Journal of Fundamental and Applied Sciences

ISSN 1112-9867

Available online at http://www.jfas.info

Research Article

Special Issue

THE CENTRAL SUBGROUP OF THE NONABELIAN TENSOR SQUARE OF BIEBERBACH GROUP WITH POINT GROUP $C_2 \times C_2$

R. Masri[1*], N. F. A. Ladi[1], N. M. Idrus[1], Y. T. Tan[1] and N. H. Sarmin[2]

[1]Universiti Pendidikan Sultan Idris

[2]Universiti Teknologi Malaysia

Published online: 24 November 2017

ABSTRACT

A Bieberbach group with point group $C_2 \times C_2$ is a free torsion crystallographic group. A central subgroup of a nonabelian tensor square of a group G, denoted by $\nabla(G)$ is a normal subgroup generated by generator $g \otimes g$ for all $g \in G$ and essentially depends on the abelianization of the group. In this paper, the formula of the central subgroup of the nonabelian tensor square of one Bieberbach group with point group $C_2 \times C_2$, of lowest dimension 3, denoted by $S_3(3)$ is generalized up to n dimension. The consistent polycyclic presentation, the derived subgroup and the abelianization of group this group of n dimension are first determined. By using these presentations, the central subgroup of the nonabelian tensor square of this group of n dimension is constructed. The findings of this research can be

Keywords: Bieberbach group; central subgroup; nonabelian tensor square.

Author Correspondence, e-mail: rohaidah@fsmt.upsi.edu.my

doi: http://dx.doi.org/10.4314/jfas.v9i7s.10

1. INTRODUCTION

1.1. Introduction

A Bieberbach group is a free torsion crystallographic group. This group is an extension of free abelian group L of finite rank by a finite group P which satisfy the short exact sequence $1 \longrightarrow L \xrightarrow{\varphi} G \xrightarrow{\phi} P \longrightarrow 1$ such that the quotient group $G/\varphi(L) \cong P$ isomorphic to group P. Here L is called as the lattice group and P is a point group. The dimension of G is also known as the rank of L. In this case, G is called as a Bieberbach group with point group P.

Many properties of this group can be explored where one of the properties is its central subgroup of the nonabelian tensor square, $\nabla(G)$. The nonabelian tensor square, $G \otimes G$ of a group G is generated by the symbols $g \otimes h$, for all $g, h \in G$, subject to relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h') \qquad [1]$$

for all $g, g', h, h' \in G$, where ${}^{g}g' = gg'g^{-1}$. The nonabelian tensor square is a specialization of the more general nonabelian tensor product introduced by Brown and Loday [1].

The computations of $\nabla(G)$ of some Bieberbach groups with certain point groups can be found in previous studies. Masri [2] has constructed the abelianization and the $\nabla(G)$ of Bieberbach groups with cyclic point group of order 2. The results of the $\nabla(G)$ of the groups were then used to compute the nonabelian tensor square of the groups. The studies of the $\nabla(G)$ of some Bieberbach groups with dihedral point group can be found in Mohd Idrus et al. [3]. She used the central subgroup of the nonabelian tensor square of the group in order to determine the presentation of the nonabelian tensor square of the group. Also recently, Tan *et al.* [4] and Masri *et al.* [5] have explored the formula of the $\nabla(G)$ of the Bieberbach group with symmetric point group of certain dimension.

The subgroup $\nabla(G)$ is normal and is generated by $g \otimes g$ for all g in G. Blyth, Fumagalli and Morigi [6] have showed that there is a relationship between the structure of $\nabla(G)$ and the abelianization of the group, G^{ab} given by the following proposition.

Proposition 1 [6]

Let G be a group such that G^{ab} is finitely generated. Assume that G^{ab} is the direct product of the cyclic groups $\langle x_i G' \rangle$, for $i = 1, \ldots, s$ and set $E(G)$ to be $\langle [x_i, x_j^{\varphi}] \mid i < j \rangle [G, G'^{\varphi}]$. Then $\nabla(G)$ is generated by the elements of the set $\{[x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] \mid 1 \le i < j \le s\}$.

In this paper, our main interest is the Bieberbach group of lowest dimension 3 with elementary abelian 2-group point group, $C_2 \times C_2$, denoted as $S_3(3)$. The presentation of $\nabla(S_3(3))$ which has been determined in Abdul Ladi *et al*. [7] will be generalized up to dimension *n*. The consistent polycyclic presentation of the group $S_3(3)$ has been constructed in Abdul Ladi *et al*. [7] as in the following :

$$S_3(3) = \left\langle a_0, a_1, l_1, l_2, l_3 \left| \begin{array}{l} a_0^2 = l_1^{-1}, a_1^2 = l_3^{-1}, {}^{a_0}a_1 = a_1 l_1^{-1}, \\ {}^{a_0}l_1 = l_1, {}^{a_0}l_2 = l_2^{-1}, {}^{a_0}l_3 = l_3, \\ {}^{a_1}l_1 = l_1^{-1}, {}^{a_1}l_2 = l_2, {}^{a_1}l_3 = l_3, \\ {}^{l_1}l_2 = l_2, {}^{l_1}l_3 = l_3, {}^{l_2}l_3 = l_3 \end{array} \right. \right\rangle (1).$$

1.2 Preliminaries

Some basic definitions and structural results related to this study are presented in this section. The consistent polycyclic presentations of group $S_3(n)$ is constructed based on the following two definitions of the polycyclic presentation of group and the consistent polycyclic presentation of group [8]. First, the definition of the polycyclic presentation is given as follows:

Definition 1 [8]

Let F_n be a free group on generators $g_1, \ldots, g_n$ and R be a set of relations of group G. The relations of a polycyclic presentation of F_n / R have the form:

$$g_i^{e_i} = g_{i+1}^{x_{i,i+1}} \cdots g_n^{x_{i,n}} \qquad \text{for } i \in I,$$

$$g_j^{-1} g_i g_j = g_{j+1}^{y_{i,j,j+1}} \cdots g_n^{y_{i,j,n}} \qquad \text{for } j < i,$$

$$g_j g_i g_j^{-1} = g_{j+1}^{z_{i,j,j+1}} \dots g_n^{z_{i,j,n}} \qquad \text{for } j < i,\ j \notin I.$$

for some $I \subseteq \{1,\dots,n\}$, certain exponents $e^i \in \mathbb{N}$, for $i \in I$, and $x_{i,j}, y_{i,j,k}, z_{i,j,k} \in \mathbb{Z}$, for all i, j and k.

Definition 2 [8]

Let G be a group generated by $g_1,\dots,g_n$ and the consistency relations in G can be determined using the following consistency relations.

$$g_k(g_j g_i) = (g_k g_j) g_i \qquad \text{for } k > j > i,$$

$$(g_j^{e_j}) g_i = g_j^{e_j - 1}(g_j g_i) \qquad \text{for } j > i, j \in I,$$

$$g_j(g_i^{e_i}) = (g_j g_i) g_i^{e_i - 1} \qquad \text{for } j > i, i \in I,$$

$$(g_i^{e_i}) g_i = g_i (g_i^{e_i}) \qquad \text{for } i \in I,$$

$$g_j = (g_j g_i^{-1}) g_i \qquad \text{for } j > i, i \notin I$$

for some $I \subseteq \{1,\dots,n\}$, for certain exponents $e^i \in \mathbb{N}$, $i \in I$. Therefore, the consistent polycyclic presentation of $S_3(n)$ can be determined by using Definition 1 and 2.

The consistency of polycyclic presentation of group $S_3(n)$ need to be determined in order to use the computational method of polycyclic groups [9]. Next, the definition of the abelianization of group is given as follows.

Definition 3

The abelianization of a group G, G^{ab} is the quotient of group G by its derived subgroup, G'

In 1991, Rocco [10] has initiated in investigating the group $\nu(G)$ which is defined as in the following.

Definition 4

Let G be a group with presentation $\langle G \mid R \rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi: g \to g^{\varphi}$ for all $g \in G$. The group $\nu(G)$ is defined to be

$$\nu(G)=\left\langle G,G^{\varphi} \mid R,R^{\varphi}, {}^{x}[g,h^{\varphi}]=[{}^{x}g,({}^{x}h)^{\varphi}]= {}^{x^{\varphi}}[g,h^{\varphi}], \forall x,g,h\in G\right\rangle.$$

Next theorem shows that $G \otimes G$ is isomorphic to a subgroup $[G,G^{\varphi}]$ of $\nu(G)$.

Theorem 1 ([10],[11])

Let G be a group. The map $\sigma: G\otimes G \to [G,G^{\varphi}] \triangleleft \nu(G)$ defined by $\sigma(g\otimes h)=[g,h^{\varphi}]$ for all g,h in G is an isomorphism.

With this theorem, all the tensor computations can be translated into the commutator computation within the subgroup $[G,G^{\varphi}]$ of $\nu(G)$.

In this paper, the subgroup $[G,G^{\varphi}]$ of $\nu(G)$ will be used to compute the presentation of the central subgroup of the nonabelian tensor square of group $S_3(n)$, denoted by $\nabla(S_3(n))$. Next, a list of commutator identities in $\nu(G)$ is given as follows. Let x, y and z be elements of group G. Then, for the left conjugation, ${}^{x}y = xyx^{-1}$ and the list of commutators are presented as in the following:

$$[xy,z]= {}^{x}[y,z]\cdot[x,z] \quad (2)$$

$$[x,yz]=[x,y]\cdot {}^{y}[x,z] \quad (3)$$

$$[x^{-1},y]=[x^{-1},[x,y]^{-1}]\cdot[x,y]^{-1} \quad (4)$$

$$[x,y^{-1}]=[y^{-1},[x,y]^{-1}]\cdot[x,y]^{-1} \quad (5)$$

$$[x^{-1},y^{-1}]=[x^{-1},[y^{-1},[x,y]]]\cdot[y^{-1},[x,y]]\cdot[x^{-1},[x,y]]\cdot[x,y] \quad (6)$$

$$ {}^{z}[x,y]=[{}^{z}x, {}^{z}y] \quad (7)$$

Proposition 2 [2]

Let G be any Bieberbach group of dimension n with point group P and lattice group L. Let $B = G \times F_m^{ab}$ where F_m^{ab} be a free abelian group of rank m. Then B is a Bieberbach group of dimension $n+m$ with point group P.

The derived subgroup $S_3(3)$, $S_3(3)'$, the abelianization of $S_3(3)$, $S_3(3)^{ab}$ and the central subgroup of the nonabelian tensor square of $S_3(3)$, $\nabla(S_3(3))$ have been determined as follows.

Proposition 3 [7]

The group $S_3(3)$ has derived subgroup, $S_3(3)' = \langle l_1^{-2}, l_2^{-2} \rangle$ and the abelianization of $S_3(3)$ is generated by cosets $l_1 S_3(3)'$ of order 2, $l_2 S_3(3)'$ of order 2 and $l_3 S_3(3)'$ of infinite order. In symbols,

$$S_3(3)^{ab} \cong \langle l_1 S_3(3)', l_2 S_3(3)', l_3 S_3(3)' \rangle \cong C_2^{\,2} \times C_0.$$

Proposition 4 [7]

The subgroup $\nabla(S_3(3))$ is generated by generators $[l_1, l_1^{\varphi}]$ and $[l_2, l_2^{\varphi}]$ of order 4, generator $[l_3, l_3^{\varphi}]$ of infinite order, generators $[l_1, l_2^{\varphi}][l_2, l_1^{\varphi}]$, $[l_1, l_3^{\varphi}][l_3, l_1^{\varphi}]$, and $[l_2, l_3^{\varphi}][l_3, l_2^{\varphi}]$, of order 2. In symbols,

$$\begin{aligned}\nabla(S_3(3)) &= \langle [l_1, l_1^{\varphi}], [l_2, l_2^{\varphi}], [l_3, l_3^{\varphi}], [l_1, l_2^{\varphi}][l_2, l_1^{\varphi}], [l_1, l_3^{\varphi}][l_3, l_1^{\varphi}], [l_2, l_3^{\varphi}][l_3, l_2^{\varphi}] \rangle \\ &\cong C_2^{\,3} \times C_4^{\,2} \times C_0.\end{aligned}$$

The following propositions are some basic identities used in this paper.

Proposition 5 [6]

Let G be any group. Then the following hold:

(i) If $g_1 \in G'$ or $g_2 \in G'$, then $[g_1, g_2^{\varphi}]^{-1} = [g_2, g_1^{\varphi}]$.

(ii) $[Z(G), (G')^{\varphi}] = 1$.

(iii) If A and B are two subgroups of G with $B \leq G'$, then $[A, B^{\varphi}] = [B, A^{\varphi}]$. In particular, $[G, G'^{\varphi}] = [G', G^{\varphi}]$.

Proposition 6 ([2], [9])

Let g and h be elements of G such that $[g,h]=1$. Then, in $\nu(G)$,

(i) $[g^n, h^{\varphi}] = [g, h^{\varphi}]^n = [g, (h^{\varphi})^n]$ for all integers n;

(ii) $[g^n, (h^m)^{\varphi}][h^m, (g^n)^{\varphi}] = ([g, h^{\varphi}][h, g^{\varphi}])^{nm}$ for all integers n, m ;

(iii) $[g, h^{\varphi}]$ is in the centre of $\nu(G)$.

Proposition 7 [2]

Let G and H be groups and let $g \in G$. Suppose ϕ is a homomorphism from G onto H. If $\phi(g)$ has finite order then $|\phi(g)|$ divides $|g|$. Otherwise the order of $\phi(g)$ equals to order of g.

Proposition 8 [12]

Let A, B and C be abelian groups. The properties of the ordinary tensor product of two abelian groups are given as in the following.

(i) $C_0 \otimes A \cong A$,

(ii) $C_0 \otimes C_0 \cong C_0$,

(iii) $C_n \otimes C_m \cong C_{\gcd(n,m)}$, for $n, m \in \mathbb{N}$, and

(iv) $A \otimes (B \times C) = (A \otimes B) \times (A \otimes C)$.

Proposition 9 [1]

Let G and H be groups such that there is an epimorphism $\varepsilon : G \to H$. Then there exists an epimorphism

$$\alpha : G \otimes G \to H \otimes H$$

defined by $\alpha(g \otimes h) = \varepsilon(g) \otimes \varepsilon(h)$.

2. RESULTS AND DISCUSSION

In this section, the central subgroup of the nonabelian tensor square of $S_3(3)$, will be generalized up to n dimension. The generalization of polycyclic presentation of $S_3(3)$ of dimension $n, S_3(n)$ is constructed first.

Lemma 1

The polycyclic presentation of $S_3(n)$,

$$S_3(n)=\left\langle a_0,a_1,l_1,\ldots,l_n \;\middle|\; \begin{array}{l} a_0^2=l_1^{-1}, a_1^2=l_3^{-1}, {}^{a_0}a_1=a_1l_1^{-1}, \\ {}^{a_0}l_1=l_1, {}^{a_0}l_2=l_2^{-1}, {}^{a_0}l_3=l_3, \\ {}^{a_0}l_k=l_k, {}^{a_1}l_1=l_1^{-1}, {}^{a_1}l_2=l_2, {}^{a_1}l_3=l_3, \\ {}^{a_1}l_k=l_k, {}^{l_i}l_j=l_j, {}^{l_i^{-1}}l_j=l_j \end{array} \right\rangle \quad (8)$$

is consistent for $1 \leq i < j \leq n$ and $4 \leq k \leq n.$

Proof. By Proposition 2, $S_3(n)=S_3(3)\times F_{n-3}^{ab}$ for $n \geq 3$ where $S_3(3)$ has the consistent polycyclic presentation as in (1) and F_{n-3}^{ab} is a free abelian group of rank $n-3$ which is generated by $l_4, l_5, \ldots l_n$. Then, l_k commutes with all elements in $S_3(3)$, which gives ${}^{a_0}l_k=l_k, {}^{a_1}l_k=l_k, {}^{l_1}l_k=l_k, {}^{l_2}l_k=l_k$ and ${}^{l_3}l_k=l_k$ for all $4 \leq k \leq n.$ Therefore, $S_3(n)$ has a polycyclic presentation as in (8).

The polycyclic presentation of $S_3(3)$ in (1) has been shown to be consistent in Abdul Ladi *et al*. (2016). Then, by Definition 2, and since ${}^{a_0}l_k=l_k, {}^{a_1}l_k=l_k, {}^{l_i}l_k=l_k$ $\forall i=1,2,3,$ the remaining consistency relations $l_k\left(l_3a_0\right)=\left(l_kl_3\right)a_0$, $l_k\left(l_3a_1\right)=\left(l_kl_3\right)a_1$, are also hold since a_0, a_1 commutes with l_3 and l_k. Then, it is showed that $l_k\left(l_3l_1\right)=\left(l_kl_3\right)l_1$, $l_k\left(l_3l_2\right)=\left(l_kl_3\right)l_2$, $l_k\left(l_2l_1\right)=\left(l_kl_2\right)l_1$, since l_1, l_2, l_3 and l_k. are commute with each other based on relation in (8). Since a_0 commutes with l_1 and a_1 commutes with l_2, then $l_k(l_1a_0)=(l_kl_1)a_0$, and $l_k(l_2a_1)=(l_kl_2)a_1$. Next, $l_k\left(l_2a_0\right)=l_k\left(a_0l_2^{-1}\right)=a_0l_2^{-1}l_k$ and

$\left(l_k l_2\right) a_0 = (l_2 l_k) a_0 = l_2 a_0 l_k = a_0 l_2^{-1} l_k$, $l_k\left(l_1 a_1\right) = l_k\left(a_1 l_1^{-1}\right) = a_1 l_1^{-1} l_k$ and $\left(l_k l_1\right) a_1 = (l_1 l_k) a_1 = l_1 a_1 l_k = a_1 l_1^{-1} l_k$, $l_k\left(a_1 a_0\right) = \left(l_k a_1\right) a_0$, $l_k\left(a_0^{\,2}\right) = l_k\left(l_1^{-1}\right) = l_1^{-1} l_k$ and $(l_k a_0) a_0 = (a_0 l_k) a_0 = a_0 a_0 l_k = l_1^{-1} l_k$, $l_k\left(a_1^{\,2}\right) = l_k\left(l_3^{-1}\right) = l_3^{-1} l_k$ and $(l_k a_1) a_1 = (a_1 l_k) a_1 = a_1 a_1 l_k = l_3^{-1} l_k$, $l_k = \left(l_k l_3^{-1}\right) l_3$, $l_k = \left(l_k l_2^{-1}\right) l_2$ and $l_k = \left(l_k l_1^{-1}\right) l_1$. Thus, the polycyclic presentation of $S_3(n)$ is consistent. □

Next, the generalization of the derived subgroup and the abelianization of the group $S_3(3)$ of dimension n is given as in the following lemma.

Lemma 2

The derived subgroup of $S_3(n)$, $S_3(n)' = \left\langle l_1^{-2}, l_2^{-2} \right\rangle$ and the abelianization of $S_3(n)$,

$$S_3(n)^{ab} \cong \left\langle l_1 S_3(n)', l_2 S_3(n)', l_3 S_3(n)', l_k S_3(n)' \right\rangle$$
$$\cong C_2^{\;2} \times C_0^{\;n-2}$$

for $4 \le k \le n$.

Proof. From relation (8), since a_0 commutes with l_1, l_3, l_k and a_1 commutes with l_2, l_3, l_k for all $4 \le k \le n$, then $[a_0, a_1] = l_1$, $[a_0, l_2] = [a_1, l_2] = l_2^{-2}$ and $[a_1, l_1] = l_1^{-2}$. However, $l_1 = (l_1^{-2})^2$. Thus, $S_3(n)' = \left\langle l_1^{-2}, l_2^{-2} \right\rangle$.

By Definition 3, the abelianization of group $S_3(n)$, $S_3(n)^{ab}$ is generated by $a_0 S_3(n)'$, $a_1 S_3(n)'$, $l_1 S_3(n)'$, $l_2 S_3(n)'$, $l_3 S_3(n)'$ and $l_k S_3(n)'$ for $4 \le k \le n$. By Proposition 3, the independent generators of $S_3(n)^{ab}$ are $l_1 S_3(3)'$, $l_2 S_3(3)'$ and $l_3 S_3(3)'$. By using similar arguments, we showed that $l_1 S_3(n)'$, $l_2 S_3(n)'$, $l_3 S_3(n)'$ and $l_k S_3(n)'$ are also the independent generators of $S_3(n)^{ab}$. Hence, by Definition 3,

$$S_3(n)^{ab} = \left\langle l_1 S_3(n)', l_2 S_3(n)', l_3 S_3(n)', l_k S_3(n)' \right\rangle$$

for $4 \le k \le n$.

By Proposition 3, it is shown that the generators in $S_3(n)^{ab}$ such as $l_1 S_3(3)'$ has order 2,

$l_2S_3(3)'$ has order 2 and $l_3S_3(3)'$ has infinite order. Next, the orders of cosets $l_1S_3(n)'$, $l_2S_3(n)'$, $l_3S_3(n)'$ and $l_kS_3(n)'$ are determined. By relations in (8), since $a_0^{\,2} = l_1^{-1}$, then $a_0^{\,4} = l_1^{-2}$. Hence, it is shown that $l_1S_3(n)'$ has order 2 since $l_1^2 \in S_3(n)'$. Since $l_2^{-2} \in S_3(n)'$, then $l_2^2 \in S_3(n)'$. It follows that $l_2S_3(n)'$ has order 2.

Suppose that the order of $l_kS_3(n)'$ is finite, then there must be $l_3^r \in S_3(n)'$. However, this is not true since there is no l_3^r in $S_3(n)'$. Therefore, $l_3S_3(n)'$ has infinite order. By using similar arguments, $l_kS_3(n)'$ is shown to have an infinite order, since there is no l_3^r in $S_3(n)'$ for any integer r. Since $4 \le k \le n$, then there are $n-3$ cosets in term of $l_kS_3(n)'$. Therefore,

$$\begin{aligned} S_3(n)^{ab} &= \langle l_1S_3(n)', l_2S_3(n)', l_3S_3(n)', l_kS_3(n)' \rangle \\ &\cong C_2 \times C_2 \times C_0 \times C_0^{\,n-3} \\ &= C_2^{\,2} \times C_0^{\,1+n-3} \\ &= C_2^{\,2} \times C_0^{\,n-2}. \end{aligned}$$

□

Next, the generalization of $\nabla(S_3(3))$ of dimension n is given in the following theorem.

Theorem 2

The subgroup of $\nabla(S_3(n))$ is given as in the following :

$$\begin{aligned} \nabla(S_3(n)) = \langle & [l_1,l_1^{\varphi}],[l_2,l_2^{\varphi}],[l_3,l_3^{\varphi}],[l_k,l_k^{\varphi}],[l_1,l_2^{\varphi}][l_2,l_1^{\varphi}], [l_1,l_3^{\varphi}][l_3,l_1^{\varphi}],[l_1,l_k^{\varphi}][l_k,l_1^{\varphi}], \\ & [l_2,l_3^{\varphi}][l_3,l_2^{\varphi}],[l_2,l_k^{\varphi}][l_k,l_2^{\varphi}],[l_3,l_k^{\varphi}][l_k,l_3^{\varphi}],[l_i,l_j^{\varphi}][l_j,l_i^{\varphi}] \rangle \\ \cong\ & C_2^{\,n-3} \times C_4^{\,2} \times C_0^{\frac{(n-1)(n-2)}{2}} \end{aligned}$$

for $k = 4,5,\ldots,n$ and $4 \le i < j \le n$.

Proof. By Lemma 2, $S_3(n)^{ab}$ is generated by the cosets $l_1S_3(n)'$, $l_2S_3(n)'$, $l_3S_3(n)'$ and

$l_k S_3(n)'$. Then, by Proposition 1, $\nabla(S_3(n)) = \langle [l_1, l_1^{\varphi}], [l_2, l_2^{\varphi}], [l_3, l_3^{\varphi}], [l_k, l_k^{\varphi}],$ $[l_1, l_2^{\varphi}][l_2, l_1^{\varphi}],\ [l_1, l_3^{\varphi}][l_3, l_1^{\varphi}],\ [l_1, l_k^{\varphi}][l_k, l_1^{\varphi}],\ [l_2, l_3^{\varphi}][l_3, l_2^{\varphi}],\ [l_2, l_k^{\varphi}][l_k, l_2^{\varphi}],\ [l_3, l_k^{\varphi}][l_k, l_3^{\varphi}],$ $[l_i, l_j^{\varphi}][l_j, l_i^{\varphi}] \rangle$ for $k = 4, 5, \ldots, n$ and $4 \le i < j \le n.$

By Proposition 4, both $[l_1, l_1^{\varphi}]$ and $[l_2, l_2^{\varphi}]$ have order 4, $[l_1, l_2^{\varphi}][l_2, l_1^{\varphi}],\ [l_1, l_3^{\varphi}][l_3, l_1^{\varphi}],\ [l_2, l_3^{\varphi}][l_3, l_2^{\varphi}]$ have order 2 and $[l_3, l_3^{\varphi}]$ has infinite order which are the same generators as $\nabla(S_3(n))$. Next, the order of the remaining generators will be determined. By Proposition 5(i) and Proposition 6(ii), then $([l_1, l_k^{\varphi}][l_k, l_1^{\varphi}])^2 = [l_1^2, l_k^{\varphi}][l_k, l_1^{2\varphi}] = [l_1^2, l_k^{\varphi}][l_1^2, l_k^{\varphi}]^{-1} = 1$. It is showed that $[l_1, l_k^{\varphi}][l_k, l_1^{\varphi}]$ has order 2. By using similar arguments, $[l_2, l_k^{\varphi}][l_k, l_2^{\varphi}]$ has order 2.

Next, we want to show that $[l_3, l_k^{\varphi}][l_k, l_3^{\varphi}]$ has infinite order. Suppose that the order of $[l_3, l_k^{\varphi}][l_k, l_3^{\varphi}]$ is finite. Then, for any integer r, s, it is shown that $[l_3^r, l_k^{s\varphi}][l_k^s, l_3^{r\varphi}] = ([l_3, l_k^{\varphi}][l_k, l_3^{\varphi}])^{rs} = 1$. Thus, $[l_k^s, l_3^{r\varphi}] = [l_3^r, l_k^{s\varphi}]^{-1}$. However, by the relations of $S_3(n)$, neither l_3^r or l_k^s is in $S_3(n)'$. Hence this is not true that the order of $[l_3, l_k^{\varphi}][l_k, l_3^{\varphi}]$ is finite. Therefore $[l_3, l_k^{\varphi}][l_k, l_3^{\varphi}]$ has infinite order. By using similar arguments, it is shown that $[l_k, l_k^{\varphi}]$ and $[l_i, l_j^{\varphi}][l_j, l_i^{\varphi}]$ also has infinite order.

Since $k = 4, 5, \ldots n$ and $4 \le i < j \le n$, then there is $n-3$ generators in terms of $[l_k, l_k^{\varphi}],\ [l_1, l_k^{\varphi}][l_k, l_1^{\varphi}],\ [l_2, l_k^{\varphi}][l_k, l_2^{\varphi}],\ [l_3, l_k^{\varphi}][l_k, l_3^{\varphi}]$ and $\dfrac{(n-3)(n-4)}{2}$ generators in terms of $[l_i, l_j^{\varphi}][l_j, l_i^{\varphi}]$. Therefore,

$$\begin{aligned}
\nabla(S_3(n)) &\cong C_4 \times C_4 \times C_0 \times C_0^{\,n-3} \times C_2 \times C_2 \times C_2 \times C_2^{\,n-3} \times C_2^{\,n-3} \times C_0^{\,n-3} \times C_0^{\frac{(n-3)(n-4)}{2}} \\
&= C_2^{\,3+(n-3)+(n-3)} \times C_4^{\,2} \times C_0^{\,1+(n-3)+(n-3)+\frac{(n-3)(n-4)}{2}} \\
&= C_2^{\,2n-3} \times C_4^{\,2} \times C_0^{\frac{(n-1)(n-2)}{2}}.
\end{aligned}$$

∎

3. CONCLUSION

In this paper, the generalization of the central subgroup of the nonabelian tensor square of the Bieberbach group $S_3(n)$ with point group $C_2 \times C_2$ is constructed up after the generalizations of the polycyclic presentation and the abelianization are determined. These results can further be used to find other useful properties of $S_3(n)$ such as the nonabelian tensor square of the group.

4. ACKNOWLEDGEMENTS

The authors would like to thank Universiti Pendidikan Sultan Idris (UPSI) for the funding of this research under the Fundamental Research Grant Scheme (FRGS) Vote no. 2016-0084-102-02. The second author would like to express her appreciation to the Ministry of Education Malaysia (MOE) for her MyMaster Scholarship. Also, the fifth author would like to acknowledge the MOE for her MyPhD Scholarship.

5. REFERENCES

[1] Brown, R. and Loday, J. L. Van Kampen Theorems for Diagram of Spaces. Topology, 1987, 26:311-335.

[2] Masri, R. (2009). The Nonabelian Tensor Squares of Certain Bieberbach Groups with Cyclic Point Group of Order Two. PhD Thesis, Universiti Teknologi Malysia, Skudai, Malaysia.

[3] Mohd Idrus, N., Wan Mohd Fauzi, W. N. F., Masri, R., Tan, Y. T., Sarmin, N. H. and Mat Hassim, H. I. The Central Subgroup of Nonabelian Tensor Square of the Third Bieberbach Group with Dihedral Point Group. International Journal of Applied Mathematics and Statistics, 2015, 53(4): 104-109.

[4] Tan, Y. T., Masri, R., Mohd Idrus, N., Wan Mohd Fauzi, W. N. F., Sarmin, N. H. and Mat Hassim, H. I. The Central Subgroup of Nonabelian Tensor Square of Bieberbach Groups of Dimension Six with Symmetric Point Group of Order Six. International Journal of Applied Mathematics and Statistics, 2015, 53(4): 98-103.

[5] Masri, R., Tan, Y. T. and Mohd Idrus, N. Generalization of the Central Subgroup of the Nonabelian Tensor Square of a Crystallographic Group with Symmetric Point Group. Journal of Informatics and Mathematical Sciences, 2016, 8(4).

[6] Blyth, R. D., Fumagalli, F. and Morigi, M. Some Structural Results on the Non-abelian Tensor Square of Groups. Journal of Group Theory, 2010, 13:83-94.

[7] Abdul Ladi, N. F., Masri, R., Mohd Idrus, N. and Tan, Y. T. (2016). The Central Subgroup of The Nonabelian Tensor Square of Bieberbach Group of Dimension Three with Point Group $C_2 \times C_2$. AIP Conference Proceedings, 2016, 020010-1-020010-6, doi: 10.1063/1.4983865.

[8] Eick, B. and Nickel, W. Computing Schur Multiplicator and Tensor Square of Polycyclic Groups. Journal of Algebra, 2008, 320(2):927-944.

[9] Blyth, R. D. and Morse, R. F. Computing the nonabelian tensor squares of polycyclic groups. Journal of Algebra, 2009, 321:2139-2148.

[10] Rocco, N. R. On a Construction Related to The Nonabelian Tensor Squares of a Group. Bol. Soc. Brasil. Mat. (N. S.), 1991, 22(1):63-79.

[11] Ellis, G. and Leonard, F. Computing Schur Multipliers and Tensor Products of Finite Groups, Vol. 2 of Proceedings Royal Irish Academy, 1995, Sect. 95A.

[12] Zomorodian, A. J. (2005). Topology for Computing. Cambridge University Press, New York, 2005, Chap. 4, pp. 79– 82.

How to cite this article:
Masri R, Ladi N F A, Idrus N M, Tan Y T, Sarmin N H.The central subgroup of the nonabelian tensor square of bieberbach group with point group $C_2 \times C_2$. J. Fundam. Appl. Sci., 2017, *9(7S), 98-110.*