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The Roller-Coaster Conjecture

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The Roller-Coaster Conjecture

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Abstract

The work presented in this thesis is rooted in graph theory and algebra. An independent set in a graph G is a set of vertices in which no two vertices are adjacent. Let s_i denote the number of independent sets of size i and define the independence polynomial of any graph G to be $S(G,z) = \sum_{i=0}^{\alpha} s_i z^i$, where α represents the size of the largest independent set in G. This paper examines the possible patterns of the coefficients of the independence polynomial for any graph G. We begin by reviewing terms and definitions necessary to discuss our work. We discuss the useful results of Alavi, Madle, Schwenk, and Erdős that states if we allow G to be any graph then the coefficients of the independence polynomial of G have no particular ordering. Further, one can order the coefficients in any desired manner. This led mathematicians to restrict themselves to a particular class of graphs, specifically those which are well-covered. Research conducted on well-covered graphs has culminated with work on the roller-coaster conjecture since the coefficients of the independence polynomial of G have a pattern up to the $s_{\lfloor \frac{\alpha}{2} \rfloor}$ term which mimics the behavior of roller-coasters. We will attempt to expand on the results of Matchett by manipulating a technique which he developed known as power magnification. Our work results in the existence of an independence polynomial for any given of the form, $s_0 + s_1 z + s_2 z^2 + \cdots + s_\alpha z^\alpha$ such that $s_1 = s_2 = \cdots = s_\alpha$ for all even α where G is not well-covered.

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Chapter 1

Introduction

Graph theory involves studying the properties of graphs and the different relationships between vertices and edges. One of the ways in which you can represent one specific type of relationship is through the use of *independence polynomials.* It was conjectured that the coefficients for the polynomials of all graphs had a particular ordering in which the coefficients would increase to a maximum integer coefficient followed by decreasing coefficients. This conjecture was proved false and in fact a group of mathematicians were able to prove that no behavior exists in terms of the ordering of the coefficients of the polynomials. Thus another approach to studying this particular relationship was necessary in which we only consider a particular class of graphs. Again we study the behavior of the coefficients of the polynomials, known as the independence sequence. Through studying this behavior, The *Roller-Coaster* conjecture was born.

Imagine that you want to predict the movement of any given roller-coaster, and the process that would be necessary to do so. First it is conjectured that all roller-coasters travel upwards to a maximum point and then decrease until the end. We know this is false, but we also know that many roller-coasters possess several of the same properties, yet each one is unique in its own way. We transfer this idea to the coefficients of polynomials and their associated graphs which is the basis of this thesis. We will review the properties of graphs and their independence sequences, state the similarities which they possess, and attempt to predict the behavior of the coefficients of the polynomials which result from any given graph. This will require us to explain work previously done by other mathematicians as well as state our results to this longstanding conjecture. The goal of this Master's Thesis will be to expand, and improve the results of the *roller-coaster* conjecture by employing techniques first introduced by Philip Matchett [5]. In order to do so, however, we will introduce the reader to several key terms and definitions.

1.1 Definitions

It is necessary to discuss several preliminary terms and definitions so that the reader becomes acquainted with the fundamental concepts involved in our work. Since we are discussing a topic rooted in graph theory it is only natural that we start with a formal definition of a graph.

Definition 1. *A* graph, *G, is set of ordered pairs such that $G = (V, E)$ where $V = V(G)$ is the set of vertices in G and $E = E(G)$ is the set of edges in G.*

Other than the definition of a graph, the most useful term that we will introduce is that of an *independent set.*

Definition 2. *An* independent set *is a set of vertices from a graph G, such that no two vertices in the set are adjacent.*

In other words an independent set contains vertices which are not connected by an edge. Two vertices which are connected by an edge are referred to as adjacent vertices. Every graph contains exactly one independent set of size zero, namely the empty set. For simplicity, many of our examples will refer to *complete graphs.*

Definition 3. *A* complete graph, *K_n, on n vertices, is a graph in which every pair of distinct vertices is connected by an edge.*

Now we will use a complete graph to clarify the definition of independent sets. Suppose $G = K_3$, refer to the figure below.

Suppose we label the vertices $1, 2$, and 3. Then G has 4 independent sets. From our statement above, the empty set of G is an independent set. The set $\{1\}$ is an independent set because it does not contain an edge with any vertex in the set, namely it does not

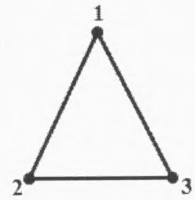


Figure 1.1: K_3, a complete graph on three vertices.

contain and edge with itself. The sets $\{2\}$ and $\{3\}$ are also independent sets for the same reason that $\{1\}$ is an independent set. We can not choose an independent set which contains more than one vertex. This is because if we add any vertex to the sets $\{1\}, \{2\}$, or $\{3\}$ the new set of size two will contain an edge and the set will no longer be independent. Recall that our work will eventually place a restriction on G. This restriction will allow us to only consider those graphs which are *well-covered.* In order to determine whether a graph is *well-covered* or not we need to know what a *maximal independent set* is for any graph G.

Definition 4. *A* maximal independent set *is an independent set that is not a subset of any other independent set.*

If we refer back to our earlier example in which $G = K_3$ (see Figure 1.1), then $\{1\}, \{2\}$, and $\{3\}$ are all maximal independent sets. This is because each set is not a subset of any other independent set. In other words, a set is maximal if it can not contain another vertex without an incident edge. If we consider the independent set $\{1\}$, we can conclude that it is maximal because if we had $\{1, 2\}$ or $\{1, 3\}$ an incident edge would exist and therefore violate the definition of independent sets. Maximal independent sets are an important concept because they help to determine the *independence number* of any graph G.

Definition 5. *For any graph, G, the* independence number *of G is the cardinality of the maximum size independent set which we will denote as $\alpha(G)$.*

Note that the above definition refers to the maximum independent set and not the maximal independent set. The meanings of these two words are not the same and thus we will briefly explain the difference. A maximal independent set is a set in which no new vertices can be added to the set without also containing an edge. It is possible for a graph to have maximal independent sets of different sizes. For example, consider the case when $G = P_3$, see Figure 1.2.

Figure 1.2: P_3, a path graph on three vertices.

Suppose we label the vertices of P_3, $1, 2$, and 3 as shown above, then we can find two maximal independent sets whose size differ. By definition, $\{2\}$ is a maximal independent set of size 1. In addition, $\{1, 3\}$ is an independent set of size 2. Therefore, we have found a graph which contains an independent set of size 1 and 2.

The maximum independent set will be the size of the largest independent set. Unlike the size of maximal independent sets, the size of the maximum independent set will always have one value. If we consider our example above then $\alpha(G) = 2$ since 2 is the largest independent set. Lastly we note that if in G every maximal independent set is of the same size, then the size of the maximal independent set will equal the size of the maximum independent set. This last statement will become clearer once we discuss the notion of *well-covered* graphs. However, from this statement we can conclude that the G we constructed above is not *well-covered*. Now that this distinction has been made we can continue our discussion of the independence number of a graph.

Suppose $G = K_3$ (see Figure 1.1), then $\alpha(G) = 1$ because we know from our previous discussion that the maximal independent set is of size 1 and no larger independent sets exist. Therefore, for any complete graph the size of the maximal independent set is equal to the size of the maximum independent set. So, for any complete graph, K_n, $\alpha(K_n) = 1$. Since every pair of vertices are adjacent, K_n can not contain an independent size larger than 1. As mentioned previously, the topics discussed in this paper will be concerned with counting the number of independent sets of a given size, i, where $0 \leq i \leq n$.

An algebraic technique used to represent the counting of independent sets is known as the *independence polynomial*.

Definition 6. *The* independence polynomial *of any graph G can be written as,*

$$S(G, z) = \sum_{i=0}^{\alpha} s_i z^i,$$

where $s_i = s_i(G)$ is the number of independent sets of size i in G and $\alpha = \alpha(G)$ is the independence number of G.

For example, the independence polynomial of K_n, a complete graph on n vertices, is $S(K_n, z) = 1 + nx$. In this example, $s_0 = 1$ counts the number of independent sets of size zero otherwise known as the empty set. If we again consider K_3 then $S(K_3, z) = 1 + 3z$ since K_3 has one empty set and three independent sets of size one. Many of the definitions mentioned above are helping the reader understand the difference between a graph which is *well-covered* or not. Using the above definitions we can now formally state what is meant for a graph to be *well-covered.*

Definition 7. *A graph G is* well-covered *if all maximal independent sets have the same size.*

We denote the size of these independent sets to be $\alpha(G)$ because the size of the maximal independent sets is equal to the size of the maximum independent sets for all well-covered graphs. We can conclude from this definition that K_n is well-covered since all maximal independent sets are of size 1. In the next section we will discuss some general cases of graphs which are well-covered as well as graphs which are not.

Before continuing further, we should mention that throughout this paper we will not be considering graphs with loops or multiple edges. The reason for this is because we do not gain any additional information about independent sets by considering these. Because of that, our discussion will be based on that of simple graphs.

1.2 Well-covered graphs

There are many examples of well-covered graphs such a several cycle graphs.

Definition 8. *A* cycle graph *is a graph, C_n with n vertices, such that C_n consists of a closed circuit or chain.*

Figure 1.3: C_7, a cycle graph on seven vertices.

Cycle graphs on $3, 4, 5, 6$, or 7 vertices are all well-covered. If one were to look at the cycle graph on seven vertices, C_7 (see Figure 1.3), the size of the maximum independent set, which is also the size of the maximal independent set, of any vertex is 3. The independence polynomial of C_7 is $S(C_7, z) = 1+7z+14z^2+7z^3$. At this point, we should make mention as to why C_8 is not well-covered. In fact, C_7 is the largest well-covered cycle graph possible. In C_7, label the vertices in order, from 1 to 7. If we want to choose an independent set of size 3 we can choose every other vertex to be in the set. Therefore, $\{1,3,5\}, \{2,4,6\}, \{3,5,7\}$ would all be maximal independent sets. If we consider the first set mentioned $\{1,3,5\}$ we note that 6 nor 7 could be in this independent set as vertex 6 is adjacent to vertex 5 and vertex 7 is adjacent to vertex 1.

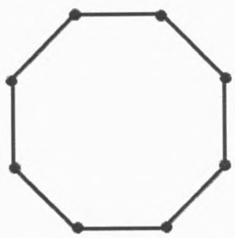


Figure 1.4: C_8, a cycle graph on eight vertices.

Next, suppose we have C_8, see Figure 1.4, and again label the vertices in order from 1 to 8. If we choose our independent sets in a particular manner, we can achieve two different maximal independent sets. The set $\{1,3,5,7\}$ is an independent set of size four. This set is maximal since every vertex in the complement contains an edge with some vertex in the independent set. However, the set $\{1,4,7\}$ is also a maximal independent set, but it is of size three. Again, every vertex in the complement of this independent set contains an edge with some vertex in the set. This contradicts our definition of well-coveredness since we have found two maximal independent sets with different cardinality. If we extend this idea to any cycle graph, C_n, in which $n \geq 8$ we see that there are two ways to choose maximal independent sets. One can either choose every other vertex to be in the maximal independent set of size $\lfloor \frac{n}{2} \rfloor$ or one can choose every third vertex which would result in a maximal independent set of size $\lfloor \frac{n}{3} \rfloor$. We note that $\lfloor \frac{n}{2} \rfloor \neq \lfloor \frac{n}{3} \rfloor$ for $n > 3$ and therefore any cycle graph on more than 7 vertices will not be well-covered.

Now that we have introduced the preliminary definitions and terms to the reader, we will discuss the history of our work. The background will include previous conjectures and theorems stated by mathematicians throughout the years. From their work we can formally state the *roller-coaster* conjecture and our work towards expanding prior results.

Chapter 2

Background

Well-covered graphs and their independence sequences are a relatively new field of study. M.D. Plummer was credited with the formal introduction of well-covered graphs in 1970 [8]. Plummer stated that a graph is said to be well-covered if the size of each maximal independent set is equal. Several years later in 1987, Alavi, Madle, Schwenk, and Erdős [1] investigated the possible orderings of the independence numbers $s_1, s_2, \ldots, s_\alpha$. This theorem does not require that our graphs be well-covered. Because of this we will delay statements for well-covered graphs until later. We will begin our discussion on graphs by introducing two operations which play an important role throughout this paper.

2.1 Graph operations: join and disjoint unions

These two operations are a key technique in our research because they allow us to combine graphs and gain additional insight into the orderings of independence numbers.

Definition 9. *Let G_1 and G_2 be two graphs with disjoint vertex sets. Then the* join *of $G_1 \vee G_2$ is the graph defined by the following:*

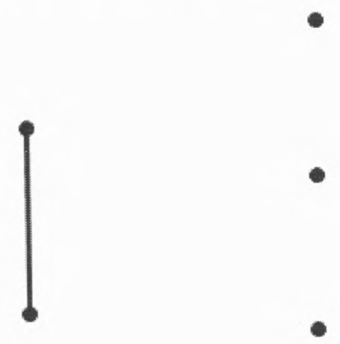
$$V(G_1 \vee G_2) = V(G_1) \cup V(G_2) \qquad \textit{and}$$

$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in \{V(G_1)\}, y \in \{V(G_2)\}\}.$$

In other words, the join operation connects each vertex in G_1 to every vertex in G_2.

In terms of the independent sets, joining G_1 and G_2 guarantees that no new independent sets have been created. Therefore, there does not exist an independent set which contains elements from both G_1 and G_2. This can be generalized in the following proposition.

To illustrate the above definition and proposition, assume we have G_1 and G_2 such that $G_1 = K_2$ and $G_2 = E_3$ where E_n is the empty set on n vertices.



(a) $G_1 = K_2$ (b) $G_2 = E_3$

Figure 2.1: G_1 and G_2 prior to the join operation.

Then the independent sets, s_i where i is the size of the set, of G_1 and G_2 are as follows:

	G_1	G_2
s_0	1	1
s_1	2	3
s_2	0	3
s_3	0	1

Therefore, the sum of the coefficients yields a new polynomial $2 + 5z + 3z^2 + z^3$, but this polynomial is not equal to $S(G_1 \vee G_2, z)$ because we have counted the empty set twice. For that reason we must subtract one, the empty set which has been over-counted to obtain, $S(G_1, z) + S(G_2, z) - 1 = 1 + 5z + 3z^2 + z^3$. Now let us consider the case of $G_1 \vee G_2$, see Figure 2.2.

Since we know that it is not possible to have an independent set which contains elements from both G_1 and G_2, the coefficients of the independence polynomial are:

Figure 2.2: $G_1 \vee G_2$, the resulting graph after K_2 is joined with E_3.

$$s_0(G_1 \vee G_2) = 1;$$

$$s_1(G_1 \vee G_2) = 5;$$

$$s_2(G_1 \vee G_2) = 3;$$

$$s_3(G_1 \vee G_2) = 1.$$

Therefore,

$$S(G_1 \vee G_2, z) = 1 + 5z + 3z^2 + z^3 = S(G_1) + S(G_2) - 1$$

and thus the result is shown. We can extend this example to prove the proposition for any two graphs.

Proposition 1. *Let G_1 and G_2 be graphs with disjoint vertex sets. Then the independence polynomial of the join of $G_1 \vee G_2$ is:*

$$S(G_1 \vee G_2, z) = S(G_1, z) + S(G_2, z) - 1.$$

Proof. Assume G_1 and G_2 are graphs with disjoint vertex sets. As previously mentioned, the join creates edges between every vertex in G_1 with every vertex in G_2. Suppose that $s_i^1 \in S(G_1, z)$ and $s_i^2 \in S(G_2, z)$, where s_i is the i^{th} coefficient of its corresponding independence polynomial, and $\alpha(G_1) = \alpha_1$ and $\alpha(G_2) = \alpha_2$ and

$$S(G_1, z) = 1 + s_1^1 z + s_2^1 z^2 + \cdots + s_{\alpha_1}^1 z^{\alpha_1}$$

and

$$S(G_2, z) = 1 + s_1^2 z + s_2^2 z^2 + \cdots + s_{\alpha_2}^2 z^{\alpha_2}$$

where s_i counts the number of independent sets in either G_1 or G_2 respectively. We connect every vertex in G_1 with every vertex in G_2 so that $G_1 \vee G_2$ now has independent sets which are entirely contained in either G_1 or G_2.

$$\begin{aligned} S(G_1 \vee G_2, z) &= s_1^1 z + s_2^1 z^2 + \cdots + s_{\alpha_1}^1 z^{\alpha_1} + s_1^2 z + s_2^2 z^2 + \cdots + s_{\alpha_2}^2 z^{\alpha_2} + 1 \\ &= 1 + s_1^1 z + s_2^1 z^2 + \cdots + s_{\alpha_1}^1 z^{\alpha_1} + 1 + s_1^2 z + s_2^2 z^2 + \cdots + s_{\alpha_2}^2 z^{\alpha_2} - 1 \\ &= S(G_1, z) + S(G_2, z) - 1. \end{aligned}$$

Note that each graph contains an independent set of size zero, namely the empty set. If we are counting the number of independent sets in the join then we will have counted the empty set of G_1 as well as the empty set of G_2. For this reason, we subtract 1 on the right hand side of the equation.

Therefore, $S(G_1 \vee G_2, z) = S(G_1, z) + S(G_2, z) - 1$. □

However, what if we want to consider the join of n graphs? We can extend Proposition 1 and prove this more generalized statement as well.

Theorem 2. *Let each G_i, $1 \leq i \leq n$, be graphs with disjoint vertex sets. Then the independence polynomial of the join of $G_1 \vee G_2 \vee \cdots \vee G_n$ is:*

$$S(G_1 \vee G_2 \vee \cdots \vee G_n) = S(G_1) + S(G_2) + \cdots + S(G_n) - (n-1).$$

Proof. We will prove Theorem 2 by induction on n. First we assume that we have two graphs G_1 and G_2 with disjoint vertex sets. Then by Proposition 1,

$$S(G_1 \vee G_2, z) = S(G_1, z) + S(G_2, z) - 1.$$

Next we assume that the polynomial of the join of any $n-1$ graphs is

$$S(G_1 \vee G_2 \vee \cdots \vee G_{n-1}, z) = S(G_1, z) + S(G_2, z) + \cdots + S(G_{n-1}, z) - (n-2).$$

If we want the join of n graphs we will join one more graph to $n-1$. By the definition of join this will create no new independent sets and connect every vertex in G_n with every vertex from $G_1, G_2, \ldots, G_{n-1}$. Therefore, the independence polynomial will be,

$$\begin{aligned} S((G_1 \vee G_2 \vee \cdots \vee G_{n-1}) \vee G_n, z) &= S(G_1 \vee G_2 \vee \cdots \vee G_{n-1}, z) + S(G_n, z) - 1 \\ &= S(G_1, z) + S(G_2, z) + \cdots \\ &+ S(G_{n-1}, z) - (n-2) + S(G_n, z) - 1 \\ &= S(G_1) + S(G_2) + \cdots + S(G_n) - (n-1). \end{aligned}$$

The first equality follows from base case and the second step from the induction hypothesis. Therefore, the result holds. □

We will now discuss another graph operation which effects the independence polynomial of graphs in a different way then the join operation. This operation is the *disjoint union*. Unlike the join operation, the *disjoint union* operation will not add edges in between graphs.

Definition 10. *Let G_1 and G_2 be graphs with disjoint edge sets. Then the* disjoint union *of these two graphs is defined as:*

$$V(G_1 \uplus G_2) = V(G_1) \cup V(G_2);$$
$$E(G_1 \uplus G_2) = E(G_1) \cup E(G_2).$$

Unlike the join operation, the disjoint union will not connect any vertices from G_1 with those edges contained in G_2. This is important as it will allow us to have independent sets which contain vertices from both G_1 and G_2. This will have a vastly different effect from that of the join on the resulting independence polynomial. Note that we must mention the resulting independence polynomial of these graph operations because the ordering of the coefficients are the basis of much of the discussion in this thesis.

Proposition 3. *Let G_1 and G_2 be graphs on disjoint vertex sets. Then*

$$S(G_1 \uplus G_2, z) = S(G_1, z)S(G_2, z).$$

To illustrate this theorem we state the following example. Assume we have two graphs G_1 and G_2 with disjoint vertex sets. Let $G_1 = K_3$ and $G_2 = K_2$. Then our graphs can be represented in the figure below

(a) $G_1 = K_3$

(b) $G_2 = K_2$

Figure 2.3: The above figure represents G_1 and G_2 prior to the disjoint union.

Because the disjoint union operation does not add any edges between G_1 and G_2, Figure 2.3 also represents $G_1 \uplus G_2$. From this we can conclude that

$$S(G_1 \uplus G_2) = 1 + 5z + 6z^2 = (1 + 3z)(1 + 2z) = S(G_1)S(G_2).$$

Before giving the form proof we will explain why the resulting independence polynomial makes sense. Suppose we want to calculate s_2. Fix a vertex in K_3. Since K_3 is a complete graph it contains no independent set of size two. This implies in order to obtain an independent set of size two, one vertex is contained in K_3 while the other vertex is contained is K_2. With our fixed vertex we have two vertices which can make an independent set. The same case will hold for all three vertices in K_3. Therefore, $s_2 = 2 \cdot 3 = 6$. Now that the reader has gained some intuition, we will state the formal proof.

Proof. Let G_1 and G_2 be two graphs with disjoint vertex sets. Assume that $\alpha(G_1) = \alpha_1$ and $\alpha(G_2) = \alpha_2$. By definition, since G_1 and G_2 are distinct vertex graphs it should be noted that they share no edges in common. Because of this, any independent sets of size i, will consist of independent sets from the union of G_1 and G_2. We will define the independent sets from G_1 to be s_i^1 and s_i^2 to be those independent sets from G_2. Further assume that we are looking for an independent set of size k in $G_1 \uplus G_2$. One can accomplish this by fixing an independent set of size i from G_1 and pairing it with all independent sets of size $k - i$

from G_2 for all $0 \leq i \leq k$. By doing this we have found independent sets of size i in G_1 plus independent sets of size $k-i$ from G_2 which results in an independent set of $i+(k-i)=k$. The number of these is the same as taking independent sets of size k and multiplying them with the number of independent sets of size $k-i$.

This implies that the coefficients of the independence polynomial can be found using the following form:

$$s_k(G_1 \uplus G_2) = \sum_{i=0}^{k} s_i(G_1)s_{k-i}(G_2).$$

Thus,

$$\begin{aligned}
&S(G_1 \uplus G_2, z) = \\
&(1 + {s_1}^1 z + {s_2}^1 z^2 + \cdots + {s_{\alpha_1}}^1 z^{\alpha_1})(1 + s_1^2 z + s_2^2 z^2 + \cdots + s_{\alpha_2}^2 z^{\alpha_2}) \\
&= S(G_1, z)S(G_2, z).
\end{aligned}$$

□

Thus far we have seen the following examples of independence polynomials:

$$\begin{aligned}
K_3 &: S(K_3, z) = 1 + 3z \\
C_7 &: S(C_7, z) = 1 + 7z + 14z^2 + 7z^3 \\
K_2 \vee K_{1,2} &: S(G_1 \vee G_2, z) = 1 + 5z + 4z^2 + z^3 \\
K_3 \uplus K_2 &: S(G_1 \uplus G_2, z) = 1 + 5z + 6z^z.
\end{aligned}$$

At a very basic level, our work starts by asking a question regarding the coefficients of independence polynomials.

Question 4. *Can we predict the behavior of the independence sequence of a graph G?*

If we look closely at the coefficients of the independence polynomials above, perhaps we see a pattern. We know that our first term, s_0, will always equal one. It appears as though after the s_0 term that our coefficient increase to a maximum integer coefficients and then the coefficients decrease until the end.

2.2 Unimodality

Many mathematicians examined countless independence polynomials and in fact conjectured that the behavior was exactly what we previously alluded to, that the coefficients increase to some maximum point and then decrease. In fact, Brown, Dilcher, and Nowakowski [2] provided an answer to the question, can we predict the behavior of the independence sequence of a graph G? The independence sequence of G refers to the coefficients of the independence polynomial. Recall our examples in which we have seen independence polynomials which are monotonically increasing, as well an example where the coefficients strictly increase to a maximum integer and then decrease. The latter case describes the behavior of *unimodality.*

Definition 11. *A sequence,* $(s_0, s_1, s_2, \ldots, s_\alpha)$ *is said to be* unimodal *if there exists some value* p*,* $0 \leq p \leq \alpha$*, such that*

$$s_0 \leq s_1 \leq \cdots \leq s_p \qquad \textit{and} \qquad s_p \geq s_{p+1} \geq \cdots \geq s_\alpha.$$

However, not all independence sequences of graphs are unimodal. Levit and Mandrescu [3] provide several counterexamples. Here we will state one such example and employ the techniques that have been previously mentioned.

$$\begin{aligned} S((K_{24} \vee (K_3 \uplus K_4 \uplus K_3)), z) &= (1 + 24z) + (1 + 3z)^2(1 + 4z) \\ &= (1 + 24z) + (1 + 10z + 33z^2 + 36z^3) - 1 \\ &= 1 + 34z + 33z^2 + 36z^3. \end{aligned}$$

We can see in the above example, that the independence sequence increases up to $s_1 = 34$ followed by a decreasing term, $s_2 = 33$, however $s_3 = 36$. This increase from s_2 to s_3 violates the definition of unimodal. In fact, Alavi, Madle, Schwenk, and Erdős [1] disprove unimodality by developing the following theorem in which one can manipulate the coefficients to achieve *any-ordered* coefficients.

Theorem 5 (Alavi et al. [1]). *For any permutation π of the set $\{1, 2, \ldots, \alpha\}$ there exists a graph whose independence sequence $\{s_0, s_1, \ldots, s_\alpha\}$ satisfies*

$$s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(\alpha)}.$$

Note that there is no $s_{\pi(0)}$ term since $s_0 = 1$. This theorem allows to create an independence polynomial in which we choose the ordering of the coefficients.

We will not prove this theorem, but rather speak of its implications in our work. Theorem 5 allows us to create an independence sequence in which we choose the desired ordering. This answers the question posed above as not only can we guess the behavior of the coefficients of an independence polynomial, but we can build those coefficients in any desired manner we choose. With this, we have answered our original question, can we determine the behavior of the coefficients of the independence polynomial of a graph, but not in a satisfying manner. This is because the answer is simply no. Although there is no apparent pattern, we can determine the ordering and have the independence sequence of a graph behave in any desired way.

This idea of any-ordering coefficients serves as a basis for the Roller-Coaster conjecture at its most elementary level. This is because Alavi, Madle, Schwenk, and Erdős considered G to be any graph with no constraints. As others began to study the topic of joins, disjoint unions, and independence polynomials more questions arose. We know that there exists a graph, G, with no constraints, which we can order the independence sequence in any way we desire.

Question 6. *Can we predict the behavior of the independence sequence of G if we restrict ourselves to a particular class of graphs?*

In order to answer Question 6 we must first decide which graphs we will restrict ourselves to. Previous work completed focuses on well-covered graphs because of a number of well behaved properties which they possess. One of the nice properties is that computing the resulting independence polynomials of well-covered graphs is not difficult and will be shown later in this paper. In the next section we will discuss path graphs. This is to show the reader that within a particular class of graphs, we can have well-coveredness or not, depending on the number of vertices.

2.3 Paths

Some the earliest research which considered well-covered graphs was done by Brown, Dilcher, and Nowakowski [2]. They stated that a graph is well-covered provided every maximal independent set has the same size, α. This definition will be useful when we discuss the graph operations on well-covered graphs. Using the above definition, all complete graphs, K_n, are well-covered since they have $n+1$ independent sets, namely one empty set and n independent sets of size one. The empty set is never maximal because one can add any vertex to it and create a larger independent set. This is a nice result to have because we can choose any K_n and not be concerned with determining whether or not it is well-covered. We have also already discussed cycle graphs in which C_n where $n \geq 8$ is not well-covered, see Figure 1.4. However, cycle graphs are not the only class of graphs with the behavior in which some graphs are well-covered while others are not. *Path* graphs, P_n are only well-covered for a few values of n.

Definition 12. *A path on n vertices, P_n, is a sequence of vertices such that*

$$V(P_n) = \{1, 2, \ldots, n\}$$

and

$$E(P_n) = \{\{i, i+1\} : 1 \leq i \leq n-1\}.$$

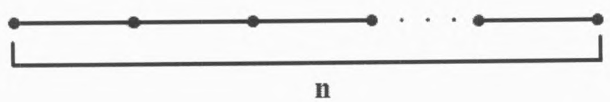


Figure 2.4: P_n, a path graph on n vertices with $n-1$ edges.

Further, a path has two terminal vertices which are connected by an edge to one vertex, unlike the internal vertices which are connected by two edges to two vertices which can be seen in Figure 2.4.

Again consider the graph, P_3 (see Figure 1.2), in which the vertices are labeled $1, 2,$ and 3. P_3 is not well-covered because all maximal independent sets do not have equal cardinality. However, P_3 is not the only the path which is not well-covered.

Theorem 7. *Any path, P_n, where $n \geq 3$ is odd, is not well-covered.*

Note that $P_1 = K_1$ is well-covered. For the proof, the reader can use the terminology and relate it to P_3. The idea will extend to any path graph, P_n, where n is odd and $n \geq 3$.

Proof. Suppose we have a path on $n \geq 3$ vertices, where n is odd. A maximal independent set will consist of every other vertex in the graph starting at a terminal vertex. The size of this independent set will be $\frac{n+1}{2}$. This set is maximal because if we attempted to add another vertex to the set, that vertex would contain an edge with a vertex in the independent set. Consider the complement of the independent set previously chosen which is nonempty since $n \geq 3$. This independent set would consist of $\frac{n-1}{2}$ vertices and would also be maximal since it contains an edge with the remaining vertices not in the independent set. However, $\frac{n-1}{2} \neq \frac{n+1}{2}$. Therefore, a path on n odd vertices is never well-covered. □

We have just shown that any graph of the form P_n, where n is odd, is not well-covered. However we have yet to mention P_n, where n is even. Unfortunately, even paths are not as straightforward because like cycle graphs, they are well-covered until the path contains 6 vertices. In the case of P_2, the size of the maximal independent set is one. You can choose either vertex and it will be contained in an independent set consisting of only itself. This means that the maximal independent set is of size one and P_2 is well-covered. If we consider P_4 then $\{1, 3\}, \{2, 4\}$, and $\{1, 4\}$ are all maximal independent sets of size two. It is not possible to get a maximal independent set greater than two. Therefore, P_4 is also well-covered. These are the only two cases in which P_n, where n is even, will be well-covered. We state and prove the following theorem to justify this statement.

Theorem 8. *Any path, P_n, where n is even and $n \geq 6$, is not well-covered.*

Proof. For this proof we will use the same argument as we did for cycle graphs of size 8 or larger, see Figure 1.4. We begin by choosing a maximal independent set which contains every other vertex starting with a terminal vertex. The size of this independent set will be $\frac{n}{2}$ as previously indicated. We will consider an independent set containing every third vertex starting from a terminal vertex. The size of this independent set will be $\lfloor \frac{n}{3} \rfloor$. Note that both of these independent sets are maximal since any vertices not in the independent set contain an edge with those vertices contained in the independent set. Since $\frac{n}{2} \neq \lfloor \frac{n}{3} \rfloor$, for $n \geq 6$, we have a contradiction to the definition of well-coveredness and the proof is complete. □

2.4 Well-covered graph operations

The purpose of the discussion on paths was to make the reader aware that not every graph in a particular class is well-covered. In general, determining whether or not a graph is well-covered is a difficult task. If it is challenging to do so for one graph, imagine the difficulty of determining if the join and disjoint unions of many graphs will result in a well-covered graph. Because of the complexity of determining well-coveredness, we will provide some equivalent statements which allow alternative methods for determining well-coveredness. These theorems will make it easier to determine whether or not a graph resulting from a join or disjoint union will be well-covered.

Theorem 9. *Let G_1 and G_2 be well-covered graphs. Then, the join of G_1 and G_2 is well-covered if and only if $\alpha(G_1) = \alpha(G_2)$.*

Proof. Assume we have two well-covered graphs G_1 and G_2 with $\alpha(G_1) = a$ and $\alpha(G_2) = b; a \neq b$. Recall our earlier proof of Proposition 1 that if we consider the independent sets contained in the join, $G_1 \vee G_2$, then the independent sets must be entirely contained in either $V(G_1)$ or $V(G_2)$. If we fix some vertex from G_1 and want to determine the size of the maximal independent set in which that vertex is is contained, we conclude that the size must be a. This is not dependent on the vertex we choose. Since we assumed G_1 is well-covered any vertex chosen will be contained in an independent set of size a. Similarly, if we fix a vertex from G_2 and want to determine the size of the maximal independent set containing it, then the size must be b. Since $a \neq b$, we have $G_1 \vee G_2$ containing maximal independent sets with different cardinalities. This implies that $G_1 \vee G_2$ is not well-covered. Therefore, $G_1 \vee G_2$ will be well-covered provided that $\alpha(G_1) = \alpha(G_2)$. To prove the other direction we again assume G_1 and G_2 are well-covered. Assume that $\alpha(G_1) = \alpha(G_2)$. Any independent set in $G_1 \vee G_2$ is either entirely in G_1 or entirely in G_2 by definition. Therefore any maximal independent set has size $\alpha(G_1) = \alpha(G_2)$. □

As an example of the above theorem, let us consider two graphs G_1 and G_2 such that $G_1 = E_k$and $G_2 = E_{n-k}$. The empty graph, E_k, refers to the graph which has k vertices and no edges. This implies that $\mid G_1 \mid = k$, $\mid G_2 \mid = n-k$, and $\alpha(G_1) = k$ and $\alpha(G_2) = n-k$. By considering $G_1 \vee G_2$, we are now investigating the graph $K_{k,n-k}$ and by Theorem 9 the only case in which this results in a well-covered graph is when $k = n-k$ or $k = \frac{n}{2}$ where n is even.

This result is helpful because we do not have to be concerned with determining whether or not the join of two graphs will be well-covered. We only need to guarantee that the graphs we are interested in joining are themselves well-covered and have equal independence numbers.

The disjoint union operation is quite similar to that of the join, however we will remove the restriction of our graphs needing to have the same independence number. Again, this is very useful because we only need to determine whether G_1 and G_2 are well-covered and that allows us to conclude that the disjoint union will be well-covered.

Theorem 10. *The disjoint union of two well-covered graphs is well-covered.*

Proof. Assume that G_1 and G_2 are both well-covered and $\alpha(G_1) = a$ and $\alpha(G_2) = b$. Then $\alpha(G_1 \uplus G_2) = a + b$ since there are no edges between G_1 and G_2. In fact, every maximal independent set in $G_1 \uplus G_2$ has size $a + b$.

Assume there exists a maximal independent set, I, with $|I| < a + b$. This implies that either $|I \cap G_1| < a$ or $|I \cap G_2| < b$. In either case this means we can make I larger in G_1 or G_2. Thus, I would not be maximal and we reach a contradiction. Therefore, the disjoint union of two well-covered graphs is well-covered. □

Note that in the above proof we did not mention whether or not $\alpha_1 = \alpha_2$. This is because the proof of this theorem does not depend upon their values. The reader might expect that at this point we will determine the resulting independence polynomials given our new restrictions on the join and disjoint union operations. However, this is not necessary because although we have made several restrictions on the graphs which may be used, the calculation of the independence polynomial is not affected.

We are now back to the same question we asked previously regarding all graphs. That is, is there a way to determine the behavior of the independence sequences associated with specifically well-covered graphs? Again, we will discuss the topic of unimodality as it is concerned with well-covered graphs.

2.5 The Unimodality Conjecture For Well-Covered Graphs

Brown, Dilcher, and Nowakowski [2] attempted to answer the question about determining the behavior of the independence sequences of well-covered graphs by stating a new conjecture using an old idea. They conjectured that for all well-covered graphs, the independence sequence is unimodal. Recall that a sequence is said to be unimodal if there exists some value p such that

$$s_0 \leq s_1 \leq \cdots \leq s_p \quad \text{and} \quad s_p \geq s_{p+1} \geq \cdots \geq s_\alpha.$$

The formal statement of the conjecture is known as *The Unimodality Conjecture.*

Conjecture 11. *Independence sequences of well-covered graphs are unimodal.*

As we saw in the case of unimodality for all graphs, well-covered graphs are not necessarily unimodal either. Again, we state a counterexample provided by Levit and Mandrescu [3]. Let us construct a well-covered graph, G, in which $4K_{10}$ is the disjoint union of 4 complete graphs on 10 vertices and $K_{n(4)} = \bigvee_{i=1}^{n} E_4$. Note that K_{10} is well-covered by definition since we are taking the disjoint union of graphs. The independence number, $\alpha(K_{10}) = 1$. Similarly, $K_{n(4)} = 1$ is well-covered since $\alpha(E_4) = 1$ and we can take the join of n of them which results in $\alpha(K_{n(4)}) = 1$. We can then calculate the independence polynomial of each of these graphs.

$$S(4K_{10},z) = S(K_{10} \uplus K_{10} \uplus K_{10} \uplus K_{10}, z) = (1+10z)^4$$

and

$$S(K_{n(4)},z) = S(E_4 \vee E_4 \vee \cdots \vee E_4, z) = n(1+z)^4 - n - 1.$$

Note that for $S(K_{n(4)},z)$ we are taking the join of n copies of E_4, hence the multiplication by n in the resulting independence polynomial.

Then,

$$G = (4K_{10}) \vee K_{n(4)}, \qquad n \geq 1.$$

G is well-covered since $\alpha(K_{10}) = 1 = \alpha(K_{n(4)})$. We can then compute the independence polynomial of G to be,

$$\begin{aligned} S(G_n,z) &= n \cdot (1+z)^4 + (1+10z)^4 - n + 1 \\ &= 1 + (40+4n)z + (600+6n)z^2 + (4000+4n)z^3 + (10000+n)z^4. \end{aligned}$$

Since $40+4n < 600+6n$ is true (for any $n \geq 1$), solving this inequality gives us a bound for when $S(G_n,z)$ is not unimodal. This occurs when $1700 < n < 2000$. Michael and Traves [6] were the first to provide a counterexample for the Unimodality conjecture in the above example for $n = 1701$. Michael and Traves [6] prove that The Unimodality Conjecture is true for $\alpha = 3$, but were able to find counterexamples in the case when $\alpha \in \{4,5,6,7\}$. However, Levit and Mandrescu [3] developed a list of criterion which will guarantee that the Unimodality Conjecture will be true for well-covered graphs with a particular given independence number.

Proposition 12 (Levit and Mandrescu). *The following sufficient conditions ensure that the independence polynomial of a graph G is unimodal:*

i.) any connected component of H of G has $\alpha(H) \leq 2$;

ii.) $\alpha(G) = 3$ and G is well-covered;

iii.) $\alpha(G) = 4$, where G is disconnected and well-covered;

iv.) $\alpha(G) = 5$, *where* $G = H_1 \cup H_2$ *with* $\alpha(H_1) = 2$ *and* H_2 *is well-covered;*

v.) $\omega(G) \leq \alpha(G) \leq 5$ *and* G *is well covered;*

vi.) $\alpha(G) = 6$, *where* G *is disconnected and any component* H *of* G *with* $\alpha(H) \in \{3, 4, 5\}$ *is well-covered and satisfies* $\omega(H) \leq \alpha(H)$.

We have gained some insight into the behavior of the coefficients of the independence polynomials of well-covered graphs, but have not yet claimed any general behavior for all well-covered graphs. Michael and Traves [6] developed the *Roller-Coaster Conjecture* to solve this problem. They claim that the independence sequence, $s_{\lceil\alpha/2\rceil}, s_{\lceil\alpha/2\rceil+1}, \ldots, s_\alpha$, of well-covered graphs are unconstrained in accordance with Proposition 5. Before the formal statement of the conjecture, we must note that there are several constraints on independence sequences of well-covered graphs.

Theorem 13 (Matchett [5]). *The independence sequence* $(s_0, s_1 \ldots, s_\alpha)$ *of a well-covered graph* G *satisfies*

$$\frac{s_0}{\binom{\alpha}{0}} \leq \frac{s_1}{\binom{\alpha}{1}} \leq \ldots \leq \frac{s_\alpha}{\binom{\alpha}{\alpha}}.$$

Proof. Let $\mathcal{S}_k$ be the independent k-sets in G. In order to prove the result we will use a double counting argument. Let

$$\mathcal{F}_k = \{(V_k, V_{k+1}) : V_k \subseteq V_{k+1}, V_k \in \mathcal{S}_k, V_{k+1} \in \mathcal{S}_{k+1}\}.$$

In order to count the size of $\mathcal{F}_k$ we will fix a V_{k+1} and count the number of sets corresponding to V_k. The way to do this is to fix V_{k+1} and delete one vertex. Continue this process and this results in

$$\mid \mathcal{F}_k \mid = (k+1) \mid \mathcal{S}_{k+1} \mid = (k+1)s_k,$$

where s_k is the size of the independent k-sets in G. On the other hand, we can also fix V_k and get to the independence number. Any $V_k \in \mathcal{S}_k$ is contained in an independent set of size α because G is well-covered. So, $|\mathcal{F}_k| \geq (\alpha - k)s_k$. This implies that

$$(\alpha - k)s_k \leq (k+1)s_{k+1} \Leftrightarrow \frac{s_k}{\frac{\alpha!}{k!(\alpha-k)!}} \leq \frac{s_{k+1}}{\frac{\alpha!}{(k+1)!(\alpha-k-1)!}} \Leftrightarrow \frac{s_k}{\binom{\alpha}{k}} \leq \frac{s_{k+1}}{\binom{\alpha}{k+1}}.$$

□

The above results led to a very useful corollary stated and proved by Michael and Traves [6].

Corollary 14. *Let* $(s_0, s_1, \dots, s_\alpha)$ *be the independence sequence of a well-covered graph. Then* $s_0 \leq s_1 \leq \dots \leq s_{\lceil \alpha/2 \rceil}$.

Proof. Let G be a well-covered graph with $s_i = s_i(G)$, where $0 \leq i \leq \lceil \frac{\alpha}{2} \rceil$ and i represents the size of the i^{th} independent set. If α is odd then the coefficients of the independence polynomial reach a maximum value when $\alpha = \lceil \alpha/2 \rceil$. This is because $\binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}$. When α is even the maximum occurs at $\binom{\alpha}{\frac{\alpha}{2}}$. Therefore, $s_0 \leq s_1 \leq \dots \leq s_{\lceil \alpha/2 \rceil}$ is the correct inequality in this case. □

The corollary above is like the first portion of a roller-coaster. Every roller-coaster, no matter how big or small, must climb up until a maximum point is reached. After this maximum is reached our coefficients are unconstrained according to Michael and Traves [6]. In other words, after you reach the maximum height in the roller-coaster car, you do not know what will happen next.

2.6 The Roller-Coaster Conjecture

These new discoveries and conclusions resulted in the statement of the *Roller-Coaster conjecture.* The *Roller-Coaster conjecture* was originally stated by Michael and Traves [6]. For the purpose of this paper we will state the *Roller-Coaster conjecture* as a blend between the statements of Michael and Traves [6] and those of Philip Matchett [5].

Conjecture 15 (The Roller-Coaster Conjecture). *Let* G *be any well-covered graph whose independence number is* $\alpha(G) = \alpha$. *Then the coefficients of* G *are strictly increasing from* s_0 *to* $s_{\lceil \alpha/2 \rceil}$, *and the independence sequence of* G *is any-ordered on* $\{\lceil \alpha/2 \rceil, \dots, \alpha\}$. *That* *is, for any permutation π of the set $\{\lceil \alpha/2 \rceil, \ldots, \alpha\}$ there exists a well-covered graph whose independence sequence $(s_0, s_1, \ldots, s_\alpha)$ satisfies*

$$\pi(s_{\lceil \alpha/2 \rceil}) < \pi(s_{\lceil \alpha/2 \rceil + 1}) < \cdots < \pi(s_\alpha).$$

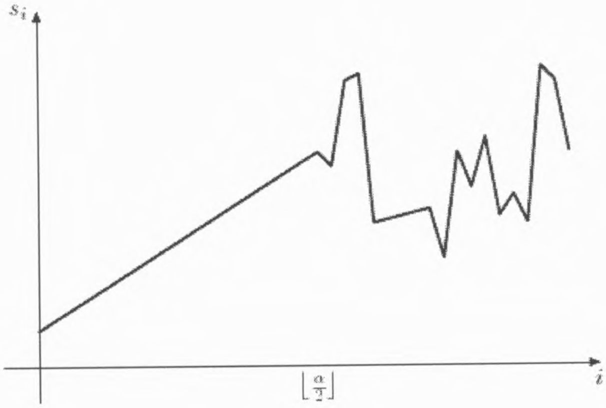


Figure 2.5: A graphical representation of the Roller-Coaster conjecture.

Michael and Traves [6] proved the Roller-Coaster conjecture for $\alpha = 1, \ldots, 7$. Matchett [5] was able to solve the conjecture for $\alpha \leq 11$. We will review several of these results. Consider the case when $\alpha = 1$. Then $\lceil \alpha/2 \rceil = \alpha = 1$ and the only criterion we need to show is that $s_0 < s_1$. Let $G = K_n$. Since K_n is well-covered we can compute its independence polynomial, $S(K_n, z) = 1 + nz$ and thus the conjecture has been shown.

In the case where $\alpha = 2$, we need to consider two separate cases since $\lceil \alpha/2 \rceil = 1$. This means that $S(G, z)$ is any ordered for s_1 and s_2. We want to show that it is possible to have an independence polynomial in which $s_1 < s_2$ and $s_1 > s_2$.

Suppose $G = K_3 \uplus K_2$, see Figure 2.3. We know that K_3 and K_2 are well-covered and thus by Theorem 3, $S(G, z) = (1 + 3z)(1 + 2z) = 1 + 5z + 6z^2$ and we have found an independence polynomial which satisfies $s_1 < s_2$. To show $s_1 > s_2$ again we take the disjoint union of two complete graphs. Let $G = K_2 \uplus K_1$, then $S(G, z) = (1+2z)(1+z) = 1+3z+2z^2$ and thus we have shown that the conjecture holds for $\alpha = 2$.

Next, consider the case $\alpha = 3$. Since $\lceil \alpha/2 \rceil = 2$ we will again consider two separate cases, namely when $s_2 < s_3$ and $s_2 > s_3$. Let $G = K_4 \uplus K_3 \uplus K_3$ then $S(G,z) = (1+4z)(1+3z)^2 = 1+10z+33z^2+36z^3$, $s_2 < s_3$. For the second case we will consider the join operation. Let $G = E_3 \vee E_3$ such that $G = K_{3,3}$. Theorem 9 tells us that G is a well-covered since we are joining two well-covered graphs with equal independence numbers. From Theorem 1 we calculate the independence polynomial, $S(G,z) = (1+3z+3z^2+z^3)+(1+3z+3z^2+z^3)-1 = 1+6z+6z^2+2z^3$, so $s_2 > s_3$ and thus $\alpha = 3$ holds.

As the attempt to solve for higher values of α we realize that it is becoming increasing more difficult to compute any-ordered coefficients by hand. Because of this Matchett [5] developed a new technique known as *power magnification*. Power magnification not only becomes useful to the Roller-Coaster conjecture, but in the case of Proposition 5 it makes it possible to prove their statement more concisely. Our goal is to prove the roller-coaster conjecture for larger values of $\alpha > 11$ using this new technique.

2.7 Power magnification

Before our formal definition, we need to discuss the notion of scaling independence polynomials by positive rational constants and what this means in terms of a graph G. It is important to note that when discussing the ordering of the coefficients of the independence polynomial, we are not concerned with their absolute sizes, but rather their relative sizes. That is, the exact values of the coefficients are unimportant, but their ordering in relation to each other is. This is where we introduce a new concept known as scaling plays a role. If we want to scale a graph we multiply it by some rational constant. This scales the coefficients of the independence polynomial which makes sense from an algebraic perspective. However, scaling does not make sense in a graph theoretical sense and thus what we are creating will be known as *pseudographs*.

Definition 13. *A* pseudograph *is an object whose independence polynomial has been scaled by $\frac{p}{q}$ for $p, q > 0$.*

Note that if $q = 1$, this polynomial represents the join of q copies of G which would represent a true graph. If $r \neq 1$ then we can multiply by r to eliminate any denominators.

What this means is that we haven't gone too far away from true graphs. We can recover a graph with the same ordering of the independence sequence by multiplying by r.

To be precise, the independence polynomial of a pseudo graph will be of the form,

$$S(\frac{p}{q}G, z) = \frac{p}{q}S(G, z).$$

The left hand side of the equation represents each coefficient being scaled by our rational constant $\frac{p}{q}$. While on the right hand side of the equation we factor out our rational constant. In either case, we can multiply through by q to get $S(pG, z) = pS(G, z)$. The reason for this aside, although not apparent at the moment, is because in order to discuss the notion of *power magnification* we need to address dividing by a constant.

Definition 14. *Let G be a well-covered graph with independence number α. Let*

$$H_c = \biguplus_{i=1}^{n} K_c$$

which is the disjoint union of n copies of K_c for large values of c. Then G power magnified *by n is defined as $H_c \uplus G$.*

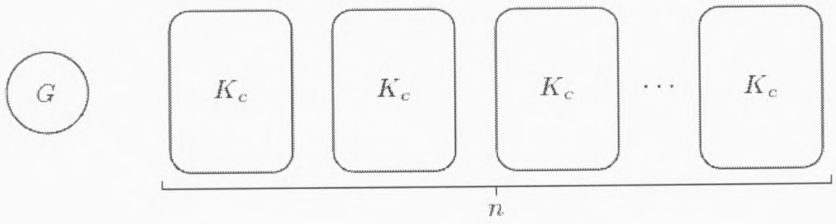


Figure 2.6: G disjoint unions with n copies of K_c.

To make it easier for the reader to follow, we will denote G power magnified by n as $G_p^n = H_p \uplus G$. Further we claim that a sequence of polynomials converges to a polynomial f if the coefficients of the sequence converge to the corresponding coefficients of f.

Proposition 16. *Let G be a well-covered graph and G_c^n be G power magnified by n. Then*

G_c is well-covered and we have

$$\lim_{c\to\infty} \frac{S(G_c^n, z)}{c^n} = z^n S(G, z). \tag{2.1}$$

Proof. Since $G_c^n = H_c \uplus G$ where H_c and G are well-covered, it follows that G_c^n is also well-covered. The resulting independence number will be $\alpha + n$. Note that $\alpha(H_c) = n$ because the largest independent set of any complete graph is 1 and we are taking n copies, therefore the independence number corresponding to that independent set will be of size n. From Theorem 3, $S(H_c \uplus G, z) = S(H_c, z)S(G, z)$. In order to compute the independence polynomial of H_c we will need to count the number of independent sets from 1 to n. Remember that by definition, H_c is the disjoint union of n copies of the complete graph on c vertices, where c is some large integer. This means that there are no edges between the copies of the K_c graphs. The coefficients for the independence polynomial will come from choosing one vertex from each K_c graph. Therefore, $S(H_c, z) = \left(\sum_{i=0}^{n} \binom{n}{i}(cz)^i\right)$. We evaluate the limit of the left hand side of equation (1) as $c \to \infty$. We will get:

$$\begin{aligned}\lim_{c\to\infty}\left(\frac{1}{c^n}\left(\sum_{i=0}^{n}\binom{n}{i}(cz)^i\right)\right) &= \lim_{c\to\infty}\left(\frac{1}{c^n} + \frac{ncz}{c^n} + \frac{\binom{n}{2}c^2z^2}{c^n} + \cdots + z^n\right) \\ &= \lim_{c\to\infty}\frac{1}{c^n} + \lim_{c\to\infty}\frac{ncz}{c^n} + \lim_{c\to\infty}\frac{\binom{n}{2}c^2z^2}{c^n} + \cdots + \lim_{c\to\infty} z^n \to z^n.\end{aligned}$$

□

It should be clearer to the reader now the necessity for discussing pseudographs. Suppose that z^nG is called the limit graph of G power magnified by n, then its independence polynomial $z^nS(G, z) = \lim_{c\to\infty} S(G_c^n, z)$. This statement calculates the independence polynomial of a pseudograph which may or may not represent a true graph. However, we can find graphs which approximate our desired polynomial as follows.

Lemma 17. *Given $\epsilon > 0$ and $z^nG = s_0 + s_1z + \cdots + s_{\alpha+n}z^{\alpha+n}$, there exists a large valued c such that the true graph, G_c, has independence polynomial $b_0 + b_1z + \cdots + b_{\alpha+n}z^{\alpha+n}$ and*

$$\left|\frac{b_i}{c^n} - s_i\right| < \epsilon$$

for all i.

The above lemma tells us that it is possible to have a true graph G_c in which if we divide each coefficient by some large valued c^n, then the coefficients of our pseudograph will be approximately equal to the coefficients of G_c. This is an important result because it implies that if we calculate a pseudograph with a desired ordering of its independence sequence, then there exists a scaled true graph which will have the same desired ordering since $\left|\frac{b_i}{c^n} - s_i\right| < \epsilon$. Note that this result holds because $\frac{1}{c^n}$ does not change the linear ordering of G_c.

We will now review Theorem 5 and reprove this theorem using power magnification and its associated properties. This is a nice application in which power magnification can be used to simplify a result. However, it is important to note that this theorem is stated for all graphs, no matter if they are well-covered or not. This allows us to add independence polynomials of different degrees using the join operation. The result of this type of join will result in a graph which is not well-covered.

Refer back to Theorem 5, then using power magnification we can prove the result.

Proof. We know the empty set has an independence polynomial of $S(E_0, z) = 1$. We begin by power magnifying the empty graph by z^n. Recall that z^n represents an independent set of size n. After power magnification, the independence polynomial will become $S(z^n E_0, z) = z^n$. Given π, some permutation of the index set $\{1, 2, \dots, \alpha\}$, we can add the polynomials of $S(z^n E_0, z)$ by the join operation to obtain:

$$\begin{aligned} P(z) &= 1 + 1S(z^{\pi(1)}E_0, z) + 2S(z^{\pi(2)}E_0, z) + \cdots + \alpha S(z^{\pi(\alpha)}E_0, z) \\ &= 1 + 1z^{\pi(1)} + 2z^{\pi(2)} + \cdots + \alpha z^{\pi(\alpha)}. \end{aligned}$$

The final result holds because $P(z)$ has the property that $s_{\pi(i)} = i$ for all $i = 1, \dots, \alpha$. Therefore $s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(\alpha)}$. □

Power magnification has significantly cut down on the amount of work needed to prove Theorem 5, but what does it mean in terms of the Roller-Coaster conjecture? Remember that as α increases new techniques were used to help solve the conjecture. We will discuss more specifically what role *power magnification* plays in this question after we state two important results shown by Matchett [5].

Theorem 18. *If the last n terms of the independence sequence for a well-covered graph are any-ordered for a given α_0, then the last n terms of the independence sequence are any-ordered for all $\alpha_0 < \alpha$.*

Proof. Let α_0 and α be given such that $\alpha_0 < \alpha$. Let $\alpha = \alpha_0 + 1$ and further suppose that G is well-covered and

$$S(G,z) = 1 + s_1 z + ... + s_{\alpha-n+1} z^{\alpha-n+1} + \cdots + s_\alpha z^\alpha,$$

where the n terms, $s_{\alpha-n+1}, \dots, s_\alpha$, have some desired ordering. We can create a power magnified graph by multiplying $z(G,z)$. In this case $\alpha = \alpha_0 + 1$ and

$$zS(G,z) = z + s_1 z^2 + \cdots + s_{\alpha-n+1} z^{\alpha-n+2} + \cdots + s_\alpha z^{\alpha+1}.$$

The desired ordering is preserved since each term is multiplied by z and thus the result is shown. $\square$

Although the above theorem may not immediately seem useful, the implication of this theorem eliminates a lot of calculations later on as we can see in the following corollary first stated by Matchett [5].

Corollary 19. *If the Roller-Coaster conjecture is true for some even value of α, then the conjecture is also true for $\alpha + 1$.*

Proof. Apply Theorem 18 with $\alpha_0 = \ell$. Given that α is even and true for the Roller-Coaster conjecture, we know that $\lceil \alpha/2 \rceil = \frac{\alpha}{2}, \dots, \alpha$. The length of this sequence is $l = \alpha - \frac{\alpha}{2} + 1$. This implies that $\lceil \alpha/2 \rceil = \frac{\alpha}{2} + 1, \cdots, \alpha + 1$ and $l = \alpha + 1 - (\frac{\alpha}{2} + 1) + 1 = \alpha - \frac{\alpha}{2} + 1$ from our original sequence. Therefore, if the last n terms of the independence sequence for well-covered graphs are any-ordered for a given α, then the last n terms of the independence sequence will be any-ordered for $\alpha + 1$ as well. $\square$

Now that we have these results, we can discuss the importance of *power magnification* in solving the Roller-Coaster conjecture. The technique which Matchett [5] uses is to power magnify independence polynomials of the form $z^i(1+cz)^{\alpha-i}$, where $c=1$. He then claims that we can perturb the coefficients to prove the conjecture for all α. This perturbed independence polynomial results in an *almost flat-roller coasters.*

Definition 15. *An* almost flat-roller coaster *graph is defined as the following:*

$$H_{\alpha,i} = z^i \left(\biguplus_{i=1}^{\alpha-i} K_1 \right) \tag{2.2}$$

for $i = 0, 1, \ldots, \alpha$.

Note that every $H_{\alpha,i}$ have independence number α. This preserves well-coveredness when taking joins of $H_{\alpha,i}$. In order to do this, Matchett [5] uses matrices, linear algebra operations, and computer technology. As α increases so do the sizes of the matrices and thus having a computer to set up these matrices and well as compute certain operations is helpful. The final result we will discuss is where we can guarantee any-ordered coefficients.

Matchett [5] proposes the following:

Theorem 20. *Let k_α be the largest integer for which the last k_α terms of the independence sequence for well-covered graphs with independence number α are any-ordered. Then for sufficiently large α,*

$$\frac{1}{2}\alpha < k_\alpha < 0.8295.$$

The above theorem gives us a bound for within which our coefficients will be any-ordered. We use this fact along with power magnification as well as the join and disjoint unions of well-covered graphs to construct a graph in which the last $(.1705)\alpha$ terms in the independence sequence are very close to being equal. We will then attempt to perturb these coefficients to obtain an any-ordered independence polynomial for $s_{\lceil \frac{\alpha}{2} \rceil}, \ldots, s_\alpha$.

The foundation for our work is complete and from here we will discuss our attempts to improve the results of Matchett. This is a difficult task and unfortunately we did not make any progress on the Roller-Coaster conjecture. However, we were able to come up with a new theorem and definition regarding *flat* roller-coaster graphs which are not necessarily well-covered. This new concept had not been considered before and yields an interesting result.

Chapter 3

Research

This section will be broken into two pieces so that we may first give a brief overview as to what techniques we will be using followed by computation and results.

3.1 Outline

We began our research by creating an algorithm which would mimic the results of Matchett [5]. Since we already knew several of the solutions Matchett was able to calculate, we posed the following question.

Question 21. *Could we re-create these solutions and use our algorithm to solve for larger cases of α?*

Although we were able to create such an algorithm, we were unable to prove the conjecture for $\alpha > 13$. With that, we developed a new strategy. Recall that Matchett [5] power magnifies independence polynomials of the form, $z^i(1+cz)^{\alpha-i}$, where $c = 1$.

Question 22. *What if $c \neq 1$? What would the independence polynomials look like if we powered magnified the alternating graphs of K_c and K_{c+1}?*

Throughout this paper we have alluded to some type of technique which Matchett [5] creates in order to solve the Roller-Coaster conjecture for $\alpha \leq 13$. We will briefly explain this technique and then discuss how by altering this technique we have graphs which are no longer well-covered. This error will allow us to create a new theorem in which we create a *flat* roller-coaster. Matchett [5] uses linear algebra and possible solutions to the equation

$$M\mathbf{x} = \mathbf{y}$$

for $\mathbf{x}$ where $\mathbf{x_i} \geq \mathbf{0}$. We will be a bit vague about these matrices' explicit values until the next section, but will give a brief description of each one. M represents an $(\alpha+1)\times(\alpha+1)$ matrix which contains exactly $\alpha+1$ different independence polynomials created by joining power magnified K_c and K_{c+1} graphs. $\mathbf{x}$ is an $(\alpha+1)\times 1$ matrix whose corresponding entries represent the number copies of each independence polynomial from M we need. And finally, $\mathbf{y}$ is an $(\alpha+1)\times 1$ matrix in which y_i represents some flat condition on the coefficients of our roller-coaster. In the equation $M\mathbf{x} = \mathbf{y}$, the matrix $\mathbf{x}$ is the one which we will need solve for. Thus our goal will be to solve

$$\mathbf{x} = M^{-1}\mathbf{y} \text{ if } M \text{ is invertible.}$$

3.2 Matrix Computation

The rest of this paper will be dedicated to solving the above equation. The first topic which we will discuss is how M is generated. We will then show the general form of any M matrix followed by an example. We then need to compute M^{-1}, which we will call A, and show that indeed $MA = I_\alpha = AM$. We already stated that the matrix M represents different independence polynomials, but did not say how this is done. For this, we introduce a new matrix $H_{\alpha,i}$, which will contain the independence polynomials of pseudographs. Note that H_α as well as M are indexed backwards.

H_α

The graph H_α is what generates any M matrix. The entries correspond to power magnified graphs which we alternate between K_c and K_{c+1}. Therefore,

$$H_\alpha = \bigvee_{i=0}^{j} \binom{j}{i} (c+1)_2^2 (z^{\alpha-i} E_0),$$

where $\binom{j}{i}$ tells us how many copies of a particular independent set we want. We will refer to each column in the above matrix as H_j where $0 \leq j \leq \alpha$. Therefore, $S(H_j, z)$ will be the independence polynomial corresponding to column j. This implies that H_α contains $\alpha + 1$ total independence polynomials. The rows, i, where $0 \leq i \leq \alpha$ count the number of independent sets of size i from which $S(H_j, z)$ is constructed. We will hold off on an example of H_α until we define M and A.

Now that we know how M is generated, we can discuss how to calculate M given any α.

3.3 M and its inverse

Before we begin our formal statement of M, we must note that we always assume that M is invertible. This is because in order to solve $\mathbf{x} = M^{-1}\mathbf{y}$, we must be able to guarantee that M^{-1} exists. Based on that assumption we will show how to calculate M given α.

Although we use Matchett's M matrix as a guide, the way in which we construct our matrix M differs in that Matchett's matrix only considers K_c with $c = 1$. In our research we considered alternating complete graphs, K_c and K_{c+1} for some large value of c. For our computation we consider c and α to always be even. The importance of this will be seen later when we prove our results.

Let M be an invertible $(\alpha+1) \times (\alpha+1)$ matrix in which $M = (m_{i,j})$. Further, assume that M is upper-triangular whose row entries correspond to

$$m_{i,j} = (c + (i+1)_2)^i \binom{j}{i}; \quad \text{where } i, j \in \{0, 1, 2, \ldots, \alpha\}$$

where

$$(i+1)_2 = \begin{cases} 0 & \text{if } i+1 \text{ is even} \\ 1 & \text{if } i+1 \text{ is odd.} \end{cases}$$

Note that $(i+1)_2$ represents the addition of $i+1$ modulo 2. Using these constraints we compute the general form of M to be,

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & c^1\binom{1}{1} & c^1\binom{2}{1} & c^1\binom{3}{1} & \dots & \binom{\alpha}{1}c^1 \\ 0 & 0 & (c+1)^2\binom{2}{2} & (c+1)^2\binom{3}{2} & \dots & \binom{\alpha}{2}(c+1)^2 \\ 0 & 0 & 0 & c^3\binom{3}{3} & \dots & \binom{\alpha}{3}c^3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \binom{\alpha-1}{\alpha}c^{\alpha-1} \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

The reader might be wondering why the final entry of the above matrix is always 1. This is because the α column represents $S(H_\alpha, z)$ is which each of the entries have been power magnified by z^α. However, if we power magnify E_0 by z^α the result does not change. We will always have one independent set of size zero, power magnification does not alter this result.

Continuing on, we compute the binomial coefficients to obtain a simplified version of M.

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & c & 2c & 3c & 4c & 5c & \dots & \alpha c \\ 0 & 0 & (c+1)^2 & 3(c+1)^2 & 6(c+1)^2 & 10(c+1)^2 & \dots & \binom{\alpha}{2}(c+1)^2 \\ 0 & 0 & 0 & c^3 & 4c^3 & 10c^3 & \dots & \binom{\alpha}{3}c^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \binom{\alpha-1}{\alpha}c^{\alpha-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 1 \end{bmatrix}.$$

Using this form of M we can give an example of M in which $c = 4$ and $\alpha = 2$. Then, M is a 3×3 matrix whose entries are calculated using K_4 and K_5 graphs.

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now that we have found the general form of any matrix, M, we will discuss the general form of its computed inverse. The reason for doing so is because our goal is to find solutions to the equation $M\mathbf{x} = \mathbf{y}$ in terms of $\mathbf{x}$. We will introduce a new matrix A which is of the form $A = (a_{i,j})$. Assume that A is a $(\alpha+1) \times (\alpha+1)$ matrix whose entries can be calculated using the M matrix above. The entries of A are defined by:

i.) $a_{i,j} = \frac{\binom{j}{i}}{k^j}$ where $i \in \{0, 1, 2, \ldots, \alpha\}$ and $j \in \{0, 1, 2, \ldots, \alpha - 1\}$ and

$$k = \begin{cases} c & \text{when } j \text{ is odd} \\ c+1 & \text{when } j \text{ is even.} \end{cases}$$

ii.) Each entry $a_{i,\alpha} = (-1)^{i+\alpha}\binom{\alpha}{i}$;

iii.) $a_{i,j} > 0,\ \forall i = j$ then row entries will alternate signs.

Using these properties, we compute the general form for all $A = (a_{i,j})$ to be

$$a_{i,j} = \frac{\binom{j}{i}}{(c + (j+1)_2)^j}(-1)^{i+j}.$$

We can then compute the general form of A as follows:

$$A = \begin{bmatrix} \frac{\binom{0}{0}}{(c+1)^0} & -\frac{\binom{1}{0}}{c^1} & \frac{\binom{2}{0}}{(c+1)^2} & \cdots & -\frac{\binom{\alpha-1}{0}}{c^{\alpha-1}} & (-1)^{0+\alpha}\binom{\alpha}{0} \\ 0 & \frac{\binom{1}{1}}{c^1} & -\frac{\binom{2}{1}}{(c+1)^2} & \cdots & \frac{\binom{\alpha-1}{1}}{c^{\alpha-1}} & (-1)^{1+\alpha}\binom{\alpha}{1} \\ 0 & 0 & \frac{\binom{2}{2}}{(c+1)^2} & \cdots & -\frac{\binom{\alpha-1}{2}}{c^{\alpha-1}} & (-1)^{2+\alpha}\binom{\alpha}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (-1)^{\alpha+\alpha}\binom{\alpha}{\alpha} \end{bmatrix}.$$

Again, we can compute the binomial coefficients to achieve,

$$A = \begin{bmatrix} 1 & -\frac{1}{c} & \frac{1}{(c+1)^2} & -\frac{1}{c^3} & \frac{1}{(c+1)^4} & -\frac{1}{c^5} & \cdots & -\frac{1}{c^{\alpha-1}} & (-1)^{\alpha} \\ 0 & \frac{1}{c} & -\frac{2}{(c+1)^2} & \frac{3}{c^3} & -\frac{4}{(c+1)^4} & \frac{5}{c^5} & \cdots & \frac{\binom{\alpha-1}{1}}{c^{\alpha-1}} & (-1)^{\alpha-1}\binom{\alpha}{1} \\ 0 & 0 & \frac{1}{(c+1)^2} & -\frac{3}{c^3} & \frac{6}{(c+1)^4} & -\frac{10}{c^5} & \cdots & -\frac{\binom{\alpha-1}{2}}{c^{\alpha-1}} & (-1)^{\alpha-2}\binom{\alpha}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the properties of A from above we calculate M^{-1} to be,

$$M^{-1} = \begin{bmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & \frac{1}{4} & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Taking the example one step further we can calculate that $MM^{-1} = I_3$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & \frac{1}{4} & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Recall that the general form of A is dependent on the assumption that A is in fact the inverse of M. We will now show that given the general form of any matrix M, we can compute a matrix A in which $MA = I_3 = AM$. This will allow us to conclude that A is the inverse of M.

Any matrix M consists of entries with the following properties,

$$m_{i,j} = \begin{cases} (c + (i+1)_2)^i \binom{j}{i} & 0 \text{ if } i \neq j \neq \alpha \\ 1 & \text{if } i = j = \alpha, \end{cases}$$

where $0 \leq i \leq \alpha$ and $0 \leq j \leq \alpha$. We claim that the inverse of M, denoted A, consists of entries in which,

$$a_{i,j} = \begin{cases} \frac{\binom{j}{i}}{(c+(j+1)_2)^j}(-1)^{i+j} & \text{if } j \neq \alpha \\ \binom{\alpha}{i}(-1)^{i+\alpha} & \text{if } j = \alpha, \end{cases}$$

where $0 \leq i \leq \alpha$ and $0 \leq j \leq \alpha$. What we will show first is that

$$(MA)_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

The first case which will consider is that of $i > j$. Since the product of 2 upper triangular matrices will result in an upper triangular matrix, it follows that if $i > j$ then $(MA)_{i,j} = 0$.

The second case is to consider $i < j \neq \alpha$.

$$\begin{aligned} (MA)_{i,j} &= \sum_{k=0}^{\alpha} m_{i,k}a_{k,j} \\ &= \sum_{k=0}^{\alpha}(c+(i+1)_2)^i \binom{k}{i} \frac{\binom{j}{k}}{(c+(j+1)_2)^j}(-1)^{i+k} \\ &= \frac{(c+(i+1)_2)^i}{(c+(j+1)_2)^j} \sum_{k=0}^{\alpha}(-1)^{j+k}\binom{k}{i}\binom{j}{k}. \end{aligned}$$

Note that the equation is found since $\frac{(c+(i+1)_2)^i}{(c+(j+1)_2)^j}$ is a constant term. To show that

$$\sum_{k=0}^{\alpha}(-1)^{j+k}\binom{k}{i}\binom{j}{k} = 0$$

we must use the Binomial Theorem which states that

$$(x+y)^k = \sum_{i=0}^{k}\binom{k}{i}x^i y^{k-i}.$$

Using the Binomial Theorem and letting $y = -1$, for the $i = 0$ case we get

$$(x-1)^j = \sum_{k=0}^{j}(-1)^{j-k}x^k\binom{j}{k}.$$

If we then let $x = 1$,

$$0 = (1-1)^j = \sum_{k=0}^{j} (-1)^{j-k} 1^k \binom{j}{k} = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k}.$$

Next we want to consider the case when $i = 1$ so we take the derivative of

$$(x-1)^j = \sum_{k=0}^{j} (-1)^{j-k} x^k \binom{j}{k} \text{ with respect to } x.$$

For each case, we will always let $y = -1$. Then

$$j(x-1)^{j-1} = \sum_{k=0}^{j} k \binom{j}{k} x^{k-1} (-1)^{j-k} = \sum_{k=0}^{j} k \binom{j}{k} x^{k-1} (-1)^{j-k}.$$

If we again let $x = 1$ then this case will also result in 0. If we want to solve for the $i = 2$ case we will again take the derivative with respect to x and get

$$j(j-1)(x-1)^{j-2} = \sum_{k=0}^{j} k(k-1) \binom{j}{k} x^{k-2} (-1)^{j-k} = \sum_{k=0}^{j} k(k-1) \binom{j}{k} x^{k-2} (-1)^{j-k}.$$

However, in this equation we do not have $\binom{k}{2}$ which we would expect. To remedy this, we can divide both sides by 2 to get,

$$\frac{j(j-1)(x-1)^{j-2}}{2} = \sum_{k=0}^{j} \frac{k(k-1)}{2} \binom{j}{k} x^{k-2} (-1)^{j-k} = \sum_{k=0}^{j} \binom{k}{2} \binom{j}{k} x^{k-2} (-1)^{j-k}.$$

Let $x = 1$ and the result holds.

Using this knowledge, we can calculate the result for any value of $i < j < \alpha$ by taking the i^{th} derivative with respect to x:

$$(j)_i (x-1)^{j-i} = \sum_{k=0}^{j} (k)_i \binom{j}{k} x^{k-i} (-1)^{j-k},$$

where $(k)_i$ is known as the falling factorial such that

$$(k)_i = k(k-1)(k-2) \cdots (k-i+1).$$

As we have done previously we multiply the above equation by $\frac{1}{i!}$ such that

$$\frac{(k)_i}{i!} = \binom{k}{i}.$$

Therefore, we obtain the equation,

$$\frac{(j)_i}{i!}(x-1)^{j-i} = \sum_{k=0}^{j} \frac{(k)_i}{i!}\binom{j}{k} x^{k-i}(-1)^{j-k}.$$

If we let $x = 1$ then $\sum_{k=0}^{j} \frac{(k)_i}{i!}\binom{j}{k} x^{k-i}(-1)^{j-k} = 0$ and we can conclude that

$$\sum_{k=0}^{j}(-1)^{j-k}\binom{k}{i}\binom{j}{k} = 0$$

provided that $i < j < \alpha$. The result holds since $(-1)^{j+k}$ and $(-1)^{j-k}$ will always have the same sign.

Next we consider $i < j = \alpha$ which is similar to what was shown above, but our constant changes. Namely,

$$\begin{aligned}(MA)_{i,\alpha} &= \sum_{k=0}^{\alpha}(c+(i+1)_2)^i\binom{k}{i}\binom{\alpha}{k}(-1)^{\alpha+k}\\ &= (c+(i+1)_2)^i\sum_{k=0}^{\alpha}\binom{k}{i}\binom{\alpha}{k}(-1)^{i+k}.\end{aligned}$$

Unlike the previous case, our constant does not have a denominator. This is because the entries in the α column do not contain any constant factors. Using a similar proof as above using the Binomial Theorem, $\sum_{k=0}^{\alpha}\binom{k}{i}\binom{\alpha}{k}(-1)^{i+k} = 0$.

The next case is when $i = j = \alpha$. The result hold because of the construction of our matrices and the fact that we have assumed that α is even:

$$(MA)_{i,j} = (0,0,\ldots,1)\cdot(0,0,\ldots,1) = 1.$$

Therefore, if $i = j = \alpha$, then $(MA)_{i,j} = 1$.

The final case we must consider is $i = j < \alpha$.

$$\begin{aligned}(MA)_{i,i} &= \sum_{k=0}^{\alpha}(c+(c+1)_2)^i \binom{k}{i} \frac{\binom{i}{k}}{(c+(i+1)_2)^i}(-1)^{i+k} \\ &= \sum_{k=0}^{\alpha}\binom{k}{i}\binom{i}{k}.\end{aligned}$$

In the above equation, if $i < k$ then $\binom{k}{i} = 0$ and if $i > k$ then $\binom{i}{k} = 0$. This implies that the only relevant term exists when $i = k$. If $i = k$ then

$$\binom{i}{i}\binom{i}{i}(-1)^{i+i} = 1.$$

Therefore, $(MA)_{i,i} = 1$. Combining the results of each of the five cases from above we can conclude that $(MA)_{i,j} = I_{\alpha+1}$.

Next we must show that

$$(AM)_{i,j} = \begin{cases} 1 & \text{of } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The first case is the same as that above. Consider $i > j$ then $(AM)_{i,j} = 0$ since the product of two upper triangular matrices is upper triangular. The next case is when $i = j = \alpha$. Again, the result is the same as the previous case,

$$(AM)_{i,j} = (0, 0, \ldots, 1) \cdot (0, 0, \ldots, 1) = 1.$$

The next case is when $i = j < \alpha$.

$$\begin{aligned}
(AM)_{i,i} &= \sum_{k=0}^{\alpha} a_{i,k} m_{k,i} \\
&= \sum_{k=0}^{\alpha-1} \frac{\binom{k}{i}}{(c+(k+1)_2)^k} (-1)^{i+k} (c+(k+1)_2)^k \binom{i}{k} \\
&+ \binom{\alpha}{i} (-1)^{i+\alpha} (c+(\alpha+1)_2)^{\alpha} \\
&= 1.
\end{aligned}$$

This result hold since each binomial coefficient will be zero with the exception of the case when $i = k$ as we saw previously.

The next case is when $i < j \neq \alpha$.

$$\begin{aligned}
(AM)_{i,j} &= \sum_{k=0}^{\alpha} a_{i,k} m_{k,j} \\
&= \sum_{k=0}^{\alpha-1} \frac{\binom{k}{i}}{(c+(k+1)_2)^k} (-1)^{i+k} (c+(k+1)_2)^k \binom{j}{k} \\
&+ \binom{\alpha}{i} (-1)^{i+k} (c+(\alpha+1)_2)^{\alpha} \binom{j}{k} \\
&= \sum_{k=0}^{j} \binom{k}{i} (-1)^{i+k} \binom{j}{k}.
\end{aligned}$$

By the Binomial Theorem we know that $\sum_{k=0}^{j} \binom{k}{i} (-1)^{i+k} \binom{j}{k} = 0$ and the result is shown.

The final case we must show is when $i < j = \alpha$.

$$\begin{aligned}
(AM)_{i,j} &= \sum_{k=0}^{\alpha-1} \frac{\binom{k}{i}}{(c+(k+1)_2)^k} (-1)^{i+k} (c+(k+1)_2)^k \binom{\alpha}{i} \\
&+ \binom{\alpha}{i} (-1)^{i+\alpha} (c+(\alpha+1)_2)^{\alpha} \binom{\alpha}{\alpha} \\
&= \sum_{k=0}^{\alpha-1} \binom{k}{i} (-1)^{i+k} \binom{\alpha}{k} + \binom{\alpha}{i} (-1)^{i+\alpha} \\
&= \sum_{k=0}^{\alpha} \binom{k}{i} (-1)^{i+k} \binom{\alpha}{k} = 0
\end{aligned}$$

by the Binomial Theorem.

Thus we have proved that $MA = I_{\alpha+1} = MA$ and $A = M^{-1}$. Therefore, for any matrix M we can compute $A = M^{-1}$ such that $MA = I_{\alpha+1}$.

One would assume that the next step in our discussion should be focused on how to compute $\mathbf{x}$ and $\mathbf{y}$, however, we have made a fundamental error in calculating M. Before we continue any further calculations we must discuss what this error is and where it occurred.

3.4 No longer well-covered

Through all our computation and work we realize that we have not truly mimicked Matchett's work nor his results. Let's refer to our example from before where $\alpha = 2$ and $c = 4$:

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix}.$$

From our definition of $S(H_j, z)$, M consists of 3 different independence polynomials:

$$S(H_0, z) = 0(z^0E_0) \vee 0(z^1E_0) \vee 1(z^2E_0) = 0z^0 + 0z^1 + 1z^2 = z^2 \tag{3.1}$$

$$S(H_1, z) = 0(z^0E_0) \vee 4(z^1E_0) \vee 1(z^2E_0) = 0z^0 + 4z + 1z^2 = 4z + z^2 \tag{3.2}$$

$$S(H_2, z) = 1(z^0E_0) \vee 8(z^1E_0) \vee 1(z^2E_0) = 1z^0 + 8z + 1z^2 = 1 + 8z + z^2 \tag{3.3}$$

These independence polynomials are created by taking p copies of well-covered graphs and then joining them together. The join of two well-covered graphs is itself well-covered if and only if they have the same independence number. If we look at the construction of equation (5), we conclude that the definition of well-coveredness has been violated. In equation (5), $1(z^0E_0) \vee 8(zE_0) \vee 1(z^2E_0)$ would not result in a graph which is well-covered. Namely, $8(zE_0)$ has $\alpha = 1$ whereas $1(z^2E_0)$ has $\alpha = 2$. Because we joined two well-covered graphs with different independence numbers our resulting polynomial represents a graph which is not well-covered. Our problem initially occurs with the construction of H_α. Had we done a bit more investigation we could have realized a bit sooner than the construction was flawed. If we look at the general form of M,

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & c^1\binom{1}{1} & c^1\binom{2}{1} & c^1\binom{3}{1} & \dots & c^1\binom{\alpha}{1} \\ 0 & 0 & (c+1)^2\binom{2}{2} & (c+1)^2\binom{3}{2} & \dots & (c+1)^2\binom{\alpha}{2} \\ 0 & 0 & 0 & c^3\binom{3}{3} & \dots & c^3\binom{\alpha}{3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

and considered the construction of the $\alpha - 2$ column, as an example, we would have seen that

$$S(\alpha-2, z) = z^\alpha E_0 \vee (2c(z^{\alpha-1}E_0) \vee [(c+1)^2(z^{\alpha-2}E_0)]$$

results in a pseudograph which is not well-covered since

$$\alpha(z^\alpha E_0) \neq \alpha((2c(z^{\alpha-1}E_0)) \neq \alpha([(c+1)^2(z^{\alpha-2}E_0)]).$$

This taught us a valuable lesson in that Matchett's construction of M with only K_1 graphs was well thought out. This clever construction guaranteed that the joins would always be well-covered since $\alpha(K_1) = 1$.

Thus we have answered the questions posed before our research began. Yes, we could create an algorithm in which we obtain similar results to Matchett. However, we were unable to solve for $\alpha > 13$. Further, we were able to construct M using K_c and K_{c+1}, but failed to maintain well-coveredness. However, we were able to come up with a new concept which we will refer to as *flat roller-coasters*.

3.5 Flat Roller-Coasters

A *flat roller-coaster* is a new term which defines a polynomial in which the coefficients are equal.

Definition 16. *A polynomial is said to be* flat *if its independence polynomial is* $q_0 + q_1 z + q_2 z^2 + \cdots + q_\alpha z^\alpha$ *in which* $\mid q_i - q_j \mid = 0$*, therefore* $q_i = q_j$ *for any* $0 < i, j \leq \alpha$.

In order to calculate a flat roller-coaster we will again refer to the equation, $M\mathbf{x} = \mathbf{y}$. In our earlier example, we did not calculate $\mathbf{x}$ or $\mathbf{y}$. Recall that x_i tells us how many copies of each H_j we need in order to satisfy some flat condition on $\mathbf{y}$. y_i represents the number of independent sets of G are of size i. As was the case earlier, these independence polynomials may not represent true graphs, however we can scale the coefficients such that we achieve an independence polynomial of a true graph.

The general form of $M\mathbf{x} = \mathbf{y}$ is

$$\begin{bmatrix} m_{\alpha,\alpha} & m_{\alpha,\alpha-1} & m_{\alpha,\alpha-2} & \cdots & m_{\alpha,0} \\ m_{\alpha-1,\alpha} & m_{\alpha-1,\alpha-1} & m_{\alpha-1,\alpha-2} & \cdots & m_{\alpha-1,0} \\ m_{\alpha-2,\alpha} & m^{\alpha-2,\alpha-1} & m_{\alpha-2,\alpha-2} & \cdots & m_{\alpha-2,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{0,\alpha} & m_{0,\alpha-1} & m_{0,\alpha-2} & \cdots & m_{0,0} \end{bmatrix} \begin{bmatrix} x_\alpha \\ x_{\alpha-1} \\ x_{\alpha-2} \\ \vdots \\ x_0 \end{bmatrix} = \begin{bmatrix} y_\alpha \\ y_{\alpha-1} \\ y_{\alpha-2} \\ \vdots \\ y_0 \end{bmatrix}.$$

Continuing with our prior example,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_1 \\ y_0 \end{bmatrix}.$$

Recall that we are trying to solve for $\mathbf{x}$ so we must consider $\mathbf{x} = M^{-1}\mathbf{y}$, as below:

$$\begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & \frac{1}{4} & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_2 \\ y_1 \\ y_0 \end{bmatrix}.$$

Therefore,

$$\mathbf{x} = \begin{bmatrix} \frac{7}{8} \\ 0 \\ \frac{1}{8} \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{8} \end{bmatrix}.$$

We should mention one important fact about $\mathbf{y}$. Recall that $m_{0,0} \neq 1$. In order for our solution to make sense for a true graph, we must have $m_{0,0} = 1$. For any $\mathbf{y}$ we will normalize the $m_{0,0}$ term by multiplying by the inverse of $m_{0,0}$. In the example above, this means we will multiply each term by 8 since $\frac{1}{8} \cdot 8 = 1$. Therefore, we can finally conclude that if we take $\frac{7}{8}$ copies of H_2, 0 copies of H_1, and $\frac{1}{8}$ copies of H_0 the resulting independence polynomial will be flat after being normalized. After this normalization occurs, we get a flat roller-coaster. Its corresponding polynomial is

$$S(G, z) = 8z^2 + 8z + 1.$$

From this, we formulate a new theorem which states that for any value of α, a flat roller-coaster exists.

Theorem 23. *Let α be even. Then there exists a pseudograph with independence polynomial $s_0 + s_1 z + \ldots + s_\alpha z^\alpha$ such that $s_i = s_j$ for all $1 \leq i, j \leq \alpha$.*

One should wonder how we are able to guarantee that $\mathbf{x} \geq 0$. Recall that we made the claim earlier in this paper that for our M matrix we would always consider α to be even. What we will show is that if we sum pairs of row entries from M^{-1} each sum will be non-negative. Since $\mathbf{y}$ is a vector containing all ones, with the exception of the last column, if the entries of M^{-1} are non-negative then $\mathbf{x}$ must also be non-negative. We will consider the last column individually by excluding it from any summations. First, we will prove that

$a_{i,j} > 0$ where $i = j$ for all $0 \leq i, j \leq \alpha$.

Proof. Fix i. Assume that α is even which implies that all diagonal entries are positive since each diagonal entry is computed by

$$a_{i,i} = (-1)^{i+i} \frac{\binom{i}{i}}{(c + (i+1)_2)^i} > 0 \text{ where } 0 \leq i \leq \alpha.$$

Next what we will show is that if we start the summation of entries of M^{-1} on the positive diagonal then each pair of entries $i, i+1$ will sum to a non-negative integer.

$$\sum_{j=i}^{\alpha-1} (-1)^{i+j} \frac{\binom{j}{i}}{(c + (j+1)_2)^j} + (-1)^{i+\alpha} \frac{\binom{\alpha}{i}}{(c + (\alpha+1)_2)^\alpha}$$
$$= (-1)^{i+i} \frac{\binom{i}{i}}{c + (i+1)_2)^i} + (-1)^{i+1+i} \frac{\binom{i+1}{i}}{(c + (i+1+1)_2)^{i+1}}$$

$$= \begin{cases} \frac{1}{c^i} - \frac{i+1}{(c+1)^{i+1}} & \text{if } i \text{ is odd} \\ \frac{1}{(c+1)^i} - \frac{i+1}{c^{i+1}} & \text{if } i \text{ is even.} \end{cases}$$

We note that,

$$\frac{1}{(c+1)^i} \geq \frac{i+1}{c^{i+1}} \iff c^{i+1} \geq (c+1)^i(i+1) = (i+1)c^i + (i+1)\binom{i}{1}c^{i-1} + \cdots + (i+1)\binom{i}{j}c^{i-j} + \cdots$$

provided that c is large enough the above result will hold.

In general, for any pairwise entires $a_{i,j}$ and $a_{i,j+1}$ we compute the sum as,

$$(-1)^{i+j} \frac{\binom{j}{i}}{(c + (j+1)_2)^j} + \frac{(-1)^{i+j+1}\binom{j+1}{i}}{(c + (j+2)_2)^{j+1}}$$
$$= (-1)^{i+j} \frac{\binom{j}{i}}{(c + (j+1)_2)^j} + \frac{(-1)^{i+j+1}\binom{j+1}{i}}{(c + (j)_2)^{j+1}}$$

Again, we note that there are two cases when solving the above equation,

$$(-1)^{i+j}\frac{\binom{j}{i}}{(c+(j+1)_2)^j}+\frac{(-1)^{i+j+1}\binom{j+1}{i}}{(c+(j)_2)^{j+1}}$$

$$=\begin{cases}\frac{\binom{j}{i}}{c^j}-\frac{\binom{j+1}{i}}{(c+1)^{j+1}} & \text{if } i \text{ is odd}\\ \frac{\binom{j}{i}}{(c+1)^j}-\frac{\binom{j+1}{i}}{c^{j+1}} & \text{if } i \text{ is even.}\end{cases}$$

If the sum of the pairwise entry contains the α column then a similar result hold since

$$\frac{\binom{\alpha-1}{i}}{c^{\alpha-1}}-\frac{\binom{\alpha}{i}}{(c+1)^{\alpha}}$$

provided that c is large enough. □

Thus we have just shown that given any even α and c there exists a flat roller coaster and our work for this paper is complete.

Chapter 4

Conclusion

Our thesis work attempted to improve and expand on the results of the Roller-Coaster conjecture originally stated by Michael and Traves [6]. Philip Matchett has gained the largest result thus far by proving the conjecture true for $\alpha \leq 11$. Although we did not reach our initial goal, we were able to construct a matrix, M, which possesses some nice properties. Rather than using Matchett's construction which used only complete graphs on one vertex, we considered graphs on c vertices where $c > 1$. The resulting matrix did yield results, however, our results regarded all graphs and not just those which were well-covered. Instead of expanding on prior results, we proved a theorem that states that there exists a graph whose coefficients are equal with the exception of the first term.

In the future we hope to work with our M matrix and attempt to gain insight into how to manipulate our construction so that we can gain results with well-covered graphs. We realize that Matchett's construction is extremely clever, but believe that there is a way to use complete graphs other than K_1s to solve the roller-coaster conjecture.

One area of work left for future research is to consider a different class of graphs. Our research was only concerned with graphs that are well-covered, but other graphs certainly could be used. We are unsure if this would yield results, but is an interesting idea for future work.

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