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A Partial Solution to the Problem of Proper Reparametrization for Rational Surfaces☆

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Abstract

Given an algebraically closed field $\mathbb{K}$, and a rational parametrization $\mathcal{P}$ of an algebraic surface $\mathcal{V} \subset \mathbb{K}^3$, we consider the problem of computing a proper rational parametrization $\mathcal{Q}$ from $\mathcal{P}$ (*reparametrization problem*). More precisely, we present an algorithm that computes a rational parametrization $\mathcal{Q}$ of $\mathcal{V}$ such that the degree of the rational map induced by $\mathcal{Q}$ is less than the degree induced by $\mathcal{P}$. The properness of the output parametrization $\mathcal{Q}$ is analyzed. In particular, if the degree of the map induced by $\mathcal{Q}$ is one, then $\mathcal{Q}$ is proper and the reparametrization problem is solved. The algorithm works if at least one of two auxiliary parametrizations defined from $\mathcal{P}$ is not proper.

Keywords: Proper reparametrization, Rational surface, Degree of a rational map
2000 MSC: 14Q10, 68W30, 14E05

1. Introduction

In this paper, we deal with the *reparametrization problem*, that is, with the problem of computing a rational proper reparametrization of a given improperly parametrized algebraic surface. More precisely, given an algebraically closed field $\mathbb{K}$, and $\mathcal{P}(\bar{t}) \in \mathbb{K}(\bar{t})^3$, $\bar{t} = (t_1, t_2)$, a rational parametrization of a surface $\mathcal{V}$, the reparametrization problem consists in computing a proper

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parametrization of $\mathcal{V}$, $\mathcal{Q}(\bar{t}\,)$, and $R(\bar{t}\,) \in (\mathbb{K}(\bar{t}\,) \setminus \mathbb{K})^2$, such that $\mathcal{P}(\bar{t}\,) = \mathcal{Q}(R(\bar{t}\,))$.

Although it is known from Castelnuovo's Theorem that unirationality and rationality are equivalent over algebraically closed fields, only partial results approaching the problem algorithmically are known (see Pérez-Díaz (2006)). In particular, given an algebraically closed field $\mathbb{K}$, and $\mathcal{P}(\bar{t}\,)$ a rational parametrization of a surface $\mathcal{V}$, an algorithm is presented in Pérez-Díaz (2006) to determine whether there exists

$$R(\bar{t}\,) = (r_1(t_1), r_2(t_2)) = \left(\frac{r_{1,1}(t_1)}{r_{1,2}(t_1)}, \frac{r_{2,1}(t_2)}{r_{2,2}(t_2)}\right) \in (\mathbb{K}(t_1) \setminus \mathbb{K}) \times (\mathbb{K}(t_2) \setminus \mathbb{K}),$$

such that $\mathcal{P}(\bar{t}\,) = \mathcal{Q}(R(\bar{t}\,))$, and $\mathcal{Q}(\bar{t}\,)$ is a proper parametrization of $\mathcal{V}$. In the affirmative case, R and $\mathcal{Q}$ are computed.

The approach presented in this paper complements the results obtained in Pérez-Díaz (2006). More precisely, in Pérez-Díaz (2006), the reparametrization problem is solved for those surfaces parametrized by $\mathcal{P}$ that admit R of the form $R(\bar{t}\,) = (r_1(t_1), r_2(t_2)) \in (\mathbb{K}(t_1) \setminus \mathbb{K}) \times (\mathbb{K}(t_2) \setminus \mathbb{K})$, and such that $\mathcal{P} = \mathcal{Q}(R)$. In this paper, we deal with surfaces not necessarily satisfying this condition. In addition, for those surfaces for which a rational proper reparametrization is not found, we show how to decrease the degree of the rational map induced by the parametrization. For this purpose, we need that at least one of two auxiliary parametrizations defined from $\mathcal{P}$ is not proper.

The reparametrization problem, in particular when the variety is a curve or a surface, is especially interesting in some practical applications in Computer Aided Geometric Design (CAGD) where objects are often given and manipulated parametrically. In addition, proper parametrizations play an important role in many practical applications in CAGD, such as in visualization (see Hoffmann et al. (1997) or Hoschek and Lasser (1993)) or rational parametrization of offsets (see Arrondo et al. (1997)). Also, it is provided an implicitization approach based on resultants (see Cox et al. (1998), and Sendra and Winkler (2001)).

A direct approach to the reparametrization problem could consist in first implicitizing the parametrization (see Busé (2003), Cox (2001), Kotsireas (2004), Sendra and Winkler (2001)), and then to apply algorithms developed for instance in Cox et al. (1997), Goldman et al. (1984), González-Vega (1997), van Hoeij (1997), Hoffmann et al. (1997), Schicho (1998), Sendra and Winkler (1991), Sendra and Winkler (1997), to parametrize the implicit equation. However, some of these implicitization methods have difficulties in the presence of base points, or deal only with special cases or, although always valid, the computing time is not totally satisfactory. In Pérez-Díaz and Sendra (2008), an algorithm is presented, based on polynomial gcds and univariate resultants, that is always valid. However, even with this approach, the solution is, in most of cases, too time consuming (see Subsection 3.1).

Therefore, we would like to approach the problem by means of rational reparametrizations. By rational reparametrization we basically mean without implicitizing, or more formally, by finding a non–constant rational change of parameter, if it exists, that transforms the input parametrization into a new parametrization of the same curve or surface. Note that any reparametrization of a rational parametrization is again a parametrization of the same variety.

It is well known that for the case of curves, it is always possible to reparametrize an improperly parametrized curve in such a way that it becomes properly parametrized. In Alonso et al. (1995), Gutierrez et al. (2002), Pérez-Díaz (2006) and Sederberg (1986), some approaches are presented to compute a proper parametrization from a given improper one.

The approach presented in this paper deals with the surface case, and it is based on polynomial gcds and univariate resultants, which always work and whose computing time is very satisfactory (see Subsection 3.1). More precisely, the algorithm presented follows from the algorithm Proper Reparametrization for Space Curves developed in Section 2 and derived from the results in Pérez-Díaz (2006). The basic idea of the approach presented in this paper is to compute a reparametrization of two auxiliary parametrizations of two space curves, $\mathcal{P}_1$, $\mathcal{P}_2$, obtained directly from a given rational parametrization of the surface $\mathcal{P}$ defined over an algebraically closed field $\mathbb{K}$ (see Definition 1). Moreover, since when we compose two rational maps we multiply their degrees, we can deduce some properties that relate the degree of the rational map induced by the given parametrization $\mathcal{P}$ to the degree of the output parametrization $\mathcal{Q}$, and the degree of the rational maps induced by the two auxiliary parametrizations, $\mathcal{P}_1$ and P_2. Furthermore, we also show the relation of the degrees of the rational maps induced by $\mathcal{P}$ and $\mathcal{Q}$ with the degree of $R(\bar{t}\,) \in \mathbb{K}(\bar{t}\,)^2$ with respect to the variables t_1, t_2.

The structure of the paper is as follows: In Section 2, we present an algorithm for computing a proper reparametrization of an algebraic space curve. The algorithm is derived from the results presented in Pérez-Díaz (2006). In Section 3, we outline the algorithm that solves (in some cases) the problem of computing a rational proper reparametrization for a given improperly parametrized algebraic surface. More precisely, we introduce some auxiliary partial parametrizations defined from the input rational parametrization $\mathcal{P}$ (see Definition 1), and we prove a theorem (see Theorem 3), where we characterize the properness of $\mathcal{P}$ in terms of the properness of its partial parametrizations. The idea provided by this theorem will be used to derive the algorithm and to characterize the properness of the output reparametrization (see Theorem 4 and Corollary 3). In addition, for those surfaces for which we cannot find a rational proper reparametrization, if at least one of two auxiliary parametrizations defined from $\mathcal{P}$ is not proper, we show how to compute a rational reparametrization such that the degree of the rational map induced by it is less than the degree induced by the input parametrization $\mathcal{P}$ (see Corollary 2). Finally, we present the actual computing times of the implementation, and we show that the algorithm presented here is much more efficient and powerful, than first finding the implicit equation (see Subsection 3.1). Section 4 is devoted to summarizing the contributions of the paper, and we comment on how the results presented in the paper can easily be extended to a variety $\mathcal{V} \subset \mathbb{K}^n$ of dimension 2, rationally parametrized by

$$\mathcal{P}(\bar{t}) = \left(\frac{p_{1,1}(\bar{t})}{p_{1,2}(\bar{t})}, \ldots, \frac{p_{n,1}(\bar{t})}{p_{n,2}(\bar{t})}\right) \in \mathbb{K}(\bar{t})^n,$$

where $\mathbb{K}$ is an algebraically closed field.

2. Proper Reparametrization for Space Curves

The problem of proper reparametrization for curves can be stated as follows: given a field K (not necessarily an algebraically closed field), and a rational parametrization $\mathcal{P}(t) \in K(t)^n$ of an algebraic curve $\mathcal{C}$, find a rational proper parametrization $\mathcal{Q}(t) \in K(t)^n$ of $\mathcal{C}$, and a rational function $R(t) \in K(t) \setminus K$ such that $\mathcal{P}(t) = \mathcal{Q}(R(t))$.

A parametrization $\mathcal{P}$ of $\mathcal{C}$ is proper if and only if the map $\mathcal{P} : K \longrightarrow \mathcal{C} \subset K^n, t \longmapsto \mathcal{P}(t)$ is birational, or equivalently, if for almost every point on $\mathcal{C}$

and for almost all values of the parameter in K the mapping $\mathcal{P}$ is rationally bijective. The notion of properness can also be stated algebraically in terms of fields of rational functions. In fact, a rational parametrization $\mathcal{P}$ is proper if and only if the induced monomorphism $\phi_{\mathcal{P}}$ on the fields of rational functions

$$\phi_{\mathcal{P}} : K(\mathcal{C}) \longrightarrow K(t); R(x_1, \ldots, x_n) \longmapsto R(\mathcal{P}(t)).$$

is an isomorphism. Therefore, $\mathcal{P}$ is proper if and only if the mapping $\phi_{\mathcal{P}}$ is surjective, that is, if and only if $\phi_{\mathcal{P}}(K(\mathcal{C})) = K(\mathcal{P}(t)) = K(t)$. Thus, Lüroth's Theorem implies that any rational curve over K can be properly parametrized (see Abhyankar and Bajaj (1988), Sendra and Winkler (2001), van Hoeij (1994)).

Under these conditions, the birationality of $\phi_{\mathcal{P}}$, i.e. the properness of $\mathcal{P}(t)$, is characterized by $\deg(\phi_{\mathcal{P}}) = 1$, where $\deg(\phi_{\mathcal{P}})$ denotes the degree of the rational map $\phi_{\mathcal{P}}$. The degree of the rational map $\phi_{\mathcal{P}}$ is defined as the degree of the finite field extension $\phi_{\mathcal{P}}(K(\mathcal{C})) \subset K(t)$; that is, $\deg(\phi_{\mathcal{P}}) = [K(t) : \phi_{\mathcal{P}}(K(\mathcal{C}))]$ (see Harris (1995) and Shafarevich (1994)). In addition, the degree of a rational map is equal to the cardinality of the fibre of a generic element. That is, $\mathcal{F}_{\mathcal{P}}(P) = \mathcal{P}^{-1}(P) = \{t \in K \,|\, \mathcal{P}(t) = P\}$, where $\mathcal{F}_{\mathcal{P}}(P)$ is the fibre of a point $P \in \mathcal{C}$ (see Theorem 7, pp. 76 in Shafarevich (1994)). In the following, we refer to the degree of the rational map induced by a parametrization $\mathcal{P}$, $\deg(\phi_{\mathcal{P}})$, as the *mapping degree of* $\mathcal{P}$.

In this section, we present a preliminary algorithm for computing a proper reparametrization of an algebraic space curve. That is, given a parametrization $\mathcal{P}(t) \in K(t)^3$ of an algebraic space curve $\mathcal{C}$ over a field K, we find a rational proper parametrization $\mathcal{Q}(t) \in K(t)^3$ of $\mathcal{C}$, and a rational function $R(t) \in K(t) \setminus K$ such that $\mathcal{P}(t) = \mathcal{Q}(R(t))$. This algorithm is obtained from the results presented in Pérez-Díaz (2006) (where the case of plane curves is solved), and it will be used to derive the algorithm for surfaces in Section 3. The results presented in this section can easily be extended to curves in $K^n, n \geq 4$.

Notation

Let K be a field, and let $K^\star = K \setminus \{0\}$. If $\mathcal{C}$ is an affine rational plane curve, and $\mathcal{P}(t)$ is a rational affine parametrization of $\mathcal{C}$ over K, we write the components of $\mathcal{P}(t)$ as

$$\mathcal{P}(t) = \left(\frac{p_{1,1}(t)}{p_{1,2}(t)}, \frac{p_{2,1}(t)}{p_{2,2}(t)}, \frac{p_{3,1}(t)}{p_{3,2}(t)}\right) \in K(t)^3, \qquad \gcd(p_{i,1}, p_{i,2}) = 1, \;\; i = 1, 2, 3.$$

For simplicity, we assume that $p_{i,1}/p_{i,2}$, $i = 1, 2, 3$ are not constant. Note that, if for instance $p_{1,1}/p_{1,2} = \lambda \in K$, then we apply results in Pérez-Díaz (2006), and we compute a proper parametrization $\mathcal{Q}$ of $(p_{2,1}(t)/p_{2,2}(t),\ p_{3,1}(t)/p_{3,2}(t))$. Then, a proper parametrization of $\mathcal{C}$ is $(\lambda, \mathcal{Q}(t))$.

Associated with the given parametrization $\mathcal{P}$, we consider the polynomials

$$H_i^{\mathcal{P}}(t, s) = p_{i,1}(t)p_{i,2}(s) - p_{i,2}(t)p_{i,1}(s) \in K[t, s],\ \ i = 1, 2, 3,$$

and

$$S^{\mathcal{P}}(t, s) = \gcd(H_1^{\mathcal{P}}(t, s), H_2^{\mathcal{P}}(t, s), H_3^{\mathcal{P}}(t, s)) \in K[t, s].$$

The polynomial $S^{\mathcal{P}}$ plays an important role in deciding whether a parametrization $\mathcal{P}$ is proper; i.e. in studying whether the parametrization is injective for almost all parameter values. More precisely, $S^{\mathcal{P}}(t, s) \in K[t, s] \setminus K[t]$, and $\deg(\phi_{\mathcal{P}}) = \deg_t(S^{\mathcal{P}})$. Thus, $\mathcal{P}$ is proper if and only if, up to constants in $K^\star$, $S^{\mathcal{P}}(t, s) = t - s$ (see Pérez-Díaz (2006), Sederberg (1986) or Sendra and Winkler (2001)).

Under these conditions, we apply the results obtained in Pérez-Díaz (2006) (Sections 2 and 3) to derive an algorithm that computes a rational proper reparametrization of an improperly parametrized algebraic space curve. It is clear that taking into account that every space curve is birationally equivalent to a plane curve (see e.g. Theorem 6.5 in Walker (1950)), the results presented in Pérez-Díaz (2006) for plane algebraic curves can be applied to space curves. However, in the following algorithm we show that one does not need to compute the birationally equivalent plane curve, and the results in Pérez-Díaz (2006) can be extended to space curves. In addition, we note that the algorithm is based on polynomial gcds and univariate resultants, which always work and whose computing time is very satisfactory.

Algorithm Proper Reparametrization for Space Curves (PRSC).

INPUT: a field K, and a rational affine parametrization

$$\mathcal{P}(t) = \left(\frac{p_{1,1}(t)}{p_{1,2}(t)}, \frac{p_{2,1}(t)}{p_{2,2}(t)}, \frac{p_{3,1}(t)}{p_{3,2}(t)}\right) \in K(t)^3, \qquad \gcd(p_{i,1}, p_{i,2}) = 1$$

of an algebraic space curve $\mathcal{C}$.

OUTPUT: *a rational proper parametrization,* $\mathcal{Q}(t) \in K(t)^3$ *of* $\mathcal{C}$*, and* $R(t) \in K(t)$ *such that* $\mathcal{P}(t) = \mathcal{Q}(R(t))$*.*

1. Compute $H_j^{\mathcal{P}}(t,s) = p_{j,1}(t)p_{j,2}(s) - p_{j,1}(s)p_{j,2}(t), \quad j = 1,2,3.$
2. Determine $S^{\mathcal{P}}(t,s) = \gcd(H_1^{\mathcal{P}}(t,s), H_2^{\mathcal{P}}(t,s), H_3^{\mathcal{P}}(t,s)) = C_m(t)s^m + \cdots + C_0(t)$.
3. If $\deg(\phi_{\mathcal{P}}) = \deg_t(S^{\mathcal{P}}) = 1$, RETURN $\mathcal{Q}(t) = \mathcal{P}(t)$, and $R(t) = t$. Otherwise go to Step 4.
4. Compute a rational function $R(t) = \frac{C_i(t)}{C_j(t)} \in K(t)$, where $C_j(t)$, $C_i(t)$ are two of the polynomials obtained in Step 2 such that $\gcd(C_j, C_i) = 1$ and $C_jC_i \notin K^*$.
5. For $i = 1,2,3$, define the polynomials $G_i^{\mathcal{P}}(t,x_i) = x_ip_{i,2}(t) - p_{i,1}(t)$, and compute

$$L_i(s,x_i) = \mathrm{Res}_t(G_i^{\mathcal{P}}(t,x_i), sC_j(t) - C_i(t)) = (q_{i,2}(s)x_i - q_{i,1}(s))^{\deg(R)}.$$

6. RETURN the rational function $R(t) = C_i(t)/C_j(t) \in K(t)$, and the proper parametrization

$$\mathcal{Q}(t) = (q_{1,1}(t)/q_{1,2}(t),\ q_{2,1}(t)/q_{2,2}(t),\ q_{3,1}(t)/q_{3,2}(t)) \in K(t)^3.$$

Remark 1. *Observe that:*

1. *If* $\mathcal{P}$ *has coefficients in a smaller subfield* L *of* K*, then both* R *and* $\mathcal{Q}$ *also have coefficients in* L*.*
2. *The above algorithm works similarly for plane curves (see Pérez-Díaz (2006)). More precisely, given a rational affine parametrization* $\mathcal{P}(t) = (p_{1,1}(t)/p_{1,2}(t),\ p_{2,1}(t)/p_{2,2}(t))$*, in reduced form, of a plane algebraic curve* $\mathcal{C}$*, compute* $H_j^{\mathcal{P}}(t,s) = p_{j,1}(t)p_{j,2}(s) - p_{j,1}(s)p_{j,2}(t)$*,* $j = 1,2$*. Then, determine the polynomial* $S^{\mathcal{P}}(t,s) = \gcd(H_1^{\mathcal{P}}(t,s), H_2^{\mathcal{P}}(t,s))$*, and apply similarly steps* 3 *to* 6 *of the algorithm.*

3. *We also may apply the algorithm for $\mathcal{P}_s(t) \in (K(s))(t)^3$ (that is, for a parametrization of a space curve defined over the field $K(s)$). The output parametrization, $\mathcal{Q}_s(t)$ is in $(K(s))(t)^3$ and its inverse ${\mathcal{Q}_s}^{-1}$ exits and lies in $K(s, x_1, x_2, x_3)$. This observation will be used in Section 3.*

In order to complete this section, we illustrate Algorithm PRSC with an example.

Example 1. *Let $\mathcal{C}$ be a rational space curve over $\mathbb{C}$ defined by the parametrization*

$$\mathcal{P}(t) = \left(\frac{p_{1,1}(t)}{p_{1,2}(t)}, \frac{p_{2,1}(t)}{p_{2,2}(t)}, \frac{p_{3,1}(t)}{p_{3,2}(t)}\right) = \left(\frac{(3t^2-4t-1+9t^4)t^4}{(3t^2-4t-1)^2},\right.$$

$$\left.\frac{9t^4-24t^3+10t^2+8t+1+t^8}{t^8}, \frac{(3t^2-4t-1)t^4}{(9t^4-24t^3+10t^2+8t+1+4t^8)}\right) \in \mathbb{C}(t)^3.$$

In Step 1 of the algorithm, we compute the polynomials

$H_1^{\mathcal{P}}(t,s) = (s-t)(3s^3t^2-4s^3t-s^3+3s^2t^3-s^2t-4s^2t^2-st^2-4st^3-t^3)(27s^4t^2-9s^4-36s^4t+27s^2t^4+9s^2t^2-12s^2t-3s^2-36st^4-12st^2+16st+4s-3t^2+4t+1-9t^4)$,

$H_2^{\mathcal{P}}(t,s) = -(s-t)(3s^3t^2-4s^3t-s^3+3s^2t^3-s^2t-4s^2t^2-st^2-4st^3-t^3)(3s^4t^2-4s^4t-s^4+3s^2t^4-4st^4-t^4)$,

$H_3^{\mathcal{P}}(t,s) = -(s-t)(3s^3t^2-4s^3t-s^3+3s^2t^3-s^2t-4s^2t^2-st^2-4st^3-t^3)(4s^4t^4+3s^2-4s-1+12s^2t-16st-4t-9s^2t^2+12st^2+3t^2)$.

Next, we compute the polynomial $S^{\mathcal{P}}(t,s)$. We obtain

$$S^{\mathcal{P}}(t,s) = C_0(t) + C_1(t)s + C_2(t)s^2 + C_3(t)s^3 + C_4(t)s^4,$$

where $C_0(t) = t^4$, $C_1(t) = 4t^4$, $C_2(t) = -3t^4$, $C_3 = 0$, and $C_4 = 3t^2-4t-1$.

Since $\deg(\phi_{\mathcal{P}}) = \deg_t(S^{\mathcal{P}}) > 1$, we go to Step 4 of the algorithm, and we compute

$$R(t) = \frac{C_4(t)}{C_0(t)} = \frac{3t^2-4t-1}{t^4} \in \mathbb{C}(t).$$

Note that $\gcd(C_0, C_1) = 1$. *Now, we compute the polynomials*

$$L_1(s,x_1) = \mathrm{Res}_t(G_1^{\mathcal{P}}(t,x_1), sC_1(t) - C_0(t)) = (s^2x_1 - s - 9)^4,$$

$$L_2(s,x_2) = \mathrm{Res}_t(G_2^{\mathcal{P}}(t,x_2), sC_1(t) - C_0(t)) = (s^2 + 1 - x_2)^4,$$

$$L_3(s,x_3) = \mathrm{Res}_t(G_3^{\mathcal{P}}(t,x_3), sC_1(t) - C_0(t)) = (-s + 4x_3 + s^2x_3)^4, \quad \textit{where}$$

$G_i^{\mathcal{P}}(t,x_i) = x_i p_{i,2}(t) - p_{i,1}(t)$ *(see Step 5). Finally, in Step 6, the algorithm outputs the proper parametrization* $\mathcal{Q}(t)$*, and the rational function* $R(t)$*:*

$$\mathcal{Q}(t) = \left(\frac{t+9}{t^2},\, t^2+1,\, \frac{t}{4+t^2}\right) \in \mathbb{C}(t)^3, \qquad R(t) = \frac{3t^2-4t-1}{t^4} \in \mathbb{C}(t).$$

3. Proper Reparametrization for Surfaces

In Section 2, we dealt with the problem of computing a rational proper reparametrization of a given improperly parametrized algebraic space curve. For the case of surfaces, although it is known from Castelnuovo's Theorem that unirationality and rationality are equivalent over algebraically closed fields, the problem is not solved computationally. That is, there does not exist an algorithm that computes the proper reparametrization.

In this section, given an algebraically closed field $\mathbb{K}$, and $\mathcal{P}(\bar{t}) \in \mathbb{K}(\bar{t})^3$, $\bar{t} = (t_1, t_2)$, a rational parametrization of a surface $\mathcal{V}$, we compute a parametrization of $\mathcal{V}$, $\mathcal{Q}(\bar{t}) \in \mathbb{K}(\bar{t})^3$, and $R(\bar{t}) = (S(\bar{t}), T(S(\bar{t}), t_2))$, $S, T \in \mathbb{K}(\bar{t})$, such that

$$\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t})), \qquad \text{and} \quad \deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{Q}})\deg_{t_1}(S)\deg_{t_2}(T)$$

(see Theorem 4). From the above equality, we show that if $\mathcal{Q}$ is not proper and $\deg_{t_1}(S)\deg_{t_2}(T) \neq 1$, then $\deg(\phi_{\mathcal{Q}}) < \deg(\phi_{\mathcal{P}})$ (see Corollary 2). In addition, we establish under which conditions $\mathcal{Q}$ is proper (see Corollary 3).

Note that this approach can be applied to other problems within the framework of algebraic manipulations of parametrized algebraic surfaces, as for instance in the decomposition problem (see for instance Gutierrez et al. (2002)).

We start by introducing the notation that we will use throughout this section. In particular, we introduce some polynomials defined in Pérez-Díaz et al. (2002), Pérez-Díaz and Sendra (2004), and Pérez-Díaz and Sendra (2005) that play an important role in deciding whether a parametrization $\mathcal{P}$ is proper. Afterwards, we state a preliminary definition (Definition 1) and a theorem (Theorem 3) that will be used to derive the algorithm and in particular, to characterize the properness of the output reparametrization (see Theorem 4 and Corollary 3).

Notation

Let $\mathbb{K}$ be an algebraically closed field, and let $\mathbb{K}^\star = \mathbb{K} \setminus \{0\}$. In addition, if $\mathcal{V}$ is an affine rational surface, and $\mathcal{P}(\bar{t}\,)$, $\bar{t}\, = (t_1, t_2)$, is a rational affine parametrization of $\mathcal{V}$ over $\mathbb{K}$, we write its components as

$$\mathcal{P}(\bar{t}\,) = \left(\frac{p_{1,1}(\bar{t}\,)}{p_{1,2}(\bar{t}\,)}, \frac{p_{2,1}(\bar{t}\,)}{p_{2,2}(\bar{t}\,)}, \frac{p_{3,1}(\bar{t}\,)}{p_{3,2}(\bar{t}\,)}\right) \in \mathbb{K}(\bar{t}\,)^3, \quad \gcd(p_{i,1}, p_{i,2}) = 1,\ i = 1, 2, 3.$$

For simplicity, we assume without loss of generality that $p_{i,1}/p_{i,2}$, $i = 1, 2, 3$ are not constant. Note that, if for instance $p_{1,1}/p_{1,2} = \lambda \in \mathbb{K}$, then a proper parametrization of $\mathcal{V}$ is $\mathcal{Q}(t_1, t_2) = (\lambda, t_1, t_2)$, and then the problem is trivial. In addition, since $\mathcal{P}$ is a surface parametrization, we may assume without loss of generality that $\{\nabla\,(p_1/q_1)\,,\, \nabla\,(p_2/q_2)\}$ are linearly independent as vectors in $\mathbb{K}(\bar{t}\,)^2$. This assumption is needed to apply results in Pérez-Díaz and Sendra (2004) (see Theorem 1).

In the following, we use the notions of content and primitive part of a polynomial. Given a non-zero polynomial $a(\,\overline{x}\,) \in I[\,\overline{x}\,]$, where $\overline{x}\, = (x_1, \ldots, x_n)$ and I is a unique factorization domain, we denote by $\mathrm{pp}_{\overline{x}}(a)$ the primitive part of a with respect to $\overline{x}$, and by $\mathrm{Content}_{\overline{x}}(a)$ the content part of a with respect to $\overline{x}$. That is, $a(\,\overline{x}\,) = \mathrm{Content}_{\overline{x}}(a)\,\mathrm{pp}_{\overline{x}}(a)$, where $\mathrm{Content}_{\overline{x}}(a) \in I$ is just the gcd of all the coefficients of $a(\,\overline{x}\,)$ with respect to $\overline{x}$. Note that the gcd of all the coefficient of $\mathrm{pp}_{\overline{x}}(a)$ with respect to $\overline{x}$ is 1 (see Winkler (1996)).

Associated with the given parametrization $\mathcal{P}$, we consider the polynomials

$$H_j^{\mathcal{P}}(\bar{t}\,, \overline{s}\,) = p_{j,1}(\bar{t}\,)p_{j,2}(\overline{s}\,) - p_{j,2}(\bar{t}\,)p_{j,1}(\overline{s}\,) \in (\mathbb{K}[\,\overline{s}\,])[\bar{t}\,], \quad j = 1, 2, 3,$$

where $\overline{s}\, = (s_1, s_2)$, and $H_4^{\mathcal{P}}(\bar{t}\,) = \mathrm{lcm}(p_{1,2}, p_{2,2}, p_{3,2}) \in \mathbb{K}[\bar{t}\,]$. In addition, we also will use the polynomials

$$T_1^{\mathcal{P}}(t_1, \overline{s}\,) = \mathrm{pp}_{\overline{s}}(\mathrm{Content}_Z(\mathrm{Res}_{t_2}(H_1^{\mathcal{P}}, H_2^{\mathcal{P}} + ZH_3^{\mathcal{P}}))) \in \mathbb{K}[t_1, \overline{s}\,],$$

$$T_2^{\mathcal{P}}(t_2, \overline{s}) = \mathrm{pp}_{\overline{s}}(\mathrm{Content}_Z(\mathrm{Res}_{t_1}(H_1^{\mathcal{P}}, H_2^{\mathcal{P}} + ZH_3^{\mathcal{P}}))) \in \mathbb{K}[t_2, \overline{s}].$$

For a field $\mathbb{L}$ we denote by $\overline{\mathbb{L}}$ its algebraic closure.

Finally, for $i = 1, 2, 3$, we also consider the polynomials

$$G_i(\bar{t}, x_i) = p_{i,1}(\bar{t}) - x_i\, p_{i,2}(\bar{t}) \in (\mathbb{K}[x_i])[\bar{t}], \quad G_4(\bar{t}) = \mathrm{lcm}(p_{1,2}, p_{2,2}, p_{3,2}) \in \mathbb{K}[\bar{t}],$$

and

$$S_1^{\mathcal{P}}(t_1, \overline{x}) = \mathrm{pp}_{\overline{x}}(\mathrm{Content}_Z(\mathrm{Res}_{t_2}(G_1, G_2 + ZG_3))) \in \mathbb{K}[t_1, \overline{x}],$$

$$S_2^{\mathcal{P}}(t_2, \overline{x}) = \mathrm{pp}_{\overline{x}}(\mathrm{Content}_Z(\mathrm{Res}_{t_1}(G_1, G_2 + ZG_3))) \in \mathbb{K}[t_2, \overline{x}],$$

where the content is taken over the field of rational functions $\mathbb{K}(\mathcal{V})$, and where $\mathbb{K}$ is an algebraically closed field. Recall that if $f(x_1, x_2, x_3)$ is the defining polynomial of $\mathcal{V}$, then $\mathbb{K}(\mathcal{V})$ is the quotient field of $\mathbb{K}[x_1, x_2, x_3]/(f)$. Furthermore, arithmetic in the field $\mathbb{K}(\mathcal{V})$ can be executed by using the defining polynomial of $\mathcal{V}$ (see Sendra et al. (2007)).

Depending on the problem we are dealing with, we will use two different concepts of degree. For a rational function in reduced form $R = M/N \in \mathbb{K}(\overline{x})$, we denote the degree of R with respect to x_i as $\deg_{x_i}(R) = \max\{\deg_{x_i}(M), \deg_{x_i}(N)\}$. In addition, we denote by $\deg(\phi_{\mathcal{P}})$ the degree of the rational map induced by $\mathcal{P}$ (in the following, we refer to the degree of the rational map $\mathcal{P}$ as the *mapping degree* of $\mathcal{P}$). That is, $\phi_{\mathcal{P}} : \mathbb{K}^2 \longrightarrow \mathcal{V};\ \bar{t} \longmapsto \mathcal{P}(\bar{t})$, and the degree of $\phi_{\mathcal{P}}$ is defined as the degree of the finite field extension $\phi_{\mathcal{P}}(\mathbb{K}(V)) \subset \mathbb{K}(\bar{t})$; i.e. $\deg(\phi_{\mathcal{P}}) = [\mathbb{K}(\bar{t}) : \phi_{\mathcal{P}}(\mathbb{K}(V))]$ (see e.g. Harris (1995) pp.80, or Shafarevich (1994) pp.143). Note that a mapping degree which is less or equal than zero is nonsense. In addition, as an important result, we recall that the properness of $\mathcal{P}(\bar{t})$ is characterized by $\deg(\phi_{\mathcal{P}}) = 1$ (see Harris (1995) and Shafarevich (1994)). Also, we recall that the mapping degree is the cardinality of the fibre of a generic element (see Theorem 7, pp. 76 in Shafarevich (1994)). That is,

$$\mathcal{F}_{\mathcal{P}}(P) = \mathcal{P}^{-1}(P) = \{\bar{t} \in \mathbb{K}^2 \mid \mathcal{P}(\bar{t}) = P\},$$

where $\mathcal{F}_{\mathcal{P}}(P)$ is the fibre of a point $P \in \mathcal{C}$.

The polynomials $S_j^{\mathcal{P}}$ and $T_j^{\mathcal{P}}$, $j = 1, 2$, play an important role in deciding whether a parametrization $\mathcal{P}$ is proper; i.e. in studying whether the parametrization is injective for almost all parameter values (see Pérez-Díaz and Sendra (2004)). More precisely, under the conditions introduced in this section (in particular, $\mathbb{K}$ is an algebraically closed field), the following theorem is proved in Pérez-Díaz and Sendra (2004).

Theorem 1. *Let* $\mathbb{F} = \overline{\mathbb{K}(\,\overline{s}\,)}$. *Then:*

1. $\phi_{\mathcal{P}}^{-1}(\mathcal{P}(\,\overline{s}\,)) = \left\{ \,\overline{t} \in \mathbb{F}^2 \;\middle|\; H_i{}^{\mathcal{P}}(\,\overline{t}\,, \overline{s}\,) = 0,\;\; i \in \{1,2,3\},\;\; H_4^{\mathcal{P}}(\,\overline{t}\,) \neq 0 \right\}$ *and*
$$\phi_{\mathcal{P}}^{-1}(\,\overline{x}\,) = \left\{ \,\overline{t}\, \in \mathbb{F}^2 \;\middle|\; G_i(\,\overline{t}\,, x_i) = 0, i \in \{1,2,3\}, G_4(\,\overline{t}\,) \neq 0 \right\}.$$

2. *The polynomial* $T_i^{\mathcal{P}}$ *defines the* t_i*–coordinates of the points in* $\phi_{\mathcal{P}}^{-1}(\mathcal{P}(\,\overline{s}\,))$, *for* $i = 1, 2$.
The polynomial $S_i^{\mathcal{P}}$ *defines the* t_i*–coordinates of the points in* $\phi_{\mathcal{P}}^{-1}(\,\overline{x}\,)$, *for* $i = 1, 2$.

3. $\deg(\phi_{\mathcal{P}}) = \mathrm{Card}(\phi_{\mathcal{P}}^{-1}(\mathcal{P}(\,\overline{s}\,))) = \deg_{t_1}(T_1^{\mathcal{P}}(t_1, \overline{s}\,)) = \deg_{t_2}(T_2^{\mathcal{P}}(t_2, \overline{s}\,))$, *and*
$$\deg(\phi_{\mathcal{P}}) = \mathrm{Card}(\phi_{\mathcal{P}}^{-1}(\,\overline{x}\,)) = \deg_{t_1}(S_1^{\mathcal{P}}(t_1, \overline{x}\,)) = \deg_{t_2}(S_2^{\mathcal{P}}(t_2, \overline{x}\,)).$$

Remark 2. *In the following, we will need to compute the mapping degree of some particular rational maps. More precisely, let* $R(\,\overline{t}\,) = (t_1, R_2(\,\overline{t}\,)) \in \mathbb{K}(\,\overline{t}\,)^2$ *and* $\phi_R : \mathbb{K}^2 \to \mathbb{K}^2$. *Then* $\deg(\phi_R) = \deg_{t_2}(R_2)$; *a similar result holds if* $(R_1(\,\overline{t}\,), t_2)$ *(see Lemma 4.32 in Sendra et al. (2007)).*

In Definition 1, given a rational parametrization of a surface, $\mathcal{N}(\,\overline{t}\,) \in \mathbb{K}(\,\overline{t}\,)^3$, we introduce some auxiliary parametrizations over $\mathbb{K}(t_i)$ defined from $\mathcal{N}$. These auxiliary parametrizations will play an important role in the computation of the reparametrization.

Definition 1. *Let* $\mathcal{N}(\,\overline{t}\,) \in \mathbb{K}(\,\overline{t}\,)^3$. *We define the* partial parametrizations associated to $\mathcal{N}$ *as the parametrizations* $\mathcal{N}_i(t_j) := \mathcal{N}(\,\overline{t}\,) \in (\mathbb{K}(t_i))(t_j)^3$ *(that is,* $\mathcal{N}$ *is defined over* $\mathbb{K}(t_i)$*), for* $i, j \in \{1, 2\}$ *and* $i \neq j$.

Remark 3. *Observe that:*

1. *The partial parametrization* $\mathcal{N}_i(t_j)$ *(*$i \neq j$*) defines a space curve over* $\overline{\mathbb{K}(t_i)}$ *(we refer to this space curve as the partial space curve). Note that, since* $\mathcal{N}(\,\overline{t}\,)$ *is a surface parametrization its jacobian has rank 2, and therefore the gradient of* $\mathcal{N}_i(t_j)$ *(with respect to* t_j*) must have rank 1.*

2. *The partial space curve introduced above depends directly on the parametrization; that is, different parametrizations of the same surface may produce different partial space curves. However, if we are working with* $\mathcal{N}_i(t_j)$ *and we perform a reparametrization that only changes* t_j*, then the corresponding curves are equal.*
3. *Definition 1 can be stated for* $R(\overline{t}\,) \in \mathbb{K}(\overline{t}\,)^2$*. More precisely, given* $R(\overline{t}\,) \in \mathbb{K}(\overline{t}\,)^2$*, one may consider* $R_i(t_j) := R(\overline{t}\,) \in (\mathbb{K}(t_i))(t_j)^2$ *(that is,* R *is defined over* $\mathbb{K}(t_i)$*), for* $i, j \in \{1, 2\}$ *and* $i \neq j$*.*

In order to illustrate Definition 1, and Remark 3 (statements 1 and 2), we consider the following example.

Example 2. *Consider the parametrization* $\mathcal{N}(t_1, t_2) = (t_1, t_2, t_1 + t_2)$ *of the plane* $z = x + y$*. Observe that* $\mathcal{N}_1(t_2)$ *parametrizes the line* $\mathcal{C}_1^{\mathcal{N}}$ *in* $\overline{\mathbb{K}(t_1)}^3$ *defined by the equations* $\{z = x + y, x = t_1\}$ *and* $\mathcal{N}_2(t_1)$ *parametrizes the line* $\mathcal{C}_2^{\mathcal{N}}$ *in* $\overline{\mathbb{K}(t_2)}^3$ *defined by equation* $\{z = x + y, y = t_2\}$*. Now, consider the new parametrization* $\mathcal{M}(t_1, t_2) = (t_1 + 1, t_2 + 1, t_1 + t_2 + 2)$ *of the same plane. Then,* $\mathcal{C}_1^{\mathcal{M}}$ *is the line defined by* $\{z = x + y, x = t_1 + 1\}$*, and hence* $\mathcal{C}_1^{\mathcal{M}} \neq \mathcal{C}_1^{\mathcal{N}}$*.*

In Theorem 3, we characterize the properness of a given parametrization $\mathcal{P}$ of a surface $\mathcal{V}$ in terms of the properness of its partial parametrizations. From Definition 1, we have that these partial parametrizations are given as $\mathcal{P}_i(t_j) := \mathcal{P}(\overline{t}\,) \in (\mathbb{K}(t_i))(t_j)^3$ (that is, $\mathcal{P}$ is defined over $\mathbb{K}(t_i)$), for $i, j \in \{1, 2\}$ and $i \neq j$. Under these conditions, it is proved that $\mathcal{P}$ is birational if and only if $\mathcal{P}_i$, $i = 1, 2$ are proper (that is, are invertible) and the inverse of each $\mathcal{P}_i$, say $\mathcal{P}_i^{-1}$, lies in $\mathbb{K}(\overline{x}\,)$ but $\mathcal{P}_i^{-1} \notin \mathbb{K}(t_i)$, where $\overline{x} = (x_1, x_2, x_3)$.

To start, we need to prove some preliminary results (Proposition 1 and Corollary 1). For this purpose, first we state the following theorem that is proved in Sendra and Winkler (2001). Which asserts that when computing a resultant, the implicit equation defining a plane curve appears to the power of the mapping degree (similar results on implicitization can be found in Chionh and Goldman (1992) and Cox et al. (1998)). Afterwards, in Lemma 1, we apply this theorem to a special given parametrization.

Theorem 2. *Let $\mathcal{P}(t) = (p_{1,1}(t)/p_{1,2}(t),\, p_{2,1}(t)/p_{2,2}(t)) \in \mathbb{K}(t)^2$ be a parametrization in reduced form of a plane curve $\mathcal{C}$ defined over the algebraically closed field $\mathbb{K}$. Then, up to multiplication by elements in $\mathbb{K}$,*

$$\mathrm{Res}_t(p_{1,1}(t) - Xp_{1,2}(t), p_{2,1}(t) - Yp_{2,2}(t)) = f(X,Y)^{\deg(\phi_{\mathcal{P}})} \in \mathbb{K}[X,Y],$$

where $f(X,Y) \in \mathbb{K}[X,Y]$ is the defining polynomial of $\mathcal{C}$.

Lemma 1. *Let*

$$\mathcal{Q}_1(t_2) = \left(\frac{p_{1,1}(\bar{t}\,)}{p_{1,2}(\bar{t}\,)}, \frac{p_{2,1}(\bar{t}\,) - x_2p_{2,2}(\bar{t}\,)}{-(p_{3,1}(\bar{t}\,) - x_3p_{3,2}(\bar{t}\,))}\right) \in (\mathbb{K}(t_1,x_2,x_3))(t_2)^2$$

be a parametrization of a plane curve $\mathcal{C}$ defined over $\mathbb{K}(t_1,x_2,x_3)$. Then,

$$\mathrm{Res}_{t_2}(G_1, G_2 + ZG_3) = f(x_1,Z,t_1,x_2,x_3)^{\deg(\phi_{\mathcal{P}_1})}h(t_1) \in \mathbb{K}[x_1,Z,t_1,x_2,x_3],$$

where $f(x_1,Z,t_1,x_2,x_3) \in (\mathbb{K}[t_1,x_2,x_3])[x_1,Z]$ is the defining polynomial of $\mathcal{C}$ (f is a polynomial in the variables x_1, Z with coefficients in $\mathbb{K}[t_1,x_2,x_3]$), and $h(t_1) \in \mathbb{K}[t_1]$.

Proof. We apply Theorem 2 to the parametrization

$$\mathcal{Q}_1(t_2) = \left(\frac{q_{1,1}(\bar{t}\,)}{q_{1,2}(\bar{t}\,)}, \frac{q_{2,1}(\bar{t}\,)}{q_{2,2}(\bar{t}\,)}\right) = \left(\frac{p_{1,1}(\bar{t}\,)}{p_{1,2}(\bar{t}\,)}, \frac{p_{2,1}(\bar{t}\,) - x_2p_{2,2}(\bar{t}\,)}{-(p_{3,1}(\bar{t}\,) - x_3p_{3,2}(\bar{t}\,))}\right) \in \mathbb{F}(t_2)^2,$$

where the algebraically closed field is given by $\mathbb{F} := \overline{\mathbb{K}(t_1,x_2,x_3)}$ (observe that $(\mathbb{K}(t_1,x_2,x_3))(t_2)^2 \subset \mathbb{F}(t_2)^2$). In addition, we consider $X = x_1$, $Y = Z$. Thus, $\mathcal{R}(x_1,Z,t_1,x_2,x_3) :=$

$$\mathrm{Res}_{t_2}(G_1, G_2 + ZG_3) = f(x_1,Z,t_1,x_2,x_3)^{\deg(\phi_{\mathcal{Q}_1})}h(t_1,x_2,x_3) \in \mathbb{F}[x_1,Z].$$

Note that

$$q_{1,1}(\bar{t}\,) - Xq_{1,2}(\bar{t}\,) = p_{1,1}(\bar{t}\,) - x_1p_{1,2}(\bar{t}\,) = G_1(t_1,t_2,x_1), \qquad \text{and}$$

$$q_{2,1}(\bar{t}\,) - Yq_{2,2}(\bar{t}\,) = (p_{2,1}(\bar{t}\,) - x_2p_{2,2}(\bar{t}\,)) + Z(p_{3,1}(\bar{t}\,) - x_3p_{3,2}(\bar{t}\,)) =$$
$$G_2(t_1,t_2,x_2) + ZG_3(t_1,t_2,x_3),$$

and both polynomials have coefficients in $\mathbb{K}$. Hence $\mathcal{R} \in \mathbb{K}[x_1,Z,t_1,x_2,x_3]$. In particular, $f(x_1,Z,t_1,x_2,x_3) \in (\mathbb{K}[t_1,x_2,x_3])[x_1,Z] = \mathbb{K}[x_1,Z,t_1,x_2,x_3]$, and $h(t_1,x_2,x_3) \in \mathbb{K}[t_1,x_2,x_3]$. Under these conditions, the following properties hold:

- $h \in \mathbb{K}[t_1]$. Indeed: let us assume that there exist $\alpha, \beta \in \overline{\mathbb{K}(t_1)}$ such that $h(t_1, \alpha, \beta) = 0$. Then, taking into account that the leading coefficient of G_1 with respect to the variable t_2 is in $\mathbb{K}[t_1, x_1]$ (and then, it does not vanish at α, β), and that $\mathcal{R}(x_1, Z, t_1, \alpha, \beta) = 0$, we get that there exists $\gamma \in \overline{\mathbb{K}(t_1, Z, x_1)}$ such that

  $$G_1(t_1, \gamma, x_1) = G_2(t_1, \gamma, \alpha) + ZG_3(t_1, \gamma, \beta) = 0.$$

  Since G_1 does not depend on Z, one deduces that $\gamma \in \overline{\mathbb{K}(t_1, x_1)}$. In addition, since $\alpha, \beta \in \overline{\mathbb{K}(t_1)}$, one gets that $G_2(t_1, \gamma, \alpha) = G_3(t_1, \gamma, \beta) = 0$, and $\gamma \in \overline{\mathbb{K}(t_1)}$. Hence, from

  $$G_1(t_1, \gamma, x_1) = p_{1,1}(t_1, \gamma) - x_1 p_{1,2}(t_1, \gamma) = 0,$$

  we obtain that $p_{1,1}(t_1, \gamma) = 0$ and $p_{1,2}(t_1, \gamma) = 0$ which implies that $\mathrm{Res}_{t_1}(p_{1,1}, p_{1,2})(\gamma) = 0$. Therefore, since $\mathrm{Res}_{t_1}(p_{1,1}, p_{1,2}) \in \mathbb{K}[t_2]$, one gets that $\gamma \in \mathbb{K}$ (note that $\mathbb{K}$ is an algebraically closed field). Thus, $t_2 - \gamma$ divides $\gcd(p_{1,1}, p_{1,2})$ which is impossible since $\gcd(p_{1,1}, p_{1,2}) = 1$.

- $\deg(\phi_{\mathcal{Q}_1}) = \deg(\phi_{\mathcal{P}_1})$. Indeed: taking into account that

  $$\mathcal{F}_{\mathcal{Q}_1}(P) = \mathcal{Q}_1^{-1}(P) = \{t \in \mathbb{K} \mid \mathcal{Q}_1(t) = P\},$$

  and that the mapping degree is equal to the cardinality of the fibre of a generic element (see Section 2), we analyze the equality $\mathcal{Q}_1(t_2) = \mathcal{Q}_1(s_2)$. For this purpose, we observe that $\mathcal{Q}_1(t_2) = \mathcal{Q}_1(s_2)$ iff

  $$p_{1,1}(\bar{t}\,)p_{1,2}(t_1, s_2) = p_{1,1}(t_1, s_2)p_{1,2}(\bar{t}\,), \qquad \text{(I)}$$

  $$p_{2,1}(\bar{t}\,)p_{3,1}(t_1, s_2) - x_3 p_{2,1}(\bar{t}\,)p_{3,2}(t_1, s_2) - x_2 p_{2,2}(\bar{t}\,)p_{3,1}(t_1, s_2) +$$
  $$x_3 x_2 p_{2,2}(\bar{t}\,)p_{3,2}(t_1, s_2) = p_{2,1}(t_1, s_2)p_{3,1}(\bar{t}\,) - x_3 p_{2,1}(t_1, s_2)p_{3,2}(\bar{t}\,) -$$
  $$x_2 p_{2,2}(t_1, s_2)p_{3,1}(\bar{t}\,) + x_3 x_2 p_{2,2}(t_1, s_2)p_{3,2}(\bar{t}\,) \qquad \text{(II)}.$$

  Equality (I) implies that the solutions on the variable t_2 are in $\overline{\mathbb{K}(t_1, s_2)}$. Thus, equality (II) is equivalent to the following three equalities:

  $$p_{2,2}(t_1, s_2)p_{3,1}(\bar{t}\,) = p_{2,2}(\bar{t}\,)p_{3,1}(t_1, s_2),$$

  $$p_{2,1}(t_1, s_2)p_{3,2}(\bar{t}\,) = p_{2,1}(\bar{t}\,)p_{3,2}(t_1, s_2),$$

$$p_{2,2}(t_1, s_2)p_{3,2}(\,\overline{t}\,) = p_{2,2}(\,\overline{t}\,)p_{3,2}(t_1, s_2).$$

Since $p_{i,j} \neq 0$, the above equalities are equivalent to

$$p_{2,1}(\,\overline{t}\,)p_{2,2}(t_1, s_2) = p_{2,1}(t_1, s_2)p_{2,2}(\,\overline{t}\,),$$

$$p_{3,1}(\,\overline{t}\,)p_{3,2}(t_1, s_2) = p_{3,1}(t_1, s_2)p_{3,2}(\,\overline{t}\,).$$

Therefore, $\mathcal{Q}_1(t_2) = \mathcal{Q}_1(s_2)$ if and only if $\mathcal{P}_1(t_2) = \mathcal{P}_1(s_2)$.

Taking into account these properties, we conclude that

$$\mathrm{Res}_{t_2}(G_1, G_2 + ZG_3) = f(x_1, Z, t_1, x_2, x_3)^{\deg(\phi_{\mathcal{P}_1})}h(t_1). \qquad \square$$

Proposition 1. *For $j = 1, 2$, $S_j^{\mathcal{P}}(t_j, \overline{x}\,) = f_j(\,\overline{x}\,)^{\deg(\phi_{\mathcal{P}_j})}$, where $f_j \in \mathbb{K}[t_j, \overline{x}\,] \setminus \mathbb{K}[t_j]$, and $d_j := \deg_{t_j}(f_j) \geq 1$.*

Proof. Let us prove the proposition for $j = 1$ (for $j = 2$, one reasons similarly). From Lemma 1, we get that $S_1^{\mathcal{P}}(t_1, \overline{x}\,) =$

$$\mathrm{pp}_{\overline{x}}(\mathrm{Content}_Z(\mathrm{Res}_{t_2}(G_1, G_2{+}ZG_3))) = \mathrm{pp}_{\overline{x}}(\mathrm{Content}_Z(f))^{\deg(\phi_{\mathcal{P}_1})} \in \mathbb{K}[t_1, \overline{x}\,],$$

where the content is taken over $\mathbb{K}(\mathcal{V})$, since $\mathcal{V}$ is the surface parametrized by $\mathcal{P}$ (see the notation in Section 3), and $\mathbb{K}$ is an algebraically closed field. Let

$$f_1 := \mathrm{pp}_{\overline{x}}(\mathrm{Content}_Z(f))^{\deg(\phi_{\mathcal{P}_1})}.$$

Clearly $f_1 \in \mathbb{K}[t_1, \overline{x}\,] \setminus \mathbb{K}[t_1]$, and let us prove that $\deg_{t_1}(f_1) \geq 1$. For this purpose, we assume that $\deg_{t_1}(f_1) = 0$ (i.e. $f_1 \in \mathbb{K}[\,\overline{x}\,]$), and we write

$$f = \mathrm{Content}_Z(f)\mathrm{pp}_Z(f) = \mathrm{pp}_{\overline{x}}(\mathrm{Content}_Z(f))\mathrm{Content}_{\overline{x}}(\mathrm{Content}_Z(f))\mathrm{pp}_Z(f)$$

$$= f_1(\,\overline{x}\,)u(t_1)g(Z, t_1, \overline{x}\,), \qquad \text{and}$$

$$\mathrm{Res}_{t_2}(G_1, G_2 + ZG_3) = f_1(\,\overline{x}\,)^{\deg(\phi_{\mathcal{P}_1})}g(Z, t_1, \overline{x}\,)^{\deg(\phi_{\mathcal{P}_1})}u(t_1)h(t_1).$$

Since $G_j(\,\overline{t}\,, \mathcal{P}) = 0$, $j = 1, 2, 3$, then $\mathrm{Res}_{t_2}(G_1, G_2 + ZG_3)(t_1, \mathcal{P}) = 0$, which implies that $f_1(\mathcal{P}) = 0$ or $g(Z, t_1, \mathcal{P}) = 0$. If $g(Z, t_1, \mathcal{P}) = 0$, the defining polynomial of the surface $\mathcal{V}$ divides g. This is impossible, because $g(Z, t_1, \overline{x}\,) = \mathrm{pp}_Z(f)$, and then all the factors of g depend on Z. Thus, $f_1(\mathcal{P}) = 0$, where $f_1(\,\overline{x}\,) \in \mathbb{K}[\overline{x}]$. Let $\tilde{f} \in \mathbb{K}[\,\overline{x}\,]$ be an irreducible factor of f_1 such that $\tilde{f}(\mathcal{P}) = 0$. Since $0 = \tilde{f}(\mathcal{P}) = \tilde{f}(\mathcal{P}_1) = \tilde{f}(\mathcal{P}_2)$, it follows that $\mathcal{P}_j(t_i) \in (\mathbb{K}(t_j))(t_i)^3$, $i, j = 1, 2$, $i \neq j$ parametrizes $\mathcal{V}$ which is impossible. $\square$

Remark 4. *We observe that:*

1. *Since* $S_j^{\mathcal{P}}(t_j, \overline{x}\,) =$

$$\mathrm{pp}_{\overline{x}}(\mathrm{Content}_Z(\mathrm{Res}_{t_i}(G_1, G_2 + ZG_3))) = f_j(\,\overline{x}\,)^{\deg(\phi_{\mathcal{P}_j})} \in \mathbb{K}[t_j, \overline{x}\,],$$

   one easily deduces that the polynomial f_j *does not have factors depending only on the variable* t_j.
2. *From the proof of Proposition 1, we have that*

$$f_j(\mathcal{P}) = f_j(\mathcal{P}_j) = f_j(\mathcal{P}_i) = 0,\ i, j \in \{1, 2\},\ i \neq j.$$

In the following, we consider the polynomials $f_j \in (\mathbb{K}[t_j])[\,\overline{x}\,]$, $j = 1, 2$, introduced in Proposition 1, and we derive a corollary where the mapping degree of $\mathcal{P}$ is related to the mapping degree of $\mathcal{P}_1$, and the mapping degree of $\mathcal{P}_2$.

Corollary 1. $\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{P}_1})d_1 = \deg(\phi_{\mathcal{P}_2})d_2$. *In addition,*

$$d_2 = r\deg(\phi_{\mathcal{P}_1}), \quad d_1 = r\deg(\phi_{\mathcal{P}_2}), \quad \textit{and} \ \ r = \frac{\gcd(d_1, d_2)}{\gcd(\deg(\phi_{\mathcal{P}_1}), \deg(\phi_{\mathcal{P}_2}))} \in \mathbb{N}.$$

Proof. From Proposition 1, $S_j^{\mathcal{P}}(t_j, \overline{x}\,) = f_j(\,\overline{x}\,)^{\deg(\phi_{\mathcal{P}_j})}$, $j = 1, 2$, where $f_j \in \mathbb{K}[t_j, \overline{x}\,] \setminus \mathbb{K}[t_j]$, and $d_j := \deg_{t_j}(f_j) \geq 1$. Then, taking into account that $\deg(\phi_{\mathcal{P}}) = \deg_{t_j}(S_j^{\mathcal{P}})$ (see Theorem 1), we deduce that

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{P}_1})d_1 = \deg(\phi_{\mathcal{P}_2})d_2.$$

Now, for $i, j \in \{1, 2\}$, $i \neq j$, we consider the polynomial (see Section 2),

$$S^{\mathcal{P}_i}(t_j, s_j) = \gcd(H_1^{\mathcal{P}_i}(t_j, s_j), H_2^{\mathcal{P}_i}(t_j, s_j), H_3^{\mathcal{P}_i}(t_j, s_j)) \in (\mathbb{K}[t_i])[t_j, s_j],$$

where

$$H_k^{\mathcal{P}_i}(t_j, s_j) = p_{k,1}(t_i, t_j)p_{k,2}(t_i, s_j) - p_{k,1}(t_i, s_j)p_{k,2}(t_i, t_j),\ k = 1, 2, 3.$$

$S^{\mathcal{P}_i}(t_j, s_j)$ divides $f_j(\mathcal{P}_i(s_j))$ in the variable t_j; indeed, let $\alpha \in \overline{\mathbb{K}(s_j)}$ be such that $S^{\mathcal{P}_i}(\alpha, s_j) = 0$. Since $S^{\mathcal{P}_i}$ does not have factors in $(\mathbb{K}[t_i])[t_j]$ (see the definition and the properties of the polynomial $S^{\mathcal{P}}$ in Section 2), then $\alpha \notin \mathbb{K}[t_i]$. In addition, note that $p_{k,2}(t_i, \alpha) \neq 0$; otherwise $p_{k,1}(t_i, \alpha)p_{k,2}(t_i, s_j) = 0$

which implies that $p_{k,1}(t_i,\alpha)=0$, and then, since the polynomials $p_{k,1},p_{k,2}$ have a common root, one gets that $\mathrm{Res}_{t_i}(p_{k,1},p_{k,2})(\alpha)=0$. This is impossible because $\mathrm{Res}_{t_i}(p_{k,1},p_{k,2})\in\mathbb{K}[t_j]$ and then all its roots are in $\mathbb{K}$ (note that $\mathbb{K}$ is an algebraically closed field) but $\alpha\notin\mathbb{K}$.
Hence, $S^{\mathcal{P}_i}(\alpha,s_j)=0$ implies that $\mathcal{P}_i(s_j)=\mathcal{P}_i(\alpha)$, and then

$$f_j(\mathcal{P}_i(s_j))_{|t_j=\alpha}=f_j(\mathcal{P}_i(t_j))_{|t_j=\alpha}=0;$$

note that $f_j(\mathcal{P}_i(t_j))=0$ for every value of t_j where $\mathcal{P}_i$ is defined (see Remark 4). Therefore, $S^{\mathcal{P}_i}(t_j,s_j)$ divides $f_j(\mathcal{P}_i(s_j))$ in the variable t_j. Thus, there exists $r_{j,i}\in\mathbb{N}$ such that

$$d_j=r_{j,i}\deg_{t_j}(S^{\mathcal{P}_i}(t_j,s_j))=r_{j,i}\deg(\phi_{\mathcal{P}_i}),\ \ i,j\in\{1,2\},\ i\neq j$$

(see Section 2). Since $\deg(\phi_{\mathcal{P}_1})d_1=\deg(\phi_{\mathcal{P}_2})d_2$, one gets that $r_{1,2}=r_{2,1}$. Let $r:=r_{1,2}=r_{2,1}$. Then, we conclude that

$$d_2=r\deg(\phi_{\mathcal{P}_1}),\quad d_1=r\deg(\phi_{\mathcal{P}_2}).$$

Finally, one easily obtains that

$$r=\frac{\gcd(d_1,d_2)}{\gcd(\deg(\phi_{\mathcal{P}_1}),\deg(\phi_{\mathcal{P}_2}))}\in\mathbb{N}.\qquad\square$$

Theorem 3. *The following statements are equivalent:*

1. *$\mathcal{P}$ is birational*
2. *$\mathcal{P}_i$ is proper, $\mathcal{P}_i^{-1}\in\mathbb{K}(\overline{x})$ and $\mathcal{P}_i^{-1}\notin\mathbb{K}(t_i)$, for $i=1,2$.*
3. *$d_i=1$, for $i=1,2$.*

If $d_j=1$, then $\mathcal{P}_i^{-1}=-a_{0,i}/a_{1,i}\in\mathbb{K}(\overline{x})$, where $f_j(\overline{x})=a_{0,i}(\overline{x})+a_{1,i}(\overline{x})t_j$, $i,j\in\{1,2\}$, $i\neq j$.

Proof.
(1) $\Rightarrow$ (2) Since $\mathcal{P}$ is birational, there exist $\mathcal{Q}(\overline{x})=(q_1(\overline{x}),q_2(\overline{x}))\in\mathbb{K}(\overline{x})^2$ such that $\mathcal{Q}(\mathcal{P}(\overline{t}))=\overline{t}$. Then,

$$q_j(\mathcal{P}_i(t_j))=t_j,\quad \text{for } i,j\in\{1,2\},\quad\text{and}\quad i\neq j.$$

Therefore, $\mathcal{P}_i$ is proper and $\mathcal{P}_i^{-1} = q_j \in \mathbb{K}(\overline{x})$, $q_j \notin \mathbb{K}(t_i)$, for $i, j \in \{1, 2\}$, and $i \neq j$.

$\boxed{(2) \Rightarrow (1)}$ Since $\mathcal{P}_i$ is proper and $\mathcal{P}_i^{-1} \in \mathbb{K}(\overline{x})$, $\mathcal{P}_i^{-1} \notin \mathbb{K}(t_i)$, $i = 1, 2$, then $\mathcal{P}_i^{-1} = -a_{0,i}(\overline{x})/a_{1,i}(\overline{x})$, where $a_{k,i} \in \mathbb{K}[\overline{x}]$ for $k = 0, 1$, $a_{0,i}a_{1,i} \neq 0$, $\gcd(a_{0,i}, a_{1,i}) = 1$, and $a_{0,i}, a_{1,i}$ are not both in $\mathbb{K}$. Thus, we may write $g_j(\mathcal{P}_i) = 0$, where

$$g_j(\overline{x}) = a_{0,i}(\overline{x}) + a_{1,i}(\overline{x})t_j, \qquad i, j \in \{1, 2\}, \quad i \neq j.$$

Under these conditions, we observe that $g_j(\mathcal{P}_i) = g_j(\mathcal{P}) = 0$. Thus, we conclude that

$$-\frac{a_{0,i}(\mathcal{P})}{a_{1,i}(\mathcal{P})} = t_j, \quad i, j \in \{1, 2\}, \ \ i \neq j.$$

Therefore,

$$\mathcal{P}^{-1}(\overline{x}) = \left(\mathcal{P}_2^{-1}(\overline{x}), \mathcal{P}_1^{-1}(\overline{x})\right) = \left(-\frac{a_{0,2}(\overline{x})}{a_{1,2}(\overline{x})}, -\frac{a_{0,1}(\overline{x})}{a_{1,1}(\overline{x})}\right).$$

$\boxed{(1) \Longleftrightarrow (3)}$ Apply Corollary 1.

Finally, we prove that if $d_j = 1$, then $\mathcal{P}_i^{-1} = -a_{0,i}/a_{1,i} \in \mathbb{K}(\overline{x})$, where $f_j(\overline{x}) = a_{0,i}(\overline{x}) + a_{1,i}(\overline{x})t_j$, $i, j \in \{1, 2\}$, $i \neq j$. Indeed, by Proposition 1, we have that

$$S_j^{\mathcal{P}}(t_j, \overline{x}) = f_j(\overline{x})^{\deg(\phi_{\mathcal{P}_j})},$$

where $f_j \in \mathbb{K}[t_j, \overline{x}] \setminus \mathbb{K}[t_j]$, and $d_j = \deg_{t_j}(f_j) = 1$. Therefore, we may write (see Remark 4)

$$f_j(\overline{x}) = a_{0,i}(\overline{x}) + a_{1,i}(\overline{x})t_j, \text{ where } a_{k,i} \in \mathbb{K}[\overline{x}],\ k = 0, 1, \ \ a_{0,i}a_{1,i} \neq 0,$$

and $a_{0,i}$, $a_{1,i}$ are not both constant. Since $f_j(\mathcal{P}_i) = 0$, we get that $\mathcal{P}_i^{-1} = -a_{0,i}/a_{1,i} \in \mathbb{K}(\overline{x})$, because $\mathcal{P}_i^{-1}(\mathcal{P}_i(t_j)) = t_j$. □

Many methods for solving the properness problem and inversion problem for a rational parametrization of a surface defined over an algebraically closed field, $\mathcal{P}$, have previously been presented (see e.g. Pérez-Díaz et al. (2002) and Pérez-Díaz and Sendra (2004); see also Theorem 1).

Observe that now, from Theorem 3, we may deal with these problems by using the auxiliary partial parametrizations, $\mathcal{P}_1, \mathcal{P}_2$ (see Definition 1). More precisely, the properness problem can be solved by checking the degree of the gcd of three polynomials (see Step 2 in Algorithm PRSC). In order to compute the inverse of these partial parametrizations, one may apply, for instance, the results presented in Sendra et al. (2007) (see Section 4). Observe that one also has to check whether $\mathcal{P}_j^{-1}$ lies in $\mathbb{K}(\overline{x})$ and $\mathcal{P}_i^{-1} \notin \mathbb{K}(t_i)$.

Using this idea, in the following we present an algorithm whose goal is to properly reparametrize the partial parametrizations, $\mathcal{P}_i$, $i = 1, 2$. If the inverse of each partial reparametrization lies in $\mathbb{K}(\overline{x})$ but not in $\mathbb{K}(t_i)$, then the algorithm outputs a proper reparametrization for $\mathcal{P}$. Otherwise, at least the mapping degree of $\mathcal{P}$ is decreased. That is, the algorithm outputs a rational parametrization $\mathcal{Q}(\bar{t}) \in \mathbb{K}(\bar{t})^3$ of $\mathcal{V}$, and $R(\bar{t}) \in \mathbb{K}(\bar{t})^2$ such that $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$, and $\deg(\phi_{\mathcal{Q}}) < \deg(\phi_{\mathcal{P}})$ (see Corollary 2). In addition, if some additional properties hold (see Corollary 3), then $\mathcal{Q}$ is proper.

In addition, in Theorem 4, we derive some properties that relate the mapping degree of $\mathcal{P}$ with the mapping degree of $\mathcal{Q}$, and the degree of R. More precisely, we prove that

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{Q}}) \deg_{t_1}(S) \deg_{t_2}(T),$$

where

$$R(\bar{t}) = (S(\bar{t}), T(S(\bar{t}), t_2)), \quad S, T \in \mathbb{K}(\bar{t}).$$

In order to state the algorithm, we remind the reader that $\mathcal{N}_i(t_j) \in (\mathbb{K}(t_i))(t_j)^3$, for $i, j \in \{1, 2\}$ and $i \neq j$, denotes the partial parametrizations associated to a rational parametrization $\mathcal{N}(\bar{t}) \in \mathbb{K}(\bar{t})^3$ of a surface (see Definition 1). We will apply to the partial parametrizations Algorithm PRSC. Observe that the partial parametrizations are defined over the field $\mathbb{K}(t_i)$ (see statement 3 in Remark 1) and then, we may apply them the Algorithm PRSC.

In addition, given $R(\bar{t}) \in \mathbb{K}(\bar{t})^2$, we will consider $R_i(t_j) \in (\mathbb{K}(t_i))(t_j)^2$, for $i, j \in \{1, 2\}$ and $i \neq j$ (see statement 3 in Remark 3). Similarly, if we have $S(\bar{t}) \in \mathbb{K}(\bar{t})$, we will consider $S_i(t_j) \in (\mathbb{K}(t_i))(t_j)$, for $i, j \in \{1, 2\}$ and $i \neq j$.

Algorithm Reparametrization for Surfaces (RS).

INPUT: an algebraically closed field $\mathbb{K}$, and a rational affine parametrization

$$\mathcal{P}(\bar{t}) = \left(\frac{p_{1,1}(\bar{t})}{p_{1,2}(\bar{t})}, \frac{p_{2,1}(\bar{t})}{p_{2,2}(\bar{t})}, \frac{p_{3,1}(\bar{t})}{p_{3,2}(\bar{t})}\right) \in \mathbb{K}(\bar{t})^3, \quad \gcd(p_{i,1}, p_{i,2}) = 1$$

of an algebraic surface $\mathcal{V}$.

OUTPUT: *a rational parametrization* $\mathcal{Q}(\bar{t}) \in \mathbb{K}(\bar{t})^3$ *of* $\mathcal{V}$, *and* $R(\bar{t}) \in \mathbb{K}(\bar{t})^2$ *such that* $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$, *and* $1 \leq \deg(\phi_{\mathcal{Q}}) < \deg(\phi_{\mathcal{P}})$.

1. Check whether $\mathcal{P}_1$ and $\mathcal{P}_2$ are proper (apply Steps 1 and 2 of Algorithm PRSC). In the affirmative case, RETURN the message "*you cannot apply the algorithm*". Otherwise, go to Step 2 if $\mathcal{P}_2$ is not proper, or go to Step 3 if $\mathcal{P}_1$ is not proper.
2. If $\mathcal{P}_2$ is not proper do:
   - 2.1. Apply Algorithm PRSC to $\mathcal{P}_2$. [*This algorithm returns a parametrization* $\mathcal{M}(\bar{t}) \in \mathbb{K}(\bar{t})^3$, *and* $S(\bar{t}) \in \mathbb{K}(\bar{t})$ *such that the partial parametrization associated to* $\mathcal{M}$, $\mathcal{M}_2(t_1) \in (\mathbb{K}(t_2))(t_1)^3$, *is proper and* $S_2(t_1) \in (\mathbb{K}(t_2))(t_1)$ *satisfies* $\mathcal{P}_2(t_1) = \mathcal{M}_2(S_2(t_1))$.]
   - 2.2. Check whether the partial parametrization associated to $\mathcal{M}$, $\mathcal{M}_1(t_2) \in (\mathbb{K}(t_1))(t_2)^3$, is proper (apply Steps 1 and 2 of Algorithm PRSC). In the affirmative case, RETURN $\mathcal{Q} := \mathcal{M}$, and $R(\bar{t}) := (S(\bar{t}), t_2)$. Otherwise, go to Step 2.3.
   - 2.3. Apply Algorithm PRSC to the partial parametrization associated to $\mathcal{M}$, $\mathcal{M}_1(t_2)$. [*This algorithm returns a parametrization* $\mathcal{Q}(\bar{t}) \in \mathbb{K}(\bar{t})^3$, *and* $T(\bar{t}) \in \mathbb{K}(\bar{t})$ *such that the partial parametrization associated to* $\mathcal{Q}$, $\mathcal{Q}_1(t_2) \in (\mathbb{K}(t_1))(t_2)^3$, *is proper and* $T_1(t_2) \in (\mathbb{K}(t_1))(t_2)$ *satisfies* $\mathcal{M}_1(t_2) = \mathcal{Q}_1(T_1(t_2))$.]
   - 2.4. Check whether the partial parametrization associated to $\mathcal{Q}$, $\mathcal{Q}_2(t_1) \in (\mathbb{K}(t_2))(t_1)^3$, is proper (apply Steps 1 and 2 of Algorithm PRSC). In the affirmative case, RETURN the reparametrization $\mathcal{Q}$, and $R(\bar{t}) := (S(\bar{t}), T(S(\bar{t}), t_2))$. Otherwise, RETURN the reparametrization $\mathcal{Q}$, $R(\bar{t}) := (S(\bar{t}), T(S(\bar{t}), t_2))$, and the message "*you may apply the algorithm again (Step 2) to* $\mathcal{Q}_2$)".
3. If $\mathcal{P}_1$ is not proper, apply Step 2.3 to $\mathcal{P}$ and $\mathcal{P}_1$.

Remark 5. *Observe that:*

1. *Note that if* $\deg(\phi_{\mathcal{P}_i}) = 1$ *for* $i = 1, 2$*, the algorithm does not start.*
2. *The algorithm returns a rational reparametrization* $\mathcal{Q}$ *of* $\mathcal{P}$*. More precisely, we get a rational parametrization* $\mathcal{Q}$ *of* $\mathcal{V}$*, and* $R(\bar{t}) \in \mathbb{K}(\bar{t})^2$ *such that* $\mathcal{P} = \mathcal{Q}(R)$*. The output parametrization* $\mathcal{Q}$ *may not be proper (see statement 3) but it holds that* $1 \leq \deg(\phi_{\mathcal{Q}}) < \deg(\phi_{\mathcal{P}})$ *(see Corollary 2), if* $\deg(\phi_{\mathcal{P}_i}) \neq 1$ *for some* $i = 1, 2$ *(see statement 1).*
3. *In Corollary 3, we show under which conditions the output parametrization is proper that is, whether* $\deg(\phi_{\mathcal{Q}}) = 1$ *.*
4. *In order to apply* Algorithm PRSC*, we need that the space curves defined by the auxiliary parametrizations* $\mathcal{P}_i$ *are over a field. Observe that in our situation, they are defined over the field* $\mathbb{K}(t_i)$*. Hence,* Algorithm RS *can be applied for an input surface defined over a field not necessarily algebraically closed. However, the results presented above (Theorems 1, 2 and 3, Lemma 1, Proposition 1, and Corollary 1) are valid over algebraically closed fields.*
5. *Note that the algorithm does not loop forever. If it starts, it always outputs a reparametrization* $\mathcal{Q}$ *satisfying* $1 \leq \deg(\phi_{\mathcal{Q}}) < \deg(\phi_{\mathcal{P}})$ *(see Corollary 2). In addition, if* $\mathcal{Q}_2$ *is not proper, one may apply the algorithm again to further decrease the mapping degree of* $\mathcal{Q}$ *(see Step 2.4). If the algorithm can be applied (see statement 1) till we get* $\deg(\phi_{\mathcal{Q}}) = 1$*, the we obtain a proper reparametrization and then, the reparametrization problem is solved.*

Next, we illustrate Algorithm RS with three examples, where different steps (depending on the input parametrization) are applied.

Example 3. *Let* $\mathcal{V}$ *be a rational surface defined over the field of the complex numbers,* $\mathbb{C}$*, by the parametrization*

$$\mathcal{P}(\bar{t}) = \left(\frac{p_{1,1}(\bar{t})}{p_{1,2}(\bar{t})}, \frac{p_{2,1}(\bar{t})}{p_{2,2}(\bar{t})}, \frac{p_{3,1}(\bar{t})}{p_{3,2}(\bar{t})}\right) = \left(\frac{t_2t_1(t_1^2+t_2^2-t_1t_2)}{(t_1+t_2)^2}, t_2, \frac{t_2(t_1^3+t_2^2t_1-t_2t_1^2+t_1^2+2t_1t_2+t_2^2)}{(t_1+t_2)^2}\right) \in \mathbb{C}(\bar{t})^3.$$

By applying Theorem 1, one may check that $\deg(\phi_{\mathcal{P}}) = 3$. *Now, we apply* Algorithm RS.
For this purpose, in Step 1, *we apply* Algorithm PRSC, *and we find that* $S^{\mathcal{P}_2}(t_1, s_1) =$

$$t_2(s_1-t_1)(s_1^2t_1^2+2s_1^2t_1t_2+s_1^2t_2^2-2s_1t_2^2t_1-s_1t_2^3+2s_1t_2t_1^2+t_2^4+t_1^2t_2^2-t_1t_2^3) \in (\mathbb{C}[t_2])[t_1, s_1]$$

which implies that $\mathcal{P}_2(t_1)$ *is not proper (in fact,* $\deg(\phi_{\mathcal{P}_2}) = \deg_{t_1}(S^{\mathcal{P}_2}) = 3$*). Thus, we go to* Step 2 *and we apply* Algorithm PRSC *to* $\mathcal{P}_2$. *We obtain*

$$S_2(t_1) = \frac{q_1(\bar{t}\,)}{q_2(\bar{t}\,)} = \frac{-t_2^2t_1(t_1^2 + t_2^2 - t_1t_2)}{(t_1 + t_2)^2} \in (\mathbb{C}[t_2])[t_1].$$

Furthermore, we determine the polynomials

$$L_i(s_1, x_i) = \mathrm{Res}_{t_1}(G_i^{\mathcal{P}_2}(\bar{t}\,, x_i), q_2(\bar{t}\,)s_1-q_1(\bar{t}\,)) = (m_{i,2}(s_1)x_i-m_{i,1}(s_1))^{\deg_{t_1}(S)},$$

where $G_i^{\mathcal{P}_2}(\bar{t}\,, x_i) = x_ip_{i,2}(\bar{t}\,) - p_{i,1}(\bar{t}\,)$, *for* $i = 1, 2, 3$, *and we get* $\mathcal{M}(\bar{t}\,) =$

$$\left(\frac{m_{1,1}(\bar{t}\,)}{m_{1,2}(\bar{t}\,)}, \frac{m_{2,1}(\bar{t}\,)}{m_{2,2}(\bar{t}\,)}, \frac{m_{3,1}(\bar{t}\,)}{m_{3,2}(\bar{t}\,)}\right) = \left(\frac{-t_1}{t_2}, t_2, \frac{t_2^2 - t_1}{t_2}\right) \in \mathbb{C}(\bar{t}\,)^3.$$

Now, in Step 2.2 *of the algorithm, we apply* Algorithm PRSC *to* $\mathcal{M}_1(t_2) \in (\mathbb{C}(t_1))(t_2)^3$, *and we find that*

$$S^{\mathcal{M}_1}(t_2, s_2) = s_2 - t_2 \in (\mathbb{C}[t_1])[t_2, s_2].$$

Thus, since $\deg(\phi_{\mathcal{M}_1}) = \deg_{t_2}(S^{\mathcal{M}_1}) = 1$, *we get that* $\mathcal{M}_1$ *is proper. Then,* Algorithm RS *outputs the parametrization* $\mathcal{Q}(\bar{t}\,) = \mathcal{M}(\bar{t}\,)$, *and*

$$R(\bar{t}\,) = (S(\bar{t}\,), t_2) = \left(\frac{-t_2^2t_1(t_1^2 + t_2^2 - t_1t_2)}{(t_1 + t_2)^2}, t_2\right) \in \mathbb{C}(\bar{t}\,)^2.$$

One may check that $\mathcal{V}$ *is the plane defined by the equation* $z = x + y$.

Example 4. *Let* $\mathcal{V}$ *be a rational surface defined over the field of the complex numbers,* $\mathbb{C}$, *by the parametrization*

$$\mathcal{P}(\bar{t}\,) = \left(\frac{p_{1,1}(\bar{t}\,)}{p_{1,2}(\bar{t}\,)}, \frac{p_{2,1}(\bar{t}\,)}{p_{2,2}(\bar{t}\,)}, \frac{p_{3,1}(\bar{t}\,)}{p_{3,2}(\bar{t}\,)}\right) = \left(\frac{-t_1^6(t_2^2 + t_1^2)^3(t_1^2t_2^2 + t_1^4 - 2)}{t_2^2(t_1^2t_2^2 + t_1^4 - 1)},\right.$$

$$-\frac{(t_1^2t_2^2+t_1^4-1)t_1^2(t_2^2+t_1^2)}{t_2^2(t_1^2t_2^2+t_1^4-2)}, \frac{(t_1^4t_2^4+2t_1^6t_2^2+t_1^8+t_2^4t_1^2+t_2^2t_1^4-t_2^2)}{t_2^2}\Bigg) \in \mathbb{C}(\,\overline{t}\,)^3.$$

By applying Theorem 1, one may check that $\deg(\phi_{\mathcal{P}}) = 8$. *Now, we apply* Algorithm RS.

For this purpose, in Step 1, *we apply* Algorithm PRSC, *and we find that*

$$S^{\mathcal{P}_2}(t_1, s_1) = (s_1 - t_1)(s_1 + t_1)(s_1^2 + t_2^2 + t_1^2) \in (\mathbb{C}[t_2])[t_1, s_1]$$

which implies that $\mathcal{P}_2(t_1)$ *is not proper (in fact,* $\deg(\phi_{\mathcal{P}_2}) = \deg_{t_1}(S^{\mathcal{P}_2}) = 4$*). Thus, we go to* Step 2 *and we apply* Algorithm PRSC *to* $\mathcal{P}_2$. *We obtain* $S_2(t_1) = -t_1^2t_2^2 - t_1^4 \in (\mathbb{C}[t_2])[t_1]$. *Furthermore, we determine the polynomials*

$$L_i(s_1, x_i) = \mathrm{Res}_{t_1}(G_i^{\mathcal{P}_2}(\,\overline{t}\,, x_i), s_1 - S_2(t_1)) = (m_{i,2}(s_1)x_i - m_{i,1}(s_1))^{\deg_{t_1}(S)},$$

where $G_i^{\mathcal{P}_2}(\,\overline{t}\,, x_i) = x_ip_{i,2}(\,\overline{t}\,) - p_{i,1}(\,\overline{t}\,)$, *for* $i = 1, 2, 3$, *and we get* $\mathcal{M}(\,\overline{t}\,) =$

$$\left(\frac{m_{1,1}(\,\overline{t}\,)}{m_{1,2}(\,\overline{t}\,)}, \frac{m_{2,1}(\,\overline{t}\,)}{m_{2,2}(\,\overline{t}\,)}, \frac{m_{3,1}(\,\overline{t}\,)}{m_{3,2}(\,\overline{t}\,)}\right) = \left(\frac{t_1^3(2+t_1)}{t_2^2(t_1+1)}, \frac{t_1(t_1+1)}{t_2^2(2+t_1)}, \frac{(-t_2^2 - t_1t_2^2 + t_1^2)}{t_2^2}\right).$$

Now, in Step 2.2 *of the algorithm, we apply* Algorithm PRSC *to* $\mathcal{M}_1(t_2) \in (\mathbb{C}(t_1))(t_2)^3$, *and we find that*

$$S^{\mathcal{M}_1}(t_2, s_2) = s_2^2 - t_2^2 \in (\mathbb{C}[t_1])[t_2, s_2].$$

Thus, since $\deg(\phi_{\mathcal{M}_1}) = \deg_{t_2}(S^{\mathcal{M}_1}) = 2$, *we get that* $\mathcal{M}_1$ *is not proper. Then, we go to* Step 2.3. *We apply* Algorithm PRSC *to* $\mathcal{M}_1$, *and we compute* $T_1(t_2) = t_2^2 \in (\mathbb{C}[t_1])[t_2]$, *and the polynomials*

$$L_i(s_2, x_i) = \mathrm{Res}_{t_2}(G_i^{\mathcal{M}_1}(\,\overline{t}\,, x_i), s_2 - T_1(t_2)) = (q_{i,2}(s_2)x_i - q_{i,1}(s_2))^{\deg_{t_2}(T)},$$

where $G_i^{\mathcal{M}_1}(\,\overline{t}\,, x_i) = x_im_{i,2}(\,\overline{t}\,) - m_{i,1}(\,\overline{t}\,)$, *for* $i = 1, 2, 3$. *We obtain* $\mathcal{Q}(\,\overline{t}\,) =$

$$\left(\frac{q_{1,1}(\,\overline{t}\,)}{q_{1,2}(\,\overline{t}\,)}, \frac{q_{2,1}(\,\overline{t}\,)}{q_{2,2}(\,\overline{t}\,)}, \frac{q_{3,1}(\,\overline{t}\,)}{q_{3,2}(\,\overline{t}\,)}\right) = \left(\frac{t_1^3(2+t_1)}{t_2(t_1+1)}, \frac{t_1(t_1+1)}{t_2(2+t_1)}, -\frac{-t_1^2 + t_2 + t_1t_2}{t_2}\right).$$

Finally, in Step 2.4 *of the algorithm, we apply* Algorithm PRSC *to* $\mathcal{Q}_2(t_1) \in (\mathbb{C}(t_2))(t_1)^3$. *We get that*

$$S^{\mathcal{Q}_2}(t_1, s_1) = s_1 - t_1 \in (\mathbb{C}[t_2])[t_1, s_1]$$

which implies that $\mathcal{Q}_2$ *is proper. Therefore,* Algorithm RS *outputs the parametrization* $\mathcal{Q}(\,\overline{t}\,)$, *and*

$$R(\,\overline{t}\,) = (S(\,\overline{t}\,), T(S(\,\overline{t}\,), t_2)) = (-t_1^2t_2^2 - t_1^4, t_2^2) \in \mathbb{C}(\,\overline{t}\,)^2.$$

In the following example, we consider a parametrization $\mathcal{P}$ of a surface such that $\mathcal{P}_1$ is not proper. Then, we go to Step 3 of the Algorithm RS, and we apply Steps 2.3 and 2.4. Afterwards, we apply the algorithm again to decrease further the mapping degree of the output parametrization.

In this situation, we have that $\mathcal{P} = \mathcal{Q}(t_1, T)$ (output of Step 2.4), and $\mathcal{Q} = \mathcal{M}(S, t_2)$ (output of Step 2.2). Therefore,

$$\mathcal{P} = \mathcal{Q}(t_1, T) = \mathcal{M}(S, t_2)(t_1, T) = \mathcal{M}(R), \quad \text{where } R = (S(t_1, T), T).$$

Example 5. *Let $\mathcal{V}$ be a rational surface defined over $\mathbb{C}$ by the parametrization*

$$\mathcal{P}(\overline{t}\,) = \left(\frac{p_{1,1}(\overline{t}\,)}{p_{1,2}(\overline{t}\,)}, \frac{p_{2,1}(\overline{t}\,)}{p_{2,2}(\overline{t}\,)}, \frac{p_{3,1}(\overline{t}\,)}{p_{3,2}(\overline{t}\,)}\right) \in \mathbb{C}(\overline{t}\,)^3, \quad \textit{where}$$

$p_{1,1} = 51t_2^6t_1^4 + 102t_2^4t_1^2 + 73t_2^2 - 90t_2^8t_1^8 - 360t_2^6t_1^6 - 540t_2^4t_1^4 - 360t_2^2t_1^2 - 90 + 93t_2^4$,
$p_{2,1} = -9000t_2^2t_1^2 - 31500t_2^4t_1^4 + 67275t_2^6t_1^4 + 29550t_2^4t_1^2 - 78750t_2^8t_1^8 - 1123 + 5475t_2^2 + 1395t_2^4 - 6789t_2^6 - 63000t_2^6t_1^6 - 63000t_2^{10}t_1^{10} - 31500t_2^{12}t_1^{12} + 3825t_2^{14}t_1^{12} + 22950t_2^{12}t_1^{10} + 59025t_2^{10}t_1^8 + 83100t_2^8t_1^6 - 9000t_2^14t_1^{14} - 1125t_2^{16}t_1^{16} - 4743t_2^{10}t_1^4 - 9486t_2^8t_1^2 + 1395t_2^{12}t_1^8 + 5580t_2^{10}t_1^6 + 8370t_2^8t_1^4 + 5580t_2^6t_1^2$,
$p_{3,2} = 25t_2^8t_1^8 + 100t_2^6t_1^6 + 150t_2^4t_1^4 + 100t_2^2t_1^2 + 25 - 31t_2^4$,
$p_{1,2} = p_{2,2} = p_{3,1} = 1$.

Using Theorem 1, one may check that $\deg(\phi_{\mathcal{P}}) = 32$. *Now, we apply* Algorithm RS.
For this purpose, in Step 1*, we apply* Algorithm PRSC*, and we find that*

$$S^{\mathcal{P}_1}(t_2, s_2) = (s_2 - t_2)(s_2 + t_2) \in (\mathbb{C}[t_1])[t_2, s_2]$$

which implies that $\mathcal{P}_1(t_2)$ is not proper (in fact, $\deg(\phi_{\mathcal{P}_1}) = \deg_{t_2}(S^{\mathcal{P}_1}) = 2$). Therefore, we go to Step 3*. Thus, in* Step 2.3*, we apply* Algorithm PRSC *to the partial parametrization $\mathcal{P}_1$. We obtain $T_1(t_2) = -t_2^2 \in (\mathbb{C}[t_1])[t_2]$. Moreover, we determine the polynomials*

$$L_i(s_2, x_i) = \mathrm{Res}_{t_1}(G_i^{\mathcal{P}_1}(\overline{t}\,, x_i), s_2 - T_1(t_2)) = (m_{i,2}(s_2)x_i - m_{i,1}(s_2))^{\deg_{t_2}(T)},$$

where $G_i^{\mathcal{P}_2}(\overline{t}\,, x_i) = x_i p_{i,2}(\overline{t}\,) - p_{i,1}(\overline{t}\,)$, for $i = 1, 2, 3$. We obtain

$$\mathcal{Q}(\overline{t}\,) = \left(\frac{q_{1,1}(\overline{t}\,)}{q_{1,2}(\overline{t}\,)}, \frac{q_{2,1}(\overline{t}\,)}{q_{2,2}(\overline{t}\,)}, \frac{q_{3,1}(\overline{t}\,)}{q_{3,2}(\overline{t}\,)}\right) \in \mathbb{C}(\overline{t}\,)^3, \qquad \textit{where}$$

$q_{1,1} = -90 - 540t_2^2t_1^4 + 360t_2t_1^2 - 90t_2^4t_1^8 + 102t_2^2t_1^2 + 360t_1^6t_2^3 + 93t_2^2 - 73t_2 - 51t_1^4t_2^3$,
$q_{2,1} = -31500t_1^{12}t_2^6 - 1125t_2^8t_1^{16} - 3825t_1^{12}t_2^7 - 31500t_2^2t_1^4 + 4743t_2^5t_1^4 + 6789t_2^3 - 5580t_2^5t_1^6 - 9486t_1^2t_2^4 - 78750t_2^4t_1^8 + 29550t_2^2t_1^2 - 5580t_1^2t_2^3 + 83100t_2^4t_1^6 + 63000t_1^6t_2^3 + 22950t_1^{10}t_2^+63000t_2^5t_1^{10} - 59025t_2^5t_1^8 + 9000t_1^{14}t_2^7 - 67275t_1^4t_2^3 - 1123 - 5475t_2 + 1395t_1^8t_2^6 + 8370t_2^4t_1^4 + 1395t_2^2 + 9000t_2t_1^2$,
$q_{3,2} = -100t_1^6t_2^3 + 25t_2^4t_1^8 + 25 - 100t_2t_1^2 + 150t_2^2t_1^4 - 31t_2^2$,
$q_{1,2} = q_{2,2} = q_{3,1} = 1$.

Now, in STEP 2.4 *of the algorithm, we apply* Algorithm PRSC *to* $\mathcal{Q}_2(t_1) \in (\mathbb{C}(t_2))(t_1)^3$*, and we find that*

$$S^{\mathcal{Q}_2}(t_1, s_1) = (t_1 - s_1)(t_1 + s_1)(t_2t_1^2 - 2 + t_2s_1^2) \in (\mathbb{C}[t_2])[t_1, s_1].$$

Thus, since $\deg(\phi_{\mathcal{Q}_2}) = \deg_{t_1}(S^{\mathcal{Q}_2}) = 4$*, we get that* $\mathcal{Q}_2$ *is not proper. Therefore, the algorithm returns the rational parametrization* $\mathcal{Q}$*, and* $R(\bar{t}) := (t_1, T(t_1, t_2)) \in \mathbb{K}(\bar{t})^2$ *with* $\mathcal{P} = \mathcal{Q}(t_1, T)$*, and the message "you may apply the algorithm again (Step 2) to* $\mathcal{Q}_2$*)".*

Under these conditions, we again apply the algorithm. We go to STEP 2.1*, and by applying* Algorithm PRSC*, we compute* $S_2(t_1) = t_2t_1^4 - 2t_1^2 \in (\mathbb{C}[t_2])[t_1]$*, and*

$$L_i(s_1, x_i) = \mathrm{Res}_{t_1}(G_i^{\mathcal{Q}_2}(\bar{t}, x_i), s_1 - S_2(t_1)) = (q_{i,2}(s_2)x_i - q_{i,1}(s_2))^{\deg_{t_1}(S)},$$

where $G_i^{\mathcal{Q}_2}(\bar{t}, x_i) = x_iq_{i,2}(\bar{t}) - q_{i,1}(\bar{t})$*, for* $i = 1, 2, 3$*. We obtain*

$$\mathcal{M}(\bar{t}) = \left(\frac{m_{1,1}(\bar{t})}{m_{1,2}(\bar{t})}, \frac{m_{2,1}(\bar{t})}{m_{2,2}(\bar{t})}, \frac{m_{3,1}(\bar{t})}{m_{3,2}(\bar{t})}\right) \in \mathbb{C}(\bar{t})^3, \qquad \text{where}$$

$m_{1,1} = -180t_2t_1 + 93t_2^2 - 73t_2 - 90 - 90t_2^2t_1^2 - 51t_1t_2^2$,
$m_{2,1} = 2790t_1t_2^3 - 5475t_2 - 1123 - 4500t_2t_1 - 4500t_1^3t_2^3 - 1125t_2^4t_1^4 - 14775t_1t_2^2 - 13125t_1^2t_2^3 + 1395t_2^2 + 1395t_2^4t_1^2 + 4743t_2^4t_1 - 3825t_2^4t_1^3 - 6750t_2^2t_1^2 + 6789t_2^3$,
$m_{3,2} = 25 - 31t_2^2 + 25t_2^2t_1^2 + 50t_2t_1$,
$m_{1,2} = m_{2,2} = m_{3,1} = 1$.

Finally, in STEP 2.3 *of the algorithm, we apply* Algorithm PRSC *to* $\mathcal{M}_1(t_2) \in (\mathbb{C}(t_1))(t_2)^3$*. We get that*

$$S^{\mathcal{M}_1}(t_2, s_2) = s_2 - t_2 \in (\mathbb{C}[t_1])[t_2, s_2]$$

which implies that $\mathcal{M}_1(t_2)$ *is proper. Thus,* Algorithm RS *outputs the parametrization* $\mathcal{M}$*, and* $R = (S, t_2) = (t_2t_1^4 - 2t_1^2, t_2) \in \mathbb{C}(\bar{t}\,)^2$ *such that* $\mathcal{Q} = \mathcal{M}(S, t_2)$*.*

Summarizing, we have that $\mathcal{P} = \mathcal{Q}(t_1, T)$ *(output of Step 2.4), and* $\mathcal{Q} = \mathcal{M}(S, t_2)$ *(output of Step 2.2). Therefore,*

$$\mathcal{P} = \mathcal{Q}(t_1, T) = \mathcal{M}(S, t_2)(t_1, T) = \mathcal{M}(R), \qquad \textit{where}$$

$$R(\bar{t}\,) = (S(t_1, T(\bar{t}\,)), T(\bar{t}\,)) = (-t_2^2t_1^4 - 2t_1^2, -t_2^2) \in \mathbb{C}(\bar{t}\,)^2.$$

In the following, we derive some properties concerning the output parametrization obtained by Algorithm RS. For this purpose, we assume that we have applied the algorithm only once, and that the output is a rational parametrization $\mathcal{Q}$, and $R(\bar{t}\,) := (S(\bar{t}\,), T(S(\bar{t}\,), t_2)) \in \mathbb{K}(\bar{t}\,)^2$ such that $\mathcal{P}(\bar{t}\,) = \mathcal{Q}(R(\bar{t}\,))$ (see Example 4). Observe that if the output is given in Step 2.2, then $\mathcal{Q} = \mathcal{M}$, and $T = t_2$ (see Example 3).

Under these conditions, and since when we compose two rational maps we multiply their degrees, we can relate the mapping degree of $\mathcal{P}$ with the mapping degree of $\mathcal{Q}$, $\mathcal{M}$, $\mathcal{P}_i$, $i = 1, 2$, and with $\deg_{t_1}(S)$ and $\deg_{t_2}(T)$. In particular, in Theorem 4 we prove that

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{Q}}) \deg_{t_1}(S) \deg_{t_2}(T).$$

For this purpose, we first present a technical lemma where we show that, in the case of a single nonconstant rational function $r(t) \in \mathbb{K}(t)$, the degree with respect to t of $r(t)$ is the mapping degree of r. This lemma is proved in Lemma 4.32. in Sendra et al. (2007).

Lemma 2. *Let* $r(t) \in \mathbb{K}(t)$ *be a nonconstant rational function in reduced form. Then,* $\deg(\phi_r) = \deg_t(r(t))$*, where* $\deg(\phi_r) = \mathit{card}(r^{-1}(a))$ *and* $r^{-1}(a) = \{t \in \mathbb{K} \,|\, r(t) = a\}$*.*

Theorem 4. $\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{Q}}) \deg_{t_1}(S(\bar{t}\,)) \deg_{t_2}(T(\bar{t}\,))$*, and*

$$\deg(\phi_{\mathcal{P}_2}) = \deg_{t_1}(S(\bar{t}\,)), \qquad \deg(\phi_{\mathcal{M}_1}) = \deg_{t_2}(T(\bar{t}\,)).$$

In addition,

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{M}}) \deg_{t_1}(S(\bar{t}\,)), \qquad \deg(\phi_{\mathcal{M}}) = \deg(\phi_{\mathcal{Q}}) \deg_{t_2}(T(\bar{t}\,)).$$

Proof. From Algorithm PRSC, we get that $\mathcal{P}_2(t_1) = \mathcal{M}_2(S_2(t_1))$. Then, since $\deg(\phi_{\mathcal{M}_2}) = 1$, we obtain (note that by Lemma 2, $\deg(\phi_{S_2(t_1)}) = \deg_{t_1}(S(\bar{t}\,)))$

$$\deg(\phi_{\mathcal{P}_2}) = \deg_{t_1}(S(\bar{t}\,)).$$

In addition, $\mathcal{P}(\bar{t}\,) = \mathcal{M}(S(\bar{t}\,), t_2)$. Then, by Remark 2,

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{M}}) \deg_{t_1}(S(\bar{t}\,)) \qquad \text{(I)}.$$

Also, from Algorithm PRSC, we get that

$$\mathcal{M}_1(t_2) = \mathcal{Q}_1(T_1(t_2)).$$

Then, since $\deg(\phi_{\mathcal{Q}_1}) = 1$, we obtain (note that by Lemma 2, $\deg(\phi_{T_1(t_2)}) = \deg_{t_2}(T(\bar{t}\,)))$

$$\deg(\phi_{\mathcal{M}_1}) = \deg_{t_2}(T(\bar{t}\,)).$$

In addition, $\mathcal{M}(\bar{t}\,) = \mathcal{Q}(t_1, T(\bar{t}\,))$. Then, by Remark 2,

$$\deg(\phi_{\mathcal{M}}) = \deg(\phi_{\mathcal{Q}}) \deg_{t_2}(T(\bar{t}\,)) \qquad \text{(II)}.$$

Therefore, from (I) and (II), we conclude that

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{M}}) \deg_{t_1}(S(\bar{t}\,)) = \deg(\phi_{\mathcal{Q}}) \deg_{t_2}(T(\bar{t}\,)) \deg_{t_1}(S(\bar{t}\,)). \qquad \square$$

In the following corollary, we show that if $\deg(\phi_{\mathcal{P}_i}) \neq 1$ for some $i = 1, 2$, Algorithm RS returns a parametrization that has a mapping degree less than the mapping degree of the input parametrization.

Corollary 2. *If* $\deg(\phi_{\mathcal{P}_i}) \neq 1$ *for some* $i = 1, 2$*, then*

$$1 \leq \deg(\phi_{\mathcal{Q}}) < \deg(\phi_{\mathcal{P}}).$$

Proof. Let us assume that $\deg(\phi_{\mathcal{P}_2}) \neq 1$. Then, from Theorem 4, we deduce that $\deg(\phi_{\mathcal{Q}}) \neq \deg(\phi_{\mathcal{P}})$; otherwise, $\deg_{t_1}(S) = 1$ and $\deg(\phi_{\mathcal{P}_2}) = \deg_{t_1}(S) = 1$ which is impossible. In addition, since

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{Q}}) \deg_{t_1}(S) \deg_{t_2}(T), \quad \text{and} \quad \deg_{t_1}(S) = \deg(\phi_{\mathcal{P}_2}) \geq 2$$

(see Theorem 4), we conclude that $\deg(\phi_{\mathcal{Q}}) < \deg(\phi_{\mathcal{P}})$. Finally, note that it always hold that $1 \leq \deg(\phi_{\mathcal{Q}})$. $\square$

Let $\mathcal{Q}$ be the output parametrization obtained by Algorithm RS, and let us assume that $\deg(\phi_{\mathcal{Q}_i}) = 1$, for $i = 1, 2$. Then, note that the algorithm cannot be applied again. Under these conditions, we cannot ensure that $\mathcal{Q}$ is proper because it could happen that $\mathcal{Q}_i^{-1} \notin \mathbb{K}(\,\overline{x}\,)$, for $i = 1, 2$ (see Theorem 3). In order to check whether $\mathcal{Q}$ is proper, one may apply Theorem 4 or the following corollary that is derived from Theorem 4.

Corollary 3. *The following statements are equivalent:*

1. $\mathcal{Q}$ *is proper.*
2. $\deg(\phi_{\mathcal{M}}) = \deg_{t_2}(T)$.
3. $\deg(\phi_{\mathcal{P}}) = \deg_{t_1}(S)\deg_{t_2}(T)$.

In the following, we apply Theorem 4 and Corollary 3 to the parametrizations introduced in Examples 3, 4 and 5.

Example 6. *In Example 3, we have that*

$$\deg_{t_2}(T) = 1, \quad \textit{and} \quad \deg_{t_1}(S) = 3.$$

Thus, since $3 = \deg(\phi_{\mathcal{P}}) = \deg_{t_1}(S)\deg_{t_2}(T)$*, by Corollary 3 we get that* $\mathcal{Q}$ *is a proper reparametrization.*

In Example 4, we have that

$$\deg_{t_2}(T) = 2, \quad \textit{and} \quad \deg_{t_1}(S) = 4.$$

Thus, since $8 = \deg(\phi_{\mathcal{P}}) = \deg_{t_1}(S)\deg_{t_2}(T)$*, by Corollary 3 we get that* $\mathcal{Q}$ *is a proper reparametrization.*

In Example 5, we have that

$$\deg_{t_1}(S) = 4, \quad \textit{and} \quad \deg_{t_2}(T) = 2.$$

Since $32 = \deg(\phi_{\mathcal{P}}) \neq \deg_{t_1}(S)\deg_{t_2}(T)$*, by Corollary 3 we get that* $\mathcal{Q}$ *is not proper. In fact, from Theorem 4, we deduce that* $\deg(\phi_{\mathcal{Q}}) = 4$*. Observe that clearly Corollary 2 holds.*

Remark 6. *In addition to the equalities derived in Theorem 4, we also have:*

- $\mathcal{P}_1(t_2) = \mathcal{M}(S_1(t_2), t_2)$ *which implies that*

$$\deg(\phi_{\mathcal{P}_1}) = \deg(\phi_{\mathcal{M}^*})\deg(\phi_{(S_1(t_2),t_2)}) = \deg(\phi_{\mathcal{M}^*}),$$

  where $\phi^*_{\mathcal{M}}$ *is the restriction of* $\phi_{\mathcal{M}}$ *to the plane curve defined, over* $\overline{\mathbb{K}(t_1)}$*, by the numerator* $x - S_1(y)$ *(S is expressed in reduced form). In addition, note that* $(S_1(t), t)$ *is proper (its inverse is* $t = y$*) and hence* $\deg(\phi_{(S_1(t_2),t_2)}) = 1$*.*

- $\mathcal{M}_2(t_1) = \mathcal{Q}(t_1, T_2(t_1))$*, which implies that*

$$1 = \deg(\phi_{\mathcal{M}_2}) = \deg(\phi_{\mathcal{Q}^*})\deg(\phi_{(t_1,T_2(t_1))}) = \deg(\phi_{\mathcal{Q}^*}),$$

  where $\phi^*_{\mathcal{Q}}$ *is the restriction of* $\phi_{\mathcal{Q}}$ *to the plane curve defined, over* $\overline{\mathbb{K}(t_2)}$*, by the numerator* $y - T_2(x)$ *(T is expressed in reduced form). In addition, note that* $(t, T_2(t))$ *is proper (its inverse is* $t = x$*) and hence* $\deg(\phi_{(t_1,T_2(t_1))}) = 1$*.*

Note that from Theorem 4, and Corollaries 1 and 3, one may derive additional equalities that may help to analyze whether a given surface parametrization can be properly reparametrized. For instance, if $\deg(\phi_{\mathcal{P}_i}) \neq 1$ for some $i = 1, 2$, and $\deg(\phi_{\mathcal{P}}) = n$ where n is a prime number, then $\deg(\phi_{\mathcal{Q}}) = 1$ (see Corollaries 1 and 3).

Finally, we also show how the equalities presented above can be stated in terms of $d_i = \deg_{t_i}(f_i)$ (see Proposition 1 and Theorem 3). More precisely, from Corollary 1, where we show that

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{P}_1})d_1 = \deg(\phi_{\mathcal{P}_2})d_2$$

and from Theorem 4, where it is proved that $\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{M}})\deg(\phi_{\mathcal{P}_2})$, and $\deg(\phi_{\mathcal{M}}) = \deg(\phi_{\mathcal{Q}})\deg_{t_2}(T)$, one deduces the following corollary.

Corollary 4. $\deg(\phi_{\mathcal{M}}) = d_2, \quad \deg(\phi_{\mathcal{Q}}) = \frac{d_2}{\deg_{t_2}(T(\bar{t}))}, \quad$ *and*

$$d_2 = \frac{\deg(\phi_{\mathcal{P}})}{\deg(\phi_{\mathcal{P}_2})} = \frac{\deg_{t_1}(S_1^{\mathcal{P}})}{\deg_{t_1}(S^{\mathcal{P}_2})}.$$

3.1. Practical Implementation

In the following table, we illustrate the performance of the implementation in Maple of Algorithm RS, showing times for some parametrizations. In the table, we also list the total degree, the degree in the variables t_1, t_2, and the maximum number of terms, of each input and output parametrization (second and fourth column, respectively) as well as R (fifth column). The last column of the table shows the mapping degrees $\deg(\phi_{\mathcal{P}})$, and $\deg(\phi_{\mathcal{Q}})$. Actual computing times, running on a PC Mobile Intel Celeron 2.4 GHz and 265 MB of RAM, are given in seconds of CPU.

Input	Degree of $\mathcal{P}[i]$	Time	Degree of $\mathcal{Q}[i]$	Degree of $R_i(\bar{t})$	$[\deg(\phi_{\mathcal{P}})$, $\deg(\phi_{\mathcal{Q}})]$
$\mathcal{P}[1]$	$[16, 16, 8, 9]$	0.022	$[4, 4, 1, 3]$	$[4, 4, 2, 2]$	[8, 1]
$\mathcal{P}[2]$	$[8, 8, 8, 7]$	0.015	$[1, 1, 1, 1]$	$[8, 8, 8, 7]$	[1, 1]
$\mathcal{P}[3]$	$[8, 8, 8, 7]$	0.031	$[4, 4, 4, 7]$	$[2, 2, 2, 1]$	[16, 4]
$\mathcal{P}[4]$	$[48, 32, 32, 31]$	0.421	$[16, 8, 16, 31]$	$[4, 4, 2, 1]$	[128, 16]
$\mathcal{P}[5]$	$[32, 16, 16, 57]$	1.139	$[8, 4, 8, 22]$	$[6, 4, 2, 3]$	[64, 8]
$\mathcal{P}[6]$	$[128, 48, 80, 156]$	0.639	$[36, 4, 32, 86]$	$[16, 12, 4, 3]$	[576, 24]
$\mathcal{P}[7]$	$[80, 48, 32, 139]$	3.183	$[16, 4, 16, 57]$	$[16, 12, 4, 5]$	[192,8]
$\mathcal{P}[8]$	$[24, 24, 24, 45]$	0.125	$[8, 4, 8, 25]$	$[6, 6, 3, 3]$	[144,8]
$\mathcal{P}[9]$	$[80, 48, 32, 109]$	0.297	$[8, 4, 8, 25]$	$[20, 12, 8, 7]$	[192,4]
$\mathcal{P}[10]$	$[80, 48, 32, 135]$	0.421	$[16, 4, 16, 35]$	$[20, 12, 8, 7]$	[288,12]
$\mathcal{P}[11]$	$[32, 12, 32, 24]$	0.063	$[16, 4, 16, 24]$	$[3, 3, 2, 1]$	[6,1]
$\mathcal{P}[12]$	$[48, 24, 48, 15]$	0.047	$[6, 4, 6, 15]$	$[8, 6, 8, 1]$	[48,1]

As we mention in the introduction, a direct approach to the reparametrization problem could consist of implicitizing the parametrization to apply afterwards algorithms to parametrize the implicit equation. In particular, in Pérez-Díaz and Sendra (2008), an implicitization method is presented based on the computation of polynomial gcds and univariate resultants for determining the implicit equation of a rational surface from a rational parametrization. The resultant based formula provides the implicit equation to a power which is the mapping degree of the parametrization.

In this paper, we approach the reparametrization problem by means of rational reparametrizations, that is without implicitizing. The algorithm presented uses polynomial gcds and univariate resultants, and it is much more efficient and powerful than first finding the implicit equation. In fact, we tried to compute the implicit equation for the above parametrizations using the method in Pérez-Díaz and Sendra (2008), and the computing time was > 3000 seconds for all the cases. The reason is that the approach in Pérez-Díaz and Sendra (2008) needs to compute several resultants of polynomials depending on 7 variables which is highly time consuming (see Theorem 10 in Pérez-Díaz and Sendra (2008)).

Finally, we observe that $\mathcal{P}[5]$ and $\mathcal{P}[7]$ take so much longer than $\mathcal{P}[6]$, but the degree of $\mathcal{P}[5]$ and $\mathcal{P}[7]$ is $[32, 16, 16, 57]$ and $[80, 48, 32, 139]$, respectively, and the degree of $\mathcal{P}[6]$ is $[128, 48, 80, 156]$. The reason is that the polynomials defining the components of the parametrization $\mathcal{P}[6]$ are not squarefree, which improves considerably the computation time of a resultant (since $\mathrm{Res}_t(p^n, q) = \mathrm{Res}_t(p, q)^n$). In fact, the denominator of the third component of $\mathcal{P}[6]$ is given as

$(25t_2^{24} - 100t_2^{28}t_1^6 - 300t_2^{30}t_1^6 - 300t_2^{32}t_1^6 - 100t_2^{34}t_1^6 + 150t_2^{28} + 100t_2^{30} + 100t_2^{26} + 25t_2^{32} + 150t_2^{32}t_1^{12} + 300t_2^{34}t_1^{12} + 150t_2^{36}t_1^{12} + 25t_2^{40}t_1^{24} - 100t_2^{38}t_1^{18} - 100t_2^{36}t_1^{18} - 31)(t_1^6t_2^4 - t_2^2 - 1)^4.$

4. Conclusion

In this paper, we study the *reparametrization problem.* That is, given an algebraically closed field $\mathbb{K}$, and a rational parametrization $\mathcal{P}$ of an algebraic surface $\mathcal{V} \subset \mathbb{K}^3$, we consider the problem of computing a proper rational parametrization $\mathcal{Q}$ from $\mathcal{P}$.

The approach presented complements some previous results developed in Pérez-Díaz (2006). More precisely, in Pérez-Díaz (2006), the reparametrization problem is solved for those surfaces parametrized by $\mathcal{P}$ such that there exists $R(\bar{t}) \in (\mathbb{K}(t_1) \setminus \mathbb{K}) \times (\mathbb{K}(t_2) \setminus \mathbb{K})$ with $\mathcal{P} = \mathcal{Q}(R)$. In this paper, we no not need to impose any condition on R, although we have to check whether at least one of the two auxiliary parametrizations defined from $\mathcal{P}$ is not proper. For those surfaces for which a rational proper reparametrization is not found, we show that the mapping degree of the reparametrization is lower; that is, $1 \leq \deg(\phi_{\mathcal{Q}}) < \deg(\phi_{\mathcal{P}})$ (see Corollary 2). Moreover, we prove that $\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{Q}}) \deg_{t_1}(S) \deg_{t_2}(T)$, where $R(\bar{t}) =$

$(S(\bar{t}), T(S(\bar{t}), t_2)),\ \ S, T \in \mathbb{K}(\bar{t})$ (see Theorem 4).

The basic idea of the algorithm is to compute a proper reparametrization of two parametric space curves defined directly from the given rational parametrization of the surface. For this purpose, we use the Algorithm PRSC presented in Section 2, and derived from the results in Pérez-Díaz (2006).

The algorithm is based on polynomial gcds and univariate resultants, which always work and whose computing time is very satisfactory.

The results presented in this paper can easily be extended to a variety $\mathcal{V} \subset \mathbb{K}^n$ of dimension 2, rationally parametrized by

$$\mathcal{P}(\bar{t}) = \left(\frac{p_{1,1}(\bar{t})}{p_{1,2}(\bar{t})}, \ldots, \frac{p_{n,1}(\bar{t})}{p_{n,2}(\bar{t})} \right) \in \mathbb{K}(\bar{t})^n,$$

where $\bar{t} = (t_1, t_2)$, $\gcd(p_{i,1}, p_{i,2}) = 1$, for $i = 1, \ldots, n$, and $\mathbb{K}$ is an algebraically closed field. For this purpose, one only has to take into account that the Algorithm PRSC presented in Section 2 can easily be extended to curves in $\mathbb{K}^n, n \geq 4$.

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