Multilevel Latent Variable Modeling

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Outline

- Generalized linear measurement model
  - Two-parameter IRT model
  - Factor model for binary indicators
  - Generalizations
- Unifying measurement and multilevel regression models
- Generalized linear latent and mixed models (GLLAMMs)
- Multilevel MIMIC model
- Maximum (pseudo) likelihood estimation
- Multilevel structural equation model
- Some extensions

Two-parameter IRT models

- Two-parameter logistic (2-PL) model, for item i, person j:

$$\begin{aligned} \text{logit}[\text{Pr}(y_{ij} = 1 \mid \eta_j)] &= a_i(\theta_j - b_i) \\ &\equiv \beta_i + \lambda_i \eta_j = \underbrace{\lambda_i}_{a_i} (\underbrace{\eta_j}_{\theta_j} - \underbrace{-\beta_i/\lambda_i}_{b_i}) \end{aligned}$$

$$\eta_j \sim \mathrm{N}(0, \psi)$$

  - β_i is an intercept for item i
  - η_j is the **ability** of person j
  - λ_i is a slope or **discrimination parameter** for item i

- Two-parameter normal orgive model:

$$\underbrace{\Phi^{-1}}_{\text{probit}} [\text{Pr}(y_{ij} = 1|\eta_j)] = \beta_i + \lambda_i \eta_j$$

Factor model for binary indicators

- Latent response formulation

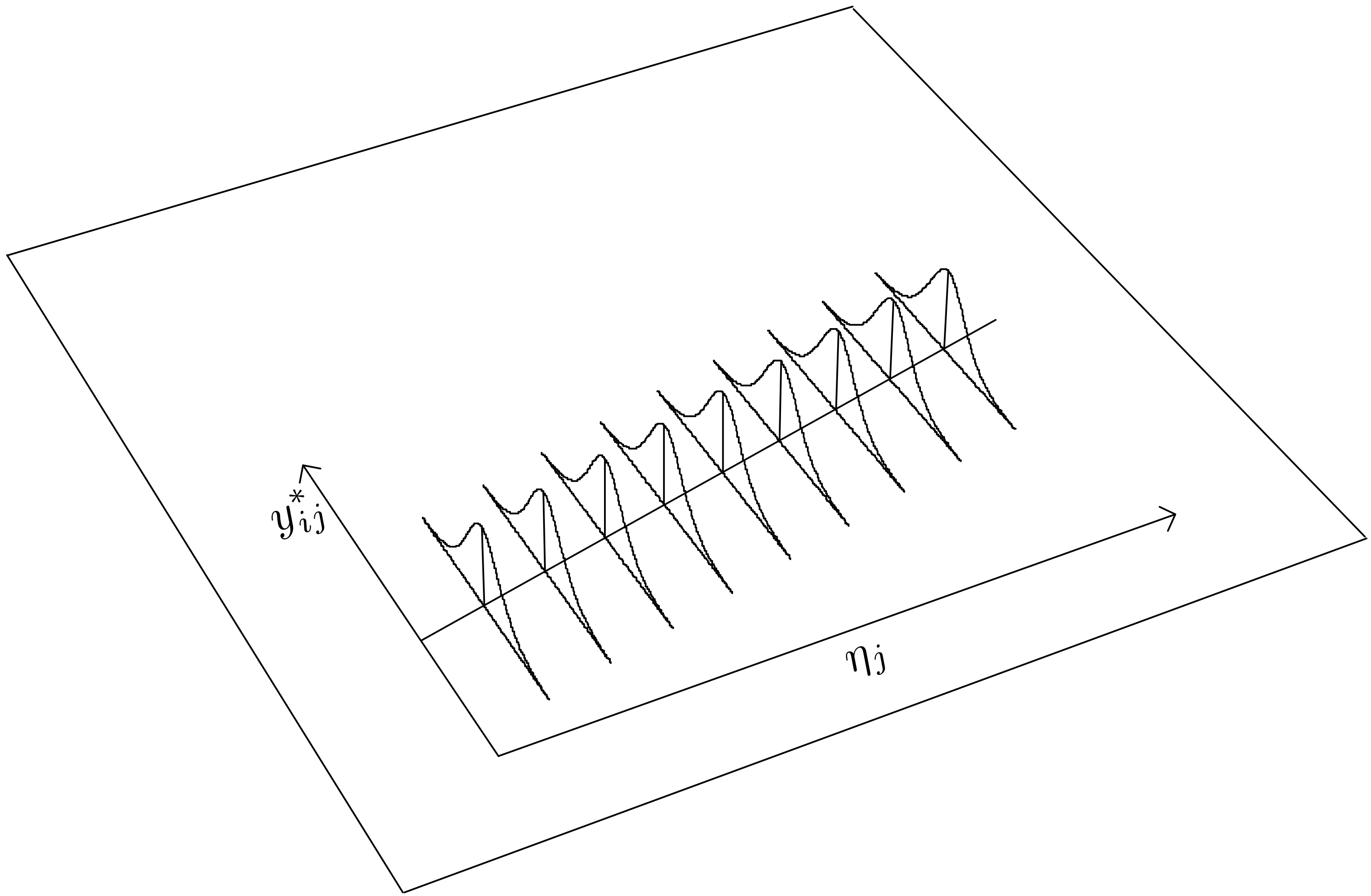
$$y_{ij}^{*} \ = \ \beta_i + \lambda_i \eta_j + \epsilon_{ij}, \qquad \eta_j \sim \mathrm{N}(0,1), \epsilon_{ij} \sim \mathrm{N}(0,1)$$

$$y_{ij} = \begin{cases} 1 & \text{if } y_{ij}^{*} > 0 \\ 0 & \text{otherwise} \end{cases}$$

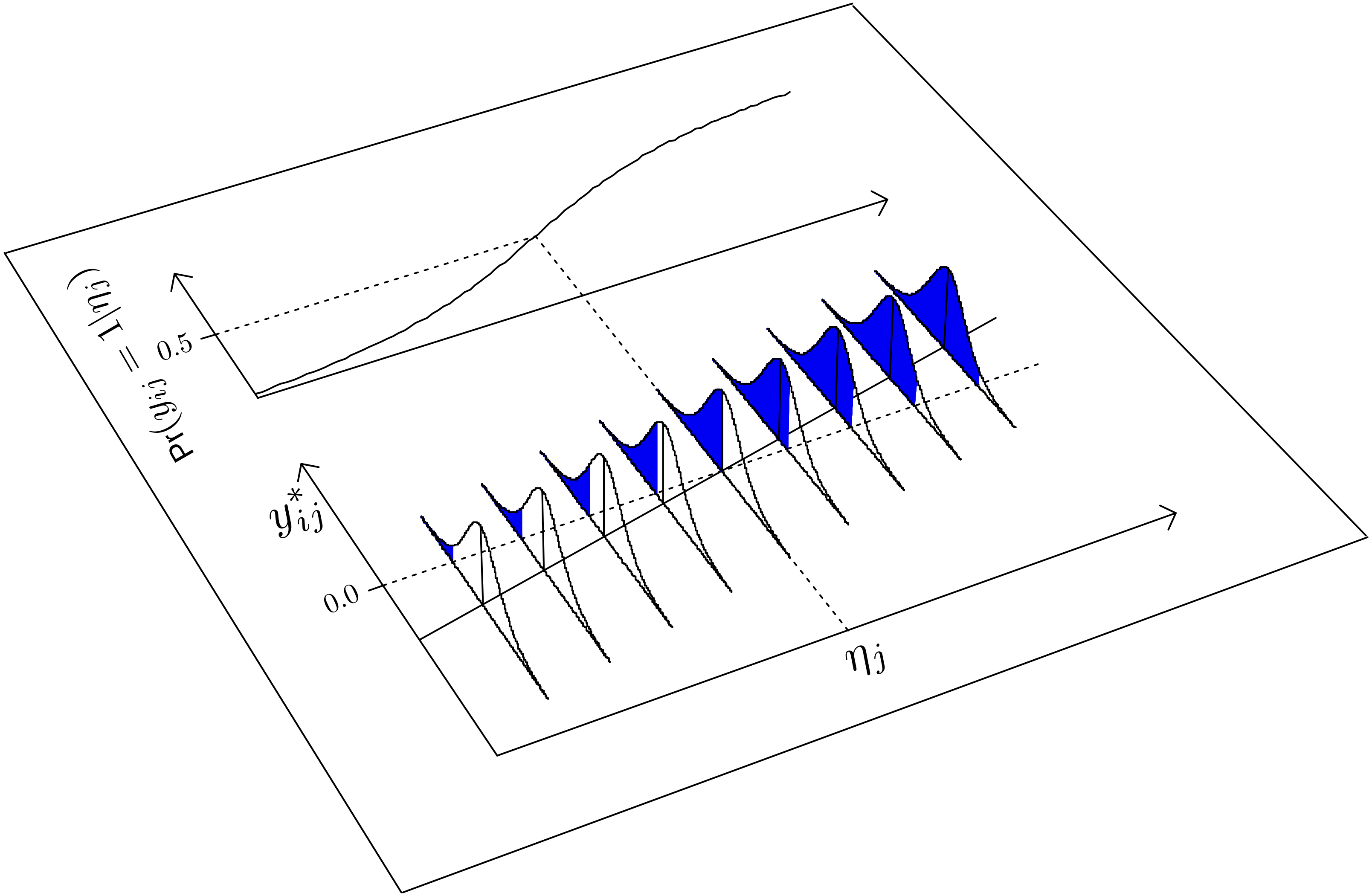
  - β_i is an intercept for item i
  - η_j is **common factor** for person j
  - λ_i is **factor loading** for indicator i
  - ϵ_{ij} is **unique factor** for item i

[Christofferson, 1975; Muthén, 1978]

Graphical illustration of latent response formulation



Graphical illustration of latent response formulation



Equivalence of IRT and factor models

- Factor model for binary indicators (latent response formulation)

$$\begin{aligned}
\text{Pr}(y_{ij}=1|\eta_j) &= \text{Pr}(y_{ij}^* > 0|\eta_j) = \text{Pr}(\beta_i + \lambda_i\eta_j + \epsilon_{ij} > 0) \\
&= \text{Pr}(\epsilon_{ij} > -(\beta_i + \lambda_i\eta_j)) \underbrace{=}_{\text{symmetry}} \text{Pr}(\epsilon_{ij} < \beta_i + \lambda_i\eta_j) \\
&= \Phi(\beta_i + \lambda_i\eta_j)
\end{aligned}$$

  - Normal ogive model (response function formulation)
- Logit IRT model:

$$\text{Pr}(y_{ij}=1|\eta_j) = \text{Pr}(\epsilon_{ij} < \beta_i + \lambda_i\eta_j) = \frac{\exp(\beta_i + \lambda_i\eta_j)}{1 + \exp(\beta_i + \lambda_i\eta_j)}$$

  - Factor model where unique factors have a logistic distribution
- For ordinal responses, latent response formulation leads Samejima's [1969] graded response model

[Takane & de Leeuw, 1987; Bartholomew, 1987]

Generalized linear measurement model

- Conditional expectation of response

$$\mu_{ij} \equiv \mathsf{E}(y_{ij} = 1|\eta_j) \quad [= \mathsf{Pr}(y_{ij} = 1|\eta_j) \text{ for binary responses}]$$

- Link function $g(\cdot)$ and distribution

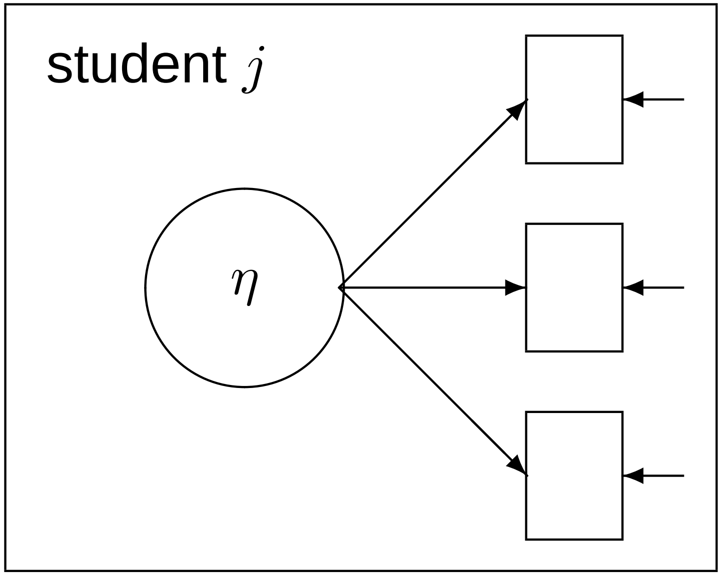
$$\begin{aligned} g(\mu_{ij}) &= \nu_{ij}, & y_{ij} &\sim \text{Exponential family}(\mu_{ij}) \\ \Phi^{-1}(\mu_{ij}) &= \beta_i + \lambda_i \eta_j, & y_{ij} &\sim \text{Bernoulli}(\mu_{ij}) \\ \text{logit}(\mu_{ij}) &= \beta_i + \lambda_i \eta_j, & y_{ij} &\sim \text{Bernoulli}(\mu_{ij}) \\ (\mu_{ij}) &= \beta_i + \lambda_i \eta_j, & y_{ij} &\sim \text{N}(\mu_{ij}, \theta_{ii}) \end{aligned}$$

- Other link functions: log, power, inverse, cumulative logit/probit, multinomial logit
- Other distributions: Poisson, gamma, multinomial

[Bartholomew, 1987; Arminger and Küsters, 1988, 1989; Mellenbergh, 1994; Rabe-Hesketh *et al.*, 2004]

Path diagram of measurement model

- Same diagram regardless of response model



- frame indicates ‘level’ at which variables vary
- ◯ encloses latent variables
- □ surrounds observed var.
- $\longrightarrow$ is regression
- $\rightarrow$ is residual variability: additive ϵ_{ij}, Poisson variability, etc.

Unifying measurement models and multilevel regression models

- One-parameter IRT model is a logistic random intercept model with
  - Items as level-1 units and persons as level-2 units
  - Ability as random intercept
  - Difficulties as regression coefficients of dummy variables for items
- Random-coefficient growth curve model is a two-factor model with fixed factor loadings
- However:
  - Cannot have factor loadings (or discrimination parameters) in multilevel models
  - Cannot have random coefficients of unbalanced covariates in measurement models
- Generalized Linear Latent and Mixed Models (GLLAMMs) overcome these limitations

[Meredith & Tisak, 1990; Rijmen *et al.*, 2003; De Boeck & Wilson, 2004]

GLLAMM response model:

Unifying measurement and multilevel models

- Units i (level 1) nested in clusters j (level 2), etc. up to level L

$$\boldsymbol{\nu} = \mathbf{X}\boldsymbol{\beta} + \sum_{l=2}^{L}\sum_{m=1}^{M_l} \eta_m^{(l)} \mathbf{Z}_m^{(l)} \boldsymbol{\lambda}_m^{(l)}$$

  - Measurement model:

$$\underbrace{\begin{bmatrix}\nu_{1j}\\ \nu_{2j}\\ \nu_{3j}\end{bmatrix}}_{\boldsymbol{\nu}_j} = \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\mathbf{X}_j}\underbrace{\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix}}_{\boldsymbol{\beta}} + \eta_{1j}^{(2)}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\mathbf{Z}_{1j}^{(2)}}\underbrace{\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}}_{\boldsymbol{\lambda}_1^{(2)}} = \begin{bmatrix}\beta_1+\eta_{1j}^{(2)}\lambda_1\\ \beta_2+\eta_{1j}^{(2)}\lambda_2\\ \beta_3+\eta_{1j}^{(2)}\lambda_3\end{bmatrix}$$

  - Multilevel regression model:

$$\underbrace{\begin{bmatrix}\nu_{1j}\\ \nu_{2j}\\ \nu_{3j}\end{bmatrix}}_{\boldsymbol{\nu}_j} = \underbrace{\begin{bmatrix}1&t_{1j}\\1&t_{2j}\\1&t_{3j}\end{bmatrix}}_{\mathbf{X}_j}\underbrace{\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}}_{\boldsymbol{\beta}} + \eta_{1j}^{(2)}\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\mathbf{Z}_{1j}^{(2)}} + \eta_{1j}^{(2)}\underbrace{\begin{bmatrix}t_{1j}\\ t_{2j}\\ t_{3j}\end{bmatrix}}_{\mathbf{Z}_{2j}^{(2)}} = \begin{bmatrix}\beta_1+\eta_{1j}^{(2)}+(\beta_2+\eta_{2j}^{(2)})t_{1j}\\ \beta_1+\eta_{1j}^{(2)}+(\beta_2+\eta_{2j}^{(2)})t_{2j}\\ \beta_1+\eta_{1j}^{(2)}+(\beta_2+\eta_{2j}^{(2)})t_{3j}\end{bmatrix}$$

[Rabe-Hesketh *et al.*, 2004]

GLLAMM structural model

$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\mathbf{w} + \boldsymbol{\zeta}$$

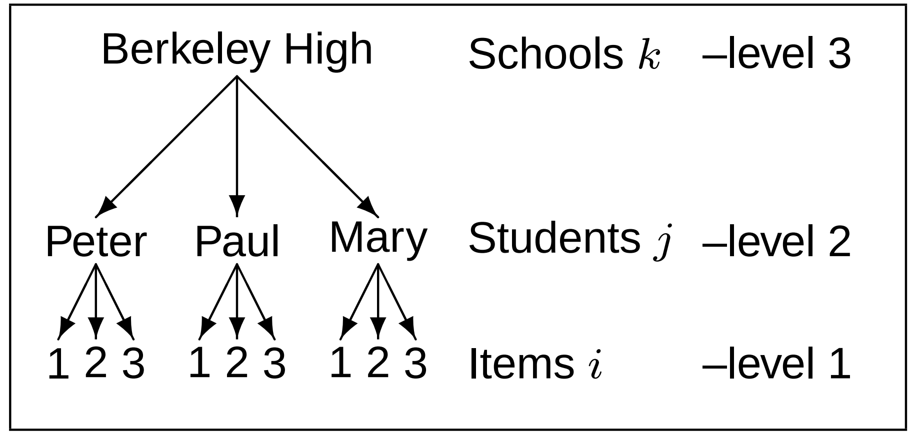
- $\boldsymbol{\eta}$ is the vector of all latent variables
- $\mathbf{B}$ is a an upper triangular matrix of regression coefficients
- $\boldsymbol{\Gamma}$ is a matrix of regression coefficients
- $\mathbf{w}$ is a vector of observed covariates
- $\boldsymbol{\zeta}$ is a vector of disturbances

$$\begin{aligned}\boldsymbol{\eta} &= (\overbrace{\eta_1^{(2)}, \eta_2^{(2)}, \ldots, \eta_{M_2}^{(2)}}^{\text{Level } 2}, \ldots, \overbrace{\eta_1^{(l)}, \ldots, \eta_{M_l}^{(l)}}^{\text{Level } l}, \ldots, \eta_{M_L}^{(L)}) \\ \boldsymbol{\zeta} &= (\zeta_1^{(2)}, \zeta_2^{(2)}, \ldots, \zeta_{M_2}^{(2)}, \ldots, \zeta_1^{(l)}, \ldots, \zeta_{M_l}^{(l)}, \ldots, \zeta_{M_L}^{(L)})\end{aligned}$$

- Multilevel version of [Muthén, 1984] (will be used later)

U.S. sample of PISA 2000 data

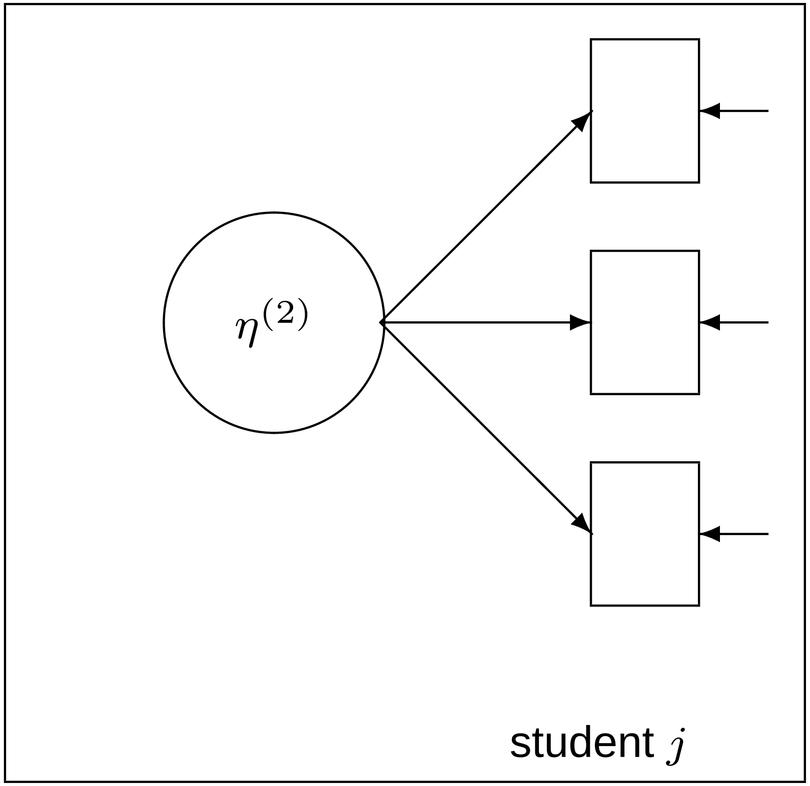
- Three-level data:
  (ignore PSUs here)



- Student-level covariates
  - [Female]: Student is female (dummy)
  - [ISEI]: International socioeconomic index
  - [Highschool]: Highest education level by either parent is high school (dummy)
  - [College]: Highest education level by either parent is college (dummy)
  - [English]: Test language (English) spoken at home (dummy)
- School-level covariate
  - [MnISEI]: School mean ISEI

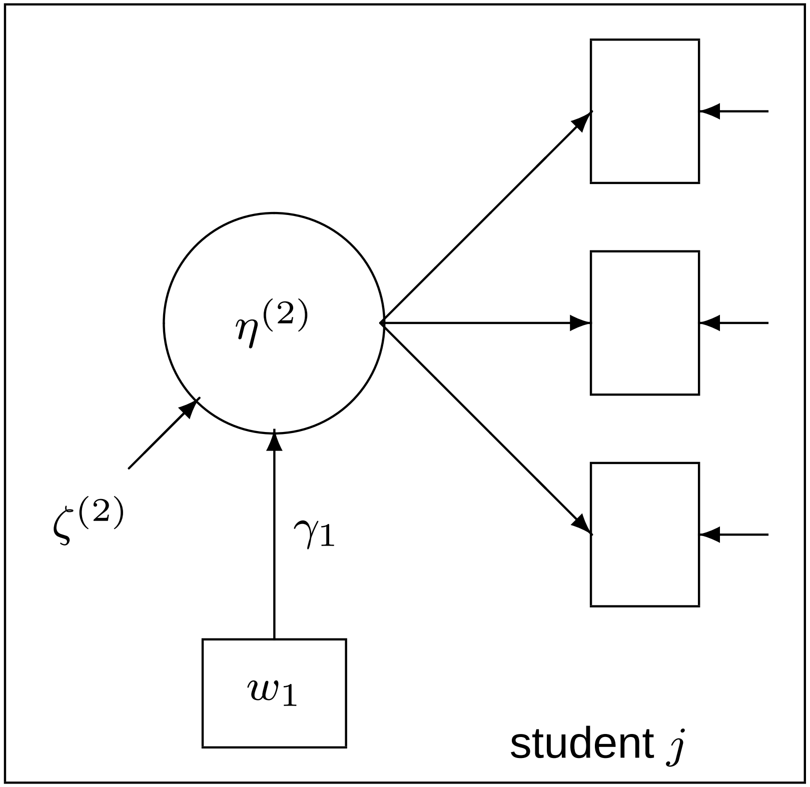
From measurement model to multilevel MIMIC model

Measurement model



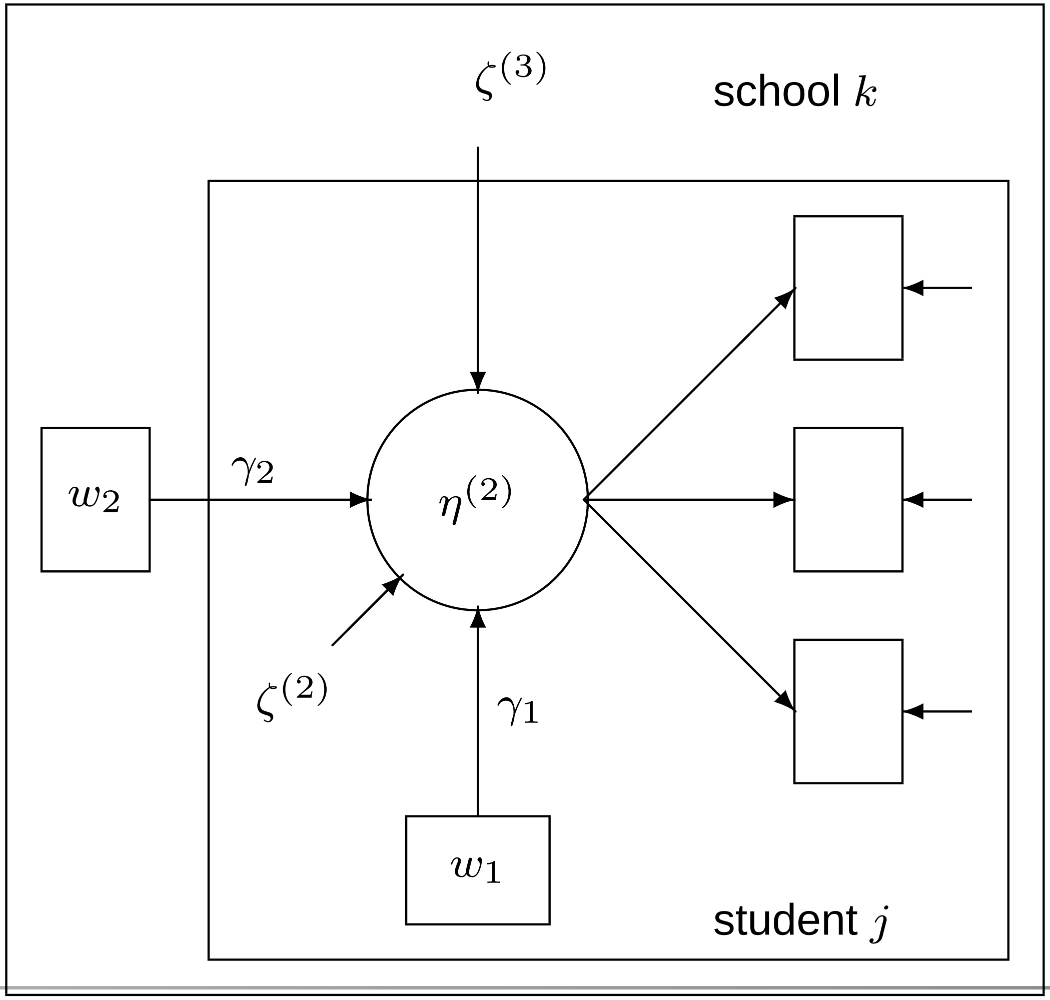
From measurement model to multilevel MIMIC model

Multiple Indicator Multiple Cause (MIMIC) model $\equiv$ IRT with latent regression



From measurement model to multilevel MIMIC model

Multilevel MIMIC model $\equiv$ IRT with multilevel latent regression



Multilevel MIMIC model

- **Response model:**
  Two-parameter logistic item response model for item i ($i\!=\!1,\ldots,7$):

$$\nu_{ijk} \;=\; \beta_i + \lambda_i \eta_{jk}^{(2)}$$

- **Structural model:**
  Two-level linear random intercept model for latent ability of student j in school k:

$$\eta_{jk}^{(2)} \;=\; \mathbf{w}'_{1jk}\boldsymbol{\gamma}_1 + \mathbf{w}'_{2k}\boldsymbol{\gamma}_2 + \zeta_{jk}^{(2)} + \zeta_k^{(3)}$$

- Disturbances at levels 2 (students j) and 3 (schools k)

$$\begin{aligned} \zeta_{jk}^{(2)} &\sim \mathrm{N}(0, \psi^{(2)}) \\ \zeta_k^{(3)} &\sim \mathrm{N}(0, \psi^{(3)}) \end{aligned}$$

Maximum likelihood estimation using adaptive quadrature

- Likelihood

$$\prod_{k=1}^{n^{(3)}} \int \left\{ \prod_{j=1}^{n_k^{(2)}} \int \left[\prod_{i=1}^{n_{jk}^{(1)}} f(y_{ijk} | \zeta_{jk}^{(2)}, \zeta_k^{(3)}) \right] \varphi(\zeta_{jk}^{(2)})\, \mathrm{d}\zeta_{jk}^{(2)} \right\} \varphi(\zeta_k^{(3)}) \mathrm{d}\zeta_k^{(3)}$$

- Integrals evaluated by adaptive quadrature
  - Ordinary quadrature: Random effects distribution is ‘kernel’ replaced by discrete distribution
  - Adaptive quadrature: Normal distribution approximating the integrand is ‘kernel’: Need cluster-specific means and standard deviations, similar to importance sampling
- Likelihood maximized by Newton-Raphson
- Implemented in Stata program `gllamm`

[Rabe-Hesketh, Skrondal & Pickles, 2005; Schilling & Bock, 2005; Pinheiro & Chao, 2006]

Taking into account PSUs and survey weights

- Three-stage survey
  - Stage 1 (Primary sampling units): Geographic areas
  - Stage 2: Schools k sampled with probabilities π_k, $w_k = 1/\pi_k$
  - Stage 3: Students j sampled with probabilities $\pi_{j|k}$, $w_{j|k} = 1/\pi_{j|k}$
- Log likelihood for three-level model

$$\sum_{k=1}^{n^{(3)}} \log \int \exp \left\{ \sum_{j=1}^{n_k^{(2)}} \log \int \exp \left[\sum_{i=1}^{n_{jk}^{(1)}} \log f(y_{ijk}|\zeta_{jk}^{(2)}, \zeta_k^{(3)}) \right] \varphi(\zeta_{jk}^{(2)})\, \mathrm{d}\zeta_{jk}^{(2)} \right\} \varphi(\zeta_k^{(3)}) \mathrm{d}\zeta_k^{(3)}$$

- Log pseudolikelihood for three-level model

$$\sum_{k=1}^{n^{(3)}} w_k \log \int \exp \left\{ \sum_{j=1}^{n_k^{(2)}} w_{jk} \log \int \exp \left[\sum_{i=1}^{n_{jk}^{(1)}} \log f(y_{ijk}|\zeta_{jk}^{(2)}, \zeta_k^{(3)}) \right] \varphi(\zeta_{jk}^{(2)})\, \mathrm{d}\zeta_{jk}^{(2)} \right\} \varphi(\zeta_k^{(3)}) \mathrm{d}\zeta_k^{(3)}$$

Taking into account PSUs and survey weights (cont'd)

- Conventional standard errors not appropriate with sampling weights
- **Sandwich estimator** of standard errors (Taylor linearization)

$$\widehat{\text{Cov}}(\widehat{\boldsymbol{\vartheta}}) = \widehat{\mathcal{I}}^{-1}\widehat{\mathcal{J}}\widehat{\mathcal{I}}^{-1}$$

  - $\mathcal{J}$: Expectation of outer product of gradients, approximated using PSU contributions to gradients
  - $\mathcal{I}$: Expected information, approximated by observed information
- Sandwich estimator accounts for
  - Sampling weights
  - Clustering at levels 'above' highest level of multilevel model
  - Stratification at stage 1
- Adaptive quadrature, pseudolikelihood, and sandwich estimator implemented in `gllamm`

[Rabe-Hesketh & Skrondal, 2006]

Estimates for multilevel MIMIC model:

Structural model

	Unweighted Maximum likelihood		Weighted Pseudo maximum likelihood		
Parameter	Est	(SE)	Est	(SE_R)	(SE_R^{PSU})
γ_1: [Female]	0.146	(0.122)	0.107	(0.201)	(0.241)
γ_2: [ISEI]	0.012	(0.004)	0.021	(0.008)	(0.007)
γ_3: [Highschool]	0.138	(0.249)	0.056	(0.472)	(0.357)
γ_4: [College]	0.411	(0.263)	0.101	(0.449)	(0.413)
γ_5: [English]	0.555	(0.227)	0.568	(0.230)	(0.252)
γ_6: [MnISEI]	0.039	(0.012)	0.020	(0.014)	(0.016)
$\psi^{(2)}$	1.244	(0.642)	1.201	(0.835)	(0.761)
$\psi^{(3)}$	0.111	(0.136)	0.051	(0.129)	(0.111)

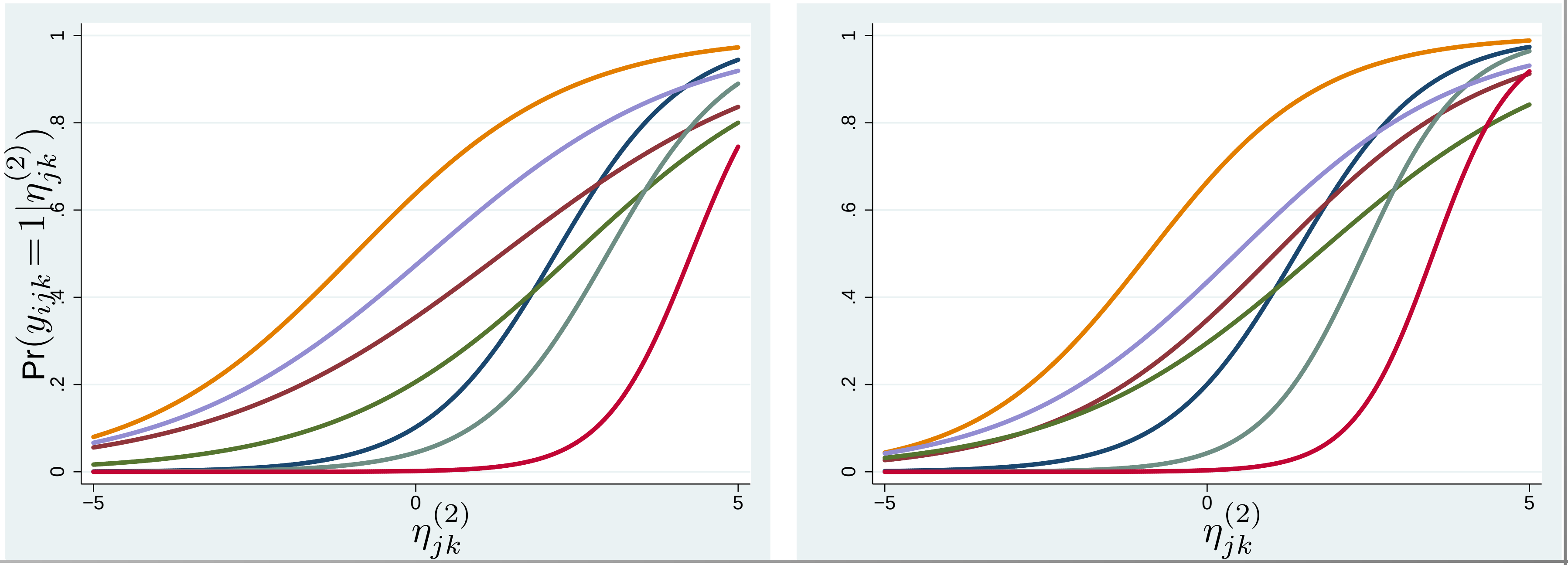
ICC=0.08 (ICC=0.14, not controlling for [MnISEI])

Estimates for multilevel MIMIC model: Measurement model

- Item characteristic curves for two-parameter logistic model

$$\text{Pr}(y_{ijk}=1|\eta_{jk}^{(2)}) = \frac{\exp(\beta_i + \lambda_i\eta_{jk}^{(2)})}{1+\exp(\beta_i + \lambda_i\eta_{jk}^{(2)})}$$

Unweighted   Weighted



GLLAMM specification of MIMIC structural model

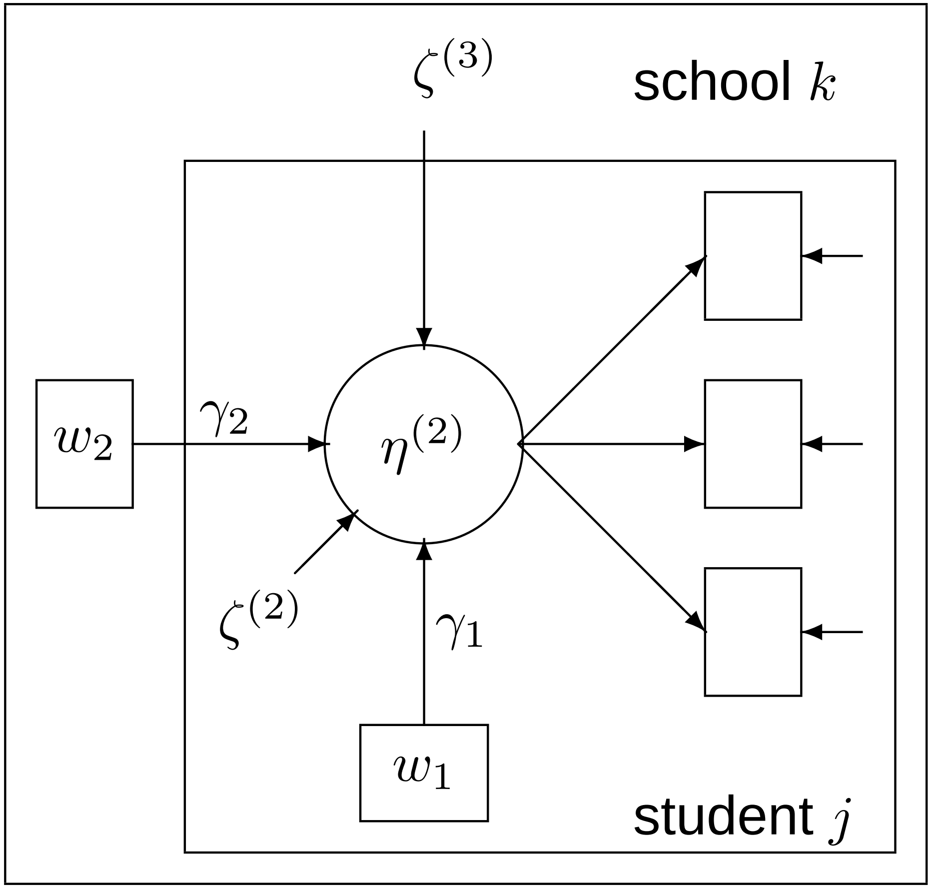
$$\boldsymbol{\eta} \;=\; \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\mathbf{w} + \boldsymbol{\zeta}$$

$$\underbrace{\begin{bmatrix} \eta_{jk}^{(2)} \\ \eta_k^{(3)} \end{bmatrix}}_{\boldsymbol{\eta}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \eta_{jk}^{(2)} \\ \eta_k^{(3)} \end{bmatrix}}_{\boldsymbol{\eta}} + \underbrace{\begin{bmatrix} \gamma_1 & \gamma_2 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{\Gamma}} \underbrace{\begin{bmatrix} w_{1jk} \\ w_{2k} \end{bmatrix}}_{\mathbf{w}} + \underbrace{\begin{bmatrix} \zeta_{jk}^{(2)} \\ \zeta_k^{(3)} \end{bmatrix}}_{\boldsymbol{\zeta}}$$

$$\eta_k^{(3)} = \zeta_k^{(3)}$$

$$\eta_{jk}^{(2)} = \gamma_1 w_{1jk} + \gamma_2 w_{2k} + \underbrace{\zeta_k^{(3)}}_{\eta_k^{(3)}} + \zeta_{jk}^{(2)}$$

Path diagram for GLLAMM formulation



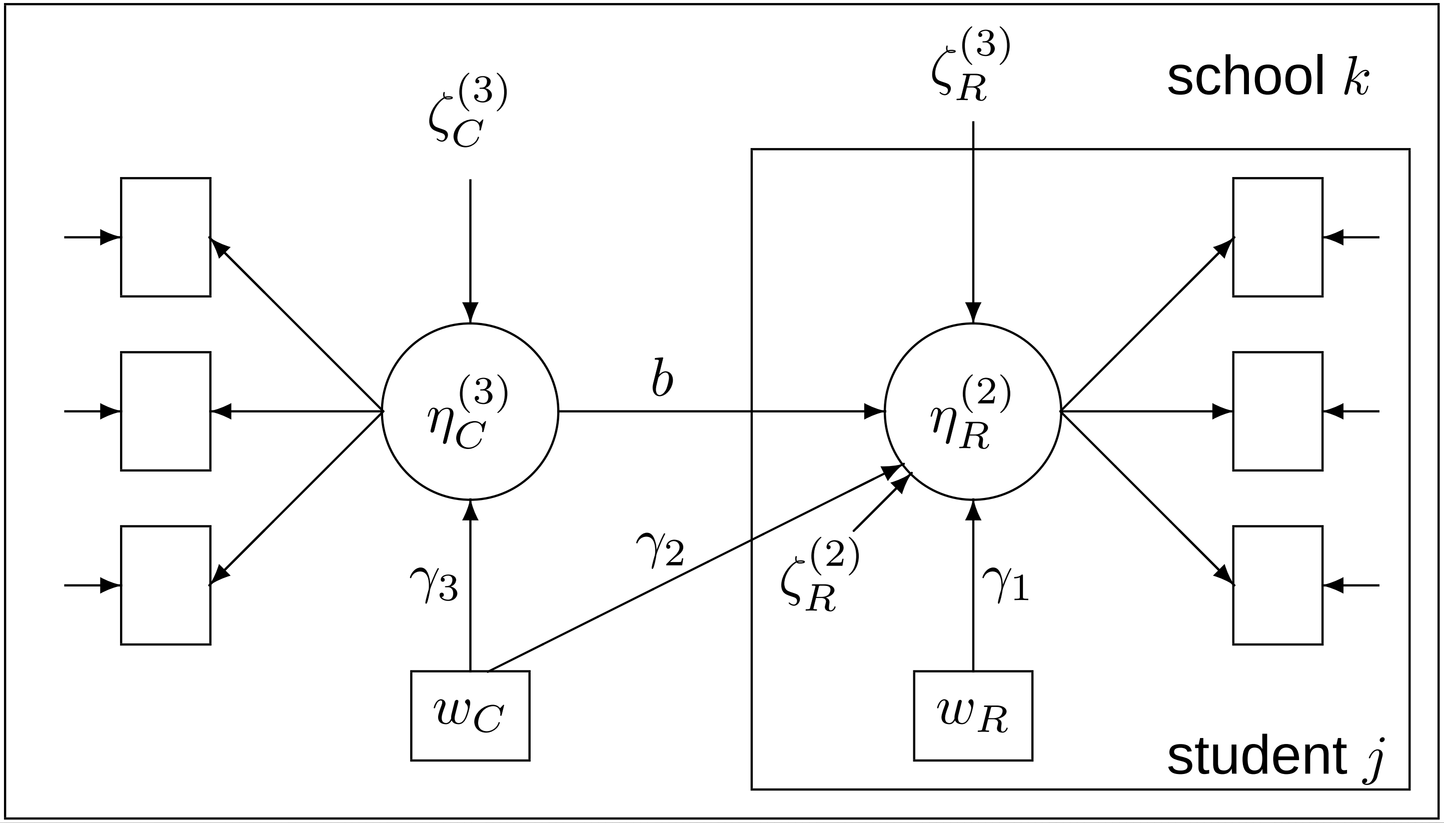
PISA 2000 data: School-level items for school-level latent covariate

- Latent covariate: Teacher excellence
- Responses from school principal: ordinal items with three categories ("satisfied", "somewhat satisfied" and "dissatisfied")
- Questions about teacher excellence:
  1. teacher expectations
  2. student-teacher relations
  3. teacher turnover
  4. teachers meeting individual students’ needs
  5. teacher absenteeism
  6. teachers’ strictness with students
  7. teacher morale
  8. teachers’ enthusiasm
  9. teachers taking pride in the school
  10. teachers valuing academic achievement

Multilevel structural equation model with school-level items

- Latent student-level response variable $\eta^{(2)}_{Rjk}$
- Latent school-level covariate $\eta^{(3)}_{Ck}$

$$\eta^{(2)}_{Rjk} = b\eta^{(3)}_{Ck} + \gamma_1 w_{Rjk} + \gamma_2 w_{Cjk} + \zeta^{(3)}_{Rk} + \zeta^{(2)}_{Rjk}, \qquad \eta^{(3)}_{Ck} = \gamma_3 w_{Ck} + \zeta^{(3)}_{Ck}$$



Maximum likelihood estimates of structural model

	MIMIC		SEM		
Parameter	Est	(SE)	Est	(SE_R)	(SE_R^{PSU})
Model for student ability					
b: [Teacher excellence]			0.109	(0.058)	(0.044)
γ_1: [Female]	0.146	(0.122)	0.148	(0.120)	(0.156)
γ_2: [ISEI]	0.012	(0.004)	0.012	(0.004)	(0.004)
γ_3: [Highschool]	0.138	(0.249)	0.133	(0.246)	(0.232)
γ_4: [College]	0.411	(0.263)	0.397	(0.259)	(0.238)
γ_5: [English]	0.555	(0.227)	0.541	(0.224)	(0.222)
γ_6: [MnISEI]	0.039	(0.012)	0.038	(0.012)	(0.014)
$\psi_R^{(2)}$	1.244	(0.642)	1.233	(0.521)	(0.606)
$\psi_R^{(3)}$	0.111	(0.136)	0.083	(0.085)	(0.121)
Model for teacher excellence					
γ_7: [MnISEI]			0.006	(0.020)	(0.019)
$\psi_C^{(3)}$			2.192	(0.427)	(0.432)

103 of the 146 schools had items on teacher excellence

Higher-level items in GLLAMM formulation

- GLLAMM response model

$$\boldsymbol{\nu} = \mathbf{X}\boldsymbol{\beta} + \sum_{l=2}^{L}\sum_{m=1}^{M_l} \eta_m^{(l)} \mathbf{Z}_m^{(l)} \boldsymbol{\lambda}_m^{(l)}$$

  - $\boldsymbol{\nu}$ is vector of linear predictors for responses at different levels
  - Design matrices $\mathbf{Z}_m^{(2)}$ assign factor loadings to student-level responses
  - Design matrices $\mathbf{Z}_m^{(3)}$ assign factor loadings to school-level responses

GLLAMM specification of multilevel SEM: Response model

$$\underbrace{\begin{bmatrix} v_{1jk} \\ \vdots \\ v_{7jk} \\ \hline v_{1k} \\ \vdots \\ v_{10,k} \end{bmatrix}}_{\boldsymbol{\nu}} = \underbrace{\begin{bmatrix} \mathbf{I}_{7\times 7} \\ \hline \mathbf{0}_{10\times 7} \end{bmatrix}}_{\mathbf{X}} \underbrace{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_7 \end{bmatrix}}_{\boldsymbol{\beta}} + \eta^{(2)}_{Rjk} \underbrace{\begin{bmatrix} \mathbf{I}_{7\times 7} \\ \hline \mathbf{0}_{10\times 7} \end{bmatrix}}_{\mathbf{Z}^{(2)}_{1k}} \underbrace{\begin{bmatrix} 1 \\ \lambda^{(2)}_2 \\ \vdots \\ \lambda^{(2)}_7 \end{bmatrix}}_{\boldsymbol{\lambda}^{(2)}_1} + \eta^{(3)}_{Ck} \underbrace{\begin{bmatrix} \mathbf{0}_{7\times 1} \\ \hline \mathbf{1}_{10\times 1} \end{bmatrix}}_{\mathbf{Z}^{(3)}_{1k}} \underbrace{1}_{\lambda^{(3)}_1}$$

$$+ \eta^{(3)}_{Rk} \underbrace{\begin{bmatrix} \mathbf{0}_{17\times 1} \end{bmatrix}}_{\mathbf{Z}^{(3)}_{2k}} \underbrace{1}_{\lambda^{(3)}_2}$$

& thresholds $\kappa_{i1}, \kappa_{i2}, \; i = 1, \ldots, 10$ for school-level items

- $\eta^{(2)}_{Rjk}$: student-level latent variable (interpretation ability)
- $\eta^{(3)}_{Ck}$: school-level latent variable (teacher excellence)
- $\eta^{(3)}_{Rk}$: school-level random intercept for interpretation ability

GLLAMM specification of multilevel SEM:

Structural model

$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\mathbf{w} + \boldsymbol{\zeta}$$

$$\underbrace{\begin{bmatrix} \eta^{(2)}_{Rjk} \\ \eta^{(3)}_{Ck} \\ \eta^{(3)}_{Rk} \end{bmatrix}}_{\boldsymbol{\eta}} = \underbrace{\begin{bmatrix} 0 & b & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \eta^{(2)}_{Rjk} \\ \eta^{(3)}_{Ck} \\ \eta^{(3)}_{Rk} \end{bmatrix}}_{\boldsymbol{\eta}} + \underbrace{\begin{bmatrix} \gamma_1 & \gamma_2 \\ 0 & \gamma_3 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{\Gamma}} \underbrace{\begin{bmatrix} w_{Rjk} \\ w_{Ck} \end{bmatrix}}_{\mathbf{w}} + \underbrace{\begin{bmatrix} \zeta^{(2)}_{Rjk} \\ \zeta^{(3)}_{Ck} \\ \zeta^{(3)}_{Rk} \end{bmatrix}}_{\boldsymbol{\zeta}}$$

$$\eta^{(3)}_{Rk} = \zeta^{(3)}_{Rk}$$

$$\eta^{(3)}_{Ck} = \gamma_3 w_{Ck} + \zeta^{(3)}_{Ck}$$

$$\eta^{(2)}_{Rjk} = b\eta^{(3)}_{Ck} + \gamma_1 w_{Rjk} + \gamma_2 w_{Ck} + \underbrace{\zeta^{(3)}_{Rk}}_{\eta^{(3)}_{Rk}} + \zeta^{(2)}_{Rjk}$$

Some extensions:

Multidimensional and discrete latent variables

- GLLAMM framework allows for multidimensional measurement models
- GLLAMM framework also allows for discrete latent variables
  - Latent class-type models
  - Nonparametric maximum likelihood estimation: latent variable distribution left unspecified
- Structural model for discrete latent variables:
  Probability that unit j belongs to class $c = 1, \ldots, C$ may depend on covariates $\mathbf{v}_j$ through a multinomial logit model

$$\pi_{jc} = \frac{\exp(\mathbf{v}_j' \boldsymbol{\alpha}^c)}{\sum_{d=1}^{C} \exp(\mathbf{v}_j' \boldsymbol{\alpha}^d)}$$

- Cannot currently combine continuous and discrete latent variables

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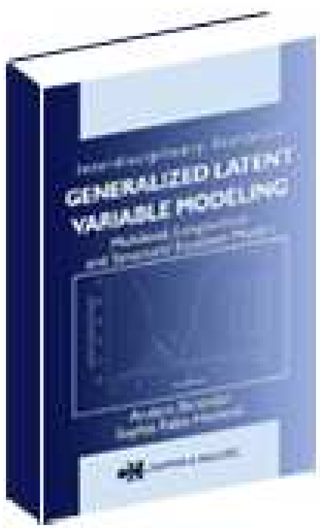
- 

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