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The Structure Of Functional Graphs For Functions From A Finite Domain To Itself For Which A Half Iterate Exists

Paweł Marcin Kozyra
Institute of Mathematics Pedagogical University of Krakow, pawelmkozyra@gmail.com

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Abstract

The notion of a replica of a nontrivial in-tree is defined. A result enabling to determine whether an in-tree is a replica of another in-tree employing an injective mapping between some subsets of sources of these in-trees is presented. There are given necessary and sufficient conditions for the existence of a functional square root of a function from a finite set to itself through presenting necessary and sufficient conditions for the existence of a square root of a component of the functional graph for the function and for the existence of a square root of the union of two components of the functional graph for the function containing cycles of the same length using the concept of the replica.

1 Introduction

The main problem of interest in this article is to determine a functional square root (half iterate) of any function $f\colon X \to X$, where X is a finite set. Generally, fractional iterates (iterative roots of n-th order) are defined as follows: $f^{\frac{1}{n}}$ is a function $g\colon X \to X$ such that $g^n = f$, for all $n \in \mathbb{N}$ and the *n-th iterate of a function* $f\colon X \to X$ is defined for non-negative integers in the following way:

- $f^0 := id_X$
- $f^{n+1} := f \circ f^n$

where id_X is the identity function on X and $f \circ g$ denotes function composition.

1.1 Historical Background

The problem of half iterates and fractional iterates has been studied since the 19th century. One of the earliest research on this topic is Charles Babbage's research from 1815 of the solutions of $f\big(f(x)\big) = x$ over $\mathbb{R}$, so-called the involutions of the real numbers [12]. For the given function h the solution Ψ of Schröder's equation

$$\Psi\big(h(x)\big) = s\Psi(x), \text{where the eigenvalue } s = h'(a) \text{ and } h(a) = a$$

enables finding arbitrary functional n-roots [35]. In general, all functional iterates of h are given by $h_t(x) = \Psi^{-1}\big(s^t\Psi(x)\big)$, for $t \in \mathbb{R}$. Kneser studied [18] the half iterate of the exponential function. Szekeres [34] dealt with regular iterations of real and complex functions. Curtright and Zachos [6] analyzed the problem of approximate solutions of functional equations.

The concept of fractional iterate is strictly connected with the notion of an nth root of a graph, since each function from a finite domain to itself can be depicted by its functional graph. The concept of the square of a finite, undirected, and without loops or multiple lines graph was introduced by Harary and Ross, [16]. They presented a criterion for a graph to be the square of a tree. This concept can be generalized. Let G be an undirected graph. The nth power of G, written G^n, is defined to be the graph having the same vertex set as G with two vertices adjacent in G^n if and only if there is a path of length at most n between them. A graph G has an nth root H if $H^n = G$. If $n = 2$, we say that H is the square root of G. Mukhopadhyay [31] characterized graphs that have at least one square-root graph and proved that a connected undirected graph G with vertex set $\{v_1, \ldots, v_n\}$ has a square root if and only if G contains a collection of n complete subgraphs $G_1, \ldots, G_n$ such that for all $i, j \in \{1, \ldots, n\}$:

1. $\bigcup_{i=1}^{n} G_i = G$,
2. $v_i \in G_i$,
3. $v_i \in G_j$ if and only if $v_j \in G_i$.

Harary et al. [15] characterized graphs whose squares are planar. Geller [10] characterized digraphs that have at least one square root. Lin and Skiena [24] invented a linear time algorithm for finding the tree square roots of a given graph and a linear time algorithm for finding the square roots of planar graphs.

Motwania and Sudanb [30] proved that computing any square root of a square graph, or deciding whether a graph is a square, is an NP-hard problem. The following results concern square root finding problem in terms of the girth of the square root. The girth of a graph G with a cycle is the length of its shortest cycle. An acyclic graph has infinite girth. Chang et al. [5] found a polynomial time algorithm to compute the tree acyclic square root. Farzad et al. [8] presented a polynomial time characterization of squares of graphs with a girth of at least 6. They also proved that the square root (if it exists) is unique up to isomorphism when the girth of the square root is at least 7 and proved the NP-completeness of the problem for square roots of girth 4. Adamaszek [1] proved that the square root of a graph is unique up to isomorphism when the girth of the square root is at least 6 if it exists. Farzad and Karimi [7] showed that the square root finding problem is NP-complete for square roots of girth 5. Thus they proved the complete dichotomy theorem for the square root problem in terms of the girth of the square roots: Square of graphs with girth g is NP-complete if and only if $g \leq 5$.

There is also vast polish literature devoted to iterative roots. Ger [11] and Kuczma [20] dealt with functional equation $\varphi^2(x) = g(x)$. Kuczma [22] dealt with iterative functional equations. Kuczma[21],[23] and Zdun [38] described fractional iterations of convex functions. Zdun [36, 37] dealt with differentiable fractional iterations and with the problem of existence and uniqueness of continuous iterative roots of homeomorphisms of the circle.

1.2 The Structure of the Article

The following section contains some basic notions and facts from Graph Theory as well as some of the previous Kozyra's results [19] used later in this article. In the third section, the concepts of a coil and a replica, as well as their properties, are described. This section lays the foundation for the main results of this article presented in the fourth section, in which our attention will be focused on necessary and sufficient conditions for the existence of a square root of a component of a functional graph or the existence of a connected square root of the union of two components of a functional graph, containing cycles of the same length. In this section, an example clarifying and using previous results is given.

2 Preliminary

In the first subsection of this section some basic, coming from [14], and occurring later in this article, terminology and facts from graph theory are presented. It can be omitted by readers acquainted with standard terminology from graph theory. The second subsection of this section contains some of Kozyra's [19] results that will be used in the further parts of this article.

2.1 Basic Definitions and Facts

A graph $G = (V, E)$ consists of two sets V and E. The elements of V are called *vertices* (or nodes). The elements of E are called *edges*. Each edge has a set of one or two vertices associated to it, which are called its *endpoints*. An edge is said to *join* its endpoints. If vertex v is an endpoint of edge e, then v is said to be *incident* on e, and e is incident on v. A *self-loop* or loop is an edge that joins a single endpoint to itself. The *graph union* of two graphs G and H is the graph $G \cup H$ whose vertex-set and edge-set are the disjoint unions, respectively, of the vertex-sets and the edge-sets of G and H. A *subgraph* of a graph $G = (V_G, E_G)$ is a graph $H = (V_H, E_H)$ such that $V_H \subset V_G$ and $E_H \subset E_G$. In a graph G, *the induced subgraph on a set of vertices* $W = \{w_1, \ldots, w_k\}$, denoted $G(W)$, has W as its vertex-set, and it contains every edge of G whose endpoints are in W. A *directed edge* (or *arc*) is an edge e, one of whose endpoints is designated as the *tail*, and whose other endpoint is designated as the *head*. An arc that is directed from vertex u to v is said to have tail u and head v. A *multi-arc* is a set of two or more arcs having the same tail and same head. If the digraphs under consideration do not have multi-arcs, then an arc that is directed from vertex u to v is represented by the ordered pair (u, v). A *digraph* (or directed graph) is a graph each of whose edges is directed. A *trivial graph* is a graph consisting of one vertex and no edges. A *simple digraph* is a digraph with no self-loops and no multi-arcs. The *underlying graph* of a directed or partially directed graph G is the graph that results from removing all the designations of head and tail from the directed edges of G. The *degree* (or valence) of a vertex v in a graph G, denoted $deg(v)$, is the number of proper edges incident on v plus twice the number of self-loops. The *indegree* of a vertex v in a digraph, denoted $in\_deg(v)$, is the number of arcs directed to v; the outdegree of vertex v, denoted $out\_deg(v)$, is the number of arcs directed from v. Each self-loop at v counts one toward the indegree of v and one toward the outdegree. A vertex u in digraph D without multi-arcs *dominates* (or beats) vertex v in D whenever (u, v) is an arc of D. The *out-set* of a vertex v in a digraph D without multi-arcs, denoted $O(v)$, is the set of all vertices that v dominates, and the in-set of v, denoted $I(v)$, is the set of all vertices that dominate v.

A *walk* in a graph G is an alternating sequence of vertices and edges,

$$W = (v_0, e_1, v_1, e_2, \ldots, e_n, v_n)$$

such that for $j = 1, \ldots, n$, the vertices v_{j-1} and v_j are the endpoints of the edge e_j. If, moreover, the edge e_j is directed from v_{j-1} to v_j , then W is a directed walk. *The length of a walk* is the number of edges (counting repetitions). A walk is *closed* if the initial vertex is also the final vertex; otherwise, it is open. A *trail* in a graph is a walk such that no edge occurs more than once. A *path* in a graph is a trail such that no internal vertex is repeated.

A *cycle* is a closed path of length at least 1. The *cycle graph* C_n is the n-vertex graph with n edges, all on a single cycle. The *distance* between two vertices in a graph is the length of the shortest walk between them. A graph is connected if between every pair of vertices there is a walk. A digraph is (weakly) *connected* if its underlying graph is connected. A *component* of a graph G is a connected subgraph H such that no subgraph of G that properly contains H is connected. In other words, a component is a maximal connected subgraph.

A *tree* is a connected graph with no cycles (i.e., acyclic). A *directed tree* is a digraph whose underlying graph is a tree. A *source* in a digraph is a vertex of indegree zero. A *basis* of a digraph is a minimal set of vertices such that every other vertex can be reached from some vertex in this set by a directed path. An *in-tree* is a directed tree with a distinguished vertex r, called the root, such that for every other vertex v, the unique path from v to r is a directed path from v toward the root r. A *functional graph* is a digraph in which each vertex has outdegree one.

Fact 2.1. *[14, Fact 19, p. 188] Let D be a functional graph, and let G be the underlying undirected graph. Then each component of G contains exactly one cycle. In D this cycle is a directed cycle, and the removal of any arc in it turns that component into an in-tree.*

The depth or level of a vertex v in in-tree, denoted by $lev(v)$, is its distance to the root, that is, the number of edges in the unique directed path from v to the root. In an in-tree a vertex w is called a *descendant* of a vertex v (and v is called an ancestor of w), if w is on the unique path from v to the root. If, in addition, $w \neq v$, then w is a proper descendant of v (and v is a proper ancestor of w).

2.2 Reformulation for the Problem of Determining a Functional Square Root

The notion of a half iterate can be translated into Graph Theory language. Any function $\varphi\colon \{1,\dots,n\} \to \{1,\dots,n\}$ can be represented as corresponding to φ the directed graph $G = (V,E)$ denoted by $G(\varphi)$, with $V = \{1,\dots,n\}$ and $E = \{(k,l)\colon k,l \in \{1,\dots,n\} \quad \& \quad \varphi(k) = l\}$.

Definition 2.1. Let $G = (V,E)$ be a functional graph. A functional graph $G' = (V,E')$ is called *a square root of graph G* iff

$$\forall_{u,v\ \in\ V}(u,v)\ \in\ E \Longleftrightarrow \exists_{w\ \in\ V}(u,w),(w,v)\ \in\ E'$$

There is straightforward relation between a functional square root of a function φ and a square root of the functional graph $G(\varphi)$. If ψ is a half iterate of φ, then $G(\psi)$ is a square root of $G(\varphi)$. Similarly, for any function φ, if G' is a square root of $G(\varphi)$, then $G' = G(\psi)$ for a half iterate ψ of φ.

Kozyra [19] presented four algorithms determining all half iterates and seven algorithms finding one functional square root of any function $f\colon X \to X$ defined on a finite set X, if these square roots exist and characterized functions which are their selves half iterates. Moreover, Kozyra found formulas for the number of all half iterates of constant and the number of all identity functions defined on an n-element set, as well as, he proved that these numbers are greater than 2^{n-1}. In this article there are used two results from [19]:

Theorem 2.1. *[19, Thm. 9, p. 7] Let α be a function from a finite domain to itself and $G(\alpha)$ be its functional graph. Then there exists a half iterate of α iff*

1. *for each component in $G(\alpha)$ there exists its square root*
   or

2. *for each component $G_1 = (V_1, E_1)$ of $G(\alpha)$ which has no square root there exists another component $G_2 = (V_2, E_2)$ of $G(\alpha)$ containing a cycle of the same length as this contained in G_1 such that there exists a square root of the graph being the union of these components – $G_3 = (V_1 \cup V_2, E_1 \cup E_2)$.*

Proposition 2.1. *[19, Prop. 10, p. 9] Assume that α is a function from an n-element set to itself and its functional graph $G(\alpha)$ is a component containing a cycle $\bar{a} := (a_0, \ldots, a_{k-1})$ of length k and $A := \{a_0, \ldots, a_{k-1}\}$. Then:*

1. *If there exists a half iterate β of α, then k is an odd number and $\beta|_A = \alpha^{(k+1)/2}|_A$.*

2. *If β is a function and $G(\alpha)$ is the cycle graph C_k containing the cycle $\bar{a}$, then $\beta^2 = \alpha$ iff $\beta = \alpha^{(n+1)/2}$.*

3 Coils and Replicas

Theorem 2.1 represents a general way how to find a square root of a functional graph. In this section, the concepts of a coil and a replica are defined as well as their properties are characterized. These concepts allow finding a square root of a component or the union of two components containing a cycle of the same length, what is described in the fourth section. Thus the problem of finding a square root of a functional graph can be completely solved by employing these tools.

Let G be a functional graph and $C = (V, E)$ be one of its component. Then by Fact 2.1, C contains exactly one cycle $\bar{c} = (c_0, \ldots, c_{k-1})$ and for each $x \in V$ sub-graph $in-tree_C(x) = (V_x, E_x)$ defined as follows:

- V_x is the set consisting of x and all vertices $v \in V \setminus \{c_0, \ldots, c_{k-1}\}$ for which there exists the unique path in the induced subgraph
  $C(V \setminus \{c_0, \ldots, c_{k-1}\} \cup \{x\})$ from v to x.

- $E_x = \{e \in E \colon \exists_{u,v \in V_x} \colon e = (u, v)\}$

is an in-tree with the root x.

Definition 3.1. Such sub-graph $in-tree_C(x)$ will be called *the in-tree generated by element x in graph C*. An in-tree $G = (V, E)$ will be called strictly nontrivial, if there exists a vertex $v \in V$ whose level is greater than 2 or equals 2.

3.1 Coils

Definition 3.2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be nontrivial in-trees with roots r_1 and r_2, respectively. *A coil* of graph G_1 with graph G_2 is a graph $G = (V_1 \cup V_2, E)$ such that the following three conditions are satisfied:

(C1) G is an in-tree with the root r_1

(C2) $\forall_{a,b \in V_1}\big[(a, b) \in E_1 \Longleftrightarrow \exists_{v \in V_2}(a, v), (v, b) \in E\big]$

(C3) $\forall_{u,v \in V_2}\big[(u, v) \in E_2 \Longleftrightarrow \exists_{a \in V_1}(u, a), (a, v) \in E\big]$

The set of all coils G_1 with G_2 of is denoted by $Coils(G_1, G_2)$.

Proposition 3.1 (Some properties of coils)**.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be nontrivial in-trees with roots r_1 and r_2, respectively. If graph $G = (V_1 \cup V_2, E)$ is a coil of graph G_1 with graph G_2, then:*

1. $(r_2, r_1) \in E$
2. $E \subseteq V_1 \times V_2 \cup V_2 \times V_1$
3. *if a is an ancestor of b in G_1, then a is an ancestor of b in G for all $a, b \in V_1$*
4. *if u is an ancestor of v in G_2, then u is an ancestor of v in G for all $u, v \in V_2$*
5. *if a is a source in G_1 and $(a, v) \in E$, then:*
   - *if v is a source in G_2, then a is a source in G*
   - *if $(u, v) \in E_2$, then u is a source in G_2 and in G*

Proof.

1. Let $u \in V_2$ be such that $(u, r_2) \in E_2$, then by (C3), there exists $a \in V_1$ such that (u, a) and $(a, r_2) \in E$. Hence by (C1) $a \neq r_1$. So there exists $b \in V_1$ such that $(a, b) \in E_1$. Then by (C2), there exists $v \in V_2$ such that (a, v), $(v, b) \in E$, hence by (C1), $v = r_2$. Suppose that $b \neq r_1$, then there exists $c \in V_1$ such that $(b, c) \in E_1$, thus by (C2), there exists $w \in V_2$ such that (b, w), $(w, c) \in E$, so by (C3), $(r_2, w) \in E_2$ – a contradiction, since r_2 is the root of G_2. Therefore $b = r_1$ and $(r_2, r_1) \in E$.

2. If $a \in V_1 \setminus \{r_1\}$, then there exists $b \in V_1$ such that $(a, b) \in E_1$, hence by (C1) and (C2), there exists the unique u such that $(a, u) \in E$ and $u \in V_2$. If $u \in V_2 \setminus \{r_2\}$, then there exists $v \in V_2$ such that $(u, v) \in E_2$, so by (C1) and (C3) there exists the unique a such that $(u, a) \in E$ and $a \in V_1$. By (C1) and (1), $(r_2, r_1) \in E$ and it is the unique path from r_2 to r_1. Summarizing, $E \subseteq V_1 \times V_2 \cup V_2 \times V_1$.

3. If a is an ancestor of b in G_1, then there exists the unique path $(a_0, \ldots, a_m)$ from a to b in G_1, since G_1 is an in-tree. By (C1) and (C2), for each $i \in \{0, \ldots, m-1\}$ there exists the unique $u_i \in V_2$ such that $(a_0, u_0, a_1, \ldots, u_{m-1}, a_m)$ is the unique path from a to b in G.

4. The proof is analogous to the proof of (3)

5. Assume that a is a source in G_1 and $(a, v) \in E$. Suppose that a is not a source in G, then by (2), there exists $u \in V_2$ such that $(u, a) \in E$, hence by (C3), $(u, v) \in E_2$, thus v is not a source in G_2. Assume now additionally that $(u, v) \in E_2$ and suppose that u is not a source in G_2. Then there exists $t \in V_2$ such that $(t, u) \in E_2$, hence by (C3), there exists $b \in V_1$ such that $(b, u) \in E$. But $(a, v) \in E$ and $(u, v) \in E_2$, therefore by (C1) and (C3), $(u, a) \in E$. So we have $(b, u) \in E$ and $(u, a) \in E$, thus by (C2), $(b, a) \in E_1$ – a contradiction, since a is a source in G_1. Therefore u is a source in G_2, and therefore if u were not a source in G, then by (2) and (C2), the vertex a wouldn't be a source in G_1.

□

3.2 The Relation between Coils and Replicas

A key notion in this article is a replica.

Definition 3.3. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be nontrivial in-trees with roots r_1 and r_2, respectively. G_2 is called *a replica* of G_1 iff there exists a function $\varphi\colon V_1 \setminus \{r_1\} \to V_2$ such that the following four conditions are satisfied:

(R1) $\forall_{a,b \in V_1 \setminus \{r_1\}}(a, b) \in E_1 \Longrightarrow \big(\varphi(a), \varphi(b)\big) \in E_2$

(R2) $\forall_{a \in V_1 \setminus \{r_1\}}(a, r_1) \in E_1 \Longrightarrow \varphi(a) = r_2$

(R3) $\forall_{v \in V_2 \setminus \varphi(V_1 \setminus \{r_1\})}\exists_{a \in V_1 \setminus \{r_1\}}\colon (v, \varphi(a)) \in E_2$

(R4) $\forall_{a_1,b_1,a_2,b_2}\big[\big((a_1, b_1) \in E_1 \wedge (a_2, b_2) \in E_1 \wedge b_1 \neq b_2\big) \Rightarrow \varphi(a_1) \neq \varphi(a_2)\big]$

As one can see, the concept of a replica is similar to an isomorphism. The reader interested in algorithms for determining the isomorphism of graphs can find more information at the following papers [2, 3, 4, 9, 13, 17, 25, 26, 27, 28, 29, 33]. Before we formulate a theorem that establishes a relation between coils and replicas, let us consider the following example.

Example 3.1. Consider three in-trees from Fig. 1. We shall show that the in-tree $G_2 = (V_2, E_2)$ with the root r_2 is a replica of the in-tree $G_1 = (V_1, E_1)$ with the root r_1: Define function φ as follows

$$\varphi = \{(a_1, r_2), (a_2, b_1), (a_3, b_1), (a_4, b_3), (a_5, r_2), (a_6, b_4)\}.$$

We shall prove that φ satisfies condition (R1) – (R4):

(R1) Note that $\big(\varphi(a_2), \varphi(a_1)\big) = (b_1, r_2) \in E_2$, $\big(\varphi(a_3), \varphi(a_1)\big) = (b_1, r_2) \in E_2$, $\big(\varphi(a_6), \varphi(a_5)\big) = (b_4, r_2) \in E_2$, $\big(\varphi(a_4), \varphi(a_2)\big) = (b_3, b_1) \in E_2$,

(R2) Note that $I_{G_1}(r_1) = \{a_1, a_5\}$ and $\varphi(a_1) = \varphi(a_5) = r_2$

(R3) Note that $V_2 \setminus \varphi(v_1 \setminus \{r_1\}) = \{b_2, b_5, b_6\}$ as well as $(b_2, \varphi(a_2)) \in E_2$, $(b_5, \varphi(a_6)) \in E_2$, and $(b_6, \varphi(a_1)) \in E_2$

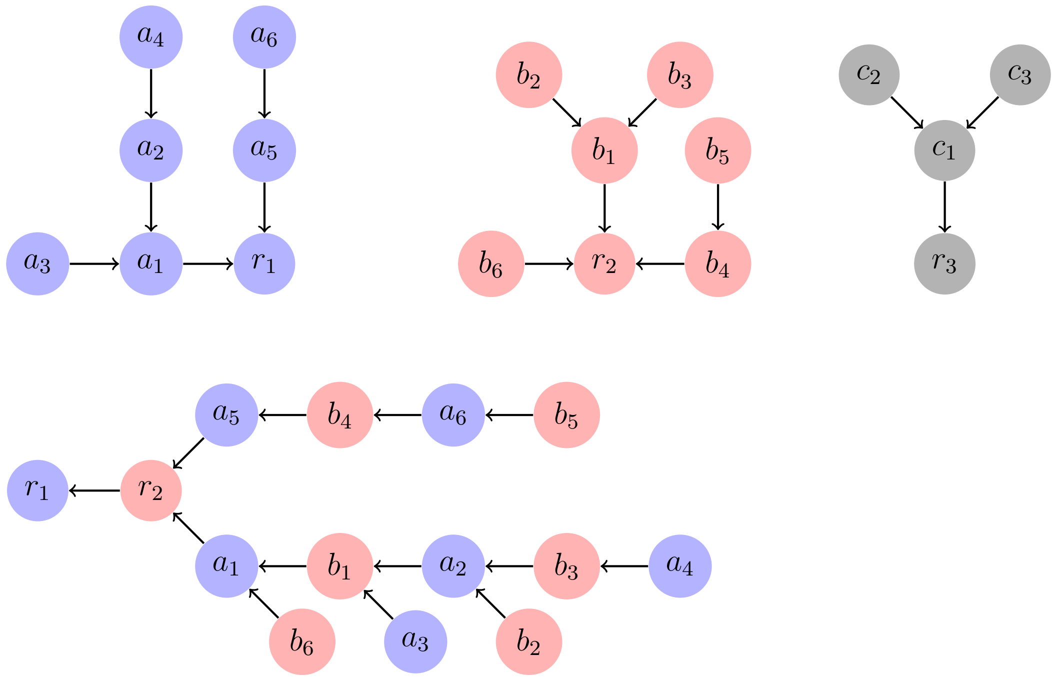

Figure 1: Three in-trees with roots r_1, r_2, r_3 and a coil of in-tree with root r_1 with in-tree with root r_2

(R4) Suppose that there exist $x_1, x_2, y_1, y_2 \in V_1$ such that $(x_1, y_1) \in E_1$, $(x_2, y_2) \in E_1$, $y_1 \neq y_2$ and $\varphi(x_1) = \varphi(x_2)$. Then $x_1 \neq x_2$, since G_1 is an in-tree. Moreover, $\{x_1, x_2\} = \varphi^{-1}(r_2) = \{a_1, a_5\}$ or $\{x_1, x_2\} = \varphi^{-1}(b_1) = \{a_2, a_3\}$. Hence we obtain that $\{y_1, y_2\} = \{r_1\}$ or $\{y_1, y_2\} = \{a_1\}$, so $y_1 = y_2$ – a contradiction.

Note also that the in-tree at the bottom of Fig. 1 is an example of a coil of the in-tree G_1 with the in-tree G_2.

We shall show that the in-tree $G_3 = (V_3, E_3)$ with the root r_3 is not a replica of the in-tree $G_1 = (V_1, E_1)$ with the root r_1: Suppose that there exists a function $\varphi\colon V_1 \setminus \{r_1\} \to V_3$ that satisfies conditions (R1) – (R4). Then by (R2), $\varphi(a_1) = \varphi(a_5) = r_3$. Moreover, by (R1), $\big(\varphi(a_2), \varphi(a_1)\big) \in E_3$ and $\big(\varphi(a_6), \varphi(a_5)\big) \in E_3$, since $(a_2, a_1) \in E_1$ and $(a_6, a_5) \in E_1$. So $\big(\varphi(a_2), r_3\big) \in E_3$ and $\big(\varphi(a_6), r_3\big) \in E_3$, thus $\varphi(a_2) = \varphi(a_6) = c_1$. Thus φ does not satisfy condition (R4) – a contradiction.

Theorem 3.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be nontrivial in-trees with roots r_1 and r_2, respectively. Then there exists $E \subseteq V_1 \times V_2 \cup V_2 \times V_1$ such that $(V_1 \cup V_2, E) \in Coils(G_1, G_2)$ iff G_2 is a replica of G_1.*

Proof. $(\Longrightarrow)$
For each $a \in V_1 \setminus \{r_1\}$ define function $\varphi\colon V_1 \setminus \{r_1\} \to V_2$ as follows:

- let $b \in V_1$ be the unique element such that $(a, b) \in E_1$, such b exists and is exactly one, since G_1 is a in-tree;
- by (C1) and (C2) from definition 3.2, there exists exactly one $v \in V_2$ such that $(a, v), (v, b) \in E$;

- put $\varphi(a) := v$.

We shall show that φ satisfies conditions (R1) – (R4) from Definition 3.3:

(R1) Fix any $a, b \in V_1 \setminus \{r_1\}$ and assume that $(a, b) \in E_1$. Then by definition of φ, $(\varphi(a), b)$, $(b, \varphi(b)) \in E$, hence by condition (C3) from definition 3.2, $(\varphi(a), \varphi(b)) \in E_2$.

(R2) Fix $a \in V_1 \setminus \{r_1\}$ and assume that $(a, r_1) \in E_1$. Then by definition of φ, $(a, \varphi(a)), (\varphi(a), r_1) \in E$. Suppose that $\varphi(a) \neq r_2$. Then there exists $v \in V_2$ such that $(\varphi(a), v) \in E_2$, so by (C3) there exists $b \in V_1$ such that $(\varphi(a), b)$ and (b, v) belong to E, thus $b = r_1$, since the outdegree of any vertex apart from r_1 in in-tree equals 1, hence $(r_1, v) \in E$ – contradiction with (C1).

(R3) Assume that $u \in V_2 \setminus Im(\varphi)$. Then $u \neq r_2$, since by (R2), $r_2 \in Im(\varphi)$. So there exists $v \in V_2$ such that $(u, v) \in E_2$, thus by (C3), there exists $a \in V_1$ such that $(u, a), (a, v) \in E$, moreover $a \neq r_1$, since $(a, v) \in E$ and r_1 is the root. Therefore by definition of φ, $\varphi(a) = v$, thus $(u, \varphi(a)) \in E_2$ and $a \in V_1 \setminus \{r_1\}$.

(R4) Fix $a_1, a_2, b_1, b_2 \in V_1$ and assume that (a_1, b_1), $(a_2, b_2) \in E_1$. Then by (C2) and definition of φ, $(a_1, \varphi(a_1))$, $(\varphi(a_1), b_1)$, $(a_2, \varphi(a_2))$, $(\varphi(a_2), b_2) \in E$. Suppose that $\varphi(a_1) = \varphi(a_2)$, then by (C1), $b_1 = b_2$.

($\Longleftarrow$) Assume that $\varphi \colon V_1 \setminus \{r_1\} \to V_2$ satisfies conditions (R1)–(R4). By (R3), for each $v \in V_2 \setminus Im(\varphi)$ there exists $a \in V_1 \setminus \{r_1\}$ such that $(v, \varphi(a)) \in E_2$. Let $\psi \colon V_2 \setminus Im(\varphi) \to V_1 \setminus \{r_1\}$ be a function of choice such that $(v, \varphi(\psi(v))) \in E_2$ for each $v \in V_2 \setminus Im(\varphi)$. Define arc set

$$
\begin{aligned}
E := &\{(a, \varphi(a)) \colon a \in V_1 \setminus \{r_1\}\} \cup \{(v, \psi(v)) \colon v \in V_2 \setminus Im(\varphi)\} \\
&\cup \{(\varphi(a), b) \colon a \in V_1 \setminus \{r_1\}, b \in V_1, (a, b) \in E_1\}.
\end{aligned}
$$

We shall prove that $G = (V_1 \cup V_2, E)$ is a coil G_1 with G_2:

(C1) Note that for any vertex $v \in V_1 \setminus \{r_1\} \cup V_2$ its outdegree in G equals 1: If $a \in V_1 \setminus \{r_1\}$, then the arc $(a, \varphi(a))$ is the unique arc belonging to E whose tail is a. Similarly if $v \in V_2 \setminus Im(\varphi)$, then $(v, \psi(v))$ is the unique arc belonging to E whose tail is v. Assume now that $v \in Im(\varphi)$. Then $v = \varphi(a)$ for a vertex $a \in V_1 \setminus \{r_1\}$, $(a, b) \in E_1$ for exactly one vertex $b \in V_1$ and $(v, b) \in E$. Suppose that $(v, b') \in E$ for a vertex $b' \in V_1$. Then there exists $a' \in V_1 \setminus \{r_1\}$ such that $(a', b') \in E_1$ and $\varphi(a') = \varphi(a) = v$, hence by (R4), $b = b'$.

Fix any $a \in V_1 \setminus \{r_1\}$. Then there exists unique path from a to r_1 in G_1 - $(a_0, a_1, \ldots, a_m)$, since G_1 is an in-tree. Moreover the sequence $(a_0, \varphi(a_0), a_1, \varphi(a_1), \ldots, r_2, r_1)$ is the unique path from a to r_1 in G. If $v \in Im(\varphi)$, then $v = \varphi(a)$ for a vertex $a \in V_1 \setminus \{r_1\}$ and if $(a_0, a_1, \ldots, a_m)$ is the unique path from a to r_1 in G_1, then $(v, a_1, \varphi(a_1), \ldots, r_2, r_1)$ is the unique path from v to r_1 in G. Similarly for any $v \in V_2 \setminus Im(\varphi)$, let $a = \psi(v) \in V_1 \setminus \{r_1\}$ and $(a_0, \ldots, a_m)$ be the unique path from a to r_1 in G_1, then $(v, a_0, \varphi(a_0), a_1, \ldots, r_2, r_1)$ is the unique path from v to r_1 in G.

(C2) Fix any vertices $a, b \in V_1$. Firstly assume that $(a, b) \in E_1$. Then by definition of E, $(a, \varphi(a))$, $(\varphi(a), b) \in E$ and $v = \varphi(a) \in V_2$. Now assume that (a, v), $(v, b') \in E$. Then by definition of E, $v = \varphi(a)$ and there exist $b \in V_1$ such that $(a, b) \in E_1$ and $a' \in V_1 \setminus \{r_1\}$ such that $(a', b') \in E_1$ and $\varphi(a') = v$. So by (R4), $b' = b$ and thus $(a, b') \in E_1$.

(C3) Fix any $u, v \in V_2$.
($\Longrightarrow$) Assume that $(u, v) \in E_2$.
We shall show that $v \in Im(\varphi)$: If $u \in V_2 \setminus Im(\varphi)$, then by (R3), $(u, \varphi(b)) \in E_2$ for a vertex $b \in V_1 \setminus \{r_1\}$, hence $v = \varphi(b)$ and $v \in Im(\varphi)$, since G_2 is an in-tree and the outdegree of u equals 1. So if $v \in V_2 \setminus Im(\varphi)$, then $u \in Im(\varphi)$, so $u = \varphi(a)$ for a vertex $a \in V_1 \setminus \{r_1\}$, and there exists unique $c \in V_1$ such that $(a, c) \in E_1$. If $c = r_1$, then by (R2), $u = \varphi(a) = r_2$, but $(u, v) \in E_2$ – a contradiction, since G_2 is an in-tree with the root r_2. So $c \in V_1 \setminus \{r_1\}$ and by (R1), $(\varphi(a), \varphi(c)) \in E_2$, so $\varphi(c) = v$, since $\varphi(a) = u$ and G_2 is an in-tree, thus $v \in Im(\varphi)$ – a contradiction.

Assume that $u \in V_2 \setminus Im(\varphi)$. Then $(u, \psi(u)) \in E$, $a := \psi(u) \in V_1 \setminus \{r_1\}$, $(a, \varphi(a)) \in E$ and $(u, \varphi(a)) \in E_2$. Hence $v = \varphi(a)$, since G_2 is an in-tree, thus $(u, a) \in E$ and $(a, v) \in E$.
Assume that $u, v \in Im(\varphi)$. Then $u = \varphi(a)$ and $v = \varphi(b)$ for some $a, b \in V_1 \setminus \{r_1\}$ and $(a, c) \in E_1$ for some $c \in V_1 \setminus \{r_1\}$. Then by (R1), $(u, \varphi(c)) \in E_2$, so $\varphi(c) = v$, since G_2 is an in-tree. Moreover, by definition of E, $(\varphi(a), c)$, $(c, \varphi(c)) \in E$, so (u, c) and $(c, v) \in E$.

($\Longleftarrow$) Assume that (u, b), $(b, v) \in E$ for some $b \in V_1$. Then by definition of E, $v = \varphi(b)$. If $u \in V_2 \setminus Im(\varphi)$, then $(u, \psi(u)) \in E$, so $\psi(u) = b$, since G is in-tree. So by definition of ψ, $(u, \varphi(\psi(u))) = (u, v) \in E_2$. If $u \in Im(\varphi)$, then $u = \varphi(a)$ for some $a \in V_1 \setminus \{r_1\}$ and $(a, c) \in E_1$ for some $c \in V_1$. We shall show that $c \neq r_1$: Suppose that $c = r_1$. Then by (R2), $\varphi(a) = r_2$, and by definition of E, $(\varphi(a), r_1) \in E$. Hence $(u, r_1) \in E$, thus $b = r_1$, since $(u, b) \in E$ and G is an in-tree. Therefore $(r_1, v) \in E$, since $(b, v) \in E$ – a contradiction with definition of E. So by (R1), $(\varphi(a), \varphi(c)) \in E_2$, hence by definition of E, $(\varphi(a), c)$, $(c, \varphi(c)) \in E$, thus $c = b$ and $v = \varphi(c)$, since G is in-tree and therefore $(u, v) \in E_2$.

□

3.3 A Characterization of Replicas by Sources and Other Properties of Replicas

Definition 3.4. Let $G = (V, E)$ be an in-tree with the root r and S be its basis (set of sources). Then:

1. An element $t \in S$ is called *trivial* iff its level equals 1, i.e., $lev(t) = 1$.

2. A source $d \in S$ is called *dominated* iff

   (a) $lev(d) \geq 2$ and

(b) $\forall_{w\,\in\,V}(d,w)\ \in\ E \Longrightarrow \exists_{u,v\,\in\,V}(u,v),(v,w)\ \in\ E$, i.e., $I(I(O(d)))\ \neq\ \emptyset$

3. A subset $B \subseteq S$ is called *a balanced set* iff

   (a) $lev(b) \geq 2$ and b is not dominated for all $b\ \in\ B$ and

   (b) $\exists_{v\,\in\,V}\forall_{b\,\in\,B}(b,v)\ \in\ E$, i.e., $\forall_{b\,\in\,B}B \subseteq I(O(b))$

4. A source $b\ \in\ S$ is called *balanced* iff $b\ \in\ B$ for a balanced set B.

5. A subset $B \subseteq S$ is called *a maximal balanced set* iff B is a balanced set and for all $B' \subseteq S$ if $B \subsetneq B'$, then B' is not a balanced set.

6. A source $m\ \in\ S$ is called *maximal* iff

   (a) $lev(m) \geq 2$ and m is not dominated as well as

   (b) $\forall_{v\,\in\,V}(m,v)\ \in\ E \Longrightarrow \neg\exists_{u\,\in\,V\,\setminus\,\{m\}}(u,v)\ \in\ E$, i.e., $I(O(m)) = \{m\}$.

7. *The first common descendant* of vertices $u,v\ \in\ V$, denoted by $FCD(u,v)$, is the vertex $k\ \in\ V$ such that:

   (a) k is a descendant of u and v and

   (b) if l is a descendant of u and v, then either $l = k$ or l is a descendant of k, for any vertex $l\ \in\ V$

8. *The important indegree of a vertex v*, denoted by $imp\_in\_deg(v)$, is the number of arcs directed to v such that their tail is not a source in G;

9. The important in-set of v, denoted by $Imp\_I_G(v)$, is the set of all vertices that dominate v and they are not sources.

Remark 3.1. Each source in an in-tree is trivial, dominated, balanced or maximal.

Proposition 3.2 (Some properties of replicas)**.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be nontrivial in-trees with roots r_1 and r_2, respectively. Assume that G_2 is a replica of G_1 and $\varphi\colon V_1 \setminus \{r_1\} \to V_2$ satisfies conditions (R1)–(R4). Let $G_2' = (Im(\varphi), E_2')$ be the induced subgraph of G_2 on the set $Im(\varphi)$, i.e. $E_2' = \{(u,v)\ \in\ E_2 \colon u,v\ \in\ Im(\varphi)\}$. Then:*

1. $\forall_{a\,\in\,V_1\,\setminus\,\{r_1\}}\big((a,r_1)\ \in\ E_1 \Longleftrightarrow \varphi(a) = r_2\big)$

2. $\forall_{u,v\,\in\,V_2}\big[\big((u,v)\ \in\ E_2 \wedge u\ \in\ Im(\varphi)\big) \Longrightarrow v\ \in\ Im(\varphi)\big]$

3. *each $v\ \in\ V_2\ \setminus\ Im(\varphi)$ is a source in G_2*

4. $\forall_{a\,\in\,V_1\,\setminus\,\{r_1\}} lev_{G_1}(a) = lev_{G_2}(\varphi(a)) + 1$

5. $\forall_{u\,\in\,Im(\varphi)} in\_deg_{G_2'}(u) \leq \sum_{a\,\in\,\varphi^{-1}(u)} in\_deg_{G_1}(a)$

6. $\forall_{u\,\in\,Im(\varphi)} imp\_in\_deg_{G_2'}(u) \leq \sum_{a\,\in\,\varphi^{-1}(u)} imp\_in\_deg_{G_1}(a)$

7. $\forall_{a \in V_1 \setminus \{r_1\}} imp\_in\_deg_{G_1}(a) \le \sum_{u \in Imp\_I_{G_2'}(\varphi(a))} in\_deg_{G_2'}(u)$

8. $\forall_{a,b \in V_1 \setminus \{r_1\}} 0 \le lev_{G_1}(FCD(a,b)) - lev_{G_2}(FCD(\varphi(a),\varphi(b))) \le 1$

9. *If* $a \in V_1 \setminus \{r_1\}$ *is a maximal or balanced source in* G_1, $u \in V_2$ *and* $(u,\varphi(a)) \in E_2$, *then* $u \in V_2 \setminus Im(\varphi)$ *and* u *is a maximal or balanced source in* G_2.

Example 3.2. As an illustration of Definition 3.4 and Proposition 3.2, let us again consider in-trees G_1 and G_2 with roots r_1 and r_2, respectively, from Figure 1. Note that G_1 has the set of sources $S_1 = \{a_3, a_4, a_6\}$. Among these sources, a_3 is dominated, however a_4 and a_6 are maximal. The in-tree G_2 has the set of sources $S_2 = \{b_2, b_3, b_5, b_6\}$. Note that $\{b_2, b_3\}$ is a maximal balanced set, b_6 is a trivial source, and b_5 is a maximal source in G_2. Let us remember that the function φ supporting that G_2 is a replica of G_1 has the following form

$$\varphi = \{(a_1, r_2), (a_2, b_1), (a_3, b_1), (a_4, b_3), (a_5, r_2), (a_6, b_4)\}.$$

Let $G_2' = (Im(\varphi), E_2')$ be the induced subgraph of G_2 on the set $Im(\varphi)$. We shall show that some statements of Proposition 3.2 hold :

- Note that $Im(\varphi) = \{r_2, b_1, b_3, b_4\} \supseteq \{r_2, b_1\} = O(b_1) \cup O(b_3) \cup O(b_4)$.
- Note that $V_2 \setminus Im(\varphi) = \{b_2, b_5, b_6\} \subseteq S_2$.
- For each $i \in \{1, 5\}$ and $j \in \{2, 3, 6\}$:
  - $lev_{G_1}(a_i) = 1$, $lev_{G_1}(a_j) = 2$, and $lev_{G_1}(a_4) = 3$;
  - $\varphi(a_i) = r_2$, $lev_{G_2}(r_2) = 0$, $\varphi(a_j) \in \{b_1, b_4\}$, $lev_{G_2}(b_1) = lev_{G_2}(b_4) = 1$, $\varphi(a_4) = b_3$, and $lev_{G_2}(b_3) = 2$

  So $lev_{G_1}(a_i) = lev_{G_2}(\varphi(a_i)) + 1$ for all $i \in \{1, \ldots, 6\}$.
- Note that $in\_deg_{G_2'}(r_2) = 2$, $\varphi^{-1}(r_2) = \{a_1, a_5\}$, $in\_deg_{G_1}(a_1) = 2$, and $in\_deg_{G_1}(a_5) = 1$, so assertion 5 from Proposition 3.2 holds, for example, for $u = r_2$.
- Similarly, statement 6 is true for $u = r_2$, since $imp\_in\_deg_{G_2'}(r_2) = 1 \le 1 = imp\_in\_deg_{G_1}(a_1) + imp\_in\_deg_{G_1}(a_5)$, and $\varphi^{-1}(r_2) = \{a_1, a_5\}$
- Note that $imp\_in\_deg_{G_1}(a_1) = 1 \le 1 = in\_deg_{G_2'}(b_1)$ and $Imp\_I_{G_2'}(\varphi(a_1)) = Imp\_I_{G_2'}(r_2) = \{b_1\}$.
- Note that $lev_{G_1}(FCD(a_3, a_4)) = lev_{G_1}(a_1) = 1$ and $lev_{G_2}(FCD(\varphi(a_3), \varphi(a_4))) = lev_{G_2}(FCD(b_1, b_3)) = lev_{G_2}(b_1) = 1$
- Note that a_6 is maximal source in G_1, $(b_5, b_4) = (b_5, \varphi(a_6)) \in E_2$, as well as $b_5 \in V_2 \setminus Im(\varphi)$ and b_5 is a maximal source in G_2.

Proof. 1. Fix any $a \in V_1 \setminus \{r_1\}$ and assume that $\varphi(a) = r_2$. Then there exists the unique $b \in V_1$ such that $(a, b) \in E_1$. Suppose that $b \neq r_1$, then by (R1), $(\varphi(a), \varphi(b)) \in E_2$ – a contradiction, since $\varphi(a) = r_2$ is the root of G_2. The converse implication is true by definition.

2. Assume that $(u,v) \in E_2$ and $u \in Im(\varphi)$. Then $u = \varphi(a)$ for some $a \in V_1 \setminus \{r_1\}$ and there exists the unique $b \in V_1$ such that $(a,b) \in E_1$. If $b = r_1$, then by (R2), $u = \varphi(a) = r_2$ – a contradiction. So $b \neq r_1$ and by (R1), $(\varphi(a), \varphi(b)) \in E_2$, hence $v = \varphi(b)$, since G_2 is an in-tree and $u = \varphi(a)$.

3. Assume that $v \in V_2$ is not a source in G_2. Then there exists $u \in V_2$ such that $(u,v) \in E_2$. If $u \in Im(\varphi)$, then by (2), $v \in Im(\varphi)$. If $u \in V_2 \setminus Im(\varphi)$, then by (R3), there exists $w \in Im(\varphi)$ such that $(u,w) \in E_2$, but G_2 is an in-tree, thus $w = v$ and $v \in Im(\varphi)$.

4. Fix any $a \in V_1 \setminus \{r_1\}$. Let $(a_0, \ldots, a_m)$ be the unique path from a to r_1 in G_1. Then by (R1) and (R2) $(\varphi(a_0), \ldots, \varphi(a_{m-2}), r_2)$ is the path from $\varphi(a_0)$ to r_2 in G_2 and this path is unique, since G_2 is an in-tree.

5. Fix any $u \in Im(\varphi)$. We shall show that

$$I_{G_2'}(u) = \varphi\left(\bigcup_{b \in \varphi_{-1}(u)} I_{G_1}(b)\right)$$

If $v \in \varphi(I_{G_1}(b))$ for some $b \in \varphi_{-1}(u)$, then $v = \varphi(c)$ for some $c \in V_1 \setminus \{r_1\}$ such that $(c,b) \in E_1$, so by (R1), $(\varphi(c), \varphi(b)) \in E_2'$, hence $(v,u) \in E_2'$, thus $v \in I_{G_2'}(u)$. On the other hand, if $v \in I_{G_2'}(u)$, then $(v,u) \in E_2'$, moreover $v = \varphi(c)$ for some $c \in V_1 \setminus \{r_1\}$, so $(c,b) \in E_1$ for some $b \in V_1$. Note that $b \neq r_1$, otherwise by (R2), $v = \varphi(c) = r_2$ - a contradiction, since $(v,u) \in E_2'$. Therefore by (R1), $(\varphi(c), \varphi(b)) \in E_2'$, thus $(v, \varphi(b)) \in E_2'$, so $\varphi(b) = u$, since G_2 is an in-tree, hence $v = \varphi(c)$, $c \in I_{G_1}(b)$ and $b \in \varphi_{-1}(u)$.
Therefore we have

$$\begin{aligned} in\_deg_{G_2'}(u) = \#I_{G_2'}(u) &\leq \# \bigcup_{b \in \varphi_{-1}(u)} I_{G_1}(b) \\ &\leq \sum_{b \in \varphi_{-1}(u)} \#I_{G_1}(b) = \sum_{b \in \varphi_{-1}(u)} in\_deg_{G_1}(b) \end{aligned}$$

6. Fix any $u \in Im(\varphi)$. We shall prove that

$$Imp\_I_{G_2'}(u) \subseteq \varphi\left(\bigcup_{b \in \varphi^{-1}(u)} Imp\_I_{G_1}(b)\right).$$

Assume that $v \in Imp\_I_{G_2'}(u)$. Then $(v,u) \in E_2'$ and there exists $w \in Im(\varphi)$ such that $(w,v) \in E_2'$, thus there exist $d, c \in V_1 \setminus \{r_1\}$ and $b \in V_1$ such that $w = \varphi(d)$, $(d,c) \in E_1$ and $(c,b) \in E_1$. Note that $b \neq r_1$, otherwise by (R2), $\varphi(c) = r_2$, but by (R1), $(w, \varphi(c)) \in E_2'$, hence $\varphi(c) = v$ and $(v,u) \in E_2'$. Therefore by (R1) and fact that G_2' is an in-tree, $\varphi(b) = u$, $v = \varphi(c)$ and $c \in Imp\_I_{G_1}(b)$. So we obtain:

$$\begin{aligned}
&imp_in_deg_{G_2'}(u) = \#Imp_I_{G_2'}(u)\\
&\le \#\varphi\left(\bigcup_{b\in\varphi^{-1}(u)} Imp_I_{G_1}(b)\right) \le \# \bigcup_{b\in\varphi^{-1}(u)} Imp_I_{G_1}(b)\\
&\le \sum_{b\in\varphi^{-1}(u)} \#Imp_I_{G_1}(b) = \sum_{b\in\varphi^{-1}(u)} imp_in_deg_{G_1}(b)
\end{aligned}$$

7. Fix any $a \in V_1 \setminus \{r_1\}$ and assume that $Imp\_I(a) = \{b_1, \ldots, b_m\}$. Then there exist $c_1, \ldots, c_m \in V_1 \setminus \{r_1\}$ such that $(c_i, b_i) \in E_1$ for all $i \in \{1, \ldots, m\}$, so by (R1), $(\varphi(c_i), \varphi(b_i)) \in E_2'$ and $(\varphi(b_i), \varphi(a)) \in E_2'$ for all $i \in \{1, \ldots, m\}$. Moreover by (R4), $\#\{\varphi(c_1), \ldots, \varphi(c_m)\} = m$. Therefore we have:

$$\begin{aligned}
&imp_in_deg_{G_1}(a) = \#Imp_I(a) = m = \#\{\varphi(c_1), \ldots, \varphi(c_m)\}\\
&\le \# \bigcup_{u \,\in\, Imp_I(\varphi(a))} I_{G_2'}(u) \le \sum_{u \,\in\, Imp_I(\varphi(a))} in_deg_{G_2'}(u)
\end{aligned}$$

8. Fix any $a, b \in V_1 \setminus \{r_1\}$ and let $(a_m, a_{m-1}, \ldots, a_1, a_0)$ be the unique path in G_1 from a to r_1 and $(b_n, \ldots, b_1, b_0)$ be the unique path in G_1 from b to r_1. Then by (R1) and (R2), $(\varphi(a_m), \ldots, \varphi(a_1))$ is the unique path in G_2 from $\varphi(a)$ to r_2 and $(\varphi(b_n), \ldots, \varphi(b_1))$ is the unique path in G_2 from $\varphi(b)$ to r_2. Suppose that $a_k = b_k = FCD(a, b)$. Then $a_i \ne b_j$ for all $i \in \{k+1, \ldots, m\}$ and $j \in \{k+1, \ldots, n\}$. So by (R4), $\varphi(a_i) \ne \varphi(b_j)$ for all $i \in \{k+2, \ldots, m\}$ and $j \in \{k+2, \ldots, n\}$, thus $FCD(\varphi(a), \varphi(b)) = \varphi(a_k)$ and $lev_{G_2}(FCD(\varphi(a), \varphi(b))) = k-1$ or $FCD(\varphi(a), \varphi(b)) = \varphi(a_{k+1})$ and $lev_{G_2}(FCD(\varphi(a), \varphi(b))) = k$.

9. Assume that $a \in V_1 \setminus \{r_1\}$ is a maximal or balanced source and $(u, \varphi(a)) \in E_2$. Then there exists $b' \in V_1$ such that $(a, b') \in E_1$. Moreover $lev_{G_1}(a) \ge 2$ and by (4), $lev_{G_2}(\varphi(a)) \ge 1$ and thus $lev_{G_2}(u) \ge 2$. Suppose that $u \in Im(\varphi)$, then $u = \varphi(e)$ for some $e \in V_1 \setminus \{r_1\}$. Thus $(e, f) \in E_1$ for some $f \in V_1$ and $f \ne r_1$, otherwise by (R2), $u = \varphi(e) = r_2$ and $(u, \varphi(a)) \in E_2$ – a contradiction. Therefore $(f, g) \in E_1$ for some $g \in V_1$ and by (R1), $(u, \varphi(f)) \in E_2$, so $\varphi(f) = \varphi(a)$, since G_2 is an in-tree. Hence by (R4), $g = b'$ and a is dominated – a contradiction. Therefore $u \in V_2 \setminus Im(\varphi)$ and by (3), u is a source in G_2.

   Suppose that u is dominated. Then there exist $s, t \in V_2$ such that $(s, t), (t, \varphi(a)) \in E_2$. Then by (3), $t \in Im(\varphi)$, since t is not a source in G_2. Thus $t = \varphi(d)$ for some $d \in V_1 \setminus \{r_1\}$ and $(d, c) \in E_1$ for some $c \in V_1$. Note that $c \ne r_1$, otherwise by (R2), $t = \varphi(d) = r_2$ and $(t, \varphi(a)) \in E_2$ – a contradiction. So there exists $b \in V_1$ such that $(c, b) \in E_1$ and by (R1), $(t, \varphi(c)) \in E_2$, thus $\varphi(c) = \varphi(a)$, since G_2 is an in-tree. Therefore by (R4), $b = b'$ and a is dominated – a contradiction.

□

The following theorem enables us to determine whether an in-tree is a replica of another in-tree through finding an injective mapping between some subsets of sources of these in-trees.

Theorem 3.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be nontrivial in-trees with roots r_1 and r_2, respectively. Let S^i be the basis of G_i, M^i be the set of all maximal sources in S^i and $\mathbb{B}^i$ be the family of all maximal balanced sets included in S^i for $i = 1, 2$. Then G_2 is a replica of G_1 if and only if there exist functions $s^{(i)} : \mathbb{B}^i \to S^i$ such that $s^{(i)}(B) \in B$ for all $B \in \mathbb{B}^i$ and $i = 1, 2$, there exist sets $S_{dux} \subseteq S^1$ and $S_{comes} \subseteq S^2$ such that $\#S_{dux} = \#S_{comes}$, $M^1 \cup s^{(1)}(\mathbb{B}^1) \subseteq S_{dux}$ and $M^2 \cup s^{(2)}(\mathbb{B}^2) \subseteq S_{comes}$, there exists an injection $\psi : S_{dux} \to S_{comes}$ such that:*

$$\forall_{a \in S_{dux}} 0 \leq lev_{G_1}(a) - lev_{G_2}(\psi(a)) \leq 1 \tag{1}$$
$$\forall_{a,b \in S_{dux}} 0 \leq lev_{G_1}(FCD(a, b)) - lev_{G_2}\big(FCD\big(\psi(a), \psi(b)\big)\big) \leq 1. \tag{2}$$

Terms *dux* and *comes* come from Theory of Music and make a reference to the necessary parts of a fugue: Subject and Answer.

Proof. ($\Longrightarrow$)

Assume that $\varphi : V_1 \setminus \{r_1\} \to V_2$ satisfies conditions (R1)–(R4).

For each $B \in \mathbb{B}^1$ define $s^{(1)} : \mathbb{B}^1 \to S^1$ in the following way:

1. if $\varphi(a) \in S^2$ for all $a \in B$, then choose $a \in B$ and put $s^{(1)}(B) := a$;
2. if $\varphi(a) \notin S^2$ for some $a \in B$, then choose such $a \in B$ that $\varphi(a) \notin S^2$ and put $s^{(1)}(B) := a$.

Put $S_{dux} := M^1 \cup s^{(1)}(\mathbb{B}_1)$, $S_{comes} := \emptyset$, $s^{(2)} := \emptyset$ and $\psi := \emptyset$. For each $a \in S_{dux}$ modify ψ, $s^{(2)}$ and S_{comes} as follows:

1. if $\varphi(a) \in S^2$, then put $\psi := \psi \cup \{(a, \varphi(a))\}$, $S_{comes} := S_{comes} \cup \{\varphi(a)\}$; if additionally $\varphi(a) \in B$ for some $B \in \mathbb{B}^2$, then put $s^{(2)} := s^{(2)} \cup \{(B, \varphi(a))\}$
2. if $\varphi(a) \notin S^2$, then choose $u \in V_2$ such that $(u, \varphi(a)) \in E_2$ and put $\psi := \psi \cup \{(a, u)\}$, $S_{comes} := S_{comes} \cup \{u\}$. Note that by (9) from Proposition 3.2, such $u \in V_2 \setminus Im(\varphi)$ and it is maximal or balanced source, so if $u \in B \in \mathbb{B}^2$, then put $s^{(2)} := s^{(2)} \cup \{(B, u)\}$.

We shall show that ψ is an injection:

- Assume that $a, b \in S_{dux}$ and $\varphi(a), \varphi(b) \in S^2$. Then $\psi(a) = \varphi(a)$, $\psi(b) = \varphi(b)$, moreover $(a, a_1) \in E_1$ and $(b, b_1) \in E_1$ for some $a_1, b_1 \in V_1 \setminus \{r_1\}$, since $lev_{G_1}(a) \geq 2$ and $lev_{G_1}(b) \geq 2$. Suppose that $\varphi(a) = \varphi(b)$, then by (R4), $a_1 = b_1$, so there exists $B \in \mathbb{B}^1$ such that $a, b \in B$, so $a = b = s^{(1)}(B)$, since $S_{dux} \cap B = s^{(1)}(B)$ for each $B \in \mathbb{B}^1$.
- Assume that $a, b \in S_{dux}$, $\varphi(a) \in S^2$ and $\varphi(b) \notin S^2$. Then by definition of ψ, $\psi(a) = \varphi(a) \neq \psi(b) \in V_2 \setminus Im(\varphi)$.
- Assume that $a, b \in S_{dux}$ and $\varphi(a), \varphi(b) \notin S^2$. Then $(\psi(a), \varphi(a)) \in E_2$ and $(\psi(b), \varphi(b)) \in E_2$. Suppose that $\psi(a) = \psi(b)$. Then $\varphi(a) = \varphi(b)$, since G_2 is an in-tree, and $(a, a_1), (b, b_1) \in E_1$ for some $a_1, b_1 \in V_1$, so by (R4), $a_1 = b_1$, thus $a = b = s^{(1)}(B)$ for some $B \in \mathbb{B}^1$.

Now we have guaranteed that $M^1 \cup s^{(1)}(\mathbb{B}^1) \subseteq S_{dux}$ but not necessary $M^2 \cup s^{(2)}(\mathbb{B}^2) \subseteq S_{comes}$. We shall show that if $u \in M^2 \setminus S_{comes}$ or there exists $B \in \mathbb{B}^2$ such that $u \in B$ and $B \cap S_{comes} = \emptyset$, then $u \in V_2 \setminus Im(\varphi)$ and there exists $a \in S^1$ such that a is dominated or balanced and $(u, \varphi(a)) \in E_2$:
Assume that $u \in M^2 \setminus S_{comes}$ or there exists $B \in \mathbb{B}^2$ such that $u \in B$ and $B \cap S_{comes} = \emptyset$.

- Suppose that $u = \varphi(a)$ for some source $a \in V_1 \setminus \{r_1\}$ (a is a source, since $\varphi(a)$ is a source). Then $a \notin S_{dux}$, since $u \notin S_{comes}$. Moreover a can not be trivial, since otherwise by (R2), $u = \varphi(a) = r_2$. Of course a is not maximal, since $M^1 \subseteq S_{dux}$. Suppose that $a \in B$ for some $B \in \mathbb{B}^1$. Then (a, b), $(s^{(1)}(B), b)$ for some $b \in V_1 \setminus \{r_1\}$. Thus by (R1), if $\psi(s^{(1)}(B))$ is trivial, then $u = \varphi(a)$ is trivial – a contradiction, if $\psi(s^{(1)}(B))$ is maximal, then u is dominated – a contradiction, if $\psi(s^{(1)}(B))$ is balanced, then u is dominated or balanced and $u \in B$ such that $B \cap S_{comes} \neq \emptyset$ – a contradiction, if $\psi(s^{(1)}(B))$ is dominated, then u is also dominated – contradiction. Therefore a is dominated. So there exist $b, c, d \in V_1 \setminus \{r_1\}$ such that (a, d),(b, c),$(c, d) \in E_1$. Thus by (R1), $(u, \varphi(d))$,$(\varphi(b), \varphi(c))$,$(\varphi(c), \varphi(d)) \in E_2$, so u is dominated in G_2 - a contradiction. Therefore $u \in V_2 \setminus Im(\varphi)$ and by (3) from Proposition 3.2, u is a source in G_2.

- By (R3), there exists $a \in V_1 \setminus \{r_1\}$ such that $(u, \varphi(a)) \in E_2$. Suppose that a is not a source. Then there exists $b \in V_1 \setminus \{r_1\}$ such that $(b, a) \in E_1$, so by (R1), $(\varphi(b), \varphi(a)) \in E_2$, thus u is not maximal, hence u must be balanced. Note that b is a nontrivial source, since if b were not a source, then u would be dominated, moreover b is not trivial, since $a \in V_1 \setminus \{r_1\}$. Moreover b is not dominated, since otherwise there would exist $c, d \in V_1 \setminus \{r_1\}$ such that (d, c), $(c, a) \in E_1$, and by (R1), u would be dominated. So b is maximal or balanced. If b is maximal, put $b' := b$, otherwise put $b' := s^{(1)}(B)$, where $b \in B \in \mathbb{B}^1$. Note also that $\varphi(b')$ is a source in G_2, since otherwise u would be dominated, so by definition of S_{comes}, $\varphi(b') \in S_{comes}$, thus there exists $B \in \mathbb{B}^2$ such that $u \in B$ and $\varphi(b') \in B \cap S_{comes}$ - a contradiction.

- Therefore $(u, \varphi(a)) \in E_2$ and $a \in V_1 \setminus \{r_1\}$ is a source in G_1. Of course a can not be trivial, since otherwise $\varphi(a) = r_2$, but $lev_{G_2}(u) \geq 2$ as u is maximal or balanced. Suppose that a is maximal, then by definition of S_{comes} there exists $v \in S_{comes}$ such that $(v, \varphi(a)) \in E_2$, moreover v is balanced, hence there exists $B \in \mathbb{B}^2$ such that $u, v \in B$, thus $u \in B$ and $v \in B \cap S_{comes}$ - a contradiction. Therefore a is dominated or balanced.

Let $M_0 := M^2 \setminus S_{comes}$ and $B_0 := \emptyset$. For each $u \in M^2 \setminus S_{comes}$ choose $a \in S^1$ such that $(u, \varphi(a)) \in E_2$ and modify S_{dux}, S_{comes} and ψ in the following way: put $S_{dux} := S_{dux} \cup \{a\}$, $S_{comes} := S_{comes} \cup \{u\}$ and $\psi := \psi \cup \{(a, u)\}$. For each $B \in \mathbb{B}^2$ such that $B \cap S_{comes} = \emptyset$ choose $u \in B$ and $a \in S^1$ such that $(u, \varphi(a)) \in E_2$ and modify B_0, S_{dux}, S_{comes}, ψ and $s^{(2)}$ as follows: put $B_0 := B_0 \cup \{u\}$, $S_{dux} := S_{dux} \cup \{a\}$, $S_{comes} := S_{comes} \cup \{u\}$, $\psi := \psi \cup \{(a, u)\}$ and $s^{(2)} := s^{(2)} \cup \{(B, u)\}$.

We shall show that ψ remains still an injection. After above modifications it suffices to show that ψ is a function. Suppose that for some $a \in S^1$ there exist $u, v \in M_0 \cup B_0$ such that (a, u),$(a, v) \in \psi$. Then $(u, \varphi(a))$,$(v, \varphi(a)) \in E_2$, so $u, v \in B_0$. Moreover, $u, v \in B$ for some $B \in \mathbb{B}^2$, thus $u = v$, since $B \cap S_{comes} = s^{(2)}(B)$ for each $B \in \mathbb{B}^2$.

By definition of ψ, $\psi(a) = \varphi(a)$ or $(\psi(a), \varphi(a)) \in E_2$ for each $a \in S_{dux}$, so $0 \leq lev_{G_2}(\psi(a)) - lev_{G_2}(\varphi(a)) \leq 1$, but by (4) from Proposition 3.2,
$lev_{G_2}(\varphi(a)) = lev_{G_1}(a) - 1$, thus $0 \leq lev_{G_1}(a) - lev_{G_2}(\psi(a)) \leq 1$ for each $a \in S_{dux}$. For the same reason, $FCD(\psi(a), \psi(b)) = FCD(\varphi(a), \varphi(b))$, so by (8) from Proposition 3.2, $0 \leq lev_{G_1}(FCD(a, b)) - lev_{G_2}(FCD(\psi(a), \psi(b))) \leq 1$ for all $a, b \in S_{dux}$.

$(\Longleftarrow)$

Assume that there exist functions $s^{(i)} \colon \mathbb{B}^i \to S^i$ such that $s^{(i)}(B) \in B$ for all $B \in \mathbb{B}^i$ and $i = 1, 2$, there exist sets $S_{dux} \subseteq S^1$ and $S_{comes} \subseteq S^2$ such that $\#S_{dux} = \#S_{comes}$, $M^1 \cup s^{(1)}(\mathbb{B}^1) \subseteq S_{dux}$ and $M^2 \cup s^{(2)}(\mathbb{B}^2) \subseteq S_{comes}$, there exists an injection $\psi \colon S_{dux} \to S_{comes}$ satisfying conditions (1) and (2).

We shall show that there exists a function φ satisfying conditions (R1)-(R4). Define $\varphi \colon V_1 \setminus \{r_1\} \to V_2$ in the following way:

- For each $a \in S_{dux}$ put:

$$\varphi(a) := \begin{cases} \psi(a), & \text{if } lev_{G_2}(\psi(a)) = lev_{G_1}(a) - 1 \\ v, & \text{if } lev_{G_2}(\psi(a)) = lev_{G_1}(a) \text{ and } \psi(a) \text{ dominates } v \text{ in } G_2 \end{cases}$$

  Let $(a_m, a_{m-1}, \ldots, a_0)$ be the unique path in G_1 from a to r_1 and $(u_{m-1}, \ldots, u_0)$ be the unique path in G_2 from $\varphi(a)$ to r_2. For each $i \in \{1, \ldots, m-1\}$ put $\varphi(a_i) := u_{i-1}$.

- For each trivial $a \in S^1 \setminus S_{dux}$ put $\varphi(a) := r_2$.

- For each dominated $a \in S^1 \setminus S_{dux}$ there exists $c \in M^1 \cup s^{(1)}(\mathbb{B}^1)$ such that a dominates a vertex belonging to the unique path $(c_m, c_{m-1}, \ldots, c_0)$ from c to r_1 in G_1, therefore $(a, c_i) \in E_1$ for some $i \in \{1, \ldots, m-2\}$. For each dominated $a \in S^1 \setminus S_{dux}$ choose such c and put $\varphi(a) := \varphi(c_{i+1})$, which is determined in the first step.

- For each $B \in \mathbb{B}^1$ and $a \in B \setminus S_{dux}$ put $\varphi(a) := \varphi(s^{(1)}(B))$.

Firstly, we shall prove that the relation φ is defined correctly, i.e., φ is actually a function. Assume that $a, b \in S_{dux}$, $c \in V_1 \setminus S^1$. Let $(a_m, \ldots, a_0)$ be the unique path from a to r_1 in G_1, $(b_n, \ldots, b_0)$ be the unique path from b to r_1 in G_1, $(u_{m-1}, \ldots, u_0)$ be the unique path from $\varphi(a)$ to r_2 in G_2, and $(v_{m-1}, \ldots, v_0)$ be the unique path from $\varphi(b)$ to r_2 in G_2. Assume that $k = lev_{G_1}(c) \leq FCD(a, b) = l < \min\{m, n\}$. Then $a_i = b_i$ for all $i \in \{0, \ldots, l\}$. By (2), $l - 1 \leq lev_{G_2}\big(FCD(\psi(a), \psi(b))\big) \leq l$. So $u_i = v_i$ for all $i \in \{0, \ldots, l-1\}$. Hence in particular, $\varphi(a_k) = u_{k-1} = v_{k-1} = \varphi(b_k)$. Thus the value $\varphi(c)$ does not depend on the choice of $a \in S_{dux}$ such that c belongs to the unique path from a to r_1.

We shall prove that φ satisfies conditions (R1)–(R4):

(R1) Fix any $a, b \in V_1 \setminus \{r_1\}$ and assume that $(a, b) \in E_1$. Then there exists $c \in S^1$ such that $a = c_j$ for some $j \in \{2, \ldots, m\}$, where $(c_m, \ldots, c_0)$ is the unique path from c to r_1 in G_1. Let $(u_{m-1}, \ldots, u_0)$ be the unique path from $\varphi(c)$ to r_2 in G_2. Then by definition of φ, $\varphi(c_i) = u_{i-1}$ for all $i \in \{1 \ldots, m\}$ and $(\varphi(a), \varphi(b)) = (\varphi(c_j), \varphi(c_{j-1})) = (u_{j-1}, u_{j-2}) \in E_2$.

(R2) By definition, φ satisfies (R2).

(R3) Fix any $v \in V_2 \setminus Im(\varphi)$. Suppose that v is not a source in G_2. Then there exists $u \in S_{comes}$ such that $v = u_i$ for some $i \in \{1, \dots, m-1\}$ where $(u_m, \dots, u_0)$ is the unique path from u to r_2 in G_2. Let $a := \psi^{-1}(u)$ and $(a_n, \dots, a_0)$ be the unique path from a to r_1 in G_1. Then $a \in S_{dux}$, and by (1), $0 \leq n - m \leq 1$. By definition of φ, $\varphi(a_i) = u_{i-1}$ for $i \in \{1, \dots, n\}$. Thus $v \in Im(\varphi)$ – a contradiction. Therefore v is a source in G_2.

Firstly assume that $v \in S_{comes}$. Then $a = \psi^{-1}(v) \in S_{dux}$ and by definition of φ, $(v, \varphi(a)) \in E_2$.

Now assume that $v \in S^2 \setminus S_{comes}$. If v is trivial, then $(v, \varphi(a)) \in E_2$ for each $a \in V_1$ whose level equals 1. If v is a dominated source, then there exists $u \in S_{comes}$ such that $(v, u_i) \in E_2$ for some $i \in \{1, \dots, m-2\}$, where $(u_m, \dots, u_0)$ is the unique path from u to r_2 in G_2. Let $a := \psi^{-1}(u)$ and $(a_n, \dots, a_0)$ be the unique path from a to r_1 in G_1. Then $a \in S_{dux}$, and by (1), $0 \leq n - m \leq 1$. By definition of φ, $\varphi(a_j) = u_{j-1}$ for $j \in \{1, \dots, n\}$. Thus $(v, \varphi(a_{i+1}))$. The last possibility is that v is balanced. Then $v \in B$ for some $B \in \mathbb{B}^2$. Let $u := s^{(2)}(B)$ and $(u_m, \dots, u_0)$ be the unique path from u to r_2 in G_2. Then $u \in S_{comes}$, $a := \psi^{-1}(u) \in S_{dux}$ and $(v, u_{m-1}) \in E_2$. Let $(a_n, \dots, a_0)$ be the unique path from a to r_1 in G_1, then by definition of φ, $0 \leq n - m \leq 1$ and $\varphi(a_j) = u_{j-1}$ for $j \in \{1, \dots, n\}$, thus $(v, \varphi(a_m)) \in E_2$.

(R4) Assume that $(a_1, b_1) \in E_1$, $(a_2, b_2) \in E_1$ and $b_1 \neq b_2$. Without loss of generality we can assume that $k = lev_{G_1}(b_1) \leq lev_{G_1}(b_2) = l$. Note that $FCD_{G_1}(b_1, b_2) = k$ if and only if there exists $c \in S_{dux}$ such that both b_1 and b_2 belong to the unique path from c to r_1 in G_1. Therefore we shall consider two cases:

(a) Assume that $FCD_{G_1}(b_1, b_2) = k$. Let $c \in S_{dux}$ be such that both b_1 and b_2 belong to the unique path $(c_m, \dots, c_0)$ from c to r_1 in G_1. Then $b_1 = c_k$, $b_2 = c_l$ and $0 \leq k < l < m$, since $b_1 \neq b_2$. Let $(u_{m-1}, \dots, u_0)$ be the unique path from $\varphi(c)$ to r_2 in G_2. Then by definition of φ, $\varphi(a_1) = \varphi(c_{k+1}) = u_k \neq u_l = \varphi(c_{l+1}) = \varphi(a_2)$.

(b) Assume that $FCD_{G_1}(b_1, b_2) = p < k$. Let $c, d \in S_{dux}$ be such that b_1 belongs to the unique path $(c_m, \dots, c_0)$ from c to r_1 and b_2 belongs to the unique path $(d_n, \dots, d_0)$ from d to r_1 in G_1. Then $b_1 = c_k$ and $b_2 = d_l$. Let $(u_{m-1}, \dots, u_0)$ be the unique path from $\varphi(c)$ to r_2 and $(v_{n-1}, \dots, v_0)$ be the unique path from $\varphi(d)$ to r_2 in G_2. Then by definition of φ, $\varphi(a_1) = \varphi(c_{k+1}) = u_k$ and $\varphi(a_2) = \varphi(d_{l+1}) = v_l$. Suppose that $\varphi(a_1) = \varphi(a_2)$. Then $k = l$ and $u_i = v_i$ for all $i \leq k$. So $FCD_{G_2}(\psi(c), \psi(d)) \geq k > p$. On the other hand, $FCD_{G_1}(c, d) = FCD_{G_1}(b_1, b_2) = p$ and by (2), $FCD_{G_2}(\psi(c), \psi(d)) \leq p$ – a contradiction.

□

4 The Main Results

Now we have tools to settle the problem of the existence of a square root of a component of a functional graph or the union of containing cycles of the same length two components of a functional graph. Together with Theorem 2.1 the following results allow solving the problem of the existence of a square root of a functional graph.

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4.1 Existence of a Square Root of a Component of a Functional Graph

Theorem 4.1. *Let $G = (V, E)$ be a component of a functional graph, containing a cycle $\bar{a} := (a_0, \ldots, a_{k-1})$ and $G_i = (V_i, E_i)$ denote the in-tree generated by element a_i of the cycle $\bar{a}$ in graph G, for $i \in \{0, \ldots, k-1\}$. Assume $\tau\colon \{0, \ldots, k-1\} \to \{0, \ldots, k-1\}$ is the function defined as follows: $\tau(i) := \left(i + \frac{k-1}{2}\right) \mod k$ for an odd number k. Then there exists a square root of G if and only if k is an odd number and there exists set $I \subset \{0, \ldots, k-1\}$ such that the following conditions are satisfied:*

1. *there exists a subgraph $G_i^D = (V_i^D, E_i^D)$ of G_i such that G_i^D is a strictly nontrivial in-tree with the root a_i for all $i \in I$;*
2. *there exists a subgraph $G_{\tau(i)}^C = (V_{\tau(i)}^C, E_{\tau(i)}^D)$ of $G_{\tau(i)}$ such that $G_{\tau(i)}^C$ is a nontrivial in-tree with the root $a_{\tau(i)}$ and $G_{\tau(i)}^C$ is a replica of G_i^D for all $i \in I$;*
3. *$V_i^D \cap V_i^C \subseteq \{a_i\}$ for all $i \in I \cap \tau(I)$;*
4. *if $t \in V \setminus \bigcup_{i \in I} \left(V_i^D \cup V_{\tau(i)}^C\right)$, then t is a trivial source in G_i for some $i \in \mathbb{Z}_i$ or $t \in \{a_0, \ldots, a_{k-1}\} \setminus \bigcup_{i \in I}\{a_i, a_{\tau(i)}\}$.*

Proof. Let k be an odd number and $\mathbb{Z}_k := \{0, \ldots, k-1\}$. Define function $\sigma\colon \mathbb{Z} \to \mathbb{Z}_k$ in the following way: $\sigma(i) := \left(i \cdot \frac{k+1}{2}\right) \mod k$ for all $i \in \mathbb{Z}$. Note that:

1. τ is an injection;
2. $\tau^{-1}(j) = \left(j + \frac{k+1}{2}\right) \mod k$;
3. $\sigma(i + k) = \sigma(i)$ for all $i \in \mathbb{Z}$;
4. $\sigma|_{\mathbb{Z}_k}$ is an injection;
5. $\sigma(i) := \begin{cases} \frac{i}{2} & \text{if } i \text{ is an even number} \\ \frac{k+i}{2} & \text{if } i \text{ is an odd number} \end{cases}$, for all $i \in \mathbb{Z}_k$;
6. $\sigma^{-1}(j) := \begin{cases} 2j & \text{if } j \leq \frac{k-1}{2} \\ 2j - k & \text{if } j \geq \frac{k+1}{2} \end{cases}$, for all $j \in \mathbb{Z}_k$;
7. $\tau(\sigma(i+1)) = \sigma(i)$ for all $i \in \mathbb{Z}$.

($\Longrightarrow$)

Assume that $G' = (V, E')$ is a square root of G. By Proposition 2.1, k is an odd number and G' is a functional graph containing the cycle $\hat{a} := (a_{\sigma(0)}, \ldots, a_{\sigma(k-1)})$. Note that G' is connected, since all vertices are connected in G, so by definition of a square root, they are also connected in G'. Let $G_i' = (V_i', E_i')$ be the in-tree generated by element $a_{\sigma(i)}$ of the cycle $\hat{a}$ in graph G', for $i \in \{0, \ldots, k-1\}$.

Define sets:

$$J := \{j \in \mathbb{Z}_k \colon \exists_{v \in V_j'} lev_{G_j'}(v) \geq 3\}$$

and $I := \{\sigma(j+1) \colon j \in J\} = \tau^{-1}(\sigma(J))$.

Suppose that $J = \emptyset$. Then $I = \emptyset$. Moreover, $lev_{G'_j}(v) \leq 2$ for all $v \in V'_j$ and all $j \in \mathbb{Z}_k$, hence each vertex $v \in V$ is a trivial source in G_j for some $j \in \mathbb{Z}_k$ or it belongs to $\{a_0, \ldots, a_{k-1}\}$. So I satisfies conditions 1–4.

Assume that $J \neq \emptyset$. For each $j \in J$ let $G''_j = (V''_j, E''_j)$ be the induced subgraph of G'_j on the set of vertices whose level in G'_j is at least equal to 3 or for which there exists an ancestor whose level in G'_j is at least equal to 3, and let graph $H_j := (W_j, F_j)$ be defined as follows: $W_j := V''_j \cup \{a_{\sigma(j+1)}\}$ and $F_j := E''_j \cup \{(a_{\sigma(j)}, a_{\sigma(j+1)})\}$. Note that G''_j is a connected subgraph of G' and an in-tree with the root $a_{\sigma(j)}$ as well as H_j is also a connected subgraph of G' and an in-tree with the root $a_{\sigma(j+1)}$. Let $G^D_{\sigma(j+1)} = (V^D_{\sigma(j+1)}, E^D_{\sigma(j+1)})$ be the induced subgraph of $G_{\sigma(j+1)}$ on the set $V_{\sigma(j+1)} \cap W_j$ and $G^C_{\sigma(j)} = (V^C_{\sigma(j)}, E^C_{\sigma(j)})$ be the induced subgraph of $G_{\sigma(j)}$ on the set $V_{\sigma(j)} \cap W_j$ for all $j \in J$. Note that $G^D_{\sigma(j+1)}$ is a strictly nontrivial in-tree with the root $a_{\sigma(j+1)}$ for all $j \in J$: Fix a vertex $v \in W_j$ such that $lev_{H_j}(v) = m \geq 4$ and let $(p_m, \ldots, p_1, p_0)$ be the unique path from v to $a_{\sigma(j+1)}$ in H_j. Then by definition of the in-tree generated by element of cycle in G' and definition of a square root of graph, $p_1 = a_{\sigma(j)}$, $p_2 \in G_{\sigma(j+1)}$, $p_3 \in G_{\sigma(j)}$, $p_4 \in G_{\sigma(j+1)}$ and (p_4, p_2, p_0) is the unique path in $G_{\sigma(j+1)}$ from p_4 to $a_{\sigma(j+1)}$, thus $G^D_{\sigma(j+1)}$ is strictly nontrivial in-tree with the root $a_{\sigma(j+1)}$. Similarly (p_3, p_1) is the unique path in $G_{\sigma(j)}$ from p_3 to $a_{\sigma(j)}$ and thus $G^C_{\sigma(j)}$ is a nontrivial in-tree with the root $a_{\sigma(j)}$. By definition of coil and square root of a graph, $H_j \in Coils(G^D_{\sigma(j+1)}, G^C_{\sigma(j)})$, so by Theorem 3.1, $G^C_{\tau(i)}$ is a replica of G^D_i, where $i := \sigma(j+1)$ for all $j \in J$.

Assume that $i \in I \cap \tau(I)$. Then $i = \sigma(j+1) = \sigma(l)$ for some $j, l \in J$. Suppose that $v \in V^D_{\sigma(j+1)} \cap V^C_{\sigma(l)}$ and $v \neq a_i$. Then $v \in G'_j \cap G'_l$, hence $j = l$, since $G'_j \cap G'_l = \emptyset$ for all $j \neq l$, therefore $\sigma(j+1) = \sigma(j)$ – a contradiction.

By definition of coil, $V^D_{\sigma(j+1)} \cup V^C_{\sigma(j)} = W_j$ for all $j \in J$, so if

$$t \in V \setminus \bigcup_{j \in J} \left(V^D_{\sigma(j+1)} \cup V^C_{\sigma(j)}\right),$$

then $lev_{G'_j}(t) \leq 2$ for some $j \in \mathbb{Z}_k$, so t is a trivial source in G_i for some $i \in \mathbb{Z}_k$ or $t \in \{a_0, \ldots, a_{k-1}\} \setminus \bigcup_{j \in J}\{a_{\sigma(j+1)}, a_{\sigma(j)}\}$.

$(\Longleftarrow)$

Assume that k is an odd number and there exists set $I \subset \{0, \ldots, k-1\}$ such that the following conditions are satisfied:

1. there exists a subgraph $G^D_i = (V^D_i, E^D_i)$ of G_i such that G^D_i is a strictly nontrivial in-tree with the root a_i for all $i \in I$;

2. there exists a subgraph $G^C_{\tau(i)} = (V^C_{\tau(i)}, E^D_{\tau(i)})$ of $G_{\tau(i)}$ such that $G^C_{\tau(i)}$ is a nontrivial in-tree with the root $a_{\tau(i)}$ and $G^C_{\tau(i)}$ is a replica of G^D_i for all $i \in I$;

3. $V^D_i \cap V^C_i \subseteq \{a_i\}$ for all $i \in I \cap \tau(I)$;

4. if $t \in V \setminus \bigcup_{i \in I}\left(V^D_i \cup V^C_{\tau(i)}\right)$, then t is a trivial source in G_i for some $i \in \mathbb{Z}_i$ or $t \in \{a_0, \ldots, a_{k-1}\} \setminus \bigcup_{i \in I}\{a_i, a_{\tau(i)}\}$.

By properties (1) and (2) and Theorem 3.1, for each $i \in I$ there exists a set of edges $E'_i \subseteq V_i^D \times V_{\tau(i)}^C \cup V_{\tau(i)}^C \times V_i^D$ such that
$H_i := (V_i^D \cup V_{\tau(i)}^C, E'_i) \in Coils(G_i^D, G_{\tau(i)}^C)$. By property (3), if $i \in I \cap \tau(I)$, then in-trees H_i and $H_{\tau^{-1}(i)}$ have only one common vertex a_i. Otherwise, if $i, j \in I$, $j \neq i$, $j \neq \tau(i)$ and $i \neq \tau(j)$, then sets of vertices of graphs H_i and H_j are disjoint. Let T_i denote the set of all trivial sources in G_i belonging to the set $V \setminus \bigcup_{i \in I} \left(V_i^D \cup V_{\tau(i)}^C\right)$ for $i \in \mathbb{Z}_k$. By (4),

$$V = \{a_0, \ldots, a_{k-1}\} \cup \bigcup_{i \in I} \left(V_i^D \cup V_{\tau(i)}^C\right) \cup \bigcup_{i \in \mathbb{Z}_k} T_i.$$

Define set

$$E' := \{(a_{\sigma(i)}, a_{\sigma(i+1)}) \colon i \in \mathbb{Z}_k\} \cup \bigcup_{i \in I} E'_i \cup \bigcup_{i \in \mathbb{Z}_k} \{(t, a_{\tau(i)}) \colon t \in T_i\}$$

We shall show that $G' = (V, E')$ is a square root of graph G: G' is a functional graph:

- If $v = a_i$ for some $i \in \mathbb{Z}_k$, then by definition of E' and properties of σ and τ, $(a_i, a_{\tau^{-1}(i)}) \in E'$. It may happen that $i \in \tau(I)$. Then by definition of coil and E', $(a_i, a_{\tau^{-1}(i)}) \in E'_{\tau^{-1}(i)} \subseteq E'$. Either way, if $(a_i, x) \in E'$ then $x = a_{\tau^{-1}(i)}$.

- If $v \in V_i^D \cup V_{\tau(i)}^C \setminus \{a_i, a_{\tau(i)}\}$ for some $i \in I$, then there exists exactly one $w \in V_i^D \cup V_{\tau(i)}^C$ such that $(v, w) \in E'_i \subseteq E'$, since H_i as a coil is an in-tree and by property (3), $v \notin V_j^D \cup V_{\tau(j)}^C$ for all $j \in I \setminus \{i\}$.

- At last if $v \in V \setminus \bigcup_{i \in I} \left(V_i^D \cup V_{\tau(i)}^C\right)$ and v is not element of the cycle $\bar{a}$, then $v \in T_i$ for exactly one $i \in \mathbb{Z}_k$ and by definition of E', $(v, a_{\tau(i)}) \in E'$.

Fix any $u, v \in V$. We shall prove that $(u, v) \in E$ iff there exists $w \in V$ such that $(u, w) \in E'$ and $(w, v) \in E'$:

- Assume that u is an element of the cycle $\bar{a}$. Then $u = a_i$ for some $i \in \mathbb{Z}_k$.

  If $(u, v) \in E$, then $v = a_{(i+1) \bmod k}$. Moreover $i = \sigma(j)$, where $j = \sigma^{-1}(i)$, $(a_i, a_{\sigma(j+1)}) \in E'$ and $(a_{\sigma(j+1)}, a_{\sigma(j+2)}) \in E'$. Note that $\sigma(j+1) = \tau^{-1}(i)$ and

$$\begin{aligned}\sigma(j+2) &= \left(j\frac{k+1}{2} + k + 1\right) \mod k = \left(j\frac{k+1}{2} + 1\right) \mod k \\ &= \left(\left(j\frac{k+1}{2}\right) \mod k + 1\right) \mod k = (i+1) \mod k.\end{aligned}$$

  Therefore $(u, a_{\tau^{-1}(i)}) \in E'$ and $(a_{\tau^{-1}(i)}, v) \in E'$.

  If (u, w),$(w, v) \in E'$, then $w = a_{\sigma(j+1)}$, $v = a_{\sigma(j+2)}$, where $j = \sigma^{-1}(i)$, so $(u, v) \in E$, since $\sigma(j+2) = (i+1) \mod k$.

- Assume that $u \in V_i^D \cup V_{\tau(i)}^C \setminus \{a_i, a_{\tau(i)}\}$ for some $i \in I$.

  If $(u, v) \in E$, then $v \in V_i^D \cup V_{\tau(i)}^C$ and by definition of the coil H_i, there exists $w \in V_i^D \cup V_{\tau(i)}^C$ such that (u, w),$(w, v) \in E'_i \subseteq E'$.

  If (u, w), $(w, v) \in E'$, then by definition of E', (u, w), $(w, v) \in E'_i$, hence by definition of H_i, $(u, v) \in E_i^D \subseteq E$ or $(u, v) \in E_{\tau(i)}^C \subset E$.

- Assume that $u \in T_i$ for some $i \in \mathbb{Z}_k$. Then $(u, a_i) \in E$ and $(u, a_{\tau(i)}) \in E'$ as well as $(a_{\tau(i)}, a_i) \in E'$, since if $\tau(i) = \sigma(j)$, then $\tau(i) = \tau(\sigma(j+1))$, hence $i = \sigma(j+1)$, since τ is an injection.

$\square$

4.2 Existence of a Square Root of the Union of Two Components of a Functional Graph

Theorem 4.2. *Let $G^1 = (V^1, E^1)$ and $G^2 = (V^2, E^2)$ be components of a functional graph, containing cycles $\boldsymbol{a}^1 = (a_0^1, \ldots, a_{k-1}^1)$ and $\boldsymbol{a}^2 = (a_0^2, \ldots, a_{k-1}^2)$, respectively, and let $G_j^i = (V_j^i, E_j^i)$ denote the in-tree generated by element a_j^i of the cycle $\boldsymbol{a}^i$ in graph G^i for $i \in \{1, 2\}$ and $j \in \mathbb{Z}_k$. For each $s \in \mathbb{Z}_k$ define functions: $\tau_s^{(1)} \colon \mathbb{Z}_k \to \mathbb{Z}_k$, $\tau_s^{(2)} \colon \mathbb{Z}_k \to \mathbb{Z}_k$ and $\sigma \colon \{1, 2\} \to \{1, 2\}$ as follows:*

$$
\begin{aligned}
\tau_s^{(1)}(j) &:= (j - 1 - s) \mod k \\
\tau_s^{(2)}(j) &:= (j + s) \mod k \\
\sigma(i) &:= 3 - i
\end{aligned}
$$

for all $j \in \mathbb{Z}_k$ and $i \in \{1, 2\}$. Then there exists a connected square root $G' = (V, E')$ of graph $G = (V^1 \cup V^2, E^1 \cup E^2)$ if and only if there exist $s \in \mathbb{Z}_k$ and subsets $I^1, I^2 \subseteq \mathbb{Z}_k$ such that the following conditions are satisfied:

1. *there exists a subgraph $G_j^{i,D} = (V_j^{i,D}, E_j^{i,D})$ of graph G_j^i such that $G_j^{i,D}$ is a strictly nontrivial in-tree with the root a_j^i for all $j \in I^i$ and for $i \in \{1, 2\}$;*
2. *there exists a subgraph $G_{\tau_s^{(i)}(j)}^{i,C} = (V_{\tau_s^{(i)}(j)}^{i,C}, E_{\tau_s^{(i)}(j)}^{i,C})$ of graph $G_{\tau_s^{(i)}(j)}^i$ such that $G_{\tau^{(i)}(j)}^{i,C}$ is a nontrivial in-tree with the root $a_{\tau_s^{(i)}(j)}^i$ and $G_{\tau_s^{(i)}(j)}^{i,C}$ is a replica of $G_j^{\sigma(i),D}$ for all $j \in I^{\sigma(i)}$ and for $i \in \{1, 2\}$;*
3. *$V_j^{i,D} \cap V_j^{i,C} \subseteq \{a_j^i\}$ for all $j \in I^i \cap \tau_s^{(i)}(I^{\sigma(i)})$ and for $i \in \{1, 2\}$;*
4. *if $t \in V \setminus \left[\bigcup_{j \in I^1} \left(V_j^{1,D} \cup V_{\tau_s^{(2)}(j)}^{2,C}\right) \cup \bigcup_{j \in I^2} \left(V_j^{2,D} \cup V_{\tau_s^{(1)}(j)}^{1,C}\right)\right]$, then t is a trivial source in G_j^i for some $i \in \{1, 2\}$ and $j \in \mathbb{Z}_k$ or t is an element of cycles $\boldsymbol{a}^1$ or $\boldsymbol{a}^2$.*

Remark 4.1. $\tau_t^{(1)}(j) = \tau_s^{(2)}(j)$ and $\tau_t^{(2)}(j) = \tau_s^{(1)}(j)$ for all $s, j \in \mathbb{Z}_k$ and $t = k - 1 - s$. Moreover $\tau_s^{(1)}$ and $\tau_s^{(2)}$ are injections for all $s \in \mathbb{Z}_k$ and ${\tau_s^{(1)}}^{-1}(j) = (j + 1 + s) \mod k$ and ${\tau_s^{(2)}}^{-1}(j) = (j - s) \mod k$ for all $s, j \in \mathbb{Z}_k$.

Proof. $(\Longrightarrow)$

Assume that $G' = (V, E')$ is a square root of $G = (V^1 \cup V^2, E^1 \cup E^2)$ and G' contains only one component.

We shall show that there exists $s \in \mathbb{Z}_k$ such that $(a^2_{\tau_s^{(2)}(j)}, a^1_j) \in E'$ and $(a^1_{\tau_s^{(1)}(j)}, a^2_j) \in E'$ for all $j \in \mathbb{Z}_k$: Assume that $(a^1_0, b) \in E'$. Then $b \in V^2$, since otherwise $b = a^1_{\frac{k+1}{2}}$ and there would have to exist two components in G'. Suppose that $b \notin \mathbf{a}^2$. Then $b \in G^2_j$ for some $j \in \mathbb{Z}_k$. Let $(b_m, \ldots, b_1, b_0)$ be the unique path from b to a^2_j. Then $(a^1_{m \mod k}, a^2_j) \in E'$, so $(a^1_0, a^2_{(j+k-m) \mod k}) \in E'$ – a contradiction, since G' is a functional graph. Therefore (a^1_0, a^2_p) for some $p \in \mathbb{Z}_k$.

Put $s := (p-1) \mod k$. $0 = \tau_s^{(1)}(j)$ for $j = \tau_s^{(1)^{-1}}(0) = p$ and $\tau_s^{(2)}(1) = p$, so in fact $(a^1_{\tau_s^{(1)}(p)}, a^2_p) \in E'$ and $(a^2_{\tau_s^{(2)}(0)}, a^1_0) \in E'$, therefore $(a^2_{\tau_s^{(2)}(j)}, a^1_j) \in E'$ and $(a^1_{\tau_s^{(1)}(j)}, a^2_j) \in E'$ for all $j \in \mathbb{Z}_k$, since $(a^1_{\tau_s^{(1)}(j)}, a^1_{\tau_s^{(1)}(j+1)}) \in E^1$ and $(a^2_{\tau_s^{(2)}(j)}, a^2_{\tau_s^{(2)}(j+1)}) \in E^2$ for all $j \in \mathbb{Z}_k$.

Let $G'^i_j = (V'^i_j, E'^i_j)$ be the in-tree generated by element a^i_j in graph G' for $i \in \{1, 2\}$ and $j \in \mathbb{Z}_k$. Define sets:

$$J^i := \{j \in \mathbb{Z}_k : \exists_{v \in V'^i_j} lev_{G'^i_j}(v) \geq 3\}$$
$$I^i := \tau_s^{(\sigma(i))^{-1}}(J^{\sigma(i)})$$

for $i \in \{1, 2\}$ and $j \in \mathbb{Z}_k$.

Suppose that $J^1 = J^2 = \emptyset$. Then $lev_{G'^i_j}(v) \leq 2$ for $i \in \{1, 2\}$, for all $j \in \mathbb{Z}_k$ and all $v \in V'^i_j$. Therefore each vertex $v \in V$ is a trivial source in G^i_j or it belongs to the cycle $\mathbf{a}^i$ for some $i \in \{1, 2\}$ and $j \in \mathbb{Z}_k$.

Assume that $J^1 \neq \emptyset$ or $J^2 \neq \emptyset$. For each $i \in \{1, 2\}$ and $j \in J^i$, let $G''^i_j = (V''^i_j, E''^i_j)$ be the induced subgraph of G'^i_j on the set of all vertices whose level in G'^i_j is at least equal to 3 or for which there exists an ancestor whose level in G'^i_j is at least equal to 3, and let graph $H^i_j = (W^i_j, F^i_j)$ be defined as follows: $W^i_j := V''^i_j \cup \{a^{\sigma(i)}_{\tau_s^{(i)^{-1}}(j)}\}$ and $F^i_j := E''^i_j \cup \{(a^i_j, a^{\sigma(i)}_{\tau_s^{(i)^{-1}}(j)})\}$. Then G''^i_j as a connected subgraph of G' is an in-tree with the root a^i_j and H^i_j as connected subgraph of G' is also an in-tree with the root $a^{\sigma(i)}_{\tau_s^{(i)^{-1}}(j)}$. Let $G^{\sigma(i),D}_{\tau_s^{(i)^{-1}}(j)} = (V^{\sigma(i),D}_{\tau_s^{(i)^{-1}}(j)}, E^{\sigma(i),D}_{\tau_s^{(i)^{-1}}(j)})$ be the induced subgraph of $G^{\sigma(i)}_{\tau_s^{(i)^{-1}}(j)}$ on the set $W^i_j \cap V^{\sigma(i)}_{\tau_s^{(i)^{-1}}(j)}$ and $G^{i,C}_j = (V^{i,C}_j, E^{i,C}_j)$ be the induced subgraph of G^i_j on the set $V^i_j \cap W^i_j$ for $i \in \{1, 2\}$ and $j \in J^i$. Then $G^{\sigma(i),D}_{\tau_s^{(i)^{-1}}(j)}$ is a strictly nontrivial in-tree with the root $a^{\sigma(i)}_{\tau_s^{(i)^{-1}}(j)}$, $G^{i,C}_j$ is a nontrivial in-tree with the root a^i_j and by definition of a coil and the square root G', $H^i_j \in Coils(G^{\sigma(i),D}_{\tau_s^{(i)^{-1}}(j)}, G^{i,C}_j)$, so by Theorem 3.1, $G^{i,C}_j$ is a replica of $G^{\sigma(i),D}_{\tau_s^{(i)^{-1}}(j)}$ for $i \in \{1, 2\}$ and $j \in J^i$.

Assume that $j \in I^i \cap \tau_s^{(i)}(I^{\sigma(i)})$ for some $i \in \{1, 2\}$. Then $V^{i,C}_j \subseteq W^i_j$ and $V^{i,D}_j \subseteq W^{\sigma(i)}_{\tau_s^{\sigma(i)}(j)}$, therefore $V^{i,C}_j \cap V^{i,D}_j \subseteq W^i_j \cap W^{\sigma(i)}_{\tau_s^{\sigma(i)}(j)} = \{a^i_j\}$.

If $t \in V \setminus \left[\bigcup_{j \in I^1} \left(V^{1,D}_j \cup V^{2,C}_{\tau_s^{(2)}(j)}\right) \cup \bigcup_{j \in I^2} \left(V^{2,D}_j \cup V^{1,C}_{\tau_s^{(1)}(j)}\right)\right]$, then by definition of coil, $t \in V \setminus \left(\bigcup_{j \in J^1} W^1_j \cup \bigcup_{j \in J^2} W^2_j\right)$, so there exist $i \in \{1, 2\}$ and $j \in \mathbb{Z}_k$ such that $lev_{G'^i_j}(t) \leq 2$, thus t is a trivial source in G^i_j for some $i \in \{1,2\}$ and $j \in \mathbb{Z}_k$ or t is element of cycles $\mathbf{a}^1$ or $\mathbf{a}^2$.

($\Longleftarrow$)

Assume that there exist $s \in \mathbb{Z}_k$ and subsets $I^1, I^2 \subseteq \mathbb{Z}_k$ such that the following conditions are satisfied:

1. there exists a subgraph $G^{i,D}_j = (V^{i,D}_j, E^{i,D}_j)$ of graph G^i_j such that $G^{i,D}_j$ is a strictly nontrivial in-tree with the root a^i_j for all $j \in I^i$ and for $i \in \{1,2\}$;

2. there exists a subgraph $G^{i,C}_{\tau^{(i)}(j)} = (V^{i,C}_{\tau^{(i)}_s(j)}, V^{i,C}_{\tau^{(i)}_s(j)})$ of graph $G^i_{\tau^{(i)}_s(j)}$ such that $G^{i,C}_{\tau^{(i)}(j)}$ is a nontrivial in-tree with the root $a^i_{\tau^{(i)}_s(j)}$ and $G^{i,C}_{\tau^{(i)}_s(j)}$ is a replica of $G^{\sigma(i),D}_j$ for all $j \in I^{\sigma(i)}$ and for $i \in \{1,2\}$;

3. $V^{i,D}_j \cap V^{i,C}_j \subseteq \{a^i_j\}$ for all $j \in I^i \cap \tau^{(i)}_s(I^{\sigma(i)})$ and for $i \in \{1,2\}$;

4. if $t \in V \setminus \left[\bigcup_{j \in I^1}\left(V^{1,D}_j \cup V^{2,C}_{\tau^{(2)}_s(j)}\right) \cup \bigcup_{j \in I^2}\left(V^{2,D}_j \cup V^{1,C}_{\tau^{(1)}_s(j)}\right)\right]$, then t is a trivial source in G^i_j for some $i \in \{1,2\}$ and $j \in \mathbb{Z}_k$ or t is an element of cycles $\mathbf{a}^1$ or $\mathbf{a}^2$.

By properties (1), (2) and Theorem 3.1, for each $j \in I^i$ and $i \in \{1,2\}$ there exists a set of edges $E'^i_j \in V^{i,D}_j \times V^{\sigma(i),C}_{\tau^{\sigma(i)}_s(j)} \cup V^{\sigma(i),C}_{\tau^{\sigma(i)}_s(j)} \times V^{i,D}_j$ such that the graph $H^i_j = (V^{i,D}_j \cup V^{\sigma(i),C}_{\tau^{\sigma(i)}_s(j)}, E'^i_j) \in Coils(G^{i,D}_j, G^{\sigma(i),C}_{\tau^{\sigma(i)}_s(j)})$. By property (3), graphs H^i_j and $H^{\sigma(i)}_{\tau^{(i)-1}_s(j)}$ have one common vertex a^i_j for $j \in I^i \cap \tau^{(i)}_s(I^{\sigma(i)})$ and for $i \in \{1,2\}$. If $j_1, j_2 \in I^i$ and $j_1 \neq j_2$, then the sets of vertices of graphs $H^i_{j_1}$ and $H^i_{j_2}$ are disjoint for $i \in \{1,2\}$. Similarly if $j_1 \in I^i$, $j_2 \in I^{\sigma(i)}$, $j_1 \neq j_2$, $j_1 \neq \tau^{(i)}_s(j_2)$ and $j_2 \neq \tau^{(i)}_s(j_1)$, then the sets of vertices of graphs $H^i_{j_1}$ and $H^i_{j_2}$ are also disjoint for $i \in \{1,2\}$, since the sets of vertices of in-trees $G^{i_1}_{j_1}$ and $G^{i_2}_{j_2}$ are disjoint, if $i_1 \neq i_2$ or $j_1 \neq j_2$. Let T^i_j be the set of all trivial sources in G^i_j which do not belong to the set $\bigcup_{j \in I^1}\left(V^{1,D}_j \cup V^{2,C}_{\tau^{(2)}_s(j)}\right) \cup \bigcup_{j \in I^2}\left(V^{2,D}_j \cup V^{1,C}_{\tau^{(1)}_s(j)}\right)$ for all $i \in \{1,2\}$ and $j \in \mathbb{Z}_k$. By property (4),

$$
\begin{aligned}
V = &\{a^1_0, \ldots, a^1_{k-1}\} \cup \{a^2_0, \ldots, a^2_{k-1}\} \\
&\cup \bigcup_{j \in I^1}\left(V^{1,D}_j \cup V^{2,C}_{\tau^{(2)}_s(j)}\right) \cup \bigcup_{j \in I^2}\left(V^{2,D}_j \cup V^{1,C}_{\tau^{(1)}_s(j)}\right) \cup \bigcup_{\substack{i\in\{1,2\}\\ j\in\mathbb{Z}_k}} T^i_j
\end{aligned}
$$

Define

$$
\begin{aligned}
E' = &\{(a^i_{\tau^{(i)}_s(j)}, a^{\sigma(i)}_j) \colon i \in \{1,2\}, j \in \mathbb{Z}_k\} \\
&\cup \bigcup_{\substack{i\in\{1,2\}\\ j \in I^i}} E'^i_j \cup \bigcup_{\substack{i\in\{1,2\}\\ j\in\mathbb{Z}_k}} \{(t, a^{\sigma(i)}_{\tau^{\sigma(i)}_s(j)}) \colon t \in T^i_j\}.
\end{aligned}
$$

It can be proved analogously like in the proof of Theorem 4.1 that $G' = (V^1 \cup V^2, E')$ is a square root of G. $\square$

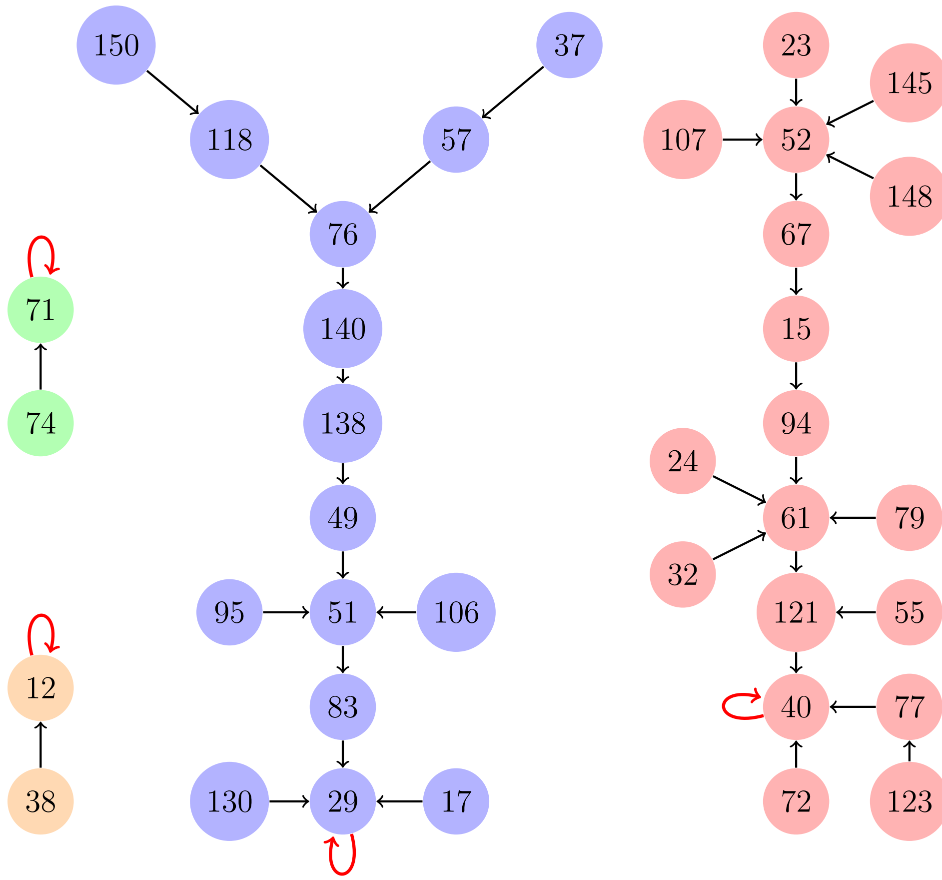

Figure 2: 4 components containing cycles of length 1

4.3 Discussion

Consider the following sequence

$$\alpha = (146, 26, 142, 27, 78, 101, 44, 70, 75, 27, 18, 12, 147, 132, 94, 142, 29, 101, 68, 25, 56, 73, 52, 61, 114, 30, 46, 4, 29, 34, 88, 61, 111, 109, 44, 149, 57, 12, 7, 40, 1, 144, 36, 85, 31, 143, 63, 44, 51, 132, 83, 67, 109, 13, 121, 69, 76, 114, 101, 20, 121, 16, 124, 93, 103, 44, 15, 131, 134, 124, 71, 40, 1, 71, 46, 140, 40, 26, 61, 134, 116, 39, 29, 22, 44, 149, 56, 99, 63, 45, 30, 59, 131, 61, 51, 68, 11, 93, 93, 137, 69, 111, 134, 114, 39, 51, 52, 81, 20, 109, 99, 86, 63, 13, 101, 143, 68, 76, 147, 1, 40, 21, 77, 134, 31, 103, 132, 78, 44, 29, 93, 139, 20, 56, 110, 108, 102, 49, 39, 138, 120, 105, 146, 80, 52, 116, 26, 52, 143, 118)$$

and its functional graph $G = (V, E)$ whose nine components are presented in figures 2 – 5. We shall find a square root G' of G using Theorems 3.2, 4.1 and 4.2.

Note that both the component with cycle 12 and the component containing cycle 71 from figure 2 have a square root separately and there also exists a connected square root of these two components. The greater in-tree generated by 40 in the fourth component (C_4) containing cycle 40 is a replica of the greater in-tree generated by 29 in the third component (C_3)

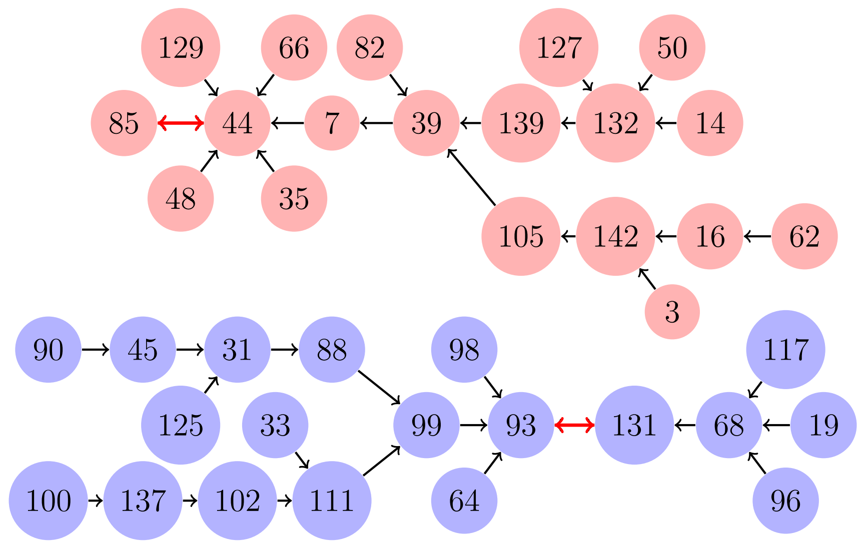

Figure 3: 2 components containing cycles of length 2

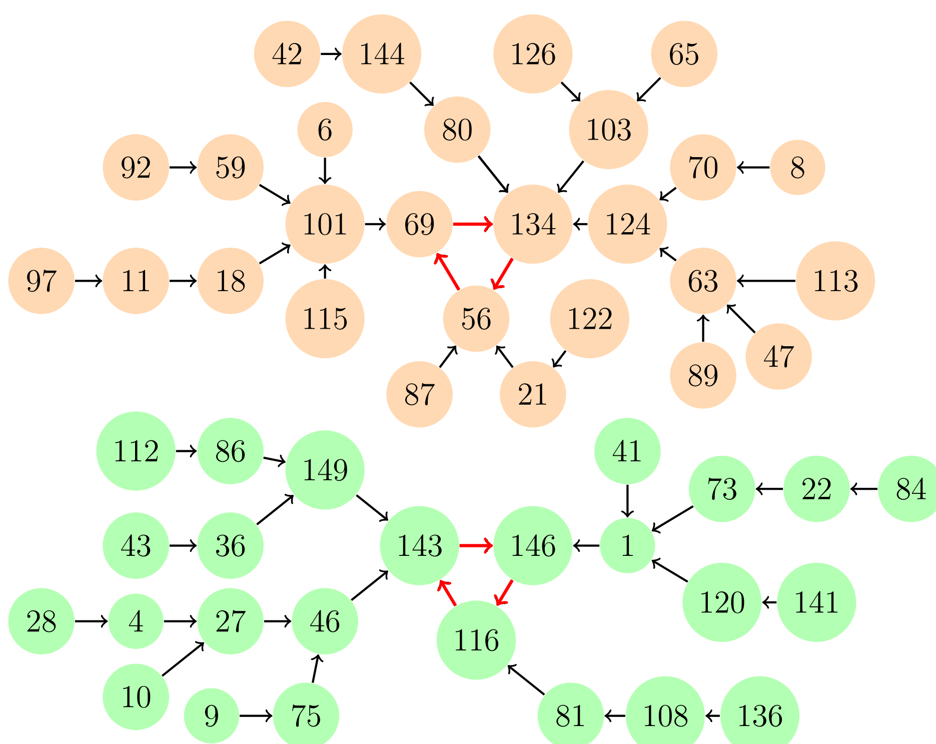

Figure 4: 2 components containing cycles of length 3

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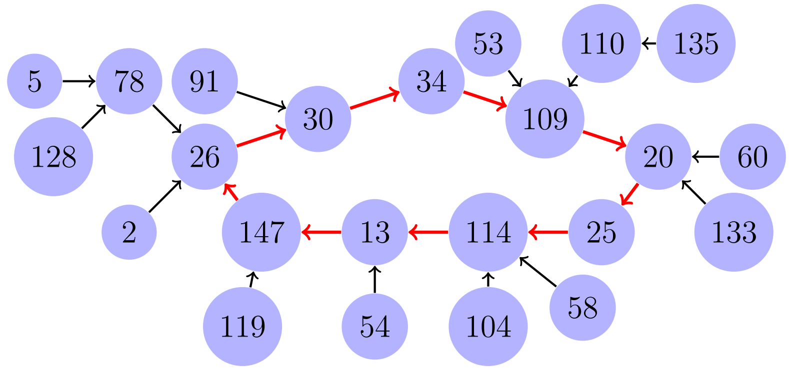

Figure 5: Component containing a cycle of length 9

containing cycle 29, since 37, 150 are maximal sources in $in-tree_{C_3}(29)$, 145, 23, 148, 107 are balanced sources in $in-tree_{C_4}(40)$, $lev(150) = lev(37) = 8$, $lev(145) = lev(23) = lev(148) = lev(107) = 7$, $lev(FCD(150, 37)) = lev(76) = 6$ and $lev(FCD(145, 23, 148, 107)) = lev(52) = 6$, so the assumptions of Theorem 3.2 are satisfied. The smaller in-tree generated by 40 in C_4 has replicas in C_3 in the form of the in-tree induced on the set $\{17, 29\}$ or the in-tree induced on the set $\{130, 29\}$.

In figure 3 the greater in-tree generated by 93 in the second component (C_2) is a replica of the greater in-tree generated by 44 in the first component (C_1), since 62 is maximal source and 50, 14, 127 are balanced sources in $in - tree_{C_1}(44)$, 90, 100 are maximal sources in $in - tree_{C_2}(93)$, $lev(62) = 6$, $lev(50) = 5$, $lev(90) = lev(100) = 5$, $lev(FCD(50, 62)) = lev(39) = 2$ and $lev(FCD(90, 100)) = lev(99) = 1$. Obviously the smaller in-tree generated by 44 in C_1 is a replica of the smaller in-tree generated by 131 in C_2.

In figure 4 the in-tree generated by 146 in the second component (C_2), has no replica in the form of the in-tree generated by 116 in the same component. Similarly this in-tree is not a replica of the in-tree generated by 143. Therefore in order to find a square root of these two components from figure 4, denoted here by C_1 and C_2, we must use Theorem 4.2 and determine their connected square root. We also use Theorem 3.2 to deduce the form of this square root. Note that $in - tree_{C_1}(56)$ has no replica in C_2 and it is a replica of $in - tree_{C_2}(116)$, since 122 and 136 are maximal sources in $in - tree_{C_1}(56)$ and $in-tree_{C_2}(116)$, respectively, $lev(136) = 3$ and $lev(122) = 2$. Therefore if a connected square root of C_1 and C_2 exists, then its cycle has the following form $(56, 116, 69, 143, 134, 146)$. So $in-tree_{C_2}(116)$ can not be a replica of $in-tree_{C_1}(69)$. Note that $in-tree_{C_1}(69)$ is a replica of the induced subgraph $C_2(\{28, 4, 10, 27, 9, 75, 46, 143\})$ of $in-tree_{C_2}(143)$, since 97, 92 are maximal sources in $in-tree_{C_1}(69)$, 28, 9 are maximal sources in $in-tree_{C_2}(143)$, $lev(97) = 4 = lev(28)$, $lev(92) = lev(9) = 3$, $lev(FCD(28, 9)) = lev(46) = 1$ and $lev(FCD(97, 92)) = lev(101) = 1$. Moreover $in - tree_{C_1}(69)$ can not be a replica of the induced subgraph $C_2(\{112, 43, 86, 36, 149, 143\})$ of $in-tree_{C_2}(143)$ since $lev(97) = 4 > 3 = lev(43) = lev(112)$. The induced subgraph $C_2(\{112, 43, 86, 36, 149, 143\})$ of $in - tree_{C_2}(143)$ is a replica of the induced subgraph $C_1(\{8, 47, 113, 89, 70, 63, 124, 134\})$ of $in - tree_{C_1}(134)$, since 112, 43 are maximal sources in $in - tree_{C_2}(143)$, 8 is a maximal source and 47, 113, 89 are balanced sources in $in-tree_{C_1}(134)$, $lev(112) = lev(43) = 3 = lev(8) = lev(47) = lev(113) = lev(89)$, $lev(FCD(8, 47)) = lev(124) = 1$ and $lev(FCD(112, 43)) = lev(149) = 1$. Moreover the

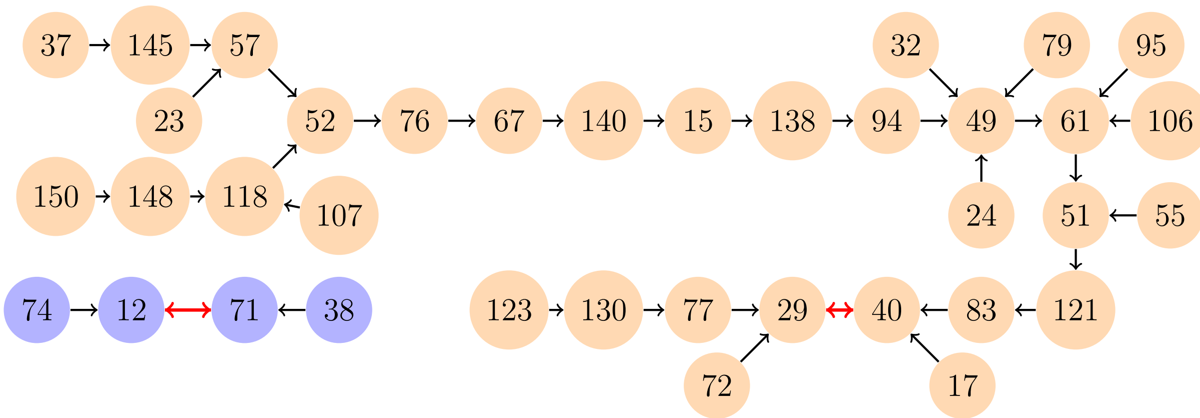

Figure 6: 2 components containing cycles of length 2

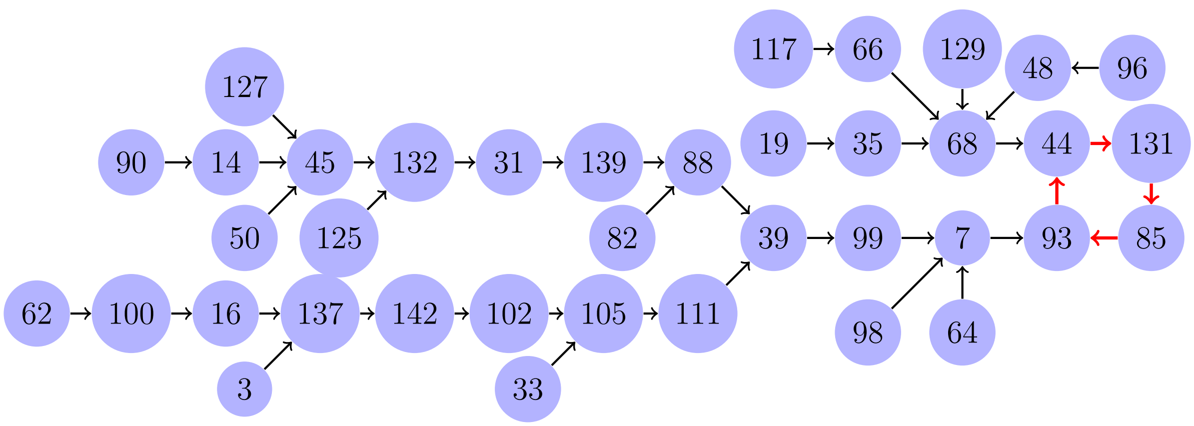

Figure 7: The unique component containing cycle of length 4

induced subgraph $C_2(\{112, 43, 86, 36, 149, 143\})$ of $in - tree_{C_2}(143)$ can not be a replica of the induced subgraph $C_1(\{42, 144, 126, 65, 80, 103, 134\})$ of $in-tree_{C_1}(134)$, since $lev(112) = lev(43) = 3 > 2 = lev(65) = lev(126)$. The induced subgraph $C_1(\{42, 144, 126, 65, 80, 103, 134\})$ of $in - tree_{C_1}(134)$ is a replica of $in - tree_{C_2}(146)$, since 42 is a maximal source, 65, 126 are balanced in $in-tree_{C_1}(134)$ and 84 and 141 are maximal sources in $in-tree_{C_2}(146)$, $lev(42) = 3 = lev(84) - 1$, $lev(65) = lev(126) = 2 = lev(141) - 1$, $lev(FCD(84, 141)) = lev(1) = 1$ and $lev(FCD(42, 65)) = lev(134) = 0$.

In figure 5 $in - tree(20)$ is a replica of $in - tree(26)$ and $in - tree(13)$ is a replica of $in - tree(109)$, therefore by Theorem 4.1, there exists a square root of component in figure 5.

In figures 6–9 components of a square root formed according to the above description are presented.

From these graphs by Theorem 2.1, one can determine the form of a half iterate β of α,

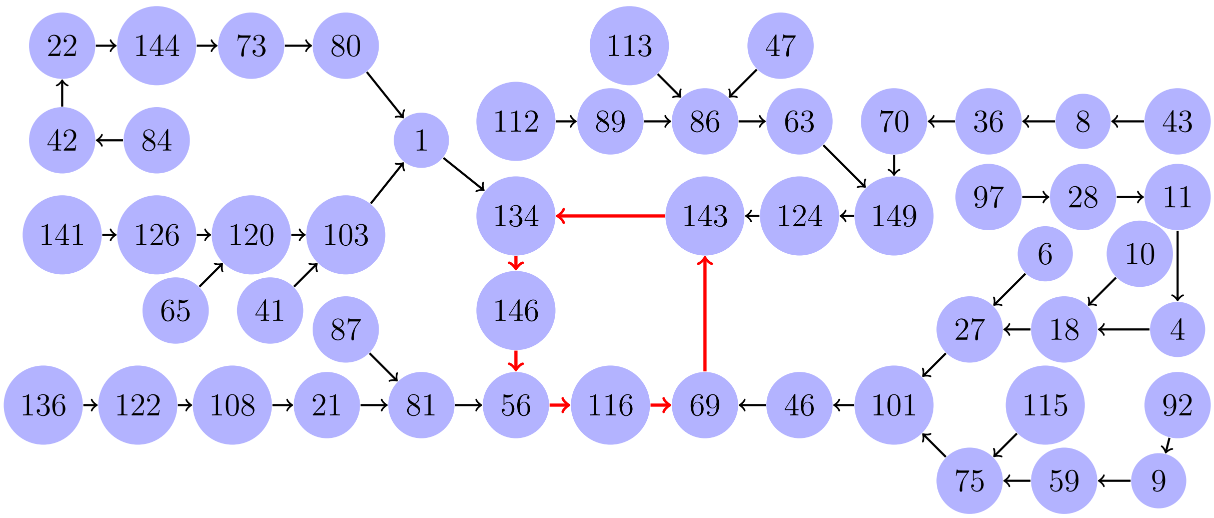

Figure 8: The unique component containing cycle of length 6

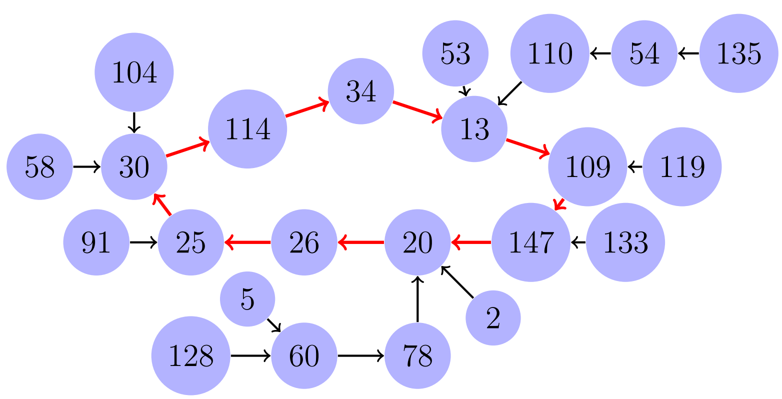

Figure 9: The unique component containing cycle of length 9

namely

$$\beta = (134, 20, 137, 18, 60, 27, 93, 36, 59, 18, 4, 71, 109, 45, 138, 137, 40, 27, 35, 26, 81, 144, 57, 49, 30, 25, 101, 11, 40, 114, 139, 49, 105, 13, 68, 70, 145, 71, 99, 29, 103, 22, 8, 131, 132, 69, 86, 68, 61, 45, 121, 76, 13, 110, 51, 116, 52, 30, 75, 78, 51, 100, 149, 7, 120, 68, 140, 44, 143, 149, 12, 29, 80, 12, 101, 67, 29, 20, 49, 1, 56, 88, 40, 42, 93, 63, 81, 39, 86, 14, 25, 9, 44, 49, 61, 48, 28, 7, 7, 16, 46, 105, 1, 30, 111, 61, 118, 21, 147, 13, 39, 89, 86, 34, 75, 69, 66, 52, 109, 103, 83, 108, 130, 143, 132, 120, 45, 60, 68, 77, 85, 31, 147, 146, 54, 122, 142, 94, 88, 15, 126, 102, 134, 73, 57, 56, 20, 118, 124, 148).$$

5 Conclusion

In the above example, it can be seen that the concept of a replica introduced in this article allows presenting results concerning the problem of the existence of half iterates of finite functions concisely and neatly. This key notion has also a deep characterization by sources of an in-tree, which facilitates checking the conditions in theorems presented in the section 'The Main Results'. Thus, this work enables us to settle, whether for a given function from a finite domain to itself there exists its half iterate, and if it exists the above theorems allow us to determine a half iterate of this function in a relatively clear and simple manner. An interesting direction for future research is to generalize the methods described in this article to identify nth roots of a functional graph.

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