GRADIENT REMEDIABILITY IN LINEAR DISTRIBUTED PARABOLIC SYSTEMS ANALYSIS, APPROXIMATIONS AND SIMULATIONS

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ABSTRACT

The aim of this paper is the introduction of a new concept that concerned the analysis of a large class of distributed parabolic systems. It is the general concept of gradient remediability. More precisely, we study with respect to the gradient observation, the existence of an input operator (gradient efficient actuators) ensuring the compensation of known or unknown disturbances acting on the considered system. Then, we introduce and we characterize the notions of exact and weak gradient remediability and their relationship with the notions of exact and weak gradient controllability. Main properties concerning the notion of gradient efficient actuators are considered. The minimum energy problem is studies, and we show how to find the optimal control, which compensates the disturbance of the system. Approximations and numerical simulations are also presented.

Keywords: actuators efficient; disturbance; gradient; parabolic systems; remediability; sensors.

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1. INTRODUCTION

Various real problems can be formulated within certain concepts of distributed systems analysis. These concepts consist of a set of notions as controllability, detectability, observability, remediability…, which enable a better knowledge and understanding of the system to be obtained. These concepts have been studies at different degrees (exact, weak, extended). Systems analysis can be done from a purely theoretical viewpoint [1,2]. However, the study may be also become concrete, in some sense, by using the structures of actuators and sensors. Thus, one can study the different concepts of controllability via actuators structures [3-5] or the different concepts of observability via sensors structures [6-8].

An extension of these concepts that is very important in practical applications is that of gradient controllability [9], gradient detectability [10] and gradient observability [11-13]. These concepts are of great interest from a more practical and control point of view since there exist systems that cannot be controllable but gradient controllable or that cannot be observable but gradient observable or that cannot be detectable but gradient detectable and they provide a means to deal with some problems from the real world, for example in the thermic isolation problem it may be that the control is only required to vanish the temperature-gradient before crossing the wall.

Hence, with the same preoccupation, we introduce in this paper, a new concept that is gradient remediability of distributed parabolic systems. We recall that the notion of remediability consists in studying the existence of a convenient input operator (efficient actuators), ensuring the remediability of any disturbance acting in the considered systems. This problem is particularly motivated by pollution problems and so called space compensation or remediability problem. The notions of remediability and efficient actuators are introduced and studied first for discrete systems and there for continuous systems of a finite time horizon and for other situations (regional and asymptotic cases, internal or boundary actions of disturbances) [14-16].

This paper is organized as follows: In the second paragraph, we start by presenting the notations and some preliminary material. After, we recall the notions of exact and weak gradient controllability. In the third paragraph, we introduce and we characterize the notions

of exact and weak gradient remadiability. We show how to find an input operator (actuators) with respect to the output (gradient observation) that ensures the gradient remediability of a disturbed parabolic system. By analogy with the relation between the remediability and the controllability examined in the finite time case, it is then natural to study, in the paragraph 4, the relationship between the gradient remediability and the gradient controllability. We show that the gradient remediability is weaker and more supple than the gradient controllability of the parabolic systems, that is to say, if any parabolic systems are gradient controllable, then it is gradient remediable. The fifth paragraph recall the notion of gradient strategic actuators and we give a characterization of gradient remediability which shows that the gradient remediability of any system may depend on the structure of the actuators and sensors. Then we introduce and we characterize the notion of gradient efficient actuators. In paragraph 6, using an extension of Hilbert Uniqueness Method (H.U.M), we examine the problem of gradient remediability with minimum energy, and we give the optimal control that compensate an arbitrary disturbance. In the last paragraph, approximations and numerical results are presented.

2. CONSIDERED SYSTEM

Let Ω be an open and bounded regular subset of IR^n $(n=1,2,3)$ with a smooth boundary $\partial\Omega$. For $T>0$, we denoted by $Q=\Omega\times]0,T[$, $\Sigma=\partial\Omega\times]0,T[$. Consider a parabolic system defined by

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t)= A\,y(x,t)+Bu(t)+ f(x,t) & Q \\ y(x,0)= y^0(x) & \Omega \\ y(\xi,t)=0 & \Sigma \end{cases} \qquad (1)$$

Where A is a second order linear differential operator with compact resolvent and which generates a strongly continuous semi-group $\left(S(t)\right)_{t\geq0}$ on the Hilbert space $L^2(\Omega)$. $\left(S^*(t)\right)_{t\geq0}$ is considered for the adjoint semi-groupe of $\left(S(t)\right)_{t\geq0}$.

$B\in\mathscr{L}(U,X), u\in L^2(0,T;U)$ where U is a Hilbert space representing the control space and $X=H_0^1(\Omega)$ the state space. The disturbance term $f\in L^2(0,T;X)$ is generally unknown.

The system (1) admits a unique solution $y \in C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$ [17] given by

$$y(t) = S(t)y^0 + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s)ds$$

and it is augmented by the output equation

$$z(t) = C\nabla y(t) \tag{2}$$

where $C \in \mathcal{L}((L^2(\Omega))^n, O)$, O is a Hilbert space (observation space). In the case of an observation on $[0,T]$ with q sensors, we take generally $O = IR^q$ .The operator ∇ is defined by

$$\nabla : H_0^1(\Omega) \to (L^2(\Omega))^n$$

$$y \to \nabla y = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots, \frac{\partial y}{\partial x_n} \right)$$

while ∇^* its adjoint operator.

3. GRADIENT REMEDIABILITY

The system (1) is disturbed by the force f assumed unknown and excited by a control u that will be chosen so as to compensate for the disturbance f . In the autonomous case, without disturbance $(f = 0)$ and without control $(u = 0)$, this same system is written

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = A\, y(x,t) & Q \\ y(x,0) = y^0(x) & \Omega \\ y(\xi,t) = 0 & \Sigma \end{cases} \tag{3}$$

The system (3) admits a unique solution $y \in C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$[17] given by $y(t) = S(t)y^0$ then the observation on $[0,T]$ is normal and it is given by $z(t) = C\nabla S(t)y^0$. But if the control $u \neq 0$ and the disturbance $f \neq 0$, the observation noted $z_{u,f}$ is disturbed such that

$$z_{u,f}(t) = C\nabla S(t)y^0 + C\nabla \int_0^t S(t-s)Bu(s)ds + C\nabla \int_0^t S(t-s)f(s)ds$$

The problem consists to study the existence of an input operator B (actuators), with respect to the output operator C (sensors), ensuring at the time T , the gradient remediability of any

disturbance on the system, that is:

For any $f \in L^2(0,T;X)$, there exists $u \in L^2(0,T;U)$ such that $z_{u,f}(T) = C\nabla S(T)y^0$ this is equivalent to $C\nabla Hu + C\nabla Ff = 0$ where H and F are two operators defined by

$$H : L^2(0,T;U) \to X$$

$$u \to Hu = \int_0^T S(T-s)Bu(s)ds$$

$$F : L^2(0,T;X) \to X$$

$$f \to Ff = \int_0^T S(T-s)f(s)ds$$

This leads to the following definitions

Definition 1:

1) We say that the system (1) augmented by the output equation (2) is exactly gradient f - remediable on $[0,T]$, if there exists a control $u \in L^2(0,T;U)$ such that

$$C\nabla Hu + C\nabla Ff = 0$$

2) We say that the system (1) augmented by the output equation (2) is weakly gradient f - remediable on $[0,T]$, if for every $\varepsilon > 0,$ there exists a control $u \in L^2(0,T;U)$ such that

$$\|C\nabla Hu + C\nabla Ff\|_O < \varepsilon$$

3) We say that the system (1) augmented by the output equation (2) is exactly (resp. weakly) gradient remediable on $[0,T]$, if for every $f \in L^2(0,T;X)$ the system (1)–(2) is exactly (resp. weakly) gradient f - remediable.

Proposition 1:

Let us consider $f \in L^2(0,T;X)$.

1) The system (1)–(2) is exactly gradient f - remediable on $[0,T]$ if and only if

$$\text{Im}\,C\nabla Ff \in \text{Im}(C\nabla H)$$

2) The system (1)–(2) is weakly gradient f - remediable on $[0,T]$ if and only if

$$\text{Im}\,C\nabla Ff \in \overline{\text{Im}(C\nabla H)}$$

Proof:

1) We assume that the system (1)–(2) is exactly gradient f - remediable on $[0,T]$, then there

exists $u \in L^2(0,T;U)$ such that $C\nabla Hu + C\nabla Ff = 0$, that is

$C\nabla Ff = -C\nabla Hu = C\nabla H(-u) = C\nabla Hu_1$ with $(u_1 = -u)$ $u_1 \in L^2(0,T;U)$ then

$C\nabla Ff \in \text{Im} C\nabla H$.

Conversely, we assume that $C\nabla Ff \in \text{Im} C\nabla H$, then there exists $u \in L^2(0,T;U)$ such that

$C\nabla Ff = C\nabla Hu$ that is $C\nabla Ff - C\nabla Hu = 0$ this gives $C\nabla Ff + C\nabla H(-u) = 0$. We put

$u_1 = -u \in L^2(0,T;U)$ where the system (1)–(2) is exactly gradient f - remediable.

2) We assume that the system (1)–(2) is weakly gradient f - remediable on $[0,T]$, then

$\forall \varepsilon > 0, \exists u \in L^2(0,T;U)$ such that $\left\| C\nabla Ff + C\nabla Hu \right\|_O < \varepsilon$ that is $\forall \varepsilon > 0, \exists u \in L^2(0,T;U)$

such that $\left\| C\nabla Ff - C\nabla H(-u) \right\|_O < \varepsilon$. We put $u_1 = -u \in L^2(0,T;U)$,

then $\forall \varepsilon > 0, \exists u_1 \in L^2(0,T;U)$ such that $\left\| C\nabla Ff - C\nabla Hu_1 \right\|_O < \varepsilon$, this gives

$C\nabla Ff \in \overline{\text{Im}(C\nabla H)}$.

Conversely, we assume that $C\nabla Ff \in \overline{\text{Im}(C\nabla H)}$, then $\forall \varepsilon > 0, \exists u_1 \in L^2(0,T;U)$ such that

$\left\| C\nabla Ff - C\nabla Hu_1 \right\|_O < \varepsilon$. We put $u_1 = -u \in L^2(0,T;U)$, then $\forall \varepsilon > 0, \exists u \in L^2(0,T;U)$ such

that $\left\| C\nabla Ff + C\nabla Hu \right\|_O < \varepsilon$ where the system (1)–(2) is weakly gradient f - remediable.

Proposition 2:

1) The system (1)–(2) is exactly gradient remediable on $[0,T]$ if and only if

$$\text{Im} C\nabla F \subset \text{Im} C\nabla H$$

2) The system (1)–(2) is weakly gradient remediable on $[0,T]$ if and only if

$$\text{Im} C\nabla F \subset \overline{\text{Im} C\nabla H}$$

Proof:

1) We assume that the system (1)–(2) is exactly gradient remediable on $[0,T]$ then

$\forall f \in L^2(0,T;X)$ the system (1)–(2) is exactly gradient f - remediable on $[0,T]$ and

from the Proposition 1 we have, $\forall f \in L^2(0,T;X): C\nabla Ff \in \text{Im} C\nabla H$, this gives

$\text{Im} C\nabla F \subset \text{Im} C\nabla H$.

Conversely, we assume that $\text{Im} C\nabla F \subset \text{Im} C\nabla H$ and we show that the system (1)–(2) is

exactly gradient remediable on $[0,T]$. Let $f \in L^2(0,T;X)$ then $C\nabla Ff \in \text{Im} C\nabla F$ since

$\mathrm{Im}\,C\nabla F \subset \mathrm{Im}\,C\nabla H$ we have $C\nabla F f \in \mathrm{Im}\,C\nabla H$ then there exists $u \in L^2(0,T;U)$ such that $C\nabla F f = C\nabla H u$ that is to say $C\nabla F f - C\nabla H u = 0$ this gives $C\nabla F f + C\nabla H(-u) = 0$. We put $u_1 = -u \in L^2(0,T;U)$ where the system (1)–(2) is exactly gradient remediable.

2) We assume that the system (1)–(2) is weakly gradient remediable on $[0,T]$ then $\forall f \in L^2(0,T;X)$ the system (1)–(2) is weakly gradient f - remediable on $[0,T]$ and from the Proposition.1 we have, $\forall f \in L^2(0,T;X) : C\nabla F f \in \overline{\mathrm{Im}\,C\nabla H}$, this gives $\mathrm{Im}\,C\nabla F \subset \overline{\mathrm{Im}\,C\nabla H}$.

Conversely, we assume that $\mathrm{Im}\,C\nabla F \subset \overline{\mathrm{Im}\,C\nabla H}$ and we show that the system (1)–(2) is weakly gradient remediable on $[0,T]$. Let $f \in L^2(0,T;X)$ then $C\nabla F f \in \mathrm{Im}\,C\nabla F$. Since $\mathrm{Im}\,C\nabla F \subset \overline{\mathrm{Im}\,C\nabla H}$ then $C\nabla F f \in \overline{\mathrm{Im}(C\nabla H)}$ this leads to $\forall \varepsilon > 0, \exists u \in L^2(0,T;U)$ such that $\|C\nabla F f - C\nabla H u\| < \varepsilon$ by putting $u_1 = -u \in L^2(0,T;U)$ this gives $\forall \varepsilon > 0, \exists u_1 \in L^2(0,T;U)$ such that $\|C\nabla F f + C\nabla H u_1\|_O < \varepsilon$ where the system (1)–(2) is weakly gradient remediable on $[0,T]$. $\square$

4. GRADIENT REMEDIABILITY AND GRADIENT CONTROLLABILITY

By analogy with the relation between the remediability and the controllability examined in the finite time case, it is then natural to study, in this paragraph, the relationship between the gradient remediability and the gradient controllability. Firstly, we recall the definitions of exact and weak controllability.

Definition 2:

1) The system (1) is said to be exactly gradient controllable on $[0,T]$ if $\forall y^d \in (L^2(\Omega))^n$ there exist a control $u \in L^2(0,T;U)$ such that $\nabla y(T) = y^d$.

2) The system (1) is said weakly gradient controllable on $[0,T]$ if $\forall y^d \in (L^2(\Omega))^n$, $\forall \varepsilon > 0$ there exist a control $u \in L^2(0,T;U)$ such that $\|\nabla y(T) - y^d\|_{(L^2(\Omega))^n} < \varepsilon$.

Proposition 3:

1) If the system (1)–(2) is exactly gradient controllable on $[0,T]$, then it is exactly gradient remediable on $[0,T]$.

2) If the system (1)–(2) is weakly gradient controllable on $[0,T]$, then it is weakly gradient remediable on $[0,T]$.

Proof:

1) We assume that the system (1)–(2) is exactly gradient controllable   and let $y^d = S(T)y^0$ then there exists $u \in L^2(0,T;U)$ such that $\nabla y(T) = \nabla S(T)y^0$ that is to say $\nabla S(T)y^0 + \nabla Hu + \nabla Ff = \nabla S(T)y^0$ this leads to $\nabla Hu + \nabla Ff = 0$ then let $C\nabla Hu + C\nabla Ff = 0$ and then the system (1) – (2) is exactly gradient remediable.

2) We assume that the system (1)–(2) is weakly gradient controllable and let $y^d = \nabla S(T)y^0$ then $\forall \varepsilon > 0$, $\exists u \in L^2(0,T;U)$ such that $\left\| \nabla y(T) - \nabla S(T)y^0 \right\|_{(L^2(\Omega))^n} < \varepsilon$ that is $\left\| \nabla Hu + \nabla Ff \right\|_{(L^2(\Omega))^n} < \varepsilon$. Since the operator C is continue, consequently we have $\left\| C\nabla Hu + C\nabla Ff \right\|_O \le k \left\| \nabla Hu + \nabla Ff \right\|_{(L^2(\Omega))^n}$ with $k > 0$ where the system (1)–(2) is weakly gradient remediable. $\qquad\square$

5. GRADIENT REMEDIABILITY, SENSORS AND ACTUATORS

We suppose that the system (1) is excited by p zone actuators $(\Omega_i, g_i)_{1 \le i \le p}$, $g_i \in L^2(\Omega_i)$, $\Omega_i = \mathrm{supp}(g_i) \subset \Omega$ in this case the control space is $U = IR^p$ and the operator:

$$B : IR^p \to X$$

$$u(t) = (u_1(t), u_2(t), \ldots, u_p(t)) \mapsto Bu = \sum_{i=1}^p \chi_{\Omega_i}(x) g_i(x) u_i(t)$$

Its adjoint is given by

$$B^* z = \left(\langle g_1, z \rangle_{\Omega_1}, \langle g_2, z \rangle_{\Omega_2}, \ldots, \langle g_p, z \rangle_{\Omega_p} \right)^T \in IR^p \qquad (4)$$

Also suppose that the output of the system (2) is given by q sensors $(D_i, h_i)_{1 \le i \le q}$, $h_i \in L^2(D_i)$,

being the spatial distribution, $D_i = \mathrm{supp}\, h_i \subset \Omega$, for $i = 1, \ldots, q$ and $D_i \cap D_j = \phi$ for $i \ne j$, then

the operator C is defined by:

$$C : \left(L^2(\Omega)\right)^n \to IR^q$$

$$Cy(t) = \left(\sum_{i=1}^n \langle h_1, y_i(t) \rangle_{D_1}, \sum_{i=1}^n \langle h_2, y_i(t) \rangle_{D_2}, \ldots, \sum_{i=1}^n \langle h_q, y_i(t) \rangle_{D_q} \right)^T$$

its adjoint is given by C^* with for $\theta = (\theta_1, \theta_2, \ldots, \theta_q)^T \in IR^q$

$$C^*\theta = \left(\sum_{i=1}^q \chi_{D_i}(x)\theta_i h_i(x), \sum_{i=1}^q \chi_{D_i}(x)\theta_i h_i(x), \ldots, \sum_{i=1}^q \chi_{D_i}(x)\theta_i h_i(x) \right) \tag{5}$$

Lemma 1 [18]:

Let V, W and Z be reflexive Banach spaces, $P \in \mathscr{L}(V, Z)$ and $Q \in \mathscr{L}(W, Z)$. Then the following properties are equivalent:

  i. $\mathrm{Im} P \subset \mathrm{Im} Q$

  ii. $\exists \gamma > 0$ such that $\left\| P^* z^* \right\|_{V'} \le \gamma \left\| Q^* z^* \right\|_{W'}$, $\quad \forall z^* \in Z'$

We have the following characterizations:

Proposition 4:

1) The system (1)–(2) is exactly gradient remediable on $[0, T]$ if and only if there exists

   $\gamma > 0$ such that for every $\theta \in IR^q$, we have

   $$\left\| S^*(T - .)\nabla^* C^* \theta \right\|_{L^2(0,T;X')} \le \gamma \left\| B^* S^*(T - .)\nabla^* C^* \theta \right\|_{L^2(0,T;IR^p)}$$

2) The system (1) – (2) is weakly gradient remediable on $[0, T]$ if and only if

   $$\ker\left(B^* F^* \nabla^* C^* \right) = \ker\left(F^* \nabla^* C^* \right)$$

Proof:

1) It follows from the fact that $F^* \nabla^* C^* = S^*(T - .)\nabla^* C^*$ and that $H^* \nabla^* C^* = B^* S^*(T - .)\nabla^* C^*$ and since the Proposition 2, we put $P = C\nabla F$ and $Q = C\nabla H$ and using the Lemma 1.

2) We assume that the system (1)–(2) is weakly gradient remediable on $[0, T]$ and we show that $\ker\left(B^* F^* \nabla^* C^* \right) = \ker\left(F^* \nabla^* C^* \right)$. Let $\theta \in IR^q$ such that $B^* F^* \nabla^* C^* \theta = 0$, and we have

$$\begin{cases} F^* = S^*(T-.) \\ H^* = B^* S^*(T-.) \end{cases}$$

then, $B^* F^* \nabla^* C^* \theta = 0 \Rightarrow H^* \nabla^* C^* \theta = 0$ this gives $\theta \in \ker(H^* \nabla^* C^*)$ and we have $\overline{\text{Im}(C\nabla H)} = \left[\ker(H^* \nabla^* C^*)\right]^{\perp}$. Since the hypothesis and the Proposition 2, we have $\text{Im} C\nabla F \subset \overline{\text{Im} C\nabla H}$ then $\text{Im} C\nabla F \subset \left[\ker(H^* \nabla^* C^*)\right]^{\perp} \Rightarrow \forall f \in L^2(0,T;X): C\nabla F f \in \left[\ker(H^* \nabla^* C^*)\right]^{\perp} \Rightarrow \langle C\nabla F f, \theta \rangle = 0$ because $\theta \in \ker(H^* \nabla^* C^*)$, this gives $\theta \in \left[\text{Im} C\nabla F\right]^{\perp} = \ker(F^* \nabla^* C^*)$, where the result.

Conversely, assume that $\ker(B^* F^* \nabla^* C^*) = \ker(F^* \nabla^* C^*)$ and we show that $\text{Im} C\nabla F \subset \overline{\text{Im} C\nabla H}$. Let $f \in L^2(0,T;X)$ such that $f \in \text{Im} C\nabla F$, we have $\overline{\text{Im} C\nabla H} = \left[\ker(H^* \nabla^* C^*)\right]^{\perp}$. For every $\theta \in IR^q$ such that $H^* \nabla^* C^* \theta = 0$, that is $B^* F^* \nabla^* C^* \theta = 0$ we have $F^* \nabla^* C^* \theta = 0$ because $\ker(B^* F^* \nabla^* C^*) = \ker(F^* \nabla^* C^*)$ then $\langle C\nabla F f, \theta \rangle = 0$, where the result. $\square$

Corollary 1:

The system (1)–(2) is exactly gradient remediable on $[0,T]$ if and only if $\exists \gamma > 0$ such that $\forall \theta \in IR^q$, we have

$$\int_0^T \left\| S^*(T-s)\nabla^* C^* \theta \right\|_{X'}^2 ds \leq \gamma \sum_{i=1}^p \int_0^T \left\langle g_i, S^*(T-s)\nabla^* C^* \theta \right\rangle_{L^2(\Omega_i)}^2 ds$$

Proof:

Since the Proposition 4, the system (1)–(2) is exactly gradient remediable on $[0,T]$ if and only if there exists $\gamma > 0$ such that for every $\theta \in IR^q$, we have

$$\left\| S^*(T-.)\nabla^* C^* \theta \right\|_{L^2(0,T;X')}^2 \leq \gamma \left\| B^* S^*(T-.)\nabla^* C^* \theta \right\|_{L^2(0,T;IR^p)}^2$$

by using (4) the formula of the operator B^*, we have

$$\int_0^T \left\| S^*(T-s)\nabla^* C^* \theta \right\|_{X'}^2 ds \leq \gamma \sum_{i=1}^p \int_0^T \left\langle g_i, S^*(T-s)\nabla^* C^* \theta \right\rangle^2 ds$$

where the result. $\square$

In the following, without loss of generality we consider, the system (1) with a dynamics A of the form

$$Ay = \sum_{m \geq 1} \lambda_m \sum_{j=1}^{r_m} \langle y, w_{mj} \rangle_{L^2(\Omega)} w_{mj}, \; \forall y \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

where $\left(w_{mj} \right)_{\substack{1 \leq j \leq r_m \\ m \geq 1}}$ is an orthogonal basis in $H_0^1(\Omega)$ of eigenvectors of A orthonormal in

$L^2(\Omega)$, associated to eigenvalues $\lambda_m < 0$ with a multiplicity r_m. Then, the operator A

generates on the Hilbert space $L^2(\Omega)$ a strongly continuous semi-group $(S(t))_{t \geq 0}$ given by

[1,19]:

$$S(t)y = \sum_{m \geq 1} e^{\lambda_m t} \sum_{j=1}^{r_m} \langle y, w_{mj} \rangle_{L^2(\Omega)} w_{mj} \tag{6}$$

Corollary 2:

The system (1)–(2) is exactly gradient remediable on $[0, T]$ if and only if $\quad \exists \gamma > 0 \quad$ such that

$\forall \theta \in IR^q$, we have

$$\sum_{m \geq 1} \frac{1}{2\lambda_m} \left(e^{2\lambda_m T} - 1 \right) \sum_{j=1}^{r_m} \langle C^* \theta, \nabla w_{mj} \rangle_{(L^2(\Omega))^n}^2 \leq \gamma \sum_{i=1}^p \int_0^T \left(\sum_{m \geq 1} e^{\lambda_m (T-s)} \sum_{j=1}^{r_m} \langle C^* \theta, \nabla w_{mj} \rangle \langle g_i, w_{mj} \rangle \right)^2 ds$$

Proof:

Since the Corollary 1, the system (1)-(2) is exactly gradient remediable on $[0, T]$ if and only

if there exists $\gamma > 0$ such that for every $\theta \in IR^q$, we have

$$\int_0^T \left\| S^*(T-s) \nabla^* C^* \theta \right\|_{X'}^2 ds \leq \gamma \sum_{i=1}^p \int_0^T \langle g_i, S^*(T-s) \nabla^* C^* \theta \rangle_{\Omega_i}^2 ds$$

By using (6) the formula of the operator S and since it is auto-adjoint, we obtain

$$\int_0^T \left\| S^*(T-s) \nabla^* C^* \theta \right\|_{L^2(\Omega)}^2 ds = \int_0^T \sum_{m \geq 1} e^{2\lambda_m (T-s)} \sum_{j=1}^{r_m} \langle \nabla^* C^* \theta, w_{mj} \rangle^2 ds$$

$$= \sum_{m \geq 1} \frac{1}{2\lambda_m} \left(e^{2\lambda_m T} - 1 \right) \sum_{j=1}^{r_m} \langle C^* \theta, \nabla w_{mj} \rangle^2$$

and

$$\sum_{i=1}^p \int_0^T \langle g_i, S^*(T-s) \nabla^* C^* \theta \rangle_{L^2(\Omega_i)}^2 ds = \sum_{i=1}^p \int_0^T \left(\sum_{m \geq 1} e^{\lambda_m (T-s)} \sum_{j=1}^{r_m} \langle \nabla^* C^* \theta, w_{mj} \rangle \langle g_i, w_{mj} \rangle \right)^2 ds$$

where the result. □

By using (5) the formula of the operator C^*, we have the following corollary:

Corollary 3:

The system (1)–(2) is exactly gradient remediable on $[0,T]$ if and only if $\exists \gamma > 0$ such that $\forall \theta \in IR^q$, we have

$$
\sum_{m \geq 1} \frac{1}{2\lambda_m} \left(e^{2\lambda_m T} - 1 \right) \sum_{j=1}^{r_m} \sum_{k=1}^{n} \sum_{l=1}^{q} \left\langle \theta_l h_l, \frac{\partial w_{mj}}{\partial x_k} \right\rangle^2_{L^2(D_l)}
$$

$$
\leq \gamma \sum_{i=1}^{p} \int_0^T \left(\sum_{m \geq 1} e^{\lambda_m (T-s)} \sum_{j=1}^{r_m} \left\langle g_i, w_{mj} \right\rangle_{L^2(\Omega_i)} \sum_{k=1}^{n} \sum_{l=1}^{q} \theta_l \left\langle h_l, \frac{\partial w_{mj}}{\partial x_k} \right\rangle \right)^2 ds
$$

Proof:

Since the Corollary 2, the system (1)-(2) is exactly gradient remediable on $[0,T]$ if and only if there exists $\gamma > 0$ such that for every $\theta \in IR^q$, we have

$$
\sum_{m \geq 1} \frac{1}{2\lambda_m} \left(e^{2\lambda_m T} - 1 \right) \sum_{j=1}^{r_m} \left\langle C^* \theta, \nabla w_{mj} \right\rangle^2_{\left(L^2(\Omega) \right)^n} \leq \gamma \sum_{i=1}^{p} \int_0^T \left(\sum_{m \geq 1} e^{\lambda_m (T-s)} \sum_{j=1}^{r_m} \left\langle C^* \theta, \nabla w_{mj} \right\rangle \left\langle g_i, w_{mj} \right\rangle \right)^2 ds
$$

and by using (5) the formula of the operator C^*, we have

$$
\left\langle C^* \theta, \nabla w_{mj} \right\rangle_{\left(L^2(\Omega) \right)^n} = \left\langle \begin{pmatrix} \sum_{l=1}^{q} \chi_{D_l} \theta_l h_l \\ \sum_{l=1}^{q} \chi_{D_l} \theta_l h_l \\ \vdots \\ \sum_{l=1}^{q} \chi_{D_l} \theta_l h_l \end{pmatrix} \begin{pmatrix} \frac{\partial w_{mj}}{\partial x_1} \\ \frac{\partial w_{mj}}{\partial x_2} \\ \vdots \\ \frac{\partial w_{mj}}{\partial x_n} \end{pmatrix} \right\rangle_{\left(L^2(\Omega) \right)^n} = \sum_{k=1}^{n} \sum_{l=1}^{q} \theta_l \left\langle h_l, \frac{\partial w_{mj}}{\partial x_k} \right\rangle_{L^2(D_l)}
$$

where the result.   □

This characterization shows that the remediability of a system may depend on the structure of the actuators and sensors.

By analogy with the concept of gradient strategic actuator, we introduce the notion of gradient efficient actuator, as follows:

Definition 3:

The actuators $(\Omega_i, g_i)_{1 \leq i \leq p}, g_i \in L^2(\Omega_i)$ are said to be gradient efficient if the system (1)–(2) so excited is weakly gradient remediable.

The actuators gradient efficient define actions with the structure (spatial distribution, location and number) can compensate the effect of disturbance distributed on the system. We have then

the following characterization of the gradient efficient actuators.

For $m \geq 1$, let M_m be the matrix of order $(p \times r_m)$ defined by $M_m = \left(\langle g_i, w_{mj} \rangle \right)_{ij}, 1 \leq i \leq p$

and $1 \leq j \leq r_m$ and let G_m be the matrix of order $(q \times r_m)$ defined by

$$G_m = \left(\sum_{k=1}^{n} \left\langle h_i, \frac{\partial w_{mj}}{\partial x_k} \right\rangle \right)_{ij}, 1 \leq i \leq q \quad \text{and} \quad 1 \leq j \leq r_m.$$

Proposition 5:

The actuators $(\Omega_i, g_i)_{1 \leq i \leq p}, g_i \in L^2(\Omega_i)$ are gradient efficient if and only if

$\ker(\nabla^* C^*) = \bigcap_{m \geq 1} \ker(M_m f_m)$. Where, for $\theta \in IR^q$ and $m \geq 1$,

$$f_m(\theta) = \left(\langle \nabla^* C^* \theta, w_{m1} \rangle, \langle \nabla^* C^* \theta, w_{m2} \rangle, \ldots, \langle \nabla^* C^* \theta, w_{m r_m} \rangle \right)^T \in IR^{r_m}$$

Proof:

We assume that the actuators $(\Omega_i, g_i)_{1 \leq i \leq p}, g_i \in L^2(\Omega_i)$ are gradient efficient and we show

that $\ker(\nabla^* C^*) = \bigcap_{m \geq 1} \ker(M_m f_m)$. Since the Proposition 4, the system (1)-(2) is weakly

gradient remediable on $[0,T]$ if and only if $\ker(B^* F^* \nabla^* C^*) = \ker(F^* \nabla^* C^*)$. Let $\theta \in IR^q$,

we have

$$B^* F^* \nabla^* C^* \theta = B^* S^* (T-.) \nabla^* C^* \theta = \begin{pmatrix} \sum_{m \geq 1} e^{\lambda_m (T-.)} \sum_{j=1}^{r_m} \langle \nabla^* C^* \theta, w_{mj} \rangle_{L^2(\Omega)} \langle g_1, w_{mj} \rangle_{L^2(\Omega_1)} \\ \sum_{m \geq 1} e^{\lambda_m (T-.)} \sum_{j=1}^{r_m} \langle \nabla^* C^* \theta, w_{mj} \rangle_{L^2(\Omega)} \langle g_2, w_{mj} \rangle_{L^2(\Omega_2)} \\ \vdots \\ \sum_{m \geq 1} e^{\lambda_m (T-.)} \sum_{j=1}^{r_m} \langle \nabla^* C^* \theta, w_{mj} \rangle_{L^2(\Omega)} \langle g_P, w_{mj} \rangle_{L^2(\Omega_p)} \end{pmatrix}$$

and we have $\forall m \geq 1$,

$$M_m f_m(\theta) = \begin{pmatrix} \sum_{j=1}^{r_m} \langle \nabla^* C^* \theta; w_{mj} \rangle_{L^2(\Omega)} \langle g_1, w_{mj} \rangle_{L^2(\Omega_1)} \\ \sum_{j=1}^{r_m} \langle \nabla^* C^* \theta; w_{mj} \rangle_{L^2(\Omega)} \langle g_2, w_{mj} \rangle_{L^2(\Omega_2)} \\ \vdots \\ \sum_{j=1}^{r_m} \langle \nabla^* C^* \theta; w_{mj} \rangle_{L^2(\Omega)} \langle g_P, w_{mj} \rangle_{L^2(\Omega_p)} \end{pmatrix}$$

If we assume that $\theta \in \bigcap_{m \geq 1} \ker(M_m f_m)$, then

$$\theta \in \ker(M_m f_m), \quad \forall m \geq 1$$

$$\Rightarrow \sum_{j=1}^{r_m} \left\langle \nabla^* C^* \theta; w_{mj} \right\rangle_{L^2(\Omega)} \left\langle g_i, w_{mj} \right\rangle = 0, \forall i \in \{1, 2, \ldots, p\}, \forall m \geq 1$$

$$\Rightarrow \sum_{m \geq 1} e^{\lambda_m (T-.)} \sum_{j=1}^{r_m} \left\langle \nabla^* C^* \theta, w_{mj} \right\rangle \left\langle g_i, w_{mj} \right\rangle = 0, \forall i \in \{1, 2, \ldots, p\}, \forall m \geq 1$$

$$\Rightarrow B^* F^* \nabla^* C^* \theta = 0 \Rightarrow \theta \in \ker\left(B^* F^* \nabla^* C^*\right) \text{where} \bigcap_{m \geq 1} \ker(M_m f_m) \subset \ker\left(B^* F^* \nabla^* C^*\right) \text{ that is}$$

$$\bigcap_{m \geq 1} \ker(M_m f_m) = \ker\left(B^* F^* \nabla^* C^*\right).$$

On the other hand, we have for every $\theta \in IR^q$,

$$F^* \nabla^* C^* \theta = S^*(T-.)\nabla^* C^* = \sum_{m \geq 1} e^{\lambda_m (T-.)} \sum_{j=1}^{r_m} \left\langle \nabla^* C^* \theta, w_{mj} \right\rangle w_{mj} \text{ We assume that}$$

$\theta \in \ker\left(F^* \nabla^* C^*\right)$, then $F^* \nabla^* C^* \theta = 0$

$$F^* \nabla^* C^* \theta = S^*(T-.)\nabla^* C^* = \sum_{m \geq 1} e^{\lambda_m (T-.)} \sum_{j=1}^{r_m} \left\langle \nabla^* C^* \theta, w_{mj} \right\rangle w_{mj} = 0$$

$$\Rightarrow \sum_{m \geq 1} \sum_{j=1}^{r_m} \left\langle \nabla^* C^* \theta, w_{mj} \right\rangle w_{mj} = 0 \Rightarrow \nabla^* C^* \theta = 0 \Leftrightarrow \theta \in \ker\left(\nabla^* C^*\right). \text{ Then,}$$

$\ker\left(F^* \nabla^* C^*\right) \subset \ker\left(\nabla^* C^*\right)$. If we assume that $\theta \in \ker\left(\nabla^* C^*\right)$, then $\nabla^* C^* \theta = 0$ that is

$$F^* \nabla^* C^* \theta = \sum_{m \geq 1} e^{\lambda_m (T-.)} \sum_{j=1}^{r_m} \left\langle \nabla^* C^* \theta, w_{mj} \right\rangle w_{mj} = 0 \Rightarrow \theta \in \ker\left(F^* \nabla^* C^*\right), \text{ then}$$

$\ker\left(\nabla^* C^*\right) \subset \ker\left(F^* \nabla^* C^*\right)$ that is $\ker\left(\nabla^* C^*\right) = \ker\left(F^* \nabla^* C^*\right)$.

Where the result.   □

Corollary 4:

If there exists $m_0 \geq 1$ such that

$$rank \, G_{m0}^T = q \tag{7}$$

then the actuators $(\Omega_i, g_i)_{1 \leq i \leq p}, g_i \in L^2(\Omega_i)$ are efficient if and only if $\bigcap_{m \geq 1} \ker\left(M_m G_m^T\right) = \{0\}$.

Proof:

Let $\theta \in IR^q$, then $\theta \in \bigcap_{m \geq 1} \ker\left(M_m G_m^T\right) \Leftrightarrow \left(M_m G_m^T\right)\theta = 0, \quad \forall m \geq 1$

$$\Leftrightarrow \sum_{l=1}^{q} \sum_{j=1}^{r_m} \left\langle g_i, w_{mj} \right\rangle \left\langle h_l, \sum_{k=1}^{n} \frac{\partial w_{mj}}{\partial x_k} \right\rangle \theta_l = 0, \forall m \geq 1, \forall i = 1, \ldots, p$$

$$\Leftrightarrow \sum_{j=1}^{r_m} \langle g_i, w_{mj} \rangle \langle \nabla^* C^*, w_{mj} \rangle_{L^2(\Omega)} = 0, \forall m \geq 1, \forall i = 1, \ldots, p$$

$$\Leftrightarrow (M_m f_m) \theta = 0, \forall m \geq 1 \Leftrightarrow \theta \in \bigcap_{m \geq 1} \ker (M_m f_m) \text{this gives} \bigcap_{m \geq 1} \ker (M_m G_m^T) = \bigcap_{m \geq 1} \ker (M_m f_m).$$

On the other hand, $\theta \in \ker (\nabla^* C^*) \Leftrightarrow \nabla^* C^* \theta = 0$, then for m_0 that appears in the hypothesis and by using (5) the formula of the operator C^*, we obtain

$$\langle \nabla^* C^* \theta, w_{m_0 j} \rangle = \sum_{l=1}^{q} \theta_l \langle h_l, \sum_{k=1}^{n} \frac{\partial w_{m_0 j}}{\partial x_k} \rangle_{L^2(D_l)} = 0, \forall j = 1, \ldots, r_{m_0} \quad \Rightarrow G_{m_0}^T \theta = 0 \Leftrightarrow \theta \in \ker G_{m_0}^T \quad \text{and}$$

since $rank\ G_{m_0}^T = q$, then $\ker G_{m_0}^T = \{0\}$ this gives $\theta = 0$. That is $\ker (\nabla^* C^*) = \{0\}$

Finally, the proof follows directly from the Proposition 5. □

Corollary 5:

If there exists $m_0 \geq 1$ such that $rank\ (G_{m_0}^T) = q$ and if

$$rank\ (M_{m_0} G_{m_0}^T) = q \tag{8}$$

Or

$$rank\ (M_{m_0}) = r_{m_0} \tag{9}$$

then the actuators $(\Omega_i, g_i)_{1 \leq i \leq p}, g_i \in L^2(\Omega_i)$ are gradient efficient.

Proof:

Assume that there exists $m_0 \geq 1$ such that $rank\ (M_{n_0} G_{n_0}^T) = q$. The matrix $(M_{m_0} G_{m_0}^T)$ is of order $p \times q$. From the theorem of rank to matrices [20], we have $rank\ (M_{m_0} G_{m_0}^T) + \dim (\ker (M_{m_0} G_{m_0}^T)) = q$, and then $\dim (\ker (M_{m_0} G_{m_0}^T)) = 0$ witch is equivalent to $\ker (M_{m_0} G_{m_0}^T) = \{0\} \Rightarrow \bigcap_{m \geq 1} \ker (M_m G_m^T) = \{0\}$. Since the Corollary 4, that is equivalent to the gradient efficient of the actuators $(\Omega_i, g_i)_{1 \leq i \leq p}, g_i \in L^2(\Omega_i)$.

Now, we suppose that $rank\ (G_{m_0}^T) = q$ and $rank\ (M_{m_0}) = r_{m_0}$. The matrix $(G_{m_0}^T)$ is of order $r_{m_0} \times q$. By using the theorem of rank to matrices [20], we have $rank\ (G_{m_0}^T) + \dim (\ker (G_{m_0}^T)) = q$ then, $\dim (\ker (G_{m_0}^T)) = 0$. That is equivalent to

$$\ker (G_{m_0}^T) = \{0\} \tag{10}$$

The same, the matrix (M_{m_0}) is of order $p \times r_{m_0}$. By using the theorem of rank for matrices

[19], we have $rang\left(M_{m_0}\right)+\dim\left(\ker\left(M_{m_0}\right)\right)=r_{m_0}$. And from (9), we obtain $\dim\left(\ker\left(M_{m_0}\right)\right)=0$ which is equivalent to

$$\ker\left(M_{m_0}\right)=\{0\} \tag{11}$$

On the other hand, let $\theta\in\ker\left(M_{m_0}G_{m_0}^T\right)$, then $\left(M_{m_0}G_{m_0}^T\right)\theta=0$ which gives $M_{m_0}\left(G_{m_0}^T\theta\right)=0$.

From (11), we obtain $G_{m_0}^T\theta=0$ and from (10), we obtain $\theta=0$ then $\ker\left(M_{m_0}G_{m_0}^T\right)=\{0\}$ and then, $\bigcap_{m\geq 1}\ker\left(M_mG_m^T\right)=\{0\}$ which is equivalent, from the Corollary 4, to the gradient efficient of the actuators $\left(\Omega_i,g_i\right)_{1\leq i\leq p},g_i\in L^2\left(\Omega_i\right)$. □

Remark 1:

1) The condition (8) $\Rightarrow q\leq p$.

2) The condition $q\leq p$ is not necessary for actuators to be gradient efficient. Indeed, in the case of a single actuator $\left(\Omega_1,g\right)$ and of q sensors $\left(D_i,h_i\right)_{1\leq i\leq q}$, with $q>1$,

$M_m=\left(\left\langle g,w_{mj}\right\rangle\right)_{1\leq j\leq r_m}$ is of order $\left(1\times r_m\right)$ and $G_m^T=\left(\sum_{k=1}^n\left\langle h_i,\dfrac{\partial w_{mj}}{\partial x_k}\right\rangle\right)_{\substack{1\leq j\leq r_m\\1\leq i\leq q}}$ is of order

$\left(r_m\times q\right)$ consequently $M_mG_m^T=\left(\sum_{j=1}^{r_m}\sum_{k=1}^n\left\langle g,w_{mj}\right\rangle\left\langle h_i,\dfrac{\partial w_{mj}}{\partial x_k}\right\rangle\right)_{1\leq i\leq q}$ is of order $\left(1\times q\right)$.

From the Corollary 4, if there exists $m_0\geq 1$ such that $rank\,G_{m_0}^T=q$, then $\left(\Omega_1,g\right)$ is gradient efficient if and only if $\bigcap_{m\geq 1}\ker\left(M_mG_m^T\right)=\{0\}$. Then there exists $n_1,n_2,\ldots,n_m$ such that $n_i\neq n_j$ for $i\neq j$ and

$$\bigcap_{i=1,m}\ker\left(M_{n_i}G_{n_i}^T\right)=\{0\} \tag{12}$$

then $\left(\Omega_1,g\right)$ is gradient efficient. In particular if $m=q$, the condition (12) is equivalent to

$$\begin{vmatrix}\sum_{j=1}^{r_{n_1}}\sum_{k=1}^n\left\langle g,w_{n_1j}\right\rangle\left\langle h_1,\dfrac{\partial w_{n_1j}}{\partial x_k}\right\rangle\cdots & \sum_{j=1}^{r_{n_1}}\sum_{k=1}^n\left\langle g,w_{n_1j}\right\rangle\left\langle h_q,\dfrac{\partial w_{n_1j}}{\partial x_k}\right\rangle\\ \vdots & \vdots\\ \sum_{j=1}^{r_{n_q}}\sum_{k=1}^n\left\langle g,w_{n_qj}\right\rangle\left\langle h_1,\dfrac{\partial w_{n_qj}}{\partial x_k}\right\rangle\cdots & \sum_{j=1}^{r_{n_q}}\sum_{k=1}^n\left\langle g,w_{n_qj}\right\rangle\left\langle h_q,\dfrac{\partial w_{n_qj}}{\partial x_k}\right\rangle\end{vmatrix}\neq 0$$

6. GRADIENT REMEDIABILITY WITH MINIMUM ENERGY

For $f \in L^2(0,T;X)$, we study the existence and the unicity of an optimal control $u \in L^2(0,T;IR^P)$ ensuring, at the time T, the gradient remediability of the disturbance f such that $C\nabla Hu + C\nabla Ff = 0$. That is the set defined by

$$D = \left\{ u \in L^2(0,T;IR^P) / C\nabla Hu + C\nabla Ff = 0 \right\} \tag{13}$$

is non empty.

We consider the function $J(u) = \left\| C\nabla Hu + C\nabla Ff \right\|_{IR^q}^2 + \left\| u \right\|_{L^2(0,T;IR^P)}^2$

The considered problem becomes $\min_{u \in D} J(u)$. For its resolution, we will use an extension of the Hilbert Uniqueness Methods (H.U.M). For $\theta \in IR^q$, let us note

$$\left\| \theta \right\|_* = \left(\int_0^T \left\| B^* S^* (T-s)\nabla^* C^* \theta \right\|_{IR^P}^2 ds \right)^{\frac{1}{2}}$$

$\| \ \|_*$ is a semi-norm on IR^q. If the condition (7) is verified then it is a norm if and only if the system (1)-(2) is weakly gradient remediable on $[0,T]$. The corresponding inner product is given by $\left\langle \theta, \sigma \right\rangle_* = \int_0^T \left\langle B^* S^* (T-s)\nabla^* C^* \theta, B^* S^* (T-s)\nabla^* C^* \sigma \right\rangle ds$

and the operator $\Lambda : IR^q \to IR^q$ defined by

$$\Lambda \theta = C\nabla HH^* \nabla^* C^* \theta$$

$$= C\nabla \int_0^T S(T-s)BB^* S^* (T-s)\nabla^* C^* \theta \, ds$$

Then, we have the following proposition:

Proposition 6:

If the condition (7) is verified, then $\| \ \|_*$ is a norm on IR^q if and only if the system (1)-(2) is weakly gradient remediable on $[0,T]$ and the operator Λ is invertible.

Proof :

We have

$$\|\theta\|_* = \left(\int_0^T \left\| B^* S^* (T-s) \nabla^* C^* \theta \right\|_{IR^P}^2 \, ds \right)^{\frac{1}{2}} = 0 \Rightarrow \left\| B^* S^* (T-s) \nabla^* C^* \theta \right\|_{L^2(0,T;IR^P)} = 0$$

$$\Rightarrow B^* S^* (T-.) \nabla^* C^* \theta = 0 \Rightarrow \theta \in \ker\left(B^* S^* (T-.) \nabla^* C^* \right) = \ker\left(B^* F^* \nabla^* C^* \right)$$

But $\ker\left(B^* F^* \nabla^* C^* \right) = \bigcap_{m \geq 1} \ker\left(M_m f_m \right)$ (see the Proof of Proposition 5) and we have also

$$\bigcap_{m \geq 1} \ker\left(M_m G_m^T \right) = \bigcap_{m \geq 1} \ker\left(M_m f_m \right) \quad \text{(see the Proof of Corollary 4)}$$

then, $\ker\left(B^* F^* \nabla^* C^* \right) = \bigcap_{m \geq 1} \ker\left(M_m G_m^T \right)$ this gives $\theta \in \bigcap_{m \geq 1} \ker\left(M_m G_m^T \right)$ and since the

Corollary 4 we obtain the result.

On the other hand the operator Λ is symmetric, indeed

$$\langle \Lambda\theta, \sigma \rangle_{IR^q} = \langle C\nabla H H^* \nabla^* C^* \theta, \sigma \rangle_{IR^q} = \langle \theta, C\nabla H H^* \nabla^* C^* \sigma \rangle_{IR^q} = \langle \theta, \Lambda\sigma \rangle_{IR^q}$$

and positive definite, indeed

$$\begin{aligned}
\langle \Lambda\theta, \theta \rangle_{IR^q} &= \langle C\nabla H H^* \nabla^* C^* \theta, \theta \rangle_{IR^q} \\
&= \langle H^* \nabla^* C^* \theta, H^* \nabla^* C^* \theta \rangle_{L^2(0,T;IR^P)} \\
&= \int_0^T \langle B^* S^* (T-s) \nabla^* C^* \theta, B^* S^* (T-s) \nabla^* C^* \theta \rangle_{IR^P} \, ds \\
&= \|\theta\|_*^2 > 0, \quad for \ \theta \neq 0
\end{aligned}$$

and then Λ is invertible. $\qquad\qquad\square$

We give hereafter the expression of the optimal control ensuring the gradient remediability of

a disturbance f at the time T.

Proposition 7:

For $f \in L^2(0,T;X)$, there exists a unique $\theta_f \in IR^q$ such that $\Lambda\theta_f = -C\nabla F f$ and the

control $u_{\theta_f}(.) = B^* S^*(.) \nabla^* C^* \theta_f$ verifies $C\nabla H u_{\theta_f} + C\nabla F f = 0$. Moreover, it is optimal and

$\left\| u_{\theta_f} \right\|_{L^2(0,T;IR^P)} = \|\theta_f\|_*$.

Proof :

From the Proposition 6, the operator Λ is invertible then, for $f \in L^2(0,T;X)$, there exists a

unique $\theta_f \in IR^q$ such that

$$\Lambda\theta_f = -C\nabla Ff$$

and if we put $u_{\theta_f}(.) = B^*S^*(.)\nabla^*C^*\theta_f$, we obtain

$$\Lambda\theta_f = C\nabla\int_0^T S(T-s)BB^*S^*(T-s)\nabla^*C^*\theta_f\,ds = C\nabla Hu_{\theta_f}$$

$$\Rightarrow -C\nabla Ff = C\nabla Hu_{\theta_f} \Rightarrow C\nabla Hu_{\theta_f} + C\nabla Ff = 0.$$

The set D defined by (13) is closed, convex and not empty. For $u \in D$, we have $J(u) = \|u\|^2_{L^2(0,T;IR^P)}$. J is strictly convex on D, and then has a unique minimum at $u^* \in D$, characterized by $\langle u^*, v - u^* \rangle_{L^2(0,T;IR^P)} \geq 0; \quad \forall v \in D$.

For $v \in D$, we have

$$\left\langle u_{\theta_f}, v - u_{\theta_f} \right\rangle_{L^2(0,T;IR^P)} = \left\langle B^*S^*(.)\nabla^*C^*\theta_f, v - B^*S^*(.)\nabla^*C^*\theta_f \right\rangle_{L^2(0,T;IR^P)}$$

$$= \left\langle \theta_f, C\nabla Hv - \Lambda\theta_f \right\rangle_{IR^q} = 0$$

Since u^* is unique, then $u^* = u_{\theta_f}$ and u_{θ_f} is optimal with

$$\left\|u_{\theta_f}\right\|^2_{L^2(0,T;IR^P)} = \left\|B^*S^*(.)\nabla^*C^*\theta_f\right\|^2_{L^2(0,T;IR^P)} = \left\|\theta_f\right\|^2_*. \qquad \Box$$

6. APPROXIMATIONS AND NUMERICAL SIMULATIONS

This section concerns approximations and numerical simulations of the problem of gradient remediability. First we give an approximation of θ_f as a solution of a finite dimension linear system $Ax = b$ and then the optimal control u_{θ_f}, with a comparison between the corresponding observation noted $z_{u,f}$ and the normal case.

7.1 Approximations

- **Coefficients of the system:** For $i, j \geq 1$, let

$a_{ij} = \left\langle \Lambda e_i, e_j \right\rangle_{IR^q}$ where $(e_i)_{1 \leq i \leq q}$ is the canonical basis of IR^q, we have

$$\Lambda e_i = C\nabla\int_0^T S(T-s)BB^*S^*(T-s)\nabla^*C^*e_i\,ds$$

And for M, N sufficiently large, we have

$$a_{ij} \approx \sum_{m=1}^{M} \sum_{l=1}^{r_m} \sum_{m'=1}^{N} \sum_{h=1}^{r_{m'}} \sum_{\tau=1}^{p} \left(\frac{e^{(\lambda_m + \lambda_{m'})T} - 1}{\lambda_m + \lambda_{m'}} \right) \langle g_\tau, w_{ml} \rangle_{L^2(\Omega_\tau)} \langle g_\tau, w_{m'h} \rangle_{L^2(\Omega_\tau)} \sum_{k'=1}^{n} \left\langle \frac{\partial w_{m'h}}{\partial x_{k'}}, h_i \right\rangle_{L^2(D_i)} \sum_{k=1}^{n} \left\langle \frac{\partial w_{ml}}{\partial x_k}, h_j \right\rangle_{L^2(D_j)}$$

and $\Lambda \theta_f = -C\nabla F f$

$$b_j = -\langle C\nabla F f, e_j \rangle_{IR^q}$$

For N sufficiently large, we have

$$b_j \approx -\sum_{m'=1}^{N} \sum_{h=1}^{r_{m'}} \sum_{k=1}^{n} \left\langle \frac{\partial w_{m'h}}{\partial x_k}, h_j \right\rangle_{L^2(D_j)} \int_0^T e^{\lambda_{m'}(T-s)} \langle f(s), w_{m'h} \rangle_{L^2(\Omega)} ds$$

- **The optimal control:** In this part, we give an

approximation of the optimal control u_{θ_f} which is defined by $u_{\theta_f}(.) = B^* S^*(T-.)\nabla^* C^* \theta_f$.

Its function coordinates $u_{j,\theta_f}(.)$ are given by

$$u_{j,\theta_f}(.) = \langle g_j, S^*(T-.)\nabla^* C^* \theta_f \rangle \approx \sum_{m'=1}^{N} \sum_{h=1}^{r_{m'}} \sum_{k=1}^{n} \sum_{i=1}^{q} \theta_{i,f} e^{\lambda_{m'}(T-.)} \langle g_j, w_{m'h} \rangle_{L^2(\Omega_j)} \left\langle \frac{\partial w_{m'h}}{\partial x_k}, h_i \right\rangle$$

for a large integer N.

- **Cost:** The minimum energy (cost) is defined by

$$\left\| u_{\theta_f} \right\|_{L^2(0,T;IR^p)} = \left(\int_0^T \left\| B^* S^*(T-s)\nabla^* C^* \theta_f \right\|_{IR^p}^2 ds \right)^{\frac{1}{2}}$$

$$\approx \left(\sum_{j=1}^{p} \int_0^T \left(\sum_{m'=1}^{N} \sum_{h=1}^{r_{m'}} \sum_{k=1}^{n} \sum_{i=1}^{q} e^{\lambda_m(T-s)} \theta_{i,f} \left\langle h_i, \frac{\partial w_{m'h}}{\partial x_k} \right\rangle \langle g_j, w_{m'h} \rangle \right)^2 ds \right)^{\frac{1}{2}}$$

for N sufficiently large.

- **The corresponding observation:** The observation corresponding to the control is
  given by

$$z_{u_{\theta_f}, f}(t) = C\nabla S(t)y^0 + C\nabla \int_0^t S(t-s)Bu_{\theta_f}(s)ds + C\nabla \int_0^t S(t-s)f(s)ds$$

Its coordinates $\left(z_{j,u_{\theta_f}, f}(.) \right)_{1 \le j \le q}$ are obtained for a large integer N, as follows:

$$z_{j,u_{\theta_f}, f}(t) \approx \sum_{m'=1}^{N} \sum_{h=1}^{r_{m'}} \sum_{k=1}^{n} e^{\lambda_{m'}t} \langle y^0, w_{m'h} \rangle_{L^2(\Omega)} \left\langle h_j, \frac{\partial w_{m'h}}{\partial x_k} \right\rangle_{L^2(D_j)} +$$

$$\sum_{m'=1}^{N} \sum_{h=1}^{r_{m'}} \sum_{k=1}^{n} \sum_{i=1}^{p} \langle g_i, w_{m'h} \rangle \left\langle \frac{\partial w_{m'h}}{\partial x_k}, h_j \right\rangle \int_0^t e^{\lambda_{m'}(t-s)} u_{i,\theta_f}(s)ds + \sum_{m'=1}^{N} \sum_{h=1}^{r_{m'}} \sum_{k=1}^{n} \left\langle \frac{\partial w_{m'h}}{\partial x_k}, h_j \right\rangle \int_0^t e^{\lambda_{m'}(t-s)} \langle f(s), w_{m'h} \rangle ds$$

7.2 Numerical simulations

We consider without loss of generally the following diffusion system

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = \Delta y(x,t) + \sum_{i=1}^{p} \chi_{\Omega_i} g_i(x) u_i(t) + f(x,t) & \Omega \times]0,T[\\ y(x,0) = y^0(x) & \Omega \\ y(\xi,t) = 0 & \partial\Omega \times]0,T[\end{cases}$$

with $\Omega =]0,1[$ and a Dirichlet boundary condition. In this case, the functions $w_m(.)$ are defined by $w_m(x) = \sqrt{2}\sin(m\pi x); m \geq 1$. The associated eigenvalues are simple and given by $\lambda_m = -m^2\pi^2; \quad m \geq 1$. Then in the case of:

- an initial state: $y^0(.) = 0$,

- a sensor: (D,h) with $D =]0,1[$ and $h(x) = \sqrt{2}x^2 \;\; (q=1)$

- an efficient actuator: (Ω, g) with $\Omega =]0,1[$ and $g(x) = 2x^3 \;\; (p=1)$

- a disturbance function: defined by $f(x,t) = 240\, e^{-\left(\frac{t}{10}+x\right)}; t > 0$

For $M = N = 1$ and $T = 70$, we obtain numerical results illustrating the theoretical results established in previous sections. Hence, in figure1, we give the representations of the discrete observation $z_{u,f}$ corresponding to the control $u = u_{\theta_f}$ and the disturbance f and $z_{0,0}$ which represent the normal observation, that is $u = 0$ and $f = 0$.
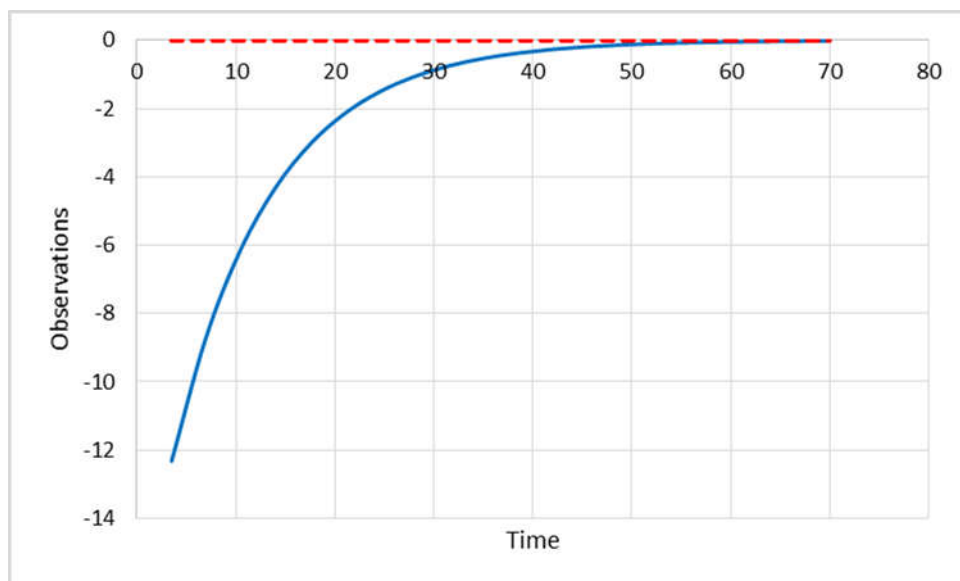

Fig.1. Representation of $z_{u,f}$ (blue line) and $z_{0,0}$ (red line).

This figure show that for t sufficiently large $(t \geq 50)$, the disturbance f is compensate by the control optimal u_{θ_f} at the time T $(T = 70)$ that is, we have $z_{u_{\theta_f},f}(t) \equiv z_{0,0}(t)$.

The optimal control u_{θ_f} ensuring the gradient remediability of the disturbance f, is represented in figure 2.
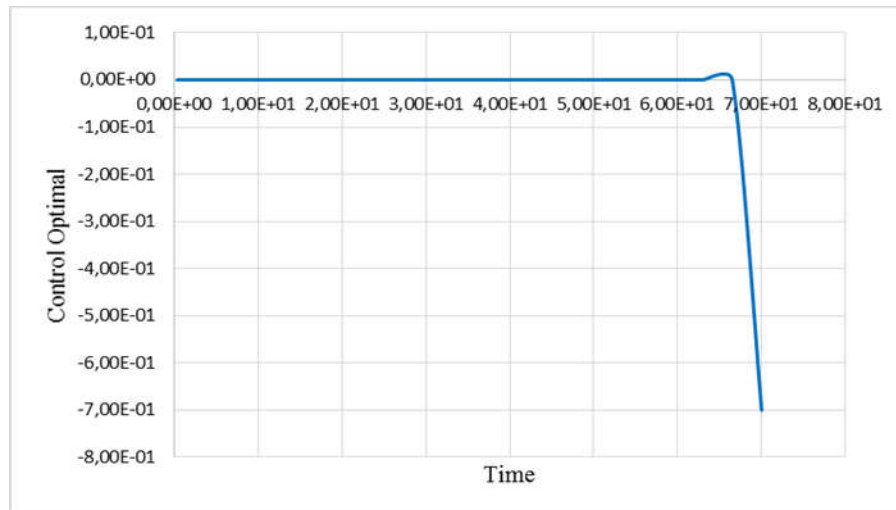

Fig.2. Representation of the optimal control u_{θ_f} (blue line)

8. CONCLUSION

In this paper, which is an extension of previous works to the analysis of the gradient of a large class of parabolic systems, new notions of weak and exact gradient remediability are introduced and characterized. The relation between the notion of gradient remediability and the notion of gradient controllability is also studied. We have shown that a parabolic system is gradient remediable if it is gradient controllable. Furthermore, we have shown that the exact and weak gradient remediability of a system may depend on the structure and the number of the actuators and sensors. Using an extension of the Hilbert Uniqueness Method, we have shown how to find the optimal control ensuring the gradient remediability of the known or unknown disturbance. The results of illustrative examples and numerical approximations are acceptable.

These results are developed for a class of discrete linear distributed parabolic systems, but the considered approach can be extended to other class of systems with a convenient choice of space.

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