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Characterization of Rational Ruled Surfaces

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Abstract

The ruled surface is a typical modeling surface in computer aided geometric design. It is usually given in the standard parametric form. However, it can also be in the forms than the standard one. For these forms, it is necessary to determine and find the standard form. In this paper, we present algorithms to determine whether a given implicit surface is a rational ruled surface. A parametrization of the surface is computed for the affirmative case. We also consider the parametric situation. More precisely, after a given rational parametric surface is determined as a ruled one, we reparameterize it to the standard form.

Key words: ruled surface, parametrization, reparametrization, birational transformation

1. Introduction

Parametric and implicit forms are two main representations of geometrical objects. In computer aided geometric design and computer graphics, people prefer the rational parametric form for modeling design [12]. On the other hand, in algebraic consideration of computer algebra and algebraic geometry; people usually use the algebraic form. Since there are different advantages of parametric and implicit forms, a nature problem is to convert the forms from one to another. Converting from the implicit form to the parametric one is the parametrization problem. On the converse direction, it is the implicitization problem. There were lots of papers focused on the implicitization problem. Some typical methods were proposed using Gröbner bases [4, 9], characteristic sets [13, 30], resultants [10, 19] and mu-bases [5, 7, 11]. However, there still lacks of a method having both completeness in theory and high efficiency in computation.

In general, the parametrization problem is more difficult than the implicitization problem. Only some of the algebraic curves and surfaces have rational parametric representations. For the curves, people have proposed different methods such as parametrization based on resolvents [14], by lines or adjoint curves [23] (see Chapter

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4) or using canonical divisor [27]. For a general surface, an efficient parametrization algorithm has not been given yet. However, to meet the practical demands, people had to design the parametrization algorithms for some commonly used surfaces. Sederberg and Snively [21] proposed four parametrization methods for cubic algebraic surfaces. One of them was based on finding two skew lines lying on the surface. Sederberg [22] and Bajaj et al. [2] expanded this method. In [28], a method to parameterize a quadric was given using a stereographic projection. Berry et al. [3] unified the implicitization and parametrization of a nonsingular cubic surface with Hilbert-Burch matrices. These methods were designed for some special surfaces. In [20], Schicho provided a general algorithm that solved the parametrization problem. However, his contributions on theoretical analysis are more than those of practicable computations. Therefore, it is still necessary to find the efficient parametrization algorithm for certain commonly used surfaces.

The ruled surface is an important surface widely used in computer aided geometric design and geometric modeling (see [1, 5, 6, 7, 8, 11, 15, 17, 16, 19, 24, 25]). Using the μ-bases method, Chen et al. [7] gave an implicitization algorithm for the rational ruled surface. The univariate resultant was also used to compute the implicit equations efficiently [19, 24]. For a given rational ruled surface, people could find a simplified reparametrization which did not contain any non-generic base point and had a pair of directrices with the lowest possible degree [6]. Busé and Dohm studied the ruled surface using μ-bases [5, 11] respectively. Li et al. [16] computed a proper reparametrization of an improper parametric ruled surface. Andradas et al. presented an algorithm to decide whether a proper rational parametrization of a ruled surface could be properly reparametrized over a real field [1]. The ruled surfaces had been used for geometric modeling of architectural freeform design in [17]. The collision and intersection of the ruled surfaces were discussed in [8, 25]. And S. Izumiya [15] studied the cylindrical helices and Bertrand curves on ruled surfaces. In these papers, the ruled surface was given in standard parametric form $\mathcal{Q}(t_1, t_2) = \mathcal{M}(t_1) + t_2\mathcal{N}(t_1) \in \mathbb{K}(t_1, t_2)^3$. It means the rational ruled surface was preassigned in the discussions. But in general modeling design, such as data fitting, the type of approximate surface may be not known. Then a problem is, for a given parametric surface not being standard form of the ruled surface, how to determine whether it is a ruled surface. If the answer is affirmative, the successive problem is then to find a standard parametric form. In this paper, we would like to consider the determination and reparametrization of the parametric ruled surface.

Go back to parametrization, the implicit surfaces are often introduced in the algebraic analysis. And they can also come directly from modeling design since they have more geometrical features and topologies than those of the parametric surfaces (see [12, 26]). As we know, there was no paper discussing the parametrization of an implicit rational ruled surface. Here, we would like to consider the parametrization problem of the ruled surface. Precisely, for a given algebraic surface, we first determine whether it is a rational ruled surface, and in the affirmative case, we compute a rational parametrization in standard form. Our discussion is benefited from the standard presentation of the rational ruled surface. Since the parameter t_2 is linear, we can construct a birational transformation to simply the given parametrization. By the linearity again, t_2 is always solvable such that we can project the surface to the rational parametric curve. And these two principal techniques help us to give the determination and (re)parametrization algorithms. The main theorems are all proved constructively, and the algorithms are then presented naturally.

The paper is organized as follows. First, some necessary preliminaries are presented in Section 2. In Section 3, we determine whether a given implicit surface is a rational ruled surface, and in the affirmative case, we compute a rational parametrization in standard form for it. In Section 4, we focus on the parametric surface, including determination and reparametrization. Finally, we conclude with Section 5, where we propose topics for further study.

2. Preliminaries on Ruled Surfaces

Let $\mathcal{V}$ be a ruled surface defined by the polynomial $f(\overline{x}) \in \mathbb{K}[\overline{x}]$, $\overline{x} = (x_1, x_2, x_3)$ where $\mathbb{K}$ is an algebraically closed field of characteristic zero.

A standard parametrization of a rational ruled surface $\mathcal{V}$ is given by a parametrization of the form

$$\mathcal{Q}(\overline{t}) = (m_1(t_1) + t_2 n_1(t_1), m_2(t_1) + t_2 n_2(t_1), m_3(t_1) + t_2 n_3(t_1)) \in \mathbb{K}(\overline{t})^3, \quad \overline{t} = (t_1, t_2) \tag{1}$$

where there exists at least one $i \in \{1, 2, 3\}$ such that $n_i \neq 0$ (otherwise, $\mathcal{V}$ degenerates to a space curve). We refer to $\mathcal{Q}$ as the standard form parametrization of $\mathcal{V}$.

Note that if $n_3 \neq 0$, the surface $\mathcal{V}$ admits a parametrization of the form

$$\mathcal{P}^3(\overline{t}) = (p_{13}(t_1) + t_2 q_{13}(t_1), p_{23}(t_1) + t_2 q_{23}(t_1), t_2) \in \mathbb{K}(\overline{t})^3, \tag{2}$$

where $q_{k3} = n_k/n_3 \neq 0$, for some $k = 1, 2$. Such a parametrization is p obtained by performing the birational transformation $(t_1, t_2) \to \left(t_1, \frac{t_2 - m_3(t_1)}{n_3(t_1)}\right)$.

One may reason similarly as above, if $n_1 \neq 0$ or $n_2 \neq 0$. Thus, in the following, we refer to parametrization $\mathcal{P}^i$ as the standard reduced form parametrization of $\mathcal{V}$.

Under these conditions, and taking into account that $\mathcal{P}^3$ parametrizes the surface $\mathcal{V}$ implicitly defined by the polynomial $f(\overline{x})$, we distinguish two different cases:

- If $n_1 n_2 n_3 \neq 0$, then $q_{13} q_{23} \neq 0$, and

$$\mathcal{P}^3(t_1, 0) = (p_{13}, p_{23}, 0), \quad \mathcal{P}^3\left(t_1, -\frac{p_{13}}{q_{13}}\right) = \left(0, p_{23} - \frac{p_{13}}{q_{13}} q_{23}, -\frac{p_{13}}{q_{13}}\right),$$

$$\mathcal{P}^3\left(t_1, -\frac{p_{23}}{q_{23}}\right) = \left(p_{13} - \frac{p_{23}}{q_{23}}q_{13}, 0, -\frac{p_{23}}{q_{23}}\right).$$

parametrize three rational planar curves with implicit equations as factors of the polynomials

$$f_0^{12}(x_1, x_2) = f(x_1, x_2, 0), \quad f_0^{23}(x_2, x_3) = f(0, x_2, x_3), \quad f_0^{13}(x_1, x_3) = f(x_1, 0, x_3),$$

respectively. We denote by $\mathcal{C}^{ij}$ these rational curves, where $ij \in \{12, 23, 13\}$.

- Let us assume that $n_2 = 0$, and $n_1 n_3 \neq 0$. Then,

$$\mathcal{Q}(\bar{t}\,) = (m_1(t_1) + t_2 n_1(t_1), m_2(t_1), m_3(t_1) + t_2 n_3(t_1)) \in \mathbb{K}(\bar{t}\,)^3.$$

Since $n_3 \neq 0$,

$$\mathcal{P}^3(\bar{t}\,) = (p_{13}(t_1) + t_2 q_{13}(t_1), m_2(t_1), t_2), \quad q_{13} = n_1/n_3 \neq 0,$$

and $\mathcal{P}^3(t_1, 0)$, $\mathcal{P}^3\left(t_1, -\frac{p_{13}}{q_{13}}\right)$ parametrize two rational planar curves with implicit equations as factors of the polynomials

$$f_0^{12}(x_1, x_2) = f(x_1, x_2, 0), \quad f_0^{23}(x_2, x_3) = f(0, x_2, x_3)$$

respectively. We denote by $\mathcal{C}^{ij}$ these rational curves, where $ij \in \{12, 23\}$.

3. Implicitly Rational Ruled Surfaces

In this section, for a surface $\mathcal{V}$ by defined by a polynomial $f(\,\overline{x}\,) \in \mathbb{K}[\,\overline{x}\,]$ implicitly, we analyze whether $\mathcal{V}$ is a rational ruled surface. In the affirmative case, we compute a rational proper parametrization of $\mathcal{V}$ in the standard reduced form given by the equation (2). For this purpose, we denote by $\mathrm{numer}(R)$, the numerator of a rational function $R \in \mathbb{K}(x_1, x_2, \ldots, x_n)$.

Theorem 1. *A surface $\mathcal{V}$ defined by a polynomial $f(\,\overline{x}\,) \in \mathbb{K}[\,\overline{x}\,]$ is a rational ruled surface if and only if the following statements hold:*

1. *At least one of the three plane algebraic curves $\mathcal{C}^{ij}$, $ij \in \{12, 13, 23\}$, is rational (see Section 2). Let $ij = 12$, and let $\mathcal{P}^{12} = (p_1, p_2) \in \mathbb{K}(t_1)^2$ be a rational proper parametrization of the rational plane curve $\mathcal{C}^{12}$ defined by the factor of f_0^{12}.*

2. *Let $g(x_1, x_2, x_3, t_2) = \mathrm{numer}(f(p_1(x_3) + t_2 x_1, p_2(x_3) + t_2 x_2, t_2))$. There exist $(q_1, q_2) \in \mathbb{K}(t_1)^2$ and $S \in \mathbb{K}(t_1) \setminus \mathbb{K}$ such that*

$$\mathcal{P}(\bar{t}\,) = (p_1(S(t_1)) + t_2 q_1(t_1), p_2(S(t_1)) + t_2 q_2(t_1), t_2)$$

is a rational proper parametrization of $\mathcal{V}$, where $\mathcal{M}(t_1) := (q_1(t_1), q_2(t_1), S(t_1))$ is proper and $g(\mathcal{M}(t_1), t_2) = 0$.

Proof. It is clear that if statements 1 and 2 hold, then $\mathcal{V}$ is a rational ruled surface. Reciprocally, let $\mathcal{V}$ be a rational ruled surface. Then a parametrization of $\mathcal{V}$ is given by the standard form parametrization (1). That is,

$$\mathcal{Q}(\bar{t}\,) = (m_1(t_1) + t_2 n_1(t_1), m_2(t_1) + t_2 n_2(t_1), m_3(t_1) + t_2 n_3(t_1)) \in \mathbb{K}(\bar{t}\,)^3.$$

We assume that $n_3 \neq 0$ (see Section 2 and Remark 1). Thus, $\mathcal{Q}(t_1, -m_3/n_3)$ parametrizes $\mathcal{C}^{12}$, and statement 1 holds. Let us prove that statement 2 holds. For this purpose, we consider $\mathcal{P}^{12} = (p_1, p_2) \in \mathbb{K}(t_1)^2$ a rational proper parametrization of $\mathcal{C}^{12}$. In addition, since $n_3 \neq 0$, $\mathcal{V}$ admits a standard reduced form parametrization given in the equation (2) (see Section 2). That is,

$$\mathcal{P}^3(\bar{t}\,) = (p_{13}(t_1) + t_2 q_{13}(t_1), p_{23}(t_1) + t_2 q_{23}(t_1), t_2) \in \mathbb{K}(\bar{t}\,)^3.$$

We assume w.l.o.g. that $\mathcal{P}^3$ is proper (otherwise, it can be easily reparametrized using the results in [16]). Observe that $\mathcal{P}^3(t_1, 0) = (p_{13}, p_{23}) \in \mathbb{K}(t_1)^2$ is a rational parametrization of $\mathcal{C}^{12}$. Then, since $\mathcal{P}^{12}$ is a proper parametrization of $\mathcal{C}^{12}$, there exists $S \in \mathbb{K}(t_1) \setminus \mathbb{K}$ such that $\mathcal{P}^{12}(S) = (p_1(S), p_2(S)) = (p_{13}, p_{23})$. Thus, since $\mathcal{P}^3$ parametrizes properly $\mathcal{V}$, we have that

$$\mathcal{P}(\bar{t}\,) := (p_1(S(t_1)) + t_2 q_1(t_1), p_2(S(t_1)) + t_2 q_2(t_1), t_2) \in \mathbb{K}(\bar{t}\,)^3, \text{ where } (q_1, q_2) = (q_{13}, q_{23})$$

is a proper parametrization of $\mathcal{V}$ (note that $\mathcal{P} = \mathcal{P}^3$), and $(q_1, q_2) \in \mathbb{K}(t_1)^2$, $S \in \mathbb{K}(t_1) \setminus \mathbb{K}$ satisfy that $g(q_1(t_1), q_2(t_1), S(t_1), t_2) = \mathrm{numer}(f(p_1(S(t_1)) + t_2 q_1(t_1), p_2(S(t_1)) + t_2 q_2(t_1), t_2)) = 0$. Observe that since $\mathcal{P}$ is proper, then $\mathcal{M}$ is proper. Indeed: if $\mathcal{M}$ is not proper, there exists $\alpha(s_1) \in \overline{\mathbb{K}(s_1)}$, $\alpha(s_1) \neq s_1$ such that $\mathcal{M}(\alpha(s_1)) = \mathcal{M}(s_1)$ ($\overline{\mathbb{K}(s_1)}$ is the algebraic closure of $\mathbb{K}(s_1)$, and s_1 is a new variable). Then, $\mathcal{P}(\alpha(s_1), s_2) = \mathcal{P}(s_1, s_2)$ and $\alpha(s_1) \neq s_1$, which is impossible since $\mathcal{P}$ is proper □

Geometrically speaking, conditions in Theorem 1 involve to determine a rational planar base curve of the ruled surface with parameterization $(p_1, p_2, 0)$ (see statement 1), and to compute the ruling direction of the ruled surface with parameterization $(q_1, q_2, 1)$ (see statement 2). The function S is for coordinating the parameterization of the base curve and the ruling direction so that the parameterization of the ruled surface is in the required reduced form.

Remark 1. *Note that in Theorem 1, we assume that $\mathcal{C}^{12}$ is the rational plane curve satisfying statement 1. In addition, in the proof of theorem, when a rational ruled surface is given, we consider a parametrization $\mathcal{Q}$ in standard form (1) such that $n_3 \neq 0$ (from this assumption, we have that $\mathcal{Q}(t_1, -m_3/n_3)$ parametrizes $\mathcal{C}^{12}$).*
Theorem 1 can be proved similarly if a different rational plane curve $\mathcal{C}^{ij}$ is considered in statement 1, and a different polynomial n_i satisfies that $n_i \neq 0$ (see Section 2). In addition, if $n_1 \neq 0$, we get that $g(\overline{x}\,, t_2) = \mathrm{numer}(f(t_2, p_1(x_3) + t_2 x_1, p_2(x_3) + t_2 x_2))$,

and if $n_2 \neq 0$*, we get that* $g(\overline{x}, t_2) = \mathrm{numer}(f(p_1(x_3) + t_2x_1, t_2, p_2(x_3) + t_2x_2))$*.*

In the following, we assume that $n_3 \neq 0$*, and then we are in the conditions of Theorem 1. This requirement can always be achieved by applying a linear transformation to* $\mathcal{V}$*, and therefore it is not a loss of generality for our purposes since one can always undo the linear transformation once* $\mathcal{P}$ *has been computed.*

In Corollary 1, we prove that statement 2 in Theorem 1 is equivalent to check the rationality of a space curve $\mathcal{D}$, and to compute, in the affirmative case, a rational proper parametrization of $\mathcal{D}$.

Corollary 1. *Let* $\mathcal{V}$ *be a surface defined by a polynomial* $f(\overline{x}) \in \mathbb{K}[\overline{x}]$ *and such that statement 1 in Theorem 1 holds.* $\mathcal{V}$ *is a rational ruled surface if and only if the coefficients of the polynomial* $g(\overline{x}, t_2)$ *w.r.t. the variable* t_2 *define a rational space curve* $\mathcal{D}$*. In this case,* $\mathcal{M}(t_1) := (q_1(t_1), q_2(t_1), S(t_1)) \in \mathbb{K}(t_1)^3$*, where* $S \notin \mathbb{K}$*, is a rational proper parametrization of* $\mathcal{D}$*.*

Proof. First, we write

$$g(\overline{x}, t_2) = h_0(\overline{x}) + h_1(\overline{x})t_2 + \cdots + h_n(\overline{x})t_2^n,$$

and we prove that $h_0 = 0$, the only factor in $\mathbb{K}[t_2]$ dividing g is t_2^r for some $r \in \mathbb{N}$, and there exist at least two different nonzero polynomials h_i and h_j. Indeed:

a. $h_0 = 0$: since $g(\overline{x}, t_2) = \mathrm{numer}(f(p_1(x_3) + t_2x_1, p_2(x_3) + t_2x_2, t_2))$, and $f(p_1(x_3), p_2(x_3), 0) = 0$, we deduce that t_2 divides g.

b. The only factor in $\mathbb{K}[t_2]$ dividing g is t_2^r, $r \in \mathbb{N}$: let $c \in \mathbb{K} \setminus \{0\}$ be such that $g(\overline{x}, c) = 0$. Since $g(\overline{x}, t_2) = \mathrm{numer}(f(p_1(x_3) + t_2x_1, p_2(x_3) + t_2x_2, t_2))$, we deduce that $f(p_1(x_3) + cx_1, p_2(x_3) + cx_2, c) = 0$, for every x_1, x_2. Then, $\mathcal{V}$ is the plane defined by the equation $x_3 - c = 0$ which is impossible because we have assumed that $n_3 \neq 0$.

c. There exist at least two different nonzero polynomials h_i and h_j. Let us assume that this statement does not hold. From statement b, this implies that $g(\overline{x}, t_2) = t_2^r h(\overline{x})$. In addition, reasoning as in statement b, we have that $h \neq 0$. From the equality

$$g(\overline{x}, t_2) = \mathrm{numer}(f(p_1(x_3) + t_2x_1, p_2(x_3) + t_2x_2, t_2)) = t_2^r h(\overline{x}),$$

and deriving w.r.t. x_1, x_2, x_3, we get that, up to factors in $\mathbb{K}[x_3] \setminus \{0\}$,

$$f_{x_1}(p_1(x_3) + t_2x_1, p_2(x_3) + t_2x_2, t_2)t_2 = t_2^r h_{x_1}(\overline{x}),$$

$$f_{x_2}(p_1(x_3) + t_2x_1, p_2(x_3) + t_2x_2, t_2)t_2 = t_2^r h_{x_2}(\overline{x}),$$

$$f_{x_1}(p_1(x_3)+t_2x_1, p_2(x_3)+t_2x_2, t_2)p_1' + f_{x_2}(p_1(x_3)+t_2x_1, p_2(x_3)+t_2x_2, t_2)p_2' = t_2^r h_{x_3}(\overline{x}),$$

where f_{var} represents the partial derivative of a polynomial f w.r.t. the variable var. Thus, up to factors in $\mathbb{K}[x_3] \setminus \{0\}$,

$$h_{x_1}(\overline{x})p_1'(x_3) + h_{x_2}(\overline{x})p_2'(x_3) = t_2 h_{x_3}(\overline{x})$$

which implies that $h_{x_3} = 0$, and then $h \in \mathbb{K}[x_1, x_2]$. Hence,

$$g(\overline{x}, t_2) = \text{numer}(f(p_1(x_3) + t_2x_1, p_2(x_3) + t_2x_2, t_2)) = t_2^r h(x_1, x_2).$$

Let $\eta_i = (a_i, b_i) \in \mathbb{K}^2$ be such that $h(\eta_i) = 0$, $i = 1, 2$, and $\eta_1 \neq \eta_2$. Then, $g(a_i, b_i, x_3, t_2) = \text{numer}(f(p_1(x_3) + t_2a_i, p_2(x_3) + t_2b_i, t_2)) = 0$, and thus $\mathcal{Q}_i = (p_1(x_3)+t_2a_i, p_2(x_3)+t_2b_i, t_2)$ parametrizes $\mathcal{V}$ which implies that $\mathcal{Q}_1(U, V) = \mathcal{Q}_2$, where $(U, V) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$. Thus, $V = t_2$ and $\eta_1 = \eta_2$ which is impossible. Therefore, there are not two different points on the curve defined by h. Hence, $h \in \mathbb{K} \setminus \{0\}$ and then, up to constants in $\mathbb{K} \setminus \{0\}$,

$$g(\overline{x}, t_2) = \text{numer}(f(p_1(x_3) + t_2x_1, p_2(x_3) + t_2x_2, t_2)) = t_2^r.$$

This is impossible, because if we consider $\eta := (a_1, a_2, a_3) \in \mathbb{K}^3$, $a_3 \neq 0$, with $f(\eta) = 0$ (observe that this point exists because $n_3 \neq 0$, and then $\mathcal{V}$ is not the plane $x_3 = 0$), we have that $f(p_1(x_3) + a_3x_1^0, p_2(x_3) + a_3x_2^0, a_3) = 0$, where $x_j^0 = (a_j - p_j(x_3))/a_3$, $j = 1, 2$. Thus, $g(x_1^0, x_2^0, x_3, a_3) = a_3^r = 0$ which is a contradiction.

Now, we are ready to prove the corollary. First, if $\mathcal{V}$ is a rational ruled surface, statement 2 in Theorem 2 holds, and then $g(\mathcal{M}(t_1), t_2) = 0$ and $\mathcal{M}$ is proper. Since $\mathcal{M}(t_1) \in \mathbb{K}(t_1)^3$ does not depend on t_2, we get $h_i(\mathcal{M}) = 0$, $i \in \{0, \ldots, n\}$ which implies that $\mathcal{M}$ is a rational proper parametrization of the space curve $\mathcal{D}$ defined by the polynomials h_i, $i \in \{0, \ldots, n\}$ (see statements a, b, c above). That is, the coefficients of the polynomial $g(\overline{x}, t_2)$ w.r.t. t_2 define a rational space curve $\mathcal{D}$, and $\mathcal{M}(t_1) = (q_1(t_1), q_2(t_1), S(t_1)) \in \mathbb{K}(t_1)^3$, where $S \notin \mathbb{K}$, is a rational proper parametrization of $\mathcal{D}$.
Reciprocally, let $(U(t_1), V(t_1), W(t_1)) \in \mathbb{K}(t_1)^3$ be a rational proper parametrization of $\mathcal{D}$. Then, $h_i(U, V, W) = 0, i \in \{0, \ldots, n\}$ which implies that $g(U(t_1), V(t_1), W(t_1), t_2) = 0$. Hence, $f(\mathcal{P}) = 0$, where

$$\mathcal{P}(\overline{t}) = (p_1(W(t_1)) + t_2U(t_1), p_2(W(t_1)) + t_2V(t_1), t_2).$$

□

Remark 2. *From the proof in Corollary 1, we have that*

$$g(\overline{x}, t_2) = t_2^r(h_1(\overline{x}) + \cdots + h_{m+1}(\overline{x})t_2^{m+1}), \quad r, m \in \mathbb{N},$$

and there exist at least two polynomials, h_i, h_j, such that $h_i \neq h_j$. Under these conditions, the space curve $\mathcal{D}$ is defined at least by the polynomials h_i and h_j. Taking into account that any space curve can be birationally projected onto a plane curve, one may apply the results in Chapter 6 in [23] to compute the rational proper parametrization of $\mathcal{D}$, $\mathcal{M}(t_1) = (q_1(t_1), q_2(t_1), S(t_1))$.

In the following corollary, we prove that the properness of the output parametrization in Theorem 1 is equivalent to the properness of the parametrization of the space curve in Corollary 1.

Corollary 2. *Let $\mathcal{V}$ be a ruled surface defined by a polynomial $f(\overline{x}) \in \mathbb{K}[\overline{x}]$. Let $\mathcal{P}^*$ be the output parametrization of Theorem 2, and $\mathcal{M}^*(t_1) := (q_1^*(t_1), q_2^*(t_1), S^*(t_1))$ a parametrization of the space curve $\mathcal{D}$ (see Corollary 1). It holds that $\mathcal{P}^*$ is proper if and only if (R^*, S^*) is proper.*

Proof. Since $\mathcal{V}$ is a ruled surface, we have that Theorem 1 and Corollary 1 hold. In particular, $\mathcal{P}(\overline{t}) = (p_1(S(t_1)) + t_2q_1(t_1), p_2(S(t_1)) + t_2q_2(t_1), t_2)$ is a rational proper parametrization of $\mathcal{V}$, and $\mathcal{M}(t_1) := (q_1(t_1), q_2(t_1), S(t_1))$ is a proper parametrization of $\mathcal{D}$. Since $\mathcal{M}^*$ is a parametrization of $\mathcal{D}$, there exists $L \in \mathbb{K}(t_1) \setminus \mathbb{K}$ such that $\mathcal{M}^*(t_1) = \mathcal{M}(L(t_1))$ and then, $\mathcal{P}(L(t_1), t_2) = \mathcal{P}^*(t_1, t_2)$, where $\mathcal{P}^*(\overline{t}) = (p_1(S^*(t_1)) + t_2q_1^*(t_1), p_2(S^*(t_1)) + t_2q_2^*(t_1), t_2)$. Using this facts and that $\mathcal{P}$ is proper, we get that $\mathcal{P}^*$ is proper if and only if $(L(t_1), t_2)$ is proper which is equivalent to $L(t_1)$ is linear (see Lemma 4.32 in [23]). Taking into account that $\mathcal{M}^* = \mathcal{M}(L)$ and that $\mathcal{M}$ is proper, we get that L is linear if and only if $\mathcal{M}^*$ is proper. Therefore, $\mathcal{P}^*$ is proper if and only if $\mathcal{M}^*$ is proper. □

Taking into account Theorem 1 and Corollaries 1 and 2, one may check whether $\mathcal{V}$ is a rational ruled surface, and in the affirmative case to compute a rational proper parametrization in standard reduced form (2).

Algorithm 1: Parametrization of a Rational Ruled Surface

- **Input:** A surface $\mathcal{V}$ defined by an irreducible polynomial $f(\overline{x}) \in \mathbb{K}[\overline{x}]$.
- **Output:** the message "`$\mathcal{V}$ is not a rational ruled surface`" or a proper parametrization $\mathcal{P}$ of "`the rational ruled surface $\mathcal{V}$ in the reduced standard form`".

1. Check whether any component of the curve defined by the polynomial f_0^{ij}, for $ij \in \{12, 13, 23\}$, is rational. In the affirmative case, assume that $ij = 12$ (see Remark 1), and go to Step 2. Otherwise, RETURN "`$\mathcal{V}$ is not a rational ruled surface`".
2. Compute $\mathcal{P}^{12} = (p_1, p_2) \in \mathbb{K}(t_1)^2$ a rational proper parametrization of $\mathcal{C}^{12}$ (see Remark 4).
3. Let $g(\overline{x}, t_2) = \text{numer}(f(p_1(x_3) + t_2x_1, p_2(x_3) + t_2x_2, t_2))$. Check whether the coefficients of $g(\overline{x}, t_2)$ w.r.t. the variable t_2 define a rational space curve $\mathcal{D}$. In the affirmative case, compute $\mathcal{M} = (q_1, q_2, S) \in \mathbb{K}(t_1)^3$ a rational proper parametrization of $\mathcal{D}$ (see Remark 2), and RETURN

$$\mathcal{P} = (p_1(S(t_1)) + t_2q_1(t_1), p_2(S(t_1)) + t_2q_2(t_1), t_2) \in \mathbb{K}(\overline{t})^3$$

"is a proper parametrization". Otherwise, go to Step 1, and consider a different rational component, and apply again the algorithm. If there have no more rational components, RETURN "$\mathcal{V}$ is not a rational ruled surface".

Remark 3. *If $\mathcal{P}^{12}$ and $\mathcal{M}$ have coefficients in a field $\mathbb{L}$, then the output parametrization $\mathcal{P}$ also has coefficients in $\mathbb{L}$. Then, in particular, if we compute the proper parametrizations, $\mathcal{P}^{12}$ and $\mathcal{M}$, in the smallest possible field extension of the ground field (see Chapter 5 in [23]), the output parametrization $\mathcal{P}$ belongs to this smallest possible field extension. For practical applications, we may consider a surface over the real field $\mathbb{R}$ and to compute $\mathcal{P}$ with real coefficients, if it is possible. For this purpose, we may apply the results in [23] (see Chapter 7), and compute $\mathcal{P}^{12}$ and $\mathcal{M}$ over the reals (if it is possible). In this sense, we observe that we get a proper parametrization in the standard reduced form over $\mathbb{R}$ (if it exists). Compare with the results in [1], where an algorithm to decide whether a proper rational parametrization of a ruled surface can be properly reparametrized over $\mathbb{R}$ is presented. The output in this paper is not necessarily given in the standard form.*

Finally we observe that if $\mathcal{P}^{12}$ and $\mathcal{M}$ are polynomials (see Section 6.2 in [23]), then $\mathcal{P}$ is also polynomial.

Remark 4. *Finding the rational parameterization of a plane algebraic curve or a space curve is frequently involved in the main algorithms of the paper. In most of cases, we are using the algorithms presented in [23] (see Chapters 4, 5 and 6). We explicitly refer the algorithm used in each computation. However, alternative parametrization algorithms can be constructed from different methods (see e.g. [14] and [27]). Some references as well as a brief comparison of the existing methods can be found in [23].*

In the following, we illustrate Algorithm 1 with two examples.

Example 1. *Consider the surface $\mathcal{V}$ over the complex field $\mathbb{C}$ defined by the polynomial $f(\overline{x}) = x_1^2 + x_2^2 + x_3^2 - 1$. Let us apply* Algorithm 1. *For this purpose, we first observe that $f(x_1, x_2, 0) = x_1^2 + x_2^2 - 1$ is a rational plane curve, and we compute a rational proper parametrization of this curve*

$$\mathcal{P}^{12}(t_1) = (p_1(t_1), p_2(t_1)) = \left(\frac{2t_1}{t_1^2+1}, \frac{t_1^2-1}{t_1^2+1}\right) \in \mathbb{R}(t_1)^2.$$

Now, we compute the polynomial $g(\overline{x}, t_2) = \mathrm{numer}(f(p_1(x_3) + t_2x_1, p_2(x_3) + t_2x_2, t_2))$, and we get

$$g(\overline{x}, t_2) = t_2(2x_3^2x_2 + t_2x_2^2x_3^2 + t_2x_3^2 + t_2x_1^2x_3^2 + 4x_3x_1 + t_2 + t_2x_1^2 + t_2x_2^2 - 2x_2).$$

The coefficients of the polynomial $g(\overline{x}, t_2)$ *w.r.t. the variable* t_2*, are*

$$h_1(\overline{x}) = 2x_2x_3^2 + 4x_3x_1 - 2x_2, \quad h_2(\overline{x}) = x_1^2x_3^2 + x_2^2x_3^2 + x_3^2 + x_1^2 + x_2^2 + 1.$$

Note that $\gcd(h_1, h_2) = 1$*. These polynomials define implicitly the rational space curve* $\mathcal{D}$*. We compute a rational proper parametrization of* $\mathcal{D}$*:*

$$\mathcal{M}(t_1) = (q_1(t_1), q_2(t_1), S(t_1)) = \left(\frac{I(1+t_1^2)}{2t_1}, \frac{-1+t_1^2}{2t_1}, -\frac{I(-1+t_1)}{1+t_1}\right) \in \mathbb{C}(t_1)^3.$$

Thus, $\mathcal{P} = (p_1(S(t_1)) + t_2q_1(t_2), p_2(S(t_1)) + t_2q_2(t_2), t_2) =$

$$\left(\frac{I(1-t_1^2+t_2+t_2t_1^2)}{2t_1}, \frac{-1-t_1^2-t_2+t_2t_1^2}{2t_1}, t_2\right) \in \mathbb{C}(\overline{t})^3$$

"`is a proper parametrization`*". Taking into account Remark 3, we conclude that* $\mathcal{V}$ *has not a parametrization over* $\mathbb{R}$ *in standard reduced form. However, one may apply results in [1] to decide whether* $\mathcal{V}$ *can be parametrized over the reals and to compute, in the affirmative case, a real parametrization.*

Example 2. *Let* $\mathcal{V}$ *be a surface defined by the polynomial*

$f(\overline{x}) = -49x_2x_1^3 - 799x_3x_2x_1^2 + 20x_2x_1^2 + 2x_2^2x_1^2 + 980x_3x_1^2 - 2205x_3^2x_1^2 + x_2^3x_1 - 33750x_3^3x_1 - 400x_3x_1 + 606x_3x_2x_1 - 5x_2^2x_1 - 68x_3x_2^2x_1 - 1747x_3^2x_2x_1 - 25x_3^2x_1 + x_2^3x_3 - 25x_2^2x_3^2 + 1396x_2x_3^2 - 1120x_3^2 - 48915x_3^4 - 5190x_3^3 - 4237x_2x_3^3 - 14x_2^2x_3 \in \mathbb{C}[\overline{x}].$

Let us apply Algorithm 1*. We observe that* $f_0^{12}(x_1, x_2) = -x_1x_2(49x_1^2 - 20x_1 - 2x_1x_2 - x_2^2 + 5x_2)$*. The curve defined by the equation* $49x_1^2 - 20x_1 - 2x_1x_2 - x_2^2 + 5x_2 = 0$ *is rational. We compute a rational proper parametrization of this curve,*

$$\mathcal{P}^{12}(t_1) = (p_1(t_1), p_2(t_1)) = \left(\frac{-\sqrt{2}t_1(-5+t_1)}{5(4t_1-10+5\sqrt{2})}, \frac{(50+5\sqrt{2})t_1(-20+100\sqrt{2}+49t)}{1225(4t_1-10+5\sqrt{2})}\right).$$

Now, we determine the polynomial $g(\overline{x}, t_2) = \operatorname{numer}(f(p_1(x_3)+t_2x_1, p_2(x_3)+t_2x_2, t_2))$*, and consider the space curve* $\mathcal{D}$ *defined by the coefficients of* g *w.r.t.* t_2*. By Remark 2,* $\mathcal{D}$ *is rational and then, we compute a rational proper parametrization of* $\mathcal{D}$*:*

$$\mathcal{M}(t_1) = (q_1(t_1), q_2(t_1), S(t_1)) = \left(\frac{25-6t_1-25\sqrt{2}+5\sqrt{2}t_1}{t_1}, \frac{-9t_1(1+\sqrt{2})}{-5+t_1}, t_1\right).$$

Thus, $\mathcal{P}(\overline{t}) = (p_1(t_1) + t_2q_1(t_2), p_2(t_1) + t_2q_2(t_2), t_2) =$

$$\left(\frac{-5\sqrt{2}t_1^2+\sqrt{2}t_1^3-1050t_2t_1+120t_2t_1^2+900t_2t_1\sqrt{2}-100t_2t_1^2\sqrt{2}+2500t_2-1875t_2\sqrt{2}}{5t_1(-4t_1+10-5\sqrt{2})},\right.$$

$$\left.\frac{t_1(15\sqrt{2}t_1-100\sqrt{2}-50t_1+10t_1^2+\sqrt{2}t_1^2-180t_2t_1-180t_2t_1\sqrt{2}+225t_2\sqrt{2})}{5(-5+t_1)(4t_1-10+5\sqrt{2})}, t_2\right)$$

"`is a proper parametrization over` $\mathbb{R}$*".*

3.1. Implicitly Rational Ruled Surfaces: A New Approach

In this section, we present a new result that characterizes rational ruled surfaces defined implicitly. From this result, we obtain a new algorithm that allows to analyze whether $\mathcal{V}$ is a rational ruled surface and in the affirmative case, to compute a rational parametrization of $\mathcal{V}$. In this new approach, instead to decide whether the polynomial g defines a rational space curve $\mathcal{D}$, we only need to decide whether a new polynomial, constructed directly from two rational parametrizations of two plane curves, defines a rational plane curve. That is, we do not need to work on the space. This new approach plays an important role to deal with surfaces defined parametrically in Section 4.

In the following new approach, we need to assume that $\mathcal{V}$ is not the plane $x_i - c = 0$, $c \in \mathbb{C}$ for $i = 1, 2, 3$, and $\mathcal{V}$ is not a cylinder over any of the coordinate planes of $\mathbb{K}^3$. That is, $\deg_{x_i}(f) > 0$, for $i = 1, 2, 3$. If $\deg_{x_3}(f) = 0$ (similarly if $\deg_{x_1}(f) = 0$ or $\deg_{x_2}(f) = 0$), we may compute a proper parametrization $(p(t_1), q(t_1))$ of the plane curve defined by the polynomial $f(x_1, x_2) = 0$. Then, $\mathcal{P}(\overline{t}\,) = (p(t_1), q(t_1), t_2) \in \mathbb{K}(\overline{t}\,)^3$ is a proper parametrization of $\mathcal{V}$.

Under these conditions, and taking into account Section 2, we get that a proper parametrization of a rational ruled surface $\mathcal{V}$, that is not a cylinder neither a plane, is given by the standard form parametrization given in the equation (1), where at least there exist $i, j \in \{1, 2, 3\}, i \neq j$, such that $n_i n_j \neq 0$. We note that if we do not assume that $\mathcal{V}$ is not a plane $x_i - c = 0$, $c \in \mathbb{C}$ for $i = 1, 2, 3$, and $\mathcal{V}$ is not a cylinder over any of the coordinate planes, we only can ensure that there exists $i \in \{1, 2, 3\}$ such that $n_i \neq 0$ (if $n_1 = n_2 = n_3 = 0$, then $\mathcal{Q}$ parametrizes a space curve).

In Theorem 2, we characterize whether a surface defined implicitly is a rational ruled surface by analyzing the rationality of two plane curves defined directly from the input surface. In addition, from the parametrization of these plane curves, we can compute a rational proper parametrization of the ruled surface.

Theorem 2. *A surface $\mathcal{V}$ defined by a polynomial $f(\overline{x}\,) \in \mathbb{K}[\,\overline{x}\,]$ is a rational ruled surface if and only if the following statements hold:*

1. *At least two of the plane algebraic curves $\mathcal{C}^{ij}$, $ij \in \{12, 13, 23\}$, are rational (see Section 2). Let us assume that $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$ are rational, and let $\mathcal{P}^{12} = (p_1, p_2) \in \mathbb{K}(t_1)^2$, $\mathcal{P}^{23} = (\tilde{p}_1, \tilde{p}_2) \in \mathbb{K}(t_1)^2$ be rational proper parametrizations of $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$, respectively.*

2. *If $p_1 \neq 0$, there exist $(R(t_1), S(t_1)) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ such that one of the following statements holds:*

   2.1. $f(\mathcal{P}) = 0$, *where*

$$\mathcal{P}(\overline{t}\,) = \left(p_1(S(t_1)) - t_2\frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2\frac{\tilde{p}_1(R(t_1)) - p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2\right)$$

is a rational proper parametrization of $\mathcal{V}$, and (R,S) is proper.

2.2. $f(\mathcal{P})=0$, *where*

$$\mathcal{P}(\bar{t}\,)=\left(p_1(S(t_1))-t_2\frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))},p_2(S(t_1)),t_2\right)$$

is a rational proper parametrization of $\mathcal{V}$, and (R,S) is proper.

3. *If $p_1=0$, there exist $R(t_1)\in\mathbb{K}(t_1)\setminus\{0\}$, and $S(t_1)\in\mathbb{K}(t_1)\setminus\mathbb{K}$ such that one of the following statements holds:*

3.1. $f(\mathcal{P})=0$, *where*

$$\begin{cases} \mathcal{P}(\bar{t}\,)=\left(t_2\frac{q_1(S(t_1))}{q_2(S(t_1))},R(t_1)-t_2\frac{R(t_1)}{q_2(S(t_1))},t_2\right), & \text{if } \mathcal{P}^{13}=(q_1,q_2),\ q_2\neq 0 \text{ is a proper parametrization of } \mathcal{C}^{13}\\ \mathcal{P}(\bar{t}\,)=(t_2S(t_1),t_2R(t_1),t_2),\ \ R\notin\mathbb{K} & \text{if } \mathcal{P}^{13}=(t_1,0) \text{ is a parametrization of } \mathcal{C}^{13}\end{cases}$$

is a rational proper parametrization of $\mathcal{V}$, and (R,S) is proper.

3.2. $f(\mathcal{P})=0$, *where*

$$\mathcal{P}(\bar{t}\,)=(t_2S(t_1),R(t_1),t_2)\in\mathbb{K}(\bar{t}\,)^3,\quad R\notin\mathbb{K}$$

is a rational proper parametrization of $\mathcal{V}$, and (R,S) is proper.

Proof. It is clear that if statements 1 and 2 (or 3) hold, then $\mathcal{V}$ is a rational ruled surface. Reciprocally, let $\mathcal{V}$ be a rational ruled surface. Then, a parametrization of $\mathcal{V}$ in the standard form is

$$\mathcal{Q}(\bar{t}\,)=(m_1(t_1)+t_2n_1(t_1),m_2(t_1)+t_2n_2(t_1),m_3(t_1)+t_2n_3(t_1))\in\mathbb{K}(\bar{t}\,)^3.$$

We assume that $n_1n_3\neq 0$ (see Section 2, and Remark 5). Thus, $\mathcal{Q}(t_1,-m_1/n_1)$ parametrizes $\mathcal{C}^{23}$, and $\mathcal{Q}(t_1,-m_3/n_3)$ parametrizes $\mathcal{C}^{12}$. Hence, statement 1 holds. Let us prove that statement 2 holds. For this purpose, we consider $\mathcal{P}^{12}=(p_1,p_2)\in\mathbb{K}(t_1)^2$, $\mathcal{P}^{23}=(\tilde{p}_1,\tilde{p}_2)\in\mathbb{K}(t_1)^2$ two rational proper parametrizations of $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$, respectively. We distinguish two different cases depending on whether $p_1\neq 0$ or $p_1=0$.

1. Let $p_1\neq 0$. Thus, again we distinguish two cases:
   a. Let $n_2\neq 0$. From the results in Section 2, the surface $\mathcal{V}$ admits a parametrization in standard reduced form

$$\mathcal{P}^3(\bar{t}\,)=(p_{13}(t_1)+t_2q_{13}(t_1),p_{23}(t_1)+t_2q_{23}(t_1),t_2)\in\mathbb{K}(\bar{t}\,)^3,$$

such that $q_{j3} = n_j/n_3 \neq 0, j = 1, 2$ (note that $n_1 n_2 n_3 \neq 0$). We assume that $\mathcal{P}^3$ is proper (otherwise, it can be easily reparametrized using the results in [16]). Observe that $\mathcal{P}^3(t_1, 0) = (p_{13}, p_{23}) \in \mathbb{K}(t_1)^2$ is a rational parametrization of $\mathcal{C}^{12}$. Then, since $\mathcal{P}^{12}$ is a rational proper parametrization of $\mathcal{C}^{12}$, there exits $S \in \mathbb{K}(t_1) \setminus \mathbb{K}$ such that $\mathcal{P}^{12}(S) = (p_1(S), p_2(S)) = (p_{13}, p_{23})$. In addition, since $\mathcal{P}^{23} = (\tilde{p}_1, \tilde{p}_2) \in \mathbb{K}(t_1)^2$ is a rational proper parametrization of $\mathcal{C}^{23}$ and

$$\mathcal{P}^3(t_1, -p_{13}/q_{13}) = \mathcal{P}^3(t_1, -p_1(S)/q_{13}) = (p_2(S) - q_{23}p_1(S)/q_{13}, -p_1(S)/q_{13})$$

is a rational parametrization of $\mathcal{C}^{23}$, there exists $R \in \mathbb{K}(t_1) \setminus \mathbb{K}$ such that

$$\tilde{p}_1(R) = p_2(S) - q_{23}p_1(S)/q_{13}, \quad \text{and} \quad \tilde{p}_2(R) = -p_1(S)/q_{13}.$$

Note that since $p_1 \neq 0$ and $S, R \in \mathbb{K}(t_1) \setminus \mathbb{K}$, then $\tilde{p}_2(R) \neq 0$. Then

$$q_{13} = \frac{-p_1(S)}{\tilde{p}_2(R)}, \quad \text{and} \quad q_{23} = \frac{\tilde{p}_1(R) - p_2(S)}{\tilde{p}_2(R)}.$$

Since $\mathcal{P}^3$ parametrizes properly $\mathcal{V}$, we have that $f(\mathcal{P}) = 0$, where $\mathcal{P}$ is the proper parametrization (note that $\mathcal{P} = \mathcal{P}^3$)

$$\mathcal{P}(\bar{t}) = \left(p_1(S(t_1)) - t_2 \frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2 \frac{\tilde{p}_1(R(t_1)) - p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2 \right).$$

We observe that since $\mathcal{P}$ is proper, then (R, S) is proper. Indeed: if (R, S) is not proper, there exists $\alpha(s_1) \in \overline{\mathbb{K}(s_1)}$, $\alpha(s_1) \neq s_1$ such that $(R(\alpha(s_1)), S(\alpha(s_1))) = (R(s_1), S(s_1))$ ($\overline{\mathbb{K}(s_1)}$ is the algebraic closure of $\mathbb{K}(s_1)$, and s_1 is a new variable). Then, $\mathcal{P}(\alpha(s_1), s_2) = \mathcal{P}(s_1, s_2)$ and $\alpha(s_1) \neq s_1$, which is impossible since $\mathcal{P}$ is proper.

b. Let $n_2 = 0$. From the results in Section 2, the surface $\mathcal{V}$ admits a parametrization

$$\mathcal{P}^3(\bar{t}) = (p_{13}(t_1) + t_2 q_{13}(t_1), m_2(t_1), t_2),$$

such that $q_{13} = n_1/n_3 \neq 0$ (note that $n_1 n_3 \neq 0$). We assume that $\mathcal{P}^3$ is proper. Observe that $\mathcal{P}^3(t_1, 0) = (p_{13}, m_2) \in \mathbb{K}(t_1)^2$ is a rational parametrization of $\mathcal{C}^{12}$. Then, since $\mathcal{P}^{12}$ is a rational proper parametrization of $\mathcal{C}^{12}$, there exists $S \in \mathbb{K}(t_1) \setminus \mathbb{K}$ such that $\mathcal{P}^{12}(S) = (p_1(S), p_2(S)) = (p_{13}, m_2)$. In addition, since $\mathcal{P}^{23} = (\tilde{p}_1, \tilde{p}_2) \in \mathbb{K}(t_1)^2$ is a rational parametrization of $\mathcal{C}^{23}$ and

$$\mathcal{P}^3(t_1, -p_{13}/q_{13}) = \mathcal{P}^3(t_1, -p_1(S)/q_{13}) = (p_2(S), -p_1(S)/q_{13})$$

is a rational parametrization of $\mathcal{C}^{23}$, there exists $R \in \mathbb{K}(t_1) \setminus \mathbb{K}$ such that

$$\tilde{p}_1(R) = p_2(S), \quad \text{and} \quad \tilde{p}_2(R) = -p_1(S)/q_{13}.$$

Note that since $p_1 \neq 0$ and $S, R \in \mathbb{K}(t_1) \setminus \mathbb{K}$, then $\tilde{p}_2(R) \neq 0$. Hence,

$$p_2(S) = \tilde{p}_1(R), \quad \text{and} \quad q_{13} = \frac{-p_1(S)}{\tilde{p}_2(R)}.$$

Since $\mathcal{P}^3$ parametrizes properly $\mathcal{V}$, we have that $f(\mathcal{P}) = 0$, where $\mathcal{P}$ is the proper parametrization ($\mathcal{P} = \mathcal{P}^3$)

$$\mathcal{P}(\bar{t}\,) = \left(p_1(S(t_1)) - t_2 \frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)), t_2 \right).$$

Finally, reasoning as in statement a above, we prove that (R, S) is proper.

2. Let $p_1 = 0$. From the above proof, we have $\tilde{p}_2 = p_{13} = 0$, and the surface $\mathcal{V}$ admits a proper parametrization

$$\mathcal{P}^3(\bar{t}\,) = (t_2 q_{13}(t_1), p_{23}(t_1) + t_2 q_{23}(t_1), t_2) \in \mathbb{K}(\bar{t}\,)^3, \quad q_{i3} = \frac{n_i}{n_3},\ i = 1, 2,\ q_{13} \neq 0.$$

We assume that $\mathcal{P}^3$ is proper. Under these conditions, we distinguish two different cases.

a. Let $n_2 \neq 0$. Then $q_{23} = n_2/n_3 \neq 0$, and $\mathcal{P}^3(t_1, -p_{23}/q_{23}) = (-q_{13}p_{23}/q_{23}, -p_{23}/q_{23}) \in \mathbb{K}(t_1)^2$ is a rational parametrization of $\mathcal{C}^{13}$. Let $\mathcal{P}^{13} = (q_1, q_2)$ be a rational proper parametrization of $\mathcal{C}^{13}$. Thus, there exists $S \in \mathbb{K}(t_1) \setminus \mathbb{K}$ such that

$$\mathcal{P}^{13}(S) = (q_1(S), q_2(S)) = (-q_{13}p_{23}/q_{23}, -p_{23}/q_{23}).$$

a.1. If $q_2 \neq 0$, since $S \notin \mathbb{K}$ we get that $q_2(S) \neq 0$, and then

$$q_{13} = \frac{q_1(S)}{q_2(S)}, \quad \text{and} \quad q_{23} = -\frac{p_{23}}{q_2(S)}.$$

Since $\mathcal{P}^3$ parametrizes properly $\mathcal{V}$, we have that $f(\mathcal{P}) = 0$, where $\mathcal{P}$ is the proper parametrization ($\mathcal{P} = \mathcal{P}^3$)

$$\mathcal{P}(\bar{t}\,) = \left(t_2 \frac{q_1(S(t_1))}{q_2(S(t_1))}, R(t_1) - t_2 \frac{R(t_1)}{q_2(S(t_1))}, t_2 \right), \quad R(t_1) := p_{23}(t_1).$$

Note that $R \neq 0$ because $\mathcal{V}$ is not the plane $x_2 = 0$ (see Section 2). Finally, reasoning as in statement 1, we prove that (R, S) is proper.

a.2. If $q_2 = 0$, then $p_{23} = 0$ and since $\mathcal{P}^3$ parametrizes properly $\mathcal{V}$, we have that $f(\mathcal{P}) = 0$, where $\mathcal{P}$ is the proper parametrization

$$\mathcal{P}(\bar{t}\,) = (t_2 S(t_1), t_2 R(t_1), t_2) \in \mathbb{K}(\bar{t}\,)^3, \quad S := q_{13},\ R := q_{23}.$$

Note that $S, R \notin \mathbb{K}$ since $\mathcal{V}$ is not a cylinder over the coordinate planes (see Section 2). Finally, reasoning as in statement 1, we prove that (R, S) is proper.

b. If $n_2 = 0$. Then $q_{23} = n_2/n_3 = 0$, and since $\mathcal{P}^3$ parametrizes properly $\mathcal{V}$, we have that $f(\mathcal{P}) = 0$, where $\mathcal{P}$ is the proper parametrization

$$\mathcal{P}(\bar{t}\,) = (t_2 S(t_1), R(t_1), t_2) \in \mathbb{K}(\bar{t}\,)^3, \quad S := q_{13},\ R := p_{23}.$$

Note that $S, R \notin \mathbb{K}$ since $\mathcal{V}$ is not a cylinder over the coordinate planes, and $\mathcal{V}$ is not the plane $x_2 - c = 0$, $c \in \mathbb{K}$ (see Section 2). Finally, reasoning as in statement 1, we prove that (R, S) is proper.

□

Geometrically speaking and similarly to Theorem 1, conditions in Theorem 2 involve to compute two planar parametrizations (see statement 1) that will be used to determine a rational planar base curve of the ruled surface and to compute the ruling direction of the ruled surface (see statement 2). The functions S, R are for coordinating the parameterization of the base curve and the ruling direction so that the parameterization of the ruled surface is in the required reduced form.

Remark 5. *Theorem 2 can be proved similarly if a different pair of rational plane curves $\mathcal{C}^{ij}$ are considered in statement 1, and if a different pair of polynomials n_i, n_j satisfies that $n_i n_j \neq 0$ (see Section 2).*

In the following, we assume that $n_1 n_3 \neq 0$, and that $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$ are the two rational plane curves satisfying statement 1 in Theorem 2. This requirement can always be achieved by applying a linear transformation to $\mathcal{V}$ without loss of generality.

In Corollary 3, we prove that statements 2 and 3 in Theorem 2 are equivalent to check the rationality of a plane curve, and to compute, in the affirmative case, a rational parametrization. For this purpose, we use the notion of content and primitive part of a polynomial. More precisely, given a nonzero polynomial $a(x_1, \ldots, x_n) \in I[x_1, \ldots, x_n]$, where I is a unique factorization domain, the content of a w.r.t. $\overline{x} := (x_1, \ldots, x_j)$, $j \leq n$ is the gcd of all the coefficients of $a(\overline{x}\,)$ w.r.t. $\overline{x}$. We denote it by $\mathrm{Content}_{\overline{x}}(a)$. Observe that $\mathrm{Content}_{\overline{x}}(a)$ divides the polynomial a. In addition, we denote by $\mathrm{pp}_{\overline{x}}(a)$ the primitive part of a w.r.t. $\overline{x}$. We have that $a(\overline{x}\,) = \mathrm{Content}_{\overline{x}}(a)\,\mathrm{pp}_{\overline{x}}(a)$, and it holds that the gcd of all coefficients of $\mathrm{pp}_{\overline{x}}(a)$ is 1 (see [29]).

Finally, using the notation introduced in Theorem 2, we consider the polynomials $N_i(x_1, x_2) = \mathrm{Content}_{t_2}(g_i)$, $i \in \{1, \ldots, 5\}$, where

$$g_1(x_1, x_2, t_2) = \mathrm{numer}\left(f\left(p_1(x_2) - t_2\frac{p_1(x_2)}{\tilde{p}_2(x_1)}, p_2(x_2) + t_2\frac{\tilde{p}_1(x_1) - p_2(x_2)}{\tilde{p}_2(x_1)}, t_2\right)\right),$$

$$g_2(x_1, x_2, t_2) = \mathrm{numer}\left(f\left(p_1(x_2) - t_2\frac{p_1(x_2)}{\tilde{p}_2(x_1)}, p_2(x_2), t_2\right)\right),$$

$$g_3(x_1,x_2,t_2) = \text{numer}\left(f\left(t_2\frac{q_1(x_2)}{q_2(x_2)}, x_1 - t_2\frac{x_1}{q_2(x_2)}, t_2\right)\right),$$

$$g_4(x_1,x_2,t_2) = f\left(t_2x_2, t_2x_1, t_2\right), \quad \text{and} \quad g_5(x_1,x_2,t_2) = f\left(t_2x_2, x_1, t_2\right).$$

Under these conditions, we prove the following corollary.

Corollary 3. *Let $\mathcal{V}$ be a surface defined by a polynomial $f(\overline{x}) \in \mathbb{K}[\overline{x}]$ and such that statement 1 in Theorem 2 holds. $\mathcal{V}$ is a rational ruled surface if and only if for some $i \in \{1,\ldots,5\}$, there exists a factor of N_i defining a rational plane curve $\mathcal{C}_{N_i}$. In this case, $(R(t_1), S(t_1)) \in \mathbb{K}(t_1)^2$, where $S \notin \mathbb{K}$, is a rational proper parametrization of $\mathcal{C}_{N_i}$.*

Proof. Let us prove that statement 2.1 in Theorem 2 is equivalent to the existence of a factor of N_1 defining a rational plane curve $\mathcal{C}_{N_1}$. For this purpose, we write

$$g_1(x_1,x_2,t_2) = h_0(x_1,x_2) + h_1(x_1,x_2)t_2 + \cdots + h_n(x_1,x_2)t_2^n.$$

Observe that since

$$g_1(x_1,x_2,t_2) = \text{numer}\left(f\left(p_1(x_2) - t_2\frac{p_1(x_2)}{\tilde{p}_2(x_1)}, p_2(x_2) + t_2\frac{\tilde{p}_1(x_1) - p_2(x_2)}{\tilde{p}_2(x_1)}, t_2\right)\right),$$

and $f(p_1(x_2), p_2(x_2), 0) = 0$, then t_2 divides g_1. Thus, $h_0 = 0$.
First, we note that if $f(\mathcal{P}) = 0$, where

$$\mathcal{P}(\overline{t}) = \left(p_1(S(t_1)) - t_2\frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2\frac{\tilde{p}_1(R(t_1)) - p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2\right),$$

then there exists $(R(t_1), S(t_1)) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ proper such that $g_1(R(t_1), S(t_1), t_2) = 0$. Since $(R(t_1), S(t_1))$ does not depend on t_2, we get that $h_j(R,S) = 0$, $j = 1,\ldots,n$. Then, $h(x_1,x_2)$ divides $N_1(x_1,x_2) = \gcd(h_1,\ldots,h_n)$, where h is the implicit polynomial defining the curve parametrized by (R,S). Therefore, $N_1(R,S) = 0$ and (R,S) is a rational proper parametrization of the plane curve $\mathcal{C}_{N_1}$ defined by a factor of the polynomial N_1.
Reciprocally, let $\mathcal{Q}(t_1) = (U(t_1), V(t_1)) \in \mathbb{K}(t_1)^2$ be such that $N_1(\mathcal{Q}) = 0$ and $\mathcal{Q}$ is proper. Since N_1 divides g_1, we deduce that $g_1(U(t_1), V(t_1), t_2) = 0$. Hence, $f(\mathcal{P}) = 0$, where

$$\mathcal{P}(\overline{t}) = \left(p_1(V(t_1)) - t_2\frac{p_1(V(t_1))}{\tilde{p}_2(U(t_1))}, p_2(V(t_1)) + t_2\frac{\tilde{p}_1(U(t_1)) - p_2(V(t_1))}{\tilde{p}_2(U(t_1))}, t_2\right).$$

One reasons similarly to prove that statement 2.2 in Theorem 2 is equivalent to the existence of a factor of N_2 defining a rational plane curve $\mathcal{C}_{N_2}$, that statement 3.1 in Theorem 2 is equivalent to the existence of a factor of N_3 or N_4 defining a rational plane curve $\mathcal{C}_{N_3}$ or $\mathcal{C}_{N_4}$, and that statement 3.2 in Theorem 2 is equivalent to the existence of a factor of N_5 defining a rational plane curve $\mathcal{C}_{N_5}$. □

In the following corollary, we prove that the properness of the output parametrization in Theorem 2 is equivalent to the properness of the parametrization of the plane curve in Corollary 3.

Corollary 4. *Let $\mathcal{V}$ be a ruled surface defined by a polynomial $f(\overline{x}) \in \mathbb{K}[\overline{x}]$. Let $\mathcal{P}^*$ be the output parametrization of Theorem 2, and $(R^*, S^*) \in \mathbb{K}(t_1)^2$ a parametrization of the corresponding plane curve $\mathcal{C}_{N_i}$, $i \in \{1, \ldots, 5\}$ (see Corollary 3). It holds that $\mathcal{P}^*$ is proper if and only if (R^*, S^*) is proper.*

Proof. Since $\mathcal{V}$ is a ruled surface, we have that Theorem 2 and Corollary 3 hold. We assume, that statement 2.1 in Theorem 2 holds (we reason similarly for the other cases). Thus, in particular, $\mathcal{P}(\bar{t}) = \left(p_1(S(t_1)) - t_2\frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2\frac{\tilde{p}_1(R(t_1))-p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2\right)$ is a rational proper parametrization of $\mathcal{V}$, and (R, S) is a proper parametrization of $\mathcal{C}_{N_1}$. Since (R^*, S^*) is a parametrization of $\mathcal{C}_{N_1}$, there exists $L \in \mathbb{K}(t_1) \setminus \mathbb{K}$ such that $(R^*(t_1), S^*(t_1)) = (R, S)(L(t_1))$ and then, $\mathcal{P}(L(t_1), t_2) = \mathcal{P}^*(t_1, t_2)$, where $\mathcal{P}^*(\bar{t}) = \left(p_1(S^*(t_1)) - t_2\frac{p_1(S^*(t_1))}{\tilde{p}_2(R^*(t_1))}, p_2(S^*(t_1)) + t_2\frac{\tilde{p}_1(R^*(t_1))-p_2(S^*(t_1))}{\tilde{p}_2(R^*(t_1))}, t_2\right)$. Using this fact and that $\mathcal{P}$ is proper, we get that $\mathcal{P}^*$ is proper if and only if $(L(t_1), t_2)$ is proper which is equivalent to $L(t_1)$ is linear (see Lemma 4.32 in [23]). Taking into account that $(R^*, S^*) = (R, S)(L)$ and that (R, S) is proper, we get that L is linear if and only if (R^*, S^*) is proper. Hence, $\mathcal{P}^*$ is proper if and only if (R^*, S^*) is proper. □

From Theorem 2 and Corollaries 3 and 4, we derive the following algorithm that decides whether $\mathcal{V}$ is a rational ruled surface and in the affirmative case, it computes a rational proper parametrization of $\mathcal{V}$.

Algorithm 2: Parametrization of a Rational Ruled Surface

- **Input:** A surface $\mathcal{V}$ defined by an irreducible polynomial $f(\overline{x}) \in \mathbb{K}[\overline{x}]$.
- **Output:** the message "$\mathcal{V}$ `is not a rational ruled surface`" or a proper parametrization $\mathcal{P}$ of "`the rational ruled surface` $\mathcal{V}$ `in the standard reduced form`".

1. If $\deg_{x_3}(f) = 0$ (similarly if $\deg_{x_1}(f) = 0$ or $\deg_{x_2}(f) = 0$), compute $(p(t_1), q(t_1))$ a parametrization of the curve defined by the polynomial $f(x_1, x_2) = 0$, and Return $\mathcal{P}(\bar{t}) = (p(t_1), q(t_1), t_2) \in \mathbb{K}(\bar{t})^3$ "`is a proper parametrization`".
2. Compute the polynomials $f_0^{ij}(x_i, x_j)$, and check whether there exist two rational plane curves $\mathcal{C}^{ij}$ and $\mathcal{C}^{k\ell}$ defined by a factor of f_0^{ij} and $f_0^{k\ell}$, respectively, for $ij \neq k\ell$, and $ij, k\ell \in \{12, 23, 13\}$. In the affirmative case, assume that $ij = 12$, and $k\ell = 23$ (see Remark 5) and go to Step 3. Otherwise, Return "$\mathcal{V}$ `is not a rational ruled surface`".

3. Compute $\mathcal{P}^{12} = (p_1, p_2) \in \mathbb{K}(t_1)^2$, and $\mathcal{P}^{23} = (\tilde{p}_1, \tilde{p}_2) \in \mathbb{K}(t_1)^2$ rational proper parametrizations of $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$, respectively (see Remark 4). If $p_1 \neq 0$ go to Step 4. Otherwise, go to Step 6.

4. Check whether there exists a rational plane curve $\mathcal{C}_{N_1}$ defined by a factor of the polynomial $N_1(x_1, x_2) = \text{Content}_{t_2}(g_1)$, where

$$g_1(x_1, x_2, t_2) = \text{numer}\left(f\left(p_1(x_2) - t_2\frac{p_1(x_2)}{\tilde{p}_2(x_1)}, p_2(x_2) + t_2\frac{\tilde{p}_1(x_1) - p_2(x_2)}{\tilde{p}_2(x_1)}, t_2\right)\right).$$

   In the affirmative case, compute $(R(t_1), S(t_1)) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ a rational proper parametrization of $\mathcal{C}_{N_1}$, and Return

$$\mathcal{P}(\bar{t}) = \left(p_1(S(t_1)) - t_2\frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2\frac{\tilde{p}_1(R(t_1)) - p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2\right),$$

   "`is a proper parametrization`". Otherwise, go to Step 5.

5. Check whether there exists a rational plane curve $\mathcal{C}_{N_2}$ defined by a factor of the polynomial $N_2(x_1, x_2) = \text{Content}_{t_2}(g_2)$, where

$$g_2(x_1, x_2, t_2) = \text{numer}\left(f\left(p_1(x_2) - t_2\frac{p_1(x_2)}{\tilde{p}_2(x_1)}, p_2(x_2), t_2\right)\right).$$

   In the affirmative case, compute $(R(t_1), S(t_1)) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ a rational proper parametrization of $\mathcal{C}_{N_2}$, and Return

$$\mathcal{P}(\bar{t}) = \left(p_1(S(t_1)) - t_2\frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)), t_2\right)$$

   "`is a proper parametrization`". Otherwise, go to Step 2, and consider different rational components and apply again the algorithm. If there have no more rational components, Return "$\mathcal{V}$ `is not a rational ruled surface`".

6. Check whether the plane curve $\mathcal{C}^{13}$ is rational. In the affirmative case, compute $\mathcal{P}^{13} = (q_1, q_2) \in \mathbb{K}(t_1)^2$ a rational proper parametrization of $\mathcal{C}^{13}$ (see Remark 4) and go to Step 7. Otherwise, go to Step 8.

7. 7.1. If $q_2 \neq 0$, check whether there exists a rational plane curve $\mathcal{C}_{N_3}$ defined by a factor of the polynomial $N_3(x_1, x_2) = \text{Content}_{t_2}(g_3)$, where

$$g_3(x_1, x_2, t_2) = \text{numer}\left(f\left(t_2\frac{q_1(x_2)}{q_2(x_2)}, x_1 - t_2\frac{x_1}{q_2(x_2)}, t_2\right)\right).$$

   In the affirmative case, compute $(R(t_1), S(t_1)) \in \mathbb{K}(t_1)^2, R \neq 0, S \notin \mathbb{K}$, a rational proper parametrization of $\mathcal{C}_{N_3}$ and Return

$$\mathcal{P}(\bar{t}) = \left(t_2\frac{q_1(S(t_1))}{q_2(S(t_1))}, R(t_1) - t_2\frac{R(t_1)}{q_2(S(t_1))}, t_2\right),$$

"`is a proper parametrization`". Otherwise, go to Step 2, and consider different rational components and apply again the algorithm. If there have no more rational components, RETURN "$\mathcal{V}$ `is not a rational ruled surface`".

7.2. If $q_2 = 0$, check whether there exists a rational plane curve $\mathcal{C}_{N_4}$ defined by a factor of the polynomial $N_4(x_1, x_2) = \text{Content}_{t_2}(g_4)$, where

$$g_4(x_1, x_2, t_2) = f(t_2x_2, t_2x_1, t_2).$$

In the affirmative case, compute $(R(t_1), S(t_1)) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ a rational proper parametrization of $\mathcal{C}_{N_4}$ and RETURN

$$\mathcal{P}(\bar{t}) = (t_2S(t_1), t_2R(t_1), t_2),$$

"`is a proper parametrization`". Otherwise, go to Step 2, and consider different rational components and apply again the algorithm. If there have no more rational components, RETURN "$\mathcal{V}$ `is not a rational ruled surface`".

8. Check whether there exists a rational plane curve $\mathcal{C}_{N_5}$ defined by a factor of the polynomial $N_5(x_1, x_2) = \text{Content}_{t_2}(g_5)$, where

$$g_5(x_1, x_2, t_2) = f(t_2x_2, x_1, t_2).$$

In the affirmative case, compute $(R(t_1), S(t_1)) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ a rational proper parametrization of $\mathcal{C}_{N_5}$ and RETURN

$$\mathcal{P}(\bar{t}) = (t_2S(t_1), R(t_1), t_2),$$

"`is a proper parametrization`". Otherwise, go to Step 2, and consider different rational components and apply again the algorithm. If there have no more rational components, RETURN "$\mathcal{V}$ `is not a rational ruled surface`".

Remark 6. *Remark 3 can be stated similarly in this new situation to the rational parametrizations* $\mathcal{P}^{12}, \mathcal{P}^{23}, \mathcal{P}^{13}$, (R, S), *and the output parametrization* $\mathcal{P}$.

In the following example, we illustrate the performance of Algorithm 2.

Example 3. *Consider the surface* $\mathcal{V}$ *over* $\mathbb{C}$ *introduced in Example 2, and defined by the polynomial*

$f(\overline{x}) = -49x_2x_1^3 - 799x_3x_2x_1^2 + 20x_2x_1^2 + 2x_2^2x_1^2 + 980x_3x_1^2 - 2205x_3^2x_1^2 + x_2^3x_1 - 33750x_3^3x_1 - 400x_3x_1 + 606x_3x_2x_1 - 5x_2^2x_1 - 68x_3x_2^2x_1 - 1747x_3^2x_2x_1 - 25x_3^2x_1 + x_2^3x_3 - 25x_2^2x_3^2 + 1396x_2x_3^2 - 1120x_3^2 - 48915x_3^4 - 5190x_3^3 - 4237x_2x_3^3 - 14x_2^2x_3 \in \mathbb{C}[\overline{x}].$

Let us apply Algorithm 2. *For this purpose, we first observe* $\deg_{x_i}(f) > 0$*, for* $i = 1, 2, 3$*. Then, in Step 2 of the algorithm, we compute*

$$f_0^{12}(x_1, x_2) = -x_1x_2(49x_1^2 - 20x_1 - 2x_1x_2 - x_2^2 + 5x_2),$$

$$f_0^{23}(x_2, x_3) = (14x_2^2 + 5190x_3^2 + 1120x_3 + 48915x_3^3 - 1396x_2x_3 + 25x_2^2x_3 + 4237x_2x_3^2 - x_2^3).$$

The plane curves defined by the equations $49x_1^2 - 20x_1 - 2x_1x_2 - x_2^2 + 5x_2 = 0$*, and* $14x_2^2 + 5190x_3^2 + 1120x_3 + 48915x_3^3 - 1396x_2x_3 + 25x_2^2x_3 + 4237x_2x_3^2 - x_2^3 = 0$ *are rational. These curves are denoted as* $\mathcal{C}^{12}$ *and* $\mathcal{C}^{23}$*, respectively.*

In Step 3 of the algorithm, we compute a rational proper parametrization of $\mathcal{C}^{12}$*,*

$$\mathcal{P}^{12}(t_1) = (p_1, p_2) = \left(\frac{-\sqrt{2}t_1(-5+t_1)}{5(4t_1 - 10 + 5\sqrt{2})}, \frac{(50+5\sqrt{2})t_1(-20+100\sqrt{2}+49t_1)}{1225(4t_1 - 10 + 5\sqrt{2})}\right) \in \mathbb{R}(t_1)^2.$$

and $\mathcal{C}^{23}$*,*

$$\mathcal{P}^{23}(t_1) = (\tilde{p}_1(t_1), \tilde{p}_2(t_1)) = \left(\frac{2(-378367t_1^2 + 10410900t_1 - 142098075 + 4102t_1^3)}{1241(t_1^3 - 25t_1^2 - 4237t_1 - 48915)},\right.$$
$$\left.\frac{2(20322550 - 513355t_1 - 28t_1^2 + 49t_1^3)}{1241(t_1^3 - 25t_1^2 - 4237t_1 - 48915)}\right) \in \mathbb{R}(t_1)^2.$$

Since $p_1 \neq 0$ *we go to Step 4 of the algorithm, and we check whether there exists a rational plane curve* $\mathcal{C}_{N_1}$ *defined by a factor of the polynomial* $N_1(x_1, x_2) = \mathrm{Content}_{t_2}(g_1)$*, where*

$$g_1(x_1, x_2, t_2) = \mathrm{numer}\left(f\left(p_1(x_2) - t_2\frac{p_1(x_2)}{\tilde{p}_2(x_1)}, p_2(x_2) + t_2\frac{\tilde{p}_1(x_1) - p_2(x_2)}{\tilde{p}_2(x_1)}, t_2\right)\right).$$

We have that

$N_1(x_1, x_2) = (73x_2x_1 - 2555x_2 + 2482x_2\sqrt{2} + 18675 - 4150\sqrt{2} - 315x_1 + 70\sqrt{2}x_1)(4x_2 - 10 + 5\sqrt{2})(x_1 - 35 + 34\sqrt{2})^3(-x_1 + 35 + 34\sqrt{2})^3(x_1 + 45)^3.$

To find $(R, S) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2$*, we consider the plane curve defined by the irreducible polynomial* $73x_2x_1 - 2555x_2 + 2482x_2\sqrt{2} + 18675 - 4150\sqrt{2} - 315x_1 + 70\sqrt{2}x_1$*, and it defines a rational plane curve* $\mathcal{C}_{N_1}$*. Then, we compute a rational proper parametrization of* $\mathcal{C}_{N_1}$ *as*

$$(R(t_1), S(t_1)) = \left(t_1, -\frac{5(-9+2\sqrt{2})(-415+7t_1)}{73(t_1 - 35 + 34\sqrt{2})}\right) \in (\mathbb{R}(t_1) \setminus \mathbb{R})^2.$$

Then,

$$\mathcal{P} = \left(p_1(S(t_1)) - t_2\frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2\frac{\tilde{p}_1(R(t_1)) - p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2\right) = \left(\frac{q_{11}}{q_{12}}, \frac{q_{21}}{q_{22}}, t_2\right),$$

where

$q_{11} = (-9+2\sqrt{2})\sqrt{2}(-490t_1^3+280t_1^2+5133550t_1-203225500+392t_1^2\sqrt{2}+7186970\sqrt{2}t_1-284515700\sqrt{2} - 686t_1^3\sqrt{2} + 6205t_2t_1^3 + 8687t_2\sqrt{2}t_1^3 - 155125t_2t_1^2 - 217175t_2t_1^2\sqrt{2} - 26290585t_2t_1 - 36806819t_2\sqrt{2}t_1 - 303517575t_2 - 424924605t_2\sqrt{2})$,

$q_{12} = 73(-415 + 7t_1)(-4866 + 106t_1 - 4199\sqrt{2} + 17\sqrt{2}t_1)(t_1 - 35 + 34\sqrt{2})$,

$q_{21} = 2068272t_1^2 - 544956812t_1 + 331704t_1^2\sqrt{2} - 87398734\sqrt{2}t_1 + 63812t_1^3 + 10234t_1^3\sqrt{2} + 11741634840 + 1883092380\sqrt{2} + 78183t_2\sqrt{2}t_1^3 - 10107945t_2t_1^2\sqrt{2} + 239474529t_2\sqrt{2}t_1 + 487494t_2t_1^3 - 63026010t_2t_1^2 + 1493194122t_2t_1 + 5038391745t_2\sqrt{2} + 31415854410t_2$,

$q_{22} = 146(t_1 + 118)(-4866 + 106t_1 - 4199\sqrt{2} + 17\sqrt{2}t_1)(t_1 - 35 + 34\sqrt{2})$

"`is a proper parametrization of the rational ruled surface` $\mathcal{V}$ `in standard reduced form`*".*

Observe that since $\mathcal{P}^{12}$, $\mathcal{P}^{23}$ and R, S have coefficients in $\mathbb{R}$, then $\mathcal{P}$ also has coefficients in $\mathbb{R}$ (see Remark 6).

4. Parametrically Ruled Surfaces

In the following, we consider a surface $\mathcal{V}$ defined by a parametrization (not necessarily proper) over $\mathbb{K}$,

$$\mathcal{M}(\bar{t}\,) = (m_1(\bar{t}\,), m_2(\bar{t}\,), m_3(\bar{t}\,)) \in \mathbb{K}(\bar{t}\,)^3.$$

In this section, we analyze whether $\mathcal{V}$ is a ruled surface, and in the affirmative case we compute a proper reparametrization in standard reduced form. More precisely, the idea is to check whether there exists a proper parametrization of the form

$$\mathcal{P}(\bar{t}\,) = (p_1(t_1) + t_2q_1(t_1), p_2(t_1) + t_2q_2(t_1), t_2) \in \mathbb{K}(\bar{t}\,)^3,$$

(where p_j, q_j are given in Theorem 2), and $(U, V) \in (\mathbb{K}(\bar{t}\,)\setminus\mathbb{K})^2$ such that $\mathcal{P}(U, V) = \mathcal{M}$. Observe that from this equality, we get that $V = m_3$.

A direct approach to this problem could consist in implicitizating the parametrization (see e.g. [19]) to apply afterwards the algorithms developed in Section 3 to the implicit equation. This solution might be too time consuming and then, we would like to approach the problem by means of rational reparametrizations which involves more satisfactory running times (compare Theorem 4, with Theorem 10 in [19]). With rational reparametrization we basically mean without implicitizating, or more formally, by finding a linear parameter transformation to reparameterized the given parametrization. Note that any reparametrization of a rational parametrization is again a parametrization of the same variety.

To start with the problem, we first assume that $\mathcal{V}$ is not a plane. Note that this assumption is not a loss of generality, because one can easily deduce whether a parametrically given surface is a plane.

Now, we deal with the cylinder case. In order to analyze whether $\mathcal{V}$ is a cylinder over any of the coordinate planes of $\mathbb{K}^3$, we apply the following result presented in [19] (Theorem 5).

Theorem 3. (Cylinder criterion) *Let* $H_i(\overline{t}\,,\overline{h}\,) = \mathrm{numer}(m_i(\overline{t}\,) - m_i(\overline{h}\,))$*, where* $\overline{h} = (h_1, h_2)$ *are new variables, and* $i \in \{1,2,3\}$*. Then,* $\mathcal{V}$ *is a cylinder over the* x_ix_j*–plane if and only if* $\gcd(H_i, H_j) \neq 1$*.*

If $\mathcal{V}$ is a cylinder over the x_1x_2-plane and $f(x_1, x_2) \in \mathbb{K}[x_1, x_2]$ is the implicit equation defining $\mathcal{V}$, we consider $a \in \mathbb{K}$ such that $(m_1, m_2)(a, t_2) \notin \mathbb{K}^2$, and we get that, up to multiplication by non-zero constants,

$$f(x_1, x_2)^r = \mathrm{Res}_{t_2}(G_1(a, t_2, x_2), G_2(a, t_2, x_2)), \quad \text{where} \quad r \in \mathbb{K} \quad \text{and}$$

$G_i(\overline{t}\,, x_i) = \mathrm{numer}(m_i(\overline{t}\,) - x_i)$, $i = 1, 2$ (see Theorem 8 in [19]). Then, one computes a proper parametrization $(p(t_1), q(t_1)) \in \mathbb{K}(t_1)^2$ of the plane curve defined by the equation $f(x_1, x_2) = 0$, and we get that $\mathcal{P}(\overline{t}\,) = (p(t_1), q(t_1), t_2)$ is a proper parametrization of $\mathcal{V}$. One reasons similarly if $\mathcal{V}$ is a cylinder over a different plane.

Once the plane case and the cylinder case are analyzed, we assume that $\mathcal{V}$ is neither a cylinder nor a plane. As we stated above, we are interested in applying Theorem 2. For this purpose, first we need to compute a rational proper parametrization of $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$ (see statement 1 of Theorem 2). We also need to determine a rational proper parametrization of $\mathcal{C}^{13}$, if we are in statement 3 of Theorem 2.

Since we do not have the implicit equation defining the surface $\mathcal{V}$, we have to compute the polynomial $f_0^{ij}(x_i, x_j)$ defining implicitly the plane curve $\mathcal{C}^{ij}$, $i < j, i, j \in \{1, 2, 3\}$, using the input parametrization $\mathcal{M}$. For this purpose, we use Theorem 10 in [19], and the fact that if $\overline{t}_0 \in \mathbb{K}^2$ is such that $m_i(\overline{t}_0) - x_1 = m_j(\overline{t}_0) - x_2 = m_k(\overline{t}_0) = 0$, for $k \in \{1, 2, 3\}$, $k \neq i$, $k \neq j$, then $(x_1, x_2) \in \mathcal{C}^{ij}$. For this purpose, in order to apply Theorem 10 in [19], we need assume that none of the projective curves defined by each numerator and denominator of m_i, $i = 1, 2, 3$ passes through the points at infinity $(0 : 1 : 0)$ and $(1 : 0 : 0)$, where the homogeneous variables are (t_1, t_2, w). Note that this requirement can always be achieved by applying a linear change of variables to $\mathcal{M}$. This assumption implies that each numerator and denominator of m_i has positive degree w.r.t. t_i, and then its leading coefficient w.r.t. t_i does not depend on $t_j, i \neq j$, $i, j \in 1, 2$. Thus, for $k = 1, 2, 3$, and $i \neq j$, $i, j \in 1, 2$, $\deg_{t_i}(G_k(\overline{t}\,, x_k)) >$

0, and the leading coefficient of $G_k(\bar{t}, x_k)$ w.r.t. t_i does not depend of t_j, where $G_k(\bar{t}, x_i) = \text{numer}(m_k(\bar{t}) - x_k)$. Finally, since $\mathcal{V}$ is neither a cylinder nor a plane, we may assume that for every $i, j \in \{1, 2, 3\}$, with $i < j$, the gradients $\{\nabla m_i(\bar{t}), \nabla m_j(\bar{t})\}$ are linearly independent.

Under these conditions, Theorem 4 shows how to compute the polynomials $f_0^{ij}(x_i, x_j)$. The theorem is obtained from Lemmas 12, 13, 14, 15, and Theorem 10 in [19].

Theorem 4. *It holds that for $j < k, i \neq j$, $i \neq k$, and $i, j, k \in \{1, 2, 3\}$,*

$$(f_0^{ij}(x_j, x_k))^r = \text{pp}_{x_k}(\text{Content}_{\{Z,W\}}(\text{Res}_{t_2}(T(t_2, x_j),\ K(t_2, Z, W, x_j, x_k)))) \in \mathbb{K}[x_j, x_k],$$

where $r \in \mathbb{K}$, and

1. $K(t_2, Z, W, x_j, x_k) = \text{Res}_{t_1}(S(t_1, x_j),\ G_{Z,W}(\bar{t}, Z, W, x_j, x_k))$,
2. $G_{Z,W}(\bar{t}, Z, W, x_j, x_k) = G_k(\bar{t}, x_k) + ZG_i(\bar{t}, 0) + WG_j(\bar{t}, x_j)$,
3. $S(t_1, x_j) = \text{pp}_{x_j}(\text{Res}_{t_2}(G_i(\bar{t}, 0), G_j(\bar{t}, x_j)))$,
4. $T(t_2, x_j) = \text{pp}_{x_j}(\text{Res}_{t_1}(G_i(\bar{t}, 0), G_j(\bar{t}, x_j)))$.

Under these conditions, we apply Theorem 2, and we obtain a procedure that computes a proper reparametrization of a given parametrization $\mathcal{M}$, if it is a ruled surface. More precisely, if $\mathcal{V}$ is a rational ruled surface, then there exists a proper parametrization given in standard reduced form

$$\mathcal{P}(\bar{t}) = (p_1(t_1) + t_2 q_1(t_1), p_2(t_1) + t_2 q_2(t_1), t_2) \in \mathbb{K}(\bar{t})^3$$

(see equation given in 2), where p_j, q_j are given in Theorem 2. Thus, we only have to check whether there exists $(U, V) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$ such that $\mathcal{P}(U, V) = \mathcal{M}$. Observe that from this equality, we get that $V = m_3$.

To start with, we prove the following theorem that is equivalent to Theorem 2 and Corollary 3, but for the parametric case. Similarly as in Theorem 2, Theorem 5 involves to compute two planar parametrizations (see statement 1) that will be used to determine a rational planar base curve of the ruled surface $\mathcal{V}$, and to compute the ruling direction of $\mathcal{V}$ (see statement 2). The functions S and R are for coordinating the parameterization of the base curve and the ruling direction so that the parameterization of $\mathcal{V}$ is in the required reduced form.

Theorem 5. *A surface $\mathcal{V}$ defined by the parametrization*

$$\mathcal{M}(\bar{t}) = (m_1(\bar{t}), m_2(\bar{t}), m_3(\bar{t})) \in \mathbb{K}(\bar{t})^3$$

is a rational ruled surface if and only if the following statements hold:

1. *At least two of the three plane algebraic curves* $\mathcal{C}^{ij}$, $ij \in \{12, 13, 23\}$, *are rational (see Section 2). Let us assume that* $\mathcal{C}^{12}$ *and* $\mathcal{C}^{23}$ *are rational, and let* $\mathcal{P}^{12} = (p_1, p_2) \in \mathbb{K}(t_1)^2$, $\mathcal{P}^{23} = (\tilde{p}_1, \tilde{p}_2) \in \mathbb{K}(t_1)^2$ *be rational proper parametrizations of* $\mathcal{C}^{12}$ *and* $\mathcal{C}^{23}$, *respectively.*

2. *If* $p_1 \neq 0$, *there exists* $(\mathcal{L}, \mathcal{T}) \in \mathbb{K}(\bar{t}\,) \setminus \mathbb{K}$ *such that one of the following statements holds:*

   2.1. $p_1(\mathcal{T}) - m_3\frac{p_1(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} - m_1 = p_2(\mathcal{T}) + m_3\frac{\tilde{p}_1(\mathcal{L}) - p_2(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} - m_2 = 0$. *In this case,*

   $$\mathcal{P}(\bar{t}\,) = \left(p_1(S(t_1)) - t_2\frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2\frac{\tilde{p}_1(R(t_1)) - p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2\right),$$

   is a rational proper parametrization of $\mathcal{V}$, *where* $(R, S) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ *is a rational proper parametrization of the curve* $\mathcal{C}_{N_1}$ *defined parametrically by* $(\mathcal{L}, \mathcal{T})$.

   2.2. $p_1(\mathcal{T}) - m_3\frac{p_1(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} - m_1 = p_2(\mathcal{T}) - m_2 = 0$. *In this case,*

   $$\mathcal{P}(\bar{t}\,) = \left(p_1(S(t_1)) - t_2\frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)), t_2\right)$$

   is a rational proper parametrization of $\mathcal{V}$, *where* $(R, S) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ *is a rational proper parametrization of the curve* $\mathcal{C}_{N_2}$ *defined parametrically by* $(\mathcal{L}, \mathcal{T})$.

3. *If* $p_1 = 0$, *there exists* $\mathcal{L} \in \mathbb{K}(\bar{t}\,) \setminus \{0\}$, *and* $\mathcal{T} \in \mathbb{K}(\bar{t}\,) \setminus \mathbb{K}$ *such that one of the following statements holds:*

   3.1.

   $$\begin{cases} m_3\frac{q_1(\mathcal{T})}{q_2(\mathcal{T})} - m_1 = \mathcal{L} - m_3\frac{\mathcal{L}}{q_2(\mathcal{T})} - m_2 = 0, & \text{if } \mathcal{P}^{13} = (q_1, q_2),\ q_2 \neq 0 \text{ is a proper parametrization of } \mathcal{C}^{13} \\ m_3\mathcal{T} - m_1 = m_3\mathcal{L} - m_2 = 0,\ \mathcal{L} \notin \mathbb{K} & \text{if } \mathcal{P}^{13} = (t_1, 0) \text{ is a parametrization of } \mathcal{C}^{13} \end{cases}$$

   In this case,

   $$\begin{cases} \mathcal{P}(\bar{t}\,) = \left(t_2\frac{q_1(S(t_1))}{q_2(S(t_1))}, R(t_1) - t_2\frac{R(t_1)}{q_2(S(t_1))}, t_2\right), & \text{if } \mathcal{P}^{13} = (q_1, q_2),\ q_2 \neq 0 \text{ is a proper parametrization of } \mathcal{C}^{13} \\ \mathcal{P}(\bar{t}\,) = (t_2 S(t_1), t_2 R(t_1), t_2),\ \ R \notin \mathbb{K} & \text{if } \mathcal{P}^{13} = (t_1, 0) \text{ is a parametrization of } \mathcal{C}^{13} \end{cases}$$

   is a rational proper parametrization of $\mathcal{V}$, *where* $(R, S) \in \mathbb{K}(t_1)^2$, $S \notin \mathbb{K}$ *is a rational proper parametrization of the curve* $\mathcal{C}_{N_3}$ *or* $\mathcal{C}_{N_4}$, *defined parametrically by* $(\mathcal{L}, \mathcal{T})$.

3.2. $m_3\mathcal{T} - m_1 = \mathcal{L} - m_2 = 0$. *In this case,* $\mathcal{P}(\bar{t}) = (t_2 S(t_1), R(t_1), t_2)$, *is a rational proper parametrization of* $\mathcal{V}$, *where* $(R, S) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ *is a rational proper parametrization of the curve* $\mathcal{C}_{N_5}$ *defined parametrically by* $(\mathcal{L}, \mathcal{T})$.

Proof. It is clear that if statements 1 and 2 (or 3) hold, then $\mathcal{V}$ is a rational ruled surface. Reciprocally, let $\mathcal{V}$ be a rational ruled surface. Then, statement 1 holds (see statement 1 in Theorem 2), and some of the statements, 2 or 3, of Theorem 2 holds. Let us assume that statement 2.1 holds (one reasons similarly if a different statement holds). That is,

$$\mathcal{P}^*(t_1, t_2) = \left(p_1(S^*(t_1)) - t_2 \frac{p_1(S^*(t_1))}{\tilde{p}_2(R^*(t_1))}, p_2(S^*(t_1)) + t_2 \frac{\tilde{p}_1(R^*(t_1)) - p_2(S^*(t_1))}{\tilde{p}_2(R^*(t_1))}, t_2\right)$$

is a proper parametrization of $\mathcal{V}$, where $(R^*, S^*) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ is a rational proper parametrization of the curve $\mathcal{C}_{N_1}$ (see Corollary 3). Since $\mathcal{M}$ is also a parametrization of $\mathcal{V}$, there exists $(U, V) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$ such that $\mathcal{P}^*(U, V) = \mathcal{M}$. From this equality, we get that $V = m_3$, and

$$p_1(S^*(U)) - m_3 \frac{p_1(S^*(U))}{\tilde{p}_2(R^*(U))} = m_1, \quad p_2(S^*(U)) + m_3 \frac{\tilde{p}_1(R^*(U)) - p_2(S^*(U))}{\tilde{p}_2(R^*(U))} = m_2.$$

That is,

$$p_1(\mathcal{T}) - m_3 \frac{p_1(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} - m_1 = p_2(\mathcal{T}) + m_3 \frac{\tilde{p}_1(\mathcal{L}) - p_2(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} - m_2 = 0,$$

where $(\mathcal{L}, \mathcal{T}) := (R^*(U), S^*(U)) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$. Observe that since (R^*, S^*) is a rational parametrization of $\mathcal{C}_{N_1}$, then $(\mathcal{L}, \mathcal{T})$ also parametrizes $\mathcal{C}_{N_1}$.
Now, we consider $(R(t_1), S(t_1)) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ a new rational proper parametrization of $\mathcal{C}_{N_1}$, and

$$\mathcal{P}(t_1, t_2) = \left(p_1(S(t_1)) - t_2 \frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2 \frac{\tilde{p}_1(R(t_1)) - p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2\right).$$

Since (R^*, S^*) and (R, S) are both rational proper parametrizations of $\mathcal{C}_{N_1}$, there exists $r \in \mathbb{K}(t_1) \setminus \mathbb{K}, \deg(r) = 1$, such that $(R^*, S^*) = (R(r), S(r))$. Then, $\mathcal{P}(r(t_1), t_2) = \mathcal{P}^*(t_1, t_2)$ which implies that $\mathcal{P}$ is a rational proper parametrization of $\mathcal{V}$ (note that $(r(t_1), t_2)$ and $\mathcal{P}^*(\bar{t})$ are both proper, and thus $\mathcal{P}(\bar{t})$ is also proper). □

In Theorem 5, one important task is to solve $(\mathcal{T}, \mathcal{L})$ from the equation systems. The systems appearing in the statements 2.2, 3.1 and 3.2 are clearly zero dimensional. We now study the system of the statement 2.1, it is defined by the equations $p_1(x) - m_3 p_1(x)/\tilde{p}_2(y) - m_1 = p_2(x) + m_3(\tilde{p}_1(y) - p_2(x))/\tilde{p}_2(y) - m_2 = 0$. We show that the system is zero dimensional if $\mathcal{M}(\bar{t})$ defines a ruled surface in the following proposition. Hence, the computations of $(\mathcal{T}, \mathcal{L}) \in \mathbb{K}(\,\bar{t}\,)^2$ are not difficult (see statement 2 in Remark 7).

To prove the proposition, we consider the equations

$$e_1(x, y, \bar{t}\,) = p_{1,1}(x)\tilde{p}_{2,1}(y) - m_3(\,\bar{t}\,)p_{1,1}(x)\tilde{p}_{2,2}(y) - m_1(\,\bar{t}\,)p_{1,2}(x)\tilde{p}_{2,1}(y)$$

$$e_2(x, y, \bar{t}\,) = p_{2,1}(x)\tilde{p}_{2,1}(y)\tilde{p}_{1,2}(y) + m_3(\,\bar{t}\,)\tilde{p}_{2,2}(y)(p_{2,2}(x)\tilde{p}_{1,1}(y) - p_{2,1}(x)\tilde{p}_{1,2}(y)) - m_2(\,\bar{t}\,)p_{2,2}(x)\tilde{p}_{2,1}(y)\tilde{p}_{1,2}(y),$$

where $p_i = p_{i,1}/p_{i,2}$, $\tilde{p}_i = \tilde{p}_{i,1}/\tilde{p}_{i,2}$, $\gcd(\tilde{p}_{i,1}, \tilde{p}_{i,2}) = \gcd(p_{i,1}, p_{i,2}) = 1$, for $i = 1, 2$. Observe that the system defined by $p_1(x) - m_3p_1(x)/\tilde{p}_2(y) - m_1 = p_2(x) + m_3(\tilde{p}_1(y) - p_2(x))/\tilde{p}_2(y) - m_2 = 0$ is equivalent to the system defined by $e_1 = e_2 = 0$. In addition, we note that $p_1(x), \tilde{p}_2(y)$ are not both constant (otherwise, $\mathcal{P}$ would parametrize a plane, and this case is excluded). Similarly, $p_2(x), (\tilde{p}_1(y) - p_2(x))/\tilde{p}_2(y)$ are not both constant.

Since the systems in Theorem 5 are in the implication from that $\mathcal{V}$ is a ruled surface, we can assume the $\mathcal{V}$ has a rational proper parametrization of the form $\mathcal{M}(\,\bar{t}\,) = (a_1(t_1) + t_2b_1(t_1), a_2(t_1) + t_2b_2(t_1), t_2)$. Under these conditions, we have the following proposition.

Proposition 1. *For the ruled surfaces, the system defined by the equations* $e_1 = e_2 = 0$ *w.r.t. the variables* $\{x, y\}$*, is zero dimensional.*

Proof. We distinguish two cases depending on the form of the parametrization $\mathcal{M}$.

1. Let $\mathcal{M}(\,\bar{t}\,) = (a_1(t_1) + t_2b_1(t_1), a_2(t_1) + t_2b_2(t_1), t_2)$ be proper, and let

$$H(x, y, \bar{t}\,) := \gcd(e_1, e_2) \in \mathbb{K}[x, y, \bar{t}\,].$$

   We assume that $p_1 \notin \mathbb{K}$, and $\tilde{p}_2 \notin \mathbb{K}$. Otherwise, the system defined by the equations $e_1 = e_2 = 0$ w.r.t. the variables $\{x, y\}$, is clearly zero dimensional. Similarly, we assume that $p_2(x) \notin \mathbb{K}$, and $(\tilde{p}_1(y) - p_2(x))/\tilde{p}_2(y) \notin \mathbb{K}$. Under these conditions, the following properties hold:

   - If $\deg_{t_2}(H) = 0$, then $\deg_{t_1}(H) = 0$. Otherwise, there exists $\alpha \in \overline{\mathbb{K}(x, y)}$ ($\overline{\mathbb{K}(x, y)}$ is the algebraic closure of $\mathbb{K}(x, y)$) such that $e_j(x, y, \alpha, t_2) = 0$ (note that $\gcd(p_{i,1}, p_{i,2}) = 1$, $i = 1, 2$). Then, $(p_1(x), p_2(x)) = (a_1(\alpha), a_2(\alpha))$ which implies that $\alpha \in \overline{\mathbb{K}(x)} \setminus \mathbb{K}$ (note that $p_i \notin \mathbb{K}$). Furthermore, $(-p_1(x)/\tilde{p}_2(y), (\tilde{p}_1(y) - p_2(x))/\tilde{p}_2(y)) = (b_1(\alpha), b_2(\alpha))$ which is impossible since $\alpha \in \overline{\mathbb{K}(x)} \setminus \mathbb{K}$ and $p_1(x)/\tilde{p}_2(y), (\tilde{p}_1(y) - p_2(x))/\tilde{p}_2(y) \notin \mathbb{K}$. Thus, we conclude that $H \in \mathbb{K}[x, y]$ or $H \in \mathbb{K}[x, y, \bar{t}\,]$ with $\deg_{t_i}(H) \geq 1$, $i = 1, 2$.
   - It holds that $C_j(x, y, t_1) := \mathrm{Content}_{t_2}(e_j) \in \mathbb{K}[x, y]$, $j = 1, 2$. If $\deg_{t_1}(C_1) \geq 1$, there exists $\alpha \in \overline{\mathbb{K}(x, y)}$ such that $p_1(x) = a_1(\alpha)$, $p_1(x)/\tilde{p}_2(y) = b_1(\alpha)$ which is impossible since $p_1(x) \notin \mathbb{K}$ and $\tilde{p}_2(y) \notin \mathbb{K}$. Thus, $C_1 \in \mathbb{K}[x, y]$. Reasoning similarly and taking into account that $p_2(x) \notin \mathbb{K}$, and $(\tilde{p}_1(y) - p_2(x))/\tilde{p}_2(y) \notin \mathbb{K}$, we conclude that $C_2 \in \mathbb{K}[x, y]$.

Under these conditions, let us assume that the system defined by the equations $e_1 = e_2 = 0$ w.r.t. the variables $\{x, y\}$, is not zero dimensional. Then, we may write $e_j = H(x, y, \bar{t})C_j(x, y)$, $j = 1, 2$, with $\deg_{t_2}(H) = 1$ (note that $\deg_{t_2}(e_j) = 1$), and $\deg_{t_1}(H) \geq 1$. Hence, $e_1C_2 - e_2C_1 = 0$ which implies that $a_1 = a_2$, $b_1 = b_2$. Thus, $\mathcal{V}$ is a plane which is impossible.

2. Let $\mathcal{M}$ be some rational parametrization of $\mathcal{V}$, and we consider $f_i(x, y, \bar{t}) := e_i(x, y, \bar{t})/C(x, y)$, $C := \gcd(C_1, C_2)$, $i = 1, 2$. Observe that if $C(\alpha, \beta) = 0$, then $(\alpha, \beta) \in \mathbb{K}^2$. Indeed: let $(\alpha, \beta) \in \overline{\mathbb{K}(s)}^2$ be such that $C(\alpha, \beta) = 0$ ($\overline{\mathbb{K}(s)}$ is the algebraic closure of $\mathbb{K}(s)$, and s is a new variable). Then, $e_1(\alpha, \beta, \bar{t}) = 0$ (note that C divides $\gcd(e_1, e_2)$). Since $p_{1,1} \neq 0$, we get that $p_{1,1}(\alpha) = p_{2,2}(\beta) = 0$ which implies that $(\alpha, \beta) \in \mathbb{K}^2$. Thus, the system defined by $e_i = 0$, $i = 1, 2$ is equivalent to the system defined by $f_i = 0$, $i = 1, 2$.
   Let $\mathcal{M}^*$ be a proper parametrization of $\mathcal{V}$ with the form of statement 1, and such that $\mathcal{M}^*(U, V) = \mathcal{M}$, where $(U, V) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$ ($\mathcal{M}^*$ exists because of the results in [16]). We denote by $f_j^{\mathcal{M}^*}$, $j = 1, 2$, the equations f_j constructed from $\mathcal{M}^*$, and $f_j^{\mathcal{M}}$, $j = 1, 2$, the equations f_j constructed from $\mathcal{M}$. From statement 1, we have that $R^*(y, \bar{t}) \neq 0$, where $R^*(y, \bar{t}) := \mathrm{Res}_x(f_1^{\mathcal{M}^*}, f_2^{\mathcal{M}^*})$. Let $R(y, \bar{t}) := \mathrm{Res}_x(f_1^{\mathcal{M}}, f_2^{\mathcal{M}})$. Taking into account the properties of resultants (see [29]), it holds that $R^*(y, U(\bar{t}), V(\bar{t})) = \ell(y, U, V)^k R(y, \bar{t})$, $k \in \mathbb{N}$, where $\ell(y, t_1, t_2)$ denotes the leading coefficient of $f_1^{\mathcal{M}^*}$ w.r.t. x (note that $\ell(y, U, V) \neq 0$). Since $R^*(y, U, V) \neq 0$, we conclude that $R(y, \bar{t}) \neq 0$ and then, $\gcd(f_1^{\mathcal{M}}, f_2^{\mathcal{M}}) = 1$. □

From Theorem 5 and Proposition 1, we obtain the following algorithm that decides whether a rational surface defined parametrically by a rational parametrization $\mathcal{M}$ is ruled. In the affirmative case, it computes a rational proper reparametrization.

Algorithm 3: Reparametrization of a Ruled Surface

- **Input:** A surface $\mathcal{V}$ defined by a rational parametrization

$$\mathcal{M}(\bar{t}) = (m_1(\bar{t}), m_2(\bar{t}), m_3(\bar{t})) \in \mathbb{K}(\bar{t})^3.$$

- **Output:** the message "`V is not a ruled surface`" or a proper parametrization $\mathcal{P}$ of "`the ruled surface V in the standard reduced form.`"

1. Check whether $\mathcal{V}$ defines a plane. In the affirmative case, compute a proper parametrization of $\mathcal{V}$. Otherwise, go to Step 2.

2. Compute the polynomials $H_i(\bar{t}, \bar{h}) = \mathrm{numer}(m_i(\bar{t}) - m_i(\bar{h}))$, where $\bar{h} = (h_1, h_2)$, and $i \in \{1, 2, 3\}$. Check whether $\gcd(H_i, H_j) = 1$, for $i, j \in \{1, 2, 3\}$ and $i < j$. In the affirmative case, go to Step 3. Otherwise, if $\gcd(H_1, H_2) \neq 1$ (similarly if $\gcd(H_1, H_3) \neq 1$ or $\gcd(H_2, H_3) \neq 1$) do:

2.1. Consider $a \in \mathbb{K}$ such that $(m_1, m_2)(a, t_2) \notin \mathbb{K}^2$.

2.2. Compute

$$f(x_1, x_2)^r = \mathrm{Res}_{t_2}(G_1(a, t_2, x_2), G_2(a, t_2, x_2)), \quad \text{where} \quad r \in \mathbb{K}, \quad \text{and}$$

$G_i(\bar{t}, x_i) = \mathrm{numer}(m_i(\bar{t}) - x_i)$, $i = 1, 2$ (see Theorem 8 in [19]).

2.3. Determine a proper parametrization $(p(t_1), q(t_1)) \in \mathbb{K}(t_1)^2$ of the plane curve defined by the equation $f(x_1, x_2) = 0$.

2.4. Return $\mathcal{P}(\bar{t}) = (p(t_1), q(t_1), t_2)$, "`is a proper parametrization`".

3. Compute the polynomials $f_0^{ij}(x_i, x_j)$ (apply Theorem 4), and check whether there exist two rational plane curves $\mathcal{C}^{ij}$ and $\mathcal{C}^{k\ell}$ defined by a factor of f_0^{ij} and $f_0^{k\ell}$, respectively, for $ij \neq k\ell$, and $ij, k\ell \in \{12, 23, 13\}$. In the affirmative case, assume that $ij = 12$, and $k\ell = 23$ (see Remark 5) and go to Step 4. Otherwise, Return "$\mathcal{V}$ `is not a ruled surface`".

4. Compute $\mathcal{P}^{12} = (p_1, p_2) \in \mathbb{K}(t_1)^2$, and $\mathcal{P}^{23} = (\tilde{p}_1, \tilde{p}_2) \in \mathbb{K}(t_1)^2$ rational proper parametrizations of $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$, respectively. If $p_1 \neq 0$ go to Step 5. Otherwise, go to Step 7.

5. Check whether there exists $(\mathcal{L}, \mathcal{T}) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$ such that

$$p_1(\mathcal{T}) - m_3 \frac{p_1(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} = m_1, \qquad p_2(\mathcal{T}) + m_3 \frac{\tilde{p}_1(\mathcal{L}) - p_2(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} = m_2.$$

In the affirmative case, compute $(R, S) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ a rational proper parametrization of the curve $\mathcal{C}_{N_1}$ defined by $(\mathcal{L}, \mathcal{T})$, and Return

$$\mathcal{P}(\bar{t}) = \left(p_1(S(t_1)) - t_2 \frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2 \frac{\tilde{p}_1(R(t_1)) - p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2 \right),$$

"`is a proper parametrization`". Otherwise, go to Step 6.

6. Check whether there exists $(\mathcal{L}, \mathcal{T}) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$ such that

$$p_1(\mathcal{T}) - m_3 \frac{p_1(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} = m_1, \qquad p_2(\mathcal{T}) = m_2.$$

In the affirmative case, compute $(R, S) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ a rational proper parametrization of the curve $\mathcal{C}_{N_2}$ defined by $(\mathcal{L}, \mathcal{T})$, and Return

$$\mathcal{P}(\bar{t}) = \left(p_1(S(t_1)) - t_2 \frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)), t_2 \right)$$

"`is a proper parametrization`". Otherwise, go to Step 3, and consider different rational components and apply again the algorithm. If there have no more rational components, Return "$\mathcal{V}$ `is not a ruled surface`".

7. Check whether the plane curve $\mathcal{C}^{13}$ is rational. In the affirmative case, compute $\mathcal{P}^{13} = (q_1, q_2) \in \mathbb{K}(t_1)^2$ a rational proper parametrization of $\mathcal{C}^{13}$, and go to Step 8. Otherwise, go to Step 9.

8. 8.1. If $q_2 \neq 0$, check whether there exists $(\mathcal{L}, \mathcal{T}) \in \mathbb{K}(\bar{t}\,)^2$, $\mathcal{T} \notin \mathbb{K}$ such that

$$m_3 \frac{q_1(\mathcal{T})}{q_2(\mathcal{T})} = m_1, \qquad \mathcal{L} - m_3 \frac{\mathcal{L}}{q_2(\mathcal{T})} = m_2.$$

In the affirmative case, compute $(R, S) \in \mathbb{K}(t_1)^2$, $S \notin \mathbb{K}$ a rational proper parametrization of the curve $\mathcal{C}_{N_3}$ defined by $(\mathcal{L}, \mathcal{T})$, and Return

$$\mathcal{P}(\bar{t}\,) = \left(t_2 \frac{q_1(S(t_1))}{q_2(S(t_1))}, R(t_1) - t_2 \frac{R(t_1)}{q_2(S(t_1))}, t_2 \right),$$

"`is a proper parametrization`". Otherwise, go to Step 3, and consider different rational components and apply again the algorithm. If there have no more rational components, Return "$\mathcal{V}$ `is not a ruled surface`".

8.2. If $q_2 = 0$, let $\mathcal{L} = m_2/m_3$, and $\mathcal{T} = m_1/m_3$, and compute $(R, S) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ a rational proper parametrization of the curve $\mathcal{C}_{N_4}$ defined by $(\mathcal{L}, \mathcal{T})$, and Return

$$\mathcal{P}(\bar{t}\,) = (t_2 S(t_1), t_2 R(t_1), t_2),$$

"`is a proper parametrization`". Otherwise, go to Step 3, and consider different rational components and apply again the algorithm. If there have no more rational components, Return "$\mathcal{V}$ `is not a ruled surface`".

9. Let $\mathcal{L} = m_2$, and $\mathcal{T} = m_1/m_3$, and compute $(R, S) \in (\mathbb{K}(t_1) \setminus \mathbb{K})^2$ a rational proper parametrization of the curve $\mathcal{C}_{N_5}$ defined by $(\mathcal{L}, \mathcal{T})$. Return

$$\mathcal{P}(\bar{t}\,) = (t_2 S(t_1), R(t_1), t_2),$$

"`is a proper parametrization`". Otherwise, go to Step 3, and consider different rational components and apply again the algorithm. If there have no more rational components, Return "$\mathcal{V}$ `is not a ruled surface`".

Remark 7. 1. *Remark 3 can be stated similarly in this new situation to the rational parametrizations $\mathcal{P}^{12}, \mathcal{P}^{23}, \mathcal{P}^{13}$, (R, S), and the output parametrization $\mathcal{P}$.*

2. *The systems appearing in Steps 6, 8 and 9 are clearly zero dimensional. The system in Step 5 is equivalent to the zero dimensional system defined by the equations $f_1 = f_2 = 0$ w.r.t. the variables $\{x, y\}$ (Proposition 1 implies that $f_j(\mathcal{T}, \mathcal{L}, \bar{t}\,) = 0$, $j = 1, 2$; note that $C(\mathcal{T}, \mathcal{L}) \neq 0$). In order to find $(\mathcal{L}, \mathcal{T}) \in (\mathbb{K}(\bar{t}\,) \setminus \mathbb{K})^2$, one may use, for instance, univariate resultants. Once $(\mathcal{L}, \mathcal{T})$ is*

determined, one computes the implicit equation of the rational plane curve $\mathcal{C}_{N_i}$ *defined by* $(\mathcal{L}, \mathcal{T})$ *as the square free part of*

$$\mathrm{Content}_{t_2}(\mathrm{Res}_{t_1}(\mathrm{numer}(\mathcal{L}-x_1), \mathrm{numer}(\mathcal{T}-x_2))) \in \mathbb{K}[x_1, x_2]$$

(see Section 4.5 in [23]). Afterwards, we parametrize $\mathcal{C}_{N_i}$ *by applying for instance the results in Sections 4.7 and 4.8 in [23].*

3. *If the system appearing in Steps 5 is not zero dimensional, then according to Proposition 1, the input surface is not a ruled surface. And we need not to solve the system.*
4. *Note that in this case, we can not apply Theorem 1. More precisely, if* $\mathcal{V}$ *is a rational ruled surface, by Theorem 1, there exists a parametrization of the form*

$$\mathcal{P}^*(\bar{t}) = (p_1(S^*(t_1)) + t_2 q_1^*(t_1), p_2(S^*(t_1)) + t_2 q_2^*(t_1), t_2) \in \mathbb{K}(\bar{t})^3$$

where (q_1^*, q_2^*, S^*) *is a rational parametrization of a space curve* $\mathcal{D}$ *(see Corollary 1). Reasoning as in Theorem 5, we have that* $\mathcal{P}^*(U, V) = \mathcal{M}$*, where* $(U, V) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$*. From this equality, we get that* $V = m_3$*, and*

$$p_1(S^*(U)) + m_3 q_1^*(U) = m_1, \quad p_2(S^*(U)) + m_3 q_2^*(U) = m_2.$$

That is,

$$p_1(\mathcal{T}) + m_3\mathcal{L}_1 - m_1 = p_2(\mathcal{T}) + t_2\mathcal{L}_2 - m_2 = 0,$$

where $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{T}) := (q_1^*(U), q_2^*(U), S^*(U)) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^3$*. Observe that we have two equations, and three unknowns* $\mathcal{L}_1, \mathcal{L}_2, \mathcal{T}$*. So, we have a consistent independent system.*

In the following example, we illustrate the performance of Algorithm 3.

Example 4. *Consider the surface* $\mathcal{V}$ *defined by the parametrization*

$$\mathcal{M}(\bar{t}) = (m_1, m_2, m_3) = \left(-\frac{2t_2^4 t_1 + 10t_2^2 t_1^3 + 5t_2 t_1^4 - 7t_2^3 t_1^2 - 5t_1^3 - 9t_1^2 t_2 + 7t_1 t_2^2 - t_2^3}{t_2(-t_1^2 - 2t_1 t_2 + t_2^2)(t_2^2 + t_1^2)}, \right.$$

$$\left. \frac{-t_2(-14t_1^2 t_2^2 + 4t_1^4 + 4t_2^3 t_1 - 14t_1^3 t_2 + 9t_1^2 + 18t_1 t_2 - 9t_2^2}{(-t_1^2 - 2t_1 t_2 + t_2^2)(t_2^2 + t_1^2)t_1}, \frac{t_1 t_2 - 1}{t_2^2 + t_1^2} \right) \in \mathbb{R}(\bar{t})^3.$$

Let us apply Algorithm 3*. We first observe that* $\mathcal{V}$ *is neither a cylinder nor a plane (see Steps 1 and 2 of the algorithm). In Step 3 of the algorithm, applying Theorem 4, we compute the polynomials* $f_0^{ij}(x_i, x_j)$ *and get*

$$f_0^{12}(x_1, x_2) = 49x_1^2 - 20x_1 - 2x_1 x_2 + 5x_2 - x_2^2,$$

$$f_0^{23}(x_2, x_3) = x_2^3 - 14x_2^2 - 25x_3 x_2^2 + 1396x_3 x_2 - 1120x_3 - 4237x_3^2 x_2 - 5190x_3^2 - 48915x_3^3,$$

define implicitly two rational plane curves $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$. Thus, in Step 4, we compute

$$\mathcal{P}^{12} = (p_1, p_2) = \left(\frac{-\sqrt{2}t_1(-5+t_1)}{5(4t_1 - 10 + 5\sqrt{2})}, \frac{(50+5\sqrt{2})t_1(-20+100\sqrt{2}+49t_1)}{1225(4t_1-10+5\sqrt{2})} \right) \in \mathbb{R}(t_1)^2,$$

and

$$\mathcal{P}^{23} = (\tilde{p}_1(t_1), \tilde{p}_2(t_1)) = \left(\frac{2(-378367t_1^2 + 10410900t_1 - 142098075 + 4102t_1^3)}{1241(t_1^3 - 25t_1^2 - 4237t_1 - 48915)}, \right.$$

$$\left. \frac{2(20322550 - 513355t_1 - 28t_1^2 + 49t_1^3)}{1241(t_1^3 - 25t_1^2 - 4237t_1 - 48915)} \right) \in \mathbb{R}(t_1)^2$$

rational proper parametrizations of $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$, respectively. Hence, we go to Step 5 of the algorithm, and we check whether there exists $(\mathcal{L}, \mathcal{T}) \in (\mathbb{C}(\bar{t}) \setminus \mathbb{C})^2$ such that

$$p_1(\mathcal{T}) - m_3 \frac{p_1(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} = m_1, \qquad p_2(\mathcal{T}) + m_3 \frac{\tilde{p}_1(\mathcal{L}) - p_2(\mathcal{T})}{\tilde{p}_2(\mathcal{L})} = m_2.$$

We obtain

$$\mathcal{L}(\bar{t}) = \frac{415t_1 - 236t_2}{2t_2 + 7t_1}, \quad \mathcal{T}(\bar{t}) = \frac{-5(-1+5\sqrt{2})t_2}{9t_1 + 4t_1\sqrt{2} - 5t_2\sqrt{2} + t_2}.$$

$(\mathcal{L}, \mathcal{T})$ *parametrizes the rational plane curve $\mathcal{C}_{N_1}$ defined by the equation*

$$-315x_1 + 18675 + 70\sqrt{2}x_1 - 4150\sqrt{2} + 2482x_2\sqrt{2} + 73x_2x_1 - 2555x_2 = 0$$

(see statement 2 in Remark 7). We compute a rational proper parametrization of $\mathcal{C}_{N_1}$, and we get

$$(R(t_1), S(t_1)) = \left(t_1, \frac{-5(-9+2\sqrt{2})(7t_1 - 415)}{73(34\sqrt{2} + t_1 - 35)} \right) \in \mathbb{R}(t_1)^2.$$

Therefore, we Return *the proper parametrization of the ruled surface $\mathcal{V}$ given by*

$$\mathcal{P}(\bar{t}) = \left(p_1(S(t_1)) - t_2 \frac{p_1(S(t_1))}{\tilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2 \frac{\tilde{p}_1(R(t_1)) - p_2(S(t_1))}{\tilde{p}_2(R(t_1))}, t_2 \right) =$$

$$= \left(\frac{q_{11}(\bar{t})}{q_{12}(\bar{t})}, \frac{q_{21}(\bar{t})}{q_{22}(\bar{t})}, t_2 \right) \in \mathbb{R}(\bar{t})^3,$$

where

$q_{11} = \sqrt{2}(-9+2\sqrt{2})(5+7\sqrt{2})(-40645100 + 1026710t_1 + 56t_1^2 - 98t_1^3 + 1241t_1^3t_2 - 31025t_1^2t_2 - 5258117t_1t_2 - 60703515t_2),$

$q_{12} = 73(7t_1 - 415)(-4199\sqrt{2} + 106t_1 - 4866 + 17t_1\sqrt{2})(34\sqrt{2} + t_1 - 35),$

$q_{21} = 78183t_2t_1^3\sqrt{2} + 63812t_1^3 + 487494t_1^3t_2 + 10234t_1^3\sqrt{2} + 331704t_1^2\sqrt{2} - 10107945t_1^2t_2\sqrt{2} - 63026010t_1^2t_2 + 2068272t_1^2 - 87398734t_1\sqrt{2} - 544956812t_1 + 239474529t_2t_1\sqrt{2} + 1493194122t_1t_2 + 11741634840 + 5038391745t_2\sqrt{2} + 31415854410t_2 + 1883092380\sqrt{2},$

$q_{22} = 146(34\sqrt{2} + t_1 - 35)(-4199\sqrt{2} + 106t_1 - 4866 + 17t_1\sqrt{2})(118 + t_1).$

Observe that since $\mathcal{P}^{12}, \mathcal{P}^{23}$ and (R, S) have coefficients in $\mathbb{R}$, then the output parametrization $\mathcal{P}$ also has coefficients in $\mathbb{R}$ (see statement 1 in Remark 7).

5. Conclusion

The parametrization of an implicit surface is a basic problem in algebraic geometry. In this paper, we focus on the problem of rational ruled surface, since the ruled surface is an important modeling surface. By the linearity of one parameter in the standard form and the birational parameter transformation, we can get a simple expression which can be projected as a planar curve. Therefore we reduce the problem to that of curve parametrization. The algorithms to determine and parameterize the implicit rational ruled surfaces are then proposed. We also have considered the determination and reparametrization for the parametric ruled surfaces not being in the standard form. More precisely, we can distinguish whether a given rational parametrization (not necessarily proper) defines a ruled surface, and in the affirmative case, we reparameterize it properly to the standard reduced form.

Besides the ruled surface, there are some other basic modeling surfaces such as sphere-swept surfaces and cyclides. They have special geometric features for modeling design. And these features are also reflected in the algebraic expressions. According to the well investigation, one can find some algorithms to determine the types of surface from a given algebraic surface, further, find a parametrization. As the further work, we would like given more discussions for those modeling surfaces.

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