# Existence of solutions and well-posedness to convection-diffusion equations in uniform spaces

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# 博 士 論 文

# Existence of Solutions and Well-posedness to Convection-Diffusion Equations in Uniform Spaces

(対流拡散方程式の一様空間における
解の存在と適切性)

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令和2年

# Existence of Solutions and Well-Posedness to Convection-Diffusion Equations in Uniform Spaces

A thesis presented

by

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# Contents

# Summary

## 1.1 Convection-diffusion equations

One of the main research areas on the theory of partial differential equations is the well-posedness issue on the Cauchy problems, that is, existence of solutions, uniqueness of solutions and continuous dependence of initial data. In particular, it is noteworthy to query whether the Cauchy problem for partial differential equations has a solution or not. Such a problem has been attracting much interest not only in mathematical but also other related fields. The purpose of this thesis is to consider the existence of solutions and the well-posedness of the Cauchy problem to the convection–diffusion equations.

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with $n \geq 1$ and $a \in \mathbb{R}^n$. We consider the following Cauchy problem to the convection-diffusion equations

$$\begin{cases} \partial_t u - \Delta u = a \cdot \nabla(|u|^{p-1}u), & t > 0, \quad x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x), & \qquad\quad\; x \in \mathbb{R}^n, \end{cases} \tag{1.1.1}$$

where $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is the unknown function, $p > 1$ is the fixed non-linear exponent, $a \in \mathbb{R}^n$ is an arbitrary fixed vector denotes the direction of the convection and $u_0 = u_0(x) : \mathbb{R}^n \to \mathbb{R}$ is given initial data. We denote $\Delta$, $\partial_t$ and $\nabla$ by the $n$-dimensional Laplacian, the partial derivative with respect to $t$ and the gradient in $x$ variables, respectively.

In this thesis, we show the existence and uniqueness of a solution and establish the well-posedness to the Cauchy problem (1.1.1) in *a uniformly local Lebesgue space and amalgam spaces.*

Semilinear parabolic problems often appear in various mathematical models such as heat transfer models, chemical reaction models, biochemical reaction models, nonlinear radiation laws, growth models of tissues in living bodies and so on. The well-posedness of the Cauchy problem to semilinear parabolic equations depends on a lot of factors such as effect of diffusion terms, nonlinear terms, boundary conditions and the shape of initial data. This is one of the reason why the issue has been attracting much attention from many mathematicians with the development of nonlinear analysis.

The convection-diffusion equations describe physical phenomena such as the dynamics of a physical substance governed by the motion derived by two processes: the convection and the diffusion. The convection term stems from the movement of molecules consisting the fluids whereas the diffusion describes the spread of particles from regions of higher concentration to regions of lower concentration through a random diffusive process of motion.

## 1.2 Nonlinear heat equations of Fujita type

One of a simplest model of nonlinear parabolic equations is the following Cauchy problem of the nonlinear heat equation. In particular, the problem is called the Fujita type nonlinear heat equation

$$\begin{cases} \partial_t u - \Delta u = u^p, & t > 0, x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.2.1}$$

where $n \geq 1$ and $p > 1$ was proposed as a simple model to know the dynamics of the solution to the Navier-Stokes equations by Fujita [29]. When we consider the Cauchy problem for a semilinear heat equation (1.2.1) with initial data is the Lebesgue spaces $u_0 \in L^r(\mathbb{R}^n)$, where $p > 1$, $1 < r < \infty$, and $L^r(\mathbb{R}^n)$ denotes the $r$-th powered integrable functions with the Legesgue measure in $x$. It is well known that if the initial data $u_0 \in L^\infty(\mathbb{R}^{n)}$, then there exists $T(u_0) > 0$ and a unique solution $u \in L^\infty(0,T; L^\infty(\mathbb{R}^n))$ of (1.2.1) (cf.[53]). On the other hand, the first result with a singular initial data is due to the pioneering work of Weissler [68, 69]. For such power type nonlinearities, the scale invariance property plays an important role. If the function $u(t,x)$ satisfies (1.2.1), then for any $\lambda > 0$, the scaled function

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$$

also satisfies the equation (1.2.1) again. Moreover, the $L^r$- norm is invariant under this scaling if and only if

$$r = r_c := \frac{n(p-1)}{2}.$$

With regard to this critical exponent, one can classify the existence and uniqueness results of the (1.2.1) into the following two cases:

Case A. If $r \geq r_c$ and $r > 1$ or $r > r_c$ and $r \geq 1$, Weissler [68] and Brezis–Cazenave [7] proved that for any $u_0 \in L^r(\mathbb{R}^n)$ there exists a positive time $T = T(u_0)$ and a unique solution to (1.2.1) in $u \in C([0,T]; L^r(\mathbb{R}^n)) \cap L^\infty_{loc}((0,T); L^\infty(\mathbb{R}^n))$.

Case B. If $r < r_c$, Weissler [68] and Brezis–Cazenave [7] indicated that there exists no positive solution in any weak sense (cf. Haraux–Weissler [39], Tayachi [66]).

Those results are considered along the theory of evolution equations using the semi-group theory developed by Hille-Yosida and it is common idea to introduce the notion of the mild solutions, i.e., $u$ is called as the mild solutions to (1.2.1) if

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \big( |u(s)|^{p-1} u(s) \big) ds \tag{1.2.2}$$

in $C([0,T]; L^r(\mathbb{R}^n))$, where $e^{t\Delta}$ is the standard heat evolution operator (see [68, 69] and to study the problem of existence and uniqueness in the larger class of functions is a basic target in this research field. Brezis-Cazenave [7] proved unconditional uniqueness for $r > r_c$, $r \geq p$ and for $r = r_c$, $r > p$ to the equation(1.2.1) In the double critical case $r = r_c = p$, namely $p = r = \frac{n}{n-2}$, Ni–Sacks [59] proved that unconditional uniqueness does not hold on the unit ball in $\mathbb{R}^n$ . In [67] Terraneo extended this non-uniqueness result to the whole space $\mathbb{R}^n$ for suitable initial data.

In the critical case $r = r_c$ and $n \geq 3$, Weissler [69] proved a global existence of a solution to (1.2.1) under a smallness assumption on the initial data of (1.2.1). Some generalization to a semilinear parabolic equation with a nonlinear gradient term was done by Snoussi–Tayachi–Weissler [65]. For the related and extended results see the references [28], [44]–[48].

On the other hand, our main problem, the Cauchy problem (1.1.1) has a very similar nature with the problem (1.2.1). It has the scaling invariant property with respect to the corresponding scale transformation, i.e., if the function $u(t, x)$ satisfies (1.1.1), then for any $\lambda > 0$, the scaled function

$$u_\lambda(t, x) = \lambda^{\frac{1}{p-1}} u(\lambda^2 t, \lambda x), \quad t > 0,\ x \in \mathbb{R}^n,$$

also satisfies the equation of (1.1.1). The space of initial data $L^r(\mathbb{R}^n)$ is invariant under this scaling if and only if

$$r = r_c = n(p - 1).$$

This exponent plays an important role for the well-posedness of the problem (1.1.1) as a limiting or critical exponent. It is pointed out by the pioneering work of Weissler [69] that the ill-posedness of the positive solution occurs on the scale invariant function space in the case of the Fujita critical exponent. These facts lead that very similar facts would hold even in a problem (1.1.1) closer to the fluid model with the nonlinear term which involves the derivative.

The problem (1.1.1) has been considered by a number of authors (see e.g. [1], [14]–[20], [36], [55], [74], [75]). Among others, Escobedo–Zuazua [16] showed that for initial data $u_0 \in L^1(\mathbb{R}^n)$, there exists a unique global classical solution $u \in C([0, \infty); L^1(\mathbb{R}^n))$ of (1.1.1) in

$$u \in C((0, \infty); W^{2,q}(\mathbb{R}^n)) \cap C^1((0, \infty); L^q(\mathbb{R}^n)),$$

for every $q \in (1, \infty)$. When $p > 1 + \frac{1}{n}$, they prove that the global existence and the large time behavior of solutions is given by the heat kernel. When $p = 1 + \frac{1}{n}$, they prove that the large time behavior of solutions (1.1.1) with initial data in $L^1(\mathbb{R}^n)$ is given by a self-similar solution. The solution to (1.1.1) preserves the initial mass $\int_{\mathbb{R}^n} u_0 dx$ for all $t > 0$, namely

$$\int_{\mathbb{R}^n} u(t, x)dx = \int_{\mathbb{R}^n} u_0(x)dx,$$

and if $u_0(x) \geq 0$ this implies the $L^1$- conservation law. In this case, the Fujita critical exponent corresponding to the nonlinear term $\nabla(|u|^{p-1}u)$ is given by $p = 1 + \frac{1}{n}$, which is expected by the scale invariance. Unlike the case of nonlinear heat equation (1.2.1) the solution does not develop the singularity within a finite time for all exponent $p > 1$. On the other hand, for the time local well-posedness of the Cauchy problem (1.2.1), the result is very similar to one of the Fujita type problem by Weissler [69]. The (1.2.1) with a power type nonlinearity is ill-posed. The vorticity Navier-Stokes equations is well-posed and the well-posedness property changes subtly depending on the structure of the problem. In order to consider the time local well-posedness of the problem for the critical exponent $p = 1 + \frac{1}{n}$, we construct the solution by the method of the integral equations via the heat semigroup in the case including the exponents before and after, in particular case of the critical exponent, the solution can be appropriately obtained even in the critical space $L^1_{uloc,\rho}(\mathbb{R}^n)$.

## 1.3 Uniformly local Lebesgue spaces

*Definition* (Uniformly local Lebesgue spaces). Let $1 \le r \le \infty$ and $\rho > 0$. The uniformly local Lebesgue spaces on $\Omega \subseteq \mathbb{R}^n$ denoted by $L^r_{\mathrm{uloc},\rho}(\Omega)$, is defined by

$$L^r_{\mathrm{uloc},\rho}(\Omega) := \Big\{ f \in L^1_{\mathrm{loc}}(\Omega) : \ \|f\|_{L^r_{\mathrm{uloc},\rho}} < \infty \Big\},$$

where for $\rho > 0$

$$\|f\|_{L^r_{\mathrm{uloc},\rho}} = \begin{cases} \sup\limits_{x\in\Omega} \left( \displaystyle\int_{B_\rho(x)\cap\Omega} |f(y)|^r dy \right)^{\frac{1}{r}}, & 1 \le r < \infty, \\ \sup\limits_{x\in\Omega} \sup\limits_{y\in B_\rho(x)\cap\Omega} |f(y)|, & r = \infty. \end{cases} \tag{1.3.1}$$

where $B_\rho(x) := \{\ y \in \Omega : |x - y| < \rho\}$ denotes a open ball in $\Omega$ with radius $\rho > 0$ and center $x \in \Omega$. Here we identify $L^\infty_{\mathrm{uloc},\rho}(\Omega)$ as $L^\infty(\Omega)$. The space $L^r_{\mathrm{uloc},\rho}(\Omega)$ is a Banach space with the norm defined in (1.3.1).

The Sobolev spaces $W^{k,r}_{\mathrm{uloc},\rho}(\Omega)$ for $1 \le r \le \infty$, $\rho > 0$ and $k = 1, 2, \ldots$ are analogously introduced. We define by

$$W^{k,r}_{\mathrm{uloc},\rho}(\Omega) := \Big\{ f \in L^r_{\mathrm{loc}}(\Omega) : \ \|f\|_{W^{k,r}_{\mathrm{uloc},\rho}} < \infty \Big\},$$

where for $\rho > 0$,

$$\|f\|_{W^{k,r}_{\mathrm{uloc},\rho}} = \|f\|_{L^r_{\mathrm{uloc},\rho}} + \sum_{|\alpha|=k} \|\partial_x^\alpha f\|_{L^r_{\mathrm{uloc},\rho}}. \tag{1.3.2}$$

We denote $W^{1,2}_{\mathrm{uloc},\rho}(\Omega)$ as $H^1_{\mathrm{uloc},\rho} = H^1_{\mathrm{uloc},\rho}(\Omega)$. for simplicity. Bounded uniformly continuous functions $BUC(\Omega)$ is not dense in uniformly local Lebesgue spaces $L^r_{\mathrm{uloc},\rho}(\Omega)$.

*Definition* ($\mathcal{L}^r_{\mathrm{uloc},\rho}(\Omega)$). We define the subspace $\mathcal{L}^r_{\mathrm{uloc},\rho}(\Omega)$ of $L^r_{\mathrm{uloc},\rho}(\Omega)$ as the closure of the space of bounded uniformly continuous functions $BUC(\Omega)$ in the space $L^r_{\mathrm{uloc},\rho}(\Omega)$, i.e.,

$$\mathcal{L}^r_{\mathrm{uloc},\rho}(\Omega) := \overline{BUC(\Omega)}^{\|\cdot\|_{L^r_{\mathrm{uloc},\rho}}}$$

and define $\mathcal{L}^\infty_{\mathrm{uloc},\rho}(\Omega) = BUC(\Omega)$. The space $\mathcal{L}^r_{\mathrm{uloc},\rho}(\Omega)$ is a Banach space with the norm defined in (1.3.1).

The authors in [4]–[6], [50] and [57] also make use spaces of functions which have the property that their elements have some uniform size when it is measured in balls of fixed radius but arbitrary center. These are the so-called uniformly local spaces. It turns out that these spaces are very natural and useful for equations in unbounded domains since, as in the case of bounded ones, they enjoy suitable inclusion properties, they have locally compact embeddings and constant functions belong to them. In particular, when trying to analyze parabolic equations in unbounded domains, these spaces will allow to consider large classes of initial data with no prescribed behavior at infinity and even allowing for local singularities.

## 1.4 Well-posedness in uniformly local spaces

We introduce mild solutions to (1.1.1) in uniformly local Lebesgue spaces $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.

*Definition* (Mild solutions). Let $p > 1$, $1 \le r < \infty$ and $T > 0$. Suppose $u_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$. The function $u$ is called a mild solution of the convection–diffusion equations (1.1.1) on $[0, T) \times \mathbb{R}^n$ if $u$ satisfies

$$u(t) = e^{t\Delta}u_0 + \int_0^t a \cdot \nabla e^{(t-s)\Delta}\big(|u(s)|^{p-1}u(s)\big)ds \tag{1.4.1}$$

in $C([0, T); \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$, where $e^{t\Delta}$ denotes the heat semigroup defined by

$$e^{t\Delta}f = G_t * f \quad \text{with} \quad G_t(x) = \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0. \tag{1.4.2}$$

Under the framework on the evolution equation, once we establish the existence and uniqueness of the mild solution the above, the standard theory of the evolution equation over any Banach space ensure that the regularity of the mild solution. Namely one can see that the following setting of standard mild solutions (the strong solution) can be derived from our main theorem for the well-posedness of the mild solution.

For the same initial data in the above definition, the function $u$ is called a strong mild solution of the convection–diffusion equations (1.1.1) on $[0, T) \times \mathbb{R}^n$ if $u$ satisfies $u \in C\big((0, T); L^\infty(\mathbb{R}^n) \cap W^{1,r}_{\mathrm{uloc},\rho}(\mathbb{R}^n)\big)$ and it satisfies

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}a \cdot \nabla\big(|u(s)|^{p-1}u(s)\big)ds \tag{1.4.3}$$

in $C([0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$. It is straightforward to see that the strong mild solution satisfies the equation (1.1.1) in $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ and hence the solution is the strong solution.

We state the well-posedness result to (1.1.1) in uniformly local Lebesgue spaces $L^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.

**Theorem 1.4.1** (The local well-posedness)**.** *Let $n \geq 1$, $p > 1$ and $1 \leq r < \infty$ with*

$$\begin{cases} r \geq n(p-1) & \text{if} \quad p > 1 + \frac{1}{n}, \\ r \geq 1 & \text{if} \quad p = 1 + \frac{1}{n}, \\ r \geq 1 & \text{if} \quad p < 1 + \frac{1}{n}. \end{cases} \tag{1.4.4}$$

*Assume that $1 \leq q < \infty$ with $p \leq q \leq pr$ and $r < q$. Then for any $u_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$, there exists $T = T(u_0) > 0$ and a unique mild solution $u \in C([0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)) \cap C((0,T);\mathcal{L}^q_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ satisfying* (1.4.3)*. Furthermore, the Cauchy problem* (1.1.1) *is well-posed in*

$$C([0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)) \cap C((0,T);\mathcal{L}^q_{\mathrm{uloc},\rho}(\mathbb{R}^n)).$$

*Namely for any initial data $u_0, v_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$ and the corresponding solutions $u(t), v(t)$ to* (1.1.1)*, $u(t) \to v(t)$ in $C([0,T);\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n))$ as $u_0 \to v_0$ in $\mathcal{L}^r_{\mathrm{uloc},\rho}(\mathbb{R}^n)$.*

*Remark.* Since our convection-diffusion equations(1.1.1) has a scale invariance property so for scaling critical class global existence result holds for small initial data.

It is interesting and meaningful to compare our result to the result for the Fujita type nonlinear heat equation. For the subcritical case $r > r_c = n(p-1)$, the problem (1.1.1) is time locally well-posed and it is the same as the case of the problem (1.2.1). However, for the Fujita critical case $p = 1 + \frac{1}{n}$, the solution exists even for the positive initial data and it does not blow up in a finite time. Such a behavior can be expected by the Escobedo-Zuazua result [16]. Moreover, the Cauchy problem remains well-posed even for the scaling critical case $r = r_c = 1$ for the Fujita critical case $p = 1 + \frac{1}{n}$. This is also expected from the result in [16] but our result support that the local well-posedness can be reflecting the local property of the problem only.

On the other hand, the difference of the local well-posedness theory is on the unconditional uniqueness that is proven by Brezis-Cazenave [7] for the Fujita type equation. In the convection-diffusion equations, the unconditional uniqueness and even unconditional well-posedness under the mild solutions frame work, it is not clear if it is true for the scaling critical case $r = r_c = 1$ if the Fujita critical case $p = 1 + \frac{1}{n}$.

Those difference indeed stems from the structure of the nonlinear term in (1.1.1), namely the nonlinear term involves the derivative. Upon this view point, it is also meaningful to compare with the result for the Cauchy problem of the incompressible Navier-Stokes equations. It is proven that the Navier-Stokes equations remains well-posed in the scaling critical spaces:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, & t > 0, x \in \mathbb{R}^n, \\ \text{div } u = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = \text{rot } u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the fluid velocity and $p = p(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ stands for the pressure. It is well-known that the Navier-Stokes equations have a similar scaling invariant structure in $L^n(\mathbb{R}^n)$ and it is well-posed in such a space (cf. Giga-Miyakawa [33], Giga-Miyakawa-Osada [34] and Giga-Giga-Saal [32]. Our result is corresponding to the Navier-Stokes case and it is possible to extend our result to the case the Navier-Stokes equations.

## 1.5 Weak solutions in uniformly local spaces

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain with uniform $C^2$ boundary. We introduce the weak solutions to (1.1.1) in uniformly local Lebesgue spaces $L^r_{\text{uloc},\rho}(\Omega)$. We consider weak solutions in uniformly local Lebesgue spaces because the weak solutions case uniqueness result hold on larger class. For this case we consider the Cauchy-Dirichlet problem of a time dependent convection-diffusion equation: For $a \in \mathbb{R}^n$,

$$\begin{cases} \partial_t u - \Delta u = a \cdot \nabla(|u|^{p-1}u), & t > 0,\ x \in \Omega, \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.5.1}$$

where $u = u(t,x)$; $\mathbb{R}_+ \times \Omega \to \mathbb{R}$ is the unknown function, and $u_0 = u_0(x)$; $\Omega \to \mathbb{R}$ is a given initial data.

Let $BUC^\infty_c(\Omega)$ denotes a class of infinitely many time differentiable uniformly continuous bounded functions whose supports are away from the boundary. Namely for $f \in BUC^\infty_c(\Omega)$, $\text{supp } f \cap \partial\Omega = \emptyset$. We should note that the class of the infinitely many differentiable compact supported functions $C^\infty_0(\Omega) \subsetneq BUC^\infty_c(\Omega)$ if $\Omega$ in an unbounded domain.

*Definition*($\mathcal{H}^1_{c,\text{uloc},\rho}(\Omega)$). Let $\mathcal{W}^{1,r}_{c,\text{uloc},\rho}(\Omega)$ be the closure of the $BUC^\infty_c(\Omega)$ in $W^{1,r}_{\text{uloc},\rho}(\Omega)$. We denote $\mathcal{H}^1_{c,\text{uloc},\rho}(\Omega)$ for $\mathcal{W}^{1,2}_{\text{uloc},\rho}(\Omega)$.

As is commented in the above $\mathcal{W}^{1,r}_{c,\text{uloc},\rho}(\Omega)$ is larger class than $W^{1,r}_{0,\text{uloc},\rho}(\Omega) \equiv \overline{C^\infty_0(\Omega)}^{W^{1,r}_{\text{uloc},\rho}(\Omega)}$ if the domain is unbounded.

*Definition* (Weak $L^r_{\text{uloc}}(\Omega)$-solutions). Let $1 \le r < \infty$, $\rho > 0$. For an initial data $u_0 \in \mathcal{L}^r_{\text{uloc},\rho}(\Omega)$ and $T > 0$, we say that $u$ is a weak $L^r_{\text{uloc}}(\Omega)$-solution of (1.5.1) in $(0,T)\times\Omega$, if

1) $u \in C([0,T) : L^r_{\mathrm{uloc},\rho}(\Omega)) \cap L^2(0,T : \mathcal{H}^1_{c,\mathrm{uloc},\rho}(\Omega) \cap \mathcal{L}^r_{\mathrm{uloc},\rho}(\Omega))$,

2) $u(t) \rightharpoonup u_0$ in $*$-weakly in $L^r_{\mathrm{uloc},\rho}(\Omega)$,

3) $u$ satisfies

$$\int_0^T \int_\Omega \big\{ - u\partial_t \phi + \nabla u \cdot \nabla \phi + a|u|^{p-1} u \cdot \nabla \phi \big\} dxdt = 0$$

for all $\phi \in C_0^\infty((0,T) \times \Omega)$.

**Theorem 1.5.1** (Existence and uniqueness of a weak solution [37] )**.** *Let $p > 1$ and $1 \le r < \infty$ with*

$$\begin{cases} r \ge n(p-1) & \text{if} \quad p > 1 + \frac{1}{n}, \\ r > 1 & \text{if} \quad p = 1 + \frac{1}{n}, \\ r \ge 1 & \text{if} \quad 1 < p < 1 + \frac{1}{n}. \end{cases} \tag{1.5.2}$$

*There exists a positive constant $\gamma_0$, depending only on $n$, $p$ and $r$, such that, if for any initial data $u_0 \in \mathcal{L}^r_{\mathrm{uloc},\rho}(\Omega)$ satisfies*

$$\rho^{\frac{1}{p-1} - \frac{n}{r}} \|u_0\|_{L^r_{\mathrm{uloc},\rho}} \le \gamma_0 \tag{1.5.3}$$

*for some $\rho > 0$, then there exists a unique weak $L^r_{\mathrm{uloc}}(\Omega)$-solution $u$ of* (1.5.1) *in* $(0, \mu\rho^2) \times \Omega$ *such that*

$$\sup_{0<t<\mu\rho^2} \|u(t)\|_{L^r_{\mathrm{uloc},\rho}} \le C \|u_0\|_{L^r_{\mathrm{uloc},\rho}}, \tag{1.5.4}$$

*where $C$ and $\mu$ are independent of $u$. Besides the solution has a uniform estimate*

$$\|u\|_{L^\infty((0,\mu\rho^2)\times\Omega)} \le C \left( \int_0^{\mu\rho^2} \|u(t)\|^r_{L^r_{\mathrm{uloc},\rho}} dt \right)^{\frac{1}{r}} \tag{1.5.5}$$

*and hence $u \in L^\infty\big((0,\mu\rho^2) \times \Omega\big)$ for some $\mu > 0$.*

In the assumption on the initial data (1.5.3), the constant $\gamma_0 > 0$ is a constant only depending on $n$, $r$ and $p$. Hence one can regard this condition on the initial data as the restriction on the choice of $\rho > 0$. Since the function class $L^r_{\mathrm{uloc},\rho}(\Omega)$ does not depend on $\rho > 0$, we have a room for the choice of $\rho > 0$ depending on the initial data. This choice is reflecting how long the local solution can be continued.

As a corollary of Theorem 1.5.1, we have:

**Corollary 1.5.2** (Global existence [37])**.** *Let $p > 1 + \frac{1}{n}$. Then there exists a constant $\gamma > 0$ such that, if $u_0 \in L^{n(p-1)}(\Omega)$ and $\|u_0\|_{L^{n(p-1)}(\Omega)} \le \gamma$, then problem* (1.5.1) *has a global solution.*

## 1.6 Weak solutions in amalgam spaces

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain with uniform $C^2$ boundary. We introduce the weak solutions to (1.5.1) in amalgam spaces $L_\rho^{r,\nu}(\Omega)$.

*Definition* (Amalgam spaces). Let $1 \leq r, \nu < \infty$. The amalgam spaces on $\Omega \subseteq \mathbb{R}^n$, $L_\rho^{r,\nu}(\Omega)$ is defined by

$$L_\rho^{r,\nu}(\Omega) := \{f : \ \|f\|_{L_\rho^{r,\nu}} < \infty\},$$

where for $\rho > 0$

$$\|f\|_{L_\rho^{r,\nu}} = \left( \sum_{x_k \in \rho\mathbb{Z}^n} \|f\|^\nu_{L^r(B_\rho(x_k)\cap\Omega)} \right)^{\frac{1}{\nu}}, \tag{1.6.1}$$

the usual adjustments being made if $r$ or $\nu$ is infinity. The space $L_\rho^{r,\nu}(\Omega)$ is a Banach space with the norm defined in (1.6.1).

The Sobolev spaces $W_\rho^{k,r,\nu}(\Omega)$ for $1 \leq r, \nu \leq \infty$, $\rho > 0$ and $k = 1, 2, \dots$ are analogously introduced. We define by

$$W_\rho^{k,r,\nu}(\Omega) := \left\{ f : \|f\|_{W_\rho^{k,r,\nu}} < \infty \right\},$$

where for $\rho > 0$,

$$\|f\|_{W_\rho^{k,r,\nu}} = \|f\|_{L_\rho^{r,\nu}} + \sum_{|\alpha|=k} \|\partial_x^\alpha f\|_{L_\rho^{r,\nu}}.$$

We denote $W_\rho^{1,2,2}(\Omega)$ as $H_\rho^1(\Omega)$ for simplicity.

*Definition* (Weak $L^{r,\nu}(\Omega)$-solutions). Let $1 \leq r, \nu < \infty$, $\rho > 0$ and $H_{0,\rho}^1(\Omega)$ be the closure of the $C_0^\infty(\Omega)$ in $H_\rho^1(\Omega)$. For an initial data $u_0 \in L_\rho^{r,\nu}(\Omega)$ and $T > 0$, we say that $u$ is a weak $L^{r,\nu}(\Omega)$-solution of (1.5.1) in $(0,T) \times \Omega$, if

1) $u \in C([0,T) : L_\rho^{r,\nu}(\Omega)) \cap L^2(0,T : H_{0,\rho}^1(\Omega) \cap L_\rho^{r,\nu}(\Omega))$,

2) $u(t) \rightharpoonup u_0$ in $*$-weakly in $L_\rho^{r,\nu}(\Omega)$,

3) $u$ satisfies

$$\int_0^T \int_\Omega \{ -u\partial_t\phi + \nabla u \cdot \nabla\phi + a|u|^{p-1}u \cdot \nabla\phi \} dxdt = 0$$

for all $\phi \in C_0^\infty((0,T) \times \Omega)$.

Finally, we state the existence and uniqueness result to (1.5.1) in amalgam spaces $L_\rho^{r,\nu}(\Omega)$.

**Theorem 1.6.1** (Existence and uniqueness of a weak solution). *Let $p > 1$ and $1 \leq r, \nu < \infty$ with*

$$\begin{cases} r \geq n(p-1) & \text{if} \quad p > 1 + \frac{1}{n}, \\ r > 1 & \text{if} \quad p = 1 + \frac{1}{n}, \\ r \geq 1 & \text{if} \quad 1 < p < 1 + \frac{1}{n}. \end{cases} \tag{1.6.2}$$

*There exist a positive constant $\gamma_0$, depending only on $n$, $p$ and $r$, such that, if for any initial data $u_0 \in L^{r,\nu}_\rho(\Omega)$ satisfies*

$$\rho^{\frac{1}{p-1}-\frac{n}{r}}\|u_0\|_{L^{r,\nu}_\rho} \le \gamma_0 \tag{1.6.3}$$

*for some $\rho > 0$, then there exists a unique weak $L^{r,\nu}(\Omega)$- solution $u$ of* (1.1.1) *in* $(0, \mu\rho^2) \times \Omega$ *such that*

$$\sup_{0<t<\mu\rho^2} \|u(t)\|_{L^{r,\nu}_\rho} \le C\|u_0\|_{L^{r,\nu}_\rho},$$

*where $C$ and $\mu$ are independent of $u$. Besides the solution has a uniform estimate*

$$\|u\|_{L^\infty((0,\mu\rho^2)\times\Omega)} \le C \left( \int_0^{\mu\rho^2} \|u(t)\|^r_{L^{r,\nu}_\rho} dt \right)^{\frac{1}{r}}$$

*and hence $u \in L^\infty\big((0, \mu\rho^2) \times \Omega\big)$ for some $\mu > 0$.*

The authors in [9]–[12], [25, 41, 51], make use the amalgam spaces. Amalgam spaces are Banach spaces of functions determined by a norm which distinguishes between local and global properties of functions. These spaces arise naturally in harmonic analysis. In 1926, Norbert Wiener, who was the first one to introduce the amalgam spaces, consider some special cases in [71]–[73]. Amalgams have been reinvented many times in the literature; the first systematic study appears by Holland in [43]; an excellent review article is [26]. H. Feichtinger [22]–[24] introduced a far-reaching generalization of amalgam spaces to general topological groups and general local/global function spaces. The amalgams distinguish between local $L^p$ and global $l^q$ properties of functions in the ways we expect. For example, rearrangements do not have identical norms in general and inclusions behave correctly. Furthermore, amalgam spaces is a space between usual Lebesgue spaces and uniformly local Lebesgue spaces.

The rest of this thesis is organized as follows. We introduce some properties of the uniform local Lebesgue spaces and show some important inequalities and lemmas used for the proof. The dissipative estimate for the heat kernel in the uniformly local space is shown which plays an important role in Chapter 3. In Chapter 3 we prove the well-posedness for the Cauchy problem (1.1.1), Theorem 1.4.1 using the Banach fixed point theorem. We then consider the weak solution for (1.5.1) in Chapter 4 in the uniformly local Lebesgue spaces and show the proof of Theorem 1.5.1. Finally, we show the proof of Theorem 1.6.1 by establishing a crucial a priori estimate in the amalgam space in Chapter 5.

**Notations**

We denote $\mathbb{N}$ and $\mathbb{R}$ be the set of natural numbers and set of real numbers respectively. For any $n \in \mathbb{N}$, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.

For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$, we regard the notation of derivative as

$$\partial_t := \frac{\partial}{\partial t}, \qquad \nabla := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}), \qquad \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}.$$

The notation $B_\rho(x)$ express the open ball in $\mathbb{R}^n$ with radius $\rho > 0$ and center $x \in \mathbb{R}^n$, that is, for any $x \in \mathbb{R}^n$ and $\rho > 0$, $B_\rho(x) := \{ y \in \mathbb{R}^n : |x - y| < \rho \}$. The beta function $B(p, q)$ is defined by

$$B(p, q) = \int_0^1 z^{p-1}(1-z)^{q-1} dz, \quad p > 0, \ q > 0.$$

We introduce basic function spaces. Let $\Omega \subset \mathbb{R}^n$ be a domain. $C^k(\Omega)$ denotes sets of all functions having continuous derivative up to order $k$ on $\Omega$. $C_0^k(\Omega)$ denotes set of all functions in $C^k(\Omega)$ whose supports are compact in $\Omega$. $C^\infty(\Omega)$ denotes set of all functions which are infinitely differentiable on $\Omega$. $BUC(\Omega)$ denotes bounded uniformly continuous functions on $\Omega$. $C_0^\infty(\Omega)$ denotes set of all functions in $C^\infty(\Omega)$ whose supports are compact in $\Omega$. $W^k(\Omega)$ denotes set of all functions having weak derivative up to order $k$ on $\Omega$.

For $1 \le q \le \infty$, the Lebesgue space on any domain $\Omega \subset \mathbb{R}^n$; $L^q(\Omega)$ is defined by

$$L^q(\Omega) := \{ f : \text{measurable function on } \Omega \text{ such that } \|f\|_{L^q(\Omega)} < \infty \},$$

where

$$\|f\|_{L^q(\Omega)} = \begin{cases} \left( \int_\Omega |f(x)|^q dx \right)^{\frac{1}{p}}, & 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|, & q = \infty. \end{cases}$$

We use an abbreviated notation $\|\cdot\|_{L^q}$ instead of $\|\cdot\|_{L^q(\mathbb{R}^n)}$ for $1 \le q \le \infty$. For $1 \le q < \infty$, $L^q_{loc}(\Omega)$ is defined by

$$L^q_{loc}(\Omega) := \{ f : \text{measurable on } \Omega \text{ such that } \|f\|_{L^q(K)} < \infty \text{ for any compact set } K \subset \Omega. \}$$

Let $X$ be a Banach space, $I$ be an interval on $\mathbb{R}$ and $1 \le r \le \infty$. The vector valued function space $L^r(I; X)$ is called Bochner space and defined by

$$L^r(I; X) := \{ f : I \to X : \text{ strongly measurable function } \ \|f\|_{L^r(I;X)} < \infty \},$$

where

$$\|f\|_{L^r(I;X)} = \begin{cases} \left( \int_I \|f(t)\|_X^r dt \right)^{\frac{1}{r}}, & 1 \le r < \infty, \\ \operatorname*{ess\,sup}_{t \in I} \|f(x)\|_X, & r = \infty. \end{cases}$$

Let $k \ge 0$ and $p \ge 1$ and let $D^\alpha f$ be the $\alpha^{\text{th}}$- weak derivative of $f$ . The Sobolev space on $\Omega$, $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\mathbb{R}^n) := \{ f \in W^k(\Omega); D^\alpha f \in L^p(\Omega), \text{ for any } \alpha \text{ with } |\alpha| < k \},$$

endowed with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left( \sum_{|\alpha| \le k} \int_{\mathbb{R}^n} |D^\alpha f|^p dx \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sum_{|\alpha| \le k} \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |D^\alpha f|, & p = \infty. \end{cases}$$

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