lengthy, and we defer it to Section 4.

- Second, Hirai defines *2-subdivision* of Γ as the directed weighted graph $\Gamma^* = (V^*, E^*, w^*)$ constructed as follows:

  (i) set $V^* = \{[p, q] \mid p \sqsubseteq q\}$;

  (ii) for each $[p, q], [p, q'] \in V^*$ with $q \to_\Gamma q'$ add edge $[p, q] \to [p, q']$ to Γ^* with weight $w(qq')$;

  (iii) for each $[p', q], [p, q] \in V^*$ with $p' \to_\Gamma p$ add edge $[p, q] \to [p', q]$ to Γ^* with weight $w(p'p)$.

  It can be seen that graph (V^*, E^*) is the Hasse diagram of the poset $(V^*, \subseteq)$ (which is the definition used in [15]). [4]

  Hirai proves that graph Γ^* is oriented modular ([14, Theorem 4.3]) and well-oriented ([14, Lemma 2.14]). Consequently, poset $\mathcal{L}_{[p,p]}^+(\Gamma^*) = \mathcal{L}_{[p,p]}^\uparrow(\Gamma^*)$ is a valuated modular semilattice for each $p \in V$. For brevity, this poset will be denoted as $\mathcal{L}_p^*(\Gamma)$, or simply as $\mathcal{L}_p^*$.

- Each function $f : V \to \overline{\mathbb{R}}$ is extended to a function $f^* : V^* \to \overline{\mathbb{R}}$ via $f^*([p, q]) = f(p) + f(q)$.

- Function $f : V \to \overline{\mathbb{R}}$ is now called *L-convex on* Γ if (i) subset $\texttt{dom}\, f \subseteq V$ is connected in $\Gamma^\sqsubset$, and (ii) for every $p \in V$, the restriction of f^* to $\mathcal{L}_p^*$ is submodular on valuated modular semilattice $\mathcal{L}_p^*$.

Minimum 0-extension problem and L-convex functions Next, we describe the relation between problem $\texttt{0-Ext}[\mu]$ and L-convex functions on Γ (where μ is an orientable modular metric and Γ is an admissible orientation of H_μ).

If Γ, Γ' are two modular complexes, then their Cartesian product $\Gamma \times \Gamma'$ is defined as the directed graph with the vertex set $V_\Gamma \times V_{\Gamma'}$ and the following edge set: there is an edge $(p, p') \to (q, q')$ iff either (i) $p = q$ and $p' \to_{\Gamma'} q'$, or (ii) $p' = q'$ and $p \to_\Gamma q$. The weight of the edge is $w_{\Gamma'}(p'q')$ in the first case and $w_\Gamma(pq)$ in the second case. The n-fold Cartesian product $\Gamma \times \ldots \times \Gamma$ is denoted as Γ^n.

A Cartesian product $\mathcal{L} \times \mathcal{L}'$ of two posets $\mathcal{L}, \mathcal{L}'$ is defined in a natural way (with $(p, p') \preceq (q, q')$ iff $p \preceq p$ and $q \preceq q'$). It is straightforward to verify from definitions that if $\mathcal{L}, \mathcal{L}'$ are modular semilattices then so is $\mathcal{L} \times \mathcal{L}'$. If $\mathcal{L}, \mathcal{L}'$ are valuated modular semilattices with valuations $v_\mathcal{L}, v_{\mathcal{L}'}$ then $\mathcal{L} \times \mathcal{L}'$ is also assumed to be valuated with the valuation $v_{\mathcal{L}\times\mathcal{L}'}(p, p') = v_\mathcal{L}(p) + v_{\mathcal{L}'}(p')$.

[4] Compared to [14, 15], we chose to scale the weights of graph Γ^* by a factor of 2. Such scaling will not affect later theorems.

Lemma 8 ([14, Lemma 4.7]). *Consider modular complexes Γ, Γ' and element $(p, p') \in \Gamma \times \Gamma'$.*

(a) *Graph $\Gamma \times \Gamma'$ is a modular complex (i.e. oriented modular).*

(b) $\mathcal{L}^{\sigma}_{(p,p')}(\Gamma \times \Gamma') = \mathcal{L}^{\sigma}_{p}(\Gamma) \times \mathcal{L}^{\sigma}_{p'}(\Gamma')$ *for $\sigma \in \{-, +\}$.*

Theorem 9 ([14, Theorem 4.8 and Lemma 4.9];[15, Lemmas 4.1 and 4.2]). *Consider modular complexes Γ, Γ' and functions $f, f' : \Gamma \to \overline{\mathbb{R}}$ and $\tilde{f} : \Gamma \times \Gamma' \to \overline{\mathbb{R}}$.*

(a) *If f, f' are L-convex on Γ then $f + f'$ and $c \cdot f$ for $c \in \mathbb{R}_+$ are also L-convex on Γ.*

(b) *If f is L-convex on Γ and $\tilde{f}(p, p') = f(p)$ for $(p, p') \in \Gamma \times \Gamma'$ then $\tilde{f}$ is L-convex on $\Gamma \times \Gamma'$.*

(c) *If $\tilde{f}$ is L-convex on $\Gamma \times \Gamma'$ and $f(p) = \tilde{f}(p, p')$ for fixed $p' \in \Gamma'$ then f is L-convex on Γ.*

(d) *The indicator function $\delta_U : V \to \{0, \infty\}$ of a d_Γ-convex set U is L-convex on Γ.*

(e) *Function $\mu_\Gamma : \Gamma \times \Gamma \to \mathbb{R}_+$ is L-convex on $\Gamma \times \Gamma$.*

It follows that an instance of 0-Ext$[\mu]$ can be reduced to the problem of minimizing function $f : \Gamma^n \to \overline{\mathbb{R}}$, $n \geq 1$ such that (i) f is L-convex on Γ^n, and (ii) $f(x_1, \ldots, x_n)$ can be written as a sum of unary and pairwise terms that are L-convex on Γ and $\Gamma \times \Gamma$, respectively. This minimization problem is considered next.

Minimizing L-convex functions A key property of L-convex functions is that local optimality implies global optimality.

Theorem 10 ([14, Lemma 2.3]). *Let $f : V \to \overline{\mathbb{R}}$ be an L-convex function on a modular complex Γ. If p is a local minimizer of f on $\mathcal{L}^{\pm}_{p}(\Gamma)$, i.e. $f(p) = \min_{q \in \mathcal{L}^{\pm}_{p}(\Gamma)} f(q)$, then it is also a global minimizer of f, i.e. $f(p) = \min_{q \in V} f(q)$.*

This theorem implies that the *Steepest Descent Algorithm* (SDA) given below is guaranteed to produce a global minimizer of f after a finite number of steps.

Algorithm 1: Steepest Descent for minimizing $f : \Gamma^n \to \overline{\mathbb{R}}$

```
1 pick arbitrary x ∈ dom f
2 while true do
3     compute y⁺ ∈ arg min {f(y) | y ∈ L⁺_x(Γⁿ)} and y⁻ ∈ arg min {f(y) | y ∈ L⁻_x(Γⁿ)}
4     pick y ∈ arg min {f(y) | y ∈ {y⁻, y⁺}}
5     if f(y) = f(x) then return x, otherwise update x := y
```

By Lemma 8, for $x = (x_1, \ldots, x_n)$ we have $\mathcal{L}^{\sigma}_{x}(\Gamma^n) = \mathcal{L}^{\sigma}_{x_1}(\Gamma) \times \ldots \times \mathcal{L}^{\sigma}_{x_n}(\Gamma)$ for $\sigma \in \{-, +\}$. The result below thus implies that the two minimization problems in line 3 can be solved in polynomial time assuming that f is an instance of 0-Ext$[\mu]$.

Theorem 11 ([14, Theorem 3.9]). *Consider VCSP instance $f : \mathcal{L}_1 \times \ldots \times \mathcal{L}_n \to \overline{\mathbb{R}}$. Suppose that $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are valuated modular semilattices and function f is submodular on $\mathcal{L} = \mathcal{L}_1 \times \ldots \times \mathcal{L}_n$. Then BLP relaxation solves f.*

To show that 0-Ext$[\mu]$ can be solved in polynomial time, a few additional definitions and results are needed. Elements p, q of a modular complex Γ are said to be $\diamond$-*neighbors* if $p, q \in [a, b]$ for some a, b with $a \sqsubseteq b$. Equivalently, p, q are $\diamond$-neighbors if $p \wedge q, p \vee q$ exist and $p \wedge q \sqsubseteq p \vee q$. Let $\Gamma^{\diamond}$ be an undirected unweighted graph on nodes V_Γ such that p, q are connected by an edge in $\Gamma^{\diamond}$ if and only if p, q are $\diamond$-neighbors. This graph is called a *thickening of* Γ. The distance from p to q in $\Gamma^{\diamond}$ will be denoted as $d^{\diamond}_{\Gamma}(p, q)$, or simply as $d^{\diamond}(p, q)$. These distances will give a bound on the number of steps of SDA.

Theorem 12 ([15, Theorem 4.7,Lemma 2.18]). *Suppose that modular complex Γ is well-oriented, and function $f : \Gamma^n \to \overline{\mathbb{R}}$ is L-convex on Γ^n. SDA terminates after at most $2 + \max_{i \in [n]} d^{\diamond}_{\Gamma}(x_i, \mathtt{opt}_i(f))$ iterations where x is the initial vertex and $\mathtt{opt}_i(f) = \{x^*_i \mid x^* \in \arg\min f\} \subseteq V_\Gamma$ is the set of minimizers of f projected to the i-th component.*

Theorem 13 ([5, Proposition 6.10],[15, Lemma 2.14])**.** *If Γ be a modular complex then graph Γ^* is well-oriented.*

Theorem 14 ([15, Proposition 4.9],[14, Lemma 4.7])**.** *If Γ is a modular complex and function $f : \Gamma^n \to \overline{\mathbb{R}}$ is L-convex on Γ^n then function $f^*_\times : (\Gamma^*)^n \to \overline{\mathbb{R}}$ defined via $f^*_\times([x_1, y_1], \ldots, [x_n, y_n]) = f(x) + f(y)$ is L-convex on $(\Gamma^*)^n$.*

From the results above one can now conclude that any instance $f : \Gamma^n \to \overline{\mathbb{R}}$ of $\texttt{0-Ext}[\mu]$ for an oriented modular metric μ can be solved in polynomial time. Indeed, apply SDA to minimize function $f^*_\times : (\Gamma^*)^n \to \overline{\mathbb{R}}$. It will produce a minimizer $([x_1, y_1], \ldots, [x_n, y_n])$ after at most $\texttt{diam}((\Gamma^*)^\diamond) + 2 \le O((\texttt{diam}(\Gamma))^2)$ iterations. Both x and y are minimizers of f.

Note that in the earlier paper [14] Hirai proved that $\texttt{0-Ext}[\mu]$ can be solved in weakly polynomial time via a different algorithm, namely the Steepest Descent Algorithm (applied to $f : \Gamma^n \to \overline{\mathbb{R}}$) but combined with a cost-scaling approach.

Remark 1. *Our terminology is slightly different from that of [15]. In particular, $\diamond$-neighbors were called* Δ-neighbors *in [15], and a path in $\Gamma^\sqsubset$ was called a* Δ'-path.

3 Extended modular complexes

To prove our main tractability result from Theorem 3, we will introduce the following definition.

Definition 15. *Let Γ be a modular complex. Binary relation $\sqsubseteq$ on $V = V_\Gamma$ is called* admissible *if it coarsens $\preceq$ (i.e. $p \sqsubseteq q$ implies $p \preceq q$), $p \sqsubseteq p$ for every $p \in V$, $p \sqsubseteq q$ for every edge $p \to q$, and the following conditions hold.*

(15a) *Suppose that $p \sqsubseteq q$, $a \preceq b$ and $a, b \in [p, q]$. Then $a \sqsubseteq b$.*
(15b) *Suppose that $p \sqsubseteq q_1$, $p \sqsubseteq q_2$ and $q_1 \vee q_2$ exists. Then $p \sqsubseteq q_1 \vee q_2$.*
(15c) *Suppose that $p_1 \sqsubseteq q$, $p_2 \sqsubseteq q$ and $p_1 \wedge p_2$ exists. Then $p_1 \wedge p_2 \sqsubseteq q$.*

A pair $(\Gamma, \sqsubseteq)$ where Γ is a modular complex and $\sqsubseteq$ is admissible will be called an *extended modular complex*. With some abuse of notation, we will use letter Γ for an extended modular complex, and treat it as an oriented modular graph when needed. Note that previously we used notation $\sqsubseteq$ for Boolean pairs in Γ. From now on we will write $p \overset{\texttt{Bp}}{\sqsubseteq} q$ if (p, q) is a Boolean pair in Γ, and reserve notation $\sqsubseteq$ (or $\sqsubseteq_\Gamma$) for the binary relation that comes with an extended modular complex Γ. As before, we write $p \sqsubset q$ to mean $p \sqsubseteq q$ and $p \neq q$.

Proposition 16. *Let Γ be a modular complex. (a) Relations $\overset{\texttt{Bp}}{\sqsubseteq}$ and $\preceq$ are admissible for Γ. (b) If relation $\sqsubseteq$ is admissible for Γ then $p \overset{\texttt{Bp}}{\sqsubseteq} q$ implies $p \sqsubseteq q$.*

This proposition shows that a modular complex is a special case of an extended modular complex (obtained by setting $\sqsubseteq$ to be $\overset{\texttt{Bp}}{\sqsubseteq}$). Also, $\overset{\texttt{Bp}}{\sqsubseteq}$ and $\preceq$ are respectively the coarsest and the finest admissible relations. Next, we will show that many of the results in Section 2 still hold if relation $\overset{\texttt{Bp}}{\sqsubseteq}$ is replaced with an arbitrary admissible relation $\sqsubseteq$.

From now on we fix an extended modular complex Γ. We define graphs $\Gamma^\sqsubset$ and $\Gamma^\diamond$ as in Section 2, and for $p \in V$ define posets $(\mathcal{L}^+_p, \mathcal{L}^-_p, \mathcal{L}^\pm_p) = (\mathcal{L}^+_p(\Gamma), \mathcal{L}^-_p(\Gamma), \mathcal{L}^\pm_p(\Gamma))$ as in eq. (8). Note, in all cases $\sqsubseteq$ now has the new meaning (it is the relation that comes with Γ).

Lemma 17. *Let Γ be an extended modular complex, and p be its element. Then $\mathcal{L}^+_p, \mathcal{L}^-_p$ are modular semilattices that are convex in Γ. Furthermore, function $v(\cdot)$ defined via $v(a) = \mu_\Gamma(p, a)$ is a valid valuation of these (semi)lattices.*

We define a *2-subdivision of* Γ exactly as in Section 2; it is the graph $\Gamma^* = (V^*, E^*, w^*)$ where $V^* = \{[p,q] : p, q \in \Gamma, p \sqsubseteq_\Gamma q\}$. Using condition (15a), it is straightforward to verify that (V^*, E^*) is the Hasse diagram of the poset $(V^*, \subseteq)$. We will show the following result.

Theorem 18. *If* Γ *is an extended modular complex then* (V^*, E^*, w^*) *is a modular complex (i.e. an oriented modular graph).*

Recall that if Γ is a modular complex (equivalently, an extended modular complex with the relation $\sqsubseteq_\Gamma = \overset{\mathtt{Bp}}{\sqsubseteq}_\Gamma$) then Γ^* is well-oriented, and thus $\mathcal{L}^+_{[p,p]}(\Gamma^*) = \mathcal{L}^\uparrow_{[p,p]}(\Gamma^*)$ for every $p \in \Gamma$. For extended modular complexes this is not necessarily the case; we may have $\mathcal{L}^+_{[p,p]}(\Gamma^*) \neq \mathcal{L}^\uparrow_{[p,p]}(\Gamma^*)$. Let us set $\mathcal{L}^*_p = \mathcal{L}^*_p(\Gamma) = \mathcal{L}^\uparrow_{[p,p]}(\Gamma^*)$. We can now define L-convex functions exactly as in the previous section:

- Function $f : V \to \overline{\mathbb{R}}$ is called *L-convex on extended modular complex* Γ if (i) subset $\mathtt{dom}\, f \subseteq V$ is connected in $\Gamma^\sqsubset$, and (ii) for every $p \in V$, the restriction of f^* to $\mathcal{L}^*_p = \mathcal{L}^\uparrow_{[p,p]}(\Gamma^*)$ is submodular on valuated modular semilattice $\mathcal{L}^*_p$.

Remark 2. *One possibility could be to define 2-subdivision* Γ^* *as the extended modular complex* $((V^*, E^*, w^*), \preceq)$*; then by construction we would have* $\mathcal{L}^+_{[p,p]}(\Gamma^*) = \mathcal{L}^\uparrow_{[p,p]}(\Gamma^*)$*, as in the non-extended case. However, under such definition the analogue of Theorem 14 is no longer true: if* f *is* L*-convex on* Γ *then* f^* *may not be* L*-convex on* $((V^*, E^*, w^*), \preceq)$*.* [5] *For this reason we avoided such definition.*

Cartesian products If $(\Gamma, \sqsubseteq), (\Gamma', \sqsubseteq')$ are two extended modular complexes, then their Cartesian product $(\Gamma, \sqsubseteq) \times (\Gamma', \sqsubseteq')$ is defined as the pair $(\Gamma \times \Gamma', \sqsubseteq_\times)$ where $(p, p') \sqsubseteq_\times (q, q')$ iff $p \sqsubseteq q$ and $p' \sqsubseteq' q'$.

Lemma 19. *Consider extended modular complexes* Γ, Γ' *and element* $(p, p') \in \Gamma \times \Gamma'$.
(a) $\Gamma \times \Gamma'$ *is an extended modular complex.*
(b) $\mathcal{L}^\sigma_{(p,p')}(\Gamma \times \Gamma') = \mathcal{L}^\sigma_p(\Gamma) \times \mathcal{L}^\sigma_{p'}(\Gamma')$ *for* $\sigma \in \{-, +\}$.

Theorem 20. *Consider extended modular complexes* Γ, Γ' *and functions* $f, f' : \Gamma \to \overline{\mathbb{R}}$ *and* $\tilde{f} : \Gamma \times \Gamma' \to \overline{\mathbb{R}}$.
(a) *If* f, f' *are* L*-convex on* Γ *then* $f + f'$ *and* $c \cdot f$ *for* $c \in \mathbb{R}_+$ *are also* L*-convex on* Γ.
(b) *If* f *is* L*-convex on* Γ *and* $\tilde{f}(p, p') = f(p)$ *for* $(p, p') \in \Gamma \times \Gamma'$ *then* $\tilde{f}$ *is* L*-convex on* $\Gamma \times \Gamma'$.
(c) *If* $\tilde{f}$ *is* L*-convex on* $\Gamma \times \Gamma'$ *and* $f(p) = \tilde{f}(p, p')$ *for fixed* $p' \in \Gamma'$ *then* f *is* L*-convex on* Γ.
(d) *The indicator function* $\delta_U : V \to \{0, \infty\}$ *is* L*-convex on* Γ *in the following cases: (i)* U *is a* d_Γ*-convex set; (ii)* $U = \{p, q\}$ *for elements* p, q *with* $p \sqsubseteq q$.
(e) *Function* $\mu_\Gamma : \Gamma \times \Gamma \to \mathbb{R}_+$ *is* L*-convex on* $\Gamma \times \Gamma$.

Theorem 21. *Let* $f : V \to \overline{\mathbb{R}}$ *be an* L*-convex function on an extended modular complex* Γ*. If* p *is a local minimizer of* f *on* $\mathcal{L}^\pm_p(\Gamma)$*, i.e.* $f(p) = \min_{q \in \mathcal{L}^\pm_p(\Gamma)} f(q)$*, then it is also a global minimizer of* f*, i.e.* $f(p) = \min_{q \in V} f(q)$.

For extended modular complex Γ let Φ_Γ be the language over domain $D = V_\Gamma$ that contains all functions $f : D^n \to \overline{\mathbb{R}}$ such that f is L-convex on Γ^n. The results above imply that any instance $\mathcal{I}$ of Φ_Γ can be solved by the Steepest Descent Algorithm (Algorithm 1), and each subproblem in line 2 can be solved in polynomial time. However, we do not know whether the number of steps would be polynomially bounded. To get a polynomial bound, we introduce an alternative algorithm, which we call *zigzag SDA*.

[5] Let Γ to be the chain $a \prec b \prec c \prec d$ with the relation $\sqsubseteq_\Gamma = \preceq$. Then any function $f : \Gamma \to \overline{\mathbb{R}}$ is L-convex on Γ, but f^* is not L-convex on $((V^*, E^*, w^*), \preceq)$ for some f.

Algorithm 2: Zigzag SDA for minimizing $f : \Gamma^n \to \overline{\mathbb{R}}$

pick arbitrary $x \in \texttt{dom}\, f$ and $\sigma \in \{-, +\}$
while `true` **do**
  compute $y \in \arg\min \{f(y) \mid y \in \mathcal{L}_x^{\sigma}(\Gamma^n)\}$
  if $f(y) < f(x)$ **then**
    update $x := y$
  else
    update $\sigma := -\sigma$ (i.e. change $+$ to $-$ and $-$ to $+$)
    if $f(x) = \min \{f(y) \mid y \in \mathcal{L}_x^{\sigma}(\Gamma^n)\}$ then return x

In order to analyze Algorithm 2, for an extended modular complex $\Gamma = ((V, E, w), \sqsubseteq)$ we define extended modular complex $\dot{\Gamma} = ((V, e, w), \preceq)$ obtained from Γ by replacing relation $\sqsubseteq$ with $\preceq$. We say that elements p, q are $\dot{\diamond}$*-neighbors* if they are $\diamond$-neighbors in $\dot{\Gamma}$. We let $\Gamma^{\dot{\diamond}}$ be the thickening of $\dot{\Gamma}$, and $d^{\dot{\diamond}}$ be the distance function in $\Gamma^{\dot{\diamond}}$. We also define $\texttt{height}^{\sqsubset}(\Gamma) = \max\{d^{\diamond}(p, q) \mid p \preceq q\}$.

Theorem 22. *Let Γ be an extended modular complex and $f : \Gamma^n \to \overline{\mathbb{R}}$ be an L-convex function on Γ^n. Zigzag SDA algorithm applied to function f terminates after generating at most $(\max_{i\in[n]} d_{\Gamma}^{\dot{\diamond}}(x, \texttt{opt}_i(f)) + 1) \cdot \texttt{height}^{\sqsubset}(\Gamma) + 1$ distinct points, where x is the initial vertex and $\texttt{opt}_i(f)$ is as defined in Theorem 12.*

Remark 3. *It can be seen that if $\sqsubset_{\Gamma} = \preceq_{\Gamma}$ then SDA and zigzag SDA are equivalent (except possibly the very first update of x); both algorithms will update $x \leftarrow \arg\min\{f(y) \mid y \in \mathcal{L}_x^{+}\}$ at even steps and $x \leftarrow \arg\min\{f(y) \mid y \in \mathcal{L}_x^{-}\}$ at odd steps, or vice versa. Furthermore, in this case we have $\texttt{height}^{\sqsubset}(\Gamma) = 1$. Thus, Theorem 22 generalizes Theorem 12 in two ways: from modular complexes to extended modular complexes, and by allowing relations $\sqsubseteq_{\Gamma}$ and $\preceq_{\Gamma}$ to be distinct.*

Note that the algorithm for $\texttt{0-Ext}[\mu]$ described in the previous section required applying SDA on 2-subdivision Γ^*. This blows up the size of the graph and the size of LPs that need to be solved at each iteration by an up to a quadratic factor. Zigzag SDA provides an alternative that avoids such blow-up.

We can now show the tractability part of Theorem 3. Suppose that graph H_μ is F-orientable modular for a metric space (V, μ) and subset $F \subseteq \binom{V}{2}$. Choose an admissible orientation of (H_μ, F), and let Γ be the corresponding extended modular complex with the relation $\sqsubseteq \, = \, \preceq$. Clearly, for any $\{x, y\} \in F$ we have either $x \preceq y$ or $y \preceq x$. From Theorem 20 we conclude that $\texttt{0-Ext}[\mu, F] \subseteq \Phi_\Gamma$, and so $\texttt{0-Ext}[\mu, F]$ can be solved in polynomial time by the zigzag SDA algorithm.

More generally, this shows tractability of VCSP(Φ_Γ) for an extended modular complex Γ assuming that a feasible solution of any Φ_Γ-instance can be computed in polynomial time. Our last result shows that VCSP(Φ_Γ) is tractable even without this assumption.

Theorem 23. *If Γ is an extended modular complex then BLP relaxation solves Φ_Γ, and an optimal solution of any Φ_Γ-instance can be computed in polynomial time.*

All proofs are given in Sections 5 and 6. For Lemmas 17, 19 and Theorems 18, 20, 21 we mostly follow the proofs of the corresponding claims in [14, 5, 15] (replacing properties of relation $\overset{\text{Bp}}{\sqsubseteq}$ with the properties of an admissible relation $\sqsubseteq$). The analysis of the zigzag SDA (Theorem 22) uses ideas from [15] combined with new arguments. Theorem 23 does not have an analogue in [14, 5, 15].

4 Submodular functions on a valuated modular semilattice

Let $\mathcal{L}$ be a valuated modular semilattice with valuation v. This section gives the definition of a *submodular function on* $\mathcal{L}$, and thus completes the definition of L-convex functions.

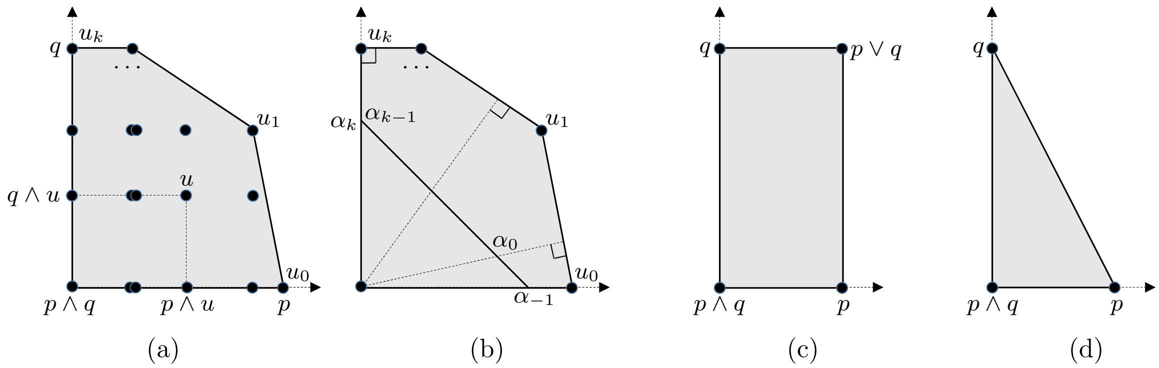

Figure 1: (a) Each point in $I(p,q)$ is assigned a coordinate in $\mathbb{R}^2$. The convex hull of these coordinates ($\operatorname{Conv} I(p,q)$) is in gray. Distinct points may have the same coordinates (as some of the points shown in the interior of the gray region), but the coordinates of points in $\mathcal{E}(p,q) = \{u_0, u_1, \ldots, u_k\}$ are guaranteed to be unique. (b) Definition of $\{\theta_i\}_i$ and $p \vee_\theta q$. First, define points $\alpha_{-1}, \alpha_0, \ldots, \alpha_k$ in $\mathbb{R}^2$ as follows: set $\alpha_{-1} = (\sqrt{2}/2, 0)$, $\alpha_k = (0, \sqrt{2}/2)$ (so that $||\alpha_k - \alpha_{-1}|| = 1$), and for $i \in [k-1]$ let α_i be the intersection of segment $[\alpha_{-1}, \alpha_k]$ and the line that goes through the origin and is perpendicular to the line passing through points $v_{pq}(u_k)$ and $v_{pq}(u_{k+1})$. Then $\theta_i = ||\alpha_i - \alpha_{-1}||$ and $p \wedge_\theta q = u_i$ for each $i \in [0,k]$ and $\theta \in (\theta_{i-1}, \theta_i)$. (c) Bounded pair (p,q). (d) Antipodal pair (p,q).

Let $\Gamma = (V, E, w)$ be the Hasse diagram of $\mathcal{L}$ where edge $p \to q$ is assigned weight $v(q) - v(p) > 0$. As discussed in Section 2, graph Γ is oriented modular. Denote $\mu = \mu_\Gamma$ and $d = d_\Gamma$. Recall that $\mathcal{L}$ is viewed as a metric space with the metric μ, and the definitions of the metric interval $I(p,q)$, convex sets, etc would be the same for $(\mathcal{L}, \mu)$ and $(\mathcal{L}, d)$.

For $p \preceq q$ let us denote $v[p,q] = v(q) - v(p)$. The following facts are known.

Lemma 24 ([14, Lemma 2.15])**.** *The following holds for each $p, q \in \mathcal{L}$ with $s = p \wedge q$.*
(a) $\mu(p,q) = \mu(s,p) + \mu(s,q) = v[s,p] + v[s,q]$.
(b) *The metric interval $I(p,q)$ is equal to the set of elements u that is represented as $u = a \vee b$ for some $(a,b) \in [s,p] \times [s,q]$, where such a representation is unique, and (a,b) equals $(u \wedge p, u \wedge q)$.*
(c) *For $u, u' \in I(p,q)$ there holds $u \wedge u' = (u \wedge u' \wedge p) \vee (u \wedge u' \wedge q)$; in particular $u \wedge u' \in I(p,q)$.*

The construction in [14] can be described as follows (see Figure 1(a) for a conceptual diagram).

(i) For $u \in I(p,q)$, let $v_{pq}(u)$ be the vector in $\mathbb{R}^2_+$ defined by

$$v_{pq}(u) = (v[s, u \wedge p], v[s, u \wedge q]) \qquad \text{where} \qquad s = p \wedge q \tag{9}$$

(ii) Let $\operatorname{Conv} I(p,q)$ denote the convex hull of vectors $v_{pq}(u)$ for all $u \in I(p,q)$.

(iii) Let $\mathcal{E}(p,q)$ be the set of elements u in $I(p,q)$ such that $v_{pq}(u)$ is a maximal extreme point of $\operatorname{Conv} I(p,q)$. (This set is called "$(p,q)$-envelope"). Note that $p, q \in \mathcal{E}(p,q)$. Hirai proves that elements of $\mathcal{E}(p,q)$ receive unique coordinates [14, Lemma 3.1]:

$$v_{pq}(u) \neq v_{pq}(u') \qquad \forall u \in \mathcal{E}(p,q), u' \in I(p,q) - \{u\} \tag{10}$$

(iv) For $\theta \in [0,1]$ define vector $c_\theta = (1-\theta, \theta)$. For points $p, q \in \mathcal{L}$ let $p \vee_\theta q$ be the point $u \in I(p,q)$ that maximizes $\langle c_\theta, v_{pq}(u) \rangle$, assuming that the maximizer is unique. If a maximizer is not unique then instead set $p \vee_\theta q = \perp$ ("undefined").

(v) By the property in eq. (10), there are only a finite number of values $\theta \in [0,1]$ such that $p \vee_\theta q = \perp$ for some $p, q \in \mathcal{L}$. Let Θ be the set of values $\theta \in [0,1]$ such that $p \vee_\theta q \neq \perp$ for all $p, q \in \mathcal{L}$, then $\Theta \subseteq [0,1]$ has measure 1.

We are now ready to define submodular functions on $\mathcal{L}$. A function $f : \mathcal{L} \to \overline{\mathbb{R}}$ is called *submodular* if every $p, q \in \mathcal{L}$ satisfies

$$f(p) + f(q) \geq f(p \wedge q) + \int_{\theta \in \Theta} f(p \vee_\theta q) d\theta \tag{11a}$$

This is equivalent to the inequality

$$f(p) + f(q) \geq f(p \wedge q) + \sum_{i=0}^{k} (\theta_i - \theta_{i-1}) f(u_i) \tag{11b}$$

where $u_0, \ldots, u_k$ are the sorted elements of $\mathcal{E}(p,q)$ with $(u_0, u_k) = (p,q)$ and $(\theta_{-1}, \theta_k) = (0,1)$,

$$\theta_i = \frac{v[s_i, u_i]}{v[s_i, u_i] + v[s_i, u_{i+1}]} \qquad \text{where} \quad s_i = u_i \wedge u_{i+1}$$

for $i \in [0, k-1]$. (See Fig. 1(b) for a geometric interpretation of values θ_i).

Hirai also gives a simplified characterization of submodularity. Pair (p,q) is called *bounded* if $p \vee q$ exists, implying $\mathcal{E}(p,q) = \{p, q, p \vee q\}$ (Fig. 1(c)). Pair (p,q) is called *antipodal* if $\mathcal{E}(p,q) = \{p,q\}$ (Fig. 1(d)). Equivalently, (p,q) is antipodal if and only if

$$v[a,p]v[b,q] \geq v[p \wedge q, a]v[p \wedge q, b] \qquad \forall (a,b) : p \succeq a \succeq p \wedge q \preceq b \preceq q, a \vee b \text{ exists} \tag{12}$$

We say that (p,q) is *special* if it is either bounded or antipodal. For special pairs inequality (11) reduces to

$$\begin{aligned} f(p) + f(q) &\geq f(p \wedge q) + f(p \vee q) && \text{if } (p,q) \text{ bounded} && (13a) \\ v[p \wedge q, q]f(p) + v[p \wedge q, p]f(q) &\geq (v[p \wedge q, p] + v[p \wedge q, q])f(p \wedge q) && \text{if } (p,q) \text{ antipodal} && (13b) \end{aligned}$$

These inequalities are called respectively *submodularity* and $\wedge$-*convexity* inequalities for (p,q).

Theorem 25 ([14, Theorem 3.5])**.** *Function $f : \mathcal{L} \to \overline{\mathbb{R}}$ is submodular if and only it satisfies*

(1) $\mathcal{E}(p,q) \subseteq \texttt{dom}\, f$ *for* $p, q \in \texttt{dom}\, f$*;*

(2) *the submodularity inequality for every bounded pair* (p,q)*;*

(3) *the* $\wedge$*-convexity inequality for every antipodal pair* (p,q)*.*

We observe that condition (1) can be strengthened further (though we will not use observation). For elements $p, q, u, u' \in \mathcal{L}$ we write $(p,q) \lhd (u,u')$ if $u, u' \in I(p,q) - \{p,q\}$ and point $\frac{1}{2}(v_{pq}(u) + v_{pq}(u'))$ lies strictly above the segment $[v_{pq}(p), v_{pq}(q)]$ in $\mathbb{R}^2$. Note that all elements $u \in \mathcal{E}(p,q) - \{p,q\}$ satisfy $(p,q) \lhd (u,u)$, and pair (p,q) is antipodal if and only if there is no $u \in \mathcal{L}$ with $(p,q) \lhd (u,u)$.

Theorem 26. *Function $f : \mathcal{L} \to \overline{\mathbb{R}}$ satisfies condition* (1) *of Theorem 25 if and only it satisfies the following condition:*

(1′) *Suppose that $p, q \in \texttt{dom}\, f$ and $\mathcal{E}(p,q) \neq \{p,q\}$. Let u^- and u^+ be the elements in $\mathcal{E}(p,q) - \{p,q\}$ that are closest to p and to q, respectively. Then there exists $t \in \texttt{dom}\, f$ such that either $(p,q) \lhd (u^-, t)$ or $(p,q) \lhd (u^+, t)$.*

The proof is given in Appendix A.

5 Proofs

Throughout the proofs we usually denote d, μ to be the distance functions in an extended modular complex Γ, and $d^{\sqsubset}$ and $d^{\diamond}$ to be the distance functions in $\Gamma^{\sqsubset}$ and in $\Gamma^{\diamond}$ respectively. We write $xy \in \Gamma^{\sqsubset}$ to indicate that nodes x, y are neighbors in $\Gamma^{\sqsubset}$ (equivalently, that either $x \sqsubset y$ or $y \sqsubset x$). Note that $xy \in \Gamma^{\sqsubset}$ if and only if $d^{\sqsubset}(x, y) = 1$.

A sequence $(u_0, u_1, \ldots, u_k)$ of nodes in Γ is called a *shortest subpath* if $\mu(u_0, u_k) = \mu(u_0, u_1) + \mu(u_1, u_2) + \ldots + \mu(u_{k-1}, u_k)$. Recall that for a modular graph Γ this is equivalent to the condition $d(u_0, u_k) = d(u_0, u_1) + d(u_1, u_2) + \ldots + d(u_{k-1}, u_k)$. We will often implicitly use the following fact.

Proposition 27. *Consider elements p, q in a modular complex Γ.* (a) *If $p \wedge q$ exists then $(p, p \wedge q)$ is a shortest subpath. Conversely, if $p \succeq x \preceq q$ and (p, x, q) is a shortest subpath then $x = p \wedge q$.* (b) *If $p \vee q$ exists then $(p, p \vee q)$ is a shortest subpath. Conversely, if $p \preceq x \succeq q$ and (p, x, q) is a shortest subpath then $x = p \vee q$.*

Proof. To see the claim, combine Lemma 7 and Lemma 24(a). □

We say that sequence (x_1, x_2, x_3, x_4) is an *isometric rectangle* if (x_{i-1}, x_i, x_{i+1}) are shortest subpaths for all $i \in [4]$, where $x_j = x_{j \bmod 4}$. Isometric rectangles have the following properties.

Proposition 28. *If (x_1, x_2, x_3, x_4) is an isometric rectangle in graph Γ in then $d(x_1, x_2) = d(x_3, x_4)$ and $d(x_1, x_4) = d(x_2, x_3)$. Furthermore, if graph Γ is modular then for any $y_1 \in I(x_1, x_4)$ there exists $y_2 = I(x_2, x_3)$ (namely, a median of y_1, x_2, x_3) such that sequences (x_1, x_2, y_2, y_1) and (y_1, y_2, x_3, x_4) are isometric rectangles.*

Proof. The definition of an isometric rectangle implies that $d(x_1, x_2) + d(x_2, x_3) = d(x_4, x_1) + d(x_3, x_4)$ and $d(x_2, x_3) + d(x_3, x_4) = d(x_1, x_2) + d(x_4, x_1)$, which in turns implies that $d(x_1, x_2) = d(x_3, x_4)$ and $d(x_1, x_4) = d(x_2, x_3)$. Now suppose that $y_1 \in I(x_1, x_4)$ and y_2 is a median of y_1, x_2, x_3. (x_1, y_1, x_4, x_3) and (y_1, y_2, x_3) are shortest subpaths, and thus so is (x_1, y_1, y_2, x_3). By a symmetric argument, (x_4, y_1, y_2, x_2) is also a shortest subpaths. This implies the claim. □

5.1 Preliminaries

First, we state some known facts about orientable modular graphs that will be needed later in the proofs.

Lemma 29 ([2, Proposition 1.7], see [34, Proposition 6.2.6, Chapter I])**.** *A connected graph Γ with distances $d = d_\Gamma$ is modular if and only if*

(1) *Γ is bipartite, and*

(2) *for vertices p, q and neighbors p_1, p_2 of p with $d(p_1, q) = d(p_2, q) = d(p, q) - 1$, there exists a common neighbor p^* of p_1, p_2 with $d(p^*, q) = d(p, q) - 2$.*

Convex sets and gated sets Consider metric space (V, μ). Recall that subset $U \subseteq V$ is called *convex* if $I(x, y) \subseteq U$ for every $x, y \in U$. It is called *gated* if for every $p \in V$ there exists unique $p^* \in U$, called the *gate of p at U*, such that $\mu(p, q) = \mu(p, p^*) + \mu(p^*, q)$ holds for every $q \in U$. The gate p^* will be denoted as $\mathrm{Pr}_U(p)$ ("projection of p onto U"). Thus, Pr_U is a map $V \to U$.

Theorem 30 ([9])**.** *Let A and A' be gated subsets of (V, μ) and let $B := \mathrm{Pr}_A(A')$ and $B' := \mathrm{Pr}_{A'}(A)$.*

(a) *Pr_A and $\mathrm{Pr}_{A'}$ induce isometries, inverse to each other, between B' and B.*

(b) *For $p \in A$ and $p' \in A'$, the following conditions are equivalent:*

  (i) *$\mu(p, p') = \mu(A, A')$.*

  (ii) *$p = \mathrm{Pr}_A(p')$ and $p' = \mathrm{Pr}_{A'}(p)$.*

(c) *B and B' are gated, and $\mathrm{Pr}_B = \mathrm{Pr}_A \circ \mathrm{Pr}_{A'}$ and $\mathrm{Pr}_{B'} = \mathrm{Pr}_{A'} \circ \mathrm{Pr}_A$.*

Now let us consider a modular graph $\Gamma = (V, E, w)$. Such graph induces two natural metrics on V, namely $\mu = \mu_\Gamma$ and $d = d_\Gamma$ (shortest path metrics w.r.t. edge lengths w and 1, respectively). In the light of Theorem 6, the definitions of convex sets, gated sets, gates and maps Pr_U would be the same for both metric spaces ((V, μ) and (V, d)). In addition, convex and gated sets for such metrics coincide.

Lemma 31 ([7], see [14, Lemma 2.9])**.** *Let Γ be a modular graph. For $U \subseteq V$, the following conditions are equivalent:*
(1) *U is convex.*
(2) *U is gated.*
(3) *$\Gamma[U]$ is connected and $I(x, y) \subseteq U$ holds for every $p, q \in U$ with $d_\Gamma(p, q) = 2$.*

Next, we review properties of modular complexes. A path $(p_0, p_1, p_2, \ldots, p_k)$ in a directed acyclic graph $\Gamma = (V, E, w)$ is said to be *ascending* if $p_0 \prec p_1 \prec \ldots \prec p_k$.

Lemma 32 ([14, Lemma 4.13])**.** *Let Γ be a modular complex. For $p, q \in V$ with $p \preceq q$, a (p, q)-path P is shortest if and only if P is an ascending path from p to q. In particular, $I(p, q) = [p, q]$, any maximal chain in $[p, q]$ has the same length, and the rank r of $[p, q]$ is given by $r(a) = d(a, p)$.*

Since set $[p, q]$ is convex in Γ by Lemma 7, one can define projection $\mathrm{Pr}_{[p,q]} : V \to [p, q]$. The lemma below describes some properties of this projection.

Lemma 33 ([14, Lemma 4.15])**.** *Let Γ be a modular complex. For elements p, q, p', q' with $p \preceq q$ and $p' \preceq q'$ define*

$$u = \mathrm{Pr}_{[p,q]}(p') \quad v = \mathrm{Pr}_{[p,q]}(q') \quad u' = \mathrm{Pr}_{[p',q']}(p) \quad v' = \mathrm{Pr}_{[p',q']}(q) \tag{14}$$

Then we have:
(1) *$u \preceq v$, $u' \preceq v'$, $\mathrm{Pr}_{[p',q']}([p, q]) = [u', v']$, and $\mathrm{Pr}_{[p,q]}([p', q']) = [u, v]$.*
(2) *$[u, v]$ is isomorphic to $[u', v']$ by map $w \mapsto \mathrm{Pr}_{[p',q']}(w)$.*

5.2 Properties of extended modular complexes

In this section Γ is always assumed to be an extended modular complex on nodes V with relation $\sqsubseteq$.

Lemma 34. *Suppose that $p \sqsubseteq q_1$, $p \sqsubseteq q_2$, $q_1 \neq q_2$ and q is a common neighbor of q_1, q_2 (implying that $d(q_1, q_2) = 2$). Then $p \sqsubseteq q$.*

Proof. Modulo symmetry, three cases are possible.

- $q_1 \to q \to q_2$. Since $p \sqsubseteq q_2$, condition (15a) gives $p \sqsubseteq q$.
- $q_1 \to q \leftarrow q_2$. Then $q = q_1 \vee q_2$ by Lemma 7(a), and condition (15b) gives $p \sqsubseteq q$.
- $q_1 \leftarrow q \to q_2$. Then $q = q_1 \wedge q_2$ by Lemma 7(a), implying $p \preceq q$ (since p lower-bounds q_1, q_2). Condition (15a) gives $p \sqsubseteq q$. □

Lemma 35. *Consider elements p, q, a, b such that (p, a, b) and (a, b, q) are shortest subpaths and $p \sqsubseteq q$. Then $a \sqsubseteq b$.*

Proof. We use induction on $d(p, q) + d(p, a) + d(q, b)$. First, assume that (a, p, q) is not a shortest subpath. Let p' be a median of a, p, q, then $p' \in [p, q] - \{p\}$ and so $p' \sqsubseteq q$. The induction hypothesis for p', q, a, b gives the claim. We can thus assume that (a, p, q) is a shortest subpath. By a symmetric argument we can also assume that (b, q, p) is a shortest subpath. Sequence (p, q, b, a) is thus an isometric rectangle, and so $d(p, q) = d(a, b)$ and $d(p, a) = d(q, b)$. We now consider 5 possible cases.

- $d(p,a) = d(q,b) = 0$. The claim then holds trivially.
- $d(p,a) = d(q,b) = 1$, $q \to b$. Then $p \preceq b$ and (p,a,b) is a shortest subpath, and so $p \to a \preceq b$ by Lemma 32. We have $a \vee q \preceq b$ and (a,b,q) is a shortest subpath; this implies that $a \vee q = b$. Since $p \sqsubseteq a$ and $p \sqsubseteq q$, we obtain $p \sqsubseteq a \vee q = b$ by condition (15b), and thus $a \sqsubseteq b$ since $a \in [p,q]$.
- $d(p,a) = d(q,b) = 1$, $p \leftarrow a$. This case is symmetric to the previous one.
- $d(p,a) = d(q,b) = 1$, $p \to a$, $b \to q$. We will show that $a \preceq b$; this will imply that (p,a,b,q) is a shortest subpath by Lemma 32, contradicting condition $d(p,q) = d(a,b)$.

  If $d(p,q) = d(a,b) = 0$ then the claim is trivial. If $d(p,q) = d(a,b) = 1$ then (p,q,b,a) is a 4-cycle, and so $p \to q$ implies that $a \to b$. Suppose that $d(p,q) = d(a,b) \geq 2$. By Proposition 28, there exist $x \in I(p,q) - \{p,q\}$ and $y \in I(a,b)$ such that (p,x,y,a) and (q,x,y,b) are isometric rectangles. We have $p \sqsubseteq x \sqsubseteq q$, so by the induction hypothesis $a \sqsubseteq y \sqsubseteq b$, implying the claim.
- $d(p,a) = d(q,b) \geq 2$. By Proposition 28, there exist $p' \in I(p,a)$ and $q' \in I(q,b)$ such that (p,q,q',p') and (p',q',b,a) are isometric rectangles. The induction hypothesis for elements p,q,p',q' gives $p' \sqsubseteq q'$. The induction hypothesis for elements p',q',a,b gives $a \sqsubseteq b$. $\square$

Corollary 36. *Suppose that $p \sqsubseteq q$ in $A' \subseteq V$ is a convex set in Γ. Let $p' = \mathrm{Pr}_{A'}(p)$ and $q' = \mathrm{Pr}_{A'}(q)$. Then $p' \sqsubseteq q'$.*

Lemma 17 (restated)**.** *Let Γ be an extended modular complex, and p be its element. Then $\mathcal{L}_p^+, \mathcal{L}_p^-$ are modular semilattices that are convex in Γ. Furthermore, function $v(\cdot)$ defined via $v(a) = \mu_\Gamma(p,a)$ is a valid valuation of these (semi)lattices.*

Proof. By symmetry, it suffices to consider only $\mathcal{L}_p^+$. Note that $\mathcal{L}_p^+$ is a subset of $\mathcal{L}_p^\uparrow$ such that for any $a,b \in \mathcal{L}_p^+$ we have (i) $a \wedge b \in \mathcal{L}_p^+$ and (ii) $a \vee b \in \mathcal{L}_p^+$ assuming that $a \vee b$ exists (by Definition 15). Thus, all claims except convexity follow from the corresponding properties of $\mathcal{L}_p^\uparrow$ (see Lemma 7). Let us show the convexity. In view of Lemma 31, it suffices to show that for any $a,b \in \mathcal{L}_p^+$ with $d(a,b) = 2$ we have $I(a,b) \subseteq \mathcal{L}_p^+$. This claim follows directly from Lemma 34. $\square$

In the next two results we use the same notation d for d_Γ and d_{Γ^*} (since they can be distinguished by the arguments). Similarly, we use μ for both μ_Γ and μ_{Γ^*}.

Lemma 37. *Let Γ^* be the 2-subdivision of Γ on nodes V^*. Then*

$$d([p,q],[p',q']) = d(p,p') + d(q,q') \qquad \forall [p,q],[p',q'] \in V^*$$

Proof. If $P = ([p_0,q_0],\ldots,[p_k,q_k])$ is a path in Γ^* from $[p_0,q_0] = [p,q]$ to $[p_k,q_k] = [p',q']$ then then the length of P in Γ^* equals $\sum_{i=0}^{k-1}(d(p_i,p_{i+1}) + d(q_i,q_{i+1})) \geq d(p,p') + d(q,q')$; hence ($\geq$) holds.

To show equality, we use induction on $D = d(p,p') + d(q,q')$. The base case $D = 0$ is trivial; suppose that $D \geq 1$. It suffices to show that one of the following holds:
(i) p has neighbor a in Γ such that $a \sqsubseteq q$ and $d(a,p') = d(p,p') - 1$.
(ii) q has neighbor b in Γ such that $p \sqsubseteq b$ and $d(b,q') = d(q,q') - 1$.
Indeed, if (i) holds then $d([a,q],[p',q']) = d(a,p') + d(q,q') = D - 1$ by induction, and hence $d([p,q],[p',q']) \leq (D-1) + 1 = D$, as required. Case (ii) is similar. Alternatively, it would also suffice to show symmetrical cases when p' or q' have appropriate neighbors.

Define u,v,u',v' by (14). Note that $p \preceq u \preceq v \preceq q$ and $p' \preceq u' \preceq v' \preceq q'$. Modulo symmetry, two cases are possible.

- $p \neq u$, implying $p \prec u$. Let $a \in [p,u] \subseteq [p,q]$ be an out-neighbor of p in Γ, then $a \sqsubseteq q$ by condition (15a). and (p,a,p') is a shortest subpath since (p,a,u) and (p,u,p') are shortest subpaths. Thus, case (i) holds.

- $(p,q,p',q') = (u,v,u',v')$. Then (p,q,q',p') is an isometric rectangle. By Proposition 28 there exist $a \in I(p,p')$ and $b \in I(q,q')$ such that (p,q,b,a) and (p',q',b,a) are isometric rectangles and $d(q,b) = d(p,a) = 1$. We have $a \sqsubseteq b$ by Lemma 35.

  If $a \to p$ then Lemma 32 for elements $a \preceq q$ gives $a \preceq b \to q$. By Lemma 7(a), $p \wedge b$ exists. Since (p,q,b) is a shortest subpath, we have $b \notin I(p,q) = [p,q]$, therefore $p \not\preceq b$ and $p \wedge b = a$. Condition (15c) gives $a \sqsubseteq q$, and so case (i) holds.

  If $q \to b$ then by a symmetric argument we conclude that case (ii) holds.

  The last remaining case $p \to a, b \to q$ is impossible by Lemma 32 for elements $p \preceq q$. □

Theorem 18 (restated)**.** *If Γ is an extended modular complex then (V^*, E^*, w^*) is a modular complex (i.e. an oriented modular graph).*

Proof. Any 4-cycle in Γ^* is represented as $([p,q],[p,q'],[p',q'],[p',q])$ for some edges $p \to p', q \to q'$ in Γ, or $([p,x],[p,y],[p,z],[p,w])$ or $([x,p],[y,p],[z,p],[w,p])$ for 4-cycle (x,y,z,w) and vertex p in Γ. This immediately implies that the orientation of Γ^* is admissible and orientation and w^* is orbit-invariant.

To show that Γ^* is modular, we are going to verify that Γ^* satisfies the two conditions of Lemma 29. If $[p,q]$ and $[p',q']$ are joined by an edge, then $d_\Gamma(p,q)$ and $d_\Gamma(p',q')$ have different parity. This implies that Γ^* is bipartite.

We next verify condition (2) of Lemma 29. Take intervals $[p,q],[p',q'] \in V^*$, and denote $D_p = d(p,p')$, $D_q = d(q,q')$, $D = d([p,q],[p',q']) = D_p + D_q$. Suppose further that we are given two distinct neighbors $[p_1,q_1],[p_2,q_2]$ of $[p,q]$ with $d([p_1,q_1],[p',q']) = d([p_2,q_2],[p',q']) = D-1$. Our goal is to show the existence of a common neighbor $[p^*,q^*]$ of $[p_1,q_1],[p_2,q_2]$ with $d([p^*,q^*],[p',q']) = D-2$.

Modulo symmetry, two cases are possible.

- $p_1 = p = p_2$. Condition $d([p_i,q_i],[p',q']) = D-1$ implies that $d(q_i,q') = D_q - 1$ for $i = 1,2$. By Lemma 29(2) for Γ, there is a common neighbor q^* of q_1,q_2 with $d(q^*,q') = D_q - 2$. By Lemma 34, $p \sqsubseteq q^*$. Thus, $[p,q^*]$ is a desired common neighbor of $[p,q_1],[p,q_2]$.

- $p_1 = p$, $q_2 = q$. For better readability let us denote $s = p_2$, $t = q_1$. To summarize, we know that (p,s,p') and (q,t,q') are shortest subpaths, $d(p,s) = d(q,t) = 1$, $s \sqsubseteq q$, $p \sqsubseteq t$, $p \sqsubseteq q$, $p' \sqsubseteq q'$. It suffices to show $s \sqsubseteq t$; this will imply that $[s,t]$ is a desired common neighbor of $[p_1,q_1],[p_2,q_2]$. Modulo symmetry, three subcases are possible.

  Case 1: $s \to p$, $t \to q$. Then $s \sqsubseteq q$ and $s \preceq p \preceq t \preceq q$, so condition (15a) gives $s \sqsubseteq t$.

  Case 2: $p \to s$, $t \to q$. By Lemma 33, $\mathrm{Pr}_{[p,q]}([p',q'])$ is equal to interval $[a,b]$ for $a = \mathrm{Pr}_{[p,q]}(p')$ and $b = \mathrm{Pr}_{[p,q]}(q')$. Note that $s,t,a,b \in [p,q]$. Since (p,s,p') and (s,a,p') are shortest subpaths, so is (p,s,a,p') and thus $s \preceq a$. Similarly $b \preceq t$. Thus $p \preceq s \preceq a \preceq b \preceq t \preceq q$ and $p \sqsubseteq q$ imply $s \sqsubseteq t$ (by condition (15a)), as desired.

  Case 3: $s \to p$, $q \to t$. Observe that $x \vee y$, $x \wedge y$ are defined and belong to $[s,t]$ for all $x,y \in [s,t]$ by Lemma 7.

  Consider set $\mathrm{Pr}_{[s,t]}([p',q'])$, which is equal to $[u,v]$ for $u = \mathrm{Pr}_{[s,t]}(p')$ and $v = \mathrm{Pr}_{[s,t]}(q')$ (Lemma 33). We must have $p \not\preceq u$ (implying $p \wedge u = s$); otherwise (s,p,u), (s,u,p') and thus (s,p,u,p'), (s,p,p') would be shortest subpaths, contradicting the assumption that (p,s,p') is a shortest subpath and $p \neq s$. Similarly, $v \vee q = t$. Note that $u \sqsubseteq v$ by Corollary 36.

Define $a = u \wedge q$. We claim that $a \sqsubseteq t$. Indeed, we have $a \sqsubseteq q$ since $s \preceq a \preceq q$ and $s \sqsubseteq q$. It now suffices to show that there exists b such that $a \sqsubseteq b$ and $b \vee q = t$; the claim will then follow by condition (15b). If $a = u$ then we can take $b = v$. Otherwise, if $a \prec u$, take b to be an out-neighbor of a in $[a, u]$; we have $b \notin [a, q]$ since $a = u \wedge q$, and hence $b \vee q = t$.

We have $p \sqsubseteq t$, $a \sqsubseteq t$ and $p \wedge a = p \wedge u \wedge q = s \wedge q = s$, so condition (15c) gives $s \sqsubseteq t$. □

By combining Lemma 37, Theorem 18 and Theorem 6(a) we obtain:

Corollary 38. *Let Γ^* be the 2-subdivision of Γ on nodes V^*. Then*

$$\mu([p,q],[p',q']) = \mu(p,p') + \mu(q,q') \qquad \forall [p,q],[p',q'] \in V^*$$

Lemma 39. *Let $f : \Gamma \to \overline{\mathbb{R}}$ be an L-convex function on an extended modular complex Γ. Then for every $p \in \Gamma$ the restrictions of f to $\mathcal{L}_p^-$ and to $\mathcal{L}_p^+$ are submodular functions.*

Proof. Let us define $\mathcal{L}_p^{*+} = \{[p,q] \ : \ p \sqsubseteq q\} \subseteq \mathcal{L}_p^*$. By Lemma 37, any vertex in any shortest path between $[p,q]$ and $[p,q']$ is of the form $[p,u]$. Hence $\mathcal{L}_p^{*+}$ is convex in Γ and in $\mathcal{L}_p^*$. Therefore, submodularity of f^* on $\mathcal{L}_p^*$ implies f^* is submodular on $\mathcal{L}_p^{*+}$ (by [14, Lemma 3.7(4)]). Obviously $\mathcal{L}_p^{*+}$ is isomorphic to $\mathcal{L}_p^+$ by $[p,q] \mapsto q$. By using relation $f(q) = f^*([p,q]) - f(p)$ $(q \in \mathcal{L}_p^+)$, we see the submodularity of f on $\mathcal{L}_p^+$. The proof of submodularity of f on $\mathcal{L}_p^-$ is symmetric. □

5.3 Proof of Proposition 16

Proposition 16 (restated)**.** *Let Γ be a modular complex. (a) Relations $\overset{\text{Bp}}{\sqsubseteq}$ and $\preceq$ are admissible for Γ. (b) If relation $\sqsubseteq$ is admissible for Γ then $p \overset{\text{Bp}}{\sqsubseteq} q$ implies $p \sqsubseteq q$.*

To prove this proposition, we will need the following result. A modular lattice $\mathcal{L}$ is called *complemented* if the maximal element $1_{\mathcal{L}}$ is a join of atoms. (This is one possible characterization of complemented modular semilattices, see [3, Chapter IV, Theorem 4.1]).

Proposition 40 ([5, Proposition 6.5])**.** *Consider elements p, q in a modular complex Γ with $p \preceq q$. Then $p \overset{\text{Bp}}{\sqsubseteq} q$ if and only if $[p,q]$ is a complemented modular lattice.*

We now proceed with the proof of Proposition 16.

(a) Checking admissibility of $\preceq$ is straightforward. Clearly, if B is a cube graph and p, q are elements of B with $p \preceq q$ then the subgraph of B induced by $[p,q]$ is also a cube graph; this implies that $\overset{\text{Bp}}{\sqsubseteq}$ satisfies condition (15a). Let us show that condition (15b) holds (condition (15c) is symmetric). Suppose that $p \overset{\text{Bp}}{\sqsubseteq} a$, $p \overset{\text{Bp}}{\sqsubseteq} b$ and $a \vee b$ exists. Let $a_1, \ldots, a_k$ be the atoms of a and $b_1, \ldots, b_\ell$ be the atoms of b, then $a = a_1 \vee \ldots \vee a_k$ and $b = b_1 \vee \ldots \vee b_\ell$ by Proposition 40. Thus, $a \vee b = a_1 \vee \ldots \vee a_k \vee b = b_1 \vee \ldots \vee b_\ell$, and so $a \vee b$ is a join of atoms of $[p, a \vee b]$. Proposition 40 gives that $p \overset{\text{Bp}}{\sqsubseteq} a \vee b$.

(b) We use induction on $d(p,q)$. Consider elements p, q with $p \overset{\text{Bp}}{\sqsubseteq} q$, $d(p,q) \geq 2$. Using Proposition 40, we conclude that $q = a \vee b$ for some $a, b \in [p,q] - \{q\}$. We have $p \overset{\text{Bp}}{\sqsubseteq} a$ and $p \overset{\text{Bp}}{\sqsubseteq} b$ by part (a), and so $p \sqsubseteq a$ and $p \sqsubseteq b$ by the induction hypothesis. Condition (15b) gives $p \sqsubseteq q$.

5.4 Properties of the Cartesian product and L-convexity of the metric function

In this section we prove several properties related to the Cartesian product of extended modular complexes $\Gamma \times \Gamma'$. We denote $(V_\Gamma, V_{\Gamma'}, V_{\Gamma \times \Gamma'}) = (V, V', V_\times)$ for brevity, and use similar notation for other objects (e.g. relations $\preceq, \sqsubseteq, \to$, distances d, μ, etc). Recall that $(p,p') \sqsubseteq_\times (q,q')$ iff $p \sqsubseteq q$ and $p' \sqsubseteq' q'$.

Lemma 19 (restated)**.** *Consider extended modular complexes Γ, Γ' and element $(p,p') \in \Gamma \times \Gamma'$.*
(a) $\Gamma \times \Gamma'$ *is an extended modular complex.*
(b) $\mathcal{L}^{\sigma}_{(p,p')}(\Gamma \times \Gamma') = \mathcal{L}^{\sigma}_{p}(\Gamma) \times \mathcal{L}^{\sigma}_{p'}(\Gamma')$ *for $\sigma \in \{-,+\}$.*

Proof. **(a)** In the light of Lemma 8(a), it suffices to verify that relation $\sqsubseteq_{\times}$ is admissible, i.e. satisfies the properties in Definition 15. We need to show the following:
- $(p,p') \sqsubseteq_{\times} (q,q')$ implies $(p,p') \preceq_{\times} (q,q')$.
- $(p,p') \sqsubseteq_{\times} (p,p')$ for every $(p,p') \in V_{\Gamma} \times V_{\Gamma'}$.
- $(p,p') \sqsubseteq_{\times} (q,q')$ for every edge $(p,p') \to_{\times} (q,q')$.
- If $(p,p') \sqsubseteq_{\times} (q,q')$, $(a,a') \preceq_{\times} (b,b')$ and $(a,a'),(b,b') \in [(p,p'),(q,q')]$, then $(a,a') \sqsubseteq_{\times} (b,b')$.
- If $(p,p') \sqsubseteq (q_1,q_1')$, $(p,p') \sqsubseteq (q_2,q_2')$ and $(q_1,q_1') \vee (q_2,q_2')$ exists, then $(p,p') \sqsubseteq_{\times} (q_1,q_1') \vee (q_2,q_2')$.

Checking each property is mechanical, and is omitted.

(b) From definitions, $\mathcal{L}^{+}_{(p,p')}(\Gamma \times \Gamma') = \{(q,q') \ : \ p \sqsubseteq q, p' \sqsubseteq' q'\} = \mathcal{L}^{+}_{p}(\Gamma) \times \mathcal{L}^{+}_{p'}(\Gamma')$ as sets. Furthermore, the partial order is the same in both cases, and so $\mathcal{L}^{+}_{(p,p')}(\Gamma \times \Gamma')$ and $\mathcal{L}^{+}_{p}(\Gamma) \times \mathcal{L}^{+}_{p'}(\Gamma')$ also equal as posets (which are modular semilattices by Lemma 17). Finally, both semilattices are assigned the same valuation, namely $v_{\times}(q,q') = \mu_{\times}((p,p'),(q,q')) = \mu(p,q) + \mu'(p',q')$. The case $\sigma = -$ is symmetric. $\square$

Lemma 41. *Consider extended modular complexes Γ, Γ'. Graphs $(\Gamma \times \Gamma')^*$ and $\Gamma^* \times \Gamma'^*$ are isomorphic with the isomorphism given by $[(p,p'),(q,q')] \mapsto ([p,q],[p',q'])$ for $p,q \in \Gamma, p',q' \in \Gamma'$ with $p \sqsubseteq q, p' \sqsubseteq' q'$. Consequently, $\mathcal{L}^{*}_{(p,p')}(\Gamma \times \Gamma')$ and $\mathcal{L}^{*}_{p}(\Gamma) \times \mathcal{L}^{*}_{p'}(\Gamma')$ are isomorphic valuated modular semilattices for any $(p,p') \in \Gamma \times \Gamma'$.*

Proof. Clearly, the mapping defined in the lemma is a bijection between the nodes of $(\Gamma \times \Gamma')^*$ and the nodes of $\Gamma^* \times \Gamma'^*$. Checking that this bijection preserves edges and edge weights is mechanical. $\square$

Lemma 42. *Consider extended modular complexes Γ, Γ' and functions $f, f' : \Gamma \to \overline{\mathbb{R}}$ and $\tilde{f} : \Gamma \times \Gamma' \to \overline{\mathbb{R}}$ such that $\texttt{dom} f$ is connected in $\Gamma^{\sqsubset}$.*
(a) *If f is L-convex on Γ and $\tilde{f}(p,p') = f(p)$ for $(p,p') \in \Gamma \times \Gamma'$ then $\tilde{f}$ is L-convex on $\Gamma \times \Gamma'$.*
(b) *If $\tilde{f}$ is L-convex on $\Gamma \times \Gamma'$ and $f(p) = \tilde{f}(p,p')$ for fixed $p' \in \Gamma'$ then f is L-convex on Γ.*

Proof. Since $(\Gamma \times \Gamma')^*$ and $\Gamma^* \times \Gamma'^*$ are isomorphic, for a function $\tilde{f}^* : (\Gamma \times \Gamma')^* \to \overline{\mathbb{R}}$ we can define function $\tilde{f}^*_{\times} : \Gamma^* \times \Gamma'^* \to \overline{\mathbb{R}}$ via $\tilde{f}^*_{\times}([p,q],[p',q']) = \tilde{f}^*([(p,p'),(q,q')])$. Clearly, $\tilde{f}^*$ is submodular on $\mathcal{L}^{*}_{(x,x')}(\Gamma \times \Gamma')$ if and only if $\tilde{f}^*_{\times}$ is submodular on $\mathcal{L}^{*}_{x}(\Gamma) \times \mathcal{L}^{*}_{x'}(\Gamma')$.

(a) Checking connectivity of $\texttt{dom} \tilde{f}$ in $\Gamma^{\sqsubset}$ is straightforward. We can write $\tilde{f}^*_{\times}([p,q],[p',q']) = \tilde{f}(p,p') + \tilde{f}(q,q') = f(p) + f(q) = f^*([p,q])$. L-convexity of f means that f^* is submodular on $\mathcal{L}^{*}_{x}$ for any $x \in \Gamma$. Therefore, by [14, Lemma 3.7(2)], function $\tilde{f}^*_{\times}$ is submodular on $\mathcal{L}^{*}_{x}(\Gamma) \times \mathcal{L}^{*}_{x'}(\Gamma')$ for any $(x,x') \in \Gamma \times \Gamma'$, and thus $\tilde{f}^*$ is submodular on $\mathcal{L}^{*}_{(x,x')}(\Gamma \times \Gamma')$.

(b) $f^*([p,q]) = f(p) + f(q) = \tilde{f}(p,p') + \tilde{f}(q,p') = \tilde{f}^*([(p,p'),(q,p')]) = \tilde{f}^*_{\times}([p,q],[p',p'])$. L-convexity of $\tilde{f}$ means that $\tilde{f}^*_{\times}$ is submodular on $\mathcal{L}^{*}_{x}(\Gamma) \times \mathcal{L}^{*}_{p'}(\Gamma')$. Therefore, by [14, Lemma 3.7(3)], function f^* is submodular on $\mathcal{L}^{*}_{x}(\Gamma)$ for any $x \in \Gamma$. $\square$

Lemma 43 ([14, Lemma 4.18])**.** *Let Λ be a modular complex. The distance function μ_{Λ} is submodular on $\mathcal{L}^{\uparrow}_{(a,b)}(\Lambda \times \Lambda) = \mathcal{L}^{\uparrow}_{a}(\Lambda) \times \mathcal{L}^{\uparrow}_{b}(\Lambda)$ for every $(a,b) \in \Lambda \times \Lambda$.*

Lemma 44. *Let Γ be an extended modular complex.* (a) *Function $\mu_{\Gamma} : \Gamma \times \Gamma \to \mathbb{R}_+$ is L-convex on $\Gamma \times \Gamma$.* (b) *For each $p \in \Gamma$, function $\mu_{\Gamma,p} : \Gamma \to \mathbb{R}_+$ defined via $\mu_{\Gamma,p}(x) = \mu(x,p)$ is L-convex on Γ.*

Proof. It suffices to prove (a); claim (b) will then follow by Lemma 42(b). To be consistent with the notation in Lemma 42, denote $\Gamma' = \Gamma$ and $\tilde{f} = \mu_\Gamma$. By Corollary 38, for each $[p,q],[p',q'] \in V^*$ we have $\tilde{f}^*_\times([p,q],[p',q']) = \tilde{f}^*([(p,p'),(q,q')]) = \mu_\Gamma(p,p') + \mu_\Gamma(q,q') = \mu_{\Gamma^*}([p,q],[p',q'])$, i.e. $\tilde{f}^*_\times = \mu_{\Gamma^*}$. Consider $(x,y) \in \Gamma \times \Gamma$. Lemma 43 for $\Lambda = \Gamma^*$ and $a = [x,x]$, $b = [y,y]$ gives that $\tilde{f}^*_\times = \mu_{\Gamma^*}$ is submodular on $\mathcal{L}^\uparrow_{[x,x]}(\Gamma^*) \times \mathcal{L}^\uparrow_{[y,y]}(\Gamma^*) = \mathcal{L}^*_x(\Gamma) \times \mathcal{L}^*_y(\Gamma)$, and therefore $\tilde{f}^*$ is submodular on $\mathcal{L}^*_{(x,y)}(\Gamma \times \Gamma)$. □

5.5 Proof of Theorem 21: Local optimality implies global optimality

In this section we prove the following result.

Theorem 21 (equivalent formulation)**.** *Let $f : V \to \overline{\mathbb{R}}$ be an L-convex function on an extended modular complex Γ. Consider element p with $\min_{q\in\Gamma} f(q) < f(p) < \infty$. There exists element u with $f(u) < f(p)$ and $pu \in \Gamma^\sqsubset$.*

We use the same argument as in [14] (slightly rearranged).

Lemma 45 ([14, Lemma 4.20])**.** *Let f be a submodular function on a modular semilattice $\mathcal{L}$. For $p,q \in \mathcal{L}$ and $\alpha \in \mathbb{R}$ with $f(p) \le \alpha$, $f(q) < \alpha$, there exists sequence $(u_0,u_1,\ldots,u_k)$ with $(u_0,u_k) = (p,q)$ such that for each $i \in [k]$ we have $f(u_i) < \alpha$ and elements u_{i-1},u_i are comparable.*

Lemma 46. *Let f be an L-convex function on an extended modular complex Γ. Consider triplet (x,y,z) and $\alpha \in \mathbb{R}$ with $x \sqsubseteq y \sqsubseteq z$, $x \not\sqsubseteq z$, $f(x) \le \alpha$, $f(y) < \infty$, $f(z) < \alpha$. There exists sequence $(u_0,u_1,\ldots,u_k)$ with $(u_0,u_k) = (x,z)$ such that for each $i \in [k]$ we have $f(u_i) < \alpha$ and $u_{i-1} \sqsubseteq u_i$.*

Proof. We denote the desired sequence as $P(x,y,z)$, if exists. We prove that $P(x,y,z)$ exists using induction on $d(x,z) + d(x,y)$. We know that function f^* is submodular on $\mathcal{L}^*_y$. Let us apply inequality (11b) to elements $p = [x,y]$ and $q = [y,z]$ (with $s = p \wedge q = [y,y]$). Since $f^*(p) + f^*(q) = f(x) + f^*(s) + f(z)$, there must exist $[a,b] \in \mathcal{E}(p,q) - \{p,q\}$ with $f(a) + f(b) = f^*([a,b]) \le f(x) + f(z)$. Since $[a,b] \in I(p,q)$, we must have $a \in [x,y]$, $b \in [y,z]$ by Lemma 37 (and also $a \sqsubseteq b$). Condition $x \not\sqsubseteq z$ implies that $(a,b) \ne (x,z)$. Therefore, three cases are possible.

- $a = x$, $y \prec b \prec z$. Then we have $f(b) \le f(z) < \alpha$ and $x \sqsubseteq b \sqsubseteq z$. Thus, we can set $P(x,y,z) = (x,b,z)$.
- $x \prec a \prec y$, $b = z$. Then we have $f(a) \le f(x) \le \alpha$ and $x \sqsubseteq a \sqsubseteq z$. Thus, we can set $P(x,y,z) = P(x,a,z)$, where we used the induction hypothesis.
- $x \prec a \preceq b \prec z$. Since $f(a) + f(b) \le f(x) + f(z) < 2\alpha$, one of the following must hold:

  $\underline{f(a) < \alpha.}$ Then we set $P(x,y,z) = (x,a,z)$ if $a \sqsubseteq z$, and $P(x,y,z) = (x, P(a,y,z))$ otherwise.

  $\underline{f(b) < \alpha.}$ Then we set $P(x,y,z) = (x,b,z)$ if $x \sqsubseteq b$, and $P(x,y,z) = (P(x,y,b),z)$ otherwise.

□

Corollary 47. *Consider elements x,y,z in an extended modular complex Γ such that $xy, yz \in \Gamma^\sqsubset$, $xz \notin \Gamma^\sqsubset$, $f(x) \le \alpha$, $f(y) < \infty$, $f(z) < \alpha$. Then there exists path $(u_0,u_1,\ldots,u_k)$ in $\Gamma^\sqsubset$ with $(u_0,u_k) = (x,z)$ such that $f(u_i) < \alpha$ for all $i \in [k]$.*

Proof. Modulo symmetry, two cases are possible:

- $x \sqsupset y \sqsubset z$. The claim then follows from Lemma 45, since function f is submodular on $\mathcal{L}^+_y$.
- $x \sqsubset y \sqsubset z$. The claim then follows from Lemma 46. □

We now proceed with the proof of Theorem 21. For value $\alpha \in \mathbb{R}$ let $\Gamma^{\sqsubset}_{\alpha}$ and $\Gamma^{\sqsubset}_{<\alpha}$ be the subgraphs of $\Gamma^{\sqsubset}$ induced by nodes p with $f(p) \le \alpha$ and with $f(p) < \alpha$, respectively.

Lemma 48. *$\Gamma^{\sqsubset}_{\alpha}$ is connected for any $\alpha \ge \alpha^* \overset{\text{def}}{=} \min_{p\in\Gamma} f(p)$.*

Proof. Suppose the claim is false. For a sufficiently large α graph $\Gamma^{\sqsubset}_{\alpha}$ is connected, since $\mathtt{dom}\, f$ is connected in $\Gamma^{\sqsubset}$. Thus, there must exist $\alpha > \alpha^*$ such that $\Gamma^{\sqsubset}_{\alpha}$ is connected but $\Gamma^{\sqsubset}_{<\alpha}$ is disconnected. There must exist a pair of vertices p, p' belonging to different components of $\Gamma^{\sqsubset}_{<\alpha}$; in particular, $f(p) < \alpha$ and $f(p') < \alpha$, and there exists path $P = (p_0, p_1, \ldots, p_k)$ in $\Gamma^{\sqsubset}$ with $(p_0, p_k) = (p, p')$, $k \ge 2$ and $f(p_i) \le \alpha$ for all $i \in [k-1]$. Pick such p, p', P so that $k \ge 2$ is minimum. The minimality of k implies that $p_{k-2}p_k \notin \Gamma^{\sqsubset}$. Apply Corollary 47 to elements $(x, y, z) = (p_{k-2}, p_{k-1}, p_k)$. We obtain a path $(u_0, \ldots, u_\ell)$ in $\Gamma^{\sqsubset}$ between $u_0 = p_{k-2}$ and $u_\ell = p_k$ with $f(u_i) < \alpha$ for all $i \in [\ell]$. Note that u_1, p' are in the same connected component of $\Gamma^{\sqsubset}_{<\alpha}$ (which is different from that of p), and path $(p_0, \ldots, p_{k-2}, u_1)$ has shorter length compared to P. This contradicts the minimality of k. $\square$

Now consider element $p \in \Gamma$ which is not a global minimum, i.e. $f(p) > \alpha^*$. Connectivity of $\Gamma^{\sqsubset}_{\alpha}$ for $\alpha = f(p)$ implies that there exists path $P = (p_0, p_1, \ldots, p_k)$ in $\Gamma^{\sqsubset}$ with $p_0 = p$, $f(p_i) \le \alpha$ for $i \in [k]$, and $f(p_k) < \alpha$. Let us pick such P so that k is minimum. It suffices to show that $k = 1$; clearly, this would imply Theorem 21. Suppose not, i.e. $k \ge 2$. The minimality of k implies that $p_{k-2}p_k \notin \Gamma^{\sqsubset}$. Applying Corollary 47 to elements $(x, y, z) = (p_{k-2}, p_{k-1}, p_k)$ gives element u_1 so that $f(u_1) < \alpha$ and $p_{k-2}u_1 \in \Gamma^{\sqsubset}$. This contradicts the minimality of k.

We can strengthen Theorem 21 as follows.

Lemma 49. *Let $f : V \to \overline{\mathbb{R}}$ be an L-convex function on an extended modular complex Γ and p, q be distinct elements in $\mathtt{dom}\, f$.* (a) *If $f(p) \ge f(q)$ then there exists element $u \in I(p, q)$ such that $f(u) \le f(p)$ and $pu \in \Gamma^{\sqsubset}$. Additionally, if $f(p) > f(q)$ then $f(u) < f(p)$.* (b) *There exists element $u \in I(p, q) \cap \mathtt{dom}\, f$ such that $pu \in \Gamma^{\sqsubset}$.*

Proof. **(a)** For element $x \in \Gamma$ define function $\mu_x : \Gamma \to \mathbb{R}$ via $\mu_x(u) = \mu(x, u)$. By Lemma 44, function μ_x is L-convex on Γ. Now for value $C \ge 0$ define function $f_C : \Gamma \to \overline{\mathbb{R}}$ via $f_C(u) = f(u) + C(\mu_p(u) + \mu_q(u) - \mu(p, q))$. It is straightforward to check that f_C is L-convex on Γ. (Note that $\mathtt{dom}\, f_C = \mathtt{dom}\, f$, since functions μ_p, μ_q are finite-valued). We have the following properties: (i) $f_C(u) = f(u)$ for all $u \in I(p, q)$; (ii) if C is sufficiently large then $f_C(u) > f(p)$ for all $u \in V_\Gamma - I(p, q)$. Applying Lemma 48 to function f_C gives the first claim, while applying Theorem 21 to function f_C gives the second claim.

(b) We can assume w.l.o.g. that $f(p) > f(q)$, since adding function of the form $C\mu_q$, $C \ge 0$ to f preserves L-convexity of f and does not affect the statement. The claim now follows from part (a). $\square$

5.6 Graph thickening and operations $\vartriangle, \triangledown, \diamond$

Let us recall some definitions from Sections 2-3. Elements p, q of an extended modular complex Γ are said to be $\diamond$-neighbors if $p, q \in [a, b]$ for some a, b with $a \sqsubseteq b$. Equivalently, p, q are $\diamond$-neighbors if $a \wedge b, a \vee b$ exist and $a \wedge b \sqsubseteq a \vee b$. Let $\Gamma^{\diamond}$ be an undirected unweighted graph on nodes V_Γ such that p, q are connected by an edge in $\Gamma^{\diamond}$ if and only if p, q are Δ-neighbors. This graph is called a *thickening of* Γ. A path in $\Gamma^{\diamond}$ is called a *$\diamond$-path*. The shortest path distance between p and q in $\Gamma^{\diamond}$ will be denoted as $d^{\diamond}(p, q)$. These concepts were introduced in [5] in the case when $\sqsubseteq = \overset{\mathtt{Bp}}{\sqsubseteq}$; we now use them for an arbitrary admissible relation $\sqsubseteq$. In order to work with $\Gamma^{\diamond}$, in this section we introduce binary operations $\vartriangle, \triangledown, \diamond$ on an extended modular complex Γ and establish some of their properties. They will be used, in particular, for proving Theorem 22 (bound on the number of steps of zigzag SDA), and for proving that the sum of L-convex functions is L-convex.

For vertices $p, q \in \Gamma$ let $\preceq_{pq}$ be a partial order on $I(p,q)$ defined as follows: $x \preceq_{pq} y$ iff (p,x,y,q) is a shortest subpath. For vertices $x, y \in I(p,q)$ this is equivalent to the condition $x \in I(p,y)$, or to the condition $y \in I(x,q)$. Clearly, $(I(p,q), \preceq_{pq})$ is a poset with the minimal element p and the maximal element q. We will need the following result.

Theorem 50 ([5, Theorem 6.1])**.** *Let Γ be a modular complex. For every $p, q \in \Gamma$ poset $(I(p,q), \preceq_{pq})$ is a modular lattice.*

We denote the meet and join operations in this poset as $\wedge_{pq}$ and $\vee_{pq}$, respectively. Clearly, for each $x, y \in I(p,q)$ the meet $x \wedge_{pq} y$ is a median of p,x,y, and in fact it is the unique median (since every median m of p,x,y belongs to $I(p,q)$ and satisfies $m \preceq_{pq} x$ and $m \preceq_{pq} y$). Similarly, $x \vee_{pq} y$ is the unique median of x,y,q.

For elements $p, q \in \Gamma$ we define $p \vartriangle q$ to be the gate of q at $\mathcal{L}_p^+$ (which is a convex subset of Γ by Lemma 17): $p \vartriangle q = \mathrm{Pr}_{\mathcal{L}_p^+}(q)$. Similarly, we let $p \triangledown q = \mathrm{Pr}_{\mathcal{L}_p^-}(q)$. Clearly, by the definition of the gate and by Lemma 32 we have

$$I(p,q) \cap \mathcal{L}_p^+ = I(p, p \vartriangle q) = [p, p \vartriangle q] = \{u \mid (p,u,p \vartriangle q, q) \text{ is a shortest subpath}\} \tag{15a}$$

$$I(p,q) \cap \mathcal{L}_p^- = I(p, p \triangledown q) = [p \triangledown q, p] = \{u \mid (p,u,p \triangledown q, q) \text{ is a shortest subpath}\} \tag{15b}$$

We also define $p \diamond q = (p \vartriangle q) \vee_{pq} (p \triangledown q)$.

Lemma 51. *Consider elements p, q of an extended modular complex Γ. There holds $p \wedge (p \diamond q) = p \triangledown q \sqsubseteq p \vartriangle q = p \vee (p \diamond q)$ (and consequently $p, p \diamond q$ are $\diamond$-neighbors). If $p \neq q$ then $p \neq p \diamond q$.*

Proof. Denote $(a,b,c) = (p \triangledown q, p \vartriangle q, p \diamond q)$. We know that $a \sqsubseteq p \sqsubseteq b$ and $(p,a,c), (p,b,c), (a,c,b)$ are shortest subpaths (since c is a median of a,b,q and $a,b \in I(p,q)$). From Lemma 35 we conclude that $a \sqsubseteq c \sqsubseteq b$. Since $p \succeq a \preceq c$ and (p,a,c) is a shortest subpath, we must have $a = p \wedge c$. Similarly, $b = p \vee c$. Condition (15b) gives $a \sqsubseteq b$. Now suppose that $p \neq q$, and consider shortest p-q path $(p,u,\ldots,q)$. If $p \to u$ then $p \vartriangle q \neq p$ and $p \diamond q \neq p$, and if $p \leftarrow u$ then $p \triangledown q \neq p$ and $p \diamond q \neq p$. $\square$

For elements p, q an integer $k \geq 0$ define $p \diamond^k q$ as follows: $p \diamond^0 q = p$, and $p \diamond^{k+1} q = p \diamond^k q$ for $k \geq 0$. Clearly, $p \diamond^k q = q$ for some index $k \geq 0$. Let k be the minimum such index. The sequence $(p \diamond^0 q, p \diamond^1 q, \ldots, p \diamond^k q)$ will be called the *normal p-q path*. By the previous lemma, it is a $\diamond$-path (i.e. a path in $\Gamma^\diamond$). Element $p \diamond^{k-1} q$ will be denoted as $p \blacklozenge q$; if $p = q$ then we define $p \blacklozenge q = p$.

For node $p \in \Gamma$ and integer $k \geq 0$ denote $B_k^\diamond(p) = \{q \mid d^\diamond(p,q) \leq k\}$ (a ball of radius k in $\Gamma^\diamond$). We will show later that $B_k^\diamond(p)$ is convex in Γ for any p and k. First, we establish this for $k = 1$.

Lemma 52. *For each $p \in \Gamma$, set $B_1^\diamond(p)$ is convex in Γ.*

Proof. By Lemma 31, it suffices to show the following: if x, x' are distinct $\diamond$-neighbors of p and y is a common neighbor of x, x' in Γ then p, y are $\diamond$-neighbors. Denote $(a,b) = (p \wedge x, p \vee x)$ and $(a', b') = (p \wedge x', p \vee x')$, then $a \sqsubseteq b$ and $a' \sqsubseteq b'$. Recall that sets $[a,b]$ and $[a',b']$ are convex in Γ by Lemma 7. If $d(p,y) = d(p,x) - 1$ then $y \in I(p,x)$, and so by convexity of $[a,b]$ we have $y \in [a,b]$, implying that p, y are $\diamond$-neighbors. We can thus assume that $d(p,y) = d(p,x) + 1$, and also that $d(p,y) = d(p,x') + 1$ (by a symmetric argument). Modulo symmetry, two cases are possible.

- $x \to y \to x'$. Since (p,a,x) and (p,x,y) are shortest subpaths, so is (p,a,y). Since (p,b',x') and (p,x',y') are shortest subpaths, so is (p,b',y). (a,p,b') and (a,y,b') are also shortest subpaths (since $a \preceq p \prec b'$ and $a \preceq y \prec b'$). Thus, we have $(a,b') = (p \wedge y, p \vee y)$ and $a \sqsubseteq y \sqsubseteq b'$ (by Lemma 35). Condition (15b) gives $a \sqsubseteq b'$, and so p, y are $\diamond$-neighbors.

- $x \to y \leftarrow x'$. By Lemma 29, nodes x, x' have a common neighbor z with $d(p,z) = d(p,x)-1 = d(p,x')-1$. Since $z \in I(p,x)$, we must have $z \in [a,b]$ by convexity of $[a,b]$. Similarly, $z \in [a',b']$. Since (x,y,x',z) is a 4-cycle, we must have $x \leftarrow z \to x'$. Since $a = p \wedge x$ and $z \in [a,x]$, we must have $a = p \wedge z$. By a symmetric argument, $a' = p \wedge z$, and so $a = a'$. We know that $p, x, x', b, b' \in \mathcal{L}_a^+$. By Lemma 17, $\mathcal{L}_a^+$ is a modular semilattice which is convex in Γ. Since $y \in I(x,x')$, we must have $y = x \vee x' \in \mathcal{L}_a^+$ by convexity. Since joins $p \vee x = b$, $p \vee x' = b'$, $x \vee x' = y$ exist, the join $\hat{b} = p \vee x \vee x'$ must also exist in $\mathcal{L}_a^+$ by definition of modular semilattices. We obtain $a \sqsubseteq \hat{b}$ and $p, y \in [a, \hat{b}]$, and so p, y are $\diamond$-neighbors. $\square$

Lemma 53. *If (p,u,q) is a shortest subpath and p, x are $\diamond$-neighbors then $(x, p \diamond q, u \diamond q, q)$ is a shortest subpath.*

Proof. It suffices to prove that $(x, p \diamond q, q)$ and $(p \diamond q, u \diamond q, q)$ are shortest subpaths.

(a) *If p, x are $\diamond$-neighbors then $(x, p \diamond q, q)$ is a shortest subpath.*

We use induction on $d(x, p \diamond q)$. Let x' be a median of $x, p \diamond q, q$. First, assume that $x' \neq x$. By Lemma 52, p, x' are $\diamond$-neighbors (since $x' \in I(x, p \diamond q)$). By the induction hypothesis, $(x', p \diamond q, q)$ is a shortest subpath. (x, x', q) is also a shortest subpath, and thus so is $(x, x', p \diamond q, q)$. Now assume that $x' = x$, i.e. $(p \diamond q, x, q)$ is a shortest subpath. $(p, p \diamond q, q)$ is a shortest subpath, and thus so is $(p, p \diamond q, x, q)$. Let $(a,b) = (p \wedge x, p \vee x)$. (p,a,x) and (p,b,x) are shortest subpaths, and thus so are (p,a,x,q) and (p,b,x,q), implying $a, b \in I(p,q)$. (a,x,b) is a shortest subpath since $a \preceq x \preceq b$, and so x is a median of a,b,q. Thus, we must have $x = a \vee_{pq} b$. We also have $a \sqsubseteq b$, and thus $a \in [p \triangledown q, p]$ by eq. (15). By a similar argument, $b \in [p, p \vartriangle q]$. We have $a \preceq_{pq} p \triangledown q$ and $b \preceq_{pq} p \vartriangle q$, and so $x = a \vee_{pq} b \preceq_{pq} (p \triangledown q) \vee_{pq} (p \vartriangle q) = p \diamond q$.

(b) *If (p,u,q) is a shortest subpath then the following sequences are also shortest subpaths:* (i) *$(p, p \vartriangle q, u \vartriangle q, q)$;* (ii) *$(p, p \triangledown q, u \triangledown q, q)$;* (iii) *$(p, p \diamond q, u \diamond q, q)$.*

First, let us show (i). For brevity, denote $b = p \vartriangle q \in I(p,q)$. Define $z = b \vee_{pq} u$. (b,z,u) is a shortest subpath (since z is the median of b,u,q). Also, $u \preceq_{pq} z$, meaning that (p,u,z,q) is a shortest subpath. Lemma 35 for elements p,b,u,z gives that $u \sqsubseteq z$. This shows that $z \in \mathcal{L}_u^+ \cap I(u,q)$, and thus $(u, z, u \vartriangle q, q)$ is a shortest subpath by eq. (15). Since $(z, u \vartriangle q, q)$ and (p,b,z,q) are shortest subpaths, so is $(p, b, z, u \vartriangle q, q)$.

A symmetric argument gives (ii). We can now show (iii) as follows: $u \diamond q = (u \triangledown q) \vee_{uq} (u \vartriangle q) = \texttt{median}(u \triangledown q, u \vartriangle q, q) = (u \triangledown q) \vee_{pq} (u \vartriangle q) \succeq_{pq} (p \triangledown q) \vee_{pq} (p \vartriangle q) = p \diamond q$. $\square$

Theorem 54. *Suppose that (p,u,q) is a shortest subpath in an extended modular complex Γ and $(p_0, p_1, \ldots, p_k)$ is a $\diamond$-path with $p_0 = p$. Then $(p_i, u \diamond^i q, q)$ is a shortest subpath for any $i \in [0,k]$.*

Consequently (by setting $p = u$), if $(p_0, p_1, \ldots, p_k)$ is a $\diamond$-path with $p_0 = p$ then $(p_i, p \diamond^i q, q)$ is a shortest subpath for any $i \in [0,k]$. Also, setting $q = p_k$ gives that $p \diamond^k q = q$, and hence the normal p-q path $(p, p \diamond^1 q, p \diamond^2 q, \ldots, q)$ is a shortest $\diamond$-path from p to q.

Proof. We use induction on i. To see the claim for index $i \in [k]$, apply Lemma 53 for shortest subpath $(p_{i-1}, u \diamond^{i-1} q, q)$ and $\diamond$-neighbors p_{i-1}, p_i. $\square$

Our next goal is to prove that set $B_k^{\diamond}(p)$ is convex in Γ for any p and k. For that we will first need a technical result.

Lemma 55. *Suppose that (p, p', q', q) is an isometric rectangle in an extended modular complex Γ and p, q are $\diamond$-neighbors. Then p', q' are also $\diamond$-neighbors.*

Proof. Note that $I(p,q) \cup I(p',q') \subseteq I(p,q') \cap I(p',q)$. For a node $x \in I(p,q)$ let $x' \in I(p',q')$ be the median of x, p', q'. (This median is unique and equals $x \vee_{pq'} p'$, since $x, p' \in I(p,q')$. Furthermore, p' is a median of p, p', q' and q' is a median of q, p', q', so this notation is consistent). Since (p, x, q, q') and (x, x', q') are shortest subpaths, so is (p, x, x', q'). Analogously, since (p', p, x, q) and (p', x', x)

are shortest subpaths, so is (p', x', x, q). We can now conclude that (p, p', x', x) and (q, q', x', x) are isometric rectangles.

Denote $(a, b) = (p \wedge q, p \vee q)$, then $a \sqsubseteq b$ and $a, b \in I(p, q)$. Since (p, p', a', a) is an isometric rectangle and $a \sqsubseteq p$, we get $a' \sqsubseteq p'$ by Lemma 35. By similar arguments we get $p' \sqsupseteq a' \sqsubseteq q'$ and $p' \sqsubseteq b' \sqsupseteq q'$. (p', a', q') and (p', b', q') are shortest subpaths, and thus $(a', b') = (p' \wedge q', p' \vee q')$. We have $a' \sqsubseteq b'$ by condition (15b), and so p', q' are $\diamond$-neighbors. $\square$

Theorem 56. *For each $p \in \Gamma$ and $k \geq 0$, set $B^{\diamond}_k(p)$ is convex in Γ.*

Proof. We use induction on k. Consider $k \geq 1$. By Lemma 29, it suffices to show that for distinct nodes $q_1, q_2 \in B^{\diamond}_k(p)$ and a common neighbor q' of q_1, q_2 we have $q' \in B^{\diamond}_k(p)$. Denote $p_1 = p \diamond^{k-1} q_1$ and $p_2 = p \diamond^{k-1} q_2$. By Theorem 54, (p_1, p_2, q_2) and (p_2, p_1, q_1) are shortest subpaths. Let p' be a median of p_1, p_2, q'. Since $p' \in I(p_1, p_2)$, we have $d^{\diamond}(p_1, p_2) \leq k - 1$ by the induction hypothesis. It thus suffices to show that p', q' are $\diamond$-neighbors. We can assume that $p_1 \neq p_2$ (otherwise $p_1 = p_2 = p'$ and the claim holds by Lemma 53). By symmetry, we can assume that $p_1 \neq p'$. We know that (p_2, p', p_1, q_1) and (p_1, p', q') are shortest subpaths. We also have $D \overset{\text{def}}{=} d(p_1, p') \geq 1$, $d(p', q') + D \leq d(p_1, q_1) + 1$ and $d(p_1, q_1) + D \leq d(p', q') + 1$. This implies that $D = 1$, $d(p_1, q_1) = d(p', q')$ and (p_1, p', q', q_1) is an isometric rectangle. Lemma 55 now gives that p', q' are $\diamond$-neighbors, as desired. $\square$

Theorem 57. *Suppose that q, u are $\diamond$-neighbors and $d^{\diamond}(p, u) \leq d^{\diamond}(p, q)$. Then $u, p \blacklozenge q$ are $\diamond$-neighbors.*

Proof. Denote $x = p \blacklozenge q$ and $y = u \diamond p$. Theorem 54 gives that (q, y, p) is a shortest subpath. From Theorem 54 we also conclude that $d^{\diamond}(u, p) = d^{\diamond}(u \diamond p, p) + 1 = d^{\diamond}(y, p) + 1$, and so the lemma's precondition gives $d^{\diamond}(p, y) \leq d^{\diamond}(p, q) - 1$. Therefore, using Theorem 54 again gives that $(y, p \blacklozenge q, q) = (y, x, q)$ is a shortest subpath. (p, y, q) is also a shortest subpath, and thus so is (p, y, x, q). We have $y, q \in B^{\diamond}_1(u)$ and $x \in I(y, q)$, so the convexity of $B^{\diamond}_1(u)$ gives $x \in B^{\diamond}_1(u)$, as claimed. $\square$

Note that operations $p \vartriangle q$, $p \triangledown q$ $p \diamond q$, $p \diamond^k q$, $p \blacklozenge q$ defined in this section implicitly depend on the relation $\sqsubseteq$ that comes with an extended modular complex $\Gamma = ((V, E, w), \sqsubseteq)$. We can also define these operations for an extended modular complex $\dot{\Gamma} \overset{\text{def}}{=} ((V, E, w), \preceq)$; in that case we will use notation $p \dot{\vartriangle} q$, $p \dot{\triangledown} q$ $p \dot{\diamond} q$, $p \dot{\diamond}^k q$, $p \dot{\blacklozenge} q$. Elements p, q are said to be $\dot{\diamond}$-neighbors if $p \wedge q, p \vee q$ exists. $\Gamma^{\dot{\diamond}}$ is the graph in which p, q are adjacent if p, q are $\dot{\diamond}$-neighbors, and $d^{\dot{\diamond}}$ is the shortest distance function in this graph. Note that all results of the previous section are applicable to extended modular complexes Γ and $\dot{\Gamma}$, since $\sqsubseteq$ and $\preceq$ are admissible relations for graph (V, E, w).

Remark 4. *It can be deduced from Theorem 54 that $p \blacklozenge q$ is the gate of q at the convex set $B^{\diamond}_{k-1}(p)$ where $k = d^{\diamond}(p, q)$. This operation was introduced in [5] under the name "Δ-gate of p at q" (in the case when $\sqsubseteq = \overset{\text{Bp}}{\sqsubseteq}$). The normal p-q path $(p_0, p_1, \ldots, p_k)$ was also used in [5] (and was shown to be a shortest $\diamond$-path from p to q). However, it was defined by a different construction, namely via a recursion $p_{i-1} = p \blacklozenge p_i$ for $i = k, k-1, \ldots, 1$. To our knowledge, operations $\triangledown$, $\vartriangle$, $\diamond$ and the first part of Theorem 54 have not appeared in [5]; instead, the proof techniques in [5] relied on the notion of "Helly graphs" and on the operation $\langle\langle x, y\rangle\rangle$ (the minimal convex set containing x, y), which we do not use.*

5.7 Connectivity of $\mathtt{dom}\, f$

In this section we will show that following.

Theorem 58. *If $f : \Gamma \to \overline{\mathbb{R}}$ is an L-convex function on an extended modular complex Γ and $p, q \in \mathtt{dom}\, f$ then $p \vartriangle q, p \triangledown q, p \dot{\vartriangle} q, p \dot{\triangledown} q \in \mathtt{dom}\, f$.*

Corollary 59. *For any $p, q \in \Gamma$ there exists a p-q path $(p_0, p_1, \ldots, p_k)$ in $\Gamma^{\sqsubset}$ which is a shortest subpath and has the following property: for any L-convex function f on Γ with $p, q \in \texttt{dom}\, f$ one has $p_i \in \texttt{dom}\, f$ for all $i \in [0, k]$. (This path is obtained by setting either $p_{i+1} = p_i \triangledown q$ or $p_{i+1} = p_i \vartriangle q$; one of these two elements is distinct from p_i if $p_i \neq q$).*

To prove Theorem 58, we start with technical observations.

Lemma 60. *Consider elements $p, q, u \in \texttt{dom}\, f$.*
(a) *If $p \sqsupseteq u \sqsubseteq q$ then $p \vartriangle q \in \texttt{dom}\, f$. Similarly, if $p \sqsubseteq u \sqsupseteq q$ and $p, q \in \texttt{dom}\, f$ then $p \triangledown q \in \texttt{dom}\, f$.*
(b) *If $p \sqsubseteq u \sqsubseteq q$ then $p \vartriangle q \in \texttt{dom}\, f$. Similarly, if $p \sqsupseteq u \sqsupseteq q$ then $p \triangledown q \in \texttt{dom}\, f$.*

Proof. By symmetry, it suffices to prove only the first claim in (a) and in (b).
(a) We know that f is submodular on $\mathcal{L}_u^+$. It can be checked that $p \vartriangle q \in \mathcal{E}(p, q)$, and therefore $p \vartriangle q \in \texttt{dom}\, f$.
(b) We know that f^* is submodular on $\mathcal{L}_u^*$. Note that $[x, y] \in I([p, u], [u, q])$ if and only if $x \sqsubseteq y$, $x \in [p, u]$, $y \in [u, q]$ (by Lemma 37). For such $[x, y]$ we have $v_{[p,u],[u,q]}([x, y]) = (\mu(x, u), \mu(u, y))$. Using these facts, it can be checked that $[u, p \vartriangle q] \in \mathcal{E}([p, u], [u, q])$, and consequently $[u, p \vartriangle q] \in \texttt{dom}\, f^*$ and $p \vartriangle q \in \texttt{dom}\, f$. $\square$

Lemma 61. *Consider elements $p, q, u \in \Gamma$ such that $u \in [p, p \vartriangle q]$. Define $u^+ = u \vartriangle q$ and $u^- = u \triangledown q$. Then $p \vartriangle q = p \vartriangle u^+$ and $p \triangledown q = p \triangledown u^-$.*

Proof. First, we make the following observation.

($\star$) *If $(p, p \vartriangle q, x, q)$ is a shortest subpath then $p \vartriangle q = p \vartriangle x$. Similarly, if $(p, p \triangledown q, x, q)$ is a shortest subpath then $p \triangledown q = p \triangledown x$.*

To see the first claim, observe that for every $y \in \mathcal{L}_p^+$ there exists a shortest x-y path going through $p \vartriangle q$, and so $p \vartriangle q$ is the gate of x at $\mathcal{L}_p^+$.

We now proceed with the proof of Lemma 61. Clearly, we have $p \vartriangle q \sqsubseteq u$, and sequence $(p, u, p \vartriangle q, q)$ is a shortest subpath. Thus, we have $p \vartriangle q \in \mathcal{L}_u^+ \cap I(u, q)$, and so $p \vartriangle q \in [u, u \vartriangle q] = [u, u^+]$ by eq. (15). Sequence $(p, u, p \vartriangle q, u^+, q)$ is thus a shortest subpath, which together with ($\star$) gives the first claim: $p \vartriangle q = p \vartriangle u^+$.

To show the second claim, denote $a = p \triangledown q$, and let $x = u \vee_{pq} (p \triangledown q)$ be the median of $u, p \triangledown q, q$. We have $x \sqsubseteq u$ (by the same argument as in the proof of Lemma 51) and $x \in I(u, q)$, therefore $x \in [u \triangledown q, u] = [u^-, u]$ by eq. (15). Sequence (u, x, u^-, q) is a thus shortest subpath. $(p, p \triangledown q, x, q)$ is also a shortest subpath, and hence so is $(p, p \triangledown q, x, u^-, q)$. Together with ($\star$) this gives that $p \triangledown q = p \triangledown u^-$. $\square$

Lemma 62. *If $p, q \in \texttt{dom}\, f$ then $p \vartriangle q, p \triangledown q \in \texttt{dom}\, f$.*

Proof. We use induction on $d(p, q)$. Suppose that $d(p, q) > 0$. By Lemma 49, there exists $u \in I(p, q) \cap \texttt{dom}\, f$ such that $pu \in \Gamma^{\sqsubset}$. By symmetry, we can assume that $p \sqsubset u$. We then have $u \in [p, p \vartriangle q] - \{p\}$. Denote $u^+ = u \vartriangle q$ and $u^- = u \triangledown q$, then we have $u^+, u^- \in \texttt{dom}\, f$ by the induction hypothesis.

Let us show that $p \triangledown q \in \texttt{dom}\, f$. We have $p \triangledown q = p \triangledown u^-$ by Lemma 61. If $u^- \neq q$ then the claim holds by the induction hypothesis. If $u^- = q$ then $u \sqsupseteq q$, and the claim follows from Lemma 60(a).

Let us show that $p \vartriangle q \in \texttt{dom}\, f$. We have $p \vartriangle q = p \vartriangle u^+$ by Lemma 61. If $u^+ \neq q$ then the claim holds by the induction hypothesis. If $u^+ = q$ then $u \sqsubseteq q$, and the claim follows from Lemma 60(b). $\square$

Lemma 63. *For each p, q there holds $(p \vartriangle q) \dot{\vartriangle} q = p \dot{\vartriangle} q$ and $(p \triangledown q) \dot{\triangledown} q = p \dot{\triangledown} q$.*

Proof. We only show the first claim. Denote $b = p \vartriangle q$ and $\dot{b} = p \mathbin{\dot{\vartriangle}} q$. We have $b \in I(p,q) \cap \mathcal{L}_p^{\uparrow}$, so $b \in [p, \dot{b}]$ and $(p, b, \dot{b}, q)$ is a shortest subpath by eq. (15). Therefore, $\dot{b} \in I(b,q) \cap \mathcal{L}_b^{\uparrow}$, and so $\dot{b} \preceq b \mathbin{\dot{\vartriangle}} q$ by eq. (15). We know that (p, b, q) and $(b, b \mathbin{\dot{\vartriangle}} q, q)$ are shortest subpaths, and thus so is $(p, b, b \mathbin{\dot{\vartriangle}} q, q)$. Therefore, $b \mathbin{\dot{\vartriangle}} q \in I(p,q) \cap \mathcal{L}_p^{\uparrow}$, and so $b \mathbin{\dot{\vartriangle}} q \preceq p \mathbin{\dot{\vartriangle}} q = \dot{b}$ by eq. (15). This shows that $b \mathbin{\dot{\vartriangle}} q = \dot{b}$. $\square$

Combining Lemmas 62 and 63 now yields the last remaining claim of Theorem 58 (via an induction argument): if $p, q \in \texttt{dom}\, f$ then $p \mathbin{\dot{\vartriangle}} q, p \mathbin{\dot{\triangledown}} q \in \texttt{dom}\, f$.

5.8 Proof of Theorem 22 (analysis of the zigzag SDA algorithm)

Theorem 22 (restated)**.** *Let Γ be an extended modular complex and $f : \Gamma^n \to \overline{\mathbb{R}}$ be an L-convex function on Γ^n. Zigzag SDA algorithm applied to function f terminates after generating at most $(\max_{i \in [n]} d^{\dot{\diamond}}_{\Gamma}(x, \texttt{opt}_i(f)) + 1) \cdot \texttt{height}^{\sqsubset}(\Gamma) + 1$ distinct points, where x is the initial vertex and $\texttt{opt}_i(f)$ is as defined in Theorem 12.*

First, we show the following fact.

Lemma 64. *Suppose that Γ is a Cartesian product of extended modular complexes: $\Gamma = \Gamma_1 \times \ldots \times \Gamma_n$. Then $d^{\diamond}_{\Gamma}(x, y) = \max_{i \in [n]} d^{\diamond}_{\Gamma_i}(x_i, y_i)$.*

Proof. It suffices to show the following fact for extended modular complexes Γ, Γ'.

($*$) *The following conditions are equivalent for elements $x, y \in \Gamma$ and $x', y' \in \Gamma'$:*

- (a) *$(x, x'), (y, y')$ are $\diamond$-neighbors;*
- (b) *x, y are $\diamond$-neighbors and x', y' are $\diamond$-neighbors.*

Let us define $(a, b) = (x \wedge y, x \vee y)$ and $(a', b') = (x' \wedge y', x' \vee y')$. (These expressions are defined if either (a) or (b) holds). By the definition of the Cartesian product $\Gamma \times \Gamma'$, we have the following implications:

$$
\begin{aligned}
\text{(a)} \quad &\Leftrightarrow \quad (x, x') \sqsupseteq (a, a') \sqsubseteq (y, y') \text{ and } (x, x') \sqsubseteq (b, b') \sqsupseteq (y, y') \\
&\Leftrightarrow \quad x \sqsupseteq a \sqsubseteq y \text{ and } x' \sqsupseteq a' \sqsubseteq y' \text{ and } x \sqsubseteq b \sqsupseteq y \text{ and } x' \sqsubseteq b' \sqsupseteq y' \quad \Leftrightarrow \quad \text{(b)}
\end{aligned}
$$

$\square$

Note that Lemma 64 applies to both Γ and to $\dot{\Gamma}$. In the light of this lemma, it suffices to prove Theorem 22 in the case when $n = 1$. Accordingly, from on we assume that the zigzag SDA algorithm is applied to minimize function $f : \Gamma \to \overline{\mathbb{R}}$ which is L-convex on an extended modular complex Γ. We will need some technical results.

Proposition 65. *If function f is L-convex on an extended modular complex Γ and $p \wedge q, p \vee q$ exist then $f(p) + f(q) \geq f(p \wedge q) + f(p \vee q)$.*

Proof. We use induction on $d(p, q)$. We can assume that $p, q \in \texttt{dom}\, f$. Denote $(a, b) = (p \wedge q, p \vee q)$. If $p \sqsupseteq a \sqsubseteq q$ then the claim holds since f is submodular on $\mathcal{L}_a^+$. Assume that $a \not\sqsubseteq p$. Clearly, $a = p \mathbin{\dot{\triangledown}} q$, so $a \in \texttt{dom}\, f$ by Theorem 58. Using Lemma 49, we conclude that there exists $x \in [a, p] - \{a, p\}$ with $x \in \texttt{dom}\, f$. Let y be a median of x, q, b. Clearly, (p, a, q, b) is an isometric rectangle, and thus so are (p, x, y, b) and (a, x, y, q) by Proposition 28. We have $x \in [a, p]$, $y \in [b, q]$ and also $x \preceq y$ (since $a \preceq b$ and (a, x, y, b) is a shortest subpath). Clearly, $(a, y) = (x \wedge q, x \vee q)$ and $(x, b) = (p \wedge y, p \vee y)$. Induction hypothesis gives

$$
f(x) + f(q) \geq f(a) + f(y) \qquad\qquad f(p) + f(y) \geq f(x) + f(b)
$$

The facts $x, a, q \in \texttt{dom}\, f$ and the first inequality give $y \in \texttt{dom}\, f$. Summing the two inequalities and subtracting finite value $f(x) + f(y)$ from both sides gives the desired claim. $\square$

The next two claims generalize the construction in [15, Section 3.1.2] and [15, Lemma 6.2].

Proposition 66. *Consider elements $p, q \in \Gamma$ such that $p \wedge q$ exists. Then there exists unique maximal element $u \preceq p$ such that $u \wedge q$ and $u \vee q$ exist.*

This element will be denoted as $p \sqcap_R q$. It satisfies, in particular, $p \sqcap_R q \in [p \wedge q, p]$.

Proof. Denote $s = p \wedge q$. Call u *valid* if $u \preceq p$ and $u \wedge q, u \vee q$ exist, and s*-valid* if additionally $s \preceq u$. Clearly, s is s-valid. It suffices to show the following: if u is s-valid and u' is valid then $u \vee u'$ exists and s-valid. Note that u, u' are upper-bounded by p, hence $u \vee u'$ exists and satisfies $u \vee u' \preceq p$. Denote $s' = u' \wedge q$, then $s' \preceq p \wedge q = s \preceq u$. We know that modular semilattice $\mathcal{L}^{\uparrow}_{s'}$ contains elements $p, q, s, u, u', u \vee u'$. Also, $u \vee q$, $u' \vee q$ and $u \vee u'$ exist. By the definition of a modular semilattice, the join $u \vee u' \vee q = (u \vee u') \vee q$ exists, as well as the meet $(u \vee u') \wedge q$. Thus, $u \vee u'$ is valid. We also have $u \vee u' \succeq u \succeq s$, and thus $u \vee u'$ is s-valid. □

Lemma 67. *Let f be an L-convex function on an extended modular complex and p, q be elements in $\mathtt{dom}\, f$ such that $p \wedge q$ exists and $f(p \wedge q) > f(q) \leq f(p)$. There exists element $u \in I(p, q)$ such that $pu \in \Gamma^{\sqsubset}$, $u \succeq p \sqcap_R q$ and the following holds:*
(i) *If $f(q) = f(p)$ then $f(u) \leq f(p)$.*
(ii) *If $f(q) < f(p)$ then $f(u) < f(p)$.*

Proof. We use induction on $d(p, q)$. We claim that condition (i) for given p, q implies (ii). Indeed, if $f(q) < f(p)$ then we can consider L-convex function $g = f + C\mu_p$ where $\mu_p(x) = \mu(x, p)$ and $C = (f(p) - f(q))/\mu(p, q) > 0$. Clearly, $g(p \wedge q) > g(q) = g(p)$. Apply (i) to function g to get element u, then $f(u) = g(u) - C\mu(p, u) < g(u) \leq g(p) = f(p)$.

As we showed, it suffices to consider the case when $f(s) > f(q) = f(p)$. Denote $x = p \sqcap_R q \in [s, p]$ and $y = x \vee q$. By Lemma 49, there exists $q' \in I(p, q)$ such that $f(q') \leq f(q)$ and $qq' \in \Gamma^{\sqsubset}$. Denote $s' = p \wedge q'$. We will show the following: (1) $f(s') \geq f(s)$, and hence $f(s') > f(q') \leq f(p)$; (2) $x \succeq s' \succeq s$, and hence $x \in [s', p]$; (3) $q' \preceq y$, and hence x, q' are lower-bounded (by s) and upper-bounded (by y) and therefore have a meet and a join. This will imply the claim. Indeed, (2,3) will then mean that $x \preceq p \sqcap_R q'$, and the induction hypothesis for p, q' will give element $u \in I(p, q') \subseteq I(p, q)$ with $f(u) \leq f(p)$, $pu \in \Gamma^{\sqsubset}$ and $u \succeq p \sqcap_R q' \succeq x$.

If $q' \sqsubset q$ then condition $q' \in I(p, q)$ gives that $s' = p \wedge q' = p \wedge q = s$, and so (1,2,3) clearly hold. Suppose that $q' \sqsupset q$. We have $(s, q') = (s' \wedge q, s' \vee q)$ (by Lemma 24 for semilattice $\mathcal{L}^{\uparrow}_s$). Proposition 65 and the fact $q \in \mathtt{dom}\, f$ give $f(s') \geq f(s) + f(q') - f(q) \geq f(s)$, and so (1) holds. Since s', q are upper-bounded (by q') and $s' \in [s, p]$, we must have $s' \preceq p \sqcap_R q = x$, and so (2) holds. Finally, we have $s' \preceq x \preceq y$, $q \preceq y$ and thus $q' = s' \vee q \preceq y$, i.e. (3) also holds. □

Lemma 68. *Consider sequence $u_0 \prec u_1 \prec \ldots \prec u_k$ such that $\infty > f(u_0) > f(u_1) > \ldots > f(u_k)$ and $f(u_i) = \min\{f(u) \,:\, u \in [u_{i-1}, u_i]\}$ for $i \in [k]$. Then $f(u_k) = \min\{f(u) \,:\, u \in [u_0, u_k]\}$.*

Proof. We use induction on k. For $k \leq 1$ the claim is trivial. Suppose that $k \geq 2$. Consider $u \in [u_0, u_k]$. Let $(a, b) = (u \wedge u_{k-1}, u \vee u_{k-1})$. Clearly, both elements are defined and $a \in [u_0, u_{k-1}]$, $b \in [u_{k-1}, u_k]$. We have $f(a) \geq f(u_{k-1})$ (by the induction hypothesis) and $f(b) \geq f(u_k)$. Proposition 65 and the fact $u_{k-1} \in \mathtt{dom}\, f$ give $f(u) \geq f(a) + f(b) - f(u_{k-1}) \geq f(u_{k-1}) + f(u_k) - f(u_{k-1}) = f(u_k)$. □

We now proceed with the proof of Theorem 22. Let $x_0, x_1, \ldots, x_m$ be a sequence generated by the zigzag SDA algorithm, with $f(x_0) > f(x_1) > \ldots > f(x_m)$. We say that element x_i of this sequence is a *corner* if either $x_{i-1} \sqsubset x_i \sqsupset x_{i+1}$ or $x_{i-1} \sqsupset x_i \sqsubset x_{i+1}$. We also define x_m to be a corner (but not x_0, if $m > 0$). Let $y_0, y_1, \ldots, y_\ell$ be the sequence of all corners listed in the order in which they appear (with $\ell \leq m$). Denote $y = y_0$. By symmetry, we can assume that $y_0 \succ y_1 \prec y_2 \succ y_3 \prec \ldots$. By the definition of zigzag SDA, $f(y_i) = \min\{f(z) \,:\, z \in \mathcal{L}^{-}_{y_i}\}$ if i is odd, and $f(y_i) = \min\{f(z) \,:\, z \in \mathcal{L}^{+}_{y_i}\}$ if i is even.

Lemma 69. *For each k we have $\dot{d}^{\diamond}(y, y_k) = k$. Also, conditions $z \in \mathcal{L}^{\uparrow}_{y_k} \cup \mathcal{L}^{\downarrow}_{y_k}$ and $f(z) < f(y_k)$ imply that $\dot{d}^{\diamond}(y, z) = k + 1$.*

Proof. We use the argument from [15]. The proof is by induction on k. For $k = 0$ the claim is trivial. Consider odd $k > 0$ (the case of even $k > 0$ will be symmetric). We have $\dot{d}^{\diamond}(y, y_{k-1}) = k-1$ and $\dot{d}^{\diamond}(y, y_k) = k$ by the induction hypothesis. Suppose that the second claim is false, then there exists $z \succ y_k$ with $f(z) < f(y_k)$ and $\dot{d}^{\diamond}(y, z) \le k$. Let $h = y \,\blacklozenge\, y_k = y \,\dot{\diamond}^{k-1} y_k$. By Theorem 54, (y_{k-1}, h, y_k) is a shortest subpath, and so $h \in [y_k, y_{k-1}]$. By Theorem 57 for elements $y_k \preceq z$, $z \vee h$ exists. By the definition of $y_{k-1} \sqcap_R z$ we have $h \preceq y_{k-1} \sqcap_R z$. Denote $s = y_{k-1} \wedge z \in [y_k, y_{k-1}]$. By Lemma 68 (with the reverse orientation) and by the definition of zigzag SDA, we have $f(s) \ge f(y_k) > f(z)$. By Lemma 67, there exists element $u \in I(y_{k-1}, z)$ such that $pu \in \Gamma^{\sqsubset}$, $u \succeq y_{k-1} \sqcap_R z \succeq h$ and $f(u) < f(y_{k-1})$. We cannot have $u \succ y_{k-1}$, therefore $u \prec y_{k-1}$. By the induction hypothesis, $\dot{d}^{\diamond}(y, u) = (k-1) + 1 = k$. Since $\dot{d}^{\diamond}(y, h) = \dot{d}^{\diamond}(y, y_{k-1}) = k - 1$ and $u \in I(h, y_{k-1}) = [h, y_{k-1}]$, we have $\dot{d}^{\diamond}(y, u) \le k - 1$ by the convexity of $\dot{B}^{\diamond}_{k-1}(y)$ (Theorem 56). This is a contradiction. □

Lemma 70. *Consider sequence $u_0 \sqsubset u_1 \sqsubset \ldots \sqsubset u_t$ such that $\infty > f(u_0) > f(u_1) > \ldots > f(u_t)$ and $f(u_k) = \min\{f(a) \,:\, a \in \mathcal{L}^{+}_{u_{k-1}}\}$ for $k \in [t]$. Then for each $k \in [0, t-1]$ we have $d^{\diamond}(u, u_k) = k$ where $u = u_0$. Also, conditions $z \in \mathcal{L}^{+}_{u_k}$ and $f(z) < f(u_k)$ imply that $d^{\diamond}(u, z) = k + 1$.*

Consequently, $t \le \max\{d^{\diamond}(p, q) \mid p \preceq q\} = \mathtt{height}^{\sqsubset}(\Gamma)$.

Proof. We use induction on k. For $k = 0$ the claim is trivial. Consider $k > 0$. Denote $(x, y) = (u_{k-1}, u_k)$. We have $d^{\diamond}(u, x) = k - 1$ and $d^{\diamond}(u, y) = k$ by the induction hypothesis. Suppose that the second claim is false, then there exists $z \in \mathcal{L}^{+}_{y}$ with $f(z) < f(y)$ and $d^{\diamond}(u, z) \le k$. Note that $x \sqsubset y \sqsubset z$. Denote $h = u \,\blacklozenge\, y$. By Theorem 54, (x, h, y) is a shortest subpath, so $h \in [x, y]$. By Theorem 57, h, z are $\diamond$-neighbors; since $h \preceq y \preceq z$, this implies that $h \sqsubseteq z$ and $y \in [h, z]$. Consider modular semilattice $\mathcal{L}^{*}_{y}$, and define elements $p = [x, y]$, $q = [y, z]$, $s = [y, y] = p \wedge q$ of this semilattice. We have $f(y) > f(z) < f(x)$, and thus $f^*(s) > f^*(q) < f^*(p)$. Lemma 67 gives element $w \in I(p, q)$ such that p, w are comparable in $\mathcal{L}^{*}_{y}$, $w \succeq p \sqcap_R q$ and $f^*(w) < f^*(p)$. Two cases are possible.

- $w \succ p$. From Lemma 37 we conclude that $w = [x, a]$ for some $a \in [y, z]$ with $x \sqsubseteq a$. Condition $f^*([x, a]) < f^*([x, y])$ means that $f(a) < f(y)$. This contradicts the assumption that $f(y) = \min\{f(a) \,:\, a \in \mathcal{L}^{+}_{x}\}$.

- $w \in [p \sqcap_R q, p] \subseteq [s, p]$. From Lemma 37 we conclude that $w = [a, y]$ for some $a \in [x, y]$. Condition $f^*([a, y]) < f^*([x, y])$ means that $f(a) < f(x)$. Define $r = [h, y] \in \mathcal{L}^{*}_{y}$. We know that $r \vee q = [h, z]$ exists in $\mathcal{L}^{*}_{y}$, and thus we must have $r \preceq p \sqcap_R q \preceq w$ in $\mathcal{L}^{*}_{y}$, i.e. $[h, y] \subseteq [a, y]$. Since implies that $a \preceq h$, and so $a \in [x, h] \subseteq [x, y]$. Condition $x \sqsubseteq y$ implies that $x \sqsubseteq a$. By the induction hypothesis we must have $d^{\diamond}(u, a) = (k-1) + 1 = k$. We also have $d^{\diamond}(u, x) = d^{\diamond}(u, h) = k - 1$ and $a \in I(x, h) = [x, h]$, which contradicts the convexity of $\dot{B}^{\diamond}_{k-1}(u)$ (Theorem 56). □

Corollary 71. *The sequences $x_0, x_1, \ldots, x_m$ and $y_0, y_1, \ldots, y_\ell$ produced by zigzag SDA satisfy $\ell \le 1 + \dot{d}^{\diamond}(x_0, \mathtt{opt}(f))$ and $m \le 1 + (\ell + 1) \cdot \mathtt{height}^{\sqsubset}(\Gamma)$.*

Proof. Denote $(x, y) = (x_0, y_0)$. Apply Lemma 69 to L-convex function $f + \delta_U$ where $U = B^{\diamond}_{r}(y)$, $r = d^{\diamond}(y, \mathtt{opt}(f)) \le 1 + d^{\diamond}(x, \mathtt{opt}(f))$. We conclude that $\ell \le r$. Combining this with Lemma 70 (with the original and reverse orientations) gives the claim. □

5.9 Proof of Theorem 20

Theorem 20 (restated). *Consider extended modular complexes Γ, Γ' and functions $f, f' : \Gamma \to \overline{\mathbb{R}}$ and $\tilde{f} : \Gamma \times \Gamma' \to \overline{\mathbb{R}}$.*

(a) *If f, f' are L-convex on Γ then $f + f'$ and $c \cdot f$ for $c \in \mathbb{R}_+$ are also L-convex on Γ.*
(b) *If f is L-convex on Γ and $\tilde{f}(p, p') = f(p)$ for $(p, p') \in \Gamma \times \Gamma'$ then $\tilde{f}$ is L-convex on $\Gamma \times \Gamma'$.*
(c) *If $\tilde{f}$ is L-convex on $\Gamma \times \Gamma'$ and $f(p) = \tilde{f}(p, p')$ for fixed $p' \in \Gamma'$ then f is L-convex on Γ.*
(d) *The indicator function $\delta_U : V \to \{0, \infty\}$ is L-convex on Γ in the following cases: (i) U is a d_Γ-convex set; (ii) $U = \{p, q\}$ for elements p, q with $p \sqsubseteq q$.*
(e) *Function $\mu_\Gamma : V \times V \to \mathbb{R}_+$ is L-convex on $\Gamma \times \Gamma$.*

(a) We need to show that $\mathtt{dom}(f + f') = (\mathtt{dom}\, f) \cap (\mathtt{dom}\, f')$ is connected in $\Gamma^\sqsubset$; checking other conditions of L-convexity is straightforward. Consider $p, q \in (\mathtt{dom}\, f) \cap (\mathtt{dom}\, f')$, and let $(p_0, p_1, \ldots, p_k)$ be the path in $\Gamma^\sqsubset$ constructed in Corollary 59. By the corollary, for each $i \in [0, k]$ we have $p_i \in \mathtt{dom}\, f$ and $p_i \in \mathtt{dom}\, f'$, which implies the claim.

(b) The claim holds by Lemma 42(a).

(c) It suffices to prove that $\mathtt{dom}\, f$ is connected in $\Gamma^\sqsubset$, the claim will then follow from Lemma 42(b). Consider $p, q \in \mathtt{dom}\, f$. We know that $(p, p'), (q, p') \in \mathtt{dom}\, \tilde{f}$, and thus there exists path $(\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_k)$ in $(\Gamma \times \Gamma')^\sqsubset$ with $(\tilde{p}_0, \tilde{p}_k) = ((p, p'), (q, p'))$ and $\tilde{p}_i \in \mathtt{dom}\, \tilde{f}$ for all i. By Corollary 59, such path can be chosen so that $\tilde{p}_i \in I((p, p'), (q, p'))$ for all i. Thus, we have $\tilde{p}_i = (p_i, p')$ for all i. Sequence $(p_0, p_1, \ldots, p_k)$ is now a path in $\Gamma^\sqsubset$ satisfying $p_i \in \mathtt{dom}\, f$ for all i.

(d) Clearly, in both cases U is connected in $\Gamma^\sqsubset$; in the case (i) this holds since U is connected in Γ, and in the case (ii) the claim is trivial.

Let U^* be the set of vertices $[p, q]$ in Γ^* with $p, q \in U$, and for $x \in \Gamma$ denote $U^*_x = U^* \cap \mathcal{L}^*_x$. Note that $\delta^*_U = \delta_{U^*}$, and the restriction of δ_{U^*} to $\mathcal{L}^*_x$ equals $\delta_{U^*_x}$. It suffices to show that U^*_x is convex in $\mathcal{L}^*_x$; the submodularity of $\delta_{U^*_x}$ on $\mathcal{L}^*_x$ will then follow from [14, Lemma 3.7(4)].

If $U = \{p, q\}$ where $p \sqsubseteq q$ then $U^*_x = \{[p, q]\}$ if $x \in [p, q]$, and $U^*_x = \varnothing$ otherwise; in both cases U^*_x is clearly convex. Suppose that U is a convex set in Γ. It suffices to show that U^* is convex in Γ^*. (Since $\mathcal{L}^*_x$ is convex in Γ^* by Lemma 17, the intersection $U^* \cap \mathcal{L}^*_x$ would then be convex in Γ^* and thus also in $\mathcal{L}^*_x$).

We use Lemma 31 to show that U^* is convex in Γ^*. Let $[u, v] \in \Gamma^*$ be common neighbor of distinct $[p, q], [p', q'] \in U^*$; we need to show that $[u, v] \in U^*$. Modulo symmetry, 4 cases are possible:

- $p = u = p'$ and $q \to v \to q'$.
- $p = u = p'$ and $q \to v \leftarrow q'$.
- $p = u = p'$ and $q \leftarrow v \to q'$.
- $p \to u = p'$ and $q \to v = q'$.

In each of these cases conditions $p, q, p', q' \in U$ and convexity of U implies that $u, v \in U$, and so $[u, v] \in U^*$.

(e) The claim holds by Lemma 44(a).

6 VCSP proofs

In this section we prove two results: Theorem 23 (BLP relaxation solves language Φ_Γ for an extended modular complex Γ) and the hardness direction of Theorem 3. Below we give some background on Valued Constraint Satisfaction Problems (VCSPs) which will be needed for these proofs.

Let us fix finite set D. Let $\mathcal{O}^{(m)}$ be the set of operations $g : D^m \to D$. Operation $g \in \mathcal{O}^{(m)}$ is called *symmetric* if $g(x_1, \ldots, x_m) = g(x_{\pi(1)}, \ldots, x_{\pi(m)})$ for any tuple $(x_1, \ldots, x_m) \in D^m$ and any permutation $\pi : [m] \to [m]$. A *fractional operation of arity* m is a probability distribution over

$\mathcal{O}^{(m)}$, i.e. vector $\omega \in [0,1]^{\mathcal{O}^{(m)}}$ with $\sum_g \omega(g) = 1$. Fractional operation ω is called *symmetric* if all operations in $\mathrm{supp}(\omega) = \{g \in \mathcal{O}^{(m)} \mid \omega(g) > 0\}$ are symmetric.

A cost function $f : D^n \to \overline{\mathbb{R}}$ is said to *admit* ω (or ω is a *fractional polymorphism of* f) if

$$\sum_{g \in \mathrm{supp}(\omega)} \omega(g) f(g(x^1, \ldots, x^m)) \leq \frac{1}{m} \sum_{i=1}^{m} f(x^m) \qquad \forall x^1, \ldots, x^m \in D^n$$

where operation $g(\cdot)$ is applied componentwise, i.e.

$$g(x^1, \ldots, x^m) = (g(x_1^1, \ldots, x_1^m), \ldots, g(x_n^1, \ldots, x_n^m)) \in D^n$$

Language Φ over D is said to admit ω if all functions $f \in \Phi$ admit ω. In this case ω is called a *fractional polymorphism of* Φ.

The *expressive power* $\langle \Phi \rangle$ *of language* Φ is defined as the set of all cost functions $f : D^n \to \overline{\mathbb{R}}$ of the form $f(x) = \min_{y \in D^k} f_{\mathcal{I}}(x, y)$ where $\mathcal{I}$ is a Φ-instance with $n+k$ variables. It is known that if Φ admits a fractional polymorphism ω then $\langle \Phi \rangle$ also admits ω.

Definition 72. *Language Φ on domain D is said to satisfy condition (MC) if the exist distinct $a, b \in D$ and binary function $f \in \langle \Phi \rangle$ such that* $\arg\min f = \{(a,b),(b,a)\}$.

Theorem 73 ([33, Theorem 3.4],[24, Theorem 5])**.** *Let Φ be a finite-valued language on domain D such that for every $a \in D$ there exists a unary cost function $g_a \in \langle \Phi \rangle$ with* $\arg\min g_a = \{a\}$. *If Φ does not satisfy (MC) then Φ admits a symmetric fractional polymorphism of every arity $m \geq 2$.*

Theorem 74 ([24, Theorem 1, Proposition 8])**.** *Let Φ be a general-valued language. BLP solves Φ if and only if Φ admits a symmetric fractional polymorphism of every arity $m \geq 2$. If this condition holds then an optimal solution of any Φ-instance can be computed in polynomial time.*

6.1 Proof of Theorem 23

Recall that Φ_Γ is the language over domain $D = V_\Gamma$ that contains all functions $f : D^n \to \overline{\mathbb{R}}$ such that f is L-convex on Γ^n.

Theorem 75. *If Γ is an extended modular complex then Φ_Γ does not satisfy (MC).*

Proof. Suppose the claim is false, then there exists an instance $f : \Gamma \times \Gamma \times \Gamma^n \to \overline{\mathbb{R}}$ of Φ_Γ with $n + 2 \geq 2$ variables such that function $g : \Gamma \times \Gamma \to \overline{\mathbb{R}}$ defined via $g(x,y) = \min_{z \in \Gamma^n} f(x,y,z)$ satisfies $\arg\min g = \{(a,b),(b,a)\}$ for some distinct $a, b \in \Gamma$. Denote $A = \arg\min f \subseteq \Gamma \times \Gamma \times \Gamma^n$, $A_{ab} = \{(a,b,z) \,|\, z \in \Gamma^n\}$ and $A_{ba} = \{(b,a,z) \,|\, z \in \Gamma^n\}$. By construction, $A \subseteq A_{ab} \cup A_{ba}$, $A \cap A_{ab} \neq \varnothing$ and $A \cap A_{ba} \neq \varnothing$. By Lemma 48, the subgraph of $(\Gamma \times \Gamma \times \Gamma^n)^{\sqsubset}$ induced by set A is connected. Thus, there must exist $p \in A_{ab}$, $q \in A_{ba}$ such that $pq \in (\Gamma \times \Gamma \times \Gamma^n)^{\sqsubset}$. By symmetry, we can assume that $p \sqsubset q$. We have $p = (a,b,x)$ and $q = (b,a,y)$ for some $x, y \in \Gamma^n$. Condition $p \sqsubset q$ thus implies that $a \sqsubset b$ and $b \sqsubset a$, which is impossible. □

We can finally prove that BLP solves Φ_Γ.

Theorem 76. *Let Γ be an extended modular complex. Language Φ_Γ admits a symmetric fractional polymorphism of every arity $m \geq 2$. Consequently, BLP relaxation solves Φ_Γ, and an optimal solution of any Φ_Γ-instance can be computed in polynomial time.*

Proof. Let us define $\Phi_\Gamma^\circ = \{f \in \langle \Phi_\Gamma \rangle \,|\, f \text{ is finite-valued}\}$. Since language Φ_Γ does not satisfy (MC), languages $\langle \Phi_\Gamma \rangle$ and Φ_Γ° also do not satisfy (MC). Clearly, for each $a \in \Gamma$ language Φ_Γ° contains unary function g_a with $\arg\min g_a = \{a\}$, namely $g_a(x) = \mu(a,x)$. By Theorem 73, Φ_Γ° admits a symmetric fractional polymorphism of every arity $m \geq 2$. Let ω_m be such fractional polymorphism.

We claim that Φ_Γ also admits ω_m. Indeed, consider function $f : \Gamma^n \to \overline{\mathbb{R}}$ in Φ_Γ. Define function $\mu_n : \Gamma^n \times \Gamma^n \to \mathbb{R}$ via $\mu_n(x,y) = \mu_{\Gamma^n}(x,y)$. By Theorem 44, μ_n is L-convex on $\Gamma^n \times \Gamma^n$, and thus $\mu_n \in \Phi_\Gamma$. For value $C > 0$ define function $f_C : \Gamma^n \to \mathbb{R}$ via $f_C(x) = \min_{y\in\Gamma^n}(f(y) + C\mu_n(x,y))$, then $f_C \in \Phi^\circ_\Gamma$ and thus f_C admits ω_m. Clearly, for any $C' > 0$ there exists $C > 0$ such that the following holds: $f_C(x) = f(x)$ if $x \in \texttt{dom}\, f$, and $f_C(x) \geq C'$ if $f(x) = \infty$. By taking the limit $C' \to \infty$ we conclude that f also admits ω_m. $\square$

6.2 Proof of the hardness direction of Theorem 3

We will use a technique from [25].

Consider language Φ on domain D. A pair of elements $(a,b) \in D \times D$ is called *conservative* if there exists a unary function $g_{ab} \in \langle\Phi\rangle$ with $\arg\min g_{ab} = \{a,b\}$. Let $\mathcal{S}(\Phi) \subseteq D \times D$ be the set of conservative pairs in Φ. For a tuple $p = (a,b) \in \mathcal{S}(\Phi)$, we denote $\bar{p} = (b,a)$; clearly, $\bar{p} \in \mathcal{S}(\Phi)$. Now consider two tuples $p = (a,b)$ and $q = (c,d)$. We say that pair (p,q) is *strictly submodular* if there exists binary cost function $f \in \langle\Phi\rangle$ such that

$$f(a,c) + f(b,d) < f(a,d) + f(b,c)$$

Clearly, if (p,q) is strictly submodular, then (q,p) is also strictly submodular, since function f' defined via $f'(x,y) = f(y,x)$ also belongs to $\langle\Phi\rangle$. We thus say that $\{p,q\}$ is strictly submodular if (p,q) is strictly submodular (or equivalently, if (q,p) is strictly submodular). Let $\mathcal{E}(\Phi) \subseteq \binom{\mathcal{S}(\Phi)}{2}$ be the set of strictly submodular pairs $\{p,q\}$, and define undirected graph $\mathcal{G}(\Phi) = (\mathcal{S}(\Phi), \mathcal{E}(\Phi))$.

Theorem 77 ([25, Theorem 3.2(a) and Lemma 5.1(b)])**.** *Suppose that language Φ is finite-valued and that, for each $a \in D$, there exists unary cost function $g_a \in \langle\Phi\rangle$ with $\arg\min g_a = \{a\}$.*

(a) *If $\{p,q\}, \{q,r\} \in \mathcal{E}(\Phi)$, then $\{p,r\} \in \mathcal{E}(\Phi)$.*

(b) *If $\{p,\bar{p}\} \in \mathcal{E}(\Phi)$ for some $p \in \mathcal{S}(\Phi)$, then Φ is NP-hard.*

Remark 5. *In [25] Kolmogorov and Živný formulated Theorem 77 only in the case when Φ contains all possible $\{0,1\}$-valued unary functions (and thus $\mathcal{S}(\Phi) = D \times D$). However, the proofs of the results above use only weaker preconditions stated in Theorem 77.*

Note that the graph defined in [25] had an edge $\{p,q\}$ if and only if our graph $\mathcal{G}(\Phi)$ has an edge $\{p,\bar{q}\}$. We translated the results from [25] accordingly.

We now apply these results to the generalized minimum 0-extension problem for metric space (V,μ) and subset $F \subseteq \binom{V}{2}$. Let us define the following language over domain $D = V$:

$$\Phi \;=\; \{\mu\} \;\cup\; \{\mu_a \;:\; a \in V\} \;\cup\; \{\mu_{ab} \;:\; \{a,b\} \in F\}$$

where unary function μ_a, μ_{ab} are defined via $\mu_a(x) = \mu(x,a)$ and $\mu_{ab}(x) = \min\{\mu(x,a), \mu(x,b)\}$. Clearly, we have $\Phi \subseteq \langle \texttt{0-Ext}[\mu, F]\rangle$.

Let $H = H_\mu = (V, E, w)$ be the graph corresponding to μ. Suppose that H is not F-orientable; our goal is to show that Φ is NP-hard. We can assume w.l.o.g. that H is modular, otherwise Φ is NP-hard by Theorem 1. Define

$$\vec{E} \;=\; \{(a,b) \;:\; \{a,b\} \in E\} \qquad\qquad \vec{F} \;=\; \{(a,b) \;:\; \{a,b\} \in F\}$$

Let us introduce relations $\|$, $\lhd$, $\approx$ for tuples $p = (a,b)$ and $q = (c,d)$ as follows:

- $p \parallel q$ if $p, q \in \vec{E}$ and (a,b,d,c) is a a 4-cycle in H;
- $p \lhd q$ if $(a,b) \in \vec{E}$, $q \in \vec{F}$, $p \neq q$, and (c,a,b,d) is a shortest subpath in H;
- $p \approx q$ if at least one of the following holds: (i) $p \parallel q$; (ii) $p \lhd q$; (iii) $q \lhd p$.

Note that relation $\approx$ is symmetric, and accordingly, $(\vec{E} \cup \vec{F}, \approx)$ is an undirected graph.

Lemma 78. *Let $p, q \in \mathcal{S}(\Phi)$.*

(a) $\vec{E} \cup \vec{F} \subseteq \mathcal{S}(\Phi)$.

(b) *If $p \parallel q$, then $\{p, q\} \in \mathcal{E}(\Phi)$.*

(c) *If $p \lhd q$, then $\{p, q\} \in \mathcal{E}(\Phi)$.*

(d) *If $p \approx q$, then $\{p, q\} \in \mathcal{E}(\Phi)$.*

Proof. **(a)** First, $\vec{F} \subseteq \mathcal{S}(\Phi)$ holds by the definition of Φ.

As for $\vec{E}$, for each $\{a, b\} \in E$, we construct a function $g_{ab} \in \langle \Phi \rangle$ by $g_{ab}(x) := \mu_a(x) + \mu_b(x)$. From the minimality of H, we have $I(a, b) = \{a, b\}$. We therefore see that $\arg\min g_{ab} = \{a, b\}$, and thus, we obtain $(a, b) \in \mathcal{S}(\Phi)$.

(b) Suppose that $p = (a, b) \in \vec{E}$, $q = (c, d) \in \vec{E}$, and (a, b, d, c) is a 4-cycle in H. The modularity of H implies that that $\{a, d\} \notin E$; otherwise a, b, d would not have a median. Similarly, $\{b, c\} \notin E$; otherwise b, d, c would not have a median. By Theorem 6(a), sequences (a, b, d) and (b, d, c) are shortest subpaths in H. This implies that $\mu(a, c) + \mu(b, d) < \mu(a, d) + \mu(b, c)$. We thereby obtain $\{p, q\} \in \mathcal{E}(\Phi)$.

(c) Suppose that $p = (a, b) \in \vec{E}$, $q = (c, d) \in \vec{F}$, and (c, a, b, d) is a shortest subpath in H. Then $\mu(a, c) + \mu(b, d) < (\mu(a, c) + \mu(a, b)) + (\mu(a, b) + \mu(b, d)) = \mu(b, c) + \mu(a, d)$, and so, we again obtain $\{p, q\} \in \mathcal{E}(\Phi)$.

(d) We decompose $p \approx q$ into cases. If $p \parallel q$, we apply (b). If $p \lhd q$, we apply (c). If $q \lhd p$, we utilize $\{p, q\} = \{q, p\}$ and apply (c) again. We conclude that $\{p, q\} \in \mathcal{E}(\Phi)$. □

We can finally finish the proof of Theorem 3.

We claim that there exists an element $p \in \vec{E} \cup \vec{F}$ such that $p, \bar{p}$ are connected by a path in $(\vec{E} \cup \vec{F}, \approx)$. Indeed, if no such p exists, then there exists a mapping $\sigma : \vec{E} \cup \vec{F} \to \{-1, +1\}$ with the following properties: (i) if $q \approx r$, then $\sigma(q) = \sigma(r)$; (ii) $\sigma(q) = -\sigma(\bar{q})$ for each $q \in \vec{E} \cup \vec{F}$. Such mapping can be constructed by greedily assigning connected components of graph $(\vec{E} \cup \vec{F}, \approx)$; the assumption ensures that no conflicts arise. Using the mapping σ, we can define an orientation of (H, F) by orienting $\{a, b\} \in E \cup F$ as $a \to b$ if $\sigma(a, b) = +1$ and as $a \leftarrow b$ if $\sigma(a, b) = -1$. Clearly, this orientation is admissible, which contradicts the assumption that H is not F-orientable.

From Theorem 77(a) and Lemma 78, we conclude that $\{p, \bar{p}\} \in \mathcal{E}(\Phi)$. Theorem 77(b) now gives that Φ is NP-hard.

A Proof of Theorem 26

Theorem 26 (restated)**.** *Let $\mathcal{L}$ be a valuated modular semilattice. The following two conditions are equivalent for a function $f : \mathcal{L} \to \overline{\mathbb{R}}$:*

(1) $\mathcal{E}(p, q) \subseteq \mathtt{dom}\, f$ *for* $p, q \in \mathtt{dom}\, f$.

(1′) *Suppose that $p, q \in \mathtt{dom}\, f$ and $\mathcal{E}(p, q) \neq \{p, q\}$. Let u^- and u^+ be the elements in $\mathcal{E}(p, q) - \{p, q\}$ that are closest to p and to q, respectively. Then there exists $t \in \mathtt{dom}\, f$ such that either $(p, q) \lhd (u^-, t)$ or $(p, q) \lhd (u^+, t)$.*

Throughout the proof we denote $s = p \wedge q$. As in Section 5, a sequence of elements $(u_0, u_1, \ldots, u_k)$ is called a *shortest subpath in $\mathcal{L}$* if $\mu(u_0, u_k) = \mu(u_0, u_1) + \mu(u_1, u_2) + \ldots + \mu(u_{k-1}, u_k)$. We will use the notion of gates defined in Section 5.1. First, we state a technical result.

Lemma 79 ([14, Lemma 3.12])**.** (a) *Let $u_0, \ldots, u_k$ be the sorted elements of $\mathcal{E}(p, q)$ with $(u_0, u_k) = (p, q)$. Then $(u_0, u_1, \ldots, u_k)$ is a shortest subpath in $\mathcal{L}$.*

(b) *Suppose that (p, a, b, q) is a shortest subpath, and $c = a \wedge b$. For any $u \in I(a, b) \subseteq I(p, q)$ there holds $v_{pq}(u) = v_{pq}(c) + v_{ab}(u)$.*

Note that claim (b) was formulated in [14] only for elements of $\mathcal{E}(p,q)$, but the proof uses only the assumption stated in (b). For convenience, we reproduce this proof here. Denote $x = b \wedge p$, $y = u \wedge p$, $z = u \wedge a$. Since (p,a,u,b,q) is a shortest subpath, from Lemma 24 we get $s \preceq x$, $x \preceq c$, $x \preceq y$, $c \preceq z$, $y \preceq x$. Applying Lemma 24(c) for $I(p,b)$ gives $z = u \wedge a = (u \wedge a \wedge b) \vee (u \wedge a \wedge p) = c \vee y$. By modularity, $v(c) + v(y) = v(c \wedge y) + v(c \vee y) = v(x) + v(z)$, which gives $v[c,z] = v(z) - v(c) = v(y) - v(x) = v[x,y]$. Therefore, $v[s,y] = v[s,x] + v[x,y] = v[s,x] + v[c,z]$, which means that equality $v_{pq}(u) = v_{pq}(c) + v_{ab}(u)$ holds on the first component. The proof of the equality on the second component is symmetric.

Corollary 80. *Let (p,a,b,q) be a shortest subpath with $v_{pq}(a) = (X_a, Y_a)$, $v_{pq}(a) = (X_b, Y_b)$.*
(a) *There holds $X_a \geq Y_a$ and $Y_a \leq Y_b$.*
(b) *The following conditions are equivalent for an element $u \in I(a,b)$:*
  (i) *$(a,b) \lhd (u,u)$;*
  (ii) *$X_a > Y_a$, $Y_a < Y_b$ and $v_{pq}(u)$ lies strictly above the segment $[v_{pq}(a), v_{pq}(b)]$.*
(c) *For elements $x,y,z \in I(a,b)$ the triangle $(v_{ab}(x), v_{ab}(y), v_{ab}(z))$ is obtained from triangle $(v_{pq}(x), v_{pq}(y), v_{pq}(z))$ by a translation, and thus these triangles are isomorphic - the corresponding side lengths and angles are the same.*

We are now ready to prove that condition (1′) implies (1). (The other direction is trivial). For brevity, we will denote $\overline{u} = v_{pq}(u)$ for $u \in I(p,q)$. A sequence of points $[\gamma_1, \ldots, \gamma_k]$ in $\mathbb{R}^2$ with $\gamma_1[1] \geq \ldots \geq \gamma_k[1]$ and $\gamma_1[2] \leq \ldots \leq \gamma_k[2]$ will be called a *polyline*; when needed, it will be viewed as a polygonal line in $\mathbb{R}^2$ obtained by connecting adjacent points with segments. In particular, if $(p, u_1, \ldots, u_{k-1}, q)$ is a shortest subpath then $[\overline{p}, \overline{u_1}, \ldots, \overline{u_{k-1}}, \overline{q}]$ is a polyline.

First, we introduce some notation. For a polyline $[\gamma,\delta]$ (i.e. a segment) we define $\texttt{angle}([\gamma,\delta]) \in [0, \pi/2]$ to be the angle between the horizontal line and the segment $[\gamma,\delta]$ (with $\texttt{angle}([\gamma,\delta]) = 0$ if $\gamma = \delta$). For elements a,b and $u \in I(a,b)$ let $\texttt{angle}_{ab}(u) = \texttt{angle}([v_{ab}(a), v_{ab}(u)])$. Note that $\texttt{angle}_{ab}(u)$ equals the angle at the first vertex in the triangle $(v_{ab}(a), v_{ab}(u), v_{ab}(a \wedge b))$. If $(a,b) = (p,q)$ then we will omit subscript pq and write simply $\texttt{angle}(u)$.

We use induction on $\mu(p,q)$ to show that $\mathcal{E}(p,q) \subseteq \texttt{dom}\, f$ for any $p,q \in \texttt{dom}\, f$. Consider $p,q \in \texttt{dom}\, f$ with $\mathcal{E}(p,q) - \{p,q\} \neq \varnothing$. Let $u = u^-$ be the element of $\mathcal{E} - \{p,q\}$ closest to p. By symmetry, we can assume that this element has the property in condition (1′), i.e. there exists $t \in \texttt{dom}\, f$ with $(p,q) \lhd (u,t)$. First, suppose that $u \in \texttt{dom}\, f$. Lemma 79(a) gives that $\mathcal{E}(p,q) - \{p\} \subseteq I(u,q)$. By applying Lemma 79(b) to shortest subpaths of the form (p,u,x,q) we can conclude that $\mathcal{E}(p,q) - \{p\} = \mathcal{E}(u,q)$. We have $\mathcal{E}(u,q) \subseteq \texttt{dom}\, f$ by the induction hypothesis, and so $\mathcal{E}(p,q) \subseteq \texttt{dom}\, f$.

We thus assume from now on that $u \notin \texttt{dom}\, f$. Pick element $t \in \texttt{dom}\, f$ with $(p,q) \lhd (u,t)$ that has the maximum value of $\texttt{angle}(t)$; if there are ties, then pick t that maximizes $\mu(p,t)$. Since $I(p,t)$ and $I(t,q)$ are convex sets, we can define projections $a = \mathrm{Pr}_{I(p,t)}(u)$ and $b = \mathrm{Pr}_{I(t,q)}(u)$. Clearly, (p,a,u), (p,a,t), (u,b,q), (t,b,q) are shortest subpaths (by the definition of gates). (p,u,q) and (p,t,q) are also shortest subpaths since $u,t \in I(p,q)$, and thus so are (p,a,u,b,q) and (p,a,t,b,q).

Lemma 81. *There holds $\overline{u} + \overline{t} = \overline{a} + \overline{b}$. Furthermore, $\bar{a} \neq \bar{t}$ and $\bar{b} \neq \bar{t}$.*

Proof. For the first claim it suffices to show that $v_{ab}(u) + v_{ab}(t) = v_{ab}(a) + v_{ab}(b)$; the desired equality will then follow by Lemma 79(b).

By Lemma 24 we can represent u,t as $u = u_1 \vee u_2$ and $t = t_1 \vee t_2$ where $u_1 = u \wedge a$, $u_2 = u \wedge b$, $t_1 = t \wedge a$, $t_2 = t \wedge b$.

Denote $c = a \wedge b$, then $c = u_1 \wedge t_1 = u_2 \wedge t_2$. Denote $a' = u_1 \vee t_1 \preceq a$. We claim that $a' = a$. Indeed, (u, u_1, a', a) and (a, a', t_1, t) are shortest subpaths by Lemma 24(a), and thus so is $(u, u_1, a', a, a', t_1, t)$ since a is the gate of u at $I(p,t)$. This is only possible if $a' = a$. By modularity, $v(u_1) + v(t_1) = v(u_1 \wedge t_1) + v(u_1 \vee t_1) = v(c) + v(a)$ and so $v[c,t_1] + v[c,u_1] = v[c,a]$. This proves equality $v_{ab}(u) + v_{ab}(t) = v_{ab}(a) + v_{ab}(b)$ on the first component. The proof of the equality on the second component is symmetric.

Next, we show that $\overline{a} \neq \overline{t}$. Suppose this is false, then $\overline{u} = \overline{b}$, and thus $u = b$ (see eq. (10)). Thus, (p,t,u,q) is a shortest subpath. By applying Lemma 79(b) to shortest subpaths of the form (p,t,x,q) we conclude that $u \in \mathcal{E}(t,q)$, and thus $u \in \texttt{dom}\, f$ by the induction hypothesis - a contradiction. A symmetrical argument shows that $\overline{b} \neq \overline{t}$. $\square$

Define $\gamma = \frac{1}{2}(\overline{u} + \overline{t}) = \frac{1}{2}(\overline{a} + \overline{b})$ and $\texttt{angle}(\gamma) = \texttt{angle}([\overline{p},\gamma])$. By assumption, γ lies strictly above the segment $[\overline{p},\overline{q}]$, i.e. $\texttt{angle}(\gamma) > \texttt{angle}(q)$.

If $\texttt{angle}(\gamma) < \texttt{angle}(t)$ then $\texttt{angle}(u) < \texttt{angle}(\gamma) < \texttt{angle}(t)$ since $\gamma = \frac{1}{2}(\overline{u} + \overline{t})$; however, condition $\texttt{angle}(u) < \texttt{angle}(t)$ is impossible by the choice of u. Therefore, $\texttt{angle}(t) \leq \texttt{angle}(\gamma)$.

Suppose $\texttt{angle}(a) > \texttt{angle}(t)$. We then have $(p,t) \lhd (a,a)$ by Corollary 80, and so $\mathcal{E}(p,t) - \{p,t\}$ is non-empty. Let a' be an element of this set, then $a' \in \texttt{dom}\, f$ by the induction hypothesis. We have $(p,t) \lhd (a',a')$, and so $\overline{a'}$ lies strictly above the segment $[\overline{p},\overline{t}]$ by Corollary 80. But then $\texttt{angle}(t) < \texttt{angle}(a')$ and $a' \in \texttt{dom}\, f$, which contradicts the choice of t.

We showed that $\texttt{angle}(a) \leq \texttt{angle}(t) \leq \texttt{angle}(\gamma)$. This implies that $\texttt{angle}(b) \geq \texttt{angle}(\gamma)$, since $\gamma = \frac{1}{2}(\overline{a} + \overline{b})$. Conditions $\texttt{angle}(a) \leq \texttt{angle}(t) \leq \texttt{angle}(b)$ mean that polyline $[\overline{p},\overline{a},\overline{t},\overline{b}]$ is convex.

We know that $\texttt{angle}(q) < \texttt{angle}(\gamma)$, $\texttt{angle}(t) \leq \texttt{angle}(\gamma) \leq \texttt{angle}(b)$ and $\overline{b} \neq \overline{t}$. Therefore, $\overline{b}$ lies strictly above the segment $[\overline{t},\overline{q}]$. By Corollary 80 we conclude that $(t,q) \lhd (b,b)$, and so $\mathcal{E}(t,q) - \{t,q\}$ is non-empty. Let $b' \neq t$ be the element of this set which is closest to t. Note that (p,t,b',q) is a shortest subpath. We have $b' \in \texttt{dom}\, f$ by the induction hypothesis. The choice of b' implies that $\texttt{angle}_{tq}(b) \leq \texttt{angle}_{tq}(b')$. By Corollary 80 we conclude that $\texttt{angle}([\overline{t},\overline{b}]) = \texttt{angle}_{tq}(b)$ and $\texttt{angle}([\overline{t},\overline{b'}]) = \texttt{angle}_{tq}(b')$.

We showed that $\texttt{angle}([\overline{t},\overline{b}]) \leq \texttt{angle}([\overline{t},\overline{b'}])$ and polyline $[\overline{p},\overline{a},\overline{t},\overline{b}]$ is convex. This implies that polyline $[\overline{p},\overline{a},\overline{t},\overline{b'}]$ is also convex. We can now conclude that $\texttt{angle}(t) \leq \texttt{angle}(b')$, $\mu(p,t) < \mu(p,b')$ and $b' \in \texttt{dom}\, f$. This contradicts the choice of t.

References

[1] H.-J. Bandelt. Networks with condorcet solutions. *European J. Oper. Res.*, 20:314–326, 1985.

[2] H.-J. Bandelt, M. van de Vel, and E. Verheul. Modular interval spaces. *Mathematische Nachrichten*, 163:177–201, 1993.

[3] G. Birkhoff. *Lattice Theory, 3rd edn.* American Mathematical Society, 1967.

[4] A. Blake, P. Kohli, and C. Rother. *Markov Random Fields for Vision and Image Processing.* MIT Press, 2011.

[5] Jérémie Chalopin, Victor Chepoi, Hiroshi Hirai, and Damian Osajda. Weakly modular graphs and nonpositive curvature. *Memoirs of the AMS*, 268(1309), 2020.

[6] Chandra Chekuri, Sanjeev Khanna, Joseph Naor, and Leonid Zosin. A linear programming formulation and approximation algorithms for the metric labeling problem. *SIAM Journal on Discrete Mathematics*, 18(3):608–625, 2004.

[7] V. Chepoi. Classification of graphs by means of metric triangles (in Russian). *Metody Diskretnogo Analiza*, 49:75–93, 1989.

[8] Julia Chuzhoy and Joseph (Seffi) Naor. The hardness of metric labeling. *SIAM Journal on Computing*, 36(2):498–515, 2006.

[9] A. W. M. Dress and R. Scharlau. Gated sets in metric spaces. *Aequationes Mathematicae*, 34:112–120, 1987.

[10] P. Felzenszwalb, G. Pap, E. Tardos, and R. Zabih. Globally optimal pixel labeling algorithms for tree metrics. In *IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, 2010.

[11] S. Fujishige. *Submodular Functions and Optimization, 2nd Edition.* Elsevier, Amsterdam, 2005.

[12] S. Fujishige and K. Murota. Notes on L-/M-convex functions and the separation theorems. *Mathematical Programming, Series A*, 88:129–146, 2000.

[13] Igor Gridchyn and Vladimir Kolmogorov. Potts model, parametric maxflow and k-submodular functions. In *Proceedings of the 14th IEEE International Conference on Computer Vision (ICCV'13)*, pages 2320–2327. IEEE, 2013.

[14] H. Hirai. Discrete convexity and polynomial solvability in minimum 0-extension problems. *Mathematical Programming, Series A*, 155:1–55, 2016.

[15] H. Hirai. L-convexity on graph structures. *Journal of the Operations Research Society of Japan*, 61:71–109, 2018.

[16] H. Hirai and R. Mizutani. Minimum 0-extension problems on directed metrics. *Discrete Optimization*, 40, 2021.

[17] Anna Huber and Vladimir Kolmogorov. Towards Minimizing k-Submodular Functions. In *Proceedings of the 2nd International Symposium on Combinatorial Optimization (ISCO'12)*, volume 7422 of *Lecture Notes in Computer Science*, pages 451–462. Springer, 2012.

[18] Anna Huber, Andrei Krokhin, and Robert Powell. Skew bisubmodularity and valued CSPs. *SIAM Journal on Computing*, 43(3):1064–1084, 2014.

[19] A. V. Karzanov. Minimum 0-extensions of graph metrics. *European Journal of Combinatorics*, 19:71–101, 1998.

[20] A. V. Karzanov. One more well-solved case of the multifacility location problem. *Discrete Optimization*, 1(1):51–66, 2004.

[21] J. Kleinberg and É. Tardos. Approximation algorithms for classification problems with pairwise relationships: metric labeling and markov random fields. *Journal of the ACM*, 49:616–639, 2002.

[22] V. Kolmogorov and A. Shioura. New algorithms for convex cost tension problem with application to computer vision. *Discrete Optimization*, 6:378–393, 2009.

[23] Vladimir Kolmogorov. Submodularity on a tree: Unifying $l^\sharp$-convex and bisubmodular functions. In *Proceedings of the 36th International Symposium on Mathematical Foundations of Computer Science (MFCS'11)*, volume 6907 of *Lecture Notes in Computer Science*, pages 400–411. Springer, 2011.

[24] Vladimir Kolmogorov, Johan Thapper, and Stanislav Živný. The power of linear programming for general-valued CSPs. *SIAM Journal on Computing*, 44(1):1–36, 2015.

[25] Vladimir Kolmogorov and Stanislav Živný. The complexity of conservative valued CSPs. *Journal of the ACM*, 60(2), 2013. Article 10.

[26] K. Murota. Discrete convex analysis. *Mathematical Programming*, 83:313–371, 1998.

[27] K. Murota. *Discrete Convex Analysis.* SIAM, Philadelphia, 2003.

[28] K. Murota. On steepest descent algorithms for discrete convex functions. *SIAM J. Optim.*, 14:699–707, 2003.

[29] K. Murota and A. Shioura. M-convex function on generalized polymatroid. *Mathematics of Operations Research*, 24:95–105, 1999.

[30] K. Murota and A. Shioura. Exact bounds for steepest descent algorithms of L-convex function minimization. *Operations Research Letters*, 42:361–366, 2014.

[31] K. Murota and A. Tamura. Proximity theorems of discrete convex functions. *Mathematical Programming, Series A*, 99:539–562, 2004.

[32] B. C. Tansel, R. L. Francis, and T. J. Lowe. Location on networks I, II. *Management Science*, 29:498–511, 1983.

[33] Johan Thapper and Stanislav Živný. The complexity of finite-valued CSPs. *Journal of the ACM (JACM)*, 63(4), 2016.

[34] M. L. J. van de Vel. *Theory of Convex Structures.* North-Holland, Amsterdam, 1993.