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“High Watermarks of Market Risks”[*]

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Abstract

We present several estimates of measures of risk amongst the most well-known, using both high and low frequency data. The aim of the article is to show which lower frequency measures can be an acceptable substitute to the high precision measures, when transaction data is unavailable for a long history. We also study the distribution of the volatility, focusing more precisely on the slope of the tail of the various risk measure distributions, in order to define the high watermarks of market risks. Based on estimates of the tail index of a Generalized Extreme Value density backed-out from the high frequency CAC40 series in the period 1997-2006, using both Maximum Likelihood and L-moment Methods, we, finally, find no evidence for the need of a specification with heavier tails than in the case of the traditional log-normal hypothesis.

Keywords. Financial Crisis, Volatility Estimator Distributions, Range-based Volatility, Extreme Value, High Frequency Data.

J.E.L. classification. G.10, G.14.

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1 Introduction

The measure of risk plays a central role in the theory and practice of finance. The most used version by professional remains today the so-called Simple Volatility. However, following for instance Barndorff-Nielsen and Shephard (2003) or Andersen *et al.* (2003), volatility should be viewed as a latent factor (namely, the quadratic variation affecting the Brownian motion in some representations, for instance) that can only be estimated using its signature on market prices. It is only when the process is known (and simulated) as in Andersen and Bollerslev (1997, 1998) that we know what the true volatility is. As shown by Barndorff-Nielsen and Shephard (2002), when the underlying process is more sophisticated, or when observed prices suffer from market microstructure distortion effects (see Brandt and Diebold, 2003), the results are less clear.

The Realized Volatility is considered, since its first use (see Andersen and Bollerslev, 1998), as the best estimator for the latent factor of risk. The daily volatility obtained from transaction data is shown to be accurate when controlling for a microstructure effect and thus empirically supports the Clark (1973) Mixture of Distribution Hypothesis. Among the high-frequency estimators, the one using all the available transactions performs better than the Realized Volatilities that use a lower sampling rate (see Bollen and Inder, 2002). Oomen (2005) empirically also shows that estimating the volatility in business-time (transaction time) is more efficient than using the traditional calendar-time, as it samples the process when it is most informative. Aït-Sahalia *et al.* (2005) argue that the most precise estimator, so far, is the mean of the Realized Volatilities chosen at the optimal frequency but measured at different phases.

However, when high-frequency data are unavailable, the best estimations of the unobservable risk factor are obtained through the Range-based (or Extreme Value) estimators. The price range, defined as the difference between the highest and lowest market prices over a fixed sampling interval, is known for a long time as a volatility estimator. Starting with Parkinson (1980), there is a wealth of literature[1] devoted to refinements of this measure (using various assumptions about the underlying process).

The aim of the present article then is to study the main properties of low and high-frequency measures of volatility, in order to find if there are glaring discrepancies between the empirical evidence and the usual assumptions

[1] Relevant literature includes Parkinson (1980), Garman and Klass (1980), Rogers and Satchell (1991), Kunitomo (1992) and Yang and Zhang (2000).

that the distribution of volatility is Gaussian. We first recall definitions and properties of six main estimates of volatility, based on daily and intra-day data. We then compute them on the French CAC40 index over a ten-year high frequency sample. We secondly study their distributional properties by testing their Goodness-of-Fit against the Gaussian hypothesis. We thirdly focus on extreme volatilities. We fit a General Extreme Value distribution to the right-hand tail of daily risk measures, in order to get estimated frequencies of high watermarks of extreme market events.

2 From Low to High Frequency Measures of Risk *via* Extreme Value Estimators of Volatility

It is well known that the amplitude of price changes is not constant, but fluctuating with time in a somewhat predictable fashion. The Integrated Variance (*i.e.*, the variance of the instantaneous returns over a period) can be approximated through estimators of the Quadratic Variation of prices. We present hereafter the main measures of daily volatility, computed from either daily data or intra-day data.

2.1 Measures of Risk and Extreme Value Estimators of Volatility

The usual indicator of risk is the variance obtained from the series of closing prices. Since this indicator is not constant over time, a way to diminish its variations in the computation is to use a rolling window with a fixed range. The general expression of the daily volatility is calculated with daily data in the following manner:

$$\hat{\sigma}_t = \left\{ \frac{1}{(N-1)} \sum_{n=t-N+1}^{t} \left[\ln\left(\frac{P_n}{P_{n-1}}\right) - \hat{\mu}_t \right]^2 \right\}^{\frac{1}{2}}, \qquad (1)$$

where N is the estimation window expressed in a number of business days, $n = [1, \ldots, T]$ and $t = [N, \ldots, T]$ are daily dates, $\{P_n\}$ is a sequence of closing prices, and $\hat{\mu}_t$ is given by (with previous notations):

$$\hat{\mu}_t = \frac{1}{N} \sum_{n=t-N+1}^{t} \left[\ln\left(\frac{P_n}{P_{n-1}}\right) \right],$$

which is an estimation of the mean log-return on the reference period.

A main critic of this daily estimator concerns the serial dependences. Indeed, the same return observations are used in the computation of many successive volatilities, which is all the more true since N is large. Moreover, Poon and Granger (2002) have noticed that the statistical properties of the sample mean make it a very inaccurate estimate of the true mean. This is particularly true for small samples since taking deviations around zero - or around a very long period mean - instead of the sample mean on a short window, increases the accuracy of the estimate (even if biased). If we consider this approach, the simplest measure of Simple Volatility should be defined by the squared return between only two observation dates (days), which is written:

$$\hat{\sigma}_t^S = \left[\frac{n_b}{\tau} \ln\left(\frac{P_t}{P_{t-\tau}}\right)^2\right]^{1/2}, \tag{2}$$

where n_b is the number of business days *per* year and τ the periodicity (one day *per* default), and $\{P_t\}$ is the series of the price of the asset at time t.

In this case, there is no hypothesis about the mean return, and also no serial dependences. Then, we will use it as our instantaneous low frequency volatility in the rest of the paper. Nevertheless, its time-variation is greatly noisy, and, therefore, this estimate is not recommended for practical applications. It is possible to reduce some of the noise, affecting the previous daily-based estimates, by using an Exponential Moving Average[2] (EMA). The EMA estimator is defined by induction with the following equation:

$$\hat{\sigma}_t^{EMA} = \left\{\rho(\hat{\sigma}_{t-1}^{EMA})^2 + (1-\rho)\left[\ln\left(\frac{P_t}{P_{t-1}}\right)\right]^2\right\}^{\frac{1}{2}}, \tag{3}$$

where ρ is the parameter governing the smoothness[3].

Moreover, the counterpart of the simplicity of the previous volatility measure computations is that they do not take into account the information given by the path of the price inside the period of reference. For example, even at the low (daily) frequency, supplementary information is often available in addition to the closing price, such as the opening price and extremal prices within the day. Parkinson (1980) proposes, then, an estimator of the

[2]Sometimes called a RiskMetrics type of measure.

[3]It has been set to .5 (mild smoothing), corresponding to a half-life of one day or so.

volatility based on this type of data, given by:

$$\hat{\sigma}_t^P = \left\{ \frac{1}{\theta_N} \sum_{n=1}^{N} \left[\ln \left(\frac{H_n}{L_n} \right) \right]^2 \right\}^{\frac{1}{2}}, \tag{4}$$

where $\theta_N = 4N \ln(2)$ is a correction parameter, and:

$$\begin{cases} H_n = \underset{P_t}{Max} \{ P_t \mid t \in [n-1, n] \} \text{ is the highest price on day } n \\ L_n = \underset{P_t}{Min} \{ P_t \mid t \in [n-1, n] \} \text{ is the lowest price on day } n. \end{cases}$$

The efficiency of Parkinson's (1980) Extreme Value Volatility estimator comes intuitively from the fact that the range of intra-daily quotes gives more information regarding the true volatility than two arbitrarily spaced points in these series (the closing prices), for the low cost of two data points *per* day. By this definition, Parkinson's (1980) estimator implicitly assumes that log-stock prices follow a geometric Brownian motion with no drift. Taking this assumption into consideration, Rogers and Satchell (1991) propose an improvement of the volatility estimator. They add a drift term in the stochastic process that can be incorporated into a volatility estimator (with previous notations):

$$\begin{aligned} \hat{\sigma}_t^{RS} = \quad & \left(\frac{1}{N} \sum_{n=t-N+1}^{t} \Big\{ \ln(H_n/O_n) \left[\ln(H_n/O_n) - \ln(C_n/O_n) \right] \right. \\ & \left. + \ln(L_n/O_n) \left[\ln(L_n/O_n) - \ln(C_n/O_n) \right] \Big\} \right)^{\frac{1}{2}}, \end{aligned} \tag{5}$$

where O_n is the open price on day n.

At last, Yang and Zhang (2000) propose another improvement by presenting an Extreme Value Volatility estimator that is unbiased, independent of any drift, and consistent in the presence of opening price jumps. Their estimator writes (with previous notations):

$$\begin{aligned} \hat{\sigma}_t^{YZ} = \quad & \left\{ \frac{1}{(N-1)} \sum_{n=t-N+1}^{t} \left[\ln(O_n/C_{n-1}) - \overline{\ln(O_n/C_{n-1})} \right]^2 \right. \\ & + \frac{\kappa}{(N-1)} \sum_{n=t-N+1}^{t} \left[\ln(C_n/O_n) - \overline{\ln(C_n/O_n)} \right]^2 \end{aligned} \tag{6}$$

$$+ (1-\kappa)\left(\hat{\sigma}_t^{RS}\right)^2\Big\}^{1/2},$$

with:

$$\kappa = \frac{.34}{\left[1.34 + \frac{N+1}{(N-1)}\right]},$$

and with C_n being the closing price on day n, the notation $\bar{X}_n$ standing for the unconditional mean of the sequence of the variable X_n, and $\hat{\sigma}_t^{RS}$ being the Rogers-Satchell (1991) estimator (see definition above).

Concerning the previous defined estimators, Alizadeh *et al.* (2002) underline that range-based estimators have many interesting properties compared to low frequency estimators or even, in some cases, to high-frequency based volatility estimators (Aït-Sahalia *et al.*, 2005, Marten and van Dijk, 2006). The range is a highly efficient volatility estimator as shown by Brandt and Diebold (2003) in a multivariate setting. For example, when the market is characterized by drops and recoveries in the same day, the classical close-to-close volatility can take low values while the daily range indicates that the volatility is truly high. Furthermore, the range is robust to microstructure biases such as the bid-ask bounce. When one measures the ratio of the variance of the Extreme Value estimators over the Close-to-Close Simple Volatility, all previous estimators provide very substantial improvements. For example, Corrado and Miller (2006) report that Parkinson's (1980) estimator allows a theoretical relative efficiency gain comprised between 2.5 and 5.

Moreover, while the extreme estimators are still dealing with traditional measures, the availability of tick-by-tick data led to a reframing of both theoretical and empirical literature on volatility. Instead of considering a constant volatility over a certain period of time (a day for instance), the continuous time model assumes a continuously varying volatility. The risk over the considered period is thus no longer a constant value, but the so-called Integrated Volatility. In a continuous framework, the most common stochastic equation of the price process is:

$$d\log(P_t) = \mu_t\, dt + \sigma_t dB_t, \tag{7}$$

with P_t the price at time t, μ_t the drift term, B_t the standard Brownian motion and σ_t the instantaneous volatility. This leads to the Integrated Volatility over the time interval τ, that is $\int_0^\tau \sigma_t^2 dt$. In this case, the empirical Integrated Volatility is in fact the Realized Volatility defined as:

$$\hat{\sigma}_{t,\tau}^{RV} = \left[\sum_{j=1}^{t/\tau} \ln \left(\frac{P_j}{P_{j-1}} \right)^2 \right]^{\frac{1}{2}}, \tag{8}$$

where t is the time interval between two successive observations and $\{P_j\}$ the sequence of high frequency (intraday) prices.

The next section is devoted to the study and the comparison of the six previously defined volatility estimators, namely the Realized, the Parkinson, the Rogers-Satchell, the Yang-Zhang, the Simple and the EMA Volatility estimators, by considering their time series and empirical distributions.

2.2 Descriptive Statistics, Correlations and Distribution Diagnoses of the Volatility

We represent in Figure 1 the various weekly estimates of daily volatility, using CAC40 French stock index intraday quotes, resampled at a 30' frequency in the period 01-01-1997 to 12-31-2006. The peaks of the variance estimates are approximately synchronous, but the general behavior of the series differs, both in the range of variances and persistence phenomenon (see next section). We remark also that Parkinson's (1980) estimator is the closest to the Realized Volatility in terms of similarity and general behavior.

Table 1 presents the four first moments of the empirical log-volatilities. The asymmetry coefficient of skewness is mostly positive (with the exception of Roger-Satchells volatility, exhibiting many very small values); the mass of probability on the right side of the distribution appears slightly larger than on the left side. The kurtosis differs across measures, with the Simple Volatility and the Rogers-Satchell measures appearing leptokurtic (due in fact to the existence of many observations close or equal to zero). Overall, as already seen in Figure 1, estimators using intra-day data are less volatile (more accurate) than the classical estimator.

Table 2 corresponds to the Pearson and Spearman correlation coefficients of risk log-estimations. It confirms once again that Parkinson's (1980) volatility is very close to the Realized Volatility.

It is generally admitted, since the seminal paper by Cizeau *et al.* (1997), that the log-volatility is approximately Gaussian for a daily integrated-horizon (see Andersen *et al.*, 2001), even if it is still discussed (see Reno and Rizza, 2003) or can be generalized (Bontemps and Meddahi, 2005).

Table 1: Statistics of the (Log-)Volatilities

	Mean (%)	Std (%)	Skewness (log vol)	Kurtosis (log vol)	Beta
Realized	16.31	9.82	.24	2.97	1
Parkinson	14.53	9.37	.10	2.98	.95
Rogers-Satchell	15.27	10.41	-1.28	7.29	.93
Yang-Zhang	20.42	12.91	.02	2.97	.87
Simple	16.65	16.16	-1.29	6.13	.80
EMA	19.72	12.27	.04	2.93	.73

Source: Euronext, 30' sampled intraday CAC40 French stock index quotes from the period 01-01-1997/12-31-2006. The Beta is computed for every estimator with respect to the Realized Volatility. Computations by the authors.

Table 2: Pearson's and Spearman's Correlations between Risk Measures

	Realized	Parkinson	Rogers-Satchell	Yang-Zhang	Simple	EMA
Realized	1	.88	.65	.79	.34	.67
Parkinson	.87	1	.62	.75	.39	.66
Rogers-Satchell	.78	.77	1	.71	.09	.32
Yang-Zhang	.77	.74	.81	1	.42	.65
Simple	.39	.44	.18	.48	1	.62
EMA	.66	.64	.42	.64	.71	1

Source: Euronext, 5' sampled intraday CAC40 French stock index quotes from the period 01-01-1997/12-31-2006. This table contains empirical Pearson (upper triangle) and Spearman (lower triangle) correlation coefficients between risk measures. Computations by the authors.

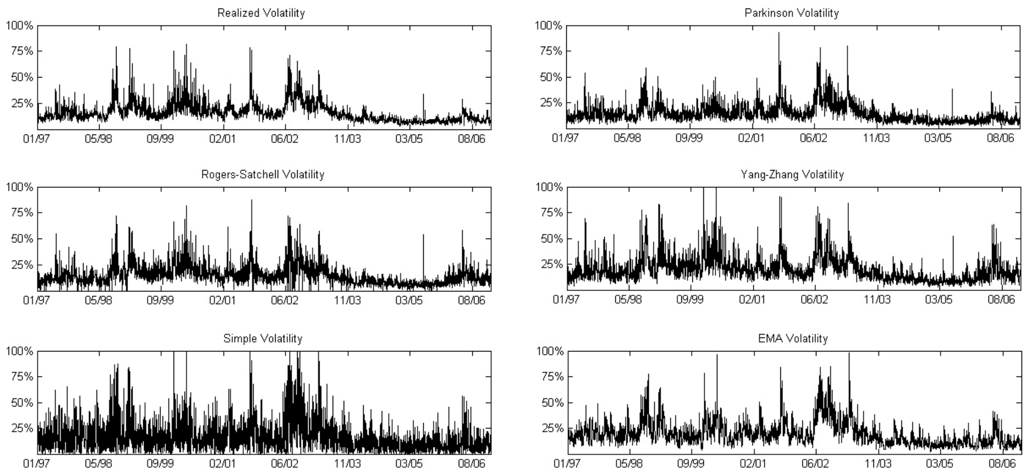

Figure 1: Daily Estimates of Annualized Volatilities (Source: Euronext, 30' sampled intraday CAC40 French stock index quotes from the period 01-01-1997/12-31-2006. Computations by the authors).

The Probability-to-Probability plots reported on Figure 2 show the empirical cumulative distributions of each volatility *versus* the Gaussian hypothesis. All of the scale and shape parameters are estimated using the Maximum Likelihood estimation method (*e.g.*, Law and Kelton, 1991).

A simple eye-ball analysis confirms the diagnosis based on higher moments: Gaussianity cannot be rejected at traditional significance levels for the most accurate estimates (namely Realized and Range-based Volatilities). Nevertheless, the differences in the left hand tails or in the mode and, likewise, small local differences in the curve can be diluted in the whole sample. A specific diagnosis of market volatility is relevant in turbulent periods and some inaccuracy in low risk periods can be tolerated provided, the estimator performs better otherwise. However, when studying volatility distributions, the area of interest is the right-hand tail where the highest volatilities are located. We have chosen to study more precisely these particular observations in the next section, by using the Parametric Block *Maxima* method for a Generalized Extreme Value (GEV) distribution of *extrema*.

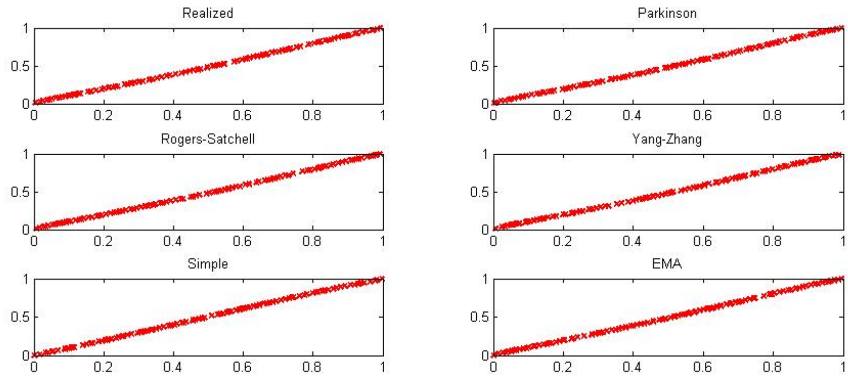

Figure 2: Goodness-of-Fit of a Gaussian Distribution (Source: Euronext; 5' resampled intraday CAC40 French stock index quotes from the period 01-01-1997/12-31-2006. Computations by the authors. The P-P Plots show the cumulative distribution of each volatility - on the x-axis - *versus* the Gaussian hypothesis - on the y-axis).

3 Extreme Values of the Daily Risk Estimates

The Generalized Extreme Value distribution (*Cf.* Jenkinson, 1955) is characterized by three parameters: $h \in \mathbb{R}$, the location parameter, $\alpha \in \mathbb{R}_+$, the scale parameter and $\xi \in \mathbb{R}$ known as the shape parameter (which is the inverse of the tail index). The last one measures the rate of decrease of the probability in the tails. The GEV distribution is given by:

$$H_\xi(\sigma) = \begin{cases} Exp\left\{-\left[1 + \xi\frac{(\sigma-h)}{\alpha}\right]^{-\xi^{-1}}\right\} & \text{if } \xi \neq 0, \\ Exp\left\{-Exp\left[-\frac{(\sigma-h)}{\alpha}\right]\right\} & \text{otherwise,} \end{cases} \tag{9}$$

for every $\sigma \in \mathbb{R}$ such that $[1 + \xi(\sigma - h)\alpha^{-1}] > 0$.

For fat-tailed distributions, the shape parameter will be significantly positive. We are then interested in the following to test the null hypothesis H_0 of the positivity of shape parameters of the noisy volatility estimators.

Among the multiple methods of estimation for the parameters of a GEV, the most common one is to use a direct numerical Maximization of the Log-likelihood. However, a natural challenger is proposed in the case of small samples (which is by definition the case when studying extreme events) and proved as one of the best methods for parameter estimations. This complementary method is based on the computation of the Probability Weighted Moments (see Greenwood *et al.*, 1979, and Hosking *et al.*, 1985). For the following empirical applications, we need to use the estimation of sample counterparts of L-moments for assessing the shape parameter as underlined hereafter. The L-moments, which are linear functions of the expectations of order statistics, were introduced by Sillitto (1951). One of the main advantages over conventional moments is that they suffer less from the effects of sampling variability because they are linear functions of the ordered data. They have been shown to provide more robust estimators of higher moments than the traditional sample moments. They can also characterize a wider range of distributions compared to the usual moments. Formally, the L-moment of order r is defined as:

$$\lambda_r = \sum_{k=1}^{r} p^*_{k-1,r-1}\, \beta_{k-1}, \tag{10}$$

with:

$$\beta_{k-1} = k^{-1}\, E(X_{[k:k]}),$$

where $p^*_{.,.}$ are the shifted Legendre polynomials coefficients, and β_{k-1} are the Probability Weighted Moments of order $k = [2, \ldots, r]$.

They can be estimated without bias from the sample Probability Weighted Moments computed such as:

$$\hat{\beta}_{k-1} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \prod_{j=1}^{k-1} \left[\frac{(i-j)}{(n-j)} \right] X_{[i:n]} \right\}, \tag{11}$$

where $X_{[i:n]}$ is the i-th order statistic of a sample of n realizations (see Appendix).

When the shape parameter is different from zero, the three first L-moments, as a function of the characteristic parameters of a GEV distribution, are given by (see Hosking and Wallis, 1997; Embrechts *et al.*, 1997; and the proof in the Appendix):

$$\begin{cases} \lambda_1 = h - \frac{\alpha}{\xi} + \frac{\alpha}{\xi}\Gamma(1-\xi) \\ \lambda_2 = \frac{\alpha}{\xi} \left(2^{\xi} - 1\right) \Gamma(1-\xi) \\ \lambda_3 = \frac{\alpha}{\xi} \left[1 - 3(2^{\xi}) + 2(3^{\xi})\right] \Gamma(1-\xi) \end{cases} \tag{12}$$

where $h \in I\!R$ is the location parameter, $\alpha \in I\!R_+$ the scale parameter, $\xi \in I\!R^*$ the shape parameter and $\Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t} dt$ is the Gamma function

Table 3: Estimates of Shape Parameters of Generalized Extreme Value Distributions of Daily Log-volatilities Period *Maxima via* Maximum Likelihood and Probability Weighted Moment Methods

Method	Frequency	Realized	Parkinson	Rogers-Satchell	Yang-Zhang	Simple	EMA
Maximum	Weekly	-.17	-.23	-.24	-.22	-.31	-.24
Likelihood	Monthly	-.24	-.21	-.35	-.24	-.24	-.22
	Quarterly	-.48	-.31	-.56	-.35	-.31	-.25
	Weekly	-.17	-.21	-.24	-.24	-.30	-.25
L-Moments	Monthly	-.20	-.24	-.30	-.26	-.23	-.20
	Quarterly	-.30	-.27	-.48	-.37	-.18	-.14

Source: Euronext, 30' sampled intraday CAC40 French stock index quotes from the period 01-01-1997/12-31-2006. Computations by the authors.

These estimates of the three first L-moments are sufficient to get an estimation of all three parameters characterizing a GEV distribution, using any classical numerical solving method. In order to check that the GEV provides a good approximation for the distribution of the *maxima* in our sample, we apply the Kolmogorov-Smirnov Goodness-of-Fit test to the resulting distributions. This test never rejects the hypothesis that the GEV fits the data, with the lowest P-value being .16 for the 5%-threshold *maxima* of the realized volatility.

The following Table 3 gives estimates of the shape parameter, based on *maxima* of the daily log-volatilities, with the Maximum Likelihood and the L-moment methods. We first notice that the shape parameter estimations obtained with the two methods are similar most of the time. Moreover, the shape parameter estimation of the Realized Volatility on a weekly basis (-.17) and the one of the EMA on a quarterly basis (-.14) are the closest to zero. We also remark that, with longer windows, estimations are smaller than with short windows. In other words, the lower the frequency for collecting the index, the more limited the presence of financial krachs, marked by extreme volatilities. Finally, and more importantly, whatever the method, the frequency and the estimate, none of the estimated shape parameters (and then tail indexes) is positive. This clearly proves that none of the underlying distributions can be considered as fat-tailed.

The previous shape parameters correspond to real market data series. In order to reinforce our previous conclusions and following Danielson and de Vries (1998), we renew the shape parameter estimation exercise using bootstrapped series of volatilities for assessing some inferences about the shape parameter estimations. More precisely, after computing the daily volatilities according to each estimator, we draw with replacement volatility series from these estimates and compute new weekly, monthly and quarterly *maxima* from these virtual samples. These new *maxima* being uncorrelated by construction, we can then fit a GEV on these series and draw the empirical distribution of the resulting shape parameters. We present hereafter in Figure 3 (respectively, in Figure 4), the GEV shape parameters estimated using the Maximum Likelihood method (respectively, the L-Moment method) based on 500 bootstrapped series of weekly *maxima* of daily volatilities. This frequency is chosen since the one for which the estimation of the shape parameter for the Realized Volatility (the benchmark) is the closest to zero[4]. Ranging from -.39 (Simple Volatility) to -.12 (Realized Volatility), it appears that none of the shape parameters of volatility estimators exhibit a positive value on the rebuilt new artificial extreme volatility series

These last results clearly indicate that the negative values of the shape parameters observed with the real series are neither exceptional nor due to the specific characteristics of the volatility time series; the sign remains the same whether these characteristics are or are not accounted for. To sum up, the resampling results allow us to assert the significance of the negativeness of the shape parameter of the log-volatilities: there is no need to use fat-tailed distributions to account for the extremes of the log-volatilities. The log-normal approximation proves adequacy, at least for the asset (the CAC40 index), the frequency (30' quotes) and the sample (1997-2006) considered and by using our methodology (Maximum Likelihood and L-Moment methods), with the chosen density (GEV distribution) and the horizon considered (daily, weekly and quarterly).

To confirm the previous observations, we present in table 4 the shape parameters estimated on bootstrapped series of volatilities and on series of

[4]Results taken into account at monthly and quarterly frequencies (not reported here) lead to the same kind of qualitative conclusions.

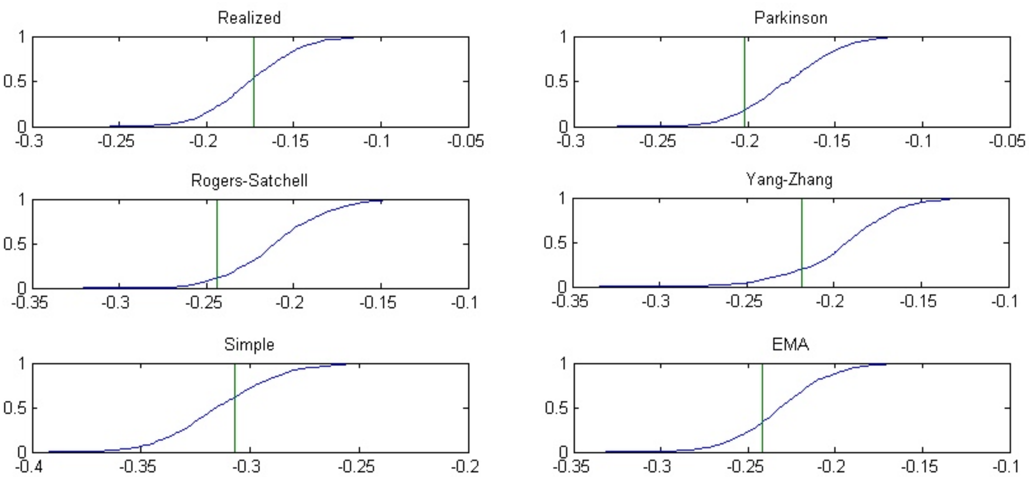

Figure 3: Bootstrapped Values of the Maximum Likelihood GEV Shape Parameters of the Daily Volatility Weekly *Maxima* (Source: Euronext; 30' resampled intraday CAC40 French stock index quotes from the period 01-01-1997/12-31-2006. Computations by the authors. The bootstrapped values of the shape parameters of the Realized Volatilities are plotted on the x-axis, the empirical cumulative distribution on the y-axis).

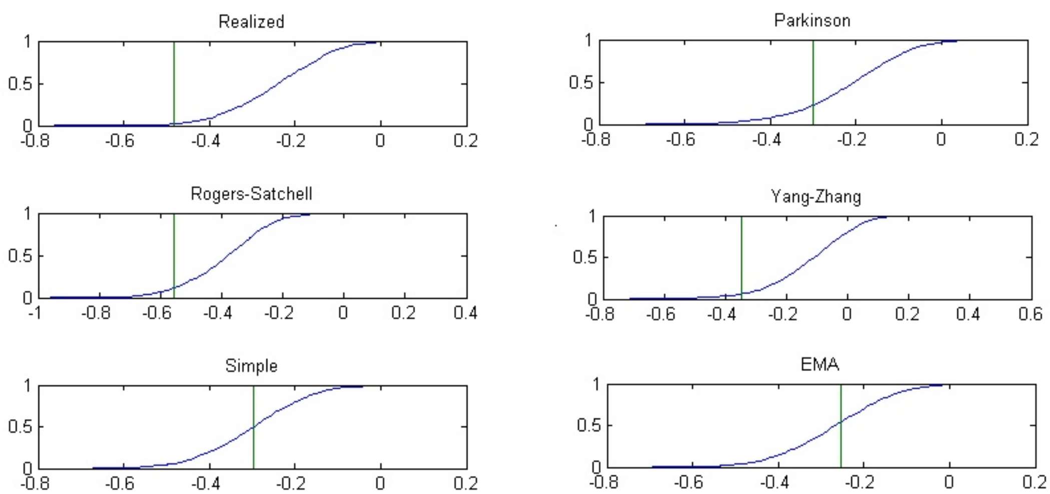

Figure 4: Bootstrapped Values of the L-Moment method GEV Shape Parameters of the Daily Volatility Weekly *Maxima* (Source: Euronext; 30' resampled intraday CAC40 French stock index quotes from the period 01-01-1997/12-31-2006. Computations by the authors. The bootstrapped values of the shape parameters of the Realized Volatilities are plotted on the x-axis, the empirical cumulative distribution on the y-axis).

volatilities obtained from bootstrapped series of returns. In order to reshuffle the series, we use four methods of bootstrap: the simple one (Efron and Tibshirani, 1986); the stationary one (Politis and Romano, 1994); the circular one (Politis and Romano, 1992, and Politis and White, 2004); the accelerated one (Efron and Tibshirani, 1993, and Gilli and Këllezi, 2006). We observe that the shape parameters obtained are similar whatever the bootstrap methods and series, and are also equivalent to the estimates obtained from the original series. This allows us to conclude that the shape parameters remain undoubtedly strictly negative.

For finally validating furthermore these results, we introduce herein a final simple reality check: given the sample estimates of the parameters, it is now possible to compute the probability of observing the historical volatility peaks under the various measures and hypotheses. The sample spans from January 1997 to December 2006, with the highest volatilities occurring in most cases at the terrorist attack on the Twin Towers (September 2001). Using the shape parameter from the GEV, estimated from the Maximum Likelihood Method, we now compute the probability of these events and their associated return-times. Table 5 presents these probabilities[5].

Though this reality check has a very limited statistical significance, it allows us to filter the results according to our subjective estimation of the likelihood of a major event. Even if the shape parameter appears relatively stable over the choice of the estimators, we obtain important different return-times, which implies large differences in the other parameters of the GEV. To illustrate this idea, we give in the following figure the density of each estimate, obtained for a GEV distribution, and we compare them to the empirical density functions. It appears the two curves are similar, which is a good sign regarding the pertinence of probabilities and return-times we obtained.

Further within the tail, the variability increases and estimates can still differ by a factor larger than two, so the choice of the measure is not insignificant. Overall, the estimates seem to give return-times which are more in line with the size of the sample (about ten years), except for the Yang-Zhang one, which gives return-times slightly larger as the sample length. We also notice the return-times decrease quickly between the first and the third ex-

[5] For information, when the return distribution is estimated by the Maximum Likelihood Method for a Normal distribution, the three largest probabilities of returns are 3.88 10^{-6}%, 5.67 10^{-6}% and 6.10 10^{-5}% which give respectively the following return-times: 103,085 years, 70,538 years and 6,567 years.

Table 4: Comparison of Estimates of Shape Parameters of Generalized Extreme Value Distributions of Daily Log-volatilities Weekly *Maxima* using Maximum Likelihood and Bootstrap Methods

Series	Methods	Statistics	Realized	Parkinson	Rogers-Satchell	Yang-Zhang	Simple	EMA
Return	Simple	Shape Param.	-.24	-.23	-.22	-.21	-.28	-.25
		[5%;95%]	[-.27;-.21]	[-.27;-.19]	[-.27;-.18]	[-.26;-.18]	[-.32;-.25]	[-.28;-.22]
		KS P-stat	(.51)	(.57)	(.52)	(.54)	(.56)	(.54)
	Stationary	Shape Param.	-.15	-.21	-.24	-.19	-.31	-.27
		[5%;95%]	[-.19;-.10]	[-.26;-.17]	[-.29;-.18]	[-.24;-.14]	[-.36;-.26]	[-.32;-.22]
		KS P-stat	(.61)	(.60)	(.54)	(.57)	(.48)	(.54)
	Circular	Shape Param.	-.19	-.21	-.22	-.20	-.29	-.25
		[5%;95%]	[-.23;-.15]	[-.27;-.16]	[-.27;-.16]	[-.25;-.15]	[-.34;-.25]	[-.29;-.2]
		KS P-stat	(.61)	(.56)	(.52)	(.58)	(.51)	(.55)
Volatility	Accelerated	Shape Param.	-.22	-.20	-.22	-.22	-.31	-.24
		[5%;95%]	[-.26;-.18]	[-.23;-.16]	[-.26;-.17]	[-.25;-.17]	[-.33;-.28]	[-.27;-.20]
		KS P-stat	(.47)	(.45)	(.50)	(.49)	(.53)	(.46)
	Stationary	Shape Param.	-.22	-.19	-.25	-.24	-.32	-.24
		[5%;95%]	[-.26;-.17]	[-.24;-.14]	[-.29;-.21]	[-.29;-.20]	[-.36;-.27]	[-.29;-.19]
		KS P-stat	(.58)	(.57)	(.53)	(.52)	(.54)	(.52)
	Circular	Shape Param.	-.22	-.21	-.22	-.22	-.30	-.24
		[5%;95%]	[-.31;-.11]	[-.27;-.15]	[-.32;-.11]	[-.30;-.12]	[-.35;-.25]	[-.29;-.19]
		KS P-stat	(.35)	(.51)	(.44)	(.41)	(.51)	(.46)

Source: Euronext; 30' resampled intraday CAC40 French stock index quotes from the period 01-01-1997/12-31-2006. The shape parameters for the various volatility measures are estimated by the Method of Maximum Likelihehood of a GEV density with a weekly frequency (the block maxima length) on 10,000 series obtained with bootstrap methods (simple: Efron and Tibshirani, 2003; stationary: Politis and Romano, 1994; circular: Politis and Romano, 1992, and Politis and White, 2004; accelerated: Efron and Tibshirani, 1993, and Gilli and Këllezi, 2006) on series of returns or on series of volatilities. The 90% confidence intervals of shape parameters are reported in brackets, whilst P-statistics of Goodness-of-Fit Kolmogorov-Smirnov tests (denoted KS P-stat.) are between parentheses. Computations by the authors.

Table 5: Probablity of Largest Negative Returns and Largest Daily Volatilities, and Return-Times using Maximum Likelihood GEV Estimates

Estimator	Crisis	Values	Weekly		Monthly		Quarterly	
			Prob. Times	Return	Prob. Times	Return	Prob. Times	Return
Returns	11/09/2001	-7.68%	.05%	37.73*	.54%	14.81*	.78%	33.34*
	15/04/2000	-7.55%	1.22%	1.64	1.33%	6.02	1.91%	13.61*
	14/03/2003	-7.00%	3.88%	0.51	4.20%	1.90	5.94%	4.38
Realized	12/09/2001	.91	.63%	3.15	2.01%	3.99	5.92%	4.40
	24/07/2002	.71	.68%	2.96	2.12%	3.77	6.27%	4.14
	03/04/2000	.70	.88%	2.31	2.69%	2.98	7.99%	3.25
Parkinson	11/09/2001	.91	.64 %	3.11	2.21 %	3.42	4.01 %	6.85
	15/03/2003	.71	.93%	2.03	2.32%	3.24	6.59%	3.92
	25/07/2002	.70	.97%	1.96	2.81%	2.78	7.52%	3.41
Rogers-Satchell	05/04/2000	.70	.68%	2.92	2.39%	3.35	4.16%	6.26
	24/07/2002	.64	1.06%	1.88	3.47%	2.31	7.22%	3.60
	21/09/2001	.63	1.10%	1.82	3.57%	2.24	7.50%	3.47
Yang-Zhang	17/04/2000	1.20	.17%	12.08*	.57%	14.07*	1.87%	13.93*
	05/01/2000	1.00	.51%	3.93	1.67%	4.79	5.65%	4.60
	21/09/2001	.90	.89%	2.26	2.79%	2.86	9.09%	2.86
Simple	11/09/2001	1.17	.19%	10.39*	1.15%	6.94	4.61%	5.64
	14/03/2003	1.14	.23%	8.55	1.30%	6.14	5.05%	5.15
	29/07/2002	1.11	.31%	6.38	1.57%	5.08	5.83%	4.46
EMA	14/03/2003	.98	.15%	13.01*	.79%	10.13*	3.59%	7.25
	15/10/2002	.85	.50%	4.01	1.84%	4.35	6.58%	3.95
	11/09/2001	.84	.54%	3.69	1.96%	4.08	6.89%	3.78

*Source: Euronext, 30' sampled intraday CAC40 French stock index quotes on the period 01-01-1997/12-31-2006. Return-times are expressed in years and they are marked with an asterisk * when they are larger than the size of the sample, i.e. 10 years. Computations by the authors.*

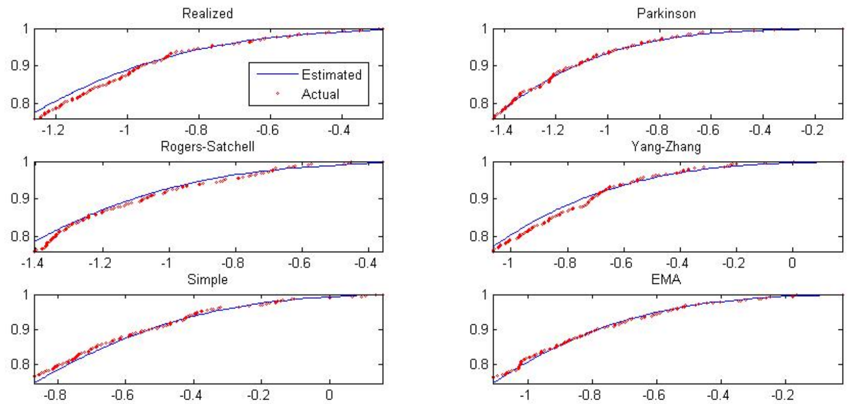

Figure 5: Empirical and Estimated Density Functions of the Volatility Estimates (Source: Euronext; 30' resampled intraday CAC40 French stock index quotes from the period 01-01-1997/12-31-2006. Are represented in this Figure, on the y-axis, the empirical cumulative functions of the log-volatilities (red dots), altogether with their GEV best fits (thin lines). Annualized Daily Volatilities are represented on the x-axis. Computations by the authors.

treme values. From these estimations of the extreme values - computed with several methods and for various volatility estimators, we can prudently infer with reasonable confidence that it is unlikely that a fat-tailed distribution is needed to fit the high volatilities. Indeed, we have obtained in most cases negative shape parameters, which corresponds to a reversed Weibull-kind of distribution. However, range-based and intra-day volatilities are not incompatible with the log-normal hypothesis and thus the standard approximation is not significantly flawed.

4 Conclusion

The Realized Volatility, despite its known shortcomings, remains a benchmark to which measures of risk should be compared. We show here that, among the low-frequency volatility measures, Parkinson's (1980) volatility was the closest to the high-frequency benchmark measure. This estimator should thus be the one used when trying to get long-horizon historical estimates, or to complement series of Realized Volatilities. Generally speaking, estimations of the whole distribution of the empirical volatilities cannot help to easily distinguish between the candidate functional forms. Given the rationale for estimating these distributions - retrieving possible risk - and the main differences between them - in the tails - it seems natural to try instead to use the Extreme Value Theory and concentrate on estimating the asymptotic distribution for the extreme measures of risk. The estimations for the Generalized Extreme Value indicate that the fat-tailed distribution is not needed to fit the sample volatilities. A log-normal process, as in the traditional stochastic volatility model, seems sufficient to reproduce the extreme empirical volatilities observed in the ten year ultra-high frequency studied sample.

However, we can think about confirming these results in line with others (see Andersen *et al.*, 2001) with a complementary analysis including additional measures (*e.g.*, Kunitomo, 1992; Garman and Klass, 1980), other samples (containing this time individual stocks), more recent observations (highlighting recent market turmoils and credit linked events in 2007, 2008 and 2009), different methodologies (Parametric Block *Maxima* and Parametric Peaks-over-the-Threshold), complementary estimation methods (other types of Moment Estimations *versus* the Maximum Likelihood method), using various distributions (Generalized Pareto Distribution *versus* GEV), other realistic sampling schemes (from one-minute to one-hour quote frequency) and other horizons (30" to a quarter). One may finally think about a Reality Check Test based on the various estimators, assets and methods of estimation (see White, 2000), for reinforcing our preliminary results on the best specification for the probability model for volatility. In particular, the final conclusion for a thin tail distribution for the volatility is deeply related to the choice of the estimation period. The recent turmoil on the financial markets, with its scope and it persistence, should have an important impact on the distribution of the volatility and then should questions its extreme behavior.

A practical application of these results will be to plug the appropriate estimates and distributions of volatilities in the Index of Market Shocks (IMS, see Maillet and Michel, 2003 and 2005) in order to get a clear ranking of the historical crises and an accurate estimation of the return-times of extreme *scenarii*, to ultimately precisely define the high watermarks of market risks.

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Appendix

Let $\{X_j\}$, with $j = [1, \dots, n]$, be a sequence of n independent and identically distributed non-degenerated random variables with a cumulative continuous distribution function $F(x)$ and with a quantile function $Q(u) = F^{-1}(u)$.

We recall that the r-th L-moment is defined, for every $r = [1, \dots, n]$, by:

$$\lambda_r = \frac{1}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E(X_{[r-j:r]}). \tag{13}$$

where $X_{[i:r]}$ is the i-th order statistic of a sample of r random variables.

Now since:

$$E(X_{[i:r]}) = \frac{r!}{(i-1)!\,(r-i)!} \int_0^1 Q(u) u^{i-1} (1-u)^{r-i} du, \tag{14}$$

we obtain:

$$\lambda_r = \frac{1}{r} \int_0^1 Q(u) \left[\sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{r!}{(r-j-1)!j!} u^{r-j-1} (1-u)^j \right] du. \tag{15}$$

It is often useful to express the r-th L-moment as a linear function of the Probability Weighted Moments, *i.e.*:

$$\lambda_r = \sum_{k=1}^{r} p^*_{k-1,r-1}\, \beta_{k-1}, \tag{16}$$

where $p^*_{.,.}$ are the Legendre polynomials coefficients, and $\beta_k = k^{-1} E(X_{[k:k]})$ are the Probability Weighted Moments of order $k = [1, \dots, r]$.

They can be estimated without bias by the following estimations:

$$\hat{\beta}_{k-1} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \prod_{j=1}^{k-1} \left[\frac{(i-j)}{(n-j)} \right] \hat{X}_{[i:n]} \right\}, \tag{17}$$

where $\hat{X}_{[i:n]}$ is the i-th order statistic of a sample of n realizations.

The three first sample L-moments can then be written in the following manner:

$$\left\{\begin{array}{ll} \lambda_1 = & \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\hat{X}_{[i:n]} \\ \lambda_2 = & \dfrac{1}{n(n-1)}\displaystyle\sum_{i=1}^{n}(2i-1-n)\hat{X}_{[i:n]} \\ \lambda_3 = & \dfrac{1}{n(n-1)(n-2)}\displaystyle\sum_{i=1}^{n}[6(i-1)(i-2)-6(i-1)(n-2) \\ & \qquad\qquad\qquad +(n-1)(n-2)]\hat{X}_{[i:n]} \end{array}\right. \tag{18}$$

Based on these previous definitions, we can now give a specific result on the L-moments for a GEV distribution.

Proposition 1 (see Hosking and Wallis, 1997, and Embrechts *et al.*, 1997). *The three first L-moments, as a function of the three characteristic parameters of a GEV distribution, are given by:*

$$\left\{\begin{array}{l} \lambda_1 = h - \frac{\alpha}{\xi} + \frac{\alpha}{\xi}\Gamma(1-\xi) \\ \lambda_2 = \frac{\alpha}{\xi}\left(2^{\xi}-1\right)\Gamma(1-\xi) \\ \lambda_3 = \frac{\alpha}{\xi}\left[1-3(2^{\xi})+2(3^{\xi})\right]\Gamma(1-\xi) \end{array}\right. \tag{19}$$

where λ_r is the r-th L-moment, $h \in \mathbb{R}$ the location parameter, $\alpha \in \mathbb{R}_+$ the scale parameter, $\xi \in \mathbb{R}^*$ the shape parameter and $\Gamma(a) = \int_0^{+\infty} t^{a-1}e^{-t}dt$ is the Gamma function.

Proof. We get the following intermediate result, for every $k \in N$:

$$\begin{aligned} \int_0^1 u^k[-\ln(u)]^{-\xi}du &= \int_0^{+\infty} x^{-\xi}e^{-(k+1)x}dx \\ &= \frac{1}{(1+k)^{1-\xi}}\int_0^{+\infty} y^{-\xi}e^{-y}dy \\ &= \frac{1}{(1+k)^{1-\xi}}\Gamma(1-\xi), \end{aligned} \tag{20}$$

thanks to simple changes of variable.

Given the quantile function of the GEV distribution such as:

$$Q(u) = h - \frac{\alpha}{\xi} + \frac{\alpha}{\xi}[-\ln(u)]^{-\xi}, \tag{21}$$

we can then compute the first L-moment:

$$\begin{aligned}\lambda_1 &= \int_0^1 Q(u)du \\ &= \int_0^1 (h - \frac{\alpha}{\xi})du + \frac{\alpha}{\xi}\int_0^1 [-\ln(u)]^{-\xi} du. \end{aligned} \tag{22}$$

Using result (20), we obtain the final expression for the first L-moment so that:

$$\lambda_1 = h - \frac{\alpha}{\xi} + \frac{\alpha}{\xi}\Gamma(1-\xi). \tag{23}$$

Similarly, the second L-moment is given by:

$$\begin{aligned}\lambda_2 &= \int_0^1 \left\{h - \frac{\alpha}{\xi} + \frac{\alpha}{\xi}[-\ln(u)]^{-\xi}\right\}[2u-1]du \\ &= \int_0^1 2u\frac{\alpha}{\xi}[-\ln(u)]^{-\xi}du - \int_0^1 \frac{\alpha}{\xi}[-\ln(u)]^{-\xi}du \end{aligned} \tag{24}$$

and applying result (20) once again is leading to the following expression of the second L-moment:

$$\lambda_2 = \frac{\alpha}{\xi}\left(2^{\xi}-1\right)\Gamma(1-\xi). \tag{25}$$

Finally, the straightforward expression of the third L-moment is:

$$\lambda_3 = \int_0^1 \left\{h - \frac{\alpha}{\xi} + \frac{\alpha}{\xi}[-\ln(u)]^{-\xi}\right\}[1-6u+6u^2]du \tag{26}$$

which is leading, still using the intermediate result (20), to the expression of the third L-moment:

$$\lambda_3 = \frac{\alpha}{\xi}\left[1-3(2^{\xi})+2(3^{\xi})\right]\Gamma(1-\xi). \tag{27}$$

□