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# Skolem Number of Cycles and Grid Graphs 

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#### Abstract

A Skolem sequence can be thought of as a labelled path where two vertices with the same label are that distance apart. This concept has naturally been generalized to labellings of other graphs, but always using at most two of any integer label. Given that more than two vertices can be mutually distance $d$ apart, we define a new generalization of a Skolem sequence on graphs that we call a proper Skolem labelling. This brings rise to the question; "what is the smallest set of consecutive positive integers we can use to proper Skolem label a graph?" This will be known as the Skolem number of the graph. In this paper we give the Skolem number for cycles and grid graphs, while also providing other related results along the way.


## 1 Introduction

A Skolem sequence is a sequence $S=\left(s_{1}, s_{2}, \ldots s_{2 n}\right)$ consisting of two of each integer in $\{1,2, \ldots, n\}$ so that whenever $s_{i}=s_{j}=k$ then $|i-j|=k$. Skolem defined these sequences while working on Steiner triple systems in [10]. The Skolem sequence has been modified by several other groups as well $[1,3,8,9]$. One modified form of a Skolem sequence contains an additional element 0 , typically called the hook of the sequence.

One can think of a Skolem sequence as a labelling on a path $G$ with labelling set $\{1,2, \ldots, n\}$ so that for each $i \in\{1,2, \ldots, n\}$, the pair of vertices $v, u \in V(G)$ labelled $i$ have $d(u, v)=i$. This naturally gave rise to the concept of Skolem labellings which was introduced in 1990 by Mendelsohn and Shalaby in [7]. In subsequent years further results have appeared about Skolem labellings of ladders and grids [2, 6], cycles [4], and Dutch windmills [5]. All the aforementioned papers work with the definition first posed by Mendelsohn and Shalaby that a $d$-Skolem labelled graph is a triple $(G, L, d)$, where
(a) $G=(V, E)$ is an undirected graph,
(b) $L: V \rightarrow\{d, d+1, \ldots, d+n-1\}$,
(c) $L(v)=L(w)=d+i$ exactly once for $i=0,1, \ldots, n-1$ and $d(v, w)=d+i$,
(d) if $G^{\prime}=\left(V, E^{\prime}\right)$ and $E^{\prime} \subset E$, then $\left(G^{\prime}, L, d\right)$ violates (c).

Note that if a triple $(G, L, d)$ satisfies parts (a), (b) and (c), but doesn't satisfy part (d), it is said to be a weak Skolem labelled graph.

Subsequent papers also considered the idea of a hook in a Skolem labelling of graphs, which is a vertex $v \in V$ such that $L(v)=0$. A labelling containing a hook is called hooked. It is often that more than one vertex is a hook, in which case hook vertices are allowed to be any distance apart from one another. Specifically, these ideas were used in [6] to find the optimal weak Skolem labellings of grid graphs using hooks. However, when attempting to abstract the idea of a Skolem sequence to a graph labelling, it also seems natural to label more than two vertices with the same label if possible. This motivates the idea we wish to establish with the following definitions. For brevity we will from here on refer to $\{1,2, \ldots, n\}$ as $[n]$ for positive integer $n$.

Definition 1.1. Given a graph $G=(V, E)$, we say $\mathcal{L}: V \rightarrow[n]$ is a proper Skolem labelling of $G$ if whenever $\mathcal{L}(v)=\mathcal{L}(w)=i$ for $v \neq w$, then $d(v, w)=i$. The proper Skolem labelling is said to be of order $n$.

Trivially every graph $G=(V, E)$ contains a proper Skolem labelling of order $|V|$ using any bijection. Hence we desire to find proper Skolem labellings of smaller orders.

Definition 1.2. The Skolem number of a graph $G$ is the smallest positive integer $s$ such that there exists a proper Skolem labelling of $G$ of order $s$.

Definition 1.3. Let $G=(V, E)$ be a graph, where $m$ divides $|V|$. Let $\mathcal{L}: V \rightarrow[|V| / m]$ be a proper Skolem labelling of $G$. If $\left|\mathcal{L}^{-1}(i)\right|=\left|\mathcal{L}^{-1}(j)\right|$ for every $i, j \in[|V| / m]$, then $\mathcal{L}$ is an m-balanced Skolem labelling on $G$, otherwise we say $\mathcal{L}$ is an unbalanced Skolem labelling on $G$.

If $\left|\mathcal{L}^{-1}(i)\right|=1$, then we call the label $i$ or the vertex $\mathcal{L}^{-1}(i)$ a hook label or hook vertex respectively; where it is clear, hook will be used instead. Note that this is different to the hook of a Skolem sequence, though mimics the function of a hook for a Skolem labelling.

Since a proper Skolem labelling has a co-domain of $[n]$, an $m$-balanced Skolem labelling uses exactly $m$ 1's in its labelling. Therefore:

Observation 1. If there exists an m-balanced Skolem labelling of $G$, then there exists a $K_{m}$ subgraph of $G$.

Using this same logic, we see that a graph that has a proper Skolem labelling with co-domain $\{1\}$ is a complete graph.

We wish to construct an $m$-balanced Skolem labelled graph with Skolem number $s$ using the minimum number of edges. To do so, first consider an $m$-balanced Skolem labelling $\mathcal{L}$ on an empty graph on $|V|$ vertices. The vertices labelled 1 must form a complete graph and each other pair of identically labelled vertices must be connected by a path. Therefore every vertex with label $k, 2 \leq k<s$, where $s$ is the order of $\mathcal{L}$ must have a degree greater than or equal to 2 and all vertices labelled $s$ must have a degree at least 1 . Figure 1 illustrates this arrangement with each column of vertices having the same label and a complete graph amongst the vertices labelled 1 .

Observation 2. If a simple connected graph $G=(V, E)$ has an m-balanced Skolem labelling, then $|E| \geq\binom{ m}{2}+|V|-m$.

Another useful observation can be derived from the fact that in a bipartite graph, any two vertices in the same part are an even distance apart and any two vertices not in the same part are an odd distance apart.

Observation 3. If $G$ is bipartite with a proper Skolem labelling $\mathcal{L}$ and $i$ is odd, then $\left|\mathcal{L}^{-1}(i)\right| \leq 2$.

The results of [4] and [6] provide upper bounds for the Skolem number of cycles and grid graphs by taking the proposed weak Skolem label and assigning each hook its own positive


Figure 1: $m$-balanced Skolem labelling with the fewest number of edges
integer. Therefore we obtain an upper bound for the Skolem number by adding the number of hooks to the highest integer used.

Many of the results in [4] rely on modular arithmetic, therefore we will often begin indices with 0 such as defining a path in the following manner.

Definition 1.4. Let $P_{n}$ be a path with vertices $\left\{v_{i} \mid 0 \leq i \leq n-1\right\}$ such that $\left\{v_{i}, v_{i+1}\right\}$ for $0 \leq i<n-2$ are the edges.

In this paper we find the Skolem number for all cycles in Section 2 and grid graphs in Section 3. In each of these sections we will make use of the following two proper Skolem labellings of a path as used in the graph in Figure 1.

Definition 1.5. If $k$ is even, there exists a proper Skolem labelling of $P_{k}$ with only the odd integers of $[k]$ such that for all $i \leq \frac{k}{2} \mathcal{L}\left(v_{i}\right)=k-(2 i-1)$ and for $i>\frac{k}{2} \mathcal{L}\left(v_{i}\right)=\mathcal{L}\left(v_{k-i+1}\right)$, called the odd Skolem labelled path as shown in Figure 2. Similarly, there exists a proper Skolem labelling of $P_{k+1}$ with only the even integers of $[k]$ and one hook such that for all $i \leq \frac{k}{2} \mathcal{L}\left(v_{i}\right)=k-(2 i-2), v_{\frac{k}{2}+1}$ is a hook, and for $i>\frac{k}{2}+1 \mathcal{L}\left(v_{i}\right)=\mathcal{L}\left(v_{k-i+2}\right)$, called the even Skolem labelled path as shown in Figure 3.


Figure 2: odd Skolem labelled path


Figure 3: even Skolem labelled path

## 2 Skolem Number of Cycles

Results found in [4] use slightly different language than we have established or used in other previous papers. Therefore we wish to summarise the relevant results using our established terminology. A Skolem circle of order $m$ is a weak Skolem labelled cycle on $2 m$ vertices. The following proposition leads to the results shown in this section.

Proposition 2.1. [4] A Skolem circle of order $m$ exists if and only if $m \equiv 0,1(\bmod 4)$.

This result is a direct consequence of the existence of a Skolem sequence shown in [10]. The existence of Skolem circles were found by noticing the "wrapping" of these sequences around a cycle produces a removable edge, which can be subdivided with a hook to create larger hooked Skolem labelled cycles. Furthermore Bubear and Hall show that for all orders of Skolem circles, there exists a labelling with a removable edge. We will take advantage of these specific removable edges in creating proper Skolem labellings in Theorem 2.2.

Let $C_{n}$ be the cycle on $n$ vertices, thus having a diameter of $\left\lfloor\frac{n}{2}\right\rfloor$. We make the following observation about the size of a set of equidistant vertices in $C_{n}$.

Observation 4. Consider the graph $C_{n}$ with a proper Skolem labelling $\mathcal{L}: V \rightarrow[m]$. For any integer $i \neq \frac{n}{3},\left|\mathcal{L}^{-1}(i)\right| \leq 2$, and $\left|\mathcal{L}^{-1}\left(\frac{n}{3}\right)\right| \leq 3$.

From this observation, we obtain a natural lower bound for the Skolem number of $C_{n}$. If $n \equiv 1,2(\bmod 3)$, then the lower bound for the Skolem number of $C_{n}$ is $\left\lceil\frac{n}{2}\right\rceil$, otherwise it is $\left\lceil\frac{n-3}{2}\right\rceil+1$. However, we can extend this a bit further and notice that for $n \equiv 0(\bmod 6)$, $\left\lceil\frac{n-3}{2}\right\rceil+1=\left\lceil\frac{n}{2}\right\rceil$. This observation, the results of [4], and a minor construction provide us with the Skolem number for $C_{n}$.

Theorem 2.2. The Skolem number of $C_{n}$ is:

- $\frac{n-3}{2}+1$ if $n \equiv 3,9,15$, or $21(\bmod 24)$;

- $\left\lceil\frac{n}{2}\right\rceil+1$ if $n \equiv 4,5,6,7,12,13,14,20,22$, or $23(\bmod 24)$.

Proof. Let the vertices of $C_{n}$ be $\left\{v_{i} \mid 0 \leq i \leq n-1\right\}$ and the edges be $\left\{v_{i}, v_{i+1}\right\}$ where the index is calculated modulo $n$.

When $n \equiv 3(\bmod 6)$ we can label the subgraph induced by $\left\{v_{i} \left\lvert\, 0 \leq i \leq\left\lfloor\frac{n}{2}\right\rfloor-1\right.\right\}$ as an odd Skolem labelled path and the subgraph induced by $\left\{v_{i}\left\lfloor\left\lfloor\frac{n}{2}\right\rfloor \leq i \leq n-1\right\}\right.$ as an even Skolem labelled path where the hook is labelled $\frac{n}{3}$, as shown in Figure 4. This achieves the lower bound $\left\lceil\frac{n-3}{2}\right\rceil+1$.


Figure 4: proper Skolem Labelling for $n \equiv 3(\bmod 6)$

So let $n \not \equiv 3(\bmod 6)$ and proceed with a case analysis modulo 8 .
First consider when $n$ is even. When $n \equiv 0$ or $2(\bmod 8)$, we see from Proposition 2.1 that $C_{n}$ is a Skolem circle, also known as a 2-balanced Skolem labelling, if and only if $\frac{n}{2} \equiv 0,1(\bmod 4)$. Hence when $\frac{n}{2} \equiv 0,1(\bmod 4)$ the Skolem number is $\frac{n}{2}$. Otherwise, if $n \equiv 4,6(\bmod 8)$, then $\frac{n}{2} \equiv 2,3(\bmod 4)$ and we find the Skolem number is $\frac{n}{2}+1$ via "wrapping" a hooked Skolem sequence around a cycle, which was shown to exist in [7].

Now consider when $n$ is odd. Suppose that $n \equiv 1$ or $3(\bmod 8)$. By the above argument, there exists a 2-balanced Skolem labelling of $C_{n-1}$. Since there is a removable edge, we can remove an edge from a labelled $C_{n-1}$ and not disturb the proper Skolem labelling [4], as shown in Figure 5a. Instead of removing such an edge, we wish to subdivide it with a single hook vertex, as shown in Figure 5b to create a $C_{n}$ with a proper Skolem labelling of order one greater than the Skolem number of $C_{n-1}$. Hence the Skolem number of $C_{n}$ is $\left\lceil\frac{n}{2}\right\rceil$ using the 2-balanced Skolem labelling of $C_{n-1}$.

(a) $C_{10}$ with a removed edge

(b) subdividing an edge of $C_{10}$

Figure 5: hooks for Skolem Labelling odd cycles

Finally, in a similar fashion, use the removable edge of a properly Skolem labelled $C_{n-1}$ for labelling $C_{n}$ when $n \equiv 5$ or $7(\bmod 8)$. This provides that the Skolem number of $C_{n}$ when $n \equiv 5$ or $7(\bmod 8)$ is $\left\lceil\frac{n}{2}\right\rceil+1$.

## 3 Skolem Number of Grid Graphs

First introduced in [2], a Skolem array of order $n$ is a $2 \times n$ array in which the distance between positions $(a, b)$ and $(c, d)$ is defined to be $|c-a|+|d-b|$, and each $i \in\{1,2, \ldots, n\}$ occurs in two positions which are distance $i$ apart.

Proposition 3.1. [2] A Skolem array of order $n$ exists if and only if $n \equiv 0,1(\bmod 4)$.
Before we discuss the similarities between types of graphs and the Skolem array, we first need to define some useful terms.

Definition 3.2. The Cartesian product of a $G$ and $H, G \square H$ is a graph with vertex set $V(G) \times V(H)$ and two vertices $\left(g_{1}, h_{1}\right),\left(g_{2}, h_{2}\right) \in V(G) \times V(H)$ are adjacent if and only if either $g_{1}=g_{2}$ and $h_{1}$ is adjacent to $h_{2}$ in $H$ or $h_{1}=h_{2}$ and $g_{1}$ is adjacent to $g_{2}$ in $G$.

A Skolem array is equivalent to a weak Skolem labelling of classes of grid graphs, hence the following propositions [6].

Proposition 3.3. [6] The ladder graph of length $n, P_{2} \square P_{n}$ has a weak Skolem labelling if and only if $n \equiv 0$ or $1(\bmod 4)$ and can be weak Skolem labelled with two hooks if and only if $n \equiv 2$ or $3(\bmod 4)$.

Furthermore, abstracting the concept to grid graphs yields:
Proposition 3.4. [6] The grid graph $P_{a} \square P_{b}$ with $a, b \geq 3$ has a weak Skolem labelling with $a b-2 a-2 b+4$ hooks.

Propositions 3.3 and 3.4 are under the assumption that at most two vertices are labelled with the same integer, but we are interested in proper Skolem labellings. However, since the number 2 is the only integer label that can exist more than twice in a proper Skolem labelling of $P_{2} \square P_{n}$, and the label 2 can appear at most three times we get the obvious corollary of Proposition 3.3.

Corollary 3.5. If $n \equiv 0$ or $1(\bmod 4)$, then there exists a 2 -balanced Skolem labelling of $P_{2} \square P_{n}$ and the Skolem number is $n$. Furthermore, the Skolem number of $P_{2} \square P_{n}$ for $n \equiv 2$ or $3(\bmod 4)$ is $n+1$.

Unfortunately a corollary for the Skolem number is not easily abstracted from Proposition 3.4. Therefore we turn our attention to finding the Skolem number of $P_{a} \square P_{b}$ by providing a design for an unbalanced Skolem labelling of $P_{a} \square P_{b}$. In addition, a python program, whose output is an array corresponding to this Skolem labelling, is given in Appendix B for precision of Theorem 3.6, which serves as the basis for the results in Theorems 3.7 and 3.8.

Since we are concerned with the distances between vertices within a grid graph, it is natural to observe the following about the taxi-cab metric applied to the integer lattice.

Observation 5. Let d be some label in a proper Skolem labelling in $P_{a} \square P_{b}$.

1. If $d$ is odd, there are at most two vertices labelled $d$ that are mutually distance $d$ apart.
2. If $d$ is even, there are at most four vertices labelled d that are mutually distance d apart.

Now we have a natural lower bound, calculated by taking the total number of vertices in $P_{a} \square P_{b}$, which is $a b$, and subtracting whenever an integer can appear multiple times according to the following observation:

Observation 6. Consider a graph $P_{a} \square P_{b}$, where $a \leq b$. Let $\mathcal{L}$ be a proper Skolem labelling of $P_{a} \square P_{b}$. Then

1. for all $d>a+b-2,\left|\mathcal{L}^{-1}(d)\right| \leq 1$,
2. for all positive odd integers $d$ such that $d \leq a+b-2,\left|\mathcal{L}^{-1}(d)\right| \leq 2$,
3. for all positive even integers $d$ such that $d \leq \frac{2}{3}(a+b-2)$ or $d \leq \min \{2 a-1, b-1\}$, $\left|\mathcal{L}^{-1}(d)\right| \leq 3$, and

## 4. for all positive even integers $d$ such that $d \leq \min (a, b),\left|\mathcal{L}^{-1}(d)\right| \leq 4$.

Without loss of generality, we will assume $a \leq b$. In the previous theorems, we have answered the question of the Skolem number for $P_{a} \square P_{b}$ whenever $a \leq 2$, Therefore for the remaining results we will assume $3 \leq a \leq b$. Appendix A illustrates a proper Skolem labeling with the Skolem number of $P_{a} \square P_{b}$ whenever $3 \leq a \leq 7$ and $a \leq b \leq 2 a+1$. These proper Skolem labelled graphs serve as the bases for finding the Skolem number of all grid graphs, since small values of $a$ and $b$ do not always follow the pattern of the labellings will show in this section.

With the exception of $(a, b)=(3,3)$ or $(3,4)$ or $(4,4)$ or $(3,6)$, whose Skolem number can obviously be extracted from the images in Appendix A, the following series of theorems determines the Skolem number for $P_{a} \square P_{b}$.

In the remaining theorems we will think of $P_{a} \square P_{b}$ as $a$ rows of $b$ vertices on the integer lattice where the lower left vertex is at the origin. Hence the vertices of $P_{a} \square P_{b}$ are $(i, j)$ for $0 \leq i \leq b-1$ and $0 \leq j \leq a-1$, where two vertices, $(k, \ell)$ and $(m, n)$, are adjacent if and only if $|k-m|+|\ell-n|=1$.

Theorem 3.6. Let $k=2\left\lfloor\frac{1}{3}(a+b-2)\right\rfloor$. If $2 \leq a-1 \leq b-1<2(a-1)$ and $b>4$, then the Skolem number of $P_{a} \square P_{b}$ is $(a-1)(b-1)+1-\frac{k}{2}-\left\lfloor\frac{a}{2}\right\rfloor$.

Proof. We have provided a labelling for all $a \leq 7$ in Appendix A as the following construction only satisfies the requirements for a proper Skolem labelling when $a>7$. For $a>7$, we construct a labelling which defines the assignment of each integer that appears more than once, then all hooks may be arbitrarily assigned with the remaining integers. The precise algorithm is provided in Appendix B, but we provide an explanation of this labeling construction here.

For our construction we begin with all even integers less than or equal to $a-1$, as these integers appear four times. Let each such integer be placed at the vertices that form horizontal and vertical axes centered at $\left(\left\lfloor\frac{a-1}{2}\right\rfloor,\left\lfloor\frac{a-1}{2}\right\rfloor\right)$; the center is denoted by an " $\times$ " in Figures 6, 7, and 8. Note that the vertex $(0,0)$ is in the bottom left corner of Figure 6 and vertex $(x, y)$ is in row $y$ column $x$ of the grid, just like the coordinate plane. Thus the vertices lying on the green cross in Figure 6 are labelled with these even integers.

Now consider the labels in $A=\{2 i \mid a-1<2 i \leq k\}$; these are the labels that that may appear on three vertices. Let two of the three vertices labelled $2 i \in A$ be ( $0, i$ ) and ( $i, 0$ ). The third vertex labelled $2 i \in A$ is the unique vertex in row $a-1$ that is exactly distance $2 i$ from the other two vertices labelled $2 i$. Hence the third vertex labelled $2 i$ is ( $x, a-1$ ), where $x=3 i-a+1$. Since $x=3 i-a+1<3\left(\left\lfloor\frac{1}{3}(a+b-2)\right\rfloor\right)-a+1<b-1$, we see that $(x, a-1) \in P_{a} \square P_{b}$ for all $2 i \leq k$. This means a subset of the vertices on the red segments in Figure 6 are assigned such labels.

The location of the labels that appear twice on the grid will be dependant on the parity of $a+b$. These labels are those from the sets $\{2 i+1 \mid 1 \leq 2 i+1 \leq a+b-2\}$ and $\{2 i \mid k<2 i \leq a+b-2\}$.

Case 1: Presume $a+b$ is odd
If $a+b$ is odd, then so is $a+b-2$, which is the diameter of the graph. Therefore $a+b-2$ has exactly two possibilities for the location of its labels; it must be in opposing corners of the grid. Since the top right corner of the grid, $(b-1, a-1)$, may already be labelled with


Figure 6: vertices labelled $2 i$ for $2 i \leq k$
some $2 i \in A$, as it belongs to the red line in Figure 6, we will label the top left, $(0, a-1)$, and lower right vertices, $(b-1,0)$ with $a+b-2$ as shown in Figure 7.


Figure 7: labels that appear twice in Case 1
Furthermore, the odd labels greater than $a-1,\{2 i+1 \mid a-1<2 i+1<a+b-2\}$, will reside on the vertices $\left\{(x, a-1) \left\lvert\, 0 \leq x<\left\lfloor\frac{a-1}{2}\right\rfloor\right.\right\} \cup\left\{\left(\left\lfloor\frac{a-1}{2}\right\rfloor-1, a-2\right),\left(\left\lfloor\frac{a-1}{2}\right\rfloor-1, a-3\right)\right\}$ and $\left\{(b-1, x) \left\lvert\, 0 \leq x<\left\lfloor\frac{a-1}{2}\right\rfloor\right.\right\} \cup\left\{\left(b-2,\left\lfloor\frac{a-1}{2}\right\rfloor-1\right) \cup\left\{\left(b-3,\left\lfloor\frac{a-1}{2}\right\rfloor-1\right)\right\}\right.$, highlighted in blue in Figure 7, in a format similar to an odd Skolem path. Finally we will let the remaining vertices of row $\left\lfloor\frac{a-1}{2}\right\rfloor-1$ form an odd Skolem path of length $b-2$ with the labels $\{2 i+1 \mid 1 \leq 2 i+1<2 a\}$ also highlighted in blue in Figure 7. Unfortunately, this path includes a vertex already labelled 2 at $\left(\left\lfloor\frac{a-1}{2}\right\rfloor,\left\lfloor\frac{a-1}{2}\right\rfloor-1\right)$, hence we will augment this labelling slightly, by moving the odd label at that vertex to one of two other nearby vertices. If $\left(\left\lfloor\frac{a-1}{2}\right\rfloor-1,\left\lfloor\frac{a-1}{2}\right\rfloor-1\right)$ is labelled 1 (this only occurs when $b=a$ or $b=a+1$ ), we will label $\left(\left\lfloor\frac{a-1}{2}\right\rfloor-1,\left\lfloor\frac{a-1}{2}\right\rfloor-2\right)$ with the label of $\left(\left\lfloor\frac{a-1}{2}\right\rfloor,\left\lfloor\frac{a-1}{2}\right\rfloor-1\right)$ in the odd Skolem path, otherwise $\left(\left\lfloor\frac{a-1}{2}\right\rfloor+1,\left\lfloor\frac{a-1}{2}\right\rfloor-2\right)$ will assume that label. This is shown by the large vertices highlighted in purple on Figure 7.

Finally, we must place the even labels that will occur twice. These are the labels $\{2 i \mid$ $k<2 i \leq a+b-2\}$. These will be assigned, in a format similar to an even Skolem path, to the vertices $\left\{(x, 0) \left\lvert\, 0 \leq x<\left\lfloor\frac{a-1}{2}\right\rfloor-1\right.\right\} \cup\left\{\left(\left\lfloor\frac{a-1}{2}\right\rfloor-1,1\right)\right\}$ and $\left\{(b-1, x) \left\lvert\,\left\lfloor\frac{a-1}{2}\right\rfloor<x<\right.\right.$ $a-2\} \cup\left\{\left(b-2,\left\lfloor\frac{a-1}{2}\right\rfloor+1\right),\left(b-3,\left\lfloor\frac{a-1}{2}\right\rfloor+1\right)\right\}$, highlighted in orange on Figure 7 .

Case 2: Presume $a+b$ is even
If $a+b$ is even, then so is $a+b-2$, which is the diameter of the graph. As in Case 1 , $a+b-2$ has exactly two possibilities for the location of its labels, but the top right corner of the grid may already be occupied, so we will label the top left and lower right vertices $a+b-2$. Figure 8 contains the same color scheme as previously described for Figure 7, where the remaining even integers less than $a+b-2$ are assigned to the orange highlighted vertices and the odd integers less than $a+b-2$ are assigned to the blue highlighted vertices, with a few exceptions described below.


Figure 8: labels that appear twicein Case 2

Similar to Case 1, the odd Skolem path shown in blue in Figure 8 includes the vertex $\left(\left\lfloor\frac{a-1}{2}\right\rfloor,\left\lfloor\frac{a-1}{2}\right\rfloor+1\right)$, which has already been labeled 2. However, in this case, the odd Skolem path also includes $\left(0,\left\lfloor\frac{a-1}{2}\right\rfloor+1\right)$ which has already been labeled $2\left\lfloor\frac{a-1}{2}\right\rfloor+2$. Therefore, we will again augment this path by moving the two labels of these vertices to $\left(\left\lfloor\frac{a-1}{2}\right\rfloor-1,\left\lfloor\frac{a-1}{2}\right\rfloor+2\right)$ or $\left(\left\lfloor\frac{a-1}{2}\right\rfloor+1,\left\lfloor\frac{a-1}{2}\right\rfloor+2\right)$, using the same condition as in Case 1, and $\left(1,\left\lfloor\frac{a-1}{2}\right\rfloor+2\right)$ respectively. This augmentation is illustrated by the large purple vertices in Figure 8.

Following this assignment of labels we have all integers less than or equal to $a+b-2$ appearing at least twice, even integers less than or equal to $k$ appearing at least three times, and even integers less than or equal to $a-1$ appearing 4 times. Therefore we have a proper Skolem labelling of order $a b-(a+b-2)-\frac{k}{2}-\left\lfloor\frac{a-1}{2}\right\rfloor$. So by Observations 5 and 6 we see that no other integer can appear more than it has already been assigned, hence the Skolem number of $P_{a} \square P_{b}$ is $(a-1)(b-1)+1-\frac{k}{2}-\left\lfloor\frac{a-1}{2}\right\rfloor$.

Theorem 3.7. If $b=2 a-1$, then the Skolem number of $P_{a} \square P_{b}$ is $a b-a-2 b+4-\left\lfloor\frac{a-1}{2}\right\rfloor$.

Proof. When $b=2 a-1$, we see that Observation 6 allows for the label $b-1$ to appear on up to three vertices. However, there is only one position up to isomorphism of vertices that suffice; $(0, a-1),(a-1,0)$, and $(b-1, a-1)$. Unfortunately this means that the top two "corner" vertices of the grid must be labelled $b-1$ which contradicts the criteria that $a+b-2$ must be the labels of opposing "corner" vertices.

By the same labelling scheme shown in Thoerem 3.6 as provided in Figures 6, 7, and 8, we assign each integer label to as many vertices as possible except for the integer $b-1$, which is only assigned to the vertices $(a-1,0)$ and $(b-1, a-1)$. Hence the Skolem number of $P_{a} \square P_{b}$ is $a b-(a+b-2)-(b-2)-\left\lfloor\frac{a-1}{2}\right\rfloor=a b-a-2 b+4-\left\lfloor\frac{a-1}{2}\right\rfloor$.
Theorem 3.8. If $a \neq 3, b \neq 6$, and $b>2 a-1$, then the Skolem number of $P_{a} \square P_{b}$ is $(a-1)(b-1)+1-2(a-1)-\left\lfloor\frac{a-1}{2}\right\rfloor$.
Proof. Let $s=(a-1)(b-1)+1-2(a-1)-\left\lfloor\frac{a-1}{2}\right\rfloor$. By Observation 6, we can see $s$ is a natural lower bound for the Skolem number of $P_{a} \square P_{b}$. Therefore, by induction, we will show there exists a proper Skolem labelling of order $s$ for $P_{a} \square P_{b}$. For $a=3$ we will use $b=7$ and 8 as the base cases, otherwise $b=2 a$ and $b=2 a+1$ serve as our base cases and show a proper Skolem labelling of order $(a-1)(b+1)+1-2(a-1)-\left\lfloor\frac{a-1}{2}\right\rfloor$ exists on $P_{a} \square P_{b+2}$ whenever a proper Skolem labelling of order $s$ exists on $P_{a} \square P_{b}$.

As before, Appendix A provides our base cases when $a \leq 7$, then Figures 9 and 10 show a proper Skolem labelling of $P_{a} \square P_{2 a}$ using the same color scheme as the labelling described in Theorem 3.6 with the vertices labelled $2 a-2$ appropriately. This labelling differs from that shown in Theorem 3.6 only in the use of the parity of $a+b$. A similar labelling is easily extracted from this design for $P_{a} \square P_{2 a+1}$, but for space purposes is not shown here.


Figure 9: proper Skolem labelling of order $s$ for $P_{a} \square P_{2 a}$ when a is even
For the inductive step, we notice that all integers that can be on three or four vertices in a proper Skolem labelling of $P_{a} \square P_{b+2}$ already exist three or four times respectively in a proper Skolem labelling of $P_{a} \square P_{b}$ whenever $b>2 a+1$. Let $s^{\prime}$ be the Skolem number of $P_{a} \square P_{b}$, where $\mathcal{L}^{\prime}$ is a proper Skolem labelling of order $s^{\prime}$.

Now create $\mathcal{L}$, a proper Skolem labelling of order $s$ of $P_{a} \square P_{b+2}$. Let the subgraph of $P_{a} \square P_{b+2}$ induced on the vertices $\{(i, j) \mid i \in\{1,2, \ldots b\}, j \in\{0,1, \ldots, a-1\}\}$ be labelled with


Figure 10: proper Skolem labelling of order $s$ for $P_{a} \square P_{2 a}$ when a is odd
$\mathcal{L}^{\prime}$ except for the vertices labelled $a+b$ and $a+b+1$. That is to say, $\mathcal{L}((i, j))=\mathcal{L}^{\prime}((i-1, j))$ for all $\{(i, j) \mid i \in\{1,2, \ldots, b\}, j \in\{0,1, \ldots, a-1\}\}$, whenever $\mathcal{L}^{\prime}((i, j)) \neq a+b+1$ or $a+b+2$. This ensures every label which can appear twice, except for $a+b+1$ and $a+b+2$ exists twice. So define $\mathcal{L}((0,0))=\mathcal{L}((b+2, a))=a+b+2$ and $\mathcal{L}((0, a))=\mathcal{L}((b+2,1))=a+b+1$. Finally, uniquely label each remaining vertex with the integers $s^{\prime}+2 \leq x \leq s$.

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## A Proper Skolem labels for small values of $a$.

A full labelling has been provided for $a=3$ and $a=4$ examples, but to showcase the pattern of these labellings that is used in Theorem 3.6 we left all hooks empty for $a=5,6$, and 7, but are each filled with a label that exists once.
$a=3$

| 4 | 2 | 5 |
| :--- | :--- | :--- |
| 2 | 3 | 2 |
| 1 | 2 | 4 |


| 4 | 2 | 6 | 5 |
| :--- | :--- | :--- | :--- |
| 2 | 3 | 2 | 1 |
| 5 | 2 | 4 | 3 |


| 5 | 2 | 4 | 1 | 6 |
| :--- | :--- | :--- | :--- | :--- |
| 2 | 3 | 2 | 1 | 5 |
| 6 | 2 | 7 | 3 | 4 |


| 4 | 6 | 9 | 2 | 4 | 7 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 1 | 2 | 3 | 2 | 8 |
| 7 | 1 | 4 | 2 | 5 | 6 |


| 7 | 5 | 2 | 4 | 3 | 6 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 2 | 10 | 2 | 1 | 5 | 7 |
| 8 | 4 | 2 | 3 | 1 | 4 | 9 |


| 9 | 7 | 2 | 4 | 3 | 6 | 8 | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 2 | 5 | 2 | 1 | 10 | 11 | 7 |
| 8 | 4 | 2 | 3 | 1 | 4 | 5 | 9 |

$a=4$

| 6 | 1 | 1 | 4 |
| :--- | :--- | :--- | :--- |
| 4 | 2 | 8 | 5 |
| 2 | 3 | 2 | 7 |
| 5 | 2 | 4 | 6 |


| 7 | 5 | 3 | 4 | 10 |
| :---: | :---: | :---: | :---: | :---: |
| 4 | 2 | 1 | 1 | 6 |
| 2 | 9 | 2 | 3 | 5 |
| 6 | 2 | 4 | 8 | 7 |


| 6 | 7 | 5 | 4 | 13 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 2 | 3 | 1 | 11 | 12 |
| 2 | 10 | 2 | 1 | 3 | 5 |
| 8 | 2 | 4 | 6 | 9 | 7 |


| 9 | 7 | 5 | 4 | 16 | 8 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 2 | 3 | 1 | 13 | 14 | 15 |
| 2 | 12 | 2 | 1 | 3 | 5 | 7 |
| 8 | 2 | 4 | 6 | 10 | 11 | 9 |


| 6 | 9 | 7 | 17 | 4 | 18 | 6 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 2 | 5 | 3 | 14 | 15 | 16 | 4 |
| 2 | 13 | 2 | 1 | 1 | 3 | 5 | 7 |
| 10 | 2 | 11 | 6 | 12 | 4 | 8 | 9 |


| 6 | 9 | 5 | 20 | 21 | 4 | 6 | 8 | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | 2 | 3 | 1 | 1 | 3 | 5 | 7 | 19 |
| 2 | 7 | 2 | 14 | 15 | 16 | 17 | 4 | 18 |
| 11 | 2 | 8 | 6 | 4 | 12 | 13 | 9 | 10 |

$a=5$

| 8 | 3 | 4 |  | 7 |
| :---: | :---: | :---: | :---: | :---: |
| 6 |  | 2 | 3 | 5 |
| 4 | 2 |  | 2 | 4 |
| 1 | 5 | 2 |  |  |
| 1 | 7 | 4 | 6 | 8 |


| 9 | 7 | 4 |  | 8 | 6 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 6 | 5 | 2 |  |  |  |
| 4 | 2 |  | 2 | 4 |  |
| 3 | 1 | 2 | 3 | 5 | 7 |
| 8 | 1 | 4 | 6 |  | 9 |


| 10 | 8 | 4 |  |  | 6 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 |  | 2 | 3 | 5 | 7 | 9 |
| 4 | 2 |  | 2 | 4 |  |  |
| 11 | 5 | 2 |  | 3 |  | 8 |
| 9 | 7 | 4 | 6 | 1 | 1 | 10 |


| 8 | 10 | 4 | 1 |  | 6 |  | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 3 | 2 | 1 | 3 |  | 7 | 9 |
| 4 | 2 |  | 2 | 4 |  |  |  |
| 5 | 7 | 2 |  |  | 5 |  |  |
| 11 | 9 | 4 | 6 | 8 | 12 | 13 | 10 |


| 8 | 11 | 4 |  |  | 6 |  |  | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 9 | 2 |  | 3 |  |  |  | 10 |
| 4 | 2 |  | 2 | 4 |  |  |  |  |
| 7 | 5 | 2 | 1 | 1 | 3 | 5 | 7 | 9 |
| 12 | 10 | 4 | 6 | 8 |  |  |  | 11 |


| 8 | 12 | 4 | 3 |  | 6 |  |  | 8 | 13 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 10 | 2 | 1 | 1 | 3 | 5 |  | 9 | 11 |
| 4 | 2 | 5 | 2 | 4 |  |  |  |  | 7 |
|  | 9 | 2 | 7 |  |  |  |  |  | 10 |
| 13 | 11 | 4 | 6 | 8 |  |  |  |  | 12 |


| 8 | 13 | 4 | 9 |  | 6 |  |  | 8 |  | 14 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 11 | 2 | 7 | 5 |  |  |  |  | 10 | 12 |
| 4 | 2 |  | 2 | 4 |  |  |  |  |  |  |
|  | 10 | 2 | 3 | 1 | 1 | 3 | 5 | 7 | 9 | 11 |
| 14 | 12 | 4 | 6 | 8 |  |  |  |  |  | 13 |

$a=6$

|  | 9 |  |  | 6 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | 7 | 4 | 1 | 1 | 8 |
| 6 | 5 | 2 |  |  |  |
| 4 | 2 |  | 2 | 4 |  |
| 3 |  | 2 | 3 | 5 | 7 |
| 10 | 8 | 4 | 6 |  | 9 |


| 10 |  | 5 |  | 6 |  | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 |  | 4 | 1 |  |  | 9 |
| 6 | 3 | 2 | 1 | 3 | 5 | 7 |
| 4 | 2 |  | 2 | 4 |  |  |
|  | 7 | 2 |  |  |  |  |
| 11 | 9 | 4 | 6 | 8 | 10 |  |


| 12 | 10 | 5 |  | 6 |  |  | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 |  | 4 | 1 |  |  |  | 11 |
| 6 | 3 | 2 | 1 | 3 | 5 | 7 | 9 |
| 4 | 2 |  | 2 | 4 |  |  |  |
|  | 7 | 2 |  |  |  |  | 10 |
| 11 | 9 | 4 | 6 | 8 |  |  | 12 |


|  | 12 | 7 |  | 6 |  |  | 8 | 13 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 10 | 4 | 3 |  |  |  |  | 11 |
| 6 | 5 | 2 | 1 | 1 | 3 | 5 | 7 | 9 |
| 4 | 2 |  | 2 | 4 |  |  |  |  |
|  | 9 | 2 |  |  |  |  |  | 10 |
| 13 | 11 | 4 | 6 | 8 |  |  |  | 12 |


| 13 | 11 |  |  | 6 |  |  | 8 |  | 14 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 9 | 4 |  |  |  |  |  |  | 12 |
| 6 | 7 | 2 | 1 | 1 |  |  |  |  | 10 |
| 4 | 2 | 3 | 2 | 4 |  |  |  |  |  |
| 5 | 10 | 2 |  | 3 | 5 | 7 | 9 | 11 | 13 |
| 14 | 12 | 4 | 6 | 8 |  |  |  |  |  |


| 10 | 14 |  |  | 6 |  |  | 8 |  |  | 15 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 12 | 4 | 5 |  |  |  |  |  | 9 | 13 |
| 6 | 7 | 2 | 3 | 1 | 1 | 3 | 5 | 7 |  | 11 |
| 4 | 2 | 9 | 2 | 4 |  |  |  |  |  |  |
|  | 11 | 2 |  |  |  |  |  |  |  | 12 |
| 15 | 13 | 4 | 6 | 8 | 10 |  |  |  |  | 14 |


| 10 | 15 |  |  | 6 |  |  | 8 |  |  | 10 | 16 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 13 | 4 | 9 |  |  |  |  |  |  |  | 14 |
| 6 | 11 | 2 | 7 | 5 |  |  |  |  |  |  | 12 |
| 4 | 2 |  | 2 | 4 |  |  |  |  |  |  |  |
|  | 12 | 2 | 3 | 1 | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
| 16 | 14 | 4 | 6 | 8 | 10 |  |  |  |  |  | 15 |


| 10 | 16 |  |  | 6 |  |  | 8 |  |  | 10 |  | 17 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 14 | 4 | 7 |  |  |  |  |  |  |  | 11 | 15 |
| 6 | 12 | 2 | 5 | 3 | 1 | 1 | 3 | 5 | 7 | 9 |  | 13 |
| 4 | 2 | 9 | 2 | 4 |  |  |  |  |  |  |  |  |
|  | 13 | 2 | 11 |  |  |  |  |  |  |  | 12 | 14 |
| 17 | 15 | 4 | 6 | 8 | 10 |  |  |  |  |  |  | 16 |

$a=7$

| 12 |  |  | 6 |  |  | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 |  | 3 | 4 |  |  | 11 |
| 8 | 1 | 1 | 2 | 5 | 7 | 9 |
| 6 | 4 | 2 |  | 2 | 4 | 6 |
|  |  | 3 | 2 |  |  |  |
|  |  | 5 | 4 |  |  |  |
| 11 | 9 | 7 | 6 | 8 | 10 | 12 |


| 12 |  |  | 6 |  |  | 8 | 13 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 |  | 1 | 4 |  |  |  | 11 |
| 8 | 3 | 1 | 2 | 3 | 5 | 7 | 9 |
| 6 | 4 | 2 |  | 2 | 4 | 6 |  |
|  |  | 5 | 2 |  |  |  |  |
|  |  | 7 | 4 |  |  |  |  |
| 13 | 11 | 9 | 6 | 8 | 10 | 12 |  |


| 12 | 13 | 11 | 6 |  |  | 8 |  | 14 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 |  | 9 | 4 |  |  |  |  |  |
| 8 |  | 7 | 2 |  |  |  |  |  |
| 6 | 4 | 2 |  | 2 | 4 | 6 |  |  |
|  | 5 | 3 | 2 | 1 | 3 | 5 | 7 | 9 |
|  |  |  | 4 | 1 |  |  |  | 11 |
| 14 |  |  | 6 | 8 | 10 | 12 |  | 13 |


| 15 | 13 | 11 | 6 |  |  | 8 |  |  | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 |  | 9 | 4 |  |  |  |  |  | 14 |
| 8 |  | 7 | 2 |  |  |  |  |  | 12 |
| 6 | 4 | 2 |  | 2 | 4 | 6 |  |  |  |
|  | 5 | 3 | 2 | 1 | 3 | 5 | 7 | 9 | 11 |
|  |  |  | 4 | 1 |  |  |  |  | 13 |
| 14 | 12 |  | 6 | 8 | 10 |  |  |  | 15 |


|  | 15 | 13 | 6 |  |  | 8 |  |  | 10 | 16 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 |  | 11 | 4 |  |  |  |  |  |  | 14 |
| 8 |  | 9 | 2 |  |  |  |  |  |  | 12 |
| 6 | 4 | 2 |  | 2 | 4 | 6 |  |  |  |  |
|  | 7 | 5 | 2 | 1 | 1 | 3 | 5 | 7 | 9 | 11 |
|  |  |  | 4 | 3 |  |  |  |  |  | 13 |
| 16 | 14 | 12 | 6 | 8 | 10 |  |  |  |  | 15 |


| 17 | 15 | 13 | 6 |  |  | 8 |  |  | 10 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 |  | 11 | 4 |  |  |  |  |  |  |  | 16 |
| 8 |  | 9 | 2 |  |  |  |  |  |  | 12 | 14 |
| 6 | 4 | 2 |  | 2 | 4 | 6 |  |  |  |  |  |
|  | 7 | 5 | 2 | 1 | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
|  |  |  | 4 | 3 |  |  |  |  |  |  | 15 |
| 16 | 14 | 12 | 6 | 8 | 10 |  |  |  |  |  | 17 |


| 12 | 17 | 15 | 6 |  |  | 8 |  |  | 10 |  |  | 18 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 |  | 13 | 4 |  |  |  |  |  |  |  |  | 16 |
| 8 |  | 11 | 2 |  |  |  |  |  |  |  |  | 14 |
| 6 | 4 | 2 |  | 2 | 4 | 6 |  |  |  |  |  |  |
|  | 9 | 7 | 2 | 3 | 1 | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
|  |  |  | 4 | 5 |  |  |  |  |  |  |  | 15 |
| 18 | 16 | 14 | 6 | 8 | 10 | 12 |  |  |  |  |  | 17 |


| 12 | 18 | 16 | 6 |  |  | 8 |  |  | 10 |  |  | 12 | 19 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 |  | 14 | 4 | 5 |  |  |  |  |  |  |  |  | 17 |
| 8 | 9 | 7 | 2 | 3 | 1 | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
| 6 | 4 | 2 |  | 2 | 4 | 6 |  |  |  |  |  |  |  |
|  |  | 11 | 2 |  |  |  |  |  |  |  |  |  | 14 |
|  |  | 13 | 4 |  |  |  |  |  |  |  |  |  | 16 |
| 19 | 17 | 15 | 6 | 8 | 10 | 12 |  |  |  |  |  |  | 18 |


| 12 | 19 | 17 | 6 |  |  | 8 |  |  | 10 |  |  | 12 |  | 20 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 |  | 15 | 4 |  |  |  |  |  |  |  |  |  |  | 18 |
| 8 |  | 13 | 2 |  |  |  |  |  |  |  |  |  | 14 | 16 |
| 6 | 4 | 2 |  | 2 | 4 | 6 |  |  |  |  |  |  |  |  |
|  | 11 | 9 | 2 | 5 | 3 | 1 | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
|  |  | 14 | 4 | 7 |  |  |  |  |  |  |  |  |  | 17 |
| 20 | 18 | 16 | 6 | 8 | 10 | 12 |  |  |  |  |  |  |  | 19 |

## B Python Program

Below is the Python code for the program used to build the labelling for the labels that appear more than once in Theorem 3.6. For simplicity all hooks are labelled as 0 . To see the full code, go to https://github.com/cecilartavion/skolem_labelling_cartesian_ products_of_paths.

Currently the code is set up to build a $30 \times 40$ matrix for a $P_{30} \square P_{40}$. Change the values of $m$ and $n$ to change the grid graph produced.

```
import numpy as np
a=30
b}=4
def topleft_blue(x):
    if x>a+b-2*int((a-1)/2)-2:
        return a-1,int ((a+b-2)/2)-int(x/2)
    elif x>a+b-2*int((a-1)/2)-3-((a+b)%2)*2:
        return a-1+int(x/2)-int ((a+b-2*int ((a-1)/2))/2), int ((a-1)/2) -1
def bottomright_blue(x):
    if x>a+b-2*int ((a-1)/2):
        return int((a+b-2)/2)-int(x/2),b-1
    else:
        return int ((a-1)/2)-1,b-1-int ((a+b-2*int ((a-1)/2))/2)+int (x/2)
def bottomleft_orange(x):
    if x>a+b-2*int((a-1)/2)-3:
        return 0,int ((a+b-2)/2)-int((x+1)/2)
    elif x>a+b-2*int((a-1)/2)-4:
        return int((a+b-2*int((a-1)/2))/2)-int((x+1)/2),int ((a-1)/2)-1
def topright_orange(x):
    if x>a+b-2*int((a-1)/2):
        return a-2-int ((a+b-2)/2)+int ((x+1)/2),b-1
    else:
        return int ((a-1)/2)+1,b-1-int ((a+b-2*int ((a-1)/2))/2)+int ((x+1)/2)
    -1+((a-1)%2)
def top_red (x):
    return a-1,int (3*x/2-a+1)
def bottom_red(x):
```

```
    return 0,int(x/2)
def topleft_red(x):
    return int(x/2),0
def leftsmall_blue(x):
    return int ((a-1)/2)-1,-2+int((a+b-2*int ((a-1)/2)-1)/2)-int (x/2)-((a+b)%2)
    *((a-1)%2)
def leftsmall_orange(x):
    return int ((a-1)/2)+1,-2+int ((a+b-2*int ((a-1)/2)-1)/2)-int (x/2)
if b}<=2*a
    k = 2*int((a+b-2)/3)
else:
    k=2*a-2
print('}\textrm{a}={},\textrm{b}={},\textrm{k}={}'.format(\textrm{a},\textrm{b},\textrm{k})
mat = np.zeros ((a,b))
for t in range(int ((a-1)/2)):
    mat[int((a-1)/2), int ((a-1)/2)-t-1]= 2*(t+1)
    mat[int((a-1)/2)-1-t,int((a-1)/2)]= 2*(t+1)
    mat[int((a-1)/2),int((a-1)/2)+2+t-1]= 2*(t+1)
    mat[int((a-1)/2)+1+t,int((a-1)/2)]= 2*(t+1)
for t in range(1,(a+b-2*int((a-1)/2)-2),2):
    if ((a+b)%2)==1:
        if int((a-1)/2)=leftsmall_blue(t)[1]:
            mat[tuple(leftsmall_blue(t)+np.array (( -1,1))) ]=t
        else:
            mat[leftsmall_blue(t)]=t
    else:
        if int((a-1)/2)=leftsmall_orange(t)[1]:
            mat[tuple(leftsmall_orange(t)+np.array ((1, 1)))]=t
        elif leftsmall_orange(t)[1]==0:
            mat[tuple(leftsmall_orange(t)+np.array ((1,1)))] = t
        else:
            mat[leftsmall_orange(t)]=t
for t in range(1-((a+b-1)%2),a+b-1,2):
    if t>a+b-2*int ((a-1)/2)-3-((a+b)%2)*2*((a-1)%2):
        if ((a+b)%2)==1:
            mat[topleft_blue(t)] = t
        if ((a+b)%2)==0:
            mat[bottomleft_orange(t-1)]=t-1
    if int ((a-1)/2)==bottomright_blue(t)[1] and ((a+b)%2)==1:
        mat[tuple(bottomright_blue(t)+np.array ((-1, -1)))] = t
    elif int((a-1)/2)==topright_orange(t)[1] - ((a+b)%2)*(a%2) and ((a+b)%2)==0
    and t>0:
        mat[tuple(topright_orange(t)+np.array ((1, - 1)))] = t-1
    else:
        if ((a+b)%2)==1:
            mat[bottomright_blue(t)] = t
        if ((a+b)%2)==0 and t>0:
            mat[topright_orange(t-1)] = t-1
    if t-1>k:
```

```
            if ((a+b)%2)==1:
        mat[bottomleft_orange(t-1)]=t-1
        mat[topright_orange(t-1)]=t-1
        if ((a+b)%2)==0:
        mat[topleft_blue(t)] = t
        mat[bottomright_blue(t)] = t
for t in range(a+(a%2),k+1,2):
    mat[top_red (t)]=t
    mat[bottom_red(t)]=t
    mat[topleft_red(t)]=t
num_digits = len(str(int(mat.max())))
for p in range(len(np.flip(mat.astype(int),0))):
    print('['+'''.join('%0*s' % (num_digits,i) for i in np.flip(mat.astype(int
    ),0)[p].astype(str))+']'')
```

