Pérez Díaz, S. 2006, "On the problem of proper reparametrization for rational curves and surfaces", Computer Aided Geometric Design, vol. 23, no. 4, pp. 307-323.

Available at https://doi.org/10.1016/j.cagd.2006.01.001

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On the Problem of Proper Reparametrization for Rational Curves and Surfaces*

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Abstract

A rational parametrization of an algebraic curve (resp. surface) establishes a rational correspondence of this curve (resp. surface) with the affine or projective line (resp. affine or projective plane). This correspondence is a birational equivalence if the parametrization is proper. So, intuitively speaking, a rational proper parametrization trace the curve or surface once. We consider the problem of computing a proper rational parametrization from a given improper one. For the case of curves we generalize, improve and reinterpret some previous results. For surfaces, we solve the problem for some special surface's parametrizations.

1 Introduction

Unirational algebraic varieties, in particular curves and surfaces, play an important role in the frame of practical applications (see [14], [15]). Many authors have addressed problems related to the construction of conversion algorithms; i.e. algorithmic methods that change from the implicit representation to the parametric one, and vice versa (see [4], [5], [12], [13], [14], [18], [22], [25], [26], [27], etc).

In addition, if one considers rational parametrizations as rational mappings from an affine space onto the variety, three natural questions appear. First, deciding whether the mapping is birational, i.e. whether the parametrization is proper or invertible (properness problem); secondly, in case of birationality, compute the inverse of the

*Author partially supported by DGES PB98-0713-C02-01 and DGES HU1999-0029.

parametrization (inversion problem). Finally, if the parametrization is not proper, the question of reparametrizing such a variety as to make it properly parametrized is considered (proper reparametrization problem).

These problems, in particular when the variety is a curve or a surface, are specially interesting in some practical applications in C.A.G.D where objects are often given and manipulated parametrically. In addition, proper parametrizations play an important role in many practical applications in computer aided geometric design, such as in visualization (see [14], [15]) or rational parametrization of offsets (see [2]). Also, they provide an implicitization approach based on resultants (see [6] and [28]).

An algorithmic approach to the two first problems based on Gröbner Basis, can be found in [23]. For plane curves, the three problems are directly related to Lüroth's Theorem, that is valid over any field, and different algorithmic procedures to solve the problem can be found in [1], [8], [9], [11], [16], [24], [28] or [29]. For rational maps between algebraic surfaces a solution of the properness problem and inversion problem can be found in [19] and [20]. For the reparametrization problem, although it is known from Castelnuovo's Theorem that unirationality and rationality are equivalent over algebraically closed fields, algorithmic questions and approaches are still required.

In this paper we deal with the proper reparametrization problem for the case of algebraic curves and surfaces. A direct approach to the reparametrization problem could consist in implicitizating the parametrization (see [4], [5], [18], [28]) to apply afterwards algorithms developed for instance in [7], [11], [12], [13], [14], [22], [26], [27], to the implicit equation. This solution might be too time consuming and then, we would like to approach the problem by means of rational reparametrizations. With rational reparametrization we basically mean without implicitizating, or more formally, by finding a non–constant rational change of parameter, if it exists, that transforms the input parametrization onto a new parametrization of the same curve or surface that solves the problem. Note that any reparametrization of a rational parametrization is again a parametrization of the same variety.

For the case of curves, it is always possible to reparametrize an improperly parametrized curve in such a way that it becomes properly parametrized. In this paper, we present a new approach to compute a proper parametrization from a given improper one. This new method improves and reinterprets some previous results (see for instance [1], [9] or [24]). We have implemented these ideas, and we have compared our method with the methods in [9] and [24].

For the case of surfaces defined over algebraically closed fields, we solve the problem of proper reparametrization for some special surface's parametrizations. The basic idea is to compute two univariate resultants of certain curves directly constructed from the given parametrization. In addition, from these results, we derive one algorithm for computing the rational proper reparametrization. We have implemented this algorithm in Maple and we show that running times are very satisfactory.

More precisely, the structure of the paper is as follows: In Section 2, we present a new approach that deal with the proper reparametrization problem for the case of curves. In Section 3, we outline the algorithm, and we illustrate it with an example. We also show the actual computing times of the implementation, and we compare the method with previous algorithms. Section 4 focuses on the problem of proper reparametrization for improperly parametrized surfaces. In particular, we show how to solve the problem for some special surface's parametrizations. Section 5 is devoted to outline the algorithm for the case of surfaces, and to illustrate it with an example. Actual computing times of the implementation are also provided. Finally, in Section 6, we summarize the main idea, contributions and advantages of the approach presented in the paper, and we describe the future work.

2 The Problem of Proper Reparametrization for Curves

The problem of proper reparametrization for curves can be stated as follows: given a field $\mathbb{K}$, and a rational parametrization $\mathcal{P}(t) \in \mathbb{K}(t)^2$ of an algebraic plane curve $\mathcal{C}$, find a rational proper parametrization $\mathcal{Q}(t) \in \mathbb{K}(t)^2$ of $\mathcal{C}$, and a rational function $R(t) \in \mathbb{K}(t) \setminus \mathbb{K}$ such that $\mathcal{P}(t) = \mathcal{Q}(R(t))$.

This section is preliminar, and we present some results that will be used to prove the correctness of the algorithm for reparametrizing a curve stated in Section 3.

For this purpose, first we introduce the notation that we will use throughout Sections 2 and 3. Afterwards, we state three lemmas that deal with properties on gcd's and resultants. From these results, we show, in Theorem 1, that the rational function $R(t) \in \mathbb{K}(t) \setminus \mathbb{K}$ satisfying that $\mathcal{P}(t) = \mathcal{Q}(R(t))$ can be determined by any pair of the coefficients of a gcd. Finally, we show how to compute the proper rational parametrization $\mathcal{Q}(t)$ (see Theorems 2 and 3).

Notation

Let $\mathbb{K}$ be a field, and let $\mathbb{K}^\star = \mathbb{K} \setminus \{0\}$. If $\mathcal{C}$ is an affine rational plane curve, and $\mathcal{P}(t)$ is a rational affine parametrization of $\mathcal{C}$ over $\mathbb{K}$, we write its components as

$$\mathcal{P}(t) = \left(\frac{p_{1,1}(t)}{p_{1,2}(t)}, \frac{p_{2,1}(t)}{p_{2,2}(t)} \right),$$

where $p_{i,1}(t), p_{i,2}(t) \in \mathbb{K}[t]$, and $\gcd(p_{i,1}, p_{i,2}) = 1$ for $i = 1, 2$. For simplicity, we assume w.l.o.g. that none of $p_{i,1}/p_{i,2}$ is constant. Note that, if for instance $p_{1,1}/p_{1,2} = \lambda \in \mathbb{K}$, then a proper parametrization of $\mathcal{C}$ is $\mathcal{Q}(t) = (\lambda, t)$, and then problem is trivial.

Furthermore, associated with the given parametrization $\mathcal{P}$, we consider the polynomials

$$H_1^{\mathcal{P}}(t,s) = p_{1,1}(t)p_{1,2}(s) - p_{1,2}(t)p_{1,1}(s), \;\; H_2^{\mathcal{P}}(t,s) = p_{2,1}(t)p_{2,2}(s) - p_{2,2}(t)p_{2,1}(s).$$

as well as $S^{\mathcal{P}}(t,s) = \gcd(H_1^{\mathcal{P}}(t,s), H_2^{\mathcal{P}}(t,s))$. The polynomial $S^{\mathcal{P}}$ plays an important role in deciding whether a parametrization $\mathcal{P}$ is proper; i.e. in studying whether the parametrization is injective for almost all parameter values. More precisely, it holds that $\mathcal{P}$ is proper if and only if, up to constants in $\mathbb{K}^\star$, $S^{\mathcal{P}}(t,s) = t - s$ (see [24], [28]).

Finally, we introduce the polynomials $G_i^{\mathcal{P}}(t, x_i) = x_i p_{i,2}(t) - p_{i,1}(t)$, for $i = 1, 2$.

Taking into account that every space curve is birationally equivalent to a plane curve (see e.g. Theorem 6.5 in [30]), we restrict the discussion to plane curves. Most of the results presented here can be easily extended to space curves.

The first lemma we state is known as the *base change formula for resultants*, and it can be found in [17].

Lemma 1 *Let P, $Q \in (\mathbb{K}[s])[t] \setminus \mathbb{K}$ be polynomials over $\mathbb{K}[s]$ with $\deg_t(P) = m$, and $\deg_t(Q) = n$. Let $R(t) = M(t)/N(t) \in \mathbb{K}(t)$ be a non–constant rational function in reduced form, such that $\deg_t(M - \beta N) = \deg_t(R)$ for every root β for the unknown t of the polynomial $P(t,s)Q(t,s)$. Let $P'(t,s)$ and $Q'(t,s)$ be the polynomials*

$$P'(t,s) = P(R(t), s)N(t)^m, \qquad Q'(t,s) = Q(R(t), s)N(t)^n.$$

Then, if a, b are the leading coefficient of Q' and Q, respectively, w.r.t the variable t,

$$\mathrm{Res}_t(P', Q') = \frac{a^{m(\deg(R)-\deg(N))}}{b^{\deg(R)m}} \cdot \mathrm{Res}_t(P,Q)^{\deg(R)} \cdot \mathrm{Res}_t(Q', N)^m. \qquad \square$$

Remark 1 *We observe that if the polynomial $P(t,s)Q(t,s)$ does not have factors in $\mathbb{K}[t]$ then, every root β for the unknown t of the polynomials $P(t,s)Q(t,s)$ is in $\overline{\mathbb{K}(s)}$ which implies that $\deg_t(M - \beta N) = \deg_t(R)$.* $\square$

Lemma 2 *Let $P, Q \in \mathbb{K}[t,s] \setminus \mathbb{K}$ be polynomials such that $\gcd(P,Q) = 1$, and let $R(s) = M(s)/N(s) \in \mathbb{K}(s)$ be a non–constant rational function in reduced form such that $\deg_s(M - \beta N) = \deg_s(R)$ for every root β for the unknown s of the polynomial $P(t,s)Q(t,s)$. Let $P^\star(t,s)$ and $Q^\star(t,s)$ be the polynomials*

$$P^\star(t,s) = P(t, R(s))N(s)^r, \qquad Q^\star(t,s) = Q(t, R(s))N(s)^l,$$

where $r := \deg_s(P)$, and $l := \deg_s(Q)$. Then, it holds that $\gcd(P^\star, Q^\star) = 1$.

Proof: First, we express the polynomials P, Q as

$$P(t,s) = a_r(t)s^r + \cdots + a_0(t), \quad Q(t,s) = b_l(t)s^l + \cdots + b_0(t).$$

Then, $P^\star(t,s) = a_r(t)M(s)^r + \cdots + a_0(t)N(s)^r, \quad Q^\star(t,s) = b_l(t)M(s)^l + \cdots + b_0(t)N(s)^l$. In these conditions, we assume that $K(t,s) := \gcd(P^\star, Q^\star) \neq 1$, and we distinguish three different cases.
(1.) If $K(t,s) \in \mathbb{K}[s]$, then $\gcd(M,N) \neq 1$ which is impossible. Thus, $K(t,s) \notin \mathbb{K}[s]$.
(2.) If $K(t,s) \in \mathbb{K}[t]$, then $K(t)$ divides $\gcd(a_r, \ldots, a_0, b_l, \ldots, b_0)$ which implies that $\gcd(P,Q) \neq 1$. This is impossible, and then $K(t,s) \notin \mathbb{K}[t]$.
(3.) Statements 1 and 2 imply that $K(t,s) \in \mathbb{K}[t,s]$ and depends on both t and s. Then, by the properties of resultants (see [3]), we have that $\mathrm{Res}_s(P^\star, Q^\star) = 0$. In these conditions, since $\deg_s(M - \beta N) = \deg_s(R)$ for every root β for the unknown s of the polynomial $P \cdot Q$, we may apply Lemma 1 (in this case for polynomials $P, Q \in (\mathbb{K}[t])[s]$), and we get that either $\mathrm{Res}_s(P,Q) = 0$ or $\mathrm{Res}_s(Q^\star, N) = 0$. The first case implies that $\gcd(P,Q) \neq 1$ (see [3]) which is impossible. The second case implies that $\gcd(Q^\star, N) \neq 1$ and then $\gcd(M,N) \neq 1$. Thus, this case is also impossible.
Hence, we conclude that $\gcd(P^\star, Q^\star) = 1$. □

By Lüroth's Theorem, we have that there exists a rational proper parametrization

$$\mathcal{U}(t) = \left(\frac{u_{1,1}(t)}{u_{1,2}(t)}, \frac{u_{2,1}(t)}{u_{2,2}(t)} \right) \in \mathbb{K}(t)^2$$

of $\mathcal{C}$ such that $\mathcal{P}(t) = \mathcal{U}(B(t))$, where $B(t) = M(t)/N(t) \in \mathbb{K}(t) \setminus \mathbb{K}$, and $\gcd(M,N) = 1$.

In the following lemma, we show the relation between the polynomial $S^{\mathcal{P}}(t,s)$, the polynomial $S^{\mathcal{U}}(t,s)$, and the rational function $B(t)$. Observe that since $\mathcal{U}$ is a proper parametrization, then $S^{\mathcal{U}}(t,s) = t - s$ (see [24] or [28]).

Lemma 3 *It holds that, up to constants in* $\mathbb{K}^\star$, $S^{\mathcal{P}}(t,s) = N(s)M(t) - M(s)N(t)$.

Proof: First, for $j = 1, 2$, we consider the polynomials

$$H_j^{\mathcal{U}}(t,s) = u_{j,1}(t)u_{j,2}(s) - u_{j,1}(s)u_{j,2}(t),$$

and we denote by $m_j := \max\{\deg(u_{j,1}(t)), \deg(u_{j,2}(t))\} = \deg_t(H_j^{\mathcal{U}})$ (note that $m_j \geq 1$ since we have assumed that none $p_{i,1}/p_{i,2}$ is constant). Now, from $\mathcal{P}(t) = \mathcal{U}(B(t))$, we deduce that

$$H_j^{\mathcal{P}}(t,s) = H_j^{\mathcal{U}}(B(t), B(s))N(t)^{m_j}N(s)^{m_j}.$$

Then, $S^{\mathcal{P}}(t,s) = \gcd(H_1^{\mathcal{P}}(t,s),\ H_2^{\mathcal{P}}(t,s)) =$

$$\gcd(H_1^{\mathcal{U}}(B(t), B(s))N(t)^{m_1}N(s)^{m_1},\ H_2^{\mathcal{U}}(B(t), B(s))N(t)^{m_2}N(s)^{m_2}) \qquad \text{(I)}.$$

On the other side, since $S^{\mathcal{U}}(t,s)=\gcd(H_1^{\mathcal{U}}(t,s),\ H_2^{\mathcal{U}}(t,s))$, we deduce that for $j=1,2$,

$$H_j^{\mathcal{U}}(t,s)=S^{\mathcal{U}}(t,s)A_j^{\mathcal{U}}(t,s)\quad \text{with } \gcd(A_1^{\mathcal{U}},\ A_2^{\mathcal{U}})=1, \text{ and } A_j^{\mathcal{U}}\in\mathbb{K}[t,s].$$

Observe that since $\mathcal{U}$ is a proper parametrization, then $\deg_t(S^{\mathcal{U}})=1$ which implies that $\deg_t(A_j^{\mathcal{U}})=m_j-1$ (note that $m_j=\deg_t(H_j^{\mathcal{U}})\geq 1$). Moreover, it holds that the polynomials $A_j^{\mathcal{U}}$ do not have factors in $\mathbb{K}[t]$ neither in $\mathbb{K}[s]$. Indeed, let us assume that $K(t)\in\mathbb{K}[t]$ is a factor of the polynomial $A_j^{\mathcal{U}}$ (similarly we reason for a factor in $\mathbb{K}[s]$). Then, K is a factor of the polynomial $H_j^{\mathcal{U}}$ which is impossible since $\gcd(u_{j,1},u_{j,2})=1$. In these conditions, and taking into account that up to constants in $\mathbb{K}^\star$, $S^{\mathcal{U}}(t,s)=t-s$, we get that

$$H_j^{\mathcal{U}}(B(t),B(s))N(t)^{m_j}N(s)^{m_j}=S^{\mathcal{U}}(B(t),B(s))\cdot A_j^{\mathcal{U}}(B(t),B(s))\cdot N(t)^{m_j}N(s)^{m_j}=$$

$$(N(s)M(t)-M(s)N(t))\cdot A_j^{\mathcal{U}}(B(t),B(s))\cdot N(t)^{m_j-1}N(s)^{m_j-1}.$$

Therefore, from (I), we deduce that $S^{\mathcal{P}}(t,s)=(N(s)M(t)-M(s)N(t))\cdot$

$$\cdot\gcd(A_1^{\mathcal{U}}(B(t),B(s))N(t)^{m_1-1}N(s)^{m_1-1},A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1}).$$

Thus, the statement of lemma holds if we prove that

$$\gcd(A_1^{\mathcal{U}}(B(t),B(s))N(t)^{m_1-1}N(s)^{m_1-1},\ A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1})=1,$$

which is equivalent to prove that

$$\mathrm{Res}_t(A_1^{\mathcal{U}}(B(t),B(s))N(t)^{m_1-1}N(s)^{m_1-1},\ A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1})\neq 0.$$

For this purpose, first note that if $m_j=1$ for some j, then $A_j^{\mathcal{U}}\in\mathbb{K}^\star$, and then the above statement follows trivially. Let us assume that $m_j\geq 2$ which implies that $\deg_t(A_j^{\mathcal{U}})=m_j-1\geq 1$. In these conditions, we apply Lemma 1 to the polynomials $P=A_1^{\mathcal{U}}(t,B(s))N(s)^{m_1-1}$, and $Q=A_2^{\mathcal{U}}(t,B(s))N(s)^{m_2-1}$, and the rational function $R(t)=B(t)$. We observe that since the polynomials $A_j^{\mathcal{U}}(t,s)$ do not have factors in $\mathbb{K}[t]$, then the polynomials $A_j^{\mathcal{U}}(t,B(s))N(s)^{m_j-1}$ do not have factors in $\mathbb{K}[t]$, which implies that $\deg_t(M-\beta N)=\deg_t(B)$ for every root β for the unknown t of PQ (see Remark 1). Hence, from Lemma 1, we deduce that

$$\mathrm{Res}_t(A_1^{\mathcal{U}}(B(t),B(s))N(t)^{m_1-1}N(s)^{m_1-1},\ A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1})=$$

$$\frac{a^{(m_1-1)(\deg(B)-\deg(N))}}{b^{(m_1-1)\deg(B)}}\cdot\mathrm{Res}_t(A_1^{\mathcal{U}}(t,B(s))N(s)^{m_1-1},\ A_2^{\mathcal{U}}(t,B(s))N(s)^{m_2-1})^{\deg(B)}\cdot$$

$$\mathrm{Res}_t(A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1},\ N(t))^{(m_1-1)},$$

where a and b are the leading coefficient of $A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1}$ and $A_2^{\mathcal{U}}(t,B(s))N(s)^{m_2-1}$, respectively, w.r.t the variable t. In these conditions, we first observe that

$$\mathrm{Res}_t(A_1^{\mathcal{U}}(t,B(s))N(s)^{m_1-1},\ A_2^{\mathcal{U}}(t,B(s))N(s)^{m_2-1})\neq 0,$$

since $\gcd(A_1^{\mathcal{U}}(t,s),\, A_2^{\mathcal{U}}(t,s)) = 1$ which implies, by Lemma 2, that $\gcd(A_1^{\mathcal{U}}(t,B(s))N(s)^{m_1-1},\ A_2^{\mathcal{U}}(t,B(s))N(s)^{m_2-1}) = 1$. Furthermore, we also have that

$$\mathrm{Res}_t(A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1},\ N(t)) \neq 0.$$

Indeed, let $A_2^{\mathcal{U}}(t,s) := a_{m_2-1}(s)t^{m_2-1} + \cdots + a_0(s)$. Then,

$$A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1} = a'_{m_2-1}(s)M(t)^{m_2-1} + \cdots + a'_0(s)N(t)^{m_2-1}.$$

Taking into account that $\gcd(M,N) = 1$, we deduce that

$$\gcd(A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1},\ N(t)) = 1.$$

Therefore, we conclude that

$$\mathrm{Res}_t(A_1^{\mathcal{U}}(B(t),B(s))N(t)^{m_1-1}N(s)^{m_1-1},\ A_2^{\mathcal{U}}(B(t),B(s))N(t)^{m_2-1}N(s)^{m_2-1}) \neq 0 \qquad \square.$$

Remark 2 *Lemma 3 can be interpreted as follows: by Lüroth's Theorem, we have that* $\mathbb{K}(p_{1,1}/p_{1,2},\, p_{2,1}/p_{2,2}) = \mathbb{K}(B(s)) \subset \mathbb{K}(s)$. *Let* $P[Z]$ *be the minimal polynomial of* s *over* $\mathbb{K}(p_{1,1}/p_{1,2},\, p_{2,1}/p_{2,2})$, *and* $P_i[Z]$ *the minimal polynomial of* s *over* $\mathbb{K}(p_{i,1}/p_{i,2})$, *for* $i = 1, 2$. *Then, Lemma 3 implies that* $P = \gcd(P_1, P_2)$ *in* $\mathbb{K}(p_{1,1}/p_{1,2},\, p_{2,1}/p_{2,2})[Z]$. $\square$

In the following, we express the polynomials N, M defining the rational function $B(t) = M(t)/N(t)$ as

$$M(t) = a_m t^m + \cdots + a_0, \quad N(t) = b_m t^m + \cdots + b_0, \qquad a_i, b_i \in \mathbb{K}$$

with $a_m \neq 0$ or $b_m \neq 0$. By Lemma 3, we deduce that, up to constants in $\mathbb{K}^\star$,

$$S^{\mathcal{P}}(t,s) = C_m(t)s^m + C_{m-1}(t)s^{m-1} + \cdots + C_0(t),$$

where $C_j(t) = a_j N(t) - b_j M(t)$, for $j = 0, \ldots, m$. In these conditions, we have the following theorem.

Theorem 1 *The following statements are equivalents*

1. $a_{j_0}b_{i_0} \neq a_{i_0}b_{j_0}$,
2. $\gcd(C_{j_0},\, C_{i_0}) = 1$,
3. $C_{j_0}(t)$, $C_{i_0}(t)$ *are not associated polynomials.*

Moreover, if $\gcd(M,\, N) = 1$, *there exist* $a_{j_0}, b_{j_0}, a_{i_0}, b_{i_0} \in \mathbb{K}$ *such that* $a_{j_0}b_{i_0} \neq a_{i_0}b_{j_0}$.

Proof: First, we prove that the assumption $\gcd(M, N) = 1$, implies that there exist $a_{j_0}, b_{j_0}, a_{i_0}, b_{i_0} \in \mathbb{K}$ satisfying that $a_{j_0} b_{i_0} \neq a_{i_0} b_{j_0}$. Let us assume that $a_j b_i - a_i b_j = 0$ for every $i, j \in \{0, \dots, m\}$. This implies, that the rank of the matrix $2 \times (m+1)$ which rows are the coefficients of the polynomials M, N is equal to 1. Thus $\gcd(M, N) \neq 1$ which is impossible. Now, let us prove the equivalence of statements.

$\boxed{(1) \Rightarrow (2)}$ Let us assume that $K(t) := \gcd(C_{j_0}, C_{i_0}) \neq 1$. Then,

$$C_{j_0}(t) = a_{j_0} N(t) - b_{j_0} M(t) = K(t) M_1(t), \quad C_{i_0}(t) = a_{i_0} N(t) - b_{i_0} M(t) = K(t) M_2(t), \quad \text{(I)}$$

with $\gcd(M_1, M_2) = 1$. Thus, $(a_{i_0} b_{j_0} - a_{j_0} b_{i_0}) M(t) = K(t)(a_{j_0} M_2(t) - a_{i_0} M_1(t))$. Since $a_{i_0} b_{j_0} - a_{j_0} b_{i_0} \neq 0$, and $M(t) \neq 0$, we deduce that $K(t)$ divides to $M(t)$. Therefore, from (I) we conclude that $\gcd(M, N) \neq 1$, which is impossible. Thus, $\gcd(C_{j_0}, C_{i_0}) = 1$.

$\boxed{(2) \Rightarrow (3)}$ If C_{j_0}, C_{i_0} are associated polynomials, then $C_{j_0} = k\, C_{i_0}$, $k \in \mathbb{K}$. Thus, $\gcd(C_{j_0}, C_{i_0}) \neq 1$ which is impossible. Therefore, C_{j_0}, C_{i_0} are not associated.

$\boxed{(3) \Rightarrow (1)}$ Let us assume that $a_{j_0} b_{i_0} = a_{i_0} b_{j_0}$, and we distinguish two different cases.

(a.) If $b_{i_0} \neq 0$, then $a_{j_0} = a_{i_0} b_{j_0} / b_{i_0}$. Thus,

$$C_{j_0}(t) = a_{j_0} N(t) - b_{j_0} M(t) = b_{j_0}/b_{i_0}(a_{i_0} N(t) - b_{i_0} M(t)) = b_{j_0}/b_{i_0} C_{i_0}(t),$$

which implies that C_{j_0}, C_{i_0} are associated. Therefore, this case is impossible.

(b.) Let $b_{i_0} = 0$. Since $a_{j_0} b_{i_0} = a_{i_0} b_{j_0}$, we deduce that $a_{i_0} = 0$ or $b_{j_0} = 0$. If $a_{i_0} = 0$ then $C_{i_0}(t) = 0$, which implies that C_{j_0}, C_{i_0} are associated polynomials. If $b_{j_0} = 0$, then $C_{j_0}(t) = a_{j_0} N(t)$, and $C_{i_0}(t) = a_{i_0} N(t)$ from where we also deduce that C_{j_0}, C_{i_0} are associated polynomials. Therefore, this case is also impossible. □

From Theorem 1, we get the next corollary that plays an important role in Section 4.

Corollary 1 *Let $H(t, s) = D_m(t) s^m + D_{m-1}(t) s^{m-1} + \cdots + D_0(t) \in (\mathbb{K}(t))[s]$ such that there exist $i_0, j_0 \in \{0, \dots, m\}$ satisfying that $\gcd(D_{i_0}, D_{j_0}) = 1$, and $D_{i_0} \in \mathbb{K}[t] \setminus \mathbb{K}$ or $D_{j_0} \in \mathbb{K}[t] \setminus \mathbb{K}$. Thus, there exists polynomials $M(t), N(t) \in \mathbb{K}[t]$, not both constant with $\gcd(M, N) = 1$, satisfying that*

$$H(t, s) = M(t) N(s) - M(s) N(t),$$

if and only if

$$D_{i_0}(t) D_{j_0}(s) - D_{i_0}(s) D_{j_0}(t) = c\, H(t, s), \qquad c \in \mathbb{K}^{\star}.$$

Proof: First, we assume that there exists polynomials $M(t), N(t) \in \mathbb{K}[t]$, with $\gcd(M, N) = 1$, satisfying that $H(t, s) = M(t) N(s) - M(s) N(t)$. In these conditions, we express the polynomials N, M as

$$M(t) = a_m t^m + \cdots + a_0, \quad N(t) = b_m t^m + \cdots + b_0,$$

with $a_m \neq 0$ or $b_m \neq 0$. Then,

$$H(t,s) = D_m(t)s^m + D_{m-1}(t)s^{m-1} + \cdots + D_0(t) \in (\mathbb{K}(t))[s],$$

where $D_j(t) = a_jN(t) - b_jM(t)$, for $j = 0,\ldots,m$. Observe that

$$D_i(t)D_j(s) - D_i(s)D_j(t) = (a_ib_j - a_jb_i)(M(t)N(s) - M(s)N(t)) = (a_ib_j - a_jb_i)H(t,s),$$

for every $i,j \in \{0,\ldots,m\}$. Thus, since $\gcd(D_{i_0}, D_{j_0}) = 1$, by applying Theorem 1 (note that the polynomial H is of the same form than the polynomial $S^{\mathcal{P}}$, and then we may apply Theorem 1), we have that $a_{i_0}b_{j_0} - a_{j_0}b_{i_0} \neq 0$. Thus

$$D_{i_0}(t)D_{j_0}(s) - D_{i_0}(s)D_{j_0}(t) = c\,H(t,s), \qquad c \in \mathbb{K}^{\star}.$$

Conversely, since $D_{i_0}(t)D_{j_0}(s) - D_{i_0}(s)D_{j_0}(t) = c\,H(t,s)$, $c \in \mathbb{K}^{\star}$, and $\gcd(D_{i_0}, D_{j_0}) = 1$, the statement follows by taking $M(t) = D_{i_0}(t)/c$, and $N(t) = D_{j_0}(t)$. Observe that since D_{i_0}, D_{j_0} are not both constant then M, N are not both constant. □

Taking into account Theorem 1, we consider the rational function

$$R(t) = \frac{C_{i_0}(t)}{C_{j_0}(t)} = \frac{a_{i_0}N(t) - b_{i_0}M(t)}{a_{j_0}N(t) - b_{j_0}M(t)} \in \mathbb{K}(t) \setminus \mathbb{K},$$

where $a_{i_0}b_{j_0} \neq a_{j_0}b_{i_0}$, and C_{i_0}, C_{j_0} are coefficients of the polynomial

$$S^{\mathcal{P}}(t,s) = C_m(t)s^m + C_{m-1}(t)s^{m-1} + \cdots + C_0(t).$$

In these conditions, we have the following theorem.

Theorem 2 *There exists a proper parametrization $\mathcal{Q}(t)$ of the curve $\mathcal{C}$ satisfying that $\mathcal{P}(t) = \mathcal{Q}(R(t))$.*

Proof: First, we observe that we may express $R(t) = g(B(t))$, where $g(t) = (b_{i_0}t - a_{i_0})/(b_{j_0}t - a_{j_0})$. Since $a_{j_0}b_{i_0} \neq a_{i_0}b_{j_0}$ (see Theorem 1), we get that $g(t)$ is invertible. In these conditions, we consider $\mathcal{Q} = \mathcal{U}(g^{-1})$, and we prove that $\mathcal{Q}$ is a proper parametrization of $\mathcal{C}$. Indeed, first note that

$$\mathcal{Q}(R(t)) = \mathcal{U}(g^{-1}(t)) \circ R(t) = \mathcal{U}(g^{-1}(t)) \circ g(B(t)) = \mathcal{U}(B(t)) = \mathcal{P}(t),$$

and then $\mathcal{Q}$ parametrizes $\mathcal{C}$. In addition, since $\mathcal{U}$ and g are invertible, we get that $\mathcal{Q} = \mathcal{U}(g^{-1})$ is proper. □

Once the rational function $R(t) = r_1(t)/r_2(t)$, with $\gcd(r_1, r_2) = 1$, is computed, one has to determine the proper rational parametrization $\mathcal{Q}(t) \in \mathbb{K}(t)^2$ of the curve $\mathcal{C}$, satisfying that $\mathcal{P} = \mathcal{Q}(R)$ (note that $\mathcal{Q}$ exists by Theorem 2). For this purpose, one may use the method of undetermined coefficients as in [9] or [24]. However in the following theorem, we show an alternative method based on univariate resultants that provides running times more satisfactory than the known algorithms (see Section 3).

Theorem 3 *For* $i = 1, 2$, *let* $L_i(s, x_i) = \mathrm{Res}_t(G_i^{\mathcal{P}}(t, x_i), sr_2(t) - r_1(t))$. *It holds that, up to constants in* $\mathbb{K}^\star$

$$L_i(s, x_i) = (q_{i,2}(s)x_i - q_{i,1}(s))^{\deg(R)},$$

where $\mathcal{Q}(s) = (q_{1,1}(s)/q_{1,2}(s),\, q_{2,1}(s)/q_{2,2}(s))$ *is the proper parametrization, in reduced form, given by Theorem 2.*

Proof: Observe that the implicit equation of the curve parametrized by $(p_{i,1}(t)/p_{i,2}(t),\, B(t))$ is given by $xq_{i,2}(y) - q_{i,1}(y)$, and apply Theorem 13 in [10]. □

3 Algorithm of Proper Reparametrization for Curves

In this section, we apply the results obtained in Section 2 to derive an algorithm that computes a rational proper reparametrization of an improperly parametrized algebraic plane curve. We outline this approach, and we illustrate it with an example. We finish this section by comparing our method with the methods in [9] and [24].

Algorithm Proper Reparametrization for Curves.

Given a rational affine parametrization $\mathcal{P}(t) = (p_{1,1}(t)/p_{1,2}(t),\, p_{2,1}(t)/p_{2,2}(t))$, in reduced form, of a plane algebraic curve $\mathcal{C}$, the algorithm computes a rational proper parametrization $\mathcal{Q}(t)$ of $\mathcal{C}$, and a rational function $R(t)$ such that $\mathcal{P}(t) = \mathcal{Q}(R(t))$.

1. Compute $H_j^{\mathcal{P}}(t, s) = p_{j,1}(t)p_{j,2}(s) - p_{j,1}(s)p_{j,2}(t), \quad j = 1, 2.$
2. Determine the polynomial $S^{\mathcal{P}}(t, s) = \gcd(H_1^{\mathcal{P}}(t, s), H_2^{\mathcal{P}}(t, s)) = C_m(t)s^m + \cdots + C_0(t)$. Let $m := \deg_t(S^{\mathcal{P}}(t, s))$.
3. If $m = 1$, then return $\mathcal{Q}(t) = \mathcal{P}(t)$, and $R(t) = t$. Otherwise go to Step 4.
4. Consider a rational function $R(t) = \frac{C_{i_0}(t)}{C_{j_0}(t)} \in \mathbb{K}(t)$, such that $C_{j_0}(t)$, $C_{i_0}(t)$ are not associated polynomials.
5. For $i = 1, 2$, determine the polynomials

$$L_i(s, x_i) = \mathrm{Res}_t(G_i^{\mathcal{P}}(t, x_i), sC_{j_0}(t) - C_{i_0}(t)) = (q_{i,2}(s)x_i - q_{i,1}(s))^{\deg(R)},$$

   where $G_i^{\mathcal{P}}(t, x_i) = x_i p_{i,2}(t) - p_{i,1}(t)$.
6. Return $\mathcal{Q}(t) = (q_{1,1}(t)/q_{1,2}(t),\, q_{2,1}(t)/q_{2,2}(t))$, and $R(t) = C_{i_0}(t)/C_{j_0}(t)$.

The algorithm follows directly from Lemma 3 in Step 2, Theorem 1 in Step 4, and Theorems 2 and 3 in Step 5. Step 3 is obtained by previous results (see [24] or [28]). In the following, we illustrate Algorithm Proper Reparametrization for Curves with an example.

Example 1

Let $\mathcal{C}$ be the rational curve defined by the parametrization

$$\mathcal{P}(t) = \left(\frac{p_{1,1}(t)}{p_{1,2}(t)}, \frac{p_{2,1}(t)}{p_{2,2}(t)}\right) = \left(\frac{3t^4+4t^3+32t^2+28t+99}{(t^2+t+7)(t^2+1)}, \frac{(t^2+t+7)^3}{(t+6)(t^2+1)^2}\right).$$

In Step 1 of the algorithm, we compute the polynomials

$H_1^{\mathcal{P}}(t,s) = -97t+97s+71t^3+568t^2+78t^4-71s^3-568s^2-78s^4-8s^4t^2-25s^4t+8s^2t^4-192s^2t+25st^4+192st^2+s^3t^4-24s^3t-s^4t^3+24st^3$,

$H_2^{\mathcal{P}}(t,s) = -539t+539s-6t^6+325t^5+428t^3+3108t^2+1914t^4+6s^6-325s^5-428s^3-3108s^2-1914s^4-s^5t^6-6s^5t^4-132s^5t^2-144s^5t-6s^4t^6+6s^4t^5-720s^4t^2-858s^4t-12s^2t^6+132s^2t^5+720s^2t^4-1596s^2t-st^6+144st^5+858st^4+1596st^2+s^6t^5+6s^6t^4+12s^6t^2+s^6t-2s^3t^6+37s^3t^5+210s^3t^4+180s^3t^2-251s^3t-37s^5t^3-210s^4t^3-180s^2t^3+251st^3+2s^6t^3$,

Now, we determine $S^{\mathcal{P}}(t,s)$. We obtain

$$S^{\mathcal{P}}(t,s) = C_0(t)+C_1(t)s+C_2(t)s^2,$$

where $C_0(t) = -(-1+6t)t,\ C_1(t) = -1-t^2,$ and $C_2(t) = t+6.$

Since $m := \deg_t(S^{\mathcal{P}}) > 1$, we go to Step 4 of algorithm, and we consider

$$R(t) = \frac{C_0(t)}{C_1(t)} = \frac{t(6t-1)}{t^2+1}.$$

Note that $\gcd(C_0, C_1) = 1$. Now, we determine the polynomials

$$L_1(s,x_1) = \mathrm{Res}_t(G_1^{\mathcal{P}}(t,x_1), sC_1(t)-C_0(t)) = 1369(2s^2+sx_1-28s-7x_1+99)^2,$$

$$L_2(s,x_2) = \mathrm{Res}_t(G_2^{\mathcal{P}}(t,x_2), sC_1(t)-C_0(t)) = 50653(-s^3+21s^2+x_2s-147s+343-6x_2)^2,$$

where $G_i^{\mathcal{P}}(t,x_i) = x_ip_{i,2}(t)-p_{i,1}(t)$ (see Step 5). Finally, in Step 6, the algorithm outputs the proper parametrization $\mathcal{Q}(t)$, and the rational function $R(t)$

$$\mathcal{Q}(t) = \left(\frac{-(2t^2-28t+99)}{t-7}, \frac{t^3-21t^2+147t-343}{t-6}\right), \qquad R(t) = \frac{t(6t-1)}{t^2+1}.$$

□

Comparison of methods

We finish this section with a comparative discussion of the existing methods that solve the proper reparametrization problem for the case of plane curves. In particular, we analyze the algorithm in [24] (A1), the algorithm in [9] (A2), and the new proper reparametrization algorithm presented in this section (A3). We base our discussion on three different aspects: density of the output parametrization, algebraic manipulation required in the algorithms, and actual computing times in the implementation.

First of all, we comment that the algorithm A1 is heuristic, and the other two algorithms, A2 and A3, are deterministic. A1 determines the rational function $R(t)$ by means of the computation of several gcd's, and solving some linear systems of equations. Algorithms A2 and A3 only require the computation of a gcd. However, the gcd, $S^{\mathcal{P}}$ (see Lemma 3), computed by A3 is more general, and it allows to determine rational function $R(t)$ simply by choosing two of the coefficients of $S^{\mathcal{P}}$ (see Theorem 2). With respect to the computation of the proper parametrization $\mathcal{Q}(t)$, only algorithm A1 avoids the use of undetermined coefficients. This implies that the outputs provided by A2 and A3 are in general more complicated, in the sense of density, than the outputs given by A1.

Concerning algebraic manipulations required to derive the rational function $R(t)$, algorithms A2 and A3 are better since they only involve the computation of a gcd. In the case of A1, evaluations and computations of solutions of some linear systems of equations generated from the parametrization are required, and therefore is not as direct. In order to compute the proper rational parametrization $\mathcal{Q}(t)$, algorithm A3 is much better since it involves the computation of an easy univariate resultant whereas algorithms A1 and A2 solve the problem by means of the undetermined coefficient process.

Algorithms A1, A2 and A3, have been implementated in Maple. In the following table we illustrate the performance of these three implementations, showing times for some parametrizations. In the table we also show the degree of each input and output parametrization. Actual computing times, running on a PC Mobile Intel Celeron 2.4 GHz and 265 MB of RAM, are given in seconds of CPU.

INPUT	Degree of Input	A1	A2	A3	Degree of Output
P_1	7	0.01	0.01	0.001	7
P_2	80	3.886	0.611	0.070	8
P_3	42	2.644	0.901	0.120	7
P_4	49	1.762	0.541	0.201	7
P_5	27	0.170	0.140	0.060	3
P_6	27	6.930	3.395	0.190	9
P_7	132	1.693	0.641	0.261	4
P_8	30	0.781	0.401	0.150	6
P_9	30	0.231	0.100	0.020	5
P_{10}	40	0.641	0.350	0.300	5

4 The Problem of Proper Reparametrization for Surfaces: An Special Case

In Sections 2 and 3, we deal with the problem of computing a rational proper reparametrization of a given improperly parametrized algebraic plane curve. For the case of surfaces, although it is known from Castelnuovo's Theorem that unirationality and rationality are equivalent over algebraically closed fields, algorithmic questions and approaches are still required in order to solve the reparametrization problem.

In this section, we solve *partially* the proper reparametrization problem for the case of surfaces. More precisely, given an algebraically closed field $\mathbb{K}$, and $\mathcal{P}(t_1, t_2)$ a rational parametrization of a surface $\mathcal{V}$, we decide whether there exists

$$R(t_1, t_2) = (r_1(t_1), r_2(t_2)) = \left(\frac{r_{1,1}(t_1)}{r_{1,2}(t_1)}, \frac{r_{2,1}(t_2)}{r_{2,2}(t_2)}\right) \in (\mathbb{K}(t_1, t_2) \setminus \mathbb{K})^2,$$

such that $\mathcal{P}(t_1, t_2) = \mathcal{Q}(R(t_1, t_2))$, and $\mathcal{Q}(t_1, t_2)$ is a proper parametrization of $\mathcal{V}$. In the affirmative case, we compute $R(t_1, t_2)$, and $\mathcal{Q}(t_1, t_2)$.

The results obtained in this section provide a new and first approach that will be used to solve the important problem of the proper reparametrization for a general parametrized surface (see Section 6). No other result approaching the problem was known up to the moment. In addition, this approach can be applied to other problems in the frame of algebraic manipulations of parametrized algebraic surfaces as for instance, in the decomposition problem (see for instance [9]).

This section is preliminar, and we present some results that will be used to prove the correctness of the algorithm for reparametrizing a surface stated in Section 5.

For this purpose, first we present the notation that we will use throughout Sections 4 and 5. In particular, we introduce some polynomials using the results in [19], [20], and [21]. Afterwards, we state some previous lemmas (Lemmas 4 and 5) that will be used to prove Theorem 5. In this theorem, we characterize the rational surfaces we reparametrize properly. Finally, in Theorem 6, we show the existence of the proper reparametrization.

Notation

Let $\mathbb{K}$ be an algebraically closed field, and let $\mathbb{K}^\star = \mathbb{K} \setminus \{0\}$. In addition, if $\mathcal{V}$ is an affine rational surface, and $\mathcal{P}(t_1, t_2)$ is a rational affine parametrization of $\mathcal{V}$ over $\mathbb{K}$, we write its components as

$$\mathcal{P}(\bar{t}\,) = \left(\frac{p_{1,1}(\bar{t}\,)}{p_{1,2}(\bar{t}\,)}, \frac{p_{2,1}(\bar{t}\,)}{p_{2,2}(\bar{t}\,)}, \frac{p_{3,1}(\bar{t}\,)}{p_{3,2}(\bar{t}\,)}\right) \in \mathbb{K}(\bar{t}\,)^3,$$

where $\bar{t} = (t_1, t_2)$, and $\gcd(p_{i,1}, p_{i,2}) = 1$ for $i = 1, 2, 3$. For simplicity, we assume w.l.o.g. that none $p_{i,1}/p_{i,2}$ is constant. Note that, if for instance $p_{1,1}/p_{1,2} = \lambda \in \mathbb{K}$, then a proper parametrization of $\mathcal{V}$ is $\mathcal{Q}(t_1, t_2) = (\lambda, t_1, t_2)$, and then problem is trivial.

Furthermore, associated with the given parametrization $\mathcal{P}$, we consider the polynomials

$H_j^{\mathcal{P}}(\bar{t}\,, \bar{s}\,) = p_{j,1}(\bar{t}\,)p_{j,2}(\bar{s}\,) - p_{j,2}(\bar{t}\,)p_{j,1}(\bar{s}\,) \in \mathbb{K}[\bar{s}\,][\bar{t}\,]$, for $j = 1, 2, 3$, where $\bar{s} = (s_1, s_2)$,

and $H_4^{\mathcal{P}}(\bar{t}\,) = \operatorname{lcm}(p_{1,2}, p_{2,2}, p_{3,2})$. In addition, we also will use the polynomials

$$S_1^{\mathcal{P}}(t_1, \bar{s}\,) = \operatorname{pp}_{\bar{s}}(\operatorname{Content}_Z(\operatorname{Res}_{t_2}(H_1^{\mathcal{P}}, H_2^{\mathcal{P}} + Z H_3^{\mathcal{P}}))),$$

$$S_2^{\mathcal{P}}(t_2, \bar{s}\,) = \operatorname{pp}_{\bar{s}}(\operatorname{Content}_Z(\operatorname{Res}_{t_1}(H_1^{\mathcal{P}}, H_2^{\mathcal{P}} + Z H_3^{\mathcal{P}}))),$$

where $\operatorname{pp}_{\bar{s}}$ denotes the primitive part w.r.t. $\bar{s}$, and $\operatorname{Content}_Z$ denotes the content w.r.t a new variable Z. We denote by $\mathbb{F}$ the algebraic closure of the field $\mathbb{K}(\bar{s}\,)$.

Finally, we introduce the polynomials $G_i^{\mathcal{P}}(\bar{t}\,, x_i) = x_i p_{i,2}(\bar{t}\,) - p_{i,1}(\bar{t}\,)$, for $i = 1, 2, 3$. The polynomials $S_j^{\mathcal{P}}$ will play an important role in deciding whether a parametrization $\mathcal{P}$ is proper; i.e. in studying whether the parametrization is injective for almost all parameter values (see [20]). More precisely, in [20] the following theorem is proved.

Theorem 4 *The following statements hold:*

1. $\mathcal{F}_{\mathcal{P}}(\bar{s}\,) = \left\{ \bar{t} \in \mathbb{F}^2 \,\middle|\, H_i^{\mathcal{P}}(\bar{t}\,, \bar{s}\,) = 0, \ i \in \{1, 2, 3\}, \quad H_4^{\mathcal{P}}(\bar{t}\,) \neq 0 \right\}$, *where* $\mathcal{F}_{\mathcal{P}}(\bar{s}\,)$ *denotes the fibre* $\mathcal{P}^{-1}(\mathcal{P}(\bar{s}\,))$*; i.e.* $\mathcal{F}_{\mathcal{P}}(\bar{s}\,) = \{\bar{t} \in \mathbb{F}^2 \,|\, \mathcal{P}(\bar{t}\,) = \mathcal{P}(\bar{s}\,)\}$.
2. $\mathcal{F}_{\mathcal{P}}(\bar{s}\,) = \{(A, B) \in \mathbb{F}^2 |\, S_1^{\mathcal{P}}(A, \bar{s}\,) = 0, \ M_A^{\mathcal{P}}(B, \bar{s}\,) = 0\}$, *where*

$$M_A^{\mathcal{P}}(t_2, \bar{s}\,) = \operatorname{pp}_{\bar{s}}(\gcd_{\mathbb{F}[t_2]}(H_1^{\mathcal{P}}(A, t_2, \bar{s}\,), H_2^{\mathcal{P}}(A, t_2, \bar{s}\,), H_3^{\mathcal{P}}(A, t_2, \bar{s}\,))).$$

3. *The polynomials* $S_1^{\mathcal{P}}$, $S_2^{\mathcal{P}}$ *define the* t_1 *and* t_2*–coordinates of the points in* $\mathcal{F}_{\mathcal{P}}(\overline{s})$*.*

4. $\mathrm{Card}(\mathcal{F}_{\mathcal{P}}(\overline{s})) = \deg(\phi_{\mathcal{P}}) = \deg_{t_1}(S_1^{\mathcal{P}}(t_1, \overline{s})) = \deg_{t_2}(S_2^{\mathcal{P}}(t_2, \overline{s}))$*, where* $\phi_{\mathcal{P}}$ *denotes the rational map* $\phi_{\mathcal{P}} : \mathbb{K}^2 \longrightarrow V; \overline{t} \longmapsto \mathcal{P}(\overline{t})$ *induced by* $\mathcal{P}$*. In particular,* $\mathcal{P}$ *is proper if and only if, up to constants in* $\mathbb{K}^\star$*, it holds that* $S_j^{\mathcal{P}} = t_j - s_j$*.* □

Remark 3 *The degree of the rational map induced by a pair of bivariate rational functions can be also computed by applying the results in [20]. More precisely, given*

$$R(\overline{t}) = (r_1(\overline{t}), r_2(\overline{t})) = (r_{1,1}(\overline{t})/r_{1,2}(\overline{t}),\, r_{2,1}(\overline{t})/r_{2,2}(\overline{t})) \in (\mathbb{K}(\overline{t}) \setminus \mathbb{K})^2,$$

where $\gcd(r_{i,1}, r_{i,2}) = 1$ *for* $i \in \{1, 2\}$*, we consider the polynomials*

$$H_i^R(\overline{t}, \overline{s}) = r_{i,1}(\overline{t})r_{i,2}(\overline{s}) - r_{i,1}(\overline{s})r_{i,2}(\overline{t}) \in \mathbb{K}[\overline{s}][\overline{t}],\ i \in \{1, 2\},$$

and $H_3^R = \mathrm{lcm}(r_{1,2}, r_{2,2})$ *(compare to* $H_j^{\mathcal{P}}$*). In these conditions, we use the polynomials*

$$S_1^R(t_1, \overline{s}) = \mathrm{pp}_{\overline{s}}(\mathrm{Res}_{t_2}(H_1^R, H_2^R)), \quad S_2^R(t_2, \overline{s}) = \mathrm{pp}_{\overline{s}}(\mathrm{Res}_{t_1}(H_1^R, H_2^R))$$

(compare to $S_j^{\mathcal{P}}$*), to obtain similar results to Theorem 4 (for more details see [21]).* □

In these conditions, using the above preliminary results, in the following we characterize whether a rational parametrization $\mathcal{P}$ of a surface $\mathcal{V}$, can be expressed as $\mathcal{P} = \mathcal{Q}(R)$, where $R(\overline{t}) = (r_1(t_1), r_2(t_2)) \in (\mathbb{K}(\overline{t}) \setminus \mathbb{K})^2$, and $\mathcal{Q}$ is a proper parametrization of $\mathcal{V}$. Note that $r_j \notin \mathbb{K}$ since we have assumed that none $p_{i,1}/p_{i,2}$ is constant. We start with the following lemmas. The first one states a well known property of resultants (see [3]).

Lemma 4 *Let* $f(x) = a_n \prod_{i=1}^{n}(x - \alpha_i)$*, and* $g(x) = b_m \prod_{i=1}^{m}(x - \beta_i)$*. Then, it holds that* $\mathrm{Res}_x(f, g) = (-1)^{nm} b_m^n \prod_{i=1}^{m} f(\beta_i)$*.* □

Lemma 5 *Let* $\mathcal{P}(\overline{t})$ *be a rational parametrization of a surface* $\mathcal{V}$ *satisfying that* $S_1^{\mathcal{P}}(t_1, \overline{s}) = H_1^R(t_1, s_1)^{\ell_2}$*, and* $S_2^{\mathcal{P}}(t_2, \overline{s}) = H_2^R(t_2, s_2)^{\ell_1}$*, where*

$$H_j^R(t_j, s_j) = r_{j,1}(t_j)r_{j,2}(s_j) - r_{j,2}(t_j)r_{j,1}(s_j), \quad \gcd(r_{j,1}, r_{j,2}) = 1, \quad j = 1, 2$$

$r_j(t_j) := r_{j,1}(t_j)/r_{j,2}(t_j) \in \mathbb{K}[t_j] \setminus \mathbb{K}$*, and* $\ell_j := \deg_{t_j}(H_j^R)$*. Let*

$$L_i(s_1, t_2, x_i) = \mathrm{Res}_{t_1}(G_i^{\mathcal{P}}(\overline{t}, x_i), s_1 r_{1,2}(t_1) - r_{1,1}(t_1)),$$

$$N_i(\overline{s}, x_i) = \mathrm{pp}_{x_i}(\mathrm{Res}_{t_2}(L_i(s_1, t_2, x_i), s_2 r_{2,2}(t_2) - r_{2,1}(t_2))), \qquad i = 1, 2, 3,$$

where pp_{x_i} *denotes the primitive part w.r.t* x_i*. It holds that*

$$N_i(\overline{s}, x_i) = (q_{i,2}(\overline{s})x_i - q_{i,1}(\overline{s}))^{\deg(r_1)\cdot\deg(r_2)}, \quad i = 1, 2, 3$$

where $q_{i,1}(\overline{s})$*,* $q_{i,2}(\overline{s}) \in \mathbb{K}[\overline{s}]$*, and* $\gcd(q_{i,1}, q_{i,2}) = 1$*.*

Proof: We prove the theorem for N_1, and similarly one reasons for $N_j,\ j=2,3$. For this purpose, first we note that the following three statements hold:
(1.) Since $\gcd(r_{j,1},\, r_{j,2})=1$ for $j=1,2$, then the polynomials H_j^R do not have factors in $\mathbb{K}[t_j]$ neither in $\mathbb{K}[s_j]$.
(2.) The polynomials H_j^R do not have multiple roots for the variable t_1. Indeed, first note that $H_j^R\in\mathbb{K}[s_j][t_j]\setminus\mathbb{K}$; if $H_j^R=c\in\mathbb{K}$, then since $H_j(s_j,s_j)=0$, we get that $c=0$. Thus, since $\gcd(r_{j,1},\, r_{j,2})=1$, we deduce that $r_{j,1},\, r_{j,2}\in\mathbb{K}$ which is impossible. Now, we assume that H_1^R has a multiple root $a(s_1)\in\overline{\mathbb{K}(s_1)}$ (similarly for H_2^R). Thus,

$$r_{1,1}(a(s_1))r_{1,2}(s_1)-r_{1,2}(a(s_1))r_{1,1}(s_1)=0,\quad r'_{1,1}(a(s_1))r_{1,2}(s_1)-r'_{1,2}(a(s_1))r_{1,1}(s_1)=0.$$

Observe that since H_1^R does not have factors in $\mathbb{K}[t_1]$ (see statement (1.)), then $a(s_1)\notin\mathbb{K}$. This implies that $r_{1,1}(a(s_1))r'_{1,1}(a(s_1))\neq 0$, and then

$$\frac{r_{1,1}(a(s_1))}{r_{1,2}(a(s_1))}=\frac{r'_{1,1}(a(s_1))}{r'_{1,2}(a(s_1))}=\frac{r_{1,1}(s_1)}{r_{1,2}(s_1)}.$$

From the above equality, one gets that $r'_{1,1}(a(s_1))r_{1,2}(a(s_1))-r_{1,1}(a(s_1))r'_{1,2}(a(s_1))=0$. Hence, $(r_{1,1}(t_1)/r_{1,2}(t_1))'\,(a(s_1))=0$, and then $a(s_1)\in\mathbb{K}$ which is impossible.
(3.) We consider

$$R(\,\overline{t}\,)=(r_1(t_1),r_2(t_2))=\left(\frac{r_{1,1}(t_1)}{r_{1,2}(t_1)},\,\frac{r_{2,1}(t_2)}{r_{2,2}(t_2)}\right)\in(\mathbb{K}(\,\overline{t}\,)\setminus\mathbb{K})^2,$$

and we prove that $\mathcal{F}_{\mathcal{P}}(\,\overline{s}\,)=\mathcal{F}_R(\,\overline{s}\,)$. For this purpose, first we note that $\mathrm{Card}(\mathcal{F}_{\mathcal{P}}(\,\overline{s}\,))=\ell_1\cdot\ell_2$ (apply Theorem 4, Statement 4). Let $a_i(s_1)\in\overline{\mathbb{K}(s_1)}$, $i=1,\ldots,\ell_1$ be the roots of the polynomial H_1^R, and let $b_j(s_2)\in\overline{\mathbb{K}(s_2)}$, $j=1,\ldots,\ell_2$ be the roots of the polynomial H_2^R. Note that by statement (2.), we have that $a_i\neq a_j$ and $b_i\neq b_j$ for every $i\neq j$. In these conditions, taking into account Theorem 4, we deduce that

$$\mathcal{F}_{\mathcal{P}}(\,\overline{s}\,)=\{(a_1(s_1),b_1(s_2)),\ldots,(a_1(s_1),b_{\ell_2}(s_2)),\cdots\cdots\cdots,(a_{\ell_1}(s_1),b_1(s_2)),\ldots,(a_{\ell_1}(s_1),b_{\ell_2}(s_2))\}.$$

Indeed, each root $a_i(s_1)$ of the polynomial $S_1^{\mathcal{P}}$ is ℓ_2 times, and $a_i\neq a_j$ for every $i\neq j$. Thus, to each $a_i(s_1)$ corresponds ℓ_2 different roots (see Lemma 5 in [20]) of the polynomial $S_2^{\mathcal{P}}$. Hence, since $b_i\neq b_j$ for every $i\neq j$, we deduce the above equality.
On the other hand, by Remark 3, we have that

$$S_1^R(t_1,\,\overline{s}\,)=\mathrm{pp}_{\overline{s}}(\mathrm{Res}_{t_2}(H_1^R,H_2^R)),\qquad S_2^R(t_2,\,\overline{s}\,)=\mathrm{pp}_{\overline{s}}(\mathrm{Res}_{t_1}(H_1^R,H_2^R)).$$

Taking into account that $\mathrm{pp}_{\overline{s}}(\mathrm{Res}_{t_2}(H_1^R,H_2^R))=(H_1^R)^{\ell_2}$, and $\mathrm{pp}_{\overline{s}}(\mathrm{Res}_{t_1}(H_1^R,H_2^R))=(H_2^R)^{\ell_1}$ (apply Lemma 4), we deduce that $S_i^R(t_i,\,\overline{s}\,)=S_i^{\mathcal{P}}(t_i,\,\overline{s}\,)$ for $i=1,2$. Thus, reasoning similarly as above with $R(\,\overline{t}\,)$, we get that $\mathcal{F}_{\mathcal{P}}(\,\overline{s}\,)=\mathcal{F}_R(\,\overline{s}\,)$.

In these conditions, and using the above statements, we prove the lemma in three different steps:

Step 1: First, we show that $\deg_{x_1}(N_1) = \deg(r_1) \cdot \deg(r_2)$. Indeed, by Lemma 4, and taking into account that the leader coefficient of $s_1 r_{1,2}(t_1) - r_{1,1}(t_1)$ w.r.t. t_1 is in $\mathbb{K}[\,\overline{s}\,]$, we get that up to factors in $\mathbb{K}[\,\overline{s}\,]$

$$L_1(s_1, t_2, x_1) = \prod_{i=1}^{\ell_1} G_1^{\mathcal{P}}(\alpha_i, t_2, x_1),$$

where $\alpha_i(s_1) \in \overline{\mathbb{K}(s_1)}, \quad i = 1, \ldots, \ell_1$, are the roots of the polynomial $s_1 r_{1,2}(t_1) - r_{1,1}(t_1)$ for t_1. Note that since $\gcd(r_{1,1}, r_{1,2}) = 1$, then $\alpha_i(s_1) \in \overline{\mathbb{K}(s_1)} \setminus \mathbb{K}$, and $r_{1,2}(\alpha_i(s_1)) \neq 0$. Thus, $r_1(\alpha_i(s_1)) = s_1$. Now, we apply Lemma 4 to the polynomial N_1. Taking into account that the leader coefficient of $s_2 r_{2,2}(t_2) - r_{2,1}(t_2)$ w.r.t. t_2 is in $\mathbb{K}[\,\overline{s}\,]$, we get that

$$N_1(\,\overline{s}\,, x_1) = \mathrm{pp}_{x_1}\left(\prod_{j=1}^{\ell_2} L_1(s_1, \beta_j, x_1)\right) = \mathrm{pp}_{x_1}\left(\prod_{j=1}^{\ell_2}\prod_{i=1}^{\ell_1} G_1^{\mathcal{P}}(\alpha_i, \beta_j, x_1)\right) =$$

$$\mathrm{pp}_{x_1}\left(\prod_{j=1}^{\ell_2}\prod_{i=1}^{\ell_1}(x_1 p_{1,2}(\alpha_i, \beta_j) - p_{1,1}(\alpha_i, \beta_j))\right), \qquad \text{(I)}$$

where $\beta_j(s_2) \in \overline{\mathbb{K}(s_2)}, \quad j = 1, \ldots, \ell_2$, are the roots of the polynomial $s_2 r_{2,2}(t_2) - r_{2,1}(t_2)$ for t_2. Note that since $\gcd(r_{2,1}, r_{2,2}) = 1$, then $\beta_j(s_2) \in \overline{\mathbb{K}(s_2)} \setminus \mathbb{K}$, and $r_{2,2}(\beta_j(s_2)) \neq 0$. Thus, $r_2(\beta_j(s_2)) = s_2$. Since we also had that $r_1(\alpha_i(s_1)) = s_1$, we deduce that $R(\alpha_i, \beta_j) = (s_1, s_2)$ for $i = 1, \ldots, \ell_1$, and $j = 1, \ldots, \ell_2$. Thus $\mathcal{F}_R(R^{-1}(\,\overline{s}\,)) =$

$$\{(\alpha_1(s_1), \beta_1(s_2)), \ldots, (\alpha_1(s_1), \beta_{\ell_2}(s_2)), \cdots\cdots, (\alpha_{\ell_1}(s_1), \beta_1(s_2)), \ldots, (\alpha_{\ell_1}(s_1), \beta_{\ell_2}(s_2))\}, \quad \text{(II)}.$$

Then, since $\mathcal{F}_{\mathcal{P}}(\,\overline{s}\,) = \mathcal{F}_R(\,\overline{s}\,)$ (see statement (3.)), we get that $\mathcal{F}_{\mathcal{P}}(R^{-1}(\,\overline{s}\,)) = \mathcal{F}_R(R^{-1}(\,\overline{s}\,))$ which implies that $\mathcal{P}$ is defined at (α_i, β_j) for $i = 1, \ldots, \ell_1$, and $j = 1, \ldots, \ell_2$. Hence, in particular, we have that $p_{1,2}(\alpha_i, \beta_j) \neq 0$ for every $i = 1, \ldots, \ell_1$, and $j = 1, \ldots, \ell_2$ which implies, from (I), that $\deg_{x_1}(N_1) = \ell_1 \cdot \ell_2 = \deg(r_1) \cdot \deg(r_2)$.

Step 2: Now, we prove that $N_1(\,\overline{s}\,, x_1)$ only has one different root for the unknown x_1. Indeed, let $A_1(\,\overline{s}\,)$, $A_2(\,\overline{s}\,)$ be two roots of $N_1(\,\overline{s}\,, x_1)$ for x_1. From (I) (see Step 1) and taking into account that $p_{1,2}(\alpha_i, \beta_j) \neq 0$ for $i = 1, \ldots, \ell_1$, and $j = 1, \ldots, \ell_2$, we get that

$$A_k = p_{1,1}(\alpha_{i_k}, \beta_{j_k}) / p_{1,2}(\alpha_{i_k}, \beta_{j_k}), \quad k = 1, 2,$$

for some $i_k \in \{1, \ldots, \ell_1\}$, and $j_k \in \{1, \ldots, \ell_2\}$. From (II) (see Step 1), we deduce that $\mathcal{P}(\alpha_{i_1}, \beta_{j_1}) = \mathcal{P}(\alpha_{i_2}, \beta_{j_2})$, which implies that $A_1 = A_2$.

Step 3: Finally, since $\deg_{x_1}(N_1) = \deg(r_1) \cdot \deg(r_2)$ (see Step 1), and $N_1(\,\overline{s}\,, x_1)$ only has one different root for the unknown x_1 (see Step 2), from the equality (I), we deduce that $N_1(\,\overline{s}\,, x_1) =$

$$\mathrm{pp}_{x_i}(\mathrm{Res}_{t_2}(L_1(s_1, t_2, x_1), s_2 r_{2,2}(t_2) - r_{2,1}(t_2))) = (q_{1,2}(\,\overline{s}\,) x_i - q_{1,1}(\,\overline{s}\,))^{\deg(r_1)\deg(r_2)} \in \mathbb{K}[\,\overline{s}\,, x_1].$$

Thus, $q_{1,1}(\,\overline{s}\,)$, $q_{1,2}(\,\overline{s}\,) \in \mathbb{K}[\,\overline{s}\,]$, and $\gcd(q_{1,1},\, q_{1,2}) = 1$. □

Applying these results one has the following theorem.

Theorem 5 *Let $\mathcal{P}(\bar{t}\,)$ be a rational parametrization of a surface $\mathcal{V}$. The following statements are equivalent:*

1. $\mathcal{P}(\bar{t}\,) = \mathcal{Q}(R(\bar{t}\,))$ *where* $R(\bar{t}\,) = (r_1(t_1), r_2(t_2)) \in (\mathbb{K}(\bar{t}\,) \setminus \mathbb{K})^2$, *and* $\mathcal{Q}(\bar{t}\,)$ *is a proper parametrization of* $\mathcal{V}$

2. $S_1^{\mathcal{P}}(t_1, \bar{s}\,) = H_1^R(t_1, s_1)^{\ell_2}$, *and* $S_2^{\mathcal{P}}(t_2, \bar{s}\,) = H_2^R(t_2, s_2)^{\ell_1}$, *where for* $j = 1, 2$,

$$H_j^R(t_j, s_j) = r_{j,1}(t_j)r_{j,2}(s_j) - r_{j,2}(t_j)r_{j,1}(s_j), \quad \gcd(r_{j,1},\, r_{j,2}) = 1,$$

$r_j(t_j) := r_{j,1}(t_j)/r_{j,2}(t_j) \in \mathbb{K}[t_j] \setminus \mathbb{K}$, *and* $\ell_j := \deg_{t_j}(H_j^R)$.

Proof: First, we prove that Statement 1 implies Statement 2. Note that since $\mathcal{Q}$ is proper, we deduce that $\mathcal{F}_{\mathcal{P}}(\bar{s}\,) = \mathcal{F}_R(\bar{s}\,)$ which implies that

$$S_1^{\mathcal{P}}(t_1, \bar{s}\,) = \mathrm{pp}_s(\mathrm{Res}_{t_2}(H_1^R, H_2^R)), \qquad S_2^{\mathcal{P}}(t_2, \bar{s}\,) = \mathrm{pp}_s(\mathrm{Res}_{t_1}(H_1^R, H_2^R)),$$

where $H_i^R(\bar{t}\,, \bar{s}\,) = r_{i,1}(\bar{t}\,)r_{i,2}(\bar{s}\,) - r_{i,2}(\bar{t}\,)r_{i,1}(\bar{s}\,)$, $i = 1, 2$ (see Theorem 4, and Remark 3). Now, by Lemma 4, we deduce that

$$S_1^{\mathcal{P}}(t_1, \bar{s}\,) = H_1^R(t_1, s_1)^{\ell_2}, \quad S_2^{\mathcal{P}}(t_2, \bar{s}\,) = H_2^R(t_2, s_2)^{\ell_1}, \quad \text{where } \ell_i := \deg_{t_i}(H_i^R), \quad i = 1, 2.$$

Now, let us prove that Statement 2 implies Statement 1. First, we consider

$$R(\bar{t}\,) = (r_1(t_1), r_2(t_2)) = \left(\frac{r_{1,1}(t_1)}{r_{1,2}(t_1)}, \frac{r_{2,1}(t_2)}{r_{2,2}(t_2)}\right) \in (\mathbb{K}(\bar{t}\,) \setminus \mathbb{K})^2,$$

and we show that there exists $\mathcal{Q}(\bar{t}\,) \in \mathbb{K}(\bar{t}\,)^3$ such that $\mathcal{P}(\bar{t}\,) = \mathcal{Q}(R(\bar{t}\,))$. By applying Lemma 5, we have that,

$$N_i(\bar{s}\,, x_i) = (q_{i,2}(\bar{s}\,)x_i - q_{i,1}(\bar{s}\,))^{\deg(r_1)\cdot\deg(r_2)}, \quad i = 1, 2, 3$$

where $q_{i,1}(\bar{s}\,)$, $q_{i,2}(\bar{s}\,) \in \mathbb{K}[\bar{s}\,]$, and $\gcd(q_{i,1}, q_{i,2}) = 1$. Let

$$\mathcal{Q}(\bar{s}\,) = (q_1(\bar{s}\,), q_2(\bar{s}\,), q_3(\bar{s}\,)) = \left(\frac{q_{1,1}(\bar{s}\,)}{q_{1,2}(\bar{s}\,)}, \frac{q_{2,1}(\bar{s}\,)}{q_{2,2}(\bar{s}\,)}, \frac{q_{3,1}(\bar{s}\,)}{q_{3,2}(\bar{s}\,)}\right),$$

and let us prove that $\mathcal{P} = \mathcal{Q}(R)$; that is $q_j(R) = p_{j,1}/p_{j,2}, \quad j = 1, 2, 3$. Since $N_j(q_j(\bar{s}\,), \bar{s}\,) = 0, \quad j = 1, 2, 3$, taking into account the behavior of resultants under specializations (see Lemma 4.3.1, pp.96 in [31]), we deduce that

$$\mathrm{Res}_{t_2}(L_j(q_j(\bar{s}\,), t_2, s_1),\, s_2r_{2,2}(t_2) - r_{2,1}(t_2)) = 0, \quad j = 1, 2, 3.$$

Since $\gcd(r_{2,1}, r_{2,2}) = 1$, we get that $s_2r_{2,2}(t_2) - r_{2,1}(t_2)$ is irreducible which implies that this factor divides $L_j(q_j(\bar{s}\,), t_2, s_1)$. Therefore, $s_2 = r_2(t_2)$ is a root of

$L_j(q_j(s_1, s_2), t_2, s_1)$, and then $L_j(q_j(s_1, r_2(t_2)), t_2, s_1) = 0$. Thus applying Lemma 4.3.1, pp.96 in [31], we deduce that

$$\mathrm{Res}_{t_1}(G_j^{\mathcal{P}}(\bar{t}\,, q_j(s_1, r_2(t_2))),\ s_1 r_{1,2}(t_1) - r_{1,1}(t_1)) = 0, \quad j = 1, 2, 3$$

Since $\gcd(r_{1,1}, r_{1,2}) = 1$, we get that $s_1 r_{1,2}(t_1) - r_{1,1}(t_1)$ is irreducible which implies that this factor divides

$$G_j^{\mathcal{P}}(\bar{t}\,, q_j(s_1, r_2(t_2))) = q_{j,1}(s_1, r_2(t_2)) p_{j,2}(\bar{t}\,) - p_{j,1}(\bar{t}\,) q_{j,2}(s_1, r_2(t_2)).$$

Hence, $s_1 = r_1(t_1)$ is a root of $q_{j,1}(s_1, r_2(t_2)) p_{j,2}(\bar{t}\,) - p_{j,1}(\bar{t}\,) q_{j,2}(s_1, r_2(t_2))$, and then

$$q_{j,1}(r_1(t_1), r_2(t_2)) p_{j,2}(\bar{t}\,) - p_{j,1}(\bar{t}\,) q_{j,2}(r_1(t_1), r_2(t_2)) = 0, \quad j = 1, 2, 3;$$

that is, $q_j(R) = p_{j,1}/p_{j,2}, \quad j = 1, 2, 3.$

Finally, we prove that $\mathcal{Q}$ is proper. Indeed, since $\mathcal{P} = \mathcal{Q}(R)$, and $\deg(\phi_{\mathcal{P}}) = \deg(\phi_R)$ (see Statement 3 of the proof in Lemma 5), and taking into account that the degree of a rational map is multiplicative under composition, we get that $\deg(\phi_{\mathcal{Q}}) = 1$ which implies that $\mathcal{Q}$ is proper (see Theorem 4). □

Remark 4 *We observe that applying Corollary 1, one may check easily if the square free parts of the polynomials $S_j^{\mathcal{P}}(t_j, s_j)$ for $j = 1, 2$, say $M_j(t_j, s_j)$, are of the form*

$$M_j(t_j, s_j) = r_{j,1}(t_j) r_{j,2}(s_j) - r_{j,2}(t_j) r_{j,1}(s_j), \qquad \gcd(r_{j,1},\, r_{j,2}) = 1,$$

with $r_j(t_j) := r_{j,1}(t_j)/r_{j,2}(t_j) \in \mathbb{K}[t_j] \setminus \mathbb{K}$. In the affirmative case, by Theorem 5, we get that $\mathcal{P}$ can be expressed as $\mathcal{P} = \mathcal{Q}(R)$ where $R(\bar{t}\,) = (r_1(t_1),\, r_2(t_2)) \in (\mathbb{K}(\bar{t}\,) \setminus \mathbb{K})^2$, and $\mathcal{Q}$ is a proper parametrization of $\mathcal{V}$. □

Let us assume that for $j = 1, 2$,

$$M_j(t_j, s_j) = r_{j,1}(t_j) r_{j,2}(s_j) - r_{j,2}(t_j) r_{j,1}(s_j), \qquad \gcd(r_{j,1},\, r_{j,2}) = 1,$$

with $r_j(t_j) := r_{j,1}(t_j)/r_{j,2}(t_j) \in \mathbb{K}[t_j] \setminus \mathbb{K}$. We express the polynomials $r_{j,1}(t_j)$, $r_{j,2}(t_j)$ as

$$r_{j,1}(t_j) = a_{m_j,j} t_j^{m_j} + \cdots + a_{0,j}, \quad r_{j,2}(t_j) = b_{m_j,j} t_j^{m_j} + \cdots + b_{0,j}, \quad j = 1, 2$$

with $a_{m_j,j} \neq 0$ or $b_{m_j,j} \neq 0$. We get that,

$$M_j(t_j, s_j) = C_{m_j,j}(t_j) s_j^{m_j} + C_{m_j-1}(t_j) s_j^{m_j-1} + \cdots + C_{0,j}(t_j),$$

where $C_{\ell,j}(t) = a_{\ell,j} r_{j,2}(t_j) - b_{\ell,j} r_{j,1}(t_j)$, for $\ell = 0, \ldots, m_j$, and $j = 1, 2$.

Now, taking into account Theorem 1 (observe that each polynomial M_j is of the same form than $S^{\mathcal{P}}$), we consider $B(\overline{t}\,) =$

$$\left(\frac{C_{i_0,1}(t_1)}{C_{j_0,1}(t_1)}, \frac{C_{i_1,2}(t_2)}{C_{j_1,2}(t_2)}\right) = \left(\frac{a_{i_0,1}r_{1,2}(t_1) - b_{i_0,1}r_{1,1}(t_1)}{a_{j_0,1}r_{1,2}(t_1) - b_{j_0,1}r_{1,1}(t_1)}, \frac{a_{i_1,2}r_{2,2}(t_2) - b_{i_1,2}r_{2,1}(t_2)}{a_{j_1,2}r_{2,2}(t_2) - b_{j_1,2}r_{2,1}(t_2)}\right) \in (\mathbb{K}(\overline{t}\,)\setminus\mathbb{K})^2,$$

where $a_{i_0,1}b_{j_0,1} \neq a_{j_0,1}b_{i_0,1}$, and $a_{i_1,2}b_{j_1,2} \neq a_{i_1,2}b_{j_1,2}$. In these conditions, we have the following theorem.

Theorem 6 *There exists a proper parametrization $\mathcal{U}(\overline{t}\,)$ of the surface $\mathcal{V}$ satisfying that $\mathcal{P}(\overline{t}\,) = \mathcal{U}(B(\overline{t}\,))$.*

Proof: First, we observe that we may express $B(\overline{t}\,) = G(R(\overline{t}\,))$, where

$$G(\overline{t}\,) = \left(\frac{b_{i_0,1}t_1 - a_{i_0,1}}{b_{j_0,1}t_1 - a_{j_0,1}}, \frac{b_{i_1,2}t_2 - a_{i_1,2}}{b_{j_1,2}t_2 - a_{j_1,2}}\right).$$

Since $a_{i_0,1}b_{j_0,1} \neq a_{j_0,1}b_{i_0,1}$, and $a_{i_1,2}b_{j_1,2} \neq a_{i_1,2}b_{j_1,2}$ (see Theorem 1), we get that $G(\overline{t}\,)$ is invertible. In these conditions, we consider $\mathcal{U} = \mathcal{Q}(G^{-1})$, and we prove that $\mathcal{U}$ is a proper parametrization of $\mathcal{V}$. Indeed, first note that

$$\mathcal{U}(B(\overline{t}\,)) = \mathcal{Q}(G^{-1}(\overline{t}\,)) \circ B(\overline{t}\,) = \mathcal{Q}(G^{-1}(\overline{t}\,)) \circ G(R(\overline{t}\,)) = \mathcal{Q}(R(\overline{t}\,)) = \mathcal{P}(\overline{t}\,),$$

and then $\mathcal{Q}$ parametrizes $\mathcal{V}$. In addition, since $\mathcal{Q}$, G are invertible, we get that $\mathcal{U} = \mathcal{Q}(G^{-1})$ is proper. □

5 Algorithm of Proper Reparametrization for Surfaces

The results obtained in Section 4 (in particular Lemma 5, and Theorems 5 and 6) can be applied to derive an algorithm that decides whether a rational parametrization $\mathcal{P}$ of a surface $\mathcal{V}$, can be expressed as $\mathcal{P} = \mathcal{Q}(R)$, where $R(\overline{t}\,) = (r_1(t_1), r_2(t_2)) \in (\mathbb{K}(\overline{t}\,)\setminus\mathbb{K})^2$, and $\mathcal{Q}$ is a proper parametrization of $\mathcal{V}$. In the affirmative case, the algorithm also computes R and $\mathcal{Q}$.

In the following, we outline this algorithm, and we illustrate it with an example. We finish this section with a brief experimental analysis of the algorithm.

Algorithm Proper Reparametrization for Surfaces.

Given a rational affine parametrization $\mathcal{P} = (p_{1,1}/p_{1,2},\, p_{2,1}/p_{2,2},\, p_{3,1}/p_{3,2})$, in reduced form, of an algebraic surface $\mathcal{V}$, the algorithm decides whether $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$, with $R(\bar{t}) = (r_1(t_1), r_2(t_2)) \in \mathbb{K}(\bar{t})^2$, and in the affirmative case, it computes $R(\bar{t})$ and the rational proper parametrization $\mathcal{Q}(\bar{t})$.

1. Compute $H_j^{\mathcal{P}}(\bar{t}, \bar{s}) = p_{j,1}(\bar{t})p_{j,2}(\bar{s}) - p_{j,2}(\bar{t})p_{j,1}(\bar{s}), \quad j = 1,2,3.$
2. Determine the polynomials
$$S_1^{\mathcal{P}} = \mathrm{pp}_{\bar{s}}(\mathrm{Content}_Z(\mathrm{Res}_{t_2}(H_1^{\mathcal{P}}, H_2^{\mathcal{P}} + ZH_3^{\mathcal{P}}))),$$
$$S_2^{\mathcal{P}} = \mathrm{pp}_{\bar{s}}(\mathrm{Content}_Z(\mathrm{Res}_{t_1}(H_1^{\mathcal{P}}, H_2^{\mathcal{P}} + ZH_3^{\mathcal{P}})))$$
3. If $\deg_{t_j}(S_j^{\mathcal{P}}) = 1$, return $\mathcal{Q}(\bar{t}) = \mathcal{P}(\bar{t})$ and $R(\bar{t}) = (t_1, t_2)$. Otherwise go to Step 4.
4. If $S_j^{\mathcal{P}}(t_j, \bar{s}) \in \mathbb{K}[t_j, s_j]$, for $j = 1,2$, go to Step 5. Otherwise, return "$\mathcal{P}$ cannot be expressed as $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$, with $R(\bar{t}) = (r_1(t_1), r_2(t_2))$".
5. Compute the polynomial
$$M_j(t_j, s_j) = \mathrm{Sqrfree}(S_j^{\mathcal{P}}(t_j, \bar{s})) = C_{m_j,j}(t_j)s_j^{m_j} + \cdots + C_{0,j}(t_j), \quad j = 1,2,$$
where Sqrfree(p) denotes the square free part of a polynomial p.

   5.1. Consider $R(\bar{t}) = (r_1(t_1), r_2(t_2))$ with
$$r_j(t_j) = C_{\ell_0,j}(t_j)/C_{n_0,j}(t_j) \in \mathbb{K}(t_j), \quad j = 1,2$$
and such that $C_{\ell_0,j}$ and $C_{n_0,j}$ are not associated polynomials.

   5.2. If $C_{\ell_0,j}(t_j)C_{n_0,j}(s_j) - C_{\ell_0,j}(s_j)C_{n_0,j}(t_j) = c_jM_j(t_j, s_j),\ c_j \in \mathbb{K}^\star,\ j = 1,2,$ then go to Step 6. Otherwise, return "$\mathcal{P}$ cannot be expressed as $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$, with $R(\bar{t}) = (r_1(t_1), r_2(t_2))$".
6. For $i = 1,2,3$, compute the polynomials
$$L_i(s_1, t_2, x_i) = \mathrm{Res}_{t_1}(G_i^{\mathcal{P}}(\bar{t}, x_i), s_1r_{1,2}(t_1) - r_{1,1}(t_1)),$$
$$N_i(\bar{s}, x_i) = \mathrm{pp}_{x_i}(\mathrm{Res}_{t_2}(L_i(s_1, t_2, x_i), s_2r_{2,2}(t_2) - r_{2,1}(t_2))) = (q_{i,2}(\bar{s})x_i - q_{i,1}(\bar{s}))^{\deg(r_1)\deg(r_2)}$$
where $G_i^{\mathcal{P}}(\bar{t}, x_i) = x_ip_{i,2}(\bar{t}) - p_{i,1}(\bar{t})$.
7. Return
$$\mathcal{Q}(\bar{t}) = (q_{1,1}(\bar{t})/q_{1,2}(\bar{t}),\, q_{2,1}(\bar{t})/q_{2,2}(\bar{t}),\, q_{3,1}(\bar{t})/q_{3,2}(\bar{t})),$$
and $R(\bar{t}) = (C_{\ell_0,1}(t_1)/C_{n_0,1}(t_1),\, C_{\ell_0,2}(t_2)/C_{n_0,2}(t_2))$.

The algorithm follows directly from Theorem 4 in Step 3, Theorem 5 in Step 4, Remark 4 in Step 5.2, Lemma 5 in Step 6, and Theorem 6 in Step 7. In the following, we illustrate Algorithm Proper Reparametrization for Surfaces with an example.

Example 2

Let $\mathcal{V}$ be the rational surface defined by the parametrization

$$\mathcal{P}(\,\bar{t}\,) = \left(\frac{{t_1}^4\,{t_2}^4 + 2\,{t_1}^4\,{t_2}^2 + 5\,{t_1}^4 + 2\,{t_2}^4 + 4\,{t_2}^2 + 11}{{t_2}^4 + 2\,{t_2}^2 + 5},\right.$$

$$\left.\frac{6 + {t_1}^4\,{t_2}^4 + 2\,{t_1}^4\,{t_2}^2 + 5\,{t_1}^4 + {t_2}^4 + 2\,{t_2}^2}{({t_2}^4 + 2\,{t_2}^2 + 5)\,({t_1}^4 + 1)}, -\frac{3 + {t_1}^4\,{t_2}^4 + 2\,{t_1}^4\,{t_2}^2 + 5\,{t_1}^4 + {t_2}^4 + 2\,{t_2}^2}{{t_2}^4 + 2\,{t_2}^2 + 5}\right).$$

In Step 1 of the algorithm, we compute the polynomials

$$H_1^{\mathcal{P}}(\,\bar{t}\,,\,\bar{s}\,) = -2t_1^4t_2^4s_2^2-25t_1^4-t_1^4t_2^4s_2^4+t_2^4+2t_2^2+25s_1^4-s_2^4-2s_2^2-2t_1^4t_2^2s_2^4+2s_1^4s_2^2t_2^4+4s_1^4s_2^2t_2^2+ 2s_1^4s_2^4t_2^2+s_1^4s_2^4t_2^4-4t_1^4t_2^2s_2^2+10s_1^4t_2^2-5t_1^4t_2^4-10t_1^4t_2^2+5s_1^4s_2^4+5s_1^4t_2^4+10s_1^4s_2^2-5t_1^4s_2^4-10t_1^4s_2^2,$$

$$H_2^{\mathcal{P}}(\,\bar{t}\,,\,\bar{s}\,) = 5t_1^4 + t_2^4 + 2t_2^2 - 5s_1^4 - s_2^4 - 2s_2^2 + t_1^4t_2^4 + 2t_1^4t_2^2 - s_1^4s_2^4 - 2s_1^4s_2^2,$$

$$H_3^{\mathcal{P}}(\,\bar{t}\,,\,\bar{s}\,) = 2t_1^4t_2^4s_2^2+25t_1^4+t_1^4t_2^4s_2^4+2t_2^4+4t_2^2-25s_1^4-2s_2^4-4s_2^2+2t_1^4t_2^2s_2^4-2s_1^4s_2^2t_2^4-4s_1^4s_2^2t_2^2- 2s_1^4s_2^4t_2^2-s_1^4s_2^4t_2^4+4t_1^4t_2^2s_2^2-10s_1^4t_2^2+5t_1^4t_2^4+10t_1^4t_2^2-5s_1^4s_2^4-5s_1^4t_2^4-10s_1^4s_2^2+5t_1^4s_2^4+10t_1^4s_2^2.$$

Now, we determine $S_j^{\mathcal{P}}(t_j,\,\bar{s}\,)$. We obtain

$$S_1^{\mathcal{P}}(t_1, s_1) = (t_1 - s_1)^4(t_1 + s_1)^4(t_1^2 + s_1^2)^4,$$

$$S_2^{\mathcal{P}}(t_2, s_2) = (t_2 + s_2)^4(t_2 - s_2)^4(t_2^2 + 2 + s_2^2)^4.$$

Since $\deg_{t_j}(S_j^{\mathcal{P}}) > 1$, and $S_j^{\mathcal{P}}(t_j,\,\bar{s}\,) \in \mathbb{K}[t_j, s_j]$ for $j = 1, 2$, we go to Step 5 of algorithm, and we determine the polynomials

$$M_1(t_1, s_1) = (t_1 - s_1)(t_1 + s_1)(t_1^2 + s_1^2) = C_{4,1}(t_1)s_1^4 + C_{0,1}(t_1),$$

where $C_{4,1} = -1$, $C_{0,1} = t_1^4$, and

$$M_2(t_2, s_2) = (t_2 + s_2)(t_2 - s_2)(t_2^2 + 2 + s_2^2) = C_{4,2}(t_2)s_2^4 + C_{2,2}(t_2)s_2^2 + C_{0,2}(t_2),$$

where $C_{4,2} = -1$, $C_{2,2} = -2$, $C_{0,2} = t_2^4 + 2t_2^2$. Now, we consider

$$R(\,\bar{t}\,) = (C_{0,1}(t_1)/C_{4,1}(t_1), C_{0,2}(t_2)/C_{4,2}(t_2)) = (-t_1^4, -t_2^2(2 + t_2^2)).$$

In Step 5.2 we check that

$$M_1 = C_{4,1}(t_1)C_{0,1}(s_1) - C_{4,1}(s_1)C_{0,1}(t_1), \quad M_2 = C_{4,2}(t_2)C_{0,2}(s_2) - C_{4,2}(s_2)C_{0,2}(t_2).$$

Note that $C_{0,i}$, $C_{4,i}$ are not associated polynomials for $i = 1, 2$. In Step 6 of algorithm, we compute the polynomials

$$L_i(s_1, x_i) = \mathrm{Res}_{t_1}(G_i^{\mathcal{P}}(\bar{t}, x_i), s_1 r_{1,2}(t_1) - r_{1,1}(t_1)),$$

$$N_i(\bar{s}, x_i) = \mathrm{pp}_{x_i}(\mathrm{Res}_{t_2}(L_i(s_1, x_i), s_2 r_{2,2}(t_2) - r_{2,1}(t_2))) = (q_{i,2}(\bar{s})x_i - q_{i,1}(\bar{s}))^{\deg_{t_1}(r_1)\cdot \deg_{t_2}(r_2)},$$

where $G_i^{\mathcal{P}}(\bar{t}, x_i) = x_i p_{i,2}(\bar{t}) - p_{i,1}(\bar{t})$. We get

$$N_1(\bar{s}, x_1) = (11 - 5x_1 - 5s_1 + s_2x_1 - 2s_2 + s_2s_1)^{16},$$

$$N_2(\bar{s}, x_2) = (-6 + 5x_2 - 5s_1x_2 + 5s_1 - s_2x_2 + s_2 + s_2s_1x_2 - s_2s_1)^{16},$$

$$N_3(\bar{s}, x_3) = (-s_2x_3 - s_2 + s_2s_1 - 5s_1 + 5x_3 + 3)^{16}.$$

Finally, in Step 7, the algorithm returns the proper parametrization

$$\mathcal{Q}(\bar{t}) = \left(-\frac{11 + t_2t_1 - 5t_1 - 2t_2}{t_2 - 5}, \frac{6 - t_2 - 5t_1 + t_2t_1}{5 - 5t_1 - t_2 + t_2t_1}, \frac{-t_2 + t_2t_1 - 5t_1 + 3}{t_2 - 5}\right),$$

and $R(\bar{t}) = (t_1^4, t_2^2(2 + t_2^2))$. □

Practical Implementation

In the following table we illustrate the performance of the implementation in Maple of Algorithm Proper Reparametrization for Surfaces, showing times for some parametrizations. In the table, we also show a list with the degree on the variables t_1, t_2, and total degree, of each input and output parametrization, and $R(\bar{t})$. Actual computing times, running on a PC Mobile Intel Celeron 2.4 GHz and 265 MB of RAM, are given in seconds of CPU.

INPUT	Degree of Input	Algorithm	Degree of Output	Degree of $R(\bar{t})$
P_1	$[1, 3, 4]$	32.437	$[2, 2, 3]$	$[1, 3, 3]$
P_2	$[2, 4, 6]$	16.744	$[1, 1, 2]$	$[2, 4, 4]$
P_3	$[3, 4, 4]$	0.030	$[1, 1, 1]$	$[3, 4, 4]$
P_4	$[4, 5, 9]$	0.162	$[2, 1, 3]$	$[2, 5, 5]$
P_5	$[4, 2, 4]$	11.657	$[4, 2, 4]$	$[1, 1, 1]$
P_6	$[6, 2, 7]$	50.463	$[6, 1, 7]$	$[1, 2, 2]$
P_7	$[5, 3, 8]$	0.370	$[1, 1, 2]$	$[5, 3, 5]$
P_8	$[5, 2, 7]$	0.081	$[1, 1, 2]$	$[5, 2, 5]$
P_9	$[6, 2, 6]$	1.672	$[3, 2, 3]$	$[2, 1, 2]$
P_{10}	$[8, 4, 8]$	14.181	$[4, 2, 4]$	$[2, 2, 2]$

We remark that the density of the input parametrization influences considerably in the processing time. In particular, in the table, we observe that for some input parametrizations having similar degrees, we obtain very different times.

6 Conclusion

In this paper, we deal with the problem of proper reparametrization for rational curves and surfaces. More precisely, for the case of curves, we present a new approach that improves and reinterprets some previous results. In particular, the proper parametrization is obtained from the coefficients of a univariate gcd, and by computing a univariate resultant.

For surfaces, no results approaching the problem algorithmically were known up to the moment. We develop a first approach that solve the problem for some special surface's parametrizations. The basic idea is to compute two univariate resultants of certain curves directly constructed from the given parametrization. These results provide effective methods that can be used to solve the general case. We will deal with this problem in a future work.

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