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# Asymptotic Behavior of an Implicit Algebraic Plane Curve<sup>☆</sup>

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## Abstract

In this paper, we introduce the notion of infinity branches as well as approaching curves. We present some properties which allow us to obtain an algorithm that compares the behavior of two implicitly defined algebraic plane curves at the infinity. As an important result, we prove that if two plane algebraic curves have the same asymptotic behavior, the Hausdorff distance between them is finite.

*Keywords:* Implicit Algebraic Plane Curve, Infinity Branches, Convergent Branches, Asymptotic Behavior, Approaching Curves  
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## 1. Introduction

Unirational algebraic varieties, play an important role in the frame of practical applications (see Hoffmann et al. (1997), and Hoschek and Lasser (1993)). In particular, many authors have studied different problems related to plane algebraic curves that are implicitly defined (see e.g. Sendra et al. (2007), and Walker (1950)). In this paper, we deal with the notion of infinity branches which, intuitively speaking, reflect the status of a curve at the points with sufficiently large coordinates. In fact, an infinity branch is associated to

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a projective place centered at an infinity point, and it can be “parametrized” by means of Puiseux series (see Section 3).

An infinity branch is a very important tool to analyze the behavior at the infinity of an implicit real algebraic plane curve. Hence, an infinity branch is applicable by itself since it faces symbolic computation to real world problems which is of special interest in the field of C.A.G.D. For instance, determining the infinity branches of an implicit plane curve is an important step in sketching its graph as well as in studying its topology (see e.g. Gao and Cheng (2012), González-Vega and Necula (2002), Hong (1996), and Zeng (2007)). In addition, in Blasco and Pérez-Díaz (2014), the notion of  $g$ -asymptote is introduced, and it is shown how these asymptotes can be computed from the infinity branches. Intuitively speaking, a curve  $\bar{\mathcal{C}}$  is a *generalized asymptote* (or  *$g$ -asymptote*) of another curve  $\mathcal{C}$  if  $\bar{\mathcal{C}}$  approaches  $\mathcal{C}$  at some infinity branch, and  $\mathcal{C}$  can not be approached by a new curve of lower degree (that is, the notion of  $g$ -asymptote generalizes the classical notion of (linear) asymptote).

The concept of infinity branch allows us to introduce the notion of convergent branches and approaching curves (see Section 4). The idea is as follows: two infinity branches converge if they get closer as they tend to infinity. In addition, we say that a curve  $\bar{\mathcal{C}}$  *approaches*  $\mathcal{C}$  at its infinity branch  $B$  if  $\bar{\mathcal{C}}$  has an infinity branch  $\bar{B}$  convergent with  $B$ . We provide some results that characterize whether two plane algebraic curves are approaching.

Using these results, we present a method to compare the asymptotic behavior of two curves; i.e., the behavior of two curves at the infinity (see Section 5). The algorithm presented will play an important role in the analysis of the generalized asymptotes. In particular, in Blasco and Pérez-Díaz (2014) (see Subsection 3.1), an algorithm for computing all the generalized asymptotes of a real plane algebraic curve  $\mathcal{C}$  implicitly defined is developed.

Finally, we prove that if two plane algebraic curves have the same *asymptotic behavior*, the Hausdorff distance between them is finite. This result will play an important role in the applications to computer aided geometric design as for instance in the approximate parametrization problem (see e.g. Pérez-Díaz et al. (2004), Pérez-Díaz et al. (2005), and Pérez-Díaz et al. (2010)). In this problem, we are given an affine curve  $\mathcal{C}$  (say that it is a perturbation of a rational curve), and one is interested in computing a rational parametrization of a rational affine curve  $\bar{\mathcal{C}}$  near it (one may state the problem also for surfaces). The effectiveness of the algorithm will depend on

the closeness of  $C$  and  $\overline{C}$ . For this purpose, at least one needs to show that the Hausdorff distance between  $C$  and  $\overline{C}$  is finite.

The results presented in this paper can be extended to curves in the affine  $n$ -space (see Blasco and Pérez-Díaz (2014b)).

The structure of the paper is as follows: In Section 2, we present the terminology that will be used throughout this paper as well as some previous results. In Section 3, the notion of *infinity branch* is introduced and some important properties are proved. In Section 4, we provide the notions of *convergent branches* and *approaching curves*. In addition, we develop some results that characterize whether two plane algebraic curves approach each other. The results presented in this section will be used in Section 5, where an algorithm to compare the asymptotic behavior of two algebraic plane curves is developed. In addition, we prove that if two plane curves have the same asymptotic behavior, the Hausdorff distance between them is finite.

## 2. Preliminaries and Terminology

In this section, we present some notions and terminology that will be used throughout the paper. In particular, we need some previous results concerning local parametrizations and Puiseux series. For further details see Duval (1989), Section 2.5 in Sendra et al. (2007), Stadelmeyer (2000), and Chapter 4 (Section 2) in Walker (1950).

We denote by  $\mathbb{C}[[t]]$  the domain of *formal power series* in the indeterminate  $t$  with coefficients in the field  $\mathbb{C}$ , i.e. the set of all sums of the form  $\sum_{i=0}^{\infty} a_i t^i$ ,  $a_i \in \mathbb{C}$ . The quotient field of  $\mathbb{C}[[t]]$  is called the field of *formal Laurent series*, and it is denoted by  $\mathbb{C}((t))$ . It is well known that every non-zero formal Laurent series  $A \in \mathbb{C}((t))$  can be written in the form  $A(t) = t^k \cdot (a_0 + a_1 t + a_2 t^2 + \dots)$ , where  $a_0 \neq 0$  and  $k \in \mathbb{Z}$ . In addition, the field  $\mathbb{C} \ll t \gg := \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$  is called the field of *formal Puiseux series*. Note that Puiseux series are power series with fractional exponents. In addition, every Puiseux series,  $\varphi$ , has a bound for the denominators of exponents with non-vanishing coefficients, which is known as *the ramification index* of the series. We denote it as  $\nu(\varphi)$  (see Duval (1989)).

The *order* of a non-zero (Puiseux or Laurent) series  $A$  is the smallest exponent of a term with non-vanishing coefficient in  $A$ . We denote it by  $\text{ord}(A)$ . We let the order of 0 be  $\infty$ .

In the following, we introduce the notion of projective local parametrization for a projective plane curve (see Definition 2.69, and Lemma 2.70 in Sendra et al. (2007)).

**Definition 2.1.** Let  $\mathcal{C}^* \subset \mathbb{P}^2(\mathbb{C})$  be a projective plane curve defined by the homogeneous polynomial  $F(x, y, z) \in \mathbb{R}[x, y, z]$ . Let  $A^*, B^*, C^*$  be series in  $\mathbb{C}((t))$  such that: (i)  $F(A^*(t) : B^*(t) : C^*(t)) = 0$  (where the three series converge), and (ii) there is no  $D \in \mathbb{C}((t)) \setminus \{0\}$  such that  $D \cdot (A^*, B^*, C^*) \in \mathbb{C}^3$ . Then  $\mathcal{P}^* = (A^* : B^* : C^*) \in \mathbb{P}^2(\mathbb{C}((t)))$  is called a projective local parametrization of  $\mathcal{C}^*$ . In addition, one can always find such a parametrization having  $\min\{\text{ord}(A^*), \text{ord}(B^*), \text{ord}(C^*)\} = 0$ , and the point  $\mathcal{P}^*(0) \in \mathcal{C}^*$  is called the center of  $\mathcal{P}^*$ .

For an affine plane curve, the above notion can be stated as follows:

**Definition 2.2.** Let  $\mathcal{C}$  be a real plane algebraic curve over  $\mathbb{C}$  implicitly defined by the irreducible polynomial  $f(x, y) \in \mathbb{R}[x, y]$ . Let  $A, B$  be series in  $\mathbb{C}((t))$  such that: (i)  $f(A(t), B(t)) = 0$  (where both series converge), and (ii) not both,  $A$  and  $B$ , are constants. Then  $\mathcal{P} = (A, B)$  is called an (affine) local parametrization of  $\mathcal{C}$ . Moreover, if  $\text{ord}(A), \text{ord}(B) \geq 0$ , the point  $\mathcal{P}(0) = (a, b) \in \mathcal{C}$  is called the center of  $\mathcal{P}$ .

In the following, we deal with affine curves. The results and notions presented can be adapted for projective curves in an obvious way.

Two local parametrizations,  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , of an algebraic plane curve  $\mathcal{C}$  are called *equivalent* if there exists  $R \in \mathbb{C}[[t]]$ , with  $\text{ord}(R) = 1$ , such that  $\mathcal{P}_1 = \mathcal{P}_2(R)$ . It can be proved that this equivalence of local parametrizations is actually an equivalence relation.

If a local parametrization  $\mathcal{P}$ , or one equivalent, satisfies that  $\mathcal{P}(t) = \mathcal{P}'(t^k)$  for some parametrization  $\mathcal{P}'$  and for some natural number  $k > 1$ , then  $\mathcal{P}$  is said to be *reducible*. Otherwise,  $\mathcal{P}(t)$  is said to be *irreducible*. Under these conditions, we introduce the notion of a *place* as follows.

**Definition 2.3.** An equivalence class of irreducible local parametrizations of the algebraic plane curve  $\mathcal{C}$  is called a *place* of  $\mathcal{C}$ . The common center of the local parametrizations (if it exists) is the center of the place.

In the following definition, we introduce the notion of *branch* of a plane curve.

**Definition 2.4.** Given a local parametrization  $(X, Y)$  of a plane curve  $\mathcal{C}$ , the set of all points  $(X(t), Y(t))$  obtained by allowing  $t$  to vary within some neighborhood of 0 where  $X(t)$  and  $Y(t)$  converge is called a branch of  $\mathcal{C}$ .

It can be shown that two equivalent local parametrizations provide the same branch. Therefore, one obtains a branch for each place of a given algebraic plane curve.

One may prove that the center of a local parametrization of  $\mathcal{C}$  is a point on  $\mathcal{C}$ . Conversely, from the following theorems, we also obtain that every point on  $\mathcal{C}$  is the center of at least one place of  $\mathcal{C}$  (see Theorems 2.77 and 2.78 in Sendra et al. (2007)).

**Theorem 2.5. (Puiseux's Theorem)** *The field  $K \ll x \gg$  is algebraically closed.*

A proof of Puiseux's Theorem can be given constructively by the Newton Polygon Method (see e.g. Section 2.5 in Sendra et al. (2007)). This method solves the construction of solutions of non-constant univariate polynomial equations over  $K \ll x \gg$ .

**Theorem 2.6.** *Let  $\mathcal{C}$  be a plane curve defined by  $f(x, y) \in \mathbb{R}[x, y]$ . To each root  $Y(x) \in \mathbb{C} \ll x \gg$  of  $f(x, y) = 0$  with  $\text{ord}(Y) > 0$  there corresponds a unique place of  $\mathcal{C}$  with center at the origin. Conversely, to each place  $(X(t), Y(t))$  of  $\mathcal{C}$  with center at the origin there correspond  $\text{ord}(X)$  roots of  $f(x, y) = 0$ , each of order greater than zero.*

If  $Y(x)$  is a Puiseux series solving  $f(x, y) = 0$ ,  $\text{ord}(Y) > 0$ , and  $n$  is the least integer for which  $Y(x) \in \mathbb{C}((x^{\frac{1}{n}}))$  (i.e.,  $\nu(Y) = n$ ), then we set  $x^{\frac{1}{n}} = t$ , and  $(t^n, Y(t^n))$  is a local parametrization with center at the origin. The solutions of  $f(x, y)$  of order 0 correspond to places with center on the  $y$ -axis but different from the origin, and the solutions of negative order correspond to places at infinity (places with center at an infinity point).

Note that several different Puiseux series may correspond to equivalent local parametrizations, and then these series provide a unique place. More precisely, let  $Y(x) = \sum_{i \geq r} a_i x^{i/n}$  be a Puiseux series with ramification index  $\nu(Y) = n$ . The series  $\sigma_\epsilon(Y)$ ,  $\epsilon^n = 1$ , are called the *conjugates* of  $Y$ , where

$$\sigma_\epsilon(Y) = \sum_{i \geq r} \epsilon^i a_i x^{i/n}.$$

The set of all (distinct) conjugates of  $Y$  is called the *conjugacy class* of  $Y$ . The number of different conjugates of  $Y$  is  $\nu(Y)$ . Two Puiseux series provide the same place if they belong to the same conjugacy class (see Duval (1989) and Verger-Gaugry (2011)).

### 3. Infinity Branches

In this section, we introduce the notion of *infinity branch* (see Definition 3.1), and we obtain some properties concerning to these algebraic entities.

For this purpose, we consider an algebraic affine plane curve  $\mathcal{C}$  over  $\mathbb{C}$ , implicitly defined by the irreducible polynomial  $f(x, y) \in \mathbb{R}[x, y]$ . Let  $\mathcal{C}^*$  be its corresponding projective curve defined by the homogeneous polynomial  $F(x, y, z) \in \mathbb{R}[x, y, z]$ . Furthermore, let  $P = (1 : m : 0)$ ,  $m \in \mathbb{C}$ , be an infinity point of  $\mathcal{C}^*$ , and we consider the curve implicitly defined by the polynomial  $g(y, z) = F(1 : y : z)$ . Observe that  $g(p) = 0$ , where  $p = (m, 0)$ .

By applying Theorem 2.5, we compute the series expansion for the solutions of  $g(y, z) = 0$ . There exist exactly  $\deg_Y(g)$  solutions given by different Puiseux series that can be grouped into conjugacy classes. Let one of these solutions be given by the following Puiseux series:

$$\varphi(t) = m + a_1 t^{N_1/N} + a_2 t^{N_2/N} + a_3 t^{N_3/N} + \dots \in \mathbb{C} \ll t \gg, \quad a_i \neq 0, \forall i \in \mathbb{N},$$

where  $\nu(\varphi) = N \in \mathbb{N}$ ,  $N_i \in \mathbb{N}$ ,  $i = 1, \dots$ , and  $0 < N_1 < N_2 < \dots$ . We have that  $g(\varphi(t), t) = 0$  in some neighborhood of  $t = 0$  where  $\varphi(t)$  converges. Then, there exists some  $M \in \mathbb{R}^+$  such that

$$F(1 : \varphi(t) : t) = g(\varphi(t), t) = 0, \quad \text{for } t \in \mathbb{C} \text{ and } |t| < M,$$

which implies that  $F(t^{-1} : t^{-1}\varphi(t) : 1) = f(t^{-1}, t^{-1}\varphi(t)) = 0$ , for  $t \in \mathbb{C}$  and  $0 < |t| < M$ . We set  $t^{-1} = z$ , and we obtain that

$$f(z, r(z)) = 0, \quad z \in \mathbb{C} \text{ and } |z| > M^{-1}, \quad \text{where}$$

$$r(z) = z\varphi(z^{-1}) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \dots, \quad a_i \neq 0, \forall i \in \mathbb{N}$$

$N, N_i \in \mathbb{N}$ ,  $i = 1, \dots$ , and  $0 < N_1 < N_2 < \dots$ .

Since  $\nu(\varphi) = N$ , we get that there are  $N$  different series in its conjugacy class. Let  $\varphi_1, \dots, \varphi_N$  be these series, and

$$r_i(z) = z\varphi_i(z^{-1}) = mz + a_1 c_i^{N_1} z^{1-N_1/N} + a_2 c_i^{N_2} z^{1-N_2/N} + a_3 c_i^{N_3} z^{1-N_3/N} + \dots \quad (1)$$

where  $c_1, \dots, c_N$  are the  $N$  complex roots of  $x^N = 1$ . Now we are ready to introduce the notion of infinity branch.

**Definition 3.1.** *The set  $B = \bigcup_{i=1}^N L_i$  where*

$$L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_i\}$$

*is called an infinity branch of the affine plane curve  $\mathcal{C}$ . The subsets  $L_1, \dots, L_N$  are called the leaves of the infinity branch  $B$ .*

**Remark 3.2.** 1. *We observe that an infinity branch is uniquely determined from one leaf, up to conjugation. That is, if  $B = \bigcup_{i=1}^N L_i$ , where  $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_i\}$ , and*

$$r_i(z) = z\varphi_i(z^{-1}) = mz + a_1z^{1-N_1/N} + a_2z^{1-N_2/N} + a_3z^{1-N_3/N} + \dots$$

*then  $r_j = r_i$ ,  $j = 1, \dots, N$ , up to conjugation; i.e.*

$$r_j(z) = z\varphi_j(z^{-1}) = mz + a_1c_j^{N_1}z^{1-N_1/N} + a_2c_j^{N_2}z^{1-N_2/N} + a_3c_j^{N_3}z^{1-N_3/N} + \dots$$

*where  $c_j^N = 1$ ,  $j = 1, \dots, N$ , and  $N, N_i \in \mathbb{N}$ .*

2. *Let  $M := \max\{M_1, \dots, M_N\}$ . In the following, we consider  $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ .*

Let  $\varphi(z) = m + a_1z^{N_1/N} + a_2z^{N_2/N} + a_3z^{N_3/N} + \dots$  be a series expansion for a solution of  $g(y, z) = 0$ . We consider  $\psi(t) := \varphi(t^N)$ , and we observe that  $(1 : \psi(t) : t^N)$  is a local projective parametrization, with center at  $P$ , of the projective curve  $\mathcal{C}^*$ .

Thus, from  $\psi_i(t) := \varphi_i(t^N)$ ,  $i = 1, \dots, N$  ( $\varphi_i$  are the  $N$  different series in the conjugacy class of  $\varphi$ ), we obtain  $N$  equivalent local projective parametrizations,  $(1 : \psi_i(t) : t^N)$  (note that they are equivalent since  $\varphi_1, \dots, \varphi_N$  belong to the same conjugacy class). Therefore, the leaves of  $B$  are all associated to a unique infinity place.

Conversely, from a given infinity place defined by a local projective parametrization  $(1 : \psi(t) : t^N)$  (see Theorem 2.5.3 in Sendra et al. (2007)), we obtain  $N$  Puiseux series,  $\varphi_j(t) = \psi(c_j t^{1/N})$ ,  $c_j^N = 1$ , that provide different expressions  $r_j(z) = z\varphi_j(z^{-1})$ ,  $j = 1, \dots, N$ . Hence, the infinity branch  $B$  is defined by the leaves  $L_j = \{(z, r_j(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ ,  $j = 1, \dots, N$ .



From the above discussion, we deduce that there exists a one-to-one relation between infinity places and infinity branches. In addition, we can say that each infinity branch is associated to a unique infinity point given by the center of the corresponding infinity place. Reciprocally, taking into account the above construction, we get that every infinity point has associated, at least, one infinity branch. Hence, every algebraic plane curve has, at least, one infinity branch. Furthermore, every algebraic plane curve has a finite number of branches.

Observe that the above construction can be applied to any infinity point of the form  $(a : b : 0)$ ,  $a \neq 0$ . In the following, we assume that  $a = 0$ ; that is, we take the infinity point  $P = (0 : 1 : 0)$ . In this case, we consider the curve implicitly defined by the polynomial  $h(x, z) = F(x : 1 : z)$ . Observe that  $h(p) = 0$ , where  $p = (0, 0)$ . In this situation, we get that there exists  $M \in \mathbb{R}^+$  such that

$$F(\varphi(t) : 1 : t) = h(\varphi(t), t) = 0, \quad \text{for } t \in \mathbb{C} \text{ and } |t| < M,$$

where

$$\varphi(t) = a_1 t^{N_1/N} + a_2 t^{N_2/N} + a_3 t^{N_3/N} + \dots \in \mathbb{C} \ll t \gg, \quad a_i \neq 0, \forall i \in \mathbb{N}$$

$N, N_i \in \mathbb{N}$ ,  $i = 1, \dots$ , and  $0 < N_1 < N_2 < \dots$ , is a series expansion for a solution of  $h(x, z) = 0$ . We set  $z = t^{-1}$ , and we get that

$$f(r(z), z) = 0, \quad z \in \mathbb{C} \text{ and } |z| > M^{-1}, \quad \text{where}$$

$$r(z) = z\varphi(z^{-1}) = a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \dots, \quad a_i \neq 0, \forall i \in \mathbb{N}$$

$N, N_i \in \mathbb{N}$ ,  $i = 1, \dots$ , and  $0 < N_1 < N_2 < \dots$ .

Thus, we obtain an infinity branch  $B = \bigcup_{i=1}^N L_i$  whose leaves have the form:

$$L_i = \{(r_i(z), z) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}.$$

Observe that we may apply this construction to any infinity point of the form  $(a : b : 0)$ ,  $b \neq 0$ .

These two approaches lead us to consider two types of infinity branches.

**Definition 3.3.** Let  $\mathcal{C}$  be an affine plane curve over  $\mathbb{C}$  defined by an irreducible polynomial  $f(x, y) \in \mathbb{R}[x, y]$ .

- An infinity branch of  $\mathcal{C}$  of type 1 associated to the infinity point  $P = (1 : m : 0)$ ,  $m \in \mathbb{C}$ , is a set  $B = \bigcup_{i=1}^N L_i$ , where  $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ ,  $i = 1, \dots, N$ ,  $M \in \mathbb{R}^+$ , and  $r_1, \dots, r_N$  are the conjugates of

$$r(z) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \dots, \quad a_i \neq 0, \forall i \in \mathbb{N}$$

$N, N_i \in \mathbb{N}$ ,  $i = 1, \dots$ , and  $0 < N_1 < N_2 < \dots$ .

- An infinity branch of  $\mathcal{C}$  of type 2 associated to the infinity point  $P = (m : 1 : 0)$ ,  $m \in \mathbb{C}$ , is a set  $B = \bigcup_{i=1}^N L_i$ , where  $L_i = \{(r_i(z), z) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ ,  $i = 1, \dots, N$ ,  $M \in \mathbb{R}^+$ , and  $r_1, \dots, r_N$  are the conjugates of

$$r(z) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \dots, \quad a_i \neq 0, \forall i \in \mathbb{N}$$

$N, N_i \in \mathbb{N}$ ,  $i = 1, \dots$ , and  $0 < N_1 < N_2 < \dots$ .

**Remark 3.4.** 1. In the following, we assume w.l.o.g that the given algebraic plane curve  $\mathcal{C}$  only has type 1 infinity branches; that is, all the infinity points are of the form  $(1 : m : 0)$ ,  $m \in \mathbb{C}$ . Otherwise, we may consider a linear change of coordinates.

2. By abuse of notation, we will say that  $N$  is the ramification index of the branch  $B$ , and we will write it as  $\nu(B) = N$ . Note that  $B$  has  $\nu(B)$  leaves.

In the following example, we compute the infinity branches for a given plane curve.

**Example 3.5.** Let  $\mathcal{C}$  be the plane curve implicitly defined by the irreducible polynomial

$$f(x, y) = y^5 - 4y^4x + 4y^3x^2 + 2y^2x - y^2x^2 + 2yx^2 + 2yx^3 + x + x^2 \in \mathbb{R}[x, y].$$

The corresponding projective curve  $\mathcal{C}^*$  is defined by  $F(x : y : z) =$

$$y^5 - 4y^4x + 4y^3x^2 + 2y^2z^2x - zy^2x^2 + 2z^2yx^2 + 2yzx^3 + z^4x + z^3x^2 \in \mathbb{R}[x, y, z].$$

Note that  $P = (1 : 0 : 0)$  is an infinity point of  $\mathcal{C}^*$ . Let us compute the infinity branches associated to  $P$ . For this purpose, we consider the curve implicitly defined by the polynomial  $g(y, z) = F(1 : y : z)$ , and we observe that  $g(p) = 0$ , where  $p = (0, 0)$ .

We compute the series expansion for the solutions of  $g(y, z) = 0$ . For this purpose, we use for instance the `algcurves` package included in the computer algebra system Maple. We get that:

$$\varphi_1(t) = -1/2t^2 + 1/8t^4 - 1/8t^5 + 1/16t^6 + 1/16t^7 + \dots \in \mathbb{C} \ll t \gg, \quad \text{and}$$

$$\varphi_2(t) = \frac{-(-2t)^{1/2}}{2} - \frac{t}{8} + \frac{27}{256}(-2t)^{3/2} - \frac{7}{32}t^2 + \frac{4057}{65536}(-2t)^{5/2} + \dots \in \mathbb{C} \ll t \gg.$$

That is,  $g(\varphi_j(t), t) = 0$ ,  $j = 1, 2$  (see e.g. Section 2.5 in Sendra et al. (2007)). Note that  $\nu(\varphi_1) = 1$ , which implies that we only have one Puiseux series in the conjugacy class of  $\varphi_1$ . However,  $\nu(\varphi_2) = 2$  and then, we have the following conjugate Puiseux series in the conjugacy class of  $\varphi_2$ :

$$\varphi_{2,1}(t) = \frac{-(-2t)^{1/2}}{2} - \frac{t}{8} + \frac{27}{256}(-2t)^{3/2} - \frac{7}{32}t^2 + \frac{4057}{65536}(-2t)^{5/2} + \dots$$

$$\varphi_{2,2}(t) = \frac{+(-2t)^{1/2}}{2} - \frac{t}{8} - \frac{27}{256}(-2t)^{3/2} - \frac{7}{32}t^2 - \frac{4057}{65536}(-2t)^{5/2} + \dots$$

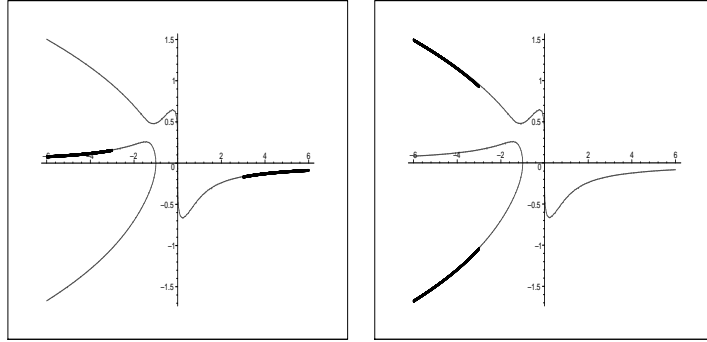


Figure 1: Infinity branches  $B_1$  (left), and  $B_2$  (right).

Thus, we obtain two infinity branches:

$$B_1 = L_1 = \{(z, r_1(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}, \quad \text{where}$$

$$r_1(z) = z\varphi_1(z^{-1}) = -1/(2z) + 1/(8z^3) - 1/(8z^4) + 1/(16z^5) + 1/(16z^6) + \dots$$

and  $B_2 = L_{2,1} \cup L_{2,2}$ , where  $L_{2,i} = \{(z, r_{2,i}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ ,  $i = 1, 2$  and

$$r_{2,1}(z) = z\varphi_{2,1}(z^{-1}) = -\frac{(-2z)^{1/2}}{2} - \frac{1}{8} + \frac{27(-2z)^{-1/2}}{64} - \frac{7z^{-1}}{32} + \frac{4057(-2z)^{-3/2}}{4096} + \dots$$

$$r_{2,2}(z) = z\varphi_{2,2}(z^{-1}) = +\frac{(-2z)^{1/2}}{2} - \frac{1}{8} - \frac{27(-2z)^{-1/2}}{64} - \frac{7z^{-1}}{32} - \frac{4057(-2z)^{-3/2}}{4096} + \dots$$

In Figure 1, we plot the curve  $\mathcal{C}$  and some points of the infinity branches  $B_1$  and  $B_2$  associated to  $P$ .

In the following, we prove that any point of the curve with sufficiently large coordinates belongs to some infinity branch. For this purpose, we recall the reader that if  $h$  is a complex-valued function of a complex variable,  $h : \mathbb{C} \rightarrow \mathbb{C}$ , we say that the limit of  $h(z)$  as  $z$  approaches  $\infty$  is  $L$ , written  $\lim_{z \rightarrow \infty} h(z) = L$ , if whenever  $\{z_n\}_{n \in \mathbb{N}}$  is a sequence of points with  $\lim_{n \rightarrow \infty} z_n = \infty$ , it holds that  $\lim_{n \rightarrow \infty} h(z_n) = L$  (see e.g. Ahlfors (1979) or Conway (1995)).

**Lemma 3.6.** *Let  $\mathcal{C}$  be an algebraic plane curve. There exists  $K \in \mathbb{R}^+$  such that for every  $p = (a, b) \in \mathcal{C}$  with  $|a| > K$ , it holds that  $p \in B_p$ , where  $B_p$  is an infinity branch of  $\mathcal{C}$ .*

**Proof:** Let us assume that the lemma does not hold, and we consider a sequence  $\{K_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$  such that  $\lim_{n \rightarrow \infty} K_n = \infty$ . Then, for every  $n \in \mathbb{N}$  there exists a point  $p_n = (a_n, b_n) \in \mathcal{C}$  such that  $|a_n| > K_n$ , and  $p_n$  does not belong to any infinity branch of  $\mathcal{C}$ .

Let  $P_n = (a_n : b_n : 1)$ . Since  $F(P_n) = f(p_n) = 0$ , then  $\lim_{n \rightarrow \infty} F(P_n) = 0$ . Thus, we distinguish two different cases:

- a) If there exists a monotone subsequence  $\{b_{n_l}/a_{n_l}\}_{l \in \mathbb{N}}$  that is not bounded, we have that  $\lim_{l \rightarrow \infty} b_{n_l}/a_{n_l} = \infty$ , and then  $\lim_{l \rightarrow \infty} a_{n_l}/b_{n_l} = 0$ . Hence,

$$\lim_{l \rightarrow \infty} F(Q_{n_l}) = F(0 : 1 : 0) = 0, \quad Q_{n_l} = (a_{n_l}/b_{n_l} : 1 : 1/b_{n_l})$$

which implies that  $P = (0 : 1 : 0)$  is an infinity point of  $\mathcal{C}^*$ .

- b) If there exists a monotone subsequence  $\{b_{n_l}/a_{n_l}\}_{l \in \mathbb{N}}$  that is bounded, we have that  $\lim_{l \rightarrow \infty} b_{n_l}/a_{n_l} = m$ . Thus,

$$\lim_{l \rightarrow \infty} F(Q_{n_l}) = F(1 : m : 0) = 0, \quad Q_{n_l} = (1 : b_{n_l}/a_{n_l} : 1/a_{n_l})$$

which implies that  $P = (1 : m : 0)$  is an infinity point of  $\mathcal{C}^*$ .

From both situations, we conclude that there exist a sequence  $\{Q_n\}_{n \in \mathbb{N}}$  that approaches to an infinity point  $P$  as  $n$  tends to infinity; that is, there exists  $M \in \mathbb{R}^+$  such that  $\|Q_n - P\| \leq \epsilon$ , for  $n \geq M$ . Thus, we deduce that  $\{Q_n\}_{n \in \mathbb{N}, n \geq M}$  can be obtained by a place centered at  $P$ . Hence,  $p_n$  belongs to some infinity branch of  $\mathcal{C}$ , which contradicts the hypothesis.  $\square$

**Remark 3.7.** Reasoning similarly as in Lemma 3.6, one has that there exists  $K \in \mathbb{R}^+$  such that for every  $p = (a, b) \in \mathcal{C}$  with  $|b| > K$ , it holds that  $p \in B_p$ , where  $B_p$  is an infinity branch of  $\mathcal{C}$ .

#### 4. Convergent Branches and Approaching Curves

In this section, we introduce the notions of convergent branches and approaching curves. Intuitively speaking, two infinity branches converge if they get closer as they tend to infinity. This concept will allow us to analyze whether two curves approach each other at the infinity.

The results presented in this section will be used in Section 5, where a method to compare the asymptotic behavior of two curves is developed.

**Definition 4.1.** Given two leaves,  $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$  and  $\bar{L} = \{(z, \bar{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \bar{M}\}$ , we say that they are convergent if  $\lim_{z \rightarrow \infty} (\bar{r}(z) - r(z)) = 0$ .

**Lemma 4.2.** Two leaves  $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$  and  $\bar{L} = \{(z, \bar{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \bar{M}\}$  are convergent if and only if the terms with non negative exponent in the series  $r(z)$  and  $\bar{r}(z)$  are the same.

**Proof:** Let

$$r(z) = mz + a_1 z^{\frac{N-N_1}{N}} + a_2 z^{\frac{N-N_2}{N}} + \dots, \quad N, N_i \in \mathbb{N}, 0 < N_1 < N_2 < \dots, a_i \neq 0$$

and

$$\bar{r}(z) = \bar{m}z + b_1 z^{\frac{\bar{N}-\bar{N}_1}{\bar{N}}} + b_2 z^{\frac{\bar{N}-\bar{N}_2}{\bar{N}}} + \dots, \quad \bar{N}, \bar{N}_i \in \mathbb{N}, \quad 0 < \bar{N}_1 < \bar{N}_2 < \dots, \quad b_i \neq 0.$$

Then,

$$r(z) - \bar{r}(z) = mz - \bar{m}z + a_1 z^{\frac{N-N_1}{N}} - b_1 z^{\frac{\bar{N}-\bar{N}_1}{\bar{N}}} + a_2 z^{\frac{N-N_2}{N}} - b_2 z^{\frac{\bar{N}-\bar{N}_2}{\bar{N}}} + \dots.$$

Note that  $\lim_{z \rightarrow \infty} (r(z) - \bar{r}(z)) = 0$  if and only if  $r(z) - \bar{r}(z)$  has no terms with non negative exponent. This situation holds if the terms with non negative exponent in both series,  $r(z)$  and  $\bar{r}(z)$ , are the same.  $\square$

**Remark 4.3.** 1. From Lemma 4.2, we deduce that  $m = \bar{m}$  and then,  $L$  and  $\bar{L}$  are associated to the same infinity point.  
2. Note that the number of terms with positive exponent in both series is finite.

**Definition 4.4.** Two infinity branches,  $B$  and  $\bar{B}$ , are convergent if there exist two convergent leaves  $L \subset B$  and  $\bar{L} \subset \bar{B}$ .

**Remark 4.5.** From Remark 4.3, statement 1, we get that two convergent infinity branches are associated to the same infinity point.

**Proposition 4.6.** Two infinity branches  $B$  and  $\bar{B}$  are convergent if and only if for each leaf  $L \subset B$  there exists a leaf  $\bar{L} \subset \bar{B}$  convergent with  $L$ , and reciprocally.

**Proof:** Let  $B$  and  $\bar{B}$  be two convergent infinity branches, and let us prove that for any  $L_i \subset B$  there exists  $\bar{L}_j \subset \bar{B}$  convergent with  $L_i$  (using Definition 4.4, we clearly have the reciprocal). From Definition 4.4, there exist two leaves  $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$ , and  $\bar{L} = \{(z, \bar{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \bar{M}\} \subset \bar{B}$  convergent. Let

$$r(z) = z\varphi(z^{-1}) = mz + u_1 z^{1-\frac{N_1}{N}} + \dots + u_k z^{1-\frac{N_k}{N}} + u_{k+1} z^{1-\frac{N_{k+1}}{N}} + \dots, \quad u_i \neq 0,$$

$$\bar{r}(z) = z\bar{\varphi}(z^{-1}) = mz + \bar{u}_1 z^{1-\frac{\bar{N}_1}{\bar{N}}} + \dots + \bar{u}_k z^{1-\frac{\bar{N}_k}{\bar{N}}} + \bar{u}_{k+1} z^{1-\frac{\bar{N}_{k+1}}{\bar{N}}} + \dots, \quad \bar{u}_i \neq 0,$$

where  $\nu(B) = N$ ,  $\nu(\bar{B}) = \bar{N}$ ,  $N_k \leq N < N_{k+1}$  and  $\bar{N}_k \leq \bar{N} < \bar{N}_{k+1}$ .

From Lemma 4.2, we deduce that the terms with non negative exponent in  $r$  and  $\bar{r}$  must coincide. Thus,  $u_l = \bar{u}_l = a_l$ , for  $l = 1, \dots, k$ , and

$$r(z) = mz + a_1 z^{1-\frac{n_1}{n}} + \dots + a_k z^{1-\frac{n_k}{n}} + u_{k+1} z^{1-\frac{N_{k+1}}{N}} + \dots, \quad a_i, u_i \neq 0$$

$$\bar{r}(z) = mz + a_1 z^{1-\frac{n_1}{n}} + \dots + a_k z^{1-\frac{n_k}{n}} + \bar{u}_{k+1} z^{1-\frac{\bar{N}_{k+1}}{\bar{N}}} + \dots, \quad a_i, \bar{u}_i \neq 0,$$

where  $n, n_i \in \mathbb{N}$ ,  $0 < n_1 < \dots < n_k < n$ ,  $N_{k+1} > N, \bar{N}_{k+1} > \bar{N}$ . Observe that we have simplified the non negative exponents such that  $\gcd(n, n_1, \dots, n_k) = 1$ . That is, for  $l = 1, \dots, k$ , there are  $b, \bar{b} \in \mathbb{N}$  such that  $N_l = bn_l$ ,  $N = bn$ ,  $\bar{N}_l = \bar{b}n_l$ , and  $\bar{N} = \bar{b}n$ .

Under these conditions, we observe that the different leaves of  $B$  and  $\bar{B}$  are obtained by conjugation on  $r(z)$  and  $\bar{r}(z)$ . That is (see equation (1)),

$$r_i(z) = mz + u_1 c_i^{N_1} z^{1-\frac{N_1}{N}} + \dots + u_k c_i^{N_k} z^{1-\frac{N_k}{N}} + u_{k+1} c_i^{N_{k+1}} z^{1-\frac{N_{k+1}}{N}} + \dots$$

$$\bar{r}_j(z) = mz + \bar{u}_1 d_j^{\bar{N}_1} z^{1-\frac{\bar{N}_1}{\bar{N}}} + \dots + \bar{u}_k d_j^{\bar{N}_k} z^{1-\frac{\bar{N}_k}{\bar{N}}} + \bar{u}_{k+1} d_j^{\bar{N}_{k+1}} z^{1-\frac{\bar{N}_{k+1}}{\bar{N}}} + \dots,$$

where  $c_1, \dots, c_N$  are the  $N$  complex roots of  $x^N = 1$ , and  $d_1, \dots, d_{\bar{N}}$  are the  $\bar{N}$  complex roots of  $x^{\bar{N}} = 1$ .

We simplify the exponents and, using that  $u_l = \bar{u}_l = a_l$ ,  $l = 1, \dots, k$ , we get that:

$$r_i(z) = mz + a_1 c_i^{N_1} z^{1-\frac{n_1}{n}} + \dots + a_k c_i^{N_k} z^{1-\frac{n_k}{n}} + u_{k+1} c_i^{N_{k+1}} z^{1-\frac{N_{k+1}}{N}} + \dots$$

$$\bar{r}_j(z) = mz + a_1 d_j^{\bar{N}_1} z^{1-\frac{n_1}{n}} + \dots + a_k d_j^{\bar{N}_k} z^{1-\frac{n_k}{n}} + \bar{u}_{k+1} d_j^{\bar{N}_{k+1}} z^{1-\frac{\bar{N}_{k+1}}{\bar{N}}} + \dots.$$

Hence, we only have to show that for each  $i \in \{1, \dots, N\}$  there exists  $j \in \{1, \dots, \bar{N}\}$  such that  $c_i^{N_l} = d_j^{\bar{N}_l}$  for every  $l = 1, \dots, k$ . Indeed: since  $c_i, i = 1, \dots, N$  are the  $N$  complex roots of  $x^N = 1$ , we have that  $c_i = e^{\frac{2(i-1)\pi I}{N}}$ , where  $I$  is the imaginary unit. Taking into account that  $N = bn$ , we deduce that  $c_i^b = e^{\frac{2(i-1)\pi I}{n}}, i = 1, \dots, N$ , and  $c_i^b = c_{i+(m-1)n}^b$  for each  $i = 1, \dots, n$ , and  $m = 1, \dots, b$ . That is,  $(c_i^b)^n = 1, i = 1, \dots, n$ . Reasoning similarly, we have that  $d_j^{\bar{b}} = e^{\frac{2(j-1)\pi I}{n}}, j = 1, \dots, \bar{N}$ , and  $d_j^{\bar{b}} = d_{j+(m-1)n}^{\bar{b}}$  for each  $j = 1, \dots, n$ , and  $m = 1, \dots, \bar{b}$ . That is,  $(d_j^{\bar{b}})^n = 1, j = 1, \dots, n$ . Therefore,  $c_i^b = d_{i+(m-1)n}^{\bar{b}}, m = 1, \dots, \bar{b}$ , and using that  $N_l = bn_l$  and  $\bar{N}_l = \bar{b}n_l, l = 1, \dots, k$ , it follows that  $c_i^{N_l} = d_j^{\bar{N}_l}, j = i + (m-1)n, m = 1, \dots, \bar{b}$ .  $\square$

**Remark 4.7.** *Two convergent infinity branches may have different ramification indexes i.e., they may have different number of leaves. However, the value  $n \in \mathbb{N}$  obtained by simplifying the non negative exponents, is the same in both branches. We refer to it the degree of the infinity branch. Observe that the proof of Proposition 4.6 implies that two convergent infinity branches have the same degree.*

*In order to illustrate this remark, we consider the curves  $\mathcal{C}$  and  $\bar{\mathcal{C}}$ , defined by the polynomials  $f(x, y) = y^4 - 2xy^2 + x^2 - y$ , and  $\bar{f}(x, y) = y^2 - x$ , respectively.  $\mathcal{C}$  has only the infinity branch  $B = \bigcup_{i=1}^4 L_i$ , where  $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ ,*

$$r_i(z) = c_i^2 z^{1/2} + \frac{1}{2} c_i^5 z^{-1/4} - \frac{1}{64} c_i^{11} z^{-7/4} + \frac{1}{128} c_i^{14} z^{-10/4} + \dots,$$

*and  $c_1 = 1, c_2 = I, c_3 = -1, c_4 = -I$ . Note that the first term of these series is  $z^{1/2}$  or  $-z^{1/2}$ . The curve  $\bar{\mathcal{C}}$  also has one infinity branch defined by  $\bar{B} = \bigcup_{i=1}^2 \bar{L}_i$ , where  $\bar{L}_i = \{(z, \bar{r}_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \bar{M}\}$ ,  $\bar{r}_i(z) = d_i z^{1/2}$ , and  $d_1 = 1, d_2 = -1$ . We get that  $B$  and  $\bar{B}$ , are convergent since  $L_1$  and  $\bar{L}_1$  converge. In fact,  $L_1$  and  $L_3$  converge with  $\bar{L}_1$  and, on the other hand,  $L_2$  and  $L_4$  converge with  $\bar{L}_2$  (see Lemma 4.2).*

Two convergent infinity branches may be contained in the same curve or they may belong to different curves. In this second case we will say that those curves *approach each other*. In order to define this concept in a more formal way, we first introduce the following distance:

**Definition 4.8.** *Given an algebraic plane curve  $\mathcal{C}$  over  $\mathbb{C}$  and a point  $p \in \mathbb{C}^2$ , we define the distance from  $p$  to  $\mathcal{C}$  as  $d(p, \mathcal{C}) = \min\{d(p, q) : q \in \mathcal{C}\}$ .*

**Remark 4.9.** *Observe that this minimum exists because  $\mathcal{C}$  is a closed set.*

**Definition 4.10.** *Let  $\mathcal{C}$  be an algebraic plane curve over  $\mathbb{C}$  with an infinity branch  $B$ . We say that a curve  $\bar{\mathcal{C}}$  approaches  $\mathcal{C}$  at its infinity branch  $B$  if there exists one leaf  $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$  such that  $\lim_{z \rightarrow \infty} d((z, r(z)), \bar{\mathcal{C}}) = 0$ .*

We will show that this condition is satisfied for one leaf of  $B$  if and only if it is satisfied for every leaf of  $B$ . It will be derived as a consequence of the following theorem.



**Theorem 4.11.** *Let  $\mathcal{C}$  be a plane algebraic curve over  $\mathbb{C}$  with an infinity branch  $B$ . A plane algebraic curve  $\bar{\mathcal{C}}$  approaches  $\mathcal{C}$  at  $B$  if and only if  $\bar{\mathcal{C}}$  has an infinity branch,  $\bar{B}$ , such that  $B$  and  $\bar{B}$  are convergent.*

**Proof:** Suppose that  $\bar{\mathcal{C}}$  approaches  $\mathcal{C}$  at  $B$ . Then, there exists a leaf  $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$  such that  $\lim_{z \rightarrow \infty} d((z, r(z)), \bar{\mathcal{C}}) = 0$ . In addition, let  $P = (1 : m : 0)$  be the infinity point associated to  $B$ , and let  $\{z_n\}_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{C}$  such that  $\lim_{n \rightarrow \infty} z_n = \infty$ . We have that  $\lim_{n \rightarrow \infty} d((z_n, r(z_n)), \bar{\mathcal{C}}) = 0$  which implies that

$$\lim_{n \rightarrow \infty} d((z_n, r(z_n)), (p_n, q_n)) = 0,$$

where, for each  $z_n$  such that  $|z_n| > M$ ,  $(p_n, q_n)$  is the point of  $\bar{\mathcal{C}}$  closest to the point  $(z_n, r(z_n))$  (this point exists because of Definition 4.8). Note that the above equality implies that

$$\lim_{n \rightarrow \infty} |p_n - z_n|^2 + |q_n - r(z_n)|^2 = 0,$$

and hence we have that:

- $\lim_{n \rightarrow \infty} (p_n - z_n) = 0$ . Then,  $\lim_{n \rightarrow \infty} p_n/z_n = 1$  which implies that  $\lim_{n \rightarrow \infty} p_n = \infty$ . Hence,  $\lim_{n \rightarrow \infty} 1/p_n = 0$ .
- $\lim_{n \rightarrow \infty} (q_n - r(z_n)) = 0$ . Then,  $\lim_{n \rightarrow \infty} (q_n/z_n - r(z_n)/z_n) = 0$  which implies that  $\lim_{n \rightarrow \infty} q_n/z_n = \lim_{n \rightarrow \infty} r(z_n)/z_n = m$ .

Therefore,

$$\lim_{n \rightarrow \infty} q_n/p_n = \lim_{n \rightarrow \infty} \frac{q_n/z_n}{p_n/z_n} = m.$$

Now, taking into account Lemma 3.6 and that  $\lim_{n \rightarrow \infty} p_n = \infty$ , we get that there exists  $n_0 \in \mathbb{N}$  such that for  $n \geq n_0$ , the points  $(p_n, q_n)$  are in some infinity branch of  $\bar{\mathcal{C}}$ . Moreover, since any curve has a finite number of infinity branches and a finite number of leaves, we can find a subsequence  $\{z_{n_l}\}_{l \in \mathbb{N}}$  and  $l_0 \in \mathbb{N}$  such that for  $l \geq l_0$ , the points  $(p_{n_l}, q_{n_l})$  are all in a same leaf  $\bar{L} = \{(z, \bar{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \bar{M}\}$ , belonging to some branch  $\bar{B} \subset \bar{\mathcal{C}}$ .

Under these conditions, we deduce that for  $l \geq l_0$ ,  $q_{n_l} = \bar{r}(p_{n_l})$ , and then

$$\lim_{l \rightarrow \infty} \bar{r}(p_{n_l})/p_{n_l} = \lim_{l \rightarrow \infty} q_{n_l}/p_{n_l} = m.$$

Since the limit  $\lim_{z \rightarrow \infty} \frac{\bar{r}(z)}{z} = \lim_{z \rightarrow \infty} \bar{\varphi}(z^{-1})$  exists, we get  $\lim_{z \rightarrow \infty} \frac{\bar{r}(z)}{z} = m$ .

In addition, note that

$$\begin{aligned} |r(z_{n_l}) - \bar{r}(z_{n_l})| &= d((z_{n_l}, r(z_{n_l})), (z_{n_l}, \bar{r}(z_{n_l}))) \leq \\ &d((z_{n_l}, r(z_{n_l})), (p_{n_l}, \bar{r}(p_{n_l}))) + d((p_{n_l}, \bar{r}(p_{n_l})), (z_{n_l}, \bar{r}(z_{n_l}))) \quad (\text{I}) \end{aligned}$$

and

$$d((z_{n_l}, r(z_{n_l})), (p_{n_l}, \bar{r}(p_{n_l}))) = d((z_{n_l}, r(z_{n_l})), (p_{n_l}, q_{n_l})) \xrightarrow[l \rightarrow \infty]{} 0.$$

Now, let us prove that  $d((p_{n_l}, \bar{r}(p_{n_l})), (z_{n_l}, \bar{r}(z_{n_l}))) \xrightarrow[l \rightarrow \infty]{} 0$ . For this purpose, we show that  $\lim_{l \rightarrow \infty} (\bar{r}(p_{n_l}) - \bar{r}(z_{n_l})) = 0$ . Indeed: let

$$\bar{r}(z) = mz + b_1 z^{\frac{s-s_1}{s}} + b_2 z^{\frac{s-s_2}{s}} + \dots, \quad s, s_i \in \mathbb{N}, 0 < s_1 < s_2 < \dots.$$

Thus,

$$\bar{r}'(z) = m + \frac{s-s_1}{s} b_1 z^{\frac{-s_1}{s}} + \frac{s-s_2}{s} b_2 z^{\frac{-s_2}{s}} + \dots \xrightarrow[z \rightarrow \infty]{} m.$$

Therefore, there exist  $K > 0$  and  $\delta > 0$  such that  $|\bar{r}'(z)| \leq K$ , for  $|z| > \delta$ . Applying the Mean Value Theorem (see Ahlfors (1979)), we have that

$$\operatorname{Re} \left( \frac{\bar{r}(p_{n_l}) - \bar{r}(z_{n_l})}{p_{n_l} - z_{n_l}} \right) = \operatorname{Re}(\bar{r}'(c_1)), \quad \operatorname{Im} \left( \frac{\bar{r}(p_{n_l}) - \bar{r}(z_{n_l})}{p_{n_l} - z_{n_l}} \right) = \operatorname{Im}(\bar{r}'(c_2)),$$

where  $\operatorname{Re}(q)$  and  $\operatorname{Im}(q)$  denote the real part and the imaginary part of  $q(z) \in \mathbb{C}(z)$ , respectively, and  $c_1, c_2 \in ]p_{n_l}, z_{n_l}[$ , where  $]p_{n_l}, z_{n_l}[ := \{z \in \mathbb{C} : z = p_{n_l} + (p_{n_l} - z_{n_l})t, t \in (0, 1)\}$ . Thus,

$$|\bar{r}(p_{n_l}) - \bar{r}(z_{n_l})|^2 = (\operatorname{Re}(\bar{r}'(c_1))^2 + \operatorname{Im}(\bar{r}'(c_2))^2) |p_{n_l} - z_{n_l}|^2.$$

Now, since  $\lim_{l \rightarrow \infty} p_{n_l} = \lim_{l \rightarrow \infty} z_{n_l} = \infty$ , and  $\lim_{l \rightarrow \infty} p_{n_l} - z_{n_l} = 0$ , we deduce that given  $\varepsilon > 0$ , there exists  $l_1 \in \mathbb{N}$  such that, for  $l \geq l_1$ ,

$$|p_{n_l}| > \delta + \varepsilon, \quad |z_{n_l}| > \delta + \varepsilon, \quad \text{and} \quad |p_{n_l} - z_{n_l}| < \varepsilon.$$

Then,  $|c_j| > \delta$  and  $|\bar{r}'(c_j)| \leq K$  for  $j = 1, 2$ , which implies that, for  $l \geq l_1$ ,

$$|\bar{r}(p_{n_l}) - \bar{r}(z_{n_l})| \leq \sqrt{2}K |p_{n_l} - z_{n_l}| \xrightarrow[l \rightarrow \infty]{} 0$$

(note that  $\operatorname{Re}(\bar{r}'(c_1)) \leq |\bar{r}'(c_1)| \leq K$ , and  $\operatorname{Im}(\bar{r}'(c_2)) \leq |\bar{r}'(c_2)| \leq K$ ). Therefore,  $\lim_{l \rightarrow \infty} (\bar{r}(p_{n_l}) - \bar{r}(z_{n_l})) = 0$ , which implies that there exists a sequence  $\{z_{n_l}\}_{l \in \mathbb{N}}$  with  $\lim_{l \rightarrow \infty} z_{n_l} = \infty$ , such that

$$\lim_{l \rightarrow \infty} (r(z_{n_l}) - \bar{r}(z_{n_l})) = 0$$

(see inequality (I)). Then, the terms with positive exponent of the series  $r(z)$  and  $\bar{r}(z)$  are the same (see the proof of Lemma 4.2). Hence, we conclude that (see Lemma 4.2)

$$\lim_{z \rightarrow \infty} (r(z) - \bar{r}(z)) = 0$$

and thus  $B$  and  $\bar{B}$  are convergent (see Definition 4.4).

Reciprocally, let us assume that  $B$  and  $\bar{B}$  are convergent. Then, by definition, there exist two leaves  $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$  and  $\bar{L} = \{(z, \bar{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \bar{M}\} \subset \bar{B}$  such that  $\lim_{z \rightarrow \infty} (r(z) - \bar{r}(z)) = 0$ . Therefore,

$$\lim_{z \rightarrow \infty} d((z, r(z)), \bar{\mathcal{C}}) \leq \lim_{z \rightarrow \infty} d((z, r(z)), (z, \bar{r}(z))) = \lim_{z \rightarrow \infty} (r(z) - \bar{r}(z)) = 0. \quad \square$$

- Remark 4.12.**
1. From Theorem 4.11, we get that “proximity” is a symmetric relation; i.e.,  $\bar{\mathcal{C}}$  approaches  $\mathcal{C}$  at some infinity branch  $B$  iff  $\mathcal{C}$  approaches  $\bar{\mathcal{C}}$  at some infinity branch  $\bar{B}$ . In the following, we say that  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  approach each other or that they are approaching curves.
  2. Theorem 4.11 and Remark 4.5 imply that two approaching curves have a common infinity point.
  3. From Theorem 4.11 and Proposition 4.6, we get that  $\bar{\mathcal{C}}$  approaches  $\mathcal{C}$  at an infinity branch  $B$  if and only if for every leaf  $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$ , it holds that  $\lim_{z \rightarrow \infty} d((z, r(z)), \bar{\mathcal{C}}) = 0$ .

**Corollary 4.13.** *Let  $\mathcal{C}$  be an algebraic plane curve with an infinity branch  $B$ . Let  $\bar{\mathcal{C}}_1$  and  $\bar{\mathcal{C}}_2$  be two different curves that approach  $\mathcal{C}$  at  $B$ . Then  $\bar{\mathcal{C}}_1$  and  $\bar{\mathcal{C}}_2$  approach each other.*

**Proof:** From Theorem 4.11, there exist two infinity branches  $B_1 \subset \bar{\mathcal{C}}_1$  and  $B_2 \subset \bar{\mathcal{C}}_2$ , convergent with  $B$ . Thus, for each leaf  $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$ , there exist two leaves  $L_1 = \{(z, r_1(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| >$

$M_1\} \subset B_1$  and  $L_2 = \{(z, r_2(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_2\} \subset B_2$  such that  $\lim_{z \rightarrow \infty} (r(z) - r_1(z)) = 0$  and  $\lim_{z \rightarrow \infty} (r(z) - r_2(z)) = 0$ . Then

$$|r_1(z) - r_2(z)| \leq |r_1(z) - r(z)| + |r(z) - r_2(z)| \xrightarrow{|z| \rightarrow \infty} 0.$$

Therefore,  $\bar{\mathcal{C}}_1$  and  $\bar{\mathcal{C}}_2$  approach each other.  $\square$

In the following, we illustrate the above results with an example.

**Example 4.14.** Let  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  be two plane curves implicitly defined by the polynomials

$$f(x, y) = 2y^3x - y^4 + 2y^2x - y^3 - 2x^3 + x^2y + 3 \in \mathbb{R}[x, y], \quad \text{and}$$

$$\bar{f}(x, y) = y^3x - y^4 + y^2x - y^3 - x^3 + x^2y + 2 \in \mathbb{R}[x, y],$$

respectively. Let us prove that  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  approach each other (see Figure 2) at the infinity branch associated to the infinity point  $P = (1 : 0 : 0)$  (note that both curves have  $P$  as an infinity point).

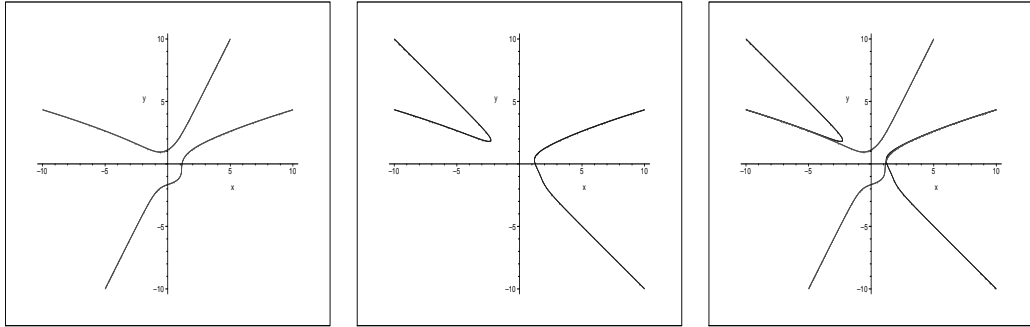


Figure 2:  $\mathcal{C}$  (left),  $\bar{\mathcal{C}}$  (center), and both approaching curves (right)

Reasoning as in Example 3.5, we get that the infinity branch of  $\mathcal{C}$  associated to  $P$  is given by  $B = L_1 \cup L_2 \cup L_3$ , where  $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ ,

$$r_i(z) = c_i^2 z^{2/3} - 1/3 + 1/9 c_i^2 z^{-2/3} - 2/81 c_i^4 z^{-4/3} - 1/2 c_i^7 z^{-7/3} + \dots$$

and  $c_i, i = 1, 2, 3$  are the complex roots of  $x^3 = 1$ . On the other hand, the infinity branch of  $\bar{\mathcal{C}}$  associated to  $P$  is given by  $\bar{B} = \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3$ , where  $\bar{L}_i = \{(z, \bar{r}_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ ,

$$\bar{r}_i(z) = c_i^2 z^{2/3} - 1/3 + 1/9 c_i^2 z^{-2/3} - 2/81 c_i^4 z^{-4/3} - 2/3 c_i^7 z^{-7/3} + \dots$$

and  $c_i$ ,  $i = 1, 2, 3$  are the complex roots of  $x^3 = 1$  (to compute  $r_i$  and  $\bar{r}_i$ , we use the `algcures` package included in Maple). From Lemma 4.2, we conclude that both branches converge, since the terms with non negative exponent in both series,  $r_i$  and  $\bar{r}_i$ , are the same.

**Remark 4.15.** *In the above example, the curves  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  are approaching curves, since  $\mathcal{C}$  approaches  $\bar{\mathcal{C}}$  at one of its infinity branches, and reciprocally. However,  $\bar{\mathcal{C}}$  has another infinity branch which is not approached by  $\mathcal{C}$  and  $\mathcal{C}$  has an infinity branch which is not approached by  $\bar{\mathcal{C}}$ . When a curve approaches another curve at all of its infinity branches and reciprocally, we say that both curves have the same asymptotic behavior. We focus on this special relation in the next section.*

## 5. Asymptotic Behavior

Using the results presented in the previous sections, in the following we present an algorithm that allows us to compare the asymptotic behavior of two curves. We say that two curves have the same *asymptotic behavior* if they approach each other at all of the infinity branches. In addition, we prove that if two plane algebraic curves have the same *asymptotic behavior*, the Hausdorff distance between them is finite. We remark that the results presented in the paper, and in particular in this section, can be stated similarly for curves in the affine  $n$ -space (see Blasco and Pérez-Díaz (2014b)).

The algorithm presented, as well as the results developed in this section, play an important role in the frame of practical applications in C.A.G.D such as approximate parametrization problems (see Section 1). In particular, estimating the Hausdorff distance between two curves is specially interesting since it is an appropriate tool for measuring the closeness between these two curves. Many authors have addressed some problems in this frame (see e.g. Bai et al. (2011), Chen et al. (2010), Kim et al. (2010), Patrikalakis and Maekawa. (2001), Rueda et al. (2014), etc).

To start with, we first introduce the following definition.

**Definition 5.1.** *We say that two algebraic plane curves,  $\mathcal{C}$  and  $\bar{\mathcal{C}}$ , have the same asymptotic behavior if every infinity branch of  $\mathcal{C}$  converges to another branch of  $\bar{\mathcal{C}}$ , and reciprocally.*

**Remark 5.2.** *From Theorem 4.11, we get that  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  have the same asymptotic behavior iff  $\mathcal{C}$  approaches  $\bar{\mathcal{C}}$  at all its infinity branches, and reciprocally.*

Now, we recall the notion of *Hausdorff distance*.

**Definition 5.3.** Given a metric space  $(E, d)$  and two subsets  $A, B \subset E \setminus \{\emptyset\}$ , the Hausdorff distance between them is defined as:

$$d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y)\}.$$

If  $E = \mathbb{C}^2$  and  $d$  is the Euclidean distance, the Hausdorff distance between two curves  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  can be expressed as:

$$d_H(\mathcal{C}, \bar{\mathcal{C}}) = \max\{\sup_{p \in \mathcal{C}} d(p, \bar{\mathcal{C}}), \sup_{\bar{p} \in \bar{\mathcal{C}}} d(\bar{p}, \mathcal{C})\}.$$

**Proposition 5.4.** Let  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  be two algebraic plane curves having the same asymptotic behavior. Then, the Hausdorff distance between them is finite.

**Proof:** Let  $r$  be the number of infinity branches of  $\mathcal{C}$ . Then,  $\mathcal{C} = B_1 \cup \dots \cup B_r \cup \widehat{B}$ , where  $\widehat{B}$  is the set of points of  $\mathcal{C}$  that do not belong to any infinity branch. Thus,

$$\sup_{p \in \mathcal{C}} d(p, \bar{\mathcal{C}}) = \max\{\sup_{p \in B_1} d(p, \bar{\mathcal{C}}), \dots, \sup_{p \in B_r} d(p, \bar{\mathcal{C}}), \sup_{p \in \widehat{B}} d(p, \bar{\mathcal{C}})\}.$$

For each  $i = 1, \dots, r$ , let  $B_i = \bigcup_{j=1}^{N_i} L_{i,j}$ , where  $L_{i,j} = \{(z, r_{i,j}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_i\}$ , and  $N_i = \nu(B_i)$ . Then,

$$\sup_{p \in B_i} d(p, \bar{\mathcal{C}}) = \max_{j=1, \dots, N_i} \left\{ \sup_{|z| > M_i} d((z, r_{i,j}(z)), \bar{\mathcal{C}}) \right\}.$$

Moreover, from Remark 5.2,  $\bar{\mathcal{C}}$  approaches  $\mathcal{C}$  at  $B_i$ , so  $\lim_{z \rightarrow \infty} d((z, r_{i,j}(z)), \bar{\mathcal{C}}) = 0$  for every  $j = 1, \dots, N_i$ . Hence, given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $d((z, r_{i,j}(z)), \bar{\mathcal{C}}) < \varepsilon$ , for  $|z| > \delta$ . Then, since  $r_{i,j}$  is a continuous function, and  $\{z \in \mathbb{C} : M_i \leq |z| \leq \delta\}$  is a compact set, we deduce that

$$\sup_{p \in B_i} d(p, \bar{\mathcal{C}}) \leq \max_{j=1, \dots, N_i} \max \left\{ \sup_{M_i \leq |z| \leq \delta} d((z, r_{i,j}(z)), \bar{\mathcal{C}}), \varepsilon \right\} < \infty.$$

Now, let  $p = (a, b) \in \widehat{B}$ . From Lemma 3.6 and Remark 3.7, we have that there exists  $K \in \mathbb{R}^+$  such that  $|a|, |b| \leq K$ . Thus,  $d(p, \mathcal{O}) \leq K$ , where  $\mathcal{O}$  is the origin and,

$$d(p, \bar{\mathcal{C}}) \leq d(p, \mathcal{O}) + d(\mathcal{O}, \bar{\mathcal{C}}) \leq K + d(\mathcal{O}, \bar{\mathcal{C}}).$$

Note that  $K < \infty$ , and  $d(\mathcal{O}, \bar{\mathcal{C}}) < \infty$ , which implies that  $\sup_{p \in \bar{\mathcal{B}}} d(p, \bar{\mathcal{C}}) < \infty$ .

Therefore, we conclude that  $\sup_{p \in \mathcal{C}} d(p, \bar{\mathcal{C}}) < \infty$ . Reasoning similarly, we deduce that  $\sup_{\bar{p} \in \bar{\mathcal{C}}} d(\bar{p}, \mathcal{C}) < \infty$ , which implies that  $d_H(\mathcal{C}, \bar{\mathcal{C}}) < \infty$ .  $\square$

The following algorithm allow us to compare the asymptotic behavior of two curves  $\mathcal{C}$  and  $\bar{\mathcal{C}}$ . We assume that we have prepared  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  such that by means of a suitable linear change of coordinates (the same change applied to both curves),  $(0 : 1 : 0)$  is not a point of infinity of  $\mathcal{C}^*$  and  $\bar{\mathcal{C}}^*$ .

**Algorithm Asymptotic Behavior.**

Given two implicit algebraic plane curves  $\mathcal{C}$  and  $\bar{\mathcal{C}}$ , the algorithm decides whether  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  have the same asymptotic behavior.

1. Compute the infinity points of  $\mathcal{C}$  and  $\bar{\mathcal{C}}$ . If they are not the same, RETURN *the curves do not have the same asymptotic behavior* (see Remark 4.5). Otherwise, let  $P_1, \dots, P_n$  be these infinity points.
2. For each  $P_k := (1 : m_k : 0), k = 1, \dots, n$  do:
  - 2.1. Compute the infinity branches of  $\mathcal{C}$  associated to  $P_k$ . Let  $B_1, \dots, B_{n_k}$  be these branches. For each  $i = 1, \dots, n_k$ , let  $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_i\}$  be any leaf of  $B_i$ .
  - 2.2. Compute the infinity branches of  $\bar{\mathcal{C}}$  associated to  $P_k$ . Let  $\bar{B}_1, \dots, \bar{B}_{l_k}$  be these branches. For each  $j = 1, \dots, l_k$ , let  $\bar{L}_j = \{(z, \bar{r}_j(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_j\}$  be any leaf of  $\bar{B}_j$ .
  - 2.3. For each  $B_i \subset \mathcal{C}$ , find  $\bar{B}_j \subset \bar{\mathcal{C}}$  such that the terms with non negative exponent in  $r_i(z)$  and  $\bar{r}_j(z)$  are the same up to conjugation. If there isn't such a branch, RETURN *the curves do not have the same asymptotic behavior* (see Lemma 4.2).
  - 2.4. For each  $\bar{B}_j \subset \bar{\mathcal{C}}$ , find  $B_i \subset \mathcal{C}$  such that the terms with non negative exponent in  $r_i(z)$  and  $\bar{r}_j(z)$  are the same up to conjugation. If there isn't such a branch, RETURN *the curves do not have the same asymptotic behavior* (see Lemma 4.2).
3. RETURN *the curves  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  have the same asymptotic behavior*.

In the following, we illustrate the performance of algorithm Asymptotic Behavior with an example.

**Example 5.5.** Let  $\mathcal{C}$ , and  $\bar{\mathcal{C}}$  be two plane curves implicitly defined by the polynomials

$$f(x, y) = 2y^3x - y^4 + 2y^2x - y^3 - 2x^3 + x^2y + 3, \quad \text{and}$$

$$\bar{f}(x, y) = 2y^3x - y^4 + 2y^2x - y^3 - 2x^3 + x^2y - 3x^2 - xy + 2x - 3y + 1,$$

respectively. We apply the algorithm **Asymptotic Behavior** to decide whether  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  have the same asymptotic behavior:

**Step 1:** Compute the infinity points of  $\mathcal{C}$  and  $\bar{\mathcal{C}}$ . We obtain that  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  have the same infinity points:  $P_1 = (1 : 0 : 0)$  and  $P_2 = (1 : 2 : 0)$ .

We start by analyzing the infinity branches associated to  $P_1$ :

**Step 2.1:** Reasoning as in Example 3.5, we get that the only infinity branch associated to  $P_1$  in  $\mathcal{C}$  is given by  $B_1 = L_{1,1} \cup L_{1,2} \cup L_{1,3}$  where  $L_{1,i} = \{(z, r_{1,i}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_1\}$ ,  $i = 1, 2, 3$ , and

$$r_{1,i}(z) = z^{2/3} - 1/3 + 1/9z^{-2/3} - 2/81z^{-4/3} + \dots,$$

up to conjugation.

**Step 2.2:** We also have that there exists only one infinity branch associated to  $P_1$  in  $\bar{\mathcal{C}}$ . It is given by  $\bar{B}_1 = \bar{L}_{1,1} \cup \bar{L}_{1,2} \cup \bar{L}_{1,3}$  where  $\bar{L}_{1,i} = \{(z, \bar{r}_{1,i}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \bar{M}_1\}$ ,  $i = 1, 2, 3$ , and

$$\bar{r}_{1,i}(z) = z^{2/3} - 1/3 + 1/2z^{-1/3} + 19/36z^{-2/3} + \dots,$$

up to conjugation.

**Step 2.3 and Step 2.4:**  $r_{1,1}(z)$  and  $\bar{r}_{1,1}(z)$  have the same terms with non negative exponent. Thus,  $B_1$  and  $\bar{B}_1$  converge.

Now we analyze the infinity branches associated to  $P_2$ :

**Step 2.1:** Reasoning as in Example 3.5, we get that the only infinity branch associated to  $P_2$  in  $\mathcal{C}$  is given by  $B_2 = L_2 = \{(z, r_2(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_2\}$ , where

$$r_2(z) = 2z + 3/8z^{-3} - 9/64z^{-4} + 27/512z^{-5} + \dots.$$



**Step 2.2:** The only infinity branch associated to  $P_2$  in  $\bar{\mathcal{C}}$  is given by  $\bar{B}_2 = \bar{L}_2 = \{(z, \bar{r}_2(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \bar{M}_2\}$ , where

$$\bar{r}_2(z) = 2z - 5/8z^{-1} - 17/64z^{-2} - 145/512z^{-3} + \dots$$

**Step 2.3 and Step 2.4:**  $r_2(z)$  and  $\bar{r}_2(z)$  have the same terms with non negative exponent. Thus,  $B_2$  and  $\bar{B}_2$  converge.

Since every infinity branch of  $\mathcal{C}$  converges to another branch of  $\bar{\mathcal{C}}$ , and reciprocally, the algorithm returns that  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  have the same asymptotic behavior (see Figure 3).

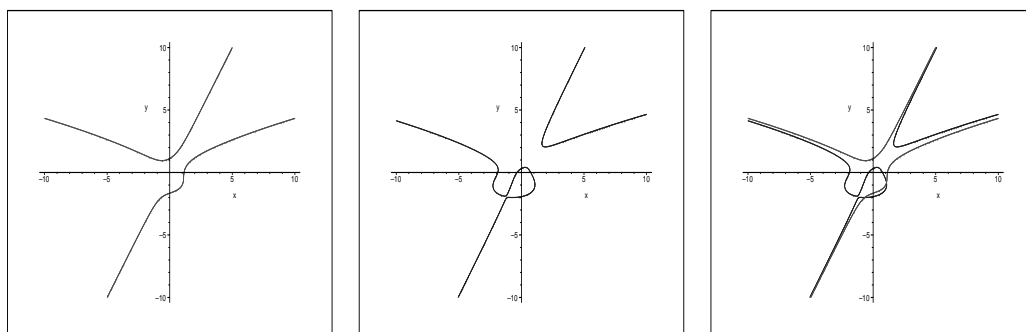


Figure 3:  $\mathcal{C}$  (left),  $\bar{\mathcal{C}}$  (center), and the asymptotic behavior of  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  (right)

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