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Algunas desigualdades integrales que involucran la función $k-$Beta usando funciones $(m, h_1, h_2)-$Convexas

Some integral inequalities involving the $k-$Beta function using $(m, h_1, h_2)-$convex functions.

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Resumen

El presente trabajo trata acerca del estudio de la integral del tipo

$$\int_a^b (x-a)^{p/k}(b-x)^{q/k} f(x)dx$$

para $p, q, k > 0$, considerando algunas desigualdades para funciones $(m, h_1, h_2)-$convexas. De estos resultados se derivan algunas otras desigualdades integrales para otras clases de funciones convexas generalizadas.

Palabras claves: Desigualdades integrales, funciones $(m, h_1, h_2)-$convexas, Función $k-$Beta
2008 MSC: 26D15, 52A01, 33E50

Abstract

The present work deals with the study of the integral of the type

$$\int_a^b (x-a)^{p/k}(b-x)^{q/k} f(x)dx$$

for $p, q, k > 0$, considering some inequalities for $(m, h_1, h_2)-$convex functions. From these results some others integral inequalities for other class of generalized convex functions are obtained.

Keywords: Integral inequalities, $(m, h_1, h_2)-$convex functions, $k-$Beta function
2008 MSC: 26D15, 52A01, 33E50

1. Introduction

Convexity is a basic notion in geometry, but it is also widely used in other areas of mathematics. The use of techniques of convexity appears in many branches of mathematics and sciences, such as Theory of Optimization and Theory of Inequalities, Functional Analysis, Mathematical Programming and Game Theory, Theory of Numbers, Variational Calculus and its interrelation with these branches shows itself day by day deeper and fruitful [9, 12, 17].

Definition 1.1. *A function $f : I \to R$ is said to be convex if for all $x, y \in I$ and $t \in [0, 1]$ the inequality*

$$f(tx + (1 - t)y) \leq t f(x) + (1 - t)f(y)$$

holds.

Over time, several problems and applications have arisen, and these have given rise to generalizations of the concept of convex function, and also numerous works of investigation have been realized extending results on inequalities for this kind of convexity: quasi-convexity [20], s-convexity in the first and second sense [2], logarithmic convexity [1], $m-$convexity [16], $h-$convexity [22, 23] and others [24].

The study of special functions has had special attention with the works of R. Diaz and E. Pariguan [6] and A. Rehman et. al. [18], in which the subject of the generalized $k-$Gamma function and generalized $k-$Beta is treated, in turn, these appear in the study of fractional integrals. Also the research field corresponding to inequalities, generalized convexity and fractional integrals has been extensively studied [10, 25].

Motivated by the works of W. Liu [14], M. E.Özdemir et. al. [15] and Hernández Hernández [11], the purpose of this paper is to study the integral

$$\int_a^b (x - a)^{p/k}(b - x)^{q/k} f(x)dx$$

for $p, q, k > 0$ for $(m, h_1, h_2)-$convex functions and establish some bounds for the aforementioned integral.

2. Preliminaries

Concerning the generalized convexity the following definitions will be necessary.

Pavić and Ardic in [16] give the following definition.

Definition 2.1. *Let $I \subset \mathbb{R}$ be an interval containing the zero, and let $m \in (0, 1]$ be a number. A function $f : I \to \mathbb{R}$ is said to be $m-$convex if the inequality*

$$f(tx + m(1 - t)y) \leq t f(x) + m(1 - t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In the same study some interesting properties of the $m-$convex functions are given.

S.S. Dragomir et. al. in [7] introduced the following definition.

Definition 2.2. *Let $I \subset \mathbb{R}$ an interval. A function $f : I \to \mathbb{R}$ is said to belong to the class $P(I)$ or to be $P-$convex if it is non negative and for all $x, y \in I$ and $t \in [0, 1]$, satisfies the following inequality*

$$f(tx + (1 - t)y) \leq f(x) + f(y)$$

It is useful to mention that the class $P(I)$ contain all the non-negative convex and quasi-convex functions.

Also, M. Alomari et. al. in [2] used the concept of $s-$convexity in the second sense as Hudzik and L. Maligranda in [13] introduced it.

Definition 2.3. *Let $s \in (0, 1]$. A real valued function f on an interval $I \subset [0, \infty)$ is $s-$convex in the second sense provided*

$$f(ta + (1 - t)b) \leq t^s f(a) + (1 - t)^s f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$. This is denoted by $f \in K_s^2$.

Some interesting properties of this class of functions were given by S.S. Dragomir and S. Fitzpatrick in [8], by example: all functions in K_s^2 are locally Hölder continuous of order s on (a, b) and therefore Riemann integrable on $[a, b]$.

In 2007, S. Varošanec in [22] introduced a new concept of generalized convexity.

Definition 2.4. *Let J an interval of $\mathbb{R}$ such that $(0, 1) \subset J$ and $h : J \to \mathbb{R}$ a non negative real valued function such that $h \not\equiv 0$. A function $f : I \to \mathbb{R}$ is said to be $h-$convex or that f belong to the clas $SX(h, I)$ if f is non negative and for all $x, y \in I$ and all $t \in (0, 1)$ the following inequality holds*

$$f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y).$$

Obviously if $h(t) = t$ all the non-negative convex function belong to the class $SX(h, I)$; if $h(t) = 1$ then $P(I) \subset SX(h, I)$, and if $h(t) = t^s$ then $K_s^2 \subset SX(h, I)$.

Numerous works have been published using $h-$convex functions applied to the Hermite-Hadamard and Hermite-Hadamard-féjer inequalities [3, 19].

In [21], Shi et. al. introduced the following definition.

Definition 2.5. *Assume $f : I \to \mathbb{R}$, $h_1, h_2 : [0, 1] \to \mathbb{R}$ and $m \in [0, 1]$. Then f is said to be $(m, h_1, h_2)-$convex if the inequality*

$$f(tx + m(1 - t)y) \leq h_1(t)f(x) + mh_2(t)f(y)$$

holds for all $x, y \in I$, and $t \in$ [0,1] .

Some special properties of the $(m, h_1, h_2)-$convexity are given in [26] and, several results regularity properties and also a study of bounds of the second degree cumulative frontier gaps of functions with this kind of generalized convexity are proved in [4, 5].

Also, in the development of this work we use the $k-$Beta function and it is useful recall some notes about it. From the work of R. Diaz and E. Pariguan [6] it is extracted the following.

Definition 2.6. *Let $x \in \mathbb{C}, k \in \mathbb{R}$ and $n \in \mathbb{N}^+$ For $k > 0$, the Pochhammer $k-$symbol is given by*

$$(x)_{n,k} = x(x + k)(x + 2k) \cdots (x + (n - 1)k).$$

Definition 2.7. *For $k > 0$, the $k-$Gamma function Γ_k is given by*

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \qquad x \in \mathbb{C} \setminus k\mathbb{Z}^-.$$

Definition 2.8. *The $k-$Beta function $B_k(x, y)$ is given by*

$$B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x + y)} \qquad Re(x) > 0, Re(y) > 0.$$

Also, the same authors established an integral representation for the $k-$Beta function as follow [6, Proposition 14]:

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt,$$

also a property follows from the definition, as it can be sawn in [18]:

$$B_k(x+k,y) = \frac{x}{x+y}B_k(x,y) \text{ and } B_k(x,y+k) = \frac{y}{x+y}B_k(x,y). \tag{1}$$

Some others properties of the $k-$Beta functions, and also for $k-$Beta function with several variables, can be found in the work of M. Rehman et. al. [18].

3. Main Results

As is proved in Lemma 1 in [11], for an integrable function $f : I \to \mathbb{R}$ on I we have

$$\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du = (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k} t^{q/k} f(ta+(1-t)b)dt \tag{2}$$

for some fixed $p, q, k > 0$.

The following results for functions whose absolute values are $(m, h_1, h_2)-$convex, including $r-$th powers of them, are established.

Theorem 3.1. *Let $f : I \to \mathbb{R}$ be an integrable function on I, and $p, q, k > 0$. If $|f|$ is $(m, h_1, h_2)-$convex on $[a, b]$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du \le (b-a)^{\frac{p+q}{k}+1}\left(I(h_1)|f(a)| + mI(h_2)|f(b)|\right), \tag{3}$$

where

$$I(h_1) = \int_0^1 (1-t)^{p/k} t^{q/k} h_1(t)dt$$

and

$$I(h_2) = \int_0^1 (1-t)^{p/k} t^{q/k} h_2(t)dt.$$

Proof. Using the $(m, h_1, h_2)-$convexity of $|f|$, we have

$$\begin{aligned}
&\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du \\
&\quad \le (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k} t^{q/k} |f(ta+(1-t)b)|\,dt \\
&\quad \le (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k} t^{q/k} \left(h_1(t)|f(a)| + mh_2(t)|f(b)|\right)dt \\
&\quad = (b-a)^{\frac{p+q}{k}+1}\left(|f(a)|\int_0^1 (1-t)^{p/k} t^{q/k} h_1(t)dt + m|f(b)|\int_0^1 (1-t)^{p/k} t^{q/k} h_2(t)dt\right).
\end{aligned}$$

Doing

$$I(h_1) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_1(t)dt$$

and

$$I(h_2) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_2(t)dt$$

we have the desired result. The proof is complete. ■

Remark 3.2. *If in Theorem 3.1 we take $m = 1$ and $h_1 = h_2$ then the Theorem 3.1 in [11] is obtained.*

Using a suitable choice of the parameter m and the functions h_1, h_2 it is possible to find others important results.

Corollary 3.3. *Let $f : I \to \mathbb{R}$ be an integrable function on I, and $p, q, k > 0$. If $|f|$ is convex on $[a, b]$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \le (b-a)^{\frac{p+q}{k}+1} \frac{kB_k(p,q)}{(p+q)_{3,k}} \left((q)_{2,k}\, p\, |f(a)| + (p)_{2,k}\, q\, |f(b)|\right).$$

Proof. Letting $m = 1$, $h_1(t) = t$ and $h_2(t) = 1 - t$, for all $t \in [0, 1]$, using Definition 2.8 and the property (1), it is obtained

$$\begin{aligned} I(h_1) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_1(t)dt &= \int_0^1 (1-t)^{p/k}\, t^{q/k+1}dt \\ &= kB_k(p+k, q+2k) \\ &= kB_k(p,q)\frac{(q)_{2,k}\, p}{(p+q)_{3,k}} \end{aligned}$$

and

$$\begin{aligned} I(h_2) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_2(t)dt &= \int_0^1 (1-t)^{p/k+1}\, t^{q/k}dt \\ &= kB_k(p+2k, q+k) \\ &= kB_k(p,q)\frac{(p)_{2,k}\, q}{(p+q)_{3,k}}. \end{aligned}$$

By replacement of these values in Theorem 3.1 it is obtained the desired result. ■

Corollary 3.4. *Let $f : I \to \mathbb{R}$ be an integrable function on I, and $p, q, k > 0$. Let $m \in (0, 1]$. If $|f|$ is $m-$convex on $[a, b]$ where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\begin{aligned} &\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \\ &\qquad \le (b-a)^{\frac{p+q}{k}+1} \frac{kB_k(p,q)}{(p+q)_{3,k}} \left((q)_{2,k}\, p\, |f(a)| + (p)_{2,k}\, qm\, |f(b)|\right). \end{aligned}$$

Proof. The proof follows by chosing $h_1(t) = t$ and $h_2(t) = 1 - t$ for all $t \in [0, 1]$. ■

Corollary 3.5. *Let $f : I \to \mathbb{R}$ be an integrable function on I, and $p, q, k > 0$. If $|f|$ is P−convex on $[a, b]$ where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \leq (b-a)^{\frac{p+q}{k}+1} \frac{kpq}{(p+q)_{2,k}} B_k(p,q) \left(|f(a)| + |f(b)|\right).$$

Proof. Letting $m = 1$,$h_1(t) = h_2(t) = 1$ for all $t \in [0, 1]$, using Definition 2.8 and property (1) it is obtained

$$I(h_1) = I(h_1) = \int_0^1 (1-t)^{p/k} t^{q/k} dt = \frac{pq}{(p+q)_{2,k}} B_k(p,q). \tag{4}$$

By replacement of these values in Theorem 3.1 it is attained the desired result. ■

Remark 3.6. *If in Corollary 3.5 we let $k = 1$ then it is obtained the Theorem 2.1 in [14].*

Corollary 3.7. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|$ is s−convex in the second sense on $[a, b]$ for some $s \in (0, 1]$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \leq k(b-a)^{\frac{p+q}{k}+1} \left(\frac{p(q+ks)}{(p+q+ks)_{2,k}} B_k(p, q+ks) |f(a)| + \frac{(p+ks)q}{(p+q+ks)_{2,k}} B_k(p+ks, q) |f(b)| \right).$$

Proof. Letting $m = 1$, $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in [0, 1]$ and some $s \in (0, 1]$, using Definition 2.8 and the property (1) we have

$$\begin{aligned} I(h_1) = \int_0^1 (1-t)^{p/k} t^{q/k} h_1(t)dt &= \int_0^1 (1-t)^{p/k} t^{q/k+s} dt \\ &= B_k(p+k, q+k(s+1)) \\ &= \frac{p(q+ks)}{(p+q+ks)_{2,k}} B_k(p, q+ks) \end{aligned}$$

and

$$\begin{aligned} I(h_2) = \int_0^1 (1-t)^{p/k} t^{q/k} h_2(t)dt &= \int_0^1 (1-t)^{p/k+s} t^{q/k} dt \\ &= B_k(p+k(s+1), q+k) \\ &= \frac{(p+ks)q}{(p+q+ks)_{2,k}} B_k(p+ks, q). \end{aligned}$$

By replacement of these values in Theorem 3.1 it is attained the desired result. ■

Theorem 3.8. *Let $f : I \to \mathbb{R}$ be an integrable function on the interval I and $p, q, k > 0$. If $|f|^r$ is $(m, h_1, h_2)-$convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\begin{aligned}
&\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \\
&\quad \leq (b-a)^{\frac{p+q}{k}+1} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq) \right)^{1/l} \left(I(h_1) |f(a)|^r + mI(h_2) |f(b)|^r \right)^{1/r},
\end{aligned} \tag{5}$$

where

$$I(h_1) = \int_0^1 h_1(t)dt \ , \ I(h_2) = \int_0^1 h_2(t)dt$$

and $(1/l) + (1/r) = 1$.

Proof. Using the Hölder inequality we have

$$\begin{aligned}
&\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \\
&\quad \leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} |f(ta + (1-t)b, \cdot)| \, dt \\
&\quad \leq (b-a)^{p+q+1} \left(\int_0^1 (1-t)^{lp/k} t^{lq/k} dt \right)^{1/l} \left(\int_0^1 |f(ta + (1-t)b, \cdot)|^r \, dt \right)^{1/r}.
\end{aligned} \tag{6}$$

Since $|f|^r$ is a $(m, h_1, h_2)-$convex function then

$$\int_0^1 |f(ta + (1-t)b, \cdot)|^r \, dt \leq |f(a)|^r \int_0^1 h_1(t)dt + m |f(b)|^r \int_0^1 h_2(t)dt,$$

doing

$$I(h_1) = \int_0^1 h_1(t)dt \ \text{ and } \ I(h_2) = \int_0^1 h_2(t)dt$$

we can write

$$\int_0^1 |f(ta + (1-t)b, \cdot)|^r \, dt \leq |f(a)|^r I(h_1) + m |f(b)|^r I(h_2). \tag{7}$$

Now, using the definition of the $k-$Beta function and the property (1), we get

$$\begin{aligned}
\int_0^1 (1-t)^{lp/k} t^{lq/k} dt &= kB_k(lp + k, lq + k) \\
&= k \frac{pq}{(lp+lq)_{2,k}} B_k(lp, lq)
\end{aligned} \tag{8}$$

So, replacing (7) and (8) in (6) it is attained the required inequality (5).
The proof is complete. ■

Remark 3.9. *If in Theorem 3.8 we take $m = 1$ and $h_1 = h_2$ then the Theorem 2 in [11] is obtained.*

Corollary 3.10. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du$$
$$\leq 2^{-1/r} (b-a)^{\frac{p+q}{k}+1} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq) \right)^{1/l} \left(|f(a)|^r + |f(b)|^r \right)^{1/r},$$

where $(1/l) + (1/r) = 1$.

Proof. Letting $m = 1, h_1(t) = t$ and $h_2(t) = 1 - t$ for all $t \in [0, 1]$ we have

$$I(h_1) = I(h_2) = \int_0^1 tdt = \frac{1}{2}.$$

So, by replacement in Theorem 3.8 it is obtained the desired result. ■

Corollary 3.11. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. Let $m \in (0, 1]$. If $|f|^r$ is $m-$convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du$$
$$\leq 2^{-1/r} (b-a)^{\frac{p+q}{k}+1} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq) \right)^{1/l} \left(|f(a)|^r + m\,|f(b)|^r \right)^{1/r},$$

where $(1/l) + (1/r) = 1$.

Proof. The proof follows by chosing $h_1(t) = t$ and $h_2(t) = 1 - t$ for all $t \in [0, 1]$ in Theorem 3.8 ■

Corollary 3.12. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is $P-$convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \tag{9}$$
$$\leq (b-a)^{\frac{p+q}{k}+1} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq) \right)^{1/l} \left(|f(a)|^r + |f(b)|^r \right)^{1/r},$$

where $(1/l) + (1/r) = 1$.

Proof. Letting $m = 1, h_1(t) = h_2(t) = 1$ for all $t \in [0, 1]$ we have $I(h_1) = I(h_2) = 1$. Then, by replacement in Theorem 3.8 it is attained the desired result. ■

Corollary 3.13. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is $s-$convex in the second sense on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du$$
$$\leq \frac{(b-a)^{\frac{p+q}{k}+1}}{(s+1)^{1/r}} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq) \right)^{1/l} \left(|f(a)|^r + |f(b)|^r \right)^{1/r},$$

where $(1/l) + (1/r) = 1$.

Proof. Letting $m = 1, h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in [0,1]$ and some $s \in (0,1]$, we have

$$I(h_1) = I(h_2) = \int_0^1 t^s dt = \frac{1}{s+1}.$$

By replacement of these values in Theorem 3.8 it is obtained the result. ∎

Theorem 3.14. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is $(m, h_1, h_2)-$convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\begin{aligned}&\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \\ &\leq k\,(b-a)^{\frac{p+q}{k}+1} \left[\frac{pq}{(p+q)_{2,k}} B_k(p,q)\right]^{1-1/r} \left(I_1(h)\,|f(a)|^r + mI_2(h)\,|f(b)|^r\right)^{1/r},\end{aligned} \tag{10}$$

where

$$I(h_1) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_1(t)dt \ \ \text{and} \ \ I(h_2) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_2(t)dt.$$

Proof. Using the power mean inequality for $r \geq 1$ we have

$$\begin{aligned}&\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du \\ &\leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k}\, t^{q/k} \left|f(ta + (1-t)b, \cdot)\right| dt \\ &\leq (b-a)^{\frac{p+q}{k}+1} \left(\int_0^1 (1-t)^{p/k}\, t^{q/k} dt\right)^{1-1/r} \times \\ &\qquad \left(\int_0^1 (1-t)^{p/k}\, t^{q/k} \left|f(ta + (1-t)b, \cdot)\right|^r dt\right)^{1/r}.\end{aligned} \tag{11}$$

Making use of the $(m, h_1, h_2)-$convexity of $|f|^r$ and the definition the $k-$Beta function, we get

$$\begin{aligned}&\int_0^1 (1-t)^{p/k} t^{q/k} \left|f(ta + (1-t)b)\right|^r dt \\ &\leq \int_0^1 (1-t)^{p/k}\, t^{q/k} \left(h_1(t)\,|f(a)|^r + mh_2(t)\,|f(b)|^r\right) dt \\ &= I(h_1)\,|f(a)|^r + mI(h_2)\,|f(b)|^r,\end{aligned} \tag{12}$$

where

$$I(h_1) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_1(t)dt \ \ \text{and} \ \ I(h_2) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_2(t)dt.$$

So, replacing (12) and (4) in the inequality (11) it is attained the desired inequality (10).
The proof is complete. ∎

Remark 3.15. *If in Theorem 3.14 we take $m = 1$ and $h_1 = h_2$ then the Theorem 3 in [11] is obtained.*

Corollary 3.16. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\begin{aligned}\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du \\ \leq k^{1+1/r}(b-a)^{\frac{p+q}{k}+1}\left[\frac{pq}{(p+q)_{2,k}}\right]^{1-1/r} B_k(p,q)\times \\ \left(\frac{(q)_{2,k}\,p}{(p+q)_{3,k}}|f(a)|^r + \frac{(p)_{2,k}\,q}{(p+q)_{3,k}}|f(b)|^r\right)^{1/r}\end{aligned}$$

Proof. Letting $m = 1, h_1(t) = t$ and $h_2 = 1 - t$ for all $t \in [0, 1]$, the values of $I(h_1)$ and $I(h_2)$ are as in the proof of Corollary 3.3. By replacement of these values in Theorem 3.14 it is attained the desired result. ■

Corollary 3.17. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. Let $m \in (0, 1]$. If $|f|^r$ is $m-$convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\begin{aligned}\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du \\ \leq k^{1+1/r}(b-a)^{\frac{p+q}{k}+1}\left[\frac{pq}{(p+q)_{2,k}}\right]^{1-1/r} B_k(p,q)\times \\ \left(\frac{(q)_{2,k}\,p}{(p+q)_{3,k}}|f(a)|^r + \frac{(p)_{2,k}\,q}{(p+q)_{3,k}}m\,|f(b)|^r\right)^{1/r}\end{aligned}$$

Proof. The proof follows by chosing $h_1(t) = t$ and $h_2(t) = 1 - t$ for all $t \in [0, 1]$ in Theorem 3.14 ■

Corollary 3.18. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is $P-$convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\begin{aligned}\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du \\ \leq k(b-a)^{\frac{p+q}{k}+1}\left[\frac{pq}{(p+q)_{2,k}} B_k(p,q)\right]\left(|f(a)|^r + |f(b)|^r\right)^{1/r},\end{aligned}$$

Proof. Letting $m = 1, h_1(t) = h_2(t) = 1$ for all $t \in [0, 1]$ we have

$$I(h_1) = I_2(h_2) = \int_0^1 (1-t)^{p/k} t^{q/k} dt = \frac{pq}{(p+q)_{2,k}} B_k(p,q).$$

By replacement of these values in Theorem 3.14 we have the desired result. ■

Corollary 3.19. *Let $f : I \to \mathbb{R}$ be an integrable function on I and $p, q, k > 0$. If $|f|^r$ is $s-$convex in the second sense on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds*

$$\begin{aligned}\int_a^b (u-a)^{p/k}(b-u)^{q/k} f(u)du \\ \leq k(b-a)^{\frac{p+q}{k}+1}\left[\frac{pq}{(p+q)_{2,k}} B_k(p,q)\right]^{1-1/r} \times \\ \left(\frac{p(q+ks)B_k(p,q+ks)}{(p+q+ks)_{2,k}}|f(a)|^r + \frac{(p+ks)qB_k(p+ks,q)}{(p+q+ks)_{2,k}}|f(b)|^r\right)^{1/r}\end{aligned}$$

Proof. Letting $m = 1, h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in [0, 1]$ and some $s \in (0, 1]$, the values of $I(h_1)$ and $I(h_2)$ are as in the proof of Corollary 3.7. By replacement of these values we find the result. ■

4. Conclusions

In the present work it is obtained some results concerning to the integral

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} f(u)du$$

for $p, q, k > 0$ and an integrable function f on the interval $[a, b]$. It was used the concept of $(m, h_1, h_2)-$convexity of a function f in order to obtain several inequalities that bound in the upper case the aforementioned integral as it is shown in Theorem 3.1, Theorem 3.8 and Theorem 3.14, and from these results some others inequalities that involve the classical concept of convexity and the generalized concept of $P-$convexity and $s-$convexity in the second sense were derived as it is shown in the presented Corollaries. Also some special functions were used. Also with the choice of $m = 1$ and $h_1 = h_2$ the principal theorems prsented in [11] are obtained.

For other kind of generalized convexity, for example: $MT-$convexity, Godunova-Levin convexity, it is possible to establish similar results with a particular choice of the functions h_1 and h_2. By example, the $MT-$convexity is related with the functions $h_1(t) = \sqrt{t}/(2\sqrt{1-t})$ and $h_2(t) = \sqrt{1-t}/(2\sqrt{t})$ for all $t \in (0, 1)$ and the Godunova-Levin convexity is related with the function $h_1(t) = 1/t$ and $h_2 = 1/(1-t)$ for all $t \in (0, 1)$.

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Referencias

[1] M. Alomari and M. Darus, *On The Hadamard's Inequality for Log-Convex Functions on the Coordinates*, J. Ineq. Appl., **2009** (2009), Article ID 283147, 13 pp,

[2] M. Alomari, M. Darus, S.S. Dragomir and P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*. Appl.Math. Lett., **23** (2010), 1071-1076

[3] M. Bombardelli and S. Varošanec. *Properties of h-convex functions related to the Hermite-Hadamard-Féjer inequalities*. Computers and Mathematics with Applications. **58**(9) (2009), 1869-1877

[4] G. Cristescu, M.A. Noor and M.U. Awan. *Bounds of the second degree cumulative frontier gaps of functions with generalized convexity*. Carpath. J. Math., 31 (2015), 173â180

[5] G. Cristescu, M. Găianu and A. M. Uzair. *Regularity properties and integral inequalities related to (k; h1; h2)-convexity of functions*. Anal. Univ. Vest, Timisoara Ser. Mat. Inf., LIII (1) (2015), 19â35

[6] R. Diaz and E. Pariguan. *On hypergeometric functions and Pochhammer k-symbol*. Divulgaciones Matemáticas, **15** (2) (2007), 179-192

[7] S. S. Dragomir, J. Pečarić and L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math., **21** (3) (1995), 335 - 341.

[8] S.S. Dragomir and S. Fitzpatrick, *The Hadamard Inequalities for $s-$convex functions in the second sense* , Demostratio Mathematica, **4** (1999), 687-696.

[9] A. Hassibi, J. P. How and S. Boyd, *Low-authority controller design via convex optimization*, AIAA Journal of Guidance, Control, and Dynamics, **22**(6) (1999), 862-872.

[10] J.E. Hernández H, *On a Hardy's inequality for a fractional integral operator*, Annals of the University of Craiova, Mathematics and Computer Science Series, **45**(2) (2018), 232-242

[11] J. E. Hernández Hernández. Some integral inequalities involving the $k-$Beta function using $h-$convex functions. Lecturas Matemáticas. 41(1)(2020), 5-17

[12] J.-B. Hiriart-Urruty, *Global optimality conditions in maximizing a convex quadratic function under convex quadratic constraints*, J. Global Optim. **21** (2001), 445-455.

[13] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. **48** (1994), 100-111

[14] W. Liu, *Some New Integral Inequalities via $P-$convexity*, arxiv:1202.0127v1 [math.FA]. **1** (2012), available in: http://arxiv.org/abs/1202.0127

[15] M. E.Özdemir, E. Set and M. Alomari, *Integral inequalities via several kinds of convexity*. Creat. Math. Inform. **20** (1) (2011), 62 - 73

[16] Z. Pavić and M. Avci Ardic, *The most important inequalities for m-convex functions*. Turk J. Math. **41** (2017), 625-635.

[17] B. T. Polyak, *Convexity of quadratic transformations and its use in control and optimization*,J. Optim. Theory Appl. **99** (1998), 553-583.

[18] A. Rehman, S. Mubeen, R. Safdar and N. Sadiq. *Properties of $k-$Beta function with several variables*, Open Math. **13** (2015) , 308-320

[19] M. Z. Sarikaya, E. Set and M. E. Özdemir, *On some new inequalities of Hadamard type involving $h-$convex functions*. Acta Math. Univ. Comenian. **79**(1) (2010), 265 - 272.

[20] E. Set, A. Akdemir and N. Uygun, *On new simpson type inequalities for generalized quasi-convex mappings*, Xth International Statistics Days Conference, 2016, Giresun, Turkey

[21] Shi, D-P., Xi B-Y., Qi, F. *Hermite–Hadamard Type Inequalities for $(m,h1,h2)-$Convex Functions Via Riemann-Liouville Fractional Integrals*. Turkish Journal of Analysis and Number Theory. **2(1)** (2014), 23-28

[22] S. Varošanec, *On $h-$convexity*, J. Math. Anal. Appl. **326(1)** (2007), 303-311.

[23] M. J. Vivas and J.E. Hernández H, *On some new generalized Hermite-Hadamard-Féjer inequalities for product of two operator $h-$convex functions*. Appl. Math. Inf. Sci. **11**(4) (2017), 983-992

[24] M.J. Vivas-Cortez and Y.C. Rangel Oliveros, *Ostrowski type inequalities for functions whose second derivatives are convex generalized*. App. Math. Inf. Sci. **12** (6) (2018), 1117-1126

[25] M.J. Vivas, C. García and J.E. Hernández H, *Ostrowski-type inequalities for functions whose derivative modulus is relatively* $(m, h_1, h_2)-$*convex*. Appl. Math. Inf. Sci. **13**(3) (2019), 369-378

[26] B. Xi, F. Qi. *Properties and Inequalities for the* (h_1, h_2) *and* $(h_1, h_2, m) -$ *GA-Convex functions.* Journal Cogent Mathematics. **3** (2016)