Massively Winning Configurations in the Convex Grabbing Game on the Plane

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Abstract

The convex grabbing game is a game where two players, Alice and Bob, alternate taking extremal points from the convex hull of a point set on the plane. Rational weights are given to the points. The goal of each player is to maximize the total weight over all points that they obtain. We restrict the setting to the case of binary weights. We show a construction of an arbitrarily large odd-sized point set that allows Bob to obtain almost 3/4 of the total weight. This construction answers a question asked by Matsumoto, Nakamigawa, and Sakuma in [Graphs and Combinatorics, 36/1 (2020)]. We also present an arbitrarily large even-sized point set where Bob can obtain the entirety of the total weight. Finally, we discuss conjectures about optimum moves in the convex grabbing game for both players in general.

1 Introduction

The graph grabbing game, which was first presented by Winkler [7], is a game where two players alternate removing non-cut vertices from a vertex-weighted graph. This game has been studied [1, 4, 6] and led to variants including the convex grabbing game by Matsumoto, Nakamigawa, and Sakuma [3], which we discuss here.

A cake $\mathcal{C}$ is determined by a set of points, which we call *cherries*, that lie in general position in the Euclidean plane. Each cherry c is given a weight $\mathrm{w}(c) \in \mathbb{Q}$. There are two players in this game: *Alice* and *Bob*. They alternate selecting cherries from the set of remaining cherries $C \subseteq \mathcal{C}$, with Alice going first. They can only select extremal points of the convex hull of C, defined by $\mathrm{Ex}(C) = \{c \in C \mid c \notin \mathrm{conv}(C \setminus \{c\})\}$, and the selected cherry is removed from the set C. The game is over when all cherries are taken.

If $|\mathcal{C}|$ is even, we say that $\mathcal{C}$ is an even-sized cake. Similarly, if $|\mathcal{C}|$ is odd, we say that $\mathcal{C}$ is an odd-sized cake.

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We denote the sequence of moves by Alice as:

$$(a_1, a_2, \ldots, a_{\lceil \frac{|\mathcal{C}|}{2} \rceil})$$

We denote the sequence of moves by Bob as:

$$(b_1, b_2, \ldots, b_{\lfloor \frac{|\mathcal{C}|}{2} \rfloor})$$

This results in a gameplay

$$\mathbf{q} = (a_1, b_1, a_2, b_2, \ldots, a_{\frac{|\mathcal{C}|}{2}}, b_{\frac{|\mathcal{C}|}{2}})$$

on an even-sized cake $\mathcal{C}$, or

$$\mathbf{q} = (a_1, b_1, a_2, b_2, \ldots, a_{\lfloor \frac{|\mathcal{C}|}{2} \rfloor}, b_{\lfloor \frac{|\mathcal{C}|}{2} \rfloor}, a_{\lceil \frac{|\mathcal{C}|}{2} \rceil})$$

on an odd-sized cake $\mathcal{C}$. In the end, each player obtains a total score equal to the sum of the weights of the cherries they selected. In particular, we define the total gain of Alice as:

$$\mathrm{A}(\mathbf{q}) = \sum_{i \in \{1, 2, \ldots, \lceil \frac{|\mathcal{C}|}{2} \rceil\}} \mathrm{w}(a_i)$$

The objective of Alice is to maximize $\mathrm{A}(\mathbf{q})$. The objective of Bob is to minimize $\mathrm{A}(\mathbf{q})$. We also define the complement:

$$\mathrm{B}(\mathbf{q}) = \sum_{i \in \{1, 2, \ldots, \lfloor \frac{|\mathcal{C}|}{2} \rfloor\}} \mathrm{w}(b_i)$$

We observe that $\mathrm{A}(\mathbf{q}) + \mathrm{B}(\mathbf{q})$ is invariant of $\mathbf{q}$; it is constant for a given cake $\mathcal{C}$. Alice wants to minimize $\mathrm{B}(\mathbf{q})$ and, naturally, Bob wants to maximize $\mathrm{B}(\mathbf{q})$. We will work with $\mathrm{B}(\mathbf{q})$ a lot because we will focus on maximizing Bob's results — which looks like a harder task, at least at first glance.

Finally, we define the *minimax result* on the cake $\mathcal{C}$, denoted by $\mathcal{M}(\mathcal{C})$, as the total gain of Bob if both players play optimally throughout the whole game.

$$\mathcal{M}(\mathcal{C}) = \min_{a_1 \in \mathrm{Ex}(\mathcal{C})} \left(\max_{\substack{b_1 \in \mathrm{Ex}(\\ \mathcal{C} \setminus \{a_1\})}} \left(\min_{\substack{a_2 \in \mathrm{Ex}(\\ \mathcal{C} \setminus \{a_1, b_1\})}} \left(\max_{\substack{b_2 \in \mathrm{Ex}(\\ \mathcal{C} \setminus \{a_1, b_1, a_2\})}} \left(\right. \right. \right. \right.$$

$$\left. \left. \left. \left. \min_{\substack{a_3 \in \mathrm{Ex}(\\ \mathcal{C} \setminus \{a_1, b_1, a_2, b_2\})}} \left(\ldots \left(\mathrm{B}((a_1, b_1, a_2, b_2, a_3, \ldots)) \right) \ldots \right) \right) \right) \right) \right)$$

We focus on a restricted version of this game where only $\{0, 1\}$ weights are considered. Any cherry $c \in \mathcal{C}$ where $\mathrm{w}(c) = 1$ is called *red*; and we define the set of red cherries $\mathrm{R}(\mathcal{C}) := \{c_r \in \mathcal{C} \mid \mathrm{w}(c_r) = 1\}$. In a similar manner, any cherry $c \in \mathcal{C}$ where $\mathrm{w}(c) = 0$ is called *green*; and we define their set $\mathrm{G}(\mathcal{C}) := \{c_g \in \mathcal{C} \mid \mathrm{w}(c_g) = 0\}$. We thereby have $\mathrm{R}(\mathcal{C}) \cup \mathrm{G}(\mathcal{C}) = \mathcal{C}$ and $\mathrm{R}(\mathcal{C}) \cap \mathrm{G}(\mathcal{C}) = \emptyset$.

For any $C \subseteq \mathcal{C}$, we define values $\mathrm{r}(C) := |\mathrm{R}(C)|$ and $\mathrm{g}(C) := |\mathrm{G}(C)|$. Note that that $\mathrm{r}(C) + \mathrm{g}(C) = |C|$ and that $\mathrm{r}(C) = \sum_{c \in C} \mathrm{w}(c)$.

Matsumoto, Nakamigawa, and Sakuma [3] posed the question of finding the maximum possible value for $\mathcal{M}(\mathcal{C}) - (\mathrm{r}(\mathcal{C}) - \mathcal{M}(\mathcal{C}))$ on an odd-sized cake, that is: How much can Bob win by? In Section 3, we present, for any natural number z, a construction of an odd-sized cake $\mathcal{C}$ such that $\mathrm{r}(\mathcal{C}) = 4z + 2$; and we provide a tactic for Bob which guarantees $\mathcal{M}(\mathcal{C}) \geq \frac{3}{4}\mathrm{r}(\mathcal{C}) - \frac{1}{2}$. Therefore, Bob can win by an arbitrarily large margin.

In Section 4, we show that there exists an even-sized cake $\mathcal{D}$ where $\mathrm{r}(\mathcal{D}) = y$ and $\mathcal{M}(\mathcal{D}) = y$ for every $y \in \mathbb{N}$; that is, Bob can obtain all red cherries.

2 Order types

In this section, we provide a combinatorial point of view on the convex grabbing game. The following definition has been adapted from [5].

Definition 1 *Given a tuple (p, q, r) of three distinct cherries, we define their* orientation ∇pqr *as $+1$ if the sequence (p, q, r) traverses the triplet $\{p, q, r\}$ in a counterclockwise direction, and as -1 if this direction is clockwise.*

Consider two cakes, P and Q, where $|P| = |Q|$. We say that a bijection $\pi : P \to Q$ is order-preserving *if $\mathrm{w}(c) = \mathrm{w}(\pi(c))$ for each cherry $c \in P$ and there exists a sign $\sigma \in \{-1, +1\}$ such that $\nabla \pi(p)\pi(q)\pi(r) = \sigma \cdot \nabla pqr$ for all sequences (p, q, r) of three distinct cherries in P.*

If such a bijection exists, we say that P and Q are order-equivalent.

Proposition 2 *If $\mathcal{C}$ and $\mathcal{D}$ are order-equivalent cakes, then $\mathcal{M}(\mathcal{C}) = \mathcal{M}(\mathcal{D})$.*

The proof is in the full version [2] of this paper.

3 Sun configuration

We present a family of odd-sized cakes which we call the sun configuration, and we show that, from any cake $\mathcal{C}$ in the family, Bob will obtain at least $\frac{3}{4}\mathrm{r}(\mathcal{C}) - \frac{1}{2}$ red cherries given that he follows a certain tactic.

Definition 3 *We define a* beam $\mathcal{y}$ *as four cherries in the order [green, red, green, red] lying on an arc (see Figure 1).*

Figure 1: Beam $(\mathcal{y})$.

Definition 4 *Let $k > 2$ be an odd integer. We define the* sun *as a cake $\mathcal{S}_k$ with k beams and an additional green cherry ζ in the centre (see Figure 2 for an example $\mathcal{S}_5$) such that:*

- *The sun is rotationally symmetric with the k beams evenly spaced around the centre ζ.*

- *Each beam is far enough from the centre such that, with the removal of any proper subset Y of the cherries on the beam, the outermost cherry on the beam will always be in $\mathrm{Ex}(\mathcal{S}_k \setminus Y)$.*

- *Consider any beam $\mathcal{y}_i$ (see Figure 3). A line drawn through any two cherries of $\mathcal{y}_i$ does not cut through any other beam and it keeps $\frac{k-1}{2}$ beams from $\mathcal{S}_k \setminus \mathcal{y}_i$ on each side. Additionally, if we consider $\mathcal{y}_i \cup \{\zeta\}$, then they are all in convex position.*

We have constructed a sun $\mathcal{S}_k$ where $\mathrm{r}(\mathcal{S}_k) = 2k$ and $\mathrm{g}(\mathcal{S}_k) = 2k + 1$.

Definition 5 *In the convex grabbing game on a sun, we say that a player follows the* Careful greedy tactic *if the player chooses a move according to these instructions:*

Is there an extremal red cherry?

  YES $\longrightarrow$ Is there an extremal red cherry on a beam that also contains a red cherry that is not extremal?

    YES $\longrightarrow$ Take extremal red cherry from this beam.
    NO $\longrightarrow$ Is there a beam that contains a single red cherry and this red cherry is extremal?

      YES $\longrightarrow$ Take this extremal red cherry.
      NO $\longrightarrow$ Take any extremal red cherry.

  NO $\longrightarrow$ Is there any beam with at least one green cherry and no red cherries on it?

    YES $\longrightarrow$ Take a green cherry from this beam. (since at least one cherry from each remaining beam is extremal in any moment)
    NO $\longrightarrow$ FAIL!

Theorem 6 *From the sun $\mathcal{S}_k$, Bob will get at least $\frac{3k-1}{2} = \frac{3}{4}\mathrm{r}(\mathcal{S}_k) - \frac{1}{2}$ red cherries by using the Careful greedy tactic, no matter how Alice plays. As a result, we obtain the desired property $\mathcal{M}(\mathcal{S}_k) \geq \frac{3}{4}\mathrm{r}(\mathcal{S}_k) - \frac{1}{2}$.*

We approach the proof as follows. We let Bob follow the Careful greedy tactic in all his moves. Alice can do anything.

We always describe the game state by the set of remaining cherries $C \subseteq \mathcal{S}_k$ and we create a lower bound for how many red cherries Bob is guaranteed to obtain from this moment until all cherries are taken. We
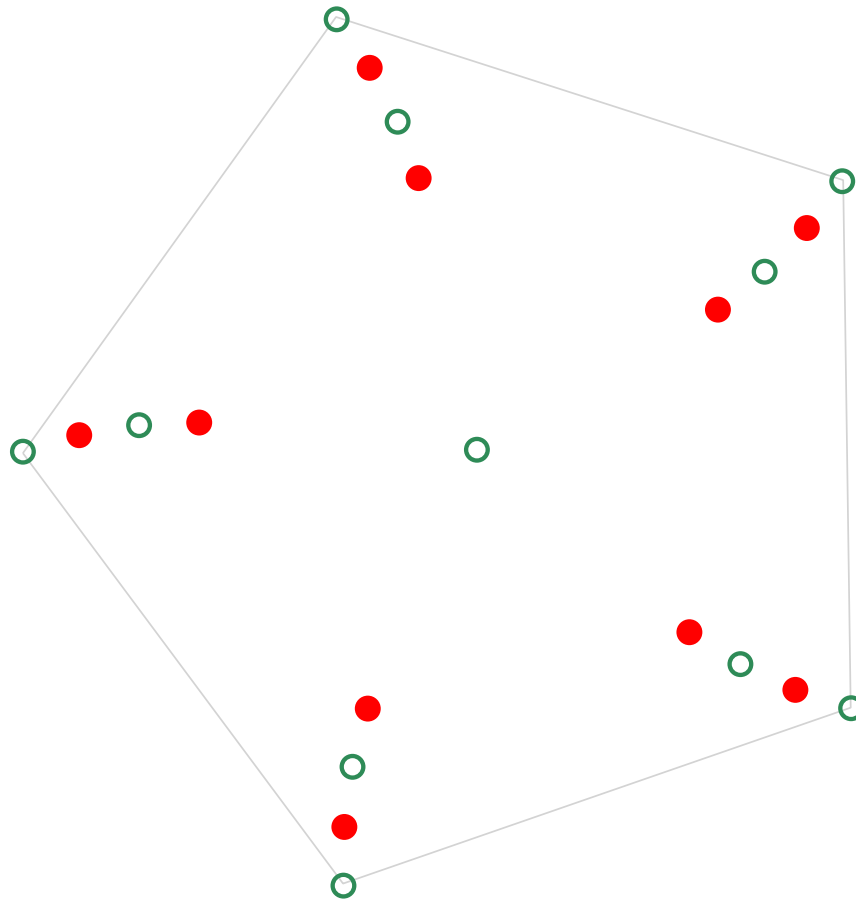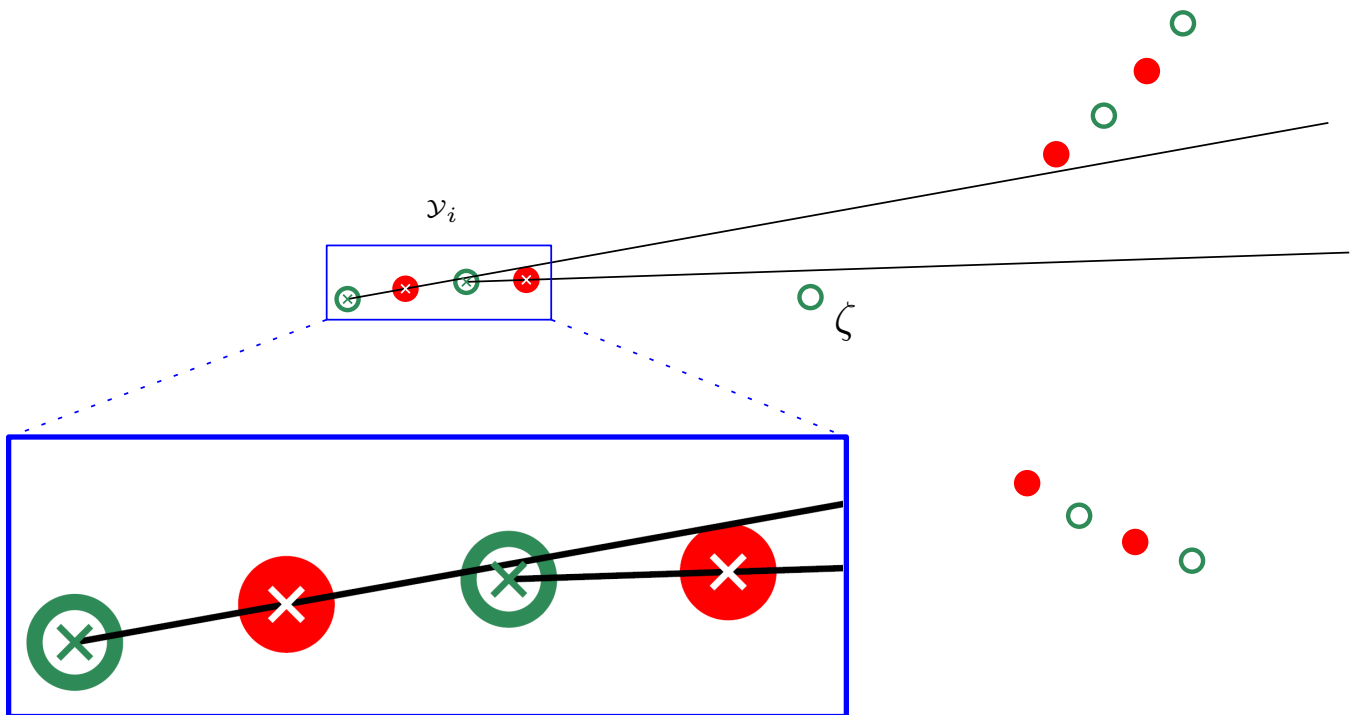
Figure 2: Sun ($\mathcal{S}_5$).

Figure 3: Highlighting details of the sun.

characterize the set C by the existence of a certain line (see Definition 7) and by quantities $r(C)$, $s(C)$, and $t(C)$ (see Definition 12).

We start our proof with Lemma 13 in order to calculate what will happen in the second phase of the game. Lemma 14 then analyzes the first phase of the game while using the result of Lemma 13 in order to obtain the sum of Bob's score over both phases. In the end, we prove Theorem 6 as a straightforward corollary of Lemma 14.

Definition 7 *In a moment of a gameplay on the sun $\mathcal{S}_k$, denote $C \subseteq \mathcal{S}_k$ as the set of remaining cherries. For all lines that pass through ζ, we define the set $\mathcal{U}_C$ as the set of all closed half-planes defined by these lines. If $C \subseteq U \in \mathcal{U}_C$, then U is called a bounding half-plane for C.*

Lemma 8 *If Bob has been following the Careful greedy tactic from the beginning of the game on the sun $\mathcal{S}_k$, then, for $C \subseteq \mathcal{S}_k$ in a moment of gameplay when it is Alice's turn and a bounding half-plane does not yet exist, we have:*

1. *$\zeta \notin \mathrm{Ex}(C)$*

2. *Each beam y is either fully remaining (that is $y \subseteq C$), or fully removed (that is $y \cap C = \emptyset$), or exactly the two innermost cherries (one green, one red) remain.*

Proof.

Item (1): From the hyperplane separation theorem; if ζ were an extremal point of C, there would be a bounding half-plane going directly through ζ.

Item (2): We proceed by induction on the number of taken cherries. Base case: $\mathcal{S}_k$ satisfies the properties since each beam is fully remaining. Assume that this holds up till some set $C_0 \subseteq \mathcal{S}_k$, and it is Alice's turn. Induction step: From C_0 Alice can only take a green cherry from some beam y. The beam y is either fully remaining, or has exactly the two innermost cherries remaining by the induction hypothesis. After Alice takes the green cherry, a red cherry will be revealed on beam y. This will be the only red cherry available, and so by following the Careful greedy tactic, Bob will take this cherry. Therefore beam y will end up either fully removed, or with the two innermost cherries remaining. These two moves will give $C_1 \subset C_0$ where $|C_1| = |C_0| - 2$ which either will maintain all properties or a bounding half-plane will have emerged. $\square$

Lemma 9 *If Bob always follows the Careful greedy tactic, he will never reach FAIL.*

Proof. Before a bounding half-plane emerges, this is clear from Lemma 8.

After a bounding half-plane emerges, leaving $C \subset \mathcal{S}_k$, there will always be a beam y such that all remaining cherries of y are in $\mathrm{Ex}(C)$; therefore, if there are no extremal red cherries, the beam y will have at least one extremal green cherry and no red cherries — Bob can take a green cherry here. Therefore, Bob will never reach the FAIL branch. $\square$

Definition 10 *A beam y is* semi-exposed *in C if it has $|y \cap \mathrm{R}(C)| = 2$ and $|y \cap \mathrm{R}(C) \cap \mathrm{Ex}(C)| = 1$.*

Lemma 11 *If Bob has been following the Careful greedy tactic from the beginning of the game on the sun $\mathcal{S}_k$, then, when it is Alice's turn, there will never be a semi-exposed beam.*

Proof. Assume to the contrary that a semi-exposed beam exists after a move by Bob and that it is the first time this happens. Lemma 8 shows it cannot happen before a bounding half plane has emerged. Suppose now we are at a point in the gameplay when a bounding half-plane exists and there is a semi-exposed beam after Bob's move.

Removing a cherry can either (1) reveal a beam in full, or (2) reveal no new cherries, or (3) reveal a single cherry. Clearly, neither (1) nor (2) can produce a semi-exposed beam.

In (3), a semi-exposed beam can be produced either by taking the centre cherry ζ, or by taking a green cherry from a beam which contains two red cherries (this beam then becomes semi-exposed). Neither of these moves can be performed by Bob because the Careful greedy tactic allows taking a green cherry only if it is from a beam and this beam does not contain any red cherries. We are left only with the option that it was Alice who generated a semi-exposed beam by (3).

Since taking a red cherry from a semi-exposed beam is the top priority in Bob's Careful greedy tactic, the only possible reason for Bob leaving a semi-exposed beam is if Alice leaves two semi-exposed beams after her turn. Alice can produce a semi-exposed beam by (3); however, (3) can only produce a single semi-exposed beam at a time. Therefore, there was at least one semi-exposed beam before Alice's move, which is a contradiction with this being the first time there is a semi-exposed beam after Bob's move. $\square$

Definition 12 *In any moment of a gameplay on the sun $\mathcal{S}_k$, denote $C \subseteq \mathcal{S}_k$ as the set of remaining cherries.*

We define $s(C)$ as the number of beams in C which have a single *remaining red cherry.*

Furthermore, we define $t(C)$ as the number of beams in C which have at least one *remaining red cherry.*

Lemma 13 *Let $C \subseteq \mathcal{S}_k$ be a remaining subset of the sun obtained by Bob following the Careful greedy tactic such that $|C|$ is odd and a bounding half-plane for C exists. It is now Alice's turn.*

From the set C, Bob will obtain at least $\frac{1}{2}(\mathrm{r}(C) - \mathrm{s}(C))$ red cherries from now until the end of the game (using the Careful greedy tactic).

Proof. This can be proved by induction on $|C|$.

If $|C| \leq 1$, then $\mathrm{r}(C) = \mathrm{s}(C)$, thus the statement holds trivially (it says that Bob will get at least 0 red cherries).

If $|C| > 1$, then we assume the statement holds for $|C'| = |C| - 2$. We proceed by case analysis.

- If Alice first takes a green cherry, then the lemma holds no matter what Bob does.

    - If Bob proceeds by taking a green cherry as well, then we see that $\mathrm{r}(C') = \mathrm{r}(C)$ and that $\mathrm{s}(C') = \mathrm{s}(C)$, thus it reduces exactly to the induction hypothesis.

    - If Bob proceeds by taking a red cherry, then we observe that $\mathrm{r}(C') = \mathrm{r}(C) - 1$ and that $|\mathrm{s}(C') - \mathrm{s}(C)| \leq 1$, thus:

    $$\frac{1}{2}(\mathrm{r}(C') - \mathrm{s}(C')) \geq \frac{1}{2}(\mathrm{r}(C) - \mathrm{s}(C)) - 1$$

    By the induction hypothesis, Bob will obtain at least $\frac{1}{2}(\mathrm{r}(C) - \mathrm{s}(C)) - 1$ red cherries in the future. And, since Bob has just taken one red cherry, Bob obtains at least $\frac{1}{2}(\mathrm{r}(C) - \mathrm{s}(C))$ red cherries in total.

- If Alice first takes a red cherry, then we need to consider the properties of the game state and Bob's strategy in order to show that the lemma holds.

    - If Alice took the red cherry from a beam with only this red cherry, then we observe that $\mathrm{r}(C') = \mathrm{r}(C) - 1$ and that $\mathrm{s}(C') = \mathrm{s}(C) - 1$, thus:

    $$\frac{1}{2}(\mathrm{r}(C) - \mathrm{s}(C)) = \frac{1}{2}(\mathrm{r}(C') - \mathrm{s}(C'))$$

    The rest follows by applying the induction hypothesis in the same way as in the previous cases.

    - If, prior to Alice's move, there were two red cherries on the beam, then Bob's Careful greedy tactic leads to taking the other red cherry from the same beam. By Lemma 11, the second red cherry is guaranteed to be extremal. This gives $\mathrm{r}(C') = \mathrm{r}(C) - 2$ and $\mathrm{s}(C') = \mathrm{s}(C)$, thus the induction hypothesis

guarantees that Bob will be able to obtain at least

$$\frac{1}{2}(\mathrm{r}(C') - \mathrm{s}(C')) = \frac{1}{2}(\mathrm{r}(C) - \mathrm{s}(C)) - 1$$

future red cherries; and, since Bob has just taken one red cherry, Bob obtains at least $\frac{1}{2}(\mathrm{r}(C) - \mathrm{s}(C))$ red cherries in total. $\square$

Lemma 14 *Let $C \subseteq \mathcal{S}_k$ be a remaining subset of the sun obtained by Bob following the Careful greedy tactic such that $|C|$ is odd and a bounding half-plane for C does not yet exist. It is now Alice's turn.*

There exists a half-plane $U \in \mathcal{U}_C$ such that Bob will obtain at least $\mathrm{r}(C) - \mathrm{t}(U \cap C)$ red cherries from now until the end of the game by using the Careful greedy tactic.

Proof. We proceed by induction on $\mathrm{r}(C)$. Note that all extremal cherries are green, by Lemma 8, and each of them lies on a beam that has a red cherry (in particular, the beam has the same number of green and red cherries).

Bob's Careful greedy tactic dictates to always take the neighboring red cherry from the same beam as Alice just took her cherry from. Finally, as our base case, we utilize Lemma 13 once a bounding half-plane emerges.

Induction step: Consider U from the induction hypothesis. Let C' be the set of cherries remaining from C after Alice's move and Bob's move, $|C'| = |C| - 2$. The induction hypothesis says that Bob will get at least $\mathrm{r}(C') - \mathrm{t}(U \cap C')$ cherries during the remainder of the game. Since $\mathrm{r}(C) = \mathrm{r}(C') + 1$ and $\mathrm{t}(U \cap C) \geq \mathrm{t}(U \cap C')$, the difference between the lemma statement and the number from the induction hypothesis is at most one. However, Bob has just taken a red cherry, so the lemma statement is satisfied.

Base case: Once a bounding half-plane emerges, Bob is guaranteed to obtain at least $\frac{1}{2}(\mathrm{r}(C) - \mathrm{s}(C))$ more red cherries by Lemma 13. We set U to be this bounding half-plane, thus $C = U \cap C$. We observe that

$$\mathrm{r}(C) = 2 \cdot \mathrm{t}(C) - \mathrm{s}(C)$$

or, in other words, that $\mathrm{s}(C) = 2 \cdot \mathrm{t}(U \cap C) - \mathrm{r}(C)$. Using this equality, Lemma 13 can be rewritten as: Bob is guaranteed to obtain at least

$$\frac{1}{2}(\mathrm{r}(C) - (2 \cdot \mathrm{t}(U \cap C) - \mathrm{r}(C)))$$

more red cherries. That is equal to $\mathrm{r}(C) - \mathrm{t}(U \cap C)$. This is what we wanted to prove. $\square$

Theorem 6 *From the sun $\mathcal{S}_k$, Bob will get at least $\frac{3k-1}{2} = \frac{3}{4}\mathrm{r}(\mathcal{S}_k) - \frac{1}{2}$ red cherries by using the Careful greedy tactic, no matter how Alice plays. As a result, we obtain the desired property $\mathcal{M}(\mathcal{S}_k) \geq \frac{3}{4}\mathrm{r}(\mathcal{S}_k) - \frac{1}{2}$. (restated)*
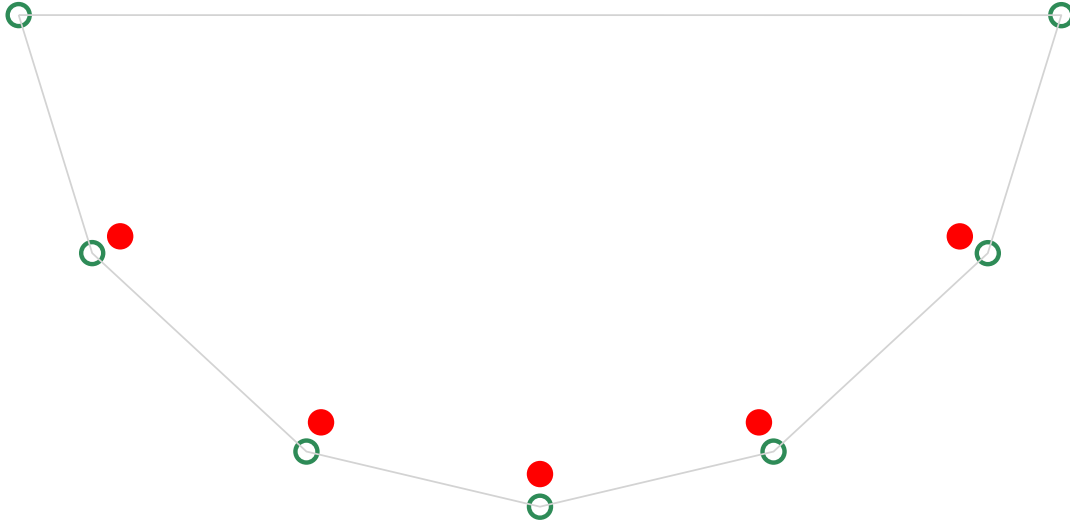
Figure 4: Moon ($\mathscr{L}_6$).

Proof. Given the properties of the sun $\mathcal{S}_k$, we see that any half-plane $U \in \mathcal{U}_{\mathcal{S}_k}$ has

$$t(U \cap \mathcal{S}_k) \leq \frac{k+1}{2}$$

and thus, by Lemma 14, Bob is guaranteed to obtain at least

$$r(\mathcal{S}_k) - t(U \cap \mathcal{S}_k) \geq 2k - \frac{k+1}{2} = \frac{3k-1}{2}$$

red cherries using his Careful greedy tactic. $\qquad\square$

4 Moon configuration

We present a family of even-sized cakes which we call the moon configuration, on which Bob can easily obtain all red cherries.

Definition 15 *Let $n \in \mathbb{N} \setminus \{0, 1\}$. We define the* moon $\mathscr{L}_n$ *(see Figure 4 for an example $\mathscr{L}_6$) as follows.*

Choose a centre point S and draw two circles; $\alpha(S, 1)$, called outer; and $\beta(S, 1-\varepsilon)$, called inner, where $0 < \varepsilon < 1 - \cos(90°/n)$. Draw n lines through S such that they are rotationally symmetric with a period of $180°/n$. Pick one line, called the main line. The main line defines two half-planes. The "upper" half-plane is discarded. The "other" half-plane will create the moon.

Place a green cherry at each intersection of any line with the outer circle α. Then place a red cherry at each intersection of any line except the main line with the inner circle β.

We have constructed a cake $\mathscr{L}_n$ where $r(\mathscr{L}_n) = n - 1$ and $g(\mathscr{L}_n) = n + 1$, while $Ex(\mathscr{L}_n) = G(\mathscr{L}_n)$.

Lemma 16 *The moon $\mathscr{L}_n$ has the following properties:*

1. *The removal of a green cherry will reveal a (single) red cherry.*

2. *The set $C \subset \mathscr{L}_n$, $|C| = |\mathscr{L}_n| - 2$, obtained by the removal of a green cherry followed by a removal of a red cherry, will be order-equivalent to $\mathscr{L}_{n-1}$.*

Proof. From the definition. $\qquad\square$

Definition 17 *In the convex grabbing game on a moon, we say that a player follows the* Simple greedy tactic *if the player chooses a move according to this rule:*

Is there any extremal red cherry?
  YES $\longrightarrow$ Take an extremal red cherry.
  NO $\longrightarrow$ Take any extremal cherry.

Theorem 18 *From the moon $\mathscr{L}_n$ if Bob follows the Simple greedy tactic he will obtain all red cherries. This results in $\mathcal{M}(\mathscr{L}_n) = r(\mathscr{L}_n) = n - 1$.*

Proof. We proceed by induction on n.

In the case with $n = 2$, the moon will have four cherries in total; three extremal green cherries and one red cherry lying inside their triangle; therefore, Alice can select any green cherry and Bob will take the only red cherry by the Simple greedy tactic.

Assume that $n > 2$ and the theorem holds for $\mathscr{L}_{n-1}$. For Alice's first move a_1, only green cherries are available, and so she will take one of them. By Lemma 16, this will reveal a single red cherry, hence Bob, by following the Simple greedy tactic, will always take this red cherry for his first move b_1.

By Lemma 16 again, the remaining set of cherries $\mathscr{L}_n \setminus \{a_1, b_1\}$ is order-equivalent to $\mathscr{L}_{n-1}$. Therefore, by the induction hypothesis, Bob will obtain all $n - 2$ red cherries from $\mathscr{L}_n \setminus \{a_1, b_1\}$; and, since he already took a red cherry in his first move, from $\mathscr{L}_n$ he obtains a total of $n - 2 + 1 = n - 1$ red cherries. $\qquad\square$

5   Miscellaneous

In order to obtain configurations which are favourable for Alice on even-sized and odd-sized cakes, a single red cherry can be placed outside the convex hull for the sun configuration and moon configuration respectively.

Adding the extra red cherry swaps the parity of our constructions. We obtain the following cakes $\mathcal{C}$ and $\mathcal{D}$ that are good for Alice.

Corollary 19 *Theorem 6 implies that there exists an even-sized cake $\mathcal{C}$ such that $\mathcal{M}(\mathcal{C}) \leq \frac{1}{4}\,\mathrm{r}(\mathcal{C}) + \frac{1}{4}$. And, in a similar manner, Theorem 18 implies that there exists an odd-sized cake $\mathcal{D}$ with any desired $\mathrm{r}(\mathcal{D}) \in \mathbb{N}$ such that $\mathcal{M}(\mathcal{D}) = 0$.*

Furthermore, we would like to know what the optimal gameplay looks like in general. We came up with the following conjectures regarding the tactics which each player could employ in order to select their next move.

Conjecture 1 *Greedy-move conjecture.*
If $\mathrm{Ex}(C) \cap \mathrm{R}(C) \neq \emptyset$, there exists a move that takes a red cherry from $\mathrm{Ex}(C) \cap \mathrm{R}(C)$ such that the move is optimal.

Note that the Careful greedy tactic (Definition 5) and the Simple greedy tactic (Definition 17) are refinements of what the Greedy-move conjecture says.

Conjecture 2 *Strong greedy-move conjecture.*
If $\mathrm{Ex}(C) \cap \mathrm{R}(C) \neq \emptyset$, then every move that takes a red cherry is optimal.

We will soon show that, even though we don't know whether the Greedy-move conjecture and the Strong greedy-move conjecture hold, we can easily prove that the former implies the latter (while the other implication holds trivially).

Conjecture 3 *No-reveal-move conjecture.*
If $\mathrm{Ex}(C) \cap \mathrm{R}(C) = \emptyset$ and we have a set of non-revealing moves $N = \{c \in \mathrm{Ex}(C) \mid \mathrm{Ex}(C \setminus \{c\}) \cap \mathrm{R}(C) = \emptyset\}$ that is not empty, then there exists $c \in N$ such that selecting c is optimal.

We later found a counterexample that disproved the No-reveal-move conjecture, which we will show soon.

Proposition 20 *The Greedy-move conjecture implies the Strong greedy-move conjecture.*

Proof. Consider the following set of red cherries $\mathrm{R}_{\mathrm{ext}}(C) = \{c \in \mathrm{R}(C) \mid c \notin \mathrm{conv}(\mathrm{G}(C))\}$. We prove the proposition by induction on $|\mathrm{R}_{\mathrm{ext}}(C)|$. If $|\mathrm{R}_{\mathrm{ext}}(C)| = 1$, both conjectures are trivially equivalent.

Assume that the Greedy-move conjecture holds in general and that the Strong greedy-move conjecture holds for up to $|\mathrm{R}_{\mathrm{ext}}(C)| = n - 1$ red cherries. We want to prove that the Strong greedy-move conjecture holds for up to $|\mathrm{R}_{\mathrm{ext}}(C)| = n$ red cherries. Seeking contradiction, assume that Alice has two possible moves taking a red cherry $c_i, c_j \in \mathrm{R}_{\mathrm{ext}}(C) \cap \mathrm{Ex}(C)$ that lead to different outcomes $\mathrm{B}(\mathbf{q})$.

If Alice starts by taking c_i, then by the induction hypothesis, c_j is among Bob's optimal moves. If Alice starts by taking c_j, then by the induction hypothesis, c_i is among Bob's optimal moves. Either way, this leaves $C' = C \setminus \{c_i, c_j\}$. In the first case, Alice ends up with $\mathrm{w}(c_i) + \mathrm{r}(C') - \mathcal{M}(C')$ points. In the second case, Alice ends up with $\mathrm{w}(c_j) + \mathrm{r}(C') - \mathcal{M}(C')$ points. Since they are both equal to $1 + \mathrm{r}(C') - \mathcal{M}(C')$, we obtain a contradiction. $\qquad\square$

Proposition 21 *The No-reveal-move conjecture is false.*

Proof. We show a sketch of the proof through the construction in Figure 5.

In this construction, the only non-revealing first move is to select $a_1 = c_2$. If, in the gameplay $\mathbf{q}$, Alice starts by taking this green cherry c_2, giving $\mathbf{q} = (c_2, b_1, \ldots, b_5)$, then Bob can select c_4, giving $\mathbf{q} = (c_2, c_4, a_2, \ldots, b_5)$, and they end up with $A(\mathbf{q}) = 1$ because the remaining part of the cake gives $\mathcal{M}(C \setminus \{c_2, c_4\}) = 3$.

However, if Alice selects $a_1 = c_1$ for her first move, she reveals two red cherries at the same time. Alice is therefore able to take a red cherry in her second move, after Bob moves. For Bob's first two moves, in order
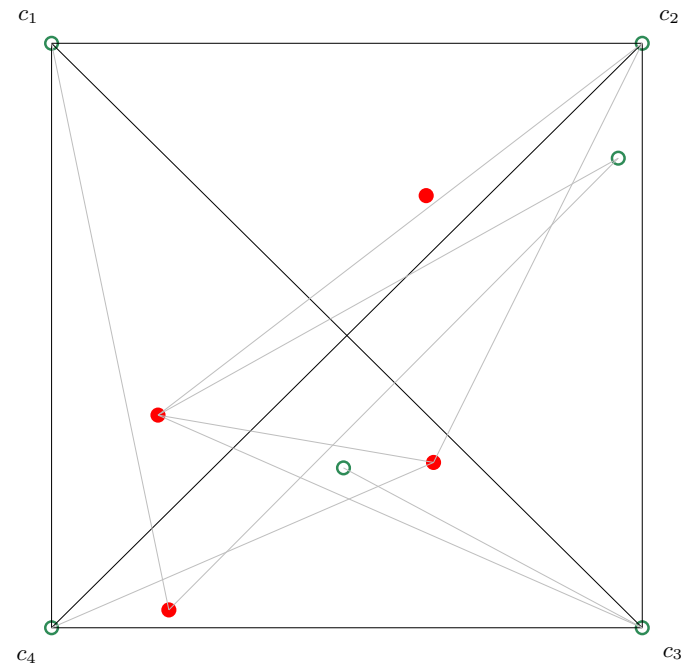

Figure 5: Our counterexample to the No-reveal-move conjecture, with lines added for visual aid.

for Alice to not obtain a second red cherry on her third move, he has to take one of the two red cherries which Alice revealed in her first move, and c_2; however, the order of Bob selecting these does not matter.

If Alice selects $a_3 = c_3$, she once again reveals two red cherries, and she is then guaranteed to be able to select a second red cherry in her fourth move. Therefore, by not selecting the non-revealing cherry in her first move, Alice is able to get a result of $A(\mathbf{q}) = 2$. □

6 Conclusion

We solved the open problem from [3] by providing a construction that builds an odd-sized cake $\mathcal{S}_k$ such that $\mathcal{M}(\mathcal{S}_k) - (\mathrm{r}(\mathcal{S}_k) - \mathcal{M}(\mathcal{S}_k)) \geq x$ for any $x \in \mathbb{N}$. It could be interesting to know whether the result can be made even stronger. Now consider the value:

$$\gamma = \limsup_{p \to \infty} \left(\max_{\substack{\text{odd-sized} \\ \text{cake } \mathcal{C}}} \left\{ \frac{\mathcal{M}(\mathcal{C})}{\mathrm{r}(\mathcal{C})} \;\middle|\; \mathrm{r}(\mathcal{C}) = p \right\} \right)$$

Our construction provides a lower bound $\gamma \geq \frac{3}{4}$. On the other hand, [3] shows that Alice can always obtain at least one red cherry on any odd-sized cake. However, this only gives the trivial upper bound $\gamma \leq 1$. We pose a new open question of determining the value γ.

Analysis of gameplays would be easier if the state space of need-to-be-considered gameplays were limited by knowing which moves are optimal (or which moves cannot be optimal) in certain situations. We therefore leave the reader with another open question: Does the Greedy-move conjecture hold?

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