Theory and Applications of Graphs

Volume 8 | Issue 2 | Article 1

July 2021

The Color Number of Cubic Graphs Having a Spanning Tree with a Bounded Number of Leaves

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Supported by DOST-ASTHRDP, Philippines Supported by JSPS KAKENHI, Grant-in-Aid for Scientific Research(C), Grant Number 18K03391

Abstract

The color number $c(G)$ of a cubic graph G is the minimum cardinality of a color class of a proper 4-edge-coloring of G. It is well-known that every cubic graph G satisfies $c(G) = 0$ if G has a Hamiltonian cycle, and $c(G) \leq 2$ if G has a Hamiltonian path. In this paper, we extend these observations by obtaining a bound for the color number of cubic graphs having a spanning tree with a bounded number of leaves.

1 Introduction

Edge-colorings of cubic graphs were extensively studied in the past few decades. Several theorems and conjectures on edge-colorings of cubic graphs were formulated.

In this paper, we only deal with simple graphs, that is, graphs without multiple edges nor loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *k-edge-coloring* of G is a mapping $f : E(G) \to \{1, 2, \ldots, k\}$. If $f(e) \neq f(e')$ for any two adjacent edges e and e', then f is a *proper k-edge-coloring* of G. A graph is *k-edge-colorable* if it has a proper k-edge-coloring. The *edge-chromatic number* $\chi'(G)$, also known as the *chromatic index*, of a graph G is the least k such that G is k-edge-colorable. Let $\Delta(G)$ be the maximum degree of G. Vizing's Theorem [8] states that the edge-chromatic number of each graph G is $\Delta(G)$ or $\Delta(G) + 1$. By Vizing's Theorem, the set of cubic graphs is divided in two classes: a cubic graph is said to be *class* 1 if $\chi'(G) = 3$ and *class* 2 if $\chi'(G) = 4$.

Class 2 cubic graphs, in particular *snarks* (which are connected class 2 cubic graphs with certain connectivity and girth conditions), are closely related to several famous conjectures such as the Cycle Double Cover Conjecture [7] and Berge-Fulkerson Conjecture [3]. For this reason, they have caught the attention of many researchers. To study such conjectures, it is important to measure how far apart a given class 2 cubic graph is from being class 1. A natural measure is the *color number* or sometimes called the *resistance* of a cubic graph G, denoted by $c(G)$, which is the minimum number of edges whose removal from G yields a 3-edge-colorable graph. The color number is equivalent to the minimum cardinality of a color class of a proper 4-edge-coloring of G. Note that a cubic graph G is 3-edge-colorable if and only if $c(G) = 0$. See [2] for more such measures.

It is well-known that every cubic graph with a Hamiltonian cycle is 3-edge-colorable, that is, $c(G) = 0$. This fact can be extended to the following, see [9, p. 37].

Theorem 1 *Let G be a cubic graph. If G has a Hamiltonian path, then $c(G) \leq 2$.*

Note that the bound $c(G) \leq 2$ is best possible: for example, the Petersen graph has a Hamiltonian path and the color number is exactly two. With this relation in mind, we are interested in a property related to Hamiltonicity such that all cubic graphs satisfying that property have a small color number. In this paper, we show that the existence of a spanning tree with a bounded number of leaves is in fact such a property. The following is the main theorem, which is an extension of Theorem 1.

Theorem 2 *Let k be an integer at least 2, and let G be a cubic graph. If G has a spanning tree with at most k leaves, then $c(G) \leq 2k - 2$.*

The proof of Theorem 1 is easy: we first color all edges in a Hamiltonian path alternately by blue and green. For each endvertex v of the Hamiltonian path, choose one edge incident to v, and color the two chosen edges by violet. All other edges form a matching, so we can color them by red. This gives a proper 4-edge-coloring with only two edges of violet color. However, the proof of Theorem 2 is not so simple since the more violet edges required, the more careful one has to be so that the chosen violet edges actually form a matching. In our proof, we solve this issue by taking a path-factor, see Lemma 4.

The paper is organized as follows: the proof of Theorem 2 is given in the next section. In Section 3, we discuss the best possibility of the bound in Theorem 2 and the relation to a well-known parameter called the *oddness* of a cubic graph.

2 Proof of the main theorem

The main idea for the proof of Theorem 2 is to use a *path-factor* of a graph, which is a spanning subgraph F such that every component in F is a path of order at least two. Note that a path-factor can also be regarded as a set of pairwise vertex-disjoint paths of order at least two, which covers the vertex set of the graph. Path-factors of graphs have been studied, for example, in [5]. We use the following lemma to find a path-factor with few paths in a tree having a bounded number of leaves.

To state our lemma, we need the following definition. A tree T is said to be *special* if it is recursively obtained by the following operations.

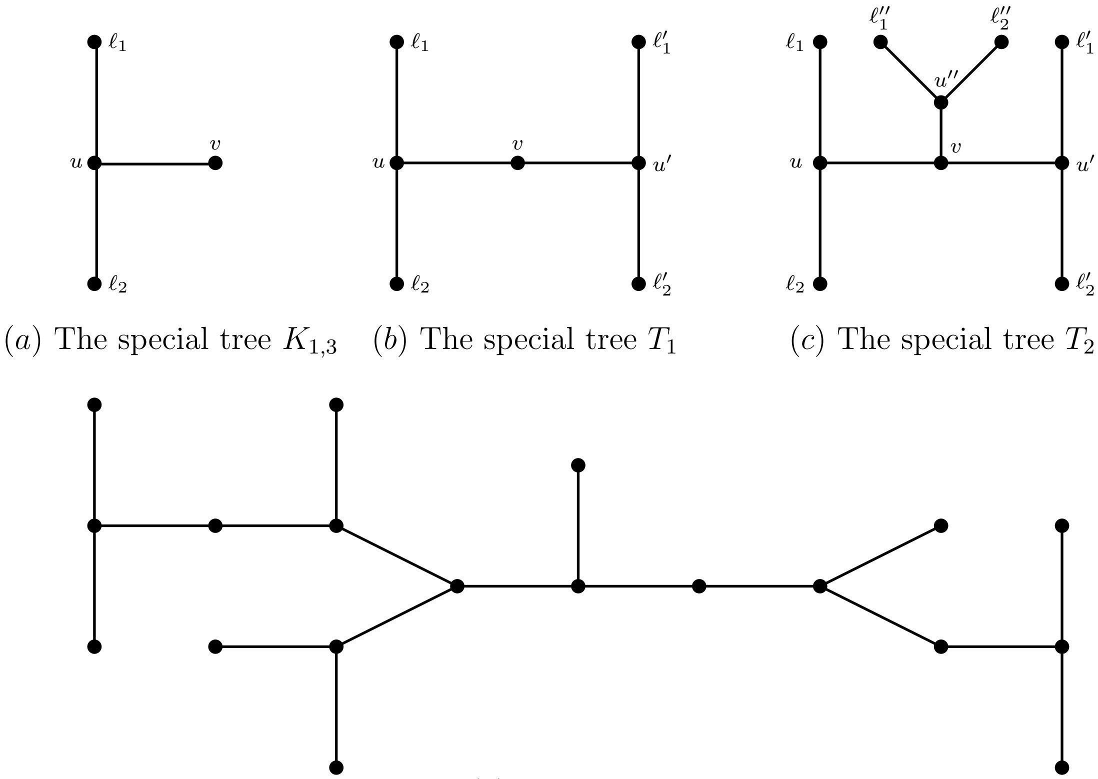
- The trivial tree, that is the tree consisting of only one vertex, is a special tree.
- Let T be a special tree and let v be a vertex in T of degree at most two. If we add three new vertices u, ℓ_1, ℓ_2 together with three new edges $uv, u\ell_1, u\ell_2$, then the obtained tree is also special.

Some examples of special trees are shown in Figure 1. The graph $K_{1,3}$ in Figure 1 (a) is the second smallest special tree. The trees T_1 and T_2 in Figure 1 (b) and (c) are special trees obtained from the trees in (a) and (b), respectively, by applying the above operation to the corresponding vertices. The tree T in Figure 1 (d) is a special tree obtained by recursively applying the above operation. Note that in the operation to construct special trees, the vertices v and u have degree at most three and exactly three in the obtained tree, respectively. In addition, it is easy to see the following fact, which will be used later.

Fact 3 *In any special tree T, no vertex of degree at most two is adjacent with a vertex of degree at most two.*

Lemma 4 *Let T be a tree of maximum degree at most three. If T is not special, then T has a path-factor with at most $k-1$ paths, where k is the number of leaves in T.*

Proof of Lemma 4. Let T be a tree of maximum degree at most three that is not special. We prove this lemma by induction on k, where k is the number of leaves in T. If $k = 2$, the statement trivially holds. Thus, we may assume that $k \geq 3$.



(a) The special tree $K_{1,3}$ (b) The special tree T_1 (c) The special tree T_2

(d) A special tree T

Figure 1: Examples of special trees

For each leaf ℓ, we take a path Q_ℓ in T from ℓ to the vertex of degree three such that Q_ℓ contains no other vertex of degree three. Such a path is uniquely determined. Since $k \geq 3$, there are two leaves ℓ_1 and ℓ_2 such that Q_{ℓ_1} and Q_{ℓ_2} share an endvertex, say u.

Suppose first that at least one of Q_{ℓ_1} and Q_{ℓ_2}, say Q_{ℓ_1}, has three or more vertices. Let $T' = T \setminus V\big(Q_{\ell_1} \setminus \{u\}\big)$. Note that u has degree two in T', and hence T' has at most $k-1$ leaves. Furthermore, since the neighbor of ℓ_2 in T' is contained in Q_{ℓ_2}, ℓ_2 is adjacent with a vertex of degree two in T'. It follows from Fact 3 that T' is not a special tree. Thus, by the induction hypothesis, T' has a path-factor F' with at most $k-2$ paths. Consequently, by adding $Q_{\ell_1} \setminus \{u\}$ to F' we obtain a path-factor of G with at most $k-1$ paths.

Therefore, we may assume that both Q_{ℓ_1} and Q_{ℓ_2} have two vertices. Let P be the path obtained by the concatenation of Q_{ℓ_1} and Q_{ℓ_2} at u. Note that P consists of the three vertices ℓ_1, u and ℓ_2. Let $T' = T \setminus V(P)$. Note that T is obtained from T' by adding three vertices u, ℓ_1, ℓ_2 together with three new edges $uv, u\ell_1, u\ell_2$, where v is the neighbor of u in T other than ℓ_1 and ℓ_2. Thus, if T' is special, then it follows from the definition of special trees that T is also special, a contradiction. Therefore, we may assume that T' is not special. By the induction hypothesis, T' has a path-factor F' with at most $k-2$ paths. As in the previous case, we obtain a path-factor with at most $k-1$ paths by adding P to F'. This completes the proof of Lemma 4. $\square$

The converse of Lemma 4 can be proved by standard induction on the number of steps we performed when we construct special trees. Since we do not use this statement for the proof of our main theorem, we leave its proof for the reader.

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. By Theorem 1, we may assume that $k \geq 3$. Let G be a cubic graph. Assume that G has no spanning tree with at most $k-1$ leaves, and let T be a spanning tree of G with k leaves. We first prove the following claims.

Claim 1 *The leaves in T are pairwise nonadjacent in G.*

Proof: Suppose that two leaves ℓ_1 and ℓ_2 are adjacent in G. Then by adding the edge $\ell_1\ell_2$ into the tree T, we obtain a spanning subgraph of G with exactly one cycle. Since $k \geq 3$, the obtained cycle contains an edge incident with a vertex of degree three. By removing this edge, we obtain a spanning tree of G with at most $k-1$ leaves, a contradiction. Thus, any two leaves are nonadjacent. ■

Claim 2 *T is not a special tree.*

Proof: Suppose on the contrary that T is a special tree. Note that T has k leaves, which implies that the number of vertices of degree three is $k-2$. By Fact 3, the vertices of degree two are adjacent to the vertices of degree three only. If there are at least $k-2$ vertices of degree two, then there is a cycle in T, a contradiction. Therefore, the number of vertices of degree two in T is at most $k-3$.

By Claim 1, exactly $2k$ edges in $G \setminus E(T)$ are incident with the leaves of T. Furthermore, by the same claim, the endvertices of each one of these $2k$ edges are a leaf of T and a vertex of degree two in T. Therefore, there are at least $2k$ edges in $G \setminus E(T)$ that are incident with vertices of degree two in T. However, since there are at most $k-3$ vertices of degree two in T, this is impossible. Hence, T is not a special tree. ■

By Lemma 4, T has a path-factor F with at most $k-1$ paths. Note that F is also a path-factor in G. We take such a path-factor of G with as few paths as possible. Let S be the set of endvertices of paths in F. Note that $|S| \leq 2k-2$. By the choice of F, no two vertices in S are adjacent in G except for the case when they are endvertices of the same path in F.

First, we color the edges of the paths in F with blue and green alternately, and let $H = G \setminus E(F)$. Note that each vertex in S (respectively not in S) has degree exactly two (respectively one) in H. Since no two vertices in S can be adjacent except for the case when they are endvertices of the same path in F, H is a vertex-disjoint union of paths each of which has at least two vertices and at most four vertices. Then we can color all edges in H with red and violet alternately, and in particular, if we use violet on as few edges as possible, every violet edge is incident to a vertex in S. Thus, there are at most $2k-2$ violet edges, and this completes the proof of Theorem 2. □

3 Conclusion

In this paper, we have shown Theorem 2 which states that the color number of any cubic graph containing a spanning tree with at most k leaves is at most $2k-2$, which is an extension of Theorem 1. Due to the next theorem, the bound "$2k-2$" is best possible.

Theorem 5 *For each $k \geq 2$, there is a 3-edge-connected cubic graph G such that G has a spanning tree with k leaves and $c(G) = 2k-2$.*

Proof: When $k = 2$, the Petersen graph satisfies all desired conditions. Hence, we assume that $k \geq 3$. Let P^* be the Petersen graph with one vertex removed. It is easy to see that P^* does not admit a 3-edge-coloring. Let $n = k-1$. For $i \in \{1, 2, \ldots, n\}$, let P_i and P'_i be copies of P^*, and let u_i, v_i, w_i (respectively u'_i, v'_i, w'_i) be the vertices of degree two in P_i (respectively P'_i). For any $i \in \{1, 2, \ldots, n\}$, add edges $w_i u_{i+1}, v_i v'_i, u'_i w'_{i+1}$, where addition in the subscript is taken modulo n, to obtain a 3-edge-connected cubic graph G as shown in Figure 2.

Consider a 4-edge-coloring of G. Since each copy of P^* must contain all of the four colors, we see that $c(G) \geq 2n$. On the other hand, it is straightforward to check that G admits a 4-edge-coloring for which there are only $2n$ edges of the fourth color (so that each copy of P^* contains one edge of the fourth color). This implies that $c(G) = 2n = 2k-2$.

Thus, it only remains to show that G has a spanning tree with k leaves. It is easy to see the following (which is also portrayed in Figure 3).

- P_1 contains a spanning subgraph Q_1 such that u_1 is a vertex of degree 0 in Q_1 and $Q_1 - u_1$ is a Hamiltonian path in $P_1 - u_1$ with endvertices v_1 and w_1.
- P'_1 contains a spanning subgraph Q'_1 corresponding to Q_1.
- For $2 \leq i \leq n$, P_i has two vertex-disjoint paths R_i and S_i such that R_i is of order 4 and its endvertices are u_i and w_i, and S_i is of order 5 and one of its endvertices is v_i.
- For $2 \leq i \leq n$, P'_i has a spanning tree T'_i such that its leaves are precisely u'_i, v'_i and w'_i.

Then, by adding the edges $w_i u_{i+1}, v_i v'_i, u'_i w'_{i+1}$, for each $i \in \{1, 2, \ldots, n\}$, to the subgraph $Q_1 \cup Q'_1 \cup \bigcup_{i=2}^{n} (R_i \cup S_i \cup T'_i)$, we obtain a spanning tree T as shown in Figure 3. Once again, we remark that addition in the subscript is taken modulo n. Note that each P_i, with $2 \leq i \leq n$, contains exactly one leaf that is an endvertex of S_i other than v_i, and there are no other leaves except for u_1 and u'_1. This implies that T has $n+1 = k$ leaves. This completes the proof. $\square$

Before closing this paper, we discuss the relation between the color number and the oddness of a cubic graph. The *oddness* of a cubic graph G, denoted by $\omega(G)$, is the smallest number of odd cycles in a 2-factor of G, where a 2-*factor* is a spanning subgraph in which every vertex has degree 2. This is a measure of how far apart a given class 2 cubic graph is from being class 1, other than the color number. Since it has been shown that cubic graphs with small oddness have several good properties (see [4] for example), the oddness has been widely studied.

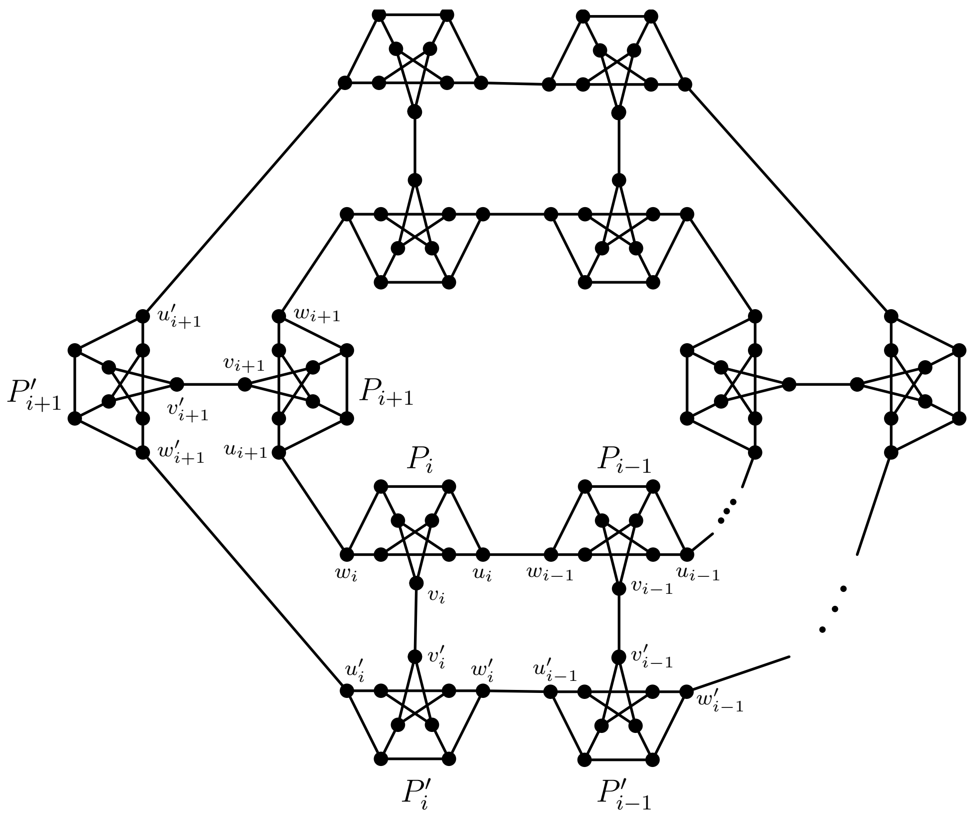


Figure 2: A 3-edge-connected cubic graph G

For a cubic graph G, it is clear that $\omega(G) \geq c(G)$ from the definition, and, in particular, it is shown that $\omega(G) = 2$ if and only if $c(G) = 2$ [6, Lemma 2.5]. Therefore, it is natural to think whether we can obtain a bound on $\omega(G)$, instead of the bound on $c(G)$ in Theorem 2, given that G has a spanning tree with at most k leaves. We leave this as an open problem.

Problem 6 *Find an upper bound on $\omega(G)$ for a cubic graph G having a spanning tree with at most k leaves.*

Note that the gap between $\omega(G)$ and $c(G)$ can be arbitrarily large even for snarks (that is, for cubic graphs assuming certain connectivity conditions). In fact, Allie [1] proved that there is no constant k such that $\omega(G) \leq kc(G)$ holds for every bridgeless cubic graph G.

On the other hand, the cubic graph G constructed in the proof of Theorem 5 satisfies $\omega(G) = 2k - 2$. This can be seen as follows (we use the same notation as in the proof of Theorem 5). Each P_i contains vertex-disjoint subgraphs C_i and Q_i such that C_i is a cycle of order 5 and Q_i is a path with endvertices u_i and w_i of order 4. A cycle of order $4n$ is obtained by taking the union of Q_i and the edges $w_i u_{i+1}$, for each $i \in \{1, \ldots, n\}$, where addition in the subscript is taken modulo n. This cycle and the n cycles C_i contain all vertices in $\bigcup_{i=1}^{n} P_i$. Similarly, we can take the cycle of order $4n$ and the n cycles of order 5 symmetrically in $\bigcup_{i=1}^{n} P'_i$, and hence G has a 2-factor with $2n$ odd cycles. This shows that $\omega(G) \leq 2n$.

Since any 2-factor of G must contain a cycle of order 5 or 9 in each P_i and P'_i, we also have $\omega(G) = 2n$. Therefore, as a possible answer to Problem 6, it might be true that $\omega(G) \leq 2k - 2$ for any cubic graph G containing a spanning tree with at most k leaves.

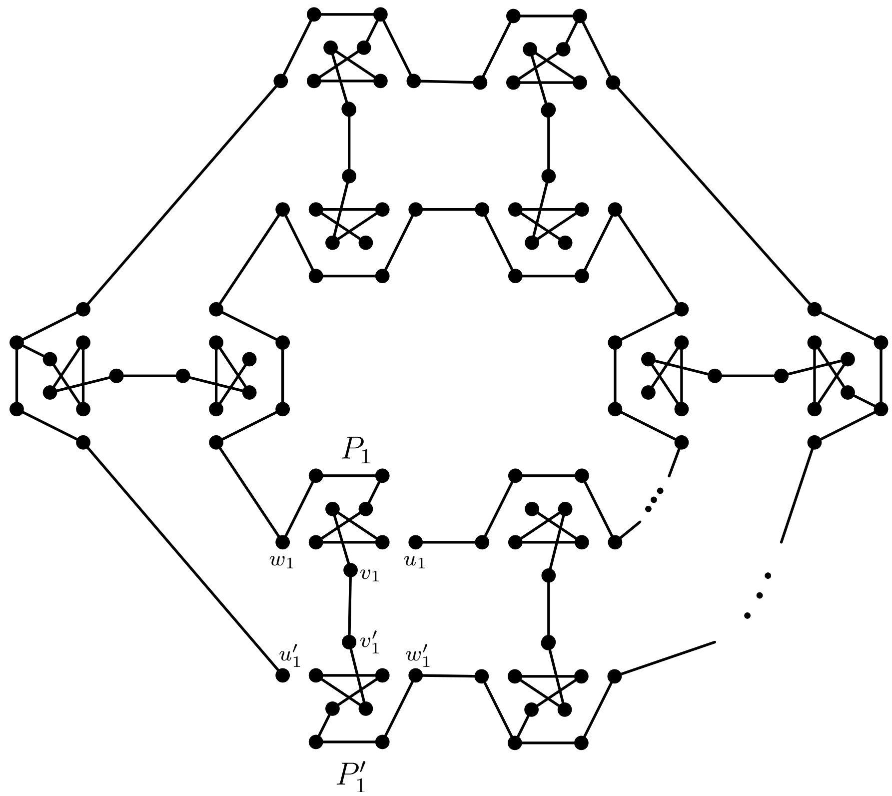


Figure 3: A spanning tree T of the cubic graph G in Figure 2

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