Pérez Díaz, S. 2018, "Analysis and construction of rational curve parametrizations with non-ordinary singularities", Computer Aided Geometric Design, vol. 66, pp. 31-51.

Available at https://doi.org/10.1016/j.cagd.2018.08.002

(Article begins on next page)

Analysis and construction of rational curve parametrizations with non–ordinary singularities

Sonia Pérez-Díaz

Grupo ASYNACS, Dpto. de Física y Matemáticas, Universidad de Alcalá, 28871-Alcalá de Henares, Madrid, Spain.

Abstract

In this paper, we provide a method that allows to construct parametric curves having (or not) non–ordinary singularities and having (or not) neighboring points. This method is based on a characterization of the non–ordinary singularities and neighboring points by means of linear equations involving the given parametrization. As a consequence, we obtain an algorithm that constructs a parametrization which contains a given point, P, as a singularity as well as some additional information as for instance, the order of P, parameters corresponding to P, multiplicity of each parameter and the singularities in the first neighborhood of the singularity P.

Keywords: Rational curve parametrization; Algebraic curve; Singularities of an algebraic curve; Non–ordinary singularity

1. Introduction

Rational algebraic curves and surfaces (i.e algebraic curves and surfaces that can be rationally parametrized) are an important topic of geometric and algebraic investigation. They have applications for instance in number theoretic problems, in models of biological shapes, in error–correcting codes, in cryptographic algorithms and of course, they are central objects in Computer Aided Geometric Design (CAGD).

A rational algebraic curve can be represented in different ways, such as implicitly by defining polynomials, parametrically by rational functions, or locally parametrically by power series expansions around a point. These representations all have their individual advantages; an implicit representation lets us decide easily whether a given point actually lies on a given curve, a parametric representation allows us to generate points of a given curve over the desired coordinate fields, and with the help of a power series expansion we can for instance overcome the numerical problems of tracing a curve through a singularity.

Many authors have studied different problems related to rational curves assuming that the original curve was given implicitly. However, the knowledge of the curve from the

Email address: sonia.perez@uah.es (Sonia Pérez-Díaz)

Preprint submitted to Elsevier *August 22, 2018*

defining parametrization is an essential point. Computer Aided Design (CAD) programs use, generally, parametric representation of curves and this is why the study and manipulation of curves from a parametrization is so interesting (see e.g. [7], [12], [13], [14], [18]). In this context, when one is plotting a curve, the presence of selfintersections and cusps could become an obstacle in CAGD. Consequently, the detection of the singular points of an algebraic curve is important and one has to understand its geometry. Here, rational parametrizations provide interesting approaches from the computational point of view. For instance, for the case of parametric plane curves, some interesting results are provided in [8], where the singular points are computed using the implicitization matrix derived from the μ–basis of the curve. In addition, a conjecture is presented which concerns the computation of the parameter values corresponding to all the singularities, from the Smith normal forms of certain Bézout resultant matrices derived from μ–bases. In [15] it is provided a technique to detect the singularities of rational parametric planar curves and to compute the correct order of each singularity including the infinitely near singularities without resorting to blow ups. The approach uses a μ–basis for the parametrization to construct two planar algebraic curves whose intersection points correspond to the parameters of the singularities including infinitely near singularities with proper multiplicity. This approach extends the Abhyankar's method presented in [1]. In [16], authors prove the conjecture of Chen et al. in [8] concerning how to calculate the parameter values corresponding to all the singularities, including the infinitely near singularities, of rational planar curves from the Smith normal forms of certain Bézout resultant matrices derived from μ–bases. In [17], it is reviewed the state-of-the-art results in μ–bases theory and applications for rational curves and surfaces, and raise unsolved problems for future research. In [29], a natural one to one correspondence is derived between the singular points of a rational planar curve and the axial moving lines that follow that curves. This correspondence is applied to compute and analyze the singular points of low degree rational planar curves by using μ–bases. In [5], it is introduced a new implicit representation of the curve which consists in the locus where the rank of a single matrix drops. From this representation, one may compute the singularities of the given curve. In [6], it is given a complete factorization of the invariant factors of resultant matrices, built from birational parameterizations of rational plane curves, in terms of the singular points of the curve and their multiplicity graph. This also allows to prove the validity of some conjectures introduced in [8]. A new technique for detecting singularities is introduced in [28]. The idea is to compute a μ–basis for the parametrization and to generate, from this μ–basis, three planar algebraic curves of different bidegrees, whose intersection points correspond to the parameters of the singularities. In order to find these intersection points, a new sparse resultant matrix for these three bivariate polynomials is constructed. Afterwards, authors compute the parameter values corresponding to the singularities by applying Gaussian elimination to the resultant matrix. In [2], authors use the projection from the rational normal curve to the given curve and its interplay with the secant varieties to the normal curve. Thus, certain zero dimensional schemes are defined, which encode all the information about the singularities. In [21], a method for detecting and analyzing the singularities of a rational curve (including the non-ordinary ones) by computing a univariate resultant, is provided. This approach generalizes some previous results presented in [1], [11], [23] and [30]. The study of singularities in parametric space curves has been addressed in [5], [24], [28] and [29].

In this paper, we provide a method that allows to construct parametric curves having (or not) non–ordinary singularities and having (or not) neighboring points. From this approach, we present an algorithm that outputs a parametrization of a rational curve having singularities at some given input points. In this algorithm, the singular point P, the order of P, the parameters corresponding to P, the multiplicity of each parameter, as well as singularities in the first neighborhood of P, are fixed as the input of the problem. As output of the algorithm, we obtain a rational curve defined parametrically with the singularities (and their properties) fixed in the input. Thus, the algorithm presented is very useful for constructing examples related to singularities, and then, the results in this paper are very important in the frame of practical designing of engineering and modeling applications.

The method developed in this paper is based on the characterization of non–ordinary singularities and neighboring points by means of linear equations involving the given parametrization. For this purpose, we first study the singularities of a given algebraic projective curve $\overline{\mathcal{C}}$ in terms of a rational projective parametrization $\overline{\mathcal{P}}(t)$ defining $\overline{\mathcal{C}}$. Since we treat with non–ordinary singularities, we analyze the basic properties of such singularities in terms of algebraic properties involving $\overline{\mathcal{P}}(t)$. It began as an attempt to understand the resolution of singularities, from the parametric point of view, without implicitizating. We remind that non–ordinary singularities have to be treated specially since a non–ordinary singularity might have other singularities in its "neighborhood". The analysis of such neighborhoods is the topic of the field of resolution of singularities (see e.g. [33]). The problem with these singularities is that they have multiple tangents, and these multiple tangents are resolved by "blowing up" the singularity (see [4], [10] or [31]).

Although the techniques used in the paper are not novel, most existing textbooks explain the problem dealt in this paper in the language of implicit equations. Here, we translate every detail of the definitions and resolutions into the language of parametric equations, which are quite helpful to CAGD.

In this paper, we treat only with rational algebraic plane curves, although the results and methods can be easily generalized to rational curves in the affine n–space. Furthermore, the approach obtained here provide some ideas that could be used to solve a similar problem for the case of surfaces. For this purpose, one could start by using, for instance, the results obtained in [22]. As a future work, we will deal with this problem and additionally, we propose the study of an important problem in the frame of practical designing of engineering and modeling applications. More precisely, we intend to analyze the free parameters obtained in the output parametrization of the algorithm presented in this paper. These free parameters will allow to model the output curve depending on the practical applied problem one is dealing with.

The structure of the paper is as follows: In Section 2, we introduce the terminology that will be used throughout this paper as well as some previous results. In Section 3, given a projective plane curve $\overline{\mathcal{C}}$ defined parametrically by $\overline{\mathcal{P}}(t)$, we show how to compute the tangents of a point $P \in \overline{\mathcal{C}}$ from $\overline{\mathcal{P}}(t)$. From this computation, we obtain Theorem 2 that characterizes non–ordinary singular points and provides an algorithm that allows us to construct a rational parametric curve that has non–ordinary singularities at some given input points. Section 4 is devoted to the analysis and construction of a rational curve parametrization with non–ordinary singularities and neighboring points. Thus, the main idea is to generalize Section 3 to the case on whether some neighboring points appear. For this purpose, we first summarize the process of blowing up a singularity (see Subsection 4.1). Afterwards, we blow up a singularity from a given parametrization, and we obtain some algebraic conditions (see Theorems 3 and 4) that allow us to characterize the form of a parametrization that defines a curve having non–ordinary singularities and neighboring points. From these results, a new algorithm is derived (see Subsection 4.2). This algorithm outputs a parametrization which contains a given point, P, as a singularity as well as some additional information as for instance, the order of P, the parameters corresponding to P, and the multiplicity of each parameter and the singularities in the first neighborhood of the singularity P. Finally, some conclusions and future challenges are presented in Section 5.

Throughout the whole paper, we outline all the results obtained with examples where we show how to construct rational parametrizations with some previously given singularities.

2. Notation and preliminaries

In this section we introduce the notation and terminology that will be used throughout the paper. In addition, we recall some basic results on parametric curves. These results will be used throughout the subsequent sections.

In the following, we assume that we give a parametric plane curve. However, all the results presented in the paper can be easily generalized to rational curves in the affine n–space.

Notation and Basic Notions

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and let $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Let $F(x_1, x_2)$ be the defining polynomial of a rational affine irreducible curve $\mathcal{C}$, and let

$$\mathcal{P}(t) = (p_1(t),\, p_2(t)) \in \mathbb{K}(t)^2,$$

be a rational parametrization of $\mathcal{C}$, where $p_i(t) = \frac{p_{i,1}(t)}{p(t)}, i = 1, 2$, and $\gcd(p_{1,1}, p_{2,1}, p) = 1$. Let $\deg(\mathcal{P}(t)) = \max\{\deg(p_{1,1}), \deg(p_{2,1}), \deg(p)\}$. We may assume w.l.o.g (without lost of generality) that $\mathcal{P}$ is proper (i.e. invertible). Otherwise, we reparametrize it using, for instance, the results in [20].

We observe that the corresponding projective curve $\overline{\mathcal{C}}$ is defined by the homogenization $\overline{F}(x_1, x_2, x_3)$ of $F(x_1, x_2)$. Therefore, if we write

$$F(x_1, x_2) = F_d(x_1, x_2) + F_{d-1}(x_1, x_2) + \cdots + F_0(x_1, x_2),$$

where $F_k(x_1, x_2)$ is a homogeneous polynomial of degree k, and $F_d \neq 0$, then

$$\overline{F}(x_1, x_2, x_3) = F_d(x_1, x_2) + F_{d-1}(x_1, x_2)x_3 + \cdots + F_0(x_1, x_2)x_3^d.$$

Thus, the projective parametrization of $\overline{\mathcal{C}}$ is defined by $\overline{\mathcal{P}}(t) = (p_{1,1}(t), p_{2,1}(t), p(t))$. Finally, we represent by $\overline{\mathcal{P}}(t, s)$ the homogeneous parametrization of $\overline{\mathcal{C}}$ obtained from the homogenization of $\overline{\mathcal{P}}(t)$.

Every affine point (a, b) on $\mathcal{C}$ corresponds to a point $(a : b : 1)$ of the projective plane $\mathbb{P}^2$ on the curve $\overline{\mathcal{C}}$, and every additional point on $\overline{\mathcal{C}}$ is a point at infinity. In other words, the first two coordinates of the additional points are the non–trivial solutions of $F_d(x, y)$. Thus, $\overline{\mathcal{C}}$ has only finitely many points at infinity.

Any rational parametrization $\mathcal{P}(t)$ induces a natural dominant rational mapping $\phi_{\mathcal{P}}$ from the affine line onto the curve. When we study the properness of $\mathcal{P}(t)$, we analyze the injectivity of $\phi_{\mathcal{P}}$ over almost all values in $\mathbb{K}$. The mapping $\phi_{\mathcal{P}}$ is dominant thus, in general, it might not be surjective, and hence some points of the algebraic set are missed. In fact, it is said that $\mathcal{P}(t)$ is *normal* if and only if $\phi_{\mathcal{P}}$ is surjective or equivalently if and only if for all $P \in \mathcal{C}$ there exists $t_0 \in \mathbb{K}$ such that $\mathcal{P}(t_0) = P$.

In [25], it is proved that any affine rational parametrization $\mathcal{P}(t)$ generates, when the parameter, t, takes values in an algebraically closed field, all affine points on the curve with the exception of at most one point which we will refer as *the critical point of* $\mathcal{P}(t)$. In fact, it is shown that any affine parametrization can always be reparametrized into a normal one (see Subsection 6.3 in [26]).

Singular Points

Singular points play an important role in the theory of algebraic curves. In the following, some basic notions and results are reviewed (see [4], [10], [26], [27] or [31]).

Definition 1. *Let $P = (a, b) \in \mathcal{C}$. We say that P is a point of multiplicity ℓ on $\mathcal{C}$ if and only if all the derivatives of F up to and including the $(\ell - 1)$-th vanish at P but at least one ℓ−th derivative does not vanish at P.*

P is called a simple point on $\mathcal{C}$ if and only if the multiplicity is 1. Otherwise, we say that P is a multiple or singular point (or singularity) of multiplicity ℓ on $\mathcal{C}$ or an ℓ-fold point.

Observe that the multiplicity of $\mathcal{C}$ at P is given as the order of the Taylor expansion of F at P. The tangents to $\mathcal{C}$ at P are the irreducible factors of the first non–vanishing form in the Taylor expansion of F at P, and the multiplicity of a tangent is the multiplicity of the corresponding factor.

For analyzing a singular point P on a curve $\mathcal{C}$ we need to know its multiplicity but also the multiplicities of the tangents at P. If all the ℓ tangents at the ℓ-fold point P are different, then this singularity is called *ordinary*, and *non–ordinary* otherwise. Thus, we say that the *character of P* is either *ordinary* or *non–ordinary*.

As far as projective curves are concerned, we observe that every point at infinity can be transformed to a point at finite distance by a change of coordinates.

In order to compute the affine singularities, one just has to find the finitely solutions of the system of algebraic equations $\{F = \frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = 0\}$, and to determine the singularities at infinity one can dehomogenize $\overline{F}(x_1, x_2, x_3)$ with respect to another variable. Also, one can look for the non–zero projective solutions of $\{\frac{\partial \overline{F}}{\partial x_1} = \frac{\partial \overline{F}}{\partial x_2} = \frac{\partial \overline{F}}{\partial x_3} = 0\}$. In this case, P be a singularity of multiplicity ℓ on $\overline{\mathcal{C}}$ if all the $(\ell-1)$–th partial derivatives of $\overline{F}(x_1, x_2, x_3)$ vanish at P and at least one ℓ–th partial derivative of $\overline{F}(x_1, x_2, x_3)$ at P does not vanish (see Theorem 1). We remark that every curve has only finitely many singularities.

From the parametric point of view that is, if the given plane curve $\mathcal{C}$ is defined parametrically, a method for computing the singularities and their multiplicities without knowing the defining polynomial of $\mathcal{C}$ is provided in [21]. In addition, it is presented a complete analysis of the non–ordinary singularities.

In the following, we discuss three equivalent definitions of a singularity of order ℓ on a rational curve. This result is proved in Theorem 21 in [32].

Theorem 1. *Let $\overline{\mathcal{C}}$ be a rational plane curve defined by the homogeneous parametrization $\overline{\mathcal{P}}(t,s)$ and its corresponding implicit equation $\overline{F}(x_1, x_2, x_3) = 0$. Let P be a singularity of multiplicity ℓ on $\overline{\mathcal{C}}$. The following statements are equivalent:*

1. *All $(\ell-1)$–th partial derivatives of $\overline{F}(x_1, x_2, x_3)$ vanish at P and at least one ℓ–th partial derivative of $\overline{F}(x_1, x_2, x_3)$ at P does not vanish.*
2. *The number of intersections, counted with multiplicity, of a generic line with the curve at the point P is ℓ.*
3. *The number of parameters (t,s) corresponding to the point P is ℓ.*

Remark 1. *Let $\mathcal{C}$ be a plane curve defined parametrically by $\mathcal{P}(t) = (p_1(t), p_2(t)), p_i(t) = \frac{p_{i,1}(t)}{p(t)}$, $i = 1, 2$, and let $P = \mathcal{P}(s_0) \in \mathcal{C}$. In order to determine the multiplicity of P, one computes $\gcd(G_1(t, s_0), G_2(t, s_0))$, where $G_i(t,s) = p_{i,1}(t)p(s) - p_{i,1}(s)p(t)$, $i = 1, 2$. One has that $\ell = \deg(\gcd(G_1(t, s_0), G_2(t, s_0)))$ (see Theorem 17 and Corollary 1 in [21]). In addition, the computation of singular points can be done by applying Theorem 11 in [21]. More precisely, if P is a singularity, then $T(s_0)M(s_0)p(s_0) = 0$, where*

$$T(s) = \mathrm{Res}_t(G_1/G,\, G_2/G),\;\; M(s) = \gcd(\mathrm{lc}(G_1, t), \mathrm{lc}(G_2, t)),\;\; G(t,s) = \gcd(G_1(t,s), G_2(t,s)),$$

and $\mathrm{Res}_t(p, q)$ denotes the univariate resultant of two polynomials, p, q, w.r.t the variable t, and $\mathrm{lc}(p, t)$ is the leading coefficient of p w.r.t the variable t.

If the point $P \in \mathcal{C}$ is a point at infinity or a critical point, one considers a new parametrization reaching the point P (see Remark 3), and one reasons similarly as before.

3. Construction of a rational curve parametrization with non–ordinary singularities

In this section, we present an algorithm that outputs a parametrization of a rational curve having singularities at some given input points. In this algorithm, the singular point P and some additional information as the order of P and the parameters corresponding to P, are fixed as the input of the problem. As the output of the algorithm, we obtain a rational curve defined parametrically with the singularities fixed in the input. Thus, the algorithm presented is very useful in the frame of practical designing of engineering and modeling applications since it allows to construct examples related to singularities.

The ideas used are based on the characterization of non–ordinary singularities and neighboring points by means of linear equations involving the given parametrization. We recall that non–ordinary singularities have to be treated specially since a non–ordinary singularity might have other singularities in its "neighborhood". The analysis of such neighborhoods is the topic of the field of resolution of singularities (see e.g. [33]) and it will be deeply treated in Section 4.

To start with and in order to check whether a singularity is ordinary or not, one has to analyze the tangents (we recall that non–ordinary singularities are singular points having multiple tangents). Hence, we first recall how to compute the tangents from a given rational parametrization $\mathcal{P}(t) = (p_1(t), p_2(t)), p_i(t) = p_{i,1}(t)/p(t)$, $i = 1, 2$, of a plane curve $\mathcal{C}$.

Consequently, we obtain a result that allows us to characterize whether a point $P \in \overline{\mathcal{C}}$ is a non–ordinary singularity by analyzing some equalities involving the parametrization $\overline{\mathcal{P}}(t) = (p_{1,1}(t), p_{2,1}(t), p(t))$ (see Theorem 2). We show how these equalities can be used to construct a rational parametric curve having non–ordinary singularities at some given input points (see *Algorithm Parametrization Construction with Singularities*).

For this purpose, we denote by

$$\mathcal{P}^{k)}(t) = (p_1^{k)}(t), p_2^{k)}(t)), \quad p_i^{k)}(t) := \frac{\partial^k p_i}{\partial^k t}(t), \; i = 1, 2, \quad \text{for } k = 1, 2, \ldots,$$

and $\mathcal{P}^{0)}(t) := \mathcal{P}(t)$. We assume that we have $r + 1$ places with center at $P \in \mathcal{C}$, given by

$$\mathcal{P}(s_j) + t^{m_j}\mathcal{P}^{m_j)}(s_j)$$

where $P = \mathcal{P}(s_j)$, $s_j \in \mathbb{K}$, $j = 0, \ldots, r$, ($s_i \neq s_k$ for $i, k \in \{0, \ldots, r\}$)

$$\mathcal{P}^{k)}(s_j) = (0, 0), \quad \text{for } j = 0, \ldots, r \;\; \text{and} \;\; k = 1, \ldots, m_j - 1 \text{ (if } m_j \geq 2),$$

and $p_1^{m_j)}(s_j) \neq 0$ or $p_2^{m_j)}(s_j) \neq 0$ for $j = 0, \ldots, r$ (see e.g. [4] or [31]). Since the curve tangents of $\mathcal{C}$ at P consist of the tangents to the places of the curve that are centered at P (see [12]), we conclude that the tangents of $\mathcal{C}$ at P are the lines defined parametrically by

$$T_j(t) = \mathcal{P}(s_j) + t^{m_j}\mathcal{P}^{m_j)}(s_j), \qquad j = 0, \ldots, r.$$

The notation can be extended analogously to the projective space and in particular, to $\overline{\mathcal{P}}(t)$.

Under these conditions, the multiplicity of P is $\ell_P = \sum_{j=0}^{r} m_j$, and the tangents of $\mathcal{C}$ at P are the lines parametrized by $T_j(t) \in \mathbb{K}(t)^2$, each of multiplicity m_j, $j = 0, \ldots, r$ (see Theorem 17 and Corollary 1 in [21]). We should note that some of these lines could be equal that is, if for instance $T_1(t)$ and $T_2(t)$ parametrize the same line, then the tangent that they define would have multiplicity $m_1 + m_2$.

Furthermore, taking into account Theorem 1, we have that P is a singularity if and only if $r \geq 1$ or $r = 0$ with $m_0 \geq 2$. In addition, from the notions of ordinary and non–ordinary singularity introduced in Section 2, we get that P is ordinary if and only if $m_j = 1$, $j = 0, \ldots, r$, and the $r+1$ tangents defined by $T_j(t)$, $j = 0, \ldots, r$, are all different.

Using the preliminaries previously introduced, we prove Theorem 2 that characterizes whether a point is a non–ordinary singularity and computes its tangents. For a simpler handling and easier understanding of the results, throughout what is left in the paper, we work in the projective space and thus, we consider the projective parametrization $\overline{\mathcal{P}}(t)$. In addition, we denote by $\overline{\mathcal{P}}^{k)}$ de derivative of order k w.r.t t of all the components of the projective parametrization $\overline{\mathcal{P}}(t)$.

Theorem 2. *A point $P \in \overline{\mathcal{C}}$ is a non–ordinary singularity if and only if one of the following statements hold:*

1. *There exists at least $s_0 \in \mathbb{K}$ such that $P = \overline{\mathcal{P}}(s_0)$ and $\overline{\mathcal{P}}^{k)}(s_0) = \lambda_{k,0}\overline{\mathcal{P}}(s_0)$, for some $\lambda_{k,0} \in \mathbb{K}$, and $k = 1, 2, \ldots, m_0 - 1, m_0 \geq 2$. In this case, we have a tangent line of multiplicity m_0 defined parametrically by*

$$\overline{T}_0(t) = \overline{\mathcal{P}}(s_0) + t\overline{\mathcal{P}}^{m_0)}(s_0),$$

and P is a non–ordinary singularity of multiplicity $\ell_P \geq m_0$.

2. *There exist at least s_0, $s_1 \in \mathbb{K}$, $s_0 \neq s_1$, such that $P = \overline{\mathcal{P}}(s_0)$ and*

$$\overline{\mathcal{P}}(s_0) = \beta_1\overline{\mathcal{P}}(s_1), \quad \text{for some } \beta_1 \in \mathbb{K}^*$$

and

$$\overline{\mathcal{P}}^{k)}(s_j) = \lambda_{k,j}\overline{\mathcal{P}}(s_j), \;\; \text{for some } \lambda_{k,j} \in \mathbb{K}, \text{ and } k = 1, 2, \ldots, m_j - 1$$

with $m_j \geq 2$ for $j = 0$ or $j = 1$, or

$$p_{i,1}^{m_0)}(s_0) - \alpha_1 p_{i,1}^{m_1)}(s_1) = (p^{m_0)}(s_0) - \alpha_1 p^{m_1)}(s_1))p_i(s_1), \text{ for some } \alpha_1 \in \mathbb{K}^* \text{ and } i = 1, 2.$$

In this case, we have two tangent lines of multiplicity m_i defined parametrically by

$$\overline{T}_i(t) = \overline{\mathcal{P}}(s_i) + t\overline{\mathcal{P}}^{m_i)}(s_i), \;\; i = 0, 1,$$

respectively, and P is a non–ordinary singularity of multiplicity $\ell_P \geq m_0 + m_1$.

Proof. Taking into account the preliminaries previously introduced, if P is a non–ordinary singularity then, one of the following statements hold:

1. There exists at least $s_0 \in \mathbb{K}$ such that $P = \mathcal{P}(s_0)$ and $\mathcal{P}^{k)}(s_0) = 0$, for $k = 1, 2, \ldots, m_0 - 1$. This last equality is equivalent to $\overline{\mathcal{P}}^{k)}(s_0) = \lambda_{k,0}\overline{\mathcal{P}}(s_0)$, where $\lambda_{k,0} = p^{k)}(s_0)/p(s_0) \in \mathbb{K}$. Indeed: it holds that
$$p_i^{1)}(s_0) = \frac{p_{1,1}^{1)}(s_0)p(s_0) - p_{1,1}(s_0)p^{1)}(s_0)}{p(s_0)^2},$$
and in general, if $p_i^{j)}(s_0) = 0$, $j = 1, \ldots, r-1$, one gets that
$$p_i^{r)}(s_0) = \frac{p_{i,1}^{r)}(s_0)p(s_0) - p_{i,1}(s_0)p^{r)}(s_0)}{p(s_0)^2}.$$
Since $p_i^{k)}(s_0) = 0$, $k = 1, 2, \ldots, m_0 - 1$, from the above equality, we get that
$$p_{i,1}^{k)}(s_0)p(s_0) = p_{i,1}(s_0)p^{k)}(s_0)$$
which implies that $p_{i,1}^{k)}(s_0) = \lambda_{k,0}p_{i,1}(s_0)$, with $\lambda_{k,0} = p^{k)}(s_0)/p(s_0) \in \mathbb{K}$.
2. There exist at least s_0, $s_1 \in \mathbb{K}$, $s_0 \neq s_1$, such that $P = \mathcal{P}(s_0) = \mathcal{P}(s_1)$ and $\mathcal{P}^{k)}(s_j) = (0, 0)$, for $k = 1, 2, \ldots, m_j - 1$ (if $m_j \geq 2$ for $j = 0$ or $j = 1$) or $\mathcal{P}^{m_0)}(s_0) = c_1\mathcal{P}^{m_1)}(s_1)$, for some $c_1 \in \mathbb{K}^*$ (note that this last equality implies that T_0 and T_1 parametrize the same tangent line; this condition is mandatory if $m_0 = m_1 = 1$).

   Hence, taking into account that $\mathcal{P}^{k)}(s_j) = (0, 0)$ for $k = 1, 2, \ldots, m_j - 1$ (if $m_j \geq 2$ for $j = 0$ or $j = 1$) and using statement 1, one deduces that $\overline{\mathcal{P}}^{k)}(s_j) = \lambda_{k,j}\overline{\mathcal{P}}(s_j)$, for some $\lambda_{k,j} \in \mathbb{K}$ and $k = 1, 2, \ldots, m_j - 1$ (if $m_j \geq 2$ for $j = 0$ or $j = 1$). In addition, since $\mathcal{P}(s_0) = \mathcal{P}(s_1)$, we get that $\overline{\mathcal{P}}(s_0) = \beta_1\overline{\mathcal{P}}(s_1)$ for $\beta_1 = p(s_0)/p(s_1) \in \mathbb{K}^*$. Finally, let us see that
$$p_{i,1}^{m_0)}(s_0) - \alpha_1 p_{i,1}^{m_1)}(s_1) = (p^{m_0)}(s_0) - \alpha_1 p^{m_1)}(s_1))p_i(s_1), \ i = 1, 2, \qquad \alpha_1 \in \mathbb{K}^*.$$
   Indeed: using the proof of the first statement, and the fact that $p_i^{k)}(s_j) = 0$, $k = 1, \ldots, m_j - 1$ (if $m_j \geq 2$), we get that
$$p_i^{m_j)}(s_j) = \frac{p_{i,1}^{m_j)}(s_j)p(s_j) - p_{i,1}(s_j)p^{m_j)}(s_j)}{p(s_j)^2}, \ i = 1, 2, \ m_j \geq 1, \ j = 0, 1.$$
   From $\mathcal{P}^{m_0)}(s_0) = c_1\mathcal{P}^{m_1)}(s_1)$, we have that
$$\frac{p_{i,1}^{m_0)}(s_0)p(s_0) - p_{i,1}(s_0)p^{m_0)}(s_0)}{p(s_0)^2} = c_1\frac{p_{i,1}^{m_1)}(s_1)p(s_1) - p_{i,1}(s_1)p^{m_1)}(s_1)}{p(s_1)^2}$$
   and using that $\overline{\mathcal{P}}(s_0) = \beta_1\overline{\mathcal{P}}(s_1)$ for $\beta_1 = p(s_0)/p(s_1) \in \mathbb{K}^*$, we deduce that
$$p_{i,1}^{m_0)}(s_0) - c_1\beta_1 p_{i,1}^{m_1)}(s_1) = (p^{m_0)}(s_0) - c_1\beta_1 p^{m_1)}(s_1))p_i(s_1), \ i = 1, 2.$$
   We consider $\alpha_1 := c_1\beta_1 \in \mathbb{K}^*$.

Reciprocally, one may easily check that if one of the statements holds then, $P \in \overline{\mathcal{C}}$ is a non–ordinary singularity. $\square$

For a parametrized curve, there are two kinds of singular points, the *self-crossing points*, which are the points that can be obtained via $\mathcal{P}(t)$ with several values of the parameter t, and the *stationary points*, for which both derivatives (with respect to the parameter t) of $\mathcal{P}(t)$ are zero. In addition, a self-crossing point may also be a stationary point for some parameter values that define the point. Note that statement 1 in Theorem 2 has to do with *stationary points*, and statement 2 in Theorem 2 has to do with *self-crossing points*.

We observe that, in Theorem 2, we get that $\ell_P \geq m_0$ or $\ell_P \geq m_0 + m_1$ (statement 1 and 2, respectively) since it could be more points $s_i \in \mathbb{K}$ satisfying equalities obtained in the theorem. Note that we state that there exists "at least" one value (statement 1) or two values (statement 2) of the parameter t satisfying the obtained conditions. If more points, $s_j \in \mathbb{K}$, are considered in statement 2 (i.e. $j \geq 2$), then we get the same equalities with s_j, $j \geq 2$, instead of s_1 (see also *Algorithm Parametrization Construction with Singularities*). In addition, we also get the following remarks concerning to Theorem 2.

Remark 2. 1. *Condition*

$$p_{i,1}^{m_0)}(s_0) - \alpha_1 p_{i,1}^{m_1)}(s_1) = (p^{m_0)}(s_0) - \alpha_1 p^{m_1)}(s_1))p_i(s_1), \text{ for some } \alpha_1 \in \mathbb{K}^* \text{ and } i = 1,2$$

must be necessarily satisfied if $m_0 = m_1 = 1$*; otherwise, this condition is not mandatory and only condition*

$$\overline{\mathcal{P}}^{k)}(s_j) = \lambda_{k,j}\overline{\mathcal{P}}(s_j) \quad \text{for } k = 1, 2, \ldots, m_j - 1, \text{ with } m_j \geq 2 \text{ for } j = 0 \text{ or } j = 1$$

has to be satisfied so that P *is non–ordinary singularity.*

In addition, note that condition

$$p_{i,1}^{m_0)}(s_0) - \alpha_1 p_{i,1}^{m_1)}(s_1) = (p^{m_0)}(s_0) - \alpha_1 p^{m_1)}(s_1))p_i(s_1), \text{ for some } \alpha_1 \in \mathbb{K}^* \text{ and } i = 1,2$$

implies that $\overline{T}_0(t)$ *and* $\overline{T}_1(t)$ *parametrize the same tangent line. In addition, condition*

$$\overline{\mathcal{P}}^{k)}(s_j) = \lambda_{k,j}\overline{\mathcal{P}}(s_j) \quad \text{for } k = 1, 2, \ldots, m_j - 1, \text{ with } m_j \geq 2 \text{ for } j = 0 \text{ or } j = 1$$

implies that the tangent line defined by $\overline{T}_j$ *(for* $j = 0$ *or* $j = 1$*) is a multiple line.*

2. *If only the equality* $\overline{\mathcal{P}}(s_0) = \beta_1\overline{\mathcal{P}}(s_1)$ *holds in statement 2 of Theorem 2, then* $P = \overline{\mathcal{P}}(s_0) \in \overline{\mathcal{C}}$ *is a singularity of multiplicity* $\ell_P \geq 2$*. Additionally, if* $\overline{\mathcal{P}}(t)$ *satisfies that*

$$\overline{\mathcal{P}}^{k)}(s_j) = \lambda_{k,j}\overline{\mathcal{P}}(s_j), \quad \text{for some } \lambda_{k,j} \in \mathbb{K},\ k = 1, 2, \ldots, m_j - 1,$$

with $m_j \geq 2$ *for* $j = 0$ *or* $j = 1$*, or*

$$p_{i,1}^{m_0)}(s_0) - \alpha_1 p_{i,1}^{m_1)}(s_1) = (p^{m_0)}(s_0) - \alpha_1 p^{m_1)}(s_1))p_i(s_1), \text{ for some } \alpha_1 \in \mathbb{K}^* \text{ and } i = 1,2$$

then, P *is non–ordinary and* $\ell_P \geq m_0 + m_1$*. In this sense, we note that if the second equality above does not hold, we are in the conditions of statement 1 in Theorem 2.*

Remark 3. *Theorem 2 characterizes whether a point,* $P = (a_1 : a_2 : 1) \in \overline{\mathcal{C}}$ *such that* $\mathcal{P}(s_0) = (a_1, a_2)$, $s_0 \in \mathbb{K}$, *is a non–ordinary singularity (i.e. we assume that* P *is an affine point and it is not the critical point). Furthermore, we also provide a method for computing the tangents of* P. *For the critical point and points at the infinity, we reason as follows:*

1. *Let* (a_1, a_2) *be a critical point of* $\mathcal{P}(t)$. *We consider a change of variable in* $\mathcal{P}(t)$ *such that* (a_1, a_2) *is generated by the new parametrization (note that the tangents of a rational curve at a point are invariant under changes of variable in the parametrization). For instance, one may take* $\mathcal{Q}(t) = \mathcal{P}(1/(t-a))$, *where* $a \in \mathbb{K}$ *and* $p(a) \neq 0$. *Note that* $\mathcal{Q}(a) = (a_1, a_2)$. *Then, we apply Theorem 2 to the projective parametrization obtained from* $\mathcal{Q}(t)$, *and* $s_0 = a$.
2. *For points at infinity,* $(a_1 : a_2 : 0)$, *we consider* $\mathcal{Q}(t) = (p_{1,1}/p_{2,1}, p/p_{2,1})$ *or* $\mathcal{Q}(t) = (p_{2,1}/p_{1,1},\, p/p_{1,1})$, *depending on whether* $a_2 \neq 0$ *or* $a_1 \neq 0$, *respectively. Then, we apply Theorem 2 to the projective parametrization obtained from* $\mathcal{Q}(t)$.

In Example 1, we apply Theorem 2 to show that $(-2 : 6 : 1)$ and $(0 : 0 : 1)$ are two non–ordinary singularities of multiplicity 3.

Example 1. *Let* $\overline{\mathcal{C}}$ *be a plane curve over* $\mathbb{C}$ *defined parametrically by* $\overline{\mathcal{P}}(t) =$

$$\left(\frac{(t-1)^2(t-2)(2t^3+2t^2+5t+2)}{2}, \frac{(t-1)^2(t-2)(2t^3-6t^2-15t-6)}{2}, t^6 - 1 + 2t^3 + \frac{13}{4}t^2 \right).$$

Observe that:

1. *For* $s_0 = 0$, *it holds that* $\overline{\mathcal{P}}^{k)}(s_0) = \lambda_{k,0}\overline{\mathcal{P}}(s_0)$, $k = 1, 2$, *where* $\lambda_1 = 0$, $\lambda_2 = -3/2$. *Thus, we have a tangent line of multiplicity* $m_0 = 3$ *defined parametrically by*
$$\overline{T}_0(t) = \overline{\mathcal{P}}(s_0) + t\overline{\mathcal{P}}^{3)}(s_0) = (2 + 3t, -6 + 39t, -1 + 12t).$$
Then, $P = \overline{\mathcal{P}}(s_0) = (-2 : 6 : 1)$ *is a non–ordinary singularity of multiplicity* $\ell_P = m_0 = 3$.
2. *For* $s_0 = 1$, $s_1 = 2$, *it holds that*
$$\overline{\mathcal{P}}(s_0) = 21/368\overline{\mathcal{P}}(s_1) = (0 : 0 : 21/4), \quad \textit{and} \quad \overline{\mathcal{P}}^{1)}(s_0) = 21/74\overline{\mathcal{P}}(s_0) = (0 : 0 : 37/2),$$
Hence, there exist two tangent lines of multiplicity $m_0 = 2$ *and* $m_1 = 1$ *defined parametrically by*
$$\overline{T}_0(t) = \overline{\mathcal{P}}(s_0) + t\overline{\mathcal{P}}^{m_0)}(s_0) = (-11t, 25t, 21/4 + 97/2t), \qquad \textit{and}$$
$$\overline{T}_1(t) = \overline{\mathcal{P}}(s_1) + t\overline{\mathcal{P}}^{m_1)}(s_1) = (18t, -22t, 92 + 229t),$$
respectively (note that $\overline{T}_0$ *and* $\overline{T}_1$ *parametrize different lines). Thus,* $P = \overline{\mathcal{P}}(s_0) = (0 : 0 : 1)$ *is a non–ordinary singularity of multiplicity* $\ell_P = m_0 + m_1 = 3$.

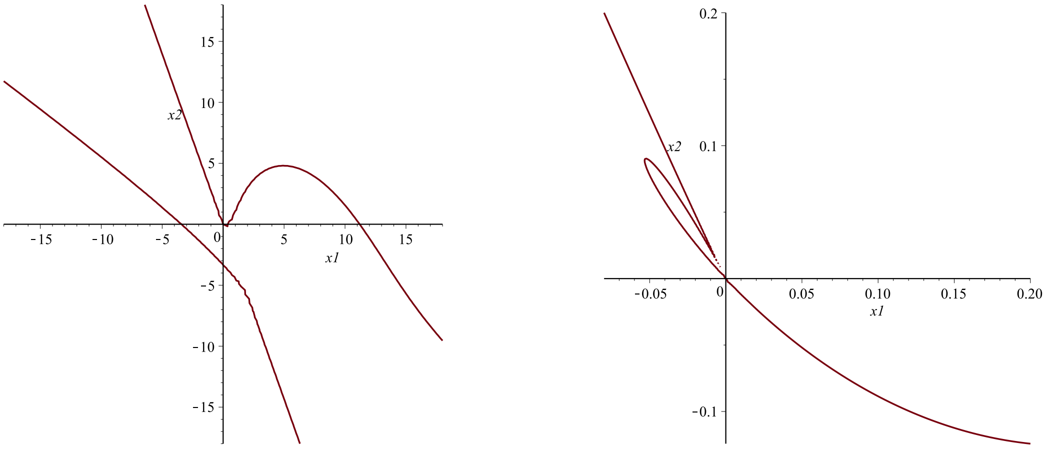


Figure 1: General view of curve $\mathcal{C}$ (left) and detailed view of the singular points $(0:0:1), (-2:6:1)$ (right)

From Theorem 2, we also obtain an algorithm that outputs a parametrization of a rational curve having singularities at some given input points. For simplicity, we outline here the algorithm when only an input affine point is considered. If more than one point is given, one may apply recursively the algorithm. For this purpose, the input considered is the output obtained the last time that the algorithm was executed.

Thus, given a point $P = (a_1 : a_2 : 1)$, three polynomials of degree $d \geq 3$ with undetermined coefficients, some (different) points $s_j \in \mathbb{K}$ and $m_j \in \mathbb{N}$, $j = 0, \ldots, r$, and some sets $\mathcal{S}_n = \{j_1, \ldots, j_{l_n}\}$, $j_k \in \{0, \ldots, r\}$, the algorithm outputs a rational curve $\overline{\mathcal{C}}$ defined by a parametrization $\overline{\mathcal{P}}(t)$ of degree d having a singularity at P of multiplicity $\ell_P \geq m_0 + \cdots + m_r$ and such that for each input set, $\mathcal{S}_n$, the tangents $\overline{T}_j(t)$, $j \in \mathcal{S}_n$, parametrize the same line.

We have that P is ordinary if and only if $m_j = 1$, $j = 0, \ldots, r$, and the tangents $\overline{T}_j(t)$, $j = 0, \ldots, r$, are all different (that is, no sets $\mathcal{S}_n$ are given in the input of the algorithm). Otherwise, P is a non–ordinary singularity.

We observe that the algorithm assumes that the given point is an affine point, and it is not a critical point (otherwise, one applies Remark 3). Finally, we remark that the algorithm can be easily generalized for space curves.

Algorithm Parametrization Construction with Singularities.

Input:

- a point $P = (a_1 : a_2 : 1)$.
- $s_j \in \mathbb{K}$ and $m_j \in \mathbb{N}$, for $j = 0, \ldots, r$ ($s_i \neq s_j$, $i, j \in \{0, \ldots, r\}$).
- $d \in \mathbb{N}$, $d \geq 3$, and polynomials $p_{i,1}(t), p(t) \in \mathbb{K}[t]$, $i = 1, 2$, of degree d, with undetermined coefficients.
- sets $\mathcal{S}_n = \{j_1, \ldots, j_{l_n}\}$, $j_k \in \{0, \ldots, r\}$, $k = 1, \ldots, l_n$ and $n = 1, 2, \ldots$.

Output: a rational curve, $\overline{\mathcal{C}}$, defined by a parametrization $\overline{\mathcal{P}}(t)$ of degree d that has a singularity at P of multiplicity $\ell_P \geq m_0 + \cdots + m_r$ and such that for each $n = 1, 2, \ldots$ the tangents $\overline{T}_j(t)$, $j \in \mathcal{S}_n$, parametrize the same line.

1. Consider the linear equations

$$p_{1,1}(s_j) - \beta_j a_1 = p_{2,1}(s_j) - \beta_j a_2 = p(s_j) - \beta_j = 0,\ j = 0, \ldots, r,$$

   and for $s_j \in \mathbb{K}$ with $m_j \geq 2$, $j = 0, \ldots, r$,

$$p_{1,1}^{k)}(s_j) - \lambda_{k,j} a_1 = p_{2,1}^{k)}(s_j) - \lambda_{k,j} a_2 = p^{k)}(s_j) - \lambda_{k,j} = 0, \quad k = 1, 2, \ldots, m_j - 1.$$

   If no sets $\mathcal{S}_n$ are given, choose a solution satisfying that for $j = 0, \ldots, r$, $\beta_j \neq 0$ and $p_{i,1}^{m_j)}(s_j) - \lambda_{m_j,j} a_i \neq 0$ (for $i = 1$ or 2) or $p^{m_j)}(s_j) - \lambda_{k,j} \neq 0$. Substitute the solution in $p_{i,1}(t), p(t)$, $i = 1, 2$, and return the parametrization $\overline{\mathcal{P}}(t) = (p_{1,1}(t),\, p_{2,1}(t),\, p(t))$. Otherwise, go to step 2.

2. For each $\mathcal{S}_n$ and $j_0 \in \mathcal{S}_n$, consider the equations

$$p_{i,1}^{m_{j_0})}(s_{j_0}) - \alpha_{j_1} p_{i,1}^{m_{j_1})}(s_{j_1}) = (p^{m_{j_0})}(s_{j_0}) - \alpha_{j_1} p^{m_{j_1})}(s_{j_1})) a_i,\ i = 1, 2,$$

   for every $j_1 \in \mathcal{S}_n$. Solve these equations and the equations obtained in step 1, and choose a solution satisfying that for $j = 0, \ldots, r$, $\alpha_j \beta_j \neq 0$ and $p_{i,1}^{m_j)}(s_j) - \lambda_{m_j,j} a_i \neq 0$ (for $i = 1$ or 2) or $p^{m_j)}(s_j) - \lambda_{k,j} \neq 0$. Substitute the solution in $p_{i,1}(t), p(t)$, $i = 1, 2$, and return the rational parametrization $\overline{\mathcal{P}}(t) = (p_{1,1}(t),\, p_{2,1}(t),\, p(t))$.

Remark 4. 1. *Note that for every $j = 0, \ldots, r$, it should be satisfied that $\alpha_j \beta_j \neq 0$ and $p_{i,1}^{m_j)}(s_j) - \lambda_{m_j,j} a_i \neq 0$ (for $i = 1$ or 2) or $p^{m_j)}(s_j) - \lambda_{k,j} \neq 0$. Under these conditions, the algorithm outputs a parametrization if and only if there is a solution to the equations obtained in steps 1 and 2, and the previous inequalities. Observe that in order to find these solutions, one may proceed as follows: one solves the equalities obtained from* *the linear equations appearing in steps 1 and 2. Afterwards one checks whether the solutions obtained satisfy the inequalities imposed.*

2. *Depending on the input of the algorithm (e.g., if only a point is considered and there are enough free unknowns), we may give particular values to the parameters* $\alpha_j, \beta_j, \lambda_{k,j} \in \mathbb{K}$*, to construct the parametrization* $\overline{\mathcal{P}}(t)$*. For instance, if we consider* $\beta_j = \lambda_{k,j} = 1$ *and* $\alpha_{j_1} = p^{m_{j_0})}(s_{j_0})/p^{m_{j_1})}(s_{j_1})$*, we obtain simpler equations.*
3. *Observe that the input of the algorithm is given over an algebraically closed field of characteristic zero,* $\mathbb{K}$*. Thus, the equations and the solutions as well as the output of the algorithm will be over* $\mathbb{K}$*. In general, for practical applications, one considers the field of complex numbers* $\mathbb{C}$*.*
4. *Theorem 2 characterizes non–ordinary singularities. In fact, there are two cases: either a multiple tangent line at an irreducible branch curve, or two irreducible branch curves having the same center and tangent line. Statement 1 of Theorem 2 corresponds to the first situation, and statement 2 of Theorem 2 mixes both situations (see statement 2 of Remark 2). The algorithm is formulated using statement 2 since this item includes the two cases. If one needs to have a multiple tangent line at an irreducible branch curve (first case), then one only has to impose that the equality in step 2 of the algorithm does not hold.*

In the following example, we illustrate the previous algorithm and we show how to construct a parametrization with a non–ordinary singular point of multiplicity 3.

Example 2. *Let us construct a rational curve,* $\overline{\mathcal{C}}$*, over* $\mathbb{C}$ *defined by a parametrization* $\overline{\mathcal{P}}(t)$ *of degree* $d = 5$ *having a non–ordinary singularity at* $P = (0 : 0 : 1)$ *of multiplicity* 3*. For this purpose, we apply* Algorithm Parametrization Construction with Singularities*. Let* $s_0 = 0$*,* $s_1 = 1$*,* $m_0 = 2$*,* $m_1 = 1$*, the generic polynomials*

$$p_{1,1}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + t^5, \qquad p_{2,1}(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + t^5,$$

$$p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + t^5,$$

and $\mathcal{S}_1 = \{0, 1\}$*. In step 1 of the algorithm, we consider the equations*

$$p_{1,1}(s_j) = p_{2,1}(s_j) = p(s_j) - \beta_j = 0,\ j = 0, 1,\ \text{and}\ p^{1)}_{1,1}(s_0) = p^{1)}_{2,1}(s_0) = p^{1)}(s_0) - \lambda_{1,0} = 0.$$

In addition, since $\mathcal{S}_1 = \{0, 1\}$*, let*

$$p^{2)}_{1,1}(s_0) - \alpha_1 p^{1)}_{1,1}(s_1) = p^{2)}_{2,1}(s_0) - \alpha_1 p^{1)}_{2,1}(s_1) = 0$$

(see step 2). We solve these equations and we get that the solution satisfies the inequalities of step 2 if $\alpha_1(a_2 - b_2) \neq 0$ *(see statement 1 in Remark 4). Thus, we substitute the solution in* $p_{i,1}(t), p(t)$*,* $i = 1, 2$*, and the algorithm* returns *the rational parametrization* $\overline{\mathcal{P}}(t) = (p_{1,1}(t),\, p_{2,1}(t),\, p(t))$*, where*

$p_{1,1}(t) = t^2(-1+t)(t^2\alpha_1 + 2a_2t + \alpha_1 a_2 t - \alpha_1 t - a_2\alpha_1)/\alpha_1$

$p_{2,1}(t) = t^2(-1+t)(t^2\alpha_1 + 2b_2t + \alpha_1 b_2 t - \alpha_1 t - b_2\alpha_1)/\alpha_1,$

$p(t) = c_0 + \lambda_{1,0}t + c_2t^2 + c_3t^3 + c_4t^4 + t^5.$

Note that $\overline{\mathcal{P}}(t)$ *satisfies that the tangent lines parametrized by* $\overline{T}_0(t)$ *and* $\overline{T}_1(t)$ *are the same line (see statement 2 in Remark 2). In addition, we observe that from the algorithm we get that* $\ell_P \geq 3$ *since it could be more points* $s_{i_0} \in \mathbb{C}$ *(*$s_{i_0} \neq 0$ *and* $s_{i_0} \neq 1$*) such that* $\mathcal{P}(s_{i_0}) = P$*. However, in this case, one may check that only* $s_0 = 0$, $s_1 = 1$ *(with* $m_0 = 2$, $m_1 = 1$*) satisfy that* $\mathcal{P}(s_0) = \mathcal{P}(s_1) = P$*. Thus, in this example, we may conclude that* $\ell_P = 3$*.*

Finally, we note that in this example, we could have applied statement 2 in Remark 4, and we would have obtained a simplified rational parametrization. Furthermore, we note that we consider the field of complex numbers $\mathbb{C}$ *although, in this particular case, since the input of the problem and the solutions are obtained in the field of real numbers* $\mathbb{R}$*, the output parametrization is defined over* $\mathbb{R}$ *(see statement 3 of Remark 4).*

In Figure 2, we plot the output curve $\mathcal{C}$ *for some particular values of the undetermined coefficients, and a detailed view of the singular point* $(0:0:1)$*.*

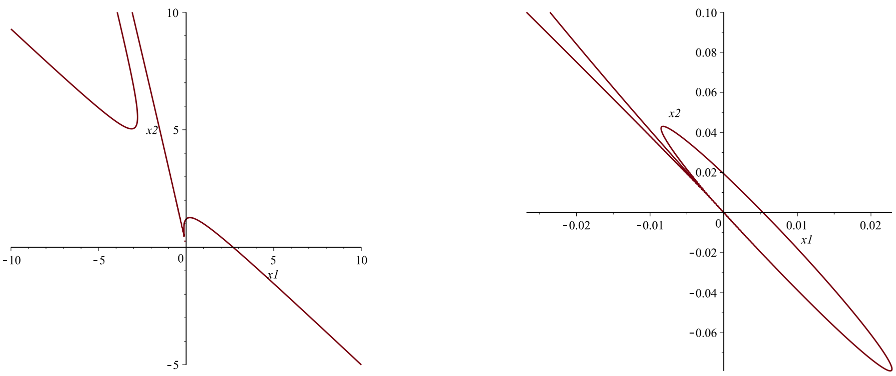


Figure 2: General view of curve $\mathcal{C}$ (left) and detailed view of the singular point $(0:0:1)$ (right)

4. Blowing up the singularities

This section is devoted to the analysis and construction of a rational curve parametrization with non–ordinary singularities and neighboring points. Thus, the main idea is to generalize Section 3 to the case on whether some neighboring points appear.

For this purpose, in Subsection 4.1, we summarize the process of blowing up a singularity and we illustrate it with an example. Afterwards, we blow up a singularity from a given parametrization, and we obtain some algebraic conditions that will allow us to characterize the form of a parametrization defining a curve that has non–ordinary singularities and neighboring points (see Subsection 4.2 and in particular, Theorems 3 and 4). Furthermore, we show how these conditions can be used to construct a rational parametric curve that has as non–ordinary singularities and neighboring points some given input points.

4.1. Description of the blowing–up process

In this subsection, we summarize the process of blowing up a given $\overline{\mathcal{C}}$ at a non–ordinary singularity P, and we introduce some notions, the *delta invariant* and *the number of local branches*, which are associated to P. We recall that the problem with non–ordinary singularities is that they have multiple tangents. One resolves these multiple tangents by "blowing up" the singularity, by achieving the blow-ups by quadratic transformations of the plane that are special birational maps of the projective plane onto itself (see [4], [10] or [31]).

We could proceed in the following way for obtaining these sequences of quadratic transformations resolving the singularities of a given irreducible curve. The method consists in recursively *blowing up* the given curve at the non–ordinary singularities:

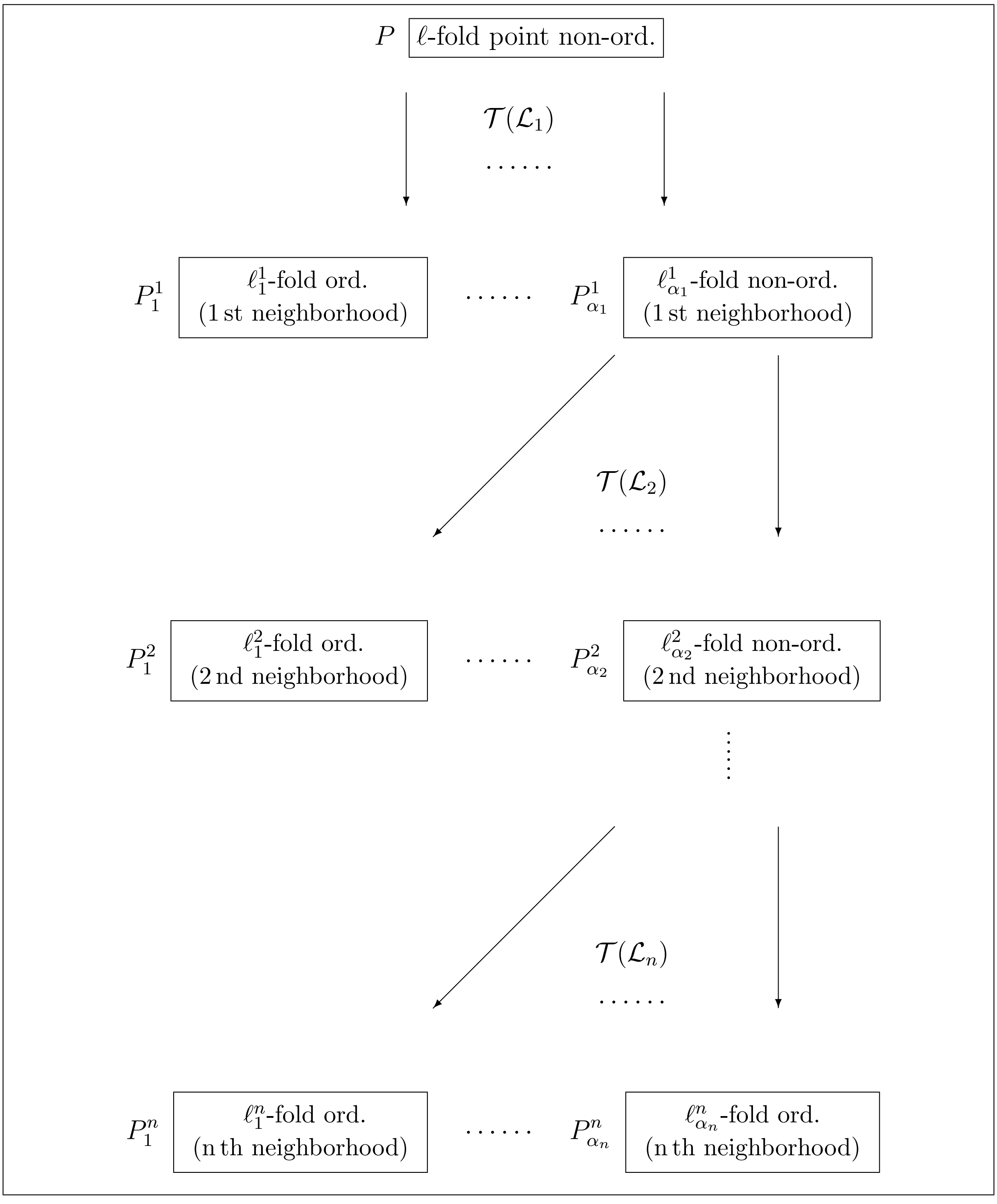
Step 1. Let P be a non–ordinary $\ell-$fold point of $\overline{\mathcal{C}}$. Apply a linear change of coordinates, $\mathcal{L}$, such that P is moved to $(0:0:1)$, none of its tangents is an irregular line (i.e. a line $x_1=0, x_2=0$ or $x_3=0$), and no other point on an irregular line is a singular point on $\overline{\mathcal{C}}$.

Step 2. Apply the quadratic transformation $\mathcal{T}=(x_2x_3, x_1x_3, x_1x_2)$ to $\overline{\mathcal{C}}$, getting the transformed curve $\overline{\mathcal{C}}_1$. It holds that:

- Outside of the irregular lines, $\mathcal{T}$ preserves the multiplicity of points and their tangents (and thus, its character).
- New ordinary singularities might be created at the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ which are called the fundamental points (observe that these points do not correspond to any singular point in the input curve since in step 1, we are assuming that and no other point on an irregular line is a singular point on the input curve).
- The new curve $\overline{\mathcal{C}}_1$ might have singularities, also non–ordinary ones, on the irregular line $x_3=0$. These singularities are obtained from P; that is, P is replaced on $\overline{\mathcal{C}}_1$ by points $(1:\gamma:0)\in\overline{\mathcal{C}}_1$, with $\gamma\neq 0$ (note that $\mathcal{T}(1,\gamma,0)=P$). Some of them might be singularities, also non–ordinary ones. We denote by $\xi_1:=\{P_1^1,\ldots,P_{\alpha_1}^1\}$, the set of points of multiplicities $\{\ell_1^1,\ldots,\ell_{\alpha_1}^1\}$, $\ell_j^1\geq 2$, where $P_i^1=(1:\gamma_i:0)$, $\gamma_i\neq 0$, $i=1,\ldots,\alpha_1$. We say that ξ_1 is the *first neighborhood* of P.

Step 3. Apply recursively steps 1 and 2 to $\overline{\mathcal{C}}_1$ until non–ordinary singularity is left. More precisely, if some of the points in ξ_1 are non–ordinary, we get the second neighborhood of P as the union of the first neighborhoods of these non–ordinary singular points.

The points in the second neighborhood of P are called the *neighboring points of* P *at its second neighborhood*, and we denote it as $\xi_2 := \{P_1^2, \ldots, P_{\alpha_2}^2\}$. The multiplicity and character of points at the second neighborhood are defined in a way analogous to the one for points in the first neighborhood. In general, we will call any point in one of the neighborhoods of P a *neighboring point* of P.



It is proved that there are at most a finite number of singular points in the neighborhoods of any point of an irreducible curve (see [31], pp. 82). Hence the analysis of a singularity in terms of neighboring singularities is a finite process and leads to a complete classification of all singular points. Observe that the process finishes when $\xi_m = \emptyset$, for some $m \in \mathbb{N}$ and thus, this method always achieves an irreducible curve having only ordinary singularities in a finite number of steps (see [10]).

Let $P \in \overline{\mathcal{C}}$ be a singularity of multiplicity ℓ_P. One can associate P its *delta invariant*, δ_P, and *the number of local branches*, r_P. The delta invariant is a very important number since for instance, the genus of an irreducible plane curve is the number $(d-1)(d-2)/2 - \sum_{P \in \mathcal{S}} \delta_P$, where d is the degree of the given curve, and $\mathcal{S}$ is the set of singular points. Intuitively speaking, the delta invariant δ_P measures the number of double points concentrated at P; i.e. δ_P concentrates δ_P ordinary double points (see Subsection 7.4.1 in [18], Subsection 2.5.4 in [3] or Subsection 8.1 in [9]). For instance, let us consider an ordinary 3–fold point. When moving one of the three lines slightly, we see that we find three ordinary double points in a small neighborhood of the original singularity.

Taking into account the above sentences and the notation previously introduced, one gets that the delta invariant is given as

$$\delta_P = \ell_P(\ell_P - 1)/2 + \sum_{j=1}^{n} \sum_{i=1}^{\alpha_j} \ell_i^j(\ell_i^j - 1)/2,$$

where n is the *number of neighborhoods*, $\xi_j := \{P_1^j, \ldots, P_{\alpha_j}^j\}$, $j = 1, \ldots, n$ (in order to compute δ_P, one also may apply the formula using intersection index of Pusieux expansion; see e.g. Subsection 2.5.4 in [3]).

In general $r_P \leq \ell_P$ and $\ell_P(\ell_P - 1)/2 \leq \delta_P$, and both of these inequalities are equalities when the singularity is an ordinary singularity of multiplicity ℓ_P. In fact, given a non–ordinary singularity $P \in \overline{\mathcal{C}}$ of multiplicity ℓ_P, three different situations could appear:

- S_1. $\ell_P(\ell_P - 1)/2 = \delta_P$ and $r_P < \ell_P$: from Theorem 1, we get that there exist ℓ_P values of the parameter t, namely $s_1, \ldots, s_{\ell_P}$, such that $\overline{\mathcal{P}}(s_j) = P$ but only r_P of them are different. In addition, from the previous paragraphs, we have that $\xi_1 = \emptyset$.
- S_2. $\ell_P(\ell_P - 1)/2 < \delta_P$ and $r_P = \ell_P$: from Theorem 1, we deduce that there exist ℓ_P different values of the parameter t, namely $s_1, \ldots, s_{\ell_P}$, such that $\overline{\mathcal{P}}(s_i) = P$. Furthermore, taking into account the previous paragraphs, we get that $\delta_{P^1} = \delta_P - \ell_P(\ell_P - 1)/2$, where $\delta_{P^1} := \sum_{j=1}^{\alpha_1} \delta_{P_j^1}$, and in general

  $$\delta_{P^i} = \delta_{P^{i-1}} - \sum_{j=1}^{\alpha_{i-1}} \ell_j^{i-1}(\ell_j^{i-1} - 1)/2, \quad \text{where } \delta_{P^i} := \sum_{j=1}^{\alpha_i} \delta_{P_j^i} \ \text{ for } i = 1, \ldots, n$$

  (we denote $P^0 := P$). We recall that the i–th neighborhood of P is given by the set of points $\xi_i := \{P_1^i, \ldots, P_{\alpha_i}^i\}$ of multiplicities $\{\ell_1^i, \ldots, \ell_{\alpha_i}^i\}$, and delta invariants $\{\delta_1^i, \ldots, \delta_{\alpha_i}^i\}$, respectively (for $i = 1, \ldots, n$). In addition, $\delta_{P^{n+1}} = 0$ and $\xi_{n+1} = \emptyset$.

S$_3$. $\ell_P(\ell_P-1)/2<\delta_P$ and $r_P<\ell_P$: in this case, S$_1$ and S$_2$ hold simultaneously.

In the following example, we illustrate the method summarized above and we describe the blowing–up process for a given irreducible plane curve defined by a parametrization $\overline{\mathcal{P}}(t)$. For each singularity P, we also compute δ_P and r_P.

Example 3. *Let $\overline{\mathcal{C}}$ be the plane curve over $\mathbb{C}$ defined by the parametrization $\overline{\mathcal{P}}(t)=(t^2,t^5,1)$. By applying [21] (see also Theorem 1 and Remark 1), we get that $\overline{\mathcal{C}}$ has two singularities: $P_1=(0:0:1)$ of multiplicity $\ell_{P_1}=2$, and $P_2=(0:1:0)$ of multiplicity $\ell_{P_2}=3$.*

Furthermore, we observe that $\gcd(G_1(t,0),G_2(t,0))=t^2$ (see Remark 1). Thus, the number of parameters, t, corresponding to P_1 is 2 ($t=0$ and $t=0$), and $r_{P_1}=1$. Hence P_1 is a non–ordinary singularity (see S_1 above). Now, we consider the reparametrization $\mathcal{M}(t)=\mathcal{P}(1/t)=(t^3,1,t^5)$, and we reason similarly for P_2 (see Remarks 1 and 3). We get that the number of parameters, t, corresponding to P_2 is 3 ($t=0$, $t=0$ and $t=0$). Hence $r_{P_2}=1$ and P_2 is a non–ordinary singularity (see S_1 above).

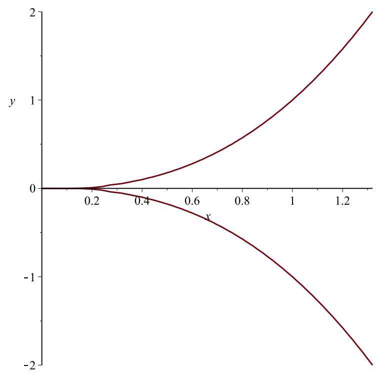


Figure 3: The curve $\mathcal{C}$ has one affine singularity at $P_1=(0:0:1)$

Now, we analyze the neighboring points for P_1 and P_2:

1. *First, we start with P_1. We take the change of coordinates $\mathcal{L}_1=(1/2x_1-1/2x_2,1/2x_2+1/2x_1,x_3)$ such that P_1 is moved to $(0:0:1)$, none of its tangents is an irregular line, and no other point on an irregular line is singular (see step 1 of the blowing–up process). Then, we consider the parametrization*

$$\overline{\mathcal{Q}}_1=\mathcal{T}(\mathcal{L}_1(\overline{\mathcal{P}}))=(1+t^3,\frac{2}{3}-t^3,-\frac{1}{6}t^5+\frac{1}{3}t^2-\frac{1}{2}t^8),$$

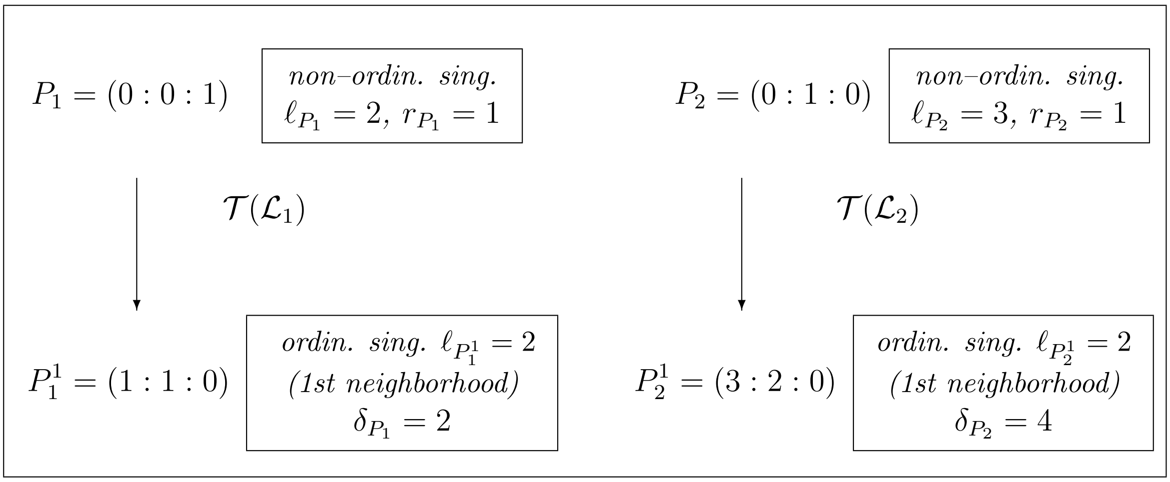
where $\mathcal{T}=(x_2x_3,x_1x_3,x_1x_2)$. We apply the results in [21] (see also Theorem 1 and Remark 1), and we get that $P_1^1=(1:1:0)$ is an ordinary singularity of multiplicity 2, the points $(1:0:0)$, $(0:1:0)$ are singularities of multiplicity 3, and $(0:0:1)$ is a singularity of multiplicity 5. Thus, the first neighborhood of P_1 is given by P_1^1 and the process finishes with this point (see step 2 of the blowing–up process). We note that $\delta_{P_1}=2$.

2. *Now, we reason similarly for* P_2. *We take the change of coordinates* $\mathcal{L}_2 = (1/2x_1 + 1/2x_3, -1/2x_3 + 1/2x_1, x_2)$ *(see step 1 of the blowing–up process), and we consider the parametrization*

$$\overline{\mathcal{Q}}_2 = \mathcal{T}(\mathcal{L}_2(\overline{\mathcal{M}})) = (6 + 6t^2, 4 - 6t^2, 2t^3 - t^5 - 3t^7)$$

(see step 2 of the blowing–up process). We apply the results in [21] (see also Theorem 1 and Remark 1), and we get that $P_2^1 = (3:2:0)$ *is an ordinary singularity of multiplicity* 2, *the points* $(1:0:0)$, $(0:1:0)$ *are singularities of multiplicity* 2. *The first neighborhood of* P_2 *is given by* P_2^1 *and thus, the process finishes with the point* P_2^1. *We note that* $\delta_{P_2} = 4$.

In the following, we show a singularity tree for this example.



4.2. Blowing up a singularity from a given parametrization

In the following, we blow up a singularity from a given rational parametrization $\overline{\mathcal{P}}(t)$ that defines an algebraic plane curve $\overline{\mathcal{C}}$. The goal is to analyze the conditions obtained in the first neighborhood of a singular point, P, so that we can decide whether, in general, one has neighboring points. In addition, we obtain equalities that allow us to construct rational parametrizations defining plane curves that have (or not) certain singular points (ordinary or non–ordinary), and such that they have (or not) neighboring points.

In the following, we apply the three steps obtained in the process presented in Subsection 4.1, and we *blow up* the given curve at the given parametrization. In particular, we will see how the parametrization after blowing up is computed from the parametrization before blowing up.

Step 1

Let P be an ℓ–fold point of $\overline{\mathcal{C}}$. We apply a linear change of coordinates, $\mathcal{L}$, such that P is moved to $(0:0:1)$. Thus, let $P = (0:0:1) \in \overline{\mathcal{C}}$ be a point of multiplicity ℓ_P. We assume w.l.o.g that P is not a critical point. From Theorem 1, we may write

$$p_{i,1}(t) = H(t)\overline{p}_{j,1}(t),\ j = 1,2,\ \text{with } \gcd(\overline{p}_{1,1}, \overline{p}_{2,1}) = 1,\ \gcd(H, p) = \gcd(p_{1,1}, p_{2,1}, p) = 1,$$

and $\deg(H) = \ell_P$. Then,

$$\boxed{\overline{\mathcal{P}}(t) = (p_{1,1}(t), p_{2,1}(t), p(t)) = (H(t)\overline{p}_{1,1}(t), H(t)\overline{p}_{2,1}(t), p(t)).}$$

In this first step of the blowing up process, we have to assume that none of the tangents of P are an irregular line (i.e. $\gcd(H, \overline{p}_{i,1}(t)) = 1$, $i = 1, 2$), and no other point on an irregular line is a singular point (i.e. $\gcd(\overline{p}_{j,1}(t), p_{i,1}(t)p(a) - p_{i,1}(a)p(t)) = t - a$, where $\overline{p}_{j,1}(a) = 0$, $i, j \in \{1, 2\}$, $i \neq j$; see Theorem 1). For this purpose, we check whether $\overline{\mathcal{P}}(t)$ satisfies these conditions. In the negative case, we apply a linear change of coordinates to $\overline{\mathcal{P}}(t)$, such that these two conditions are satisfied.

Under these conditions, we adapt Theorem 2 to this particular parametrization and we get the following theorem where it is characterized whether $P = (0 : 0 : 1)$ is a non–ordinary singularity.

Theorem 3. *The following statements hold:*

1. $P = (0 : 0 : 1) \in \overline{\mathcal{C}}$ *is a singularity of multiplicity* $\ell_P = m_0 + m_1 + \cdots + m_r$ *if and only if*

   $$\boxed{H(t) = (t - s_0)^{m_0}(t - s_1)^{m_1} \cdots (t - s_r)^{m_r}} \qquad (C_1)$$

   $s_i \neq s_j$, $i, j \in \{0, \ldots, r\}$. *In this case,* $r_P = r + 1$.
2. *Let us assume that statement 1 holds. Then,* $P = (0 : 0 : 1) \in \overline{\mathcal{C}}$ *is a non–ordinary singularity if and only if* $m_j \geq 2$ *for some* $j = 0, \ldots, r$, *or there exist* $s_{j_0}, s_{j_1} \in \mathbb{K}$, *where* $j_0, j_1 \in \{0, \ldots, r\}$, $j_0 \neq j_1$, *such that*

   $$\boxed{\overline{p}_{i,1}(s_{j_0}) = \alpha_{j_1} \frac{H^{m_{j_1})}(s_{j_1})}{H^{m_{j_0})}(s_{j_0})} \overline{p}_{i,1}(s_{j_1}),\ i = 1, 2,\ \alpha_{j_1} \in \mathbb{K}^*} \qquad (C_2)$$

   where $m_{j_k} = m_k$, $k = 0, 1$.

Proof. Statement 1 is proved by applying Theorem 1. Statement 2 is deduced by applying Theorem 2, and taking into account that if statement 1 holds then, $H^{k)}(s_j) = 0$, $0 \leq k \leq m_j - 1$ for $j = 0, \ldots, r$, and thus $\overline{\mathcal{P}}^{m_j)}(s_j) = (H^{m_j)}(s_j)\overline{p}_{1,1}(s_j), H^{m_j)}(s_j)\overline{p}_{2,1}(s_j), p^{m_j)}(s_j))$. □

Remark 5. *Condition* (C_2) *in Theorem 3 must be necessarily satisfied if* $m_j = 1$, $j = 0, \ldots, r$; *otherwise, this condition is not mandatory so that* P *is a non–ordinary singularity. Furthermore, if condition* (C_2) *holds, we get that the parametrizations* $\overline{T}_{j_0}(t)$ *and* $\overline{T}_{j_1}(t)$ *define the same tangent line (see statement 1 in Remark 2).*

In the following, we illustrate Theorem 3 with an example.

Example 4. *Let $\overline{\mathcal{C}}$ be a plane curve defined over $\mathbb{C}$ by the rational parametrization*

$$\overline{\mathcal{P}}(t) = (p_{1,1}(t),\, p_{2,1}(t),\, p(t)) =$$

$$\left(\frac{1}{5}t^2(t-1)(t-2)^3(10-47t+5t^2), \frac{1}{5}t^2(t-1)(t-2)^3(-1+5t)(t-5), -2+2t^2+t^6\right).$$

Observe that we may write

$$p_{i,1}(t) = H(t)\overline{p}_{i,1}(t),\ j=1,2,\ \textit{with}\ \gcd(\overline{p}_{1,1},\overline{p}_{2,1})=1,\ \gcd(H,p)=\gcd(p_{1,1},p_{2,1},p)=1,$$

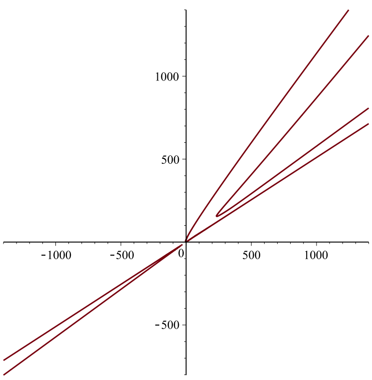
where $H(t)=t^2(t-1)(t-2)^3$. By statement 1 of Theorem 3, we get that $P=(0:0:1)$ is a singularity of multiplicity $\ell_P = m_0+m_1+m_2=6$ (where $m_0=2$, $m_1=1$, $m_2=3$) and $r_P=3$. In addition, using statement 2 of Theorem 3, we deduce that P is non–ordinary since $m_j\geq 2$, $j=0,2$.

Additionally, one may check that (C_2) holds:

$$\overline{p}_{i,1}(s_0) = 5\frac{H^{m_1)}(s_1)}{H^{m_0)}(s_0)}\overline{p}_{i,1}(s_1),\ i=1,2$$

where $s_0=0$, $s_1=1$, $s_2=2$. Hence, $\overline{T}_0(t)$ and $\overline{T}_1(t)$ parametrize the same tangent line (see Remark 5). In fact, we may compute the three tangent lines, $\overline{T}_j(t)=\overline{\mathcal{P}}(s_j)+t\overline{\mathcal{P}}^{m_j)}(s_j), j=0,1,2$, and we have that

$$\overline{T}_0(t)=(32t,16t,-2+4t),\quad \overline{T}_1(t)=(\frac{32}{5}t,\frac{16}{5}t,1+10t),\quad \overline{T}_2(t)=(-\frac{1536}{5}t,-\frac{648}{5}t,70+960t).$$



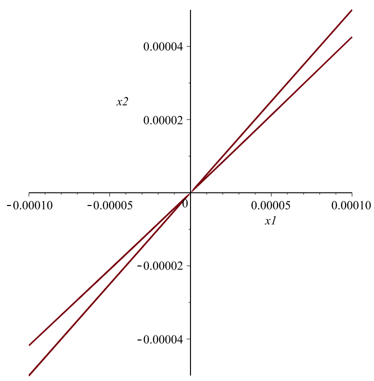


Figure 4: General view of curve $\mathcal{C}$ (left) and detailed view of the singular point P (right)

Given some (different) points $s_j \in \mathbb{K}$ and $m_j \in \mathbb{N}$, $j = 0, \ldots, r$, three polynomials with undetermined coefficients, and some sets $\mathcal{S}_n = \{j_1, \ldots, j_{l_n}\}$, $j_k \in \{0, \ldots, r\}$, from Theorem 3, we obtain an algorithm that outputs a curve, $\overline{\mathcal{C}}$, defined by a rational parametrization $\overline{\mathcal{P}}(t)$ having a singularity at $P = (0 : 0 : 1)$ of multiplicity $\ell_P = m_0 + \cdots + m_r$ and such that for each input set, $\mathcal{S}_n$, the tangents $\overline{T}_j(t)$, $j \in \mathcal{S}_n$, parametrize the same line. This algorithm is a particular case of *Algorithm Parametrization Construction with Singularities* for the case of the singularity $P = (0 : 0 : 1)$. One may check that for this case, this new algorithm simplifies algorithm presented in Section 3.

We recall that P is ordinary if and only if $m_j = 1$, $j = 0, \ldots, r$, and the tangents $\overline{T}_j(t)$, $j = 0, \ldots, r$, are all different (that is, no sets $\mathcal{S}_n$ are given in the input of the algorithm). Otherwise, P is a non–ordinary singularity.

Algorithm Parametrization Construction with Singularity at $(0 : 0 : 1)$.

Input:

- $s_j \in \mathbb{K}$ and $m_j \in \mathbb{N}$, for $j = 0, \ldots, r$ ($s_i \neq s_j$, $i, j \in \{0, \ldots, r\}$).
- $d \in \mathbb{N}$, $d \geq m_0 + \cdots + m_r$, and polynomials $\overline{p}_{i,1}(t) \in \mathbb{K}[t]$, $i = 1, 2$, of degree $d - m_0 - \cdots - m_r$, and $p(t) \in \mathbb{K}[t]$ of degree d, with undetermined coefficients.
- sets $\mathcal{S}_n = \{j_1, \ldots, j_{l_n}\}$, $j_k \in \{0, \ldots, r\}$, $k = 1, \ldots, l_n$ and $n = 1, 2, \ldots$.

Output: a rational curve, $\overline{\mathcal{C}}$, defined by a parametrization $\overline{\mathcal{P}}(t)$ of degree d that has a singularity at $P = (0 : 0 : 1)$ of multiplicity $\ell_P = m_0 + \cdots + m_r$ and such that for each $n = 1, 2, \ldots$ the tangents $\overline{T}_j(t)$, $j \in \mathcal{S}_n$, parametrize the same line.

1. Let $H(t) = (t - s_0)^{m_0}(t - s_1)^{m_1} \cdots (t - s_r)^{m_r}$. If no sets $\mathcal{S}_n$ are given, return the parametrization $\overline{\mathcal{P}}(t) = (p_{1,1}(t), p_{2,1}(t), p(t)) = (H(t)\overline{p}_{1,1}(t), H(t)\overline{p}_{2,1}(t), p(t))$, with $\gcd(\overline{p}_{1,1}, \overline{p}_{2,1}) = \gcd(H, p) = \gcd(p_{1,1}, p_{2,1}, p) = 1$. Otherwise, go to step 2.

2. For each $\mathcal{S}_n$ and $j_0 \in \mathcal{S}_n$, consider the equations

$$\overline{p}_{i,1}(s_{j_0}) = \alpha_{j_1} \frac{H^{m_{j_1})}(s_{j_1})}{H^{m_{j_0})}(s_{j_0})} \overline{p}_{i,1}(s_{j_1}),\ i = 1, 2$$

for every $j_1 \in \mathcal{S}_n$. Solve these equations, and choose a solution satisfying that $\alpha_j \neq 0$ and $\gcd(\overline{p}_{1,1}, \overline{p}_{2,1}) = \gcd(H, p) = \gcd(p_{1,1}, p_{2,1}, p) = 1$. Substitute the solution in $\overline{p}_{i,1}(t), p(t)$, $i = 1, 2$, and return the rational parametrization

$$\overline{\mathcal{P}}(t) = (p_{1,1}(t), p_{2,1}(t), p(t)) = (H(t)\overline{p}_{1,1}(t), H(t)\overline{p}_{2,1}(t), p(t)).$$

Remark 6. 1. *Note that for every $j = 0, \dots, r$, it should be satisfied that $\alpha_j \neq 0$ and $\gcd(\overline{p}_{1,1}, \overline{p}_{2,1}) = \gcd(H, p) = \gcd(p_{1,1}, p_{2,1}, p) = 1$. Under these conditions, the algorithm outputs a parametrization if and only if there is solution to the equations considered in step 2 satisfying the above conditions. Observe that in order to find these solutions, one may proceed as follows: one solves the equalities obtained from the linear equations appearing in step 2. Afterwards, one checks whether the solutions obtained satisfy the conditions imposed.*

We also observe that $\gcd(\overline{p}_{1,1}, \overline{p}_{2,1}) = \gcd(H, p) = 1$ implies that $\ell_P = m_0 + \cdots + m_r$ (see statement 1 in Theorem 3 and compare with Theorem 2).

2. *Depending on the input of the algorithm (e.g., if there are enough free unknowns), we may give particular values to the parameter $\alpha_{j_1} \in \mathbb{K}^*$, to construct $\overline{\mathcal{P}}(t)$. For instance, let us take $\alpha_{j_1} = \frac{H^{m_{j_0})}(s_{j_0})}{H^{m_{j_1})}(s_{j_1})} \in \mathbb{K}^*$. Then, (C_2) is simplified since we get the equivalent equations $\overline{p}_{i,1}(s_{j_0}) = \overline{p}_{i,1}(s_{j_1}),\ j_0, j_1 \in \{0, \dots, r\},\ j_0 \neq j_1,\ i = 1, 2$.*
3. *Observe that the input of the algorithm is given over an algebraically closed field of characteristic zero, $\mathbb{K}$. Thus, the equations and the solutions as well as the output of the algorithm will be over $\mathbb{K}$. In general, for practical applications, one considers the field of complex numbers $\mathbb{C}$.*
4. *If the user needs to have a multiple tangent line at an irreducible branch curve, then he/she only has to impose that the equality outlined in step 2 of the algorithm does not hold (see statement 4 of Remark 4).*

In Example 5, we illustrate the previous algorithm and we show how to construct a parametrization with $P = (0 : 0 : 1)$ a non–ordinary singularity of multiplicity 7.

Example 5. *We construct a rational curve $\overline{\mathcal{C}}$ over $\mathbb{C}$ defined by a parametrization $\overline{\mathcal{P}}(t)$ of degree $d = 10$ such that it has a non–ordinary singularity at $P = (0 : 0 : 1)$ of multiplicity 7. For this purpose, let $s_0 = 0,\ s_1 = 1,\ s_2 = 2,\ \ m_0 = 2, m_1 = 2, m_2 = 3$, and $\mathcal{S}_1 = \{0, 1, 2\}$. In addition, let*

$$\overline{p}_{1,1}(t) = a_0 + a_1 t + a_2 t^2 + t^3, \qquad \overline{p}_{2,1}(t) = b_0 + b_1 t + b_2 t^2 + t^3,$$

and $p(t)$ is any polynomial of degree 10. Then, in the first step of the algorithm we consider $H(t) = t^2(t-1)^2(t-2)^3$. In step 2, we solve the equations

$$\overline{p}_{1,1}(s_j) = \alpha_j \frac{H^{m_j)}(s_j)}{H^{m_0)}(s_0)} \overline{p}_{1,1}(s_0),\ j = 1, 2$$

(see statement 1 in Remark 6). Note that, in this case, equations obtained in step 2 are equal for $p_{1,1}$ and $p_{2,1}$ (these polynomials are equal except for the name of the unknown coefficients). Thus, we first may compute $p_{1,1}$, and $p_{2,1}$ is obtained similarly although with different unknown coefficients. One may check that the solution satisfies the conditions of step 2 (see statement 1 in Remark 6). Thus, we substitute the solution in $\overline{p}_{i,1}(t)$, $i = 1, 2$, and we obtain the parametrization

$$\overline{\mathcal{P}}(t) = (p_{1,1}(t), p_{2,1}(t), p(t)) = (H(t)\overline{p}_{1,1}(t), H(t)\overline{p}_{2,1}(t), p(t))$$

where,

$$\overline{p}_{1,1}(t) = \frac{-8a_0+12ta_0-16t-2t\alpha_1 a_0-6t\alpha_2 a_0-4t^2a_0+24t^2+t^2\alpha_1 a_0+6t^2\alpha_2 a_0-8t^3}{-8(c_0+c_1t+c_2t^2+c_3t^3+c_4t^4+c_5t^5)}$$

$$\overline{p}_{2,1}(t) = \frac{-8b_0+12tb_0-16t-2t\alpha_1 b_0-6t\alpha_2 b_0-4t^2b_0+24t^2+t^2\alpha_1 b_0+6t^2\alpha_2 b_0-8t^3}{-8(c_0+c_1t+c_2t^2+c_3t^3+c_4t^4+c_5t^5)}$$

and $p(t)$ *is any polynomial of degree* 10. *The curve* $\overline{\mathcal{C}}$ *has a singularity at* P *of multiplicity* $\ell_P = m_0 + m_1 + m_2 = 7$ *and* $\overline{T}_j(t) = \overline{\mathcal{P}}(s_j) + t\overline{\mathcal{P}}^{m_j)}(s_j)$, $j = 0, 1, 2$ *define the same tangent line (see Remark 5).*

Observe that in this example, we could have applied statement 2 in Remark 6, and we would have obtained a simplified rational parametrization. Furthermore, we note that we consider the field of complex numbers $\mathbb{C}$ *although, in this particular case, since the input of the problem and the solutions are obtained in the field of real numbers* $\mathbb{R}$*, the output parametrization is defined over* $\mathbb{R}$ *(see statement 3 of Remark 6).*

In Figure 5, we plot the output curve $\mathcal{C}$ *for some particular values of the undetermined coefficients. In particular, we plot a detailed view of the singular point* $(0:0:1)$.

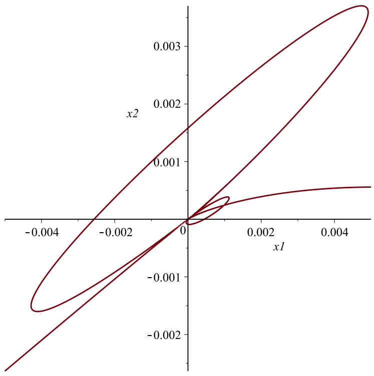


Figure 5: Detailed view of the curve $\mathcal{C}$ at the singular point $(0:0:1)$

Step 2

The second step in the blowing up process consists in applying the quadratic transformation $\mathcal{T} = (x_2x_3, x_1x_3, x_1x_2)$ to $\overline{\mathcal{C}}$, getting the transformed curve $\overline{\mathcal{C}}_1$. Thus, by applying $\mathcal{T}$ to $\overline{\mathcal{P}}(t)$, we get the projective parametrization of $\overline{\mathcal{C}}_1$ defined by

$$\boxed{\overline{\mathcal{Q}}_1(t) = (p_{2,1}(t)p(t), p_{1,1}(t)p(t), p_{1,1}(t)p_{2,1}(t)) = (\overline{p}_{2,1}(t)p(t), \overline{p}_{1,1}(t)p(t), H(t)\overline{p}_{1,1}(t)\overline{p}_{2,1}(t)).}$$

Now, we should check whether there exist singular points $(1:\gamma:0) \in \overline{\mathcal{C}}_1$, with $\gamma \neq 0$ (note that if these points exist, it should be reached by the values of t being roots of the polynomial $H(t)$). For this purpose, we first take a change of coordinates $\mathcal{L}$, such that $(1:\gamma:0)$ is moved to $P^1=(0:0:1)$. Then, we get the parametrization

$$\mathcal{L}(\overline{\mathcal{Q}}_1)=(q_{1,1}(t),q_{2,1}(t),q(t)):=(\overline{p}_{1,1}(t)p(t)-\gamma\overline{p}_{2,1}(t)p(t),H(t)\overline{p}_{1,1}(t)\overline{p}_{2,1}(t),\overline{p}_{2,1}(t)p(t))$$

and we characterize whether P^1 is a singularity (which is equivalent to $(1:\gamma:0)\in\xi_1$). We also analyze if P^1 is a non–ordinary singularity. For this purpose, we apply Theorem 3, and we get the following result.

Theorem 4. *The following statements hold:*

1. $P^1=(0:0:1)$ *is a singularity of multiplicity* $\ell_{P^1}=m_{j_0,1}+\cdots+m_{j_l,1}$ *if and only if*

   $$\mathcal{L}(\overline{\mathcal{Q}}_1)=(q_{1,1}(t),q_{2,1}(t),q(t)):=(\overline{p}_{1,1}(t)p(t)-\gamma\overline{p}_{2,1}(t)p(t),H(t)\overline{p}_{1,1}(t)\overline{p}_{2,1}(t),\overline{p}_{2,1}(t)p(t)),$$

   where

   $$q_{i,1}(t)=H_1(t)\overline{q}_{i,1}(t),\ i=1,2,\ \textit{with}\ \gcd(H_1,q)=\gcd(\overline{q}_{1,1},\overline{q}_{2,1})=\gcd(q_{1,1},q_{2,1},q)=1,$$

   and $H_1(t)=(t-s_{j_0})^{m_{j_0,1}}\cdots(t-s_{j_l})^{m_{j_l,1}}$, $1\le m_{j_k,1}\le m_{j_k}$, *with* $j_k\in\{0,\ldots,r\}$.
2. *Let us assume that statement 1 holds. Then,* $P^1=(0:0:1)$ *is non–ordinary if and only if* $m_{j_k,1}\ge 2$ *for some* $k\in\{0,\ldots,l\}$ *or there exist* $s_{j_u},s_{j_v}\in\mathbb{K}$, $u,v\in\{0,\ldots,l\}$ *such that*

   $$\overline{q}_{i,1}(s_{j_u})=\alpha_{j_v}\frac{H_1^{m_{j_v,1})}(s_{j_v})}{H_1^{m_{j_u,1})}(s_{j_u})}\overline{q}_{i,1}(s_{j_v}),\ i=1,2,\ \alpha_{j_v}\in\mathbb{K}^*.$$

Remark 7.
1. *Since* $q_{2,1}(t)=H(t)\overline{p}_{1,1}(t)\overline{p}_{2,1}(t)$, *and* $q(t)=\overline{p}_{2,1}(t)p(t)$, *it already holds that*

   $$q_{2,1}(t)=H_1(t)\overline{q}_{2,1}(t),\ \textit{and}\ \gcd(H_1,q)=1,$$

   for any

   $$H_1(t)=(t-s_{j_0})^{m_{j_0,1}}\cdots(t-s_{j_l})^{m_{j_l,1}},\ 1\le m_{j_k,1}\le m_{j_k},$$

   with $j_k\in\{0,\ldots,r\}$. *Thus, in statement 1 of Theorem 4, we only have to check whether*

   $$q_{1,1}(t)=H_1(t)\overline{q}_{1,1}(t),\quad \textit{with}\ \gcd(\overline{q}_{1,1},\overline{q}_{2,1})=1.$$

   For this purpose, note that $(t-s_j)$ *divides* $q_{1,1}(t)$ *(i.e.,* $q_{1,1}(s_j)=0$*) if and only if* $\overline{p}_{1,1}(s_j)=\gamma\overline{p}_{2,1}(s_j)$. *Note that this condition is equivalent to condition* (C_2) *for the point* P *(see statement 2 in Theorem 3). Thus,* (C_2) *holds for* P *if and only if* P^1 *is a singularity.*
2. *Let* $\overline{\mathcal{P}}(t)$ *be the output of the Algorithm Parametrization Construction with Singularity at* $(0:0:1)$. *One may apply again this algorithm to*

   $$\mathcal{L}(\overline{\mathcal{Q}}_1)=(q_{1,1}(t),q_{2,1}(t),q(t)):=(\overline{p}_{1,1}(t)p(t)-\gamma\overline{p}_{2,1}(t)p(t),H(t)\overline{p}_{1,1}(t)\overline{p}_{2,1}(t),\overline{p}_{2,1}(t)p(t)),$$

   where $\gamma\in\mathbb{K}^*$. *Substituting the solution of the equalities obtained in* $\overline{\mathcal{P}}(t)$, *we obtain that the rational curve defined by* $\overline{\mathcal{P}}(t)$ *has a non–ordinary singularity at* $P=(0:0:1)$ *of multiplicity* $\ell_P=m_0+\cdots+m_r$ *and* $(1:\gamma:0)\in\xi_1$. *Depending on the input of the algorithm, we get that* $(1:\gamma:0)$ *is a non–ordinary singularity or not.*

3. *Since* $q_{2,1}(t) = H(t)\overline{p}_{1,1}(t)\overline{p}_{2,1}(t)$*, and we write* $q_{2,1}(t) = H_1(t)\overline{q}_{2,1}(t)$*, we have that* $\overline{q}_{2,1}(s_{j_u}) = 0$ *if and only if* $m_{j_u,1} < m_{j_u}$ *(for some* $j_u \in \{0, \ldots, r\}$*). Thus, if* $m_{j_u,1} < m_{j_u}$ *and* $m_{j_v,1} = m_{j_v}$ *condition in statement 2 of Theorem 4 never hold since* $\overline{q}_{2,1}(s_{j_u}) = 0$ *and* $\overline{q}_{2,1}(s_{j_v}) \neq 0$*.*

In the following, we illustrate Theorem 4 and Remark 7 with Example 6. In particular, we show how to construct a rational curve defined by a parametrization $\overline{\mathcal{P}}(t)$ of degree 7 having a non–ordinary singularity at $P = (0 : 0 : 1)$ of multiplicity 5, and such that there exists a point, $(1 : \gamma : 0) \in \xi_1$, $\gamma \neq 0$, of multiplicity 2.

Example 6. *We construct a rational curve* $\overline{\mathcal{C}}$ *over* $\mathbb{C}$ *defined by a parametrization* $\overline{\mathcal{P}}(t)$ *of degree* $d = 7$ *having a non–ordinary singularity at* $P = (0 : 0 : 1)$ *of multiplicity* 5*, and* $(1 : \gamma : 0) \in \xi_1$, $\gamma \neq 0$, *non–ordinary of multiplicity* 2*. For this purpose, we apply Algorithm Parametrization Construction with Singularity at* $(0 : 0 : 1)$*, with* $s_0 = 0, s_1 = 1, s_2 = 2$, $m_0 = m_1 = 2$, $m_2 = 1$*, the generic polynomials*

$$\overline{p}_{1,1}(t) = a_0 + a_1 t + t^2, \quad \overline{p}_{2,1}(t) = b_0 + b_1 t + t^2,$$

and

$$p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c5t^5 + c_6 t^6 + t^7.$$

The algorithm returns

$$\overline{\mathcal{P}}(t) = (p_{1,1}(t), p_{2,1}(t), p(t)),$$

where

$$p_{i,1}(t) = H(t)\overline{p}_{i,1}(t),\ i = 1, 2,\ \text{with } \gcd(\overline{p}_{1,1}, \overline{p}_{2,1}) = \gcd(H, p) = \gcd(p_{1,1}, p_{2,1}, p) = 1,$$

and

$$H(t) = t^2(t-1)^2(t-2).$$

Observe that, P *is a non–ordinary singularity of* $\overline{\mathcal{C}}$*, and* $\ell_P = m_0 + m_1 + m_2 = 5$ *and* $r_P = 3$*.*

Now, let us make that $(1 : \gamma : 0) \in \xi_1$, $\gamma \neq 0$*, is a non–ordinary point of multiplicity* 2*. For this purpose, we consider the parametrization*

$$\mathcal{L}(\overline{\mathcal{Q}}_1) = (q_{1,1}(t), q_{2,1}(t), q(t)) := (\overline{p}_{1,1}(t)p(t) - \gamma\overline{p}_{2,1}(t)p(t), H(t)\overline{p}_{1,1}(t)\overline{p}_{2,1}(t), \overline{p}_{2,1}(t)p(t)),$$

and we apply Algorithm Parametrization Construction with Singularity at $(0 : 0 : 1)$ *(see statement 2 of Remark 7), with* $s_{0,1} = 0, s_{1,1} = 1$, $m_{0,1} = m_{1,1} = 1$, $\mathcal{S}_1 = \{0, 1\}$*, and the polynomials* $\overline{q}_{1,1}(t), \overline{q}_{2,1}(t)$*, where*

$$q_{i,1}(t) = H_1(t)\overline{q}_{i,1}(t),\ i = 1, 2, \qquad H_1(t) = t(t-1)$$

(note that one has to check that $\gcd(H_1, q) = \gcd(\overline{q}_{1,1}, \overline{q}_{2,1}) = \gcd(q_{1,1}, q_{2,1}, q) = 1$*). Observe that this condition implies that* $(1 : \gamma : 0) \in \xi_1$, $\gamma \neq 0$*, and* $\ell_{P^1} = m_{0,1} + m_{1,1} = 2$*.*

Now, we apply step 2 of the algorithm. Taking into account statement 3 in Remark 7, since $m_{i,1} < m_i$, $i = 0, 1$*, we have that* $\overline{q}_{2,1}(s_{i,1}) = 0$, $i = 0, 1$*, and thus we only get the* *equality*

$$\overline{q}_{1,1}(s_{0,1}) = \alpha_1 \frac{H^{1)}(s_{1,1})}{H^{1)}(s_{0,1})} \overline{q}_{1,1}(s_{1,1}).$$

Solving the above equation, we obtain the parametrization

$$\overline{\mathcal{P}}(t) = (p_{1,1}(t), p_{2,1}(t), p(t)), \text{ where } p_{i,1}(t) = H(t)\overline{p}_{i,1}(t),\ i = 1, 2,$$

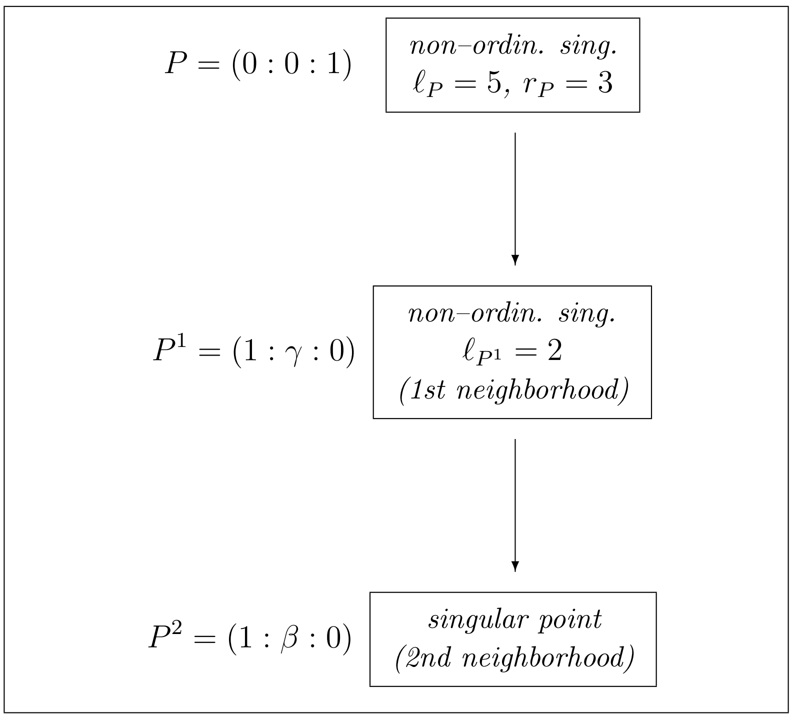
and

$$\overline{p}_{1,1}(t) = (-2\alpha_1 b_0^2 c_0^2 \gamma + 2tb_0 c_0^2 \alpha_1 + t\gamma + 2t\gamma\alpha_1 b_0^2 c_0^2 - 2t^2\alpha_1 b_0 c_0^2)/(-2\alpha_1 b_0 c_0^2)$$

$$\overline{p}_{2,1}(t) = (-2\alpha_1 b_0^2 c_0^2 + t + 2tb_0^2 c_0^2 \alpha_1 + 2tb_0 c_0^2 \alpha_1 - 2t^2\alpha_1 b_0 c_0^2)/(-2\alpha_1 b_0 c_0^2)$$

$$p(t) = -t + c_0 + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + c_6t^6 - c_2t + t^7 - 2tc_0\alpha_1 - tc_0 - tc_4 - tc_5 - tc_6 - tc_3.$$

The curve $\overline{\mathcal{C}}$ *defined by* $\overline{\mathcal{P}}(t)$ *has a non–ordinary singularity at* $P = (0 : 0 : 1)$ *of multiplicity* 5*, and* $P^1 = (1 : \gamma : 0) \in \xi_1$*,* $\gamma \neq 0$*, non–ordinary of multiplicity* 2*. We note that from statement* 1 *of Remark 7, and since condition of step 2 of the algorithm holds for* P^1*, we deduce that there exists* $P^2 \in \xi_2$*. In the following, we show a singularity tree for this example.*



In Figure 6, we plot the output curve $\mathcal{C}$ *for some particular values of the undetermined coefficients, and a detailed view of the singular point* $(0 : 0 : 1)$*.*

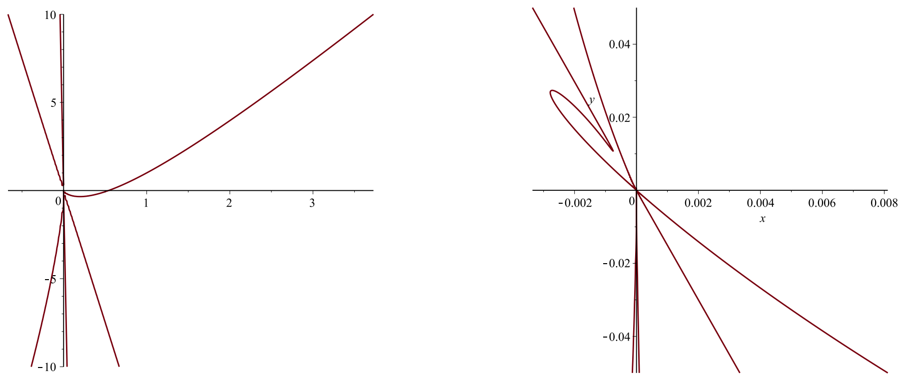


Figure 6: General view of curve $\mathcal{C}$ (left) and detailed view of the singular point $(0:0:1)$ (right)

Step 3

This last step is concerned to the study of the second neighborhood of P and thus, one has to analyze the singularities $(1:\gamma:0) \in \overline{\mathcal{C}}_2$, $\gamma \neq 0$ (see step 3 of the blowing–up process). For this purpose, we consider the parametrization $\mathcal{L}(\overline{\mathcal{Q}}_1)$ and the point P^1, and we reason similarly as we did in step 2 with the parametrization $\overline{\mathcal{P}}(t)$ and the point P. That is, by applying $\mathcal{T}$ to $\mathcal{L}(\overline{\mathcal{Q}}_1)$, we get the projective parametrization of $\overline{\mathcal{C}}_2$ defined by $\overline{\mathcal{Q}}_2(t)$, and we analyze the singularities $(1:\gamma:0) \in \overline{\mathcal{C}}_2$, with $\gamma \neq 0$.

5. Conclusion and future challenges

In this paper, we provide a method that allows to construct parametric curves having (or not) non–ordinary singularities and having (or not) neighboring points. From this approach, we present an algorithm that outputs a parametrization of a rational curve having singularities at some given input points with some additional input information as the order of the singularities, parameters corresponding to the singularities, multiplicity of each parameter, and singularities in the first neighborhood of the given singularities. For simplicity, we outline the algorithm when only an input affine point is considered. If more than one point is given, one may apply recursively the algorithm.

The method presented in this paper is based on the characterization of non–ordinary singularities and neighboring points by means of linear equations involving a given parametrization. We translate every detail of the definitions and resolutions of singularities into the language of parametric equations, which are quite helpful to CAGD group in understanding the singular points.

As a future work, we propose the study of an important problem in the frame of practical designing of engineering and modeling applications. More precisely, we intend to analyze the free parameters obtained in the output parametrization of the algorithm presented. These free parameters allow to model the output curve depending on the practical applied problem one is dealing with (see Examples 2, 5 and 6). To illustrate this idea, one can have a look, for instance, to the approach presented in [19]. Here, one deals with the parametric blending problem for surfaces although the same ideas can be used for the case of curves (we recall that a blending variety is a variety that provides a smooth transition between distinct geometric features of an object). The method presented in [19] provides families of solutions (depending on free parameters) that join smoothly several input parametric varieties. One could use the results obtained in that paper to analyze whether the families of the output solutions we get in this work could be use to provide a smooth transition between some input parametric curves. In addition, in [19], it is also analyzed the structure of the space of blending solutions. As a future work, and using these ideas, we also propose the study of the solution space of the problem considered in this paper. The existence of solutions, the dimension and in general, the analysis of the structure of the solution space obtained in this work, could be interesting parameters to be analyzed in order to apply these results to modeling applications.

Finally, we should mention that the approach obtained here provides effective methods that could be used to address a similar problem to the case of surfaces. For this purpose, one could start by using, for instance, the method developed in [22].

Acknowledgements

This work has been partially funded by Ministerio de Economía y Competitividad under the Project MTM2017-88796-P. The author belongs to the Research Group ASYNACS (Ref. CCEE2011/R34).

We thank the reviewers for their valuable comments, which have improved the manuscript substantially.

[1] Abhyankar, S. (1990). *Algebraic Geometry for Scientists and Engineers.* Mathematical Surveys and Monographs, Vol. 35. American Mathematical Society.

[2] Bernardi, A., Gimigliano, A., Idà, M. (2018). *Singularities of Plane Rational Curves via Projections.* Journal of Symbolic Computation. Vol. 86. pp. 189–214.

[3] Bobenko, A.I., Klein, C. (2011). *Computational Approach to Riemann Surfaces.* Lecture Notes in Mathematics Volume 2013.

[4] Brieskorn, E., Knoerrer, H. (1986). *Plane Algebraic Curves.* Birkhäuser Verlag Basel.

[5] Busé, L., Thang, L.B. (2010). *Matrix–Based Implicit Representations of Rational Algebraic Curves and Applications.* Computer Aided Geometric Design. Vol. 27 (9). pp. 681–699.

[6] Busé, L., D'Andrea, C. (2012). *Singular Factors of Rational Plane Curves.* Journal of Algebra. Vol. 357. , pp. 322–346.

[7] Chen, F. (2006). *Applications of μ–Basis in Geometric Modelling.* In Algebraic Geometry and Geometric Modelling, Barcelona. pp. 9–13.

[8] Chen, F., Wang, W., Liu, Y. (2008) *Computing Singular Points of Plane Rational Curves.* J. Symbolic Comput. Vol. 43 (2). pp. 92–117.

[9] Emiris, I.Z., Sottile, F., Theobald, T. (2010). *Nonlinear Computational Geometry.* The IMA Volumes in Mathematics and its Applications. Vol. 151.

[10] Fulton, W. (1989). *Algebraic Curves – An Introduction to Algebraic Geometry.* Addison-Wesley, Redwood City CA.

[11] Gutierrez, J., Rubio, R., Yu, J-T. (2002). *D-Resultant for Rational Functions.* Proceedings of the American Mathematical Society. Vol. 130(8). pp. 2237–2246.

[12] Hoffmann, C. (1989). *Geometric and Solid Modeling: An Introduction.* Morgan Kaufmann Publishers, Inc. California.

[13] Hoffmann, C.M., Sendra, J.R., Winkler, F. (1997). *Parametric Algebraic Curves and Applications.* J. Symbolic Computation. Vol. 23.

[14] Hoschek, J., Lasser, D. (1993). *Fundamentals of Computer Aided Geometric Design.* A.K. Peters Wellesley MA., Ltd.

[15] Jia, X., Goldman, R. (2009). *μ–Bases and Singularities of Rational Planar Curves.* Computer Aided Geometric Design. Vol. 26(9). pp. 970–988.

[16] Jia, X., Goldman, R. (2012). *Using Smith forms and μ–Bases to Compute all the Singularities of Rational Planar Curves.* Computer Aided Geometric Design. Vol. 29. pp. 296–314.

[17] Jia, X., Goldman, R., Chen, F. (2018). *Survey on the Theory and Applications of μ–Bases for Rational Curves and Surfaces.* Journal of Computational and Applied Mathematics. Vol. 329. pp. 2–23.

[18] Jüttler, B., Piene, R. (2007). *Geometric Modeling and Algebraic Geometry.* Springer Science and Business Media.

[19] Pérez-Díaz, S., Sendra, J.R. (2003). *Computing all Parametric Solutions for Blending Parametric Surfaces.* Journal of Symbolic Computation. Vol. 36/6. pp. 925–964.

[20] Pérez-Díaz, S. (2006). *On the Problem of Proper Reparametrization for Rational Curves and Surfaces.* Computer Aided Geometric Design. Vol. 23(4). pp. 307–323.

[21] Pérez-Díaz, S. (2007). *Computation of the Singularities of Parametric Plane Curves.* Journal of Symbolic Computation. Vol. 42/8. pp. 835–857.

[22] Pérez-Díaz, S., Sendra, J.R., Villarino, C. (2015). *Computing the Singularities of Rational Surfaces.* Mathematics of Computation. Vol. 84 (294), pp. 1991–2021.

[23] Park, H. (2002). *Effective Computation of Singularities of Parametric Affine Curves.* Journal of Pure and Applied Algebra. Vol. 173, pp. 49–58.

[24] Rubio, R., Serradilla, J.M., Vélez, M.P. (2009). *Detecting Real Singularities of a Space Curve from a Real Rational Parametrization.* Journal of Symbolic Computation. Volume 44 (5). pp. 490–498.

[25] Sendra, J.R. (2002). *Normal Parametrizations of Algebraic Plane Curves.* Journal of Symbolic Computation. Vol. 33. pp. 863–885.

[26] Sendra, J.R., Winkler, F., Pérez-Díaz, S. (2007). *Rational Algebraic Curves: A Computer Algebra Approach.* Series: Algorithms and Computation in Mathematics. Vol. 22. Springer Verlag.

[27] Shafarevich, I.R. (1994). *Basic Algebraic Geometry Schemes; 1 Varieties in Projective Space.* Berlin New York: Springer–Verlag. Vol. 1.

[28] Shi, X., Jia, X., Goldman, R. (2013). *Using a Bihomogeneous Resultant to Find the Singularities of Rational Space Curves.* Journal of Symbolic Computation. Vol. 53. pp. 1–25.

[29] Song, N., Chen, F., Goldman, R. (2007). *Axial Moving Lines and Singularities of rational Planar Curves.* Comput. Aided Geom. Design. Vol. 24(4). pp. 200–209.

[30] Van den Essen, A., Yu, J.-T. (1997). *The D–Resultant, Singularities and the Degree of Unfaithfulness.* Proceedings of the American Mathematical Society. Vol. 125 (3), pp. 689–695.

[31] Walker, R.J. (1950). *Algebraic Curves.* Princeton Univ. Press.

[32] Wang, X., Goldman, R. (2015). *Quaternion Rational Surfaces: Rational Surfaces Generated from the Quaternion Product of two Rational Space Curves.* Graphical Models. Vol. 81, Issue C. pp. 18–32.

[33] Zariski, O. (1939). *The Reduction of Singularities of Algebraic Surfaces.* Annals of Mathematics. Vol. 40. pp. 639–689.