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APPLICATION OF NEW ITERATIVE METHOD FOR SOLVING LINEAR AND NONLINEAR INITIAL BOUNDARY VALUE PROBLEMS WITH NON LOCAL CONDITIONS

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ABSTRACT
In this paper, a new iterative method to solve linear and nonlinear initial boundary value problem (IBVP) with non local conditions was developed. The new method is an elegant combination of traditional variational iteration and the decomposition methods. Several examples are presented to verify the accuracy and efficiency of this new method with the exact solution. This new iterative method developed may be suitable for teaching and better understanding of some advanced undergraduate courses on analytical and classical mechanics.

Keywords: Variational iteration method, decomposition method, initial boundary value problems, iterative method, non-local conditions

INTRODUCTION
With the rapid development of linear and nonlinear science and engineering, many analytical and numerical methods have been developed by various researcher for solving differential equation with non local conditions, (Cheniguel, 2012; Cheniguel, 2011; Cheniguel & Ayadi (2010), Siddique, 2010; Rahman, 2009) . This development was driven by the needs from application in physics, astrophysics, experimental and mathematical physics, nuclear charge in heavy atoms, thermodynamics and fluid mechanics. The widely use methods for solving these equations are perturbation method. Cheniguel & Reghious (2013) have studied IBVP using Homotopy Perturbation method (HPM) formulated by merging the standard homotopy with perturbation. The objectives of these studies were mostly to determine the analytical and numerical solutions where a considerable volume of calculations is usually needed. In this paper, we apply the new iterative method (NIM) by Versha & Sachin (2008) and modified new iterative method (MNIM) by Yaseen & Samraiz (2012) to linear and nonlinear homogeneous and inhomogeneous IBVP with non classical condition. These new techniques minimize the amount of calculations introduced by HPM. The general form of equation is given as

$$u_t = G(x,t,u,u_x,u_{xx})$$
$$a < x < b,\ 0 < t \le T \tag{1}$$

Subject to the initial condition:

$$u(x,0) = f(x), \quad 0 \le t \le T \tag{2}$$

and the non-local boundary conditions

$$u(a,t) = \int_a^b \varphi(x,t)u(x,t)dx + g_0(t), \quad 0 < t \le T \tag{3}$$

$$u(b,t) = \int_a^b \psi(x,t)u(x,t)dx + g_1(t), \quad 0 < t \le T \tag{4}$$

where $f, g_0, g_1, \varphi, \psi$ are sufficiently smooth known functions and T is a given constant.

NEW ITERATIVE METHOD
Consider the following general functional equation

$$u(x) = f(x) + N(u(x)) \tag{5}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function. $x = (x_1, x_2, ..., x_n)$. Looking for a solution u of (5) having the form

$$u(x) = \sum_{i=0}^{\infty} u_i(x) \tag{6}$$

The nonlinear operator N can be decomposed as (7)
From equations (6) and (7), equation (5) is equivalent to

$$\sum_{n=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{ N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j)\right\}$$

$$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty}\left\{ N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j)\right\} \tag{8}$$

The recurrence relation are define as

$$u_0 = f \tag{9}$$

$$u_1 = N(u_0) \tag{10}$$

$$u_{n+1} = N(u_0 + ...u_n) - N(u_0 + ... + u_{n-1}) \tag{11}$$

Then

$$(u_1 + ... + u_n) = N(u_0 + ...u_n) \tag{12}$$

and

$$\sum_{i=0}^{\infty} u_i = f + N(\sum_{i=0}^{\infty} u_i) \tag{13}$$

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The k - term approximate solution of (5) is given by $\sum_{i=0}^{k-1} u_i$. We refer to Varsha and Jafar (2006), Bhalekar & Varsha (2006) and Bhalekar & Varsha (2010) for details of the convergence.
The MNIM is based on including particular terms of the source term of inhomogeneous IBVP into the integral representing $N(u)$ in NIM.

NUMERICAL EXAMPLES

Example 1
Consider the IBVP

$$u_{tt} + u_t - u_{xx} - u_x = 4t^3 + 12t^2 - 4x^3 - 12x^2$$
$$0 < x < 1, \quad 0 < t < T \tag{14}$$

with the initial condition:

$$u(x,0) = x^4, \quad u_t(x,0) = 0, \quad 0 < x < 1, \ 0 < t < T \tag{15}$$

and the boundary conditions:

$$u(0,t) = \int_0^1 \varphi(x,t)\, u(x,t)\, dt + g_0(t) = 1 + \frac{1}{5}t^4 \tag{16}$$

where $\varphi(x,t) = \dfrac{1}{5}$ and $g_0(t) = \dfrac{24}{25}$

$$u(1,t) = \int_0^1 \psi(x,t)\, u(x,t)\, dt + g_1(t) = 1 + \frac{1}{6}t^4 \tag{17}$$

where $\psi(x,t) = \dfrac{1}{6}$ and $g_1(t) = \dfrac{29}{30}$

To solve this problem, equation (14) is equivalent to the following integral equation

$$u = x^4 + \int_0^t \int_0^t \left(u_{xx} + u_x - u_t + 4t^3 + 12t^2 - 4x^3 - 12x^2\right) dt\, dt \tag{18}$$

Set $u_0 = x^4$ and

$$N(u) = \int_0^t \int_0^t \left(u_{xx} + u_x - u_t + 4t^3 + 12t^2 - 4x^3 - 12x^2\right) dt\, dt$$

. Following the algorithm (9) – (11), the successive approximations are

$$u_1 = N(u_0) = \frac{1}{5}t^5 + t^4 \tag{19}$$

$$u_2 = N(u_0 + u_1) - N(u_0) = t^4 - \frac{1}{30}t^6 - x^4 \tag{20}$$

$$\begin{aligned} u_3 &= N(u_0 + u_1 + u_2) - N(u_0 + u_1) \\ &= \frac{1}{210}t^7 - \frac{1}{30}t^6 - \frac{2}{5}t^5 + \frac{1}{2}(-4x^3 - 12x^2)t^2 - x^4 \end{aligned} \tag{21}$$

Thus, the approximate solution of Equations (14) - (17) after 3rd iteration is

$$\sum_{i=0}^{3} u_i = \frac{1}{210}t^7 - \frac{1}{15}t^6 - \frac{1}{5}t^5 + 2t^4 + (-2x^3 - 6x^2)t^2 - x^4 \tag{22}$$

which is very close to exact solution $u(x,t) = x^4 + t^4$ obtained by Cheniguel and Reghioua (2013)

Example 2
Consider the following nonlinear reaction-diffusion equation:

$$u_t - u_{xx} = u^2 - u_x^2 \qquad 0 < x < 1, \ 0 < t < T \tag{23}$$

Subject to the initial condition:

$$u(x,0) = e^x, \quad 0 < x < 1, \ 0 < t < T \tag{24}$$

and the boundary conditions:

$$u(0,t) = \int_0^1 \varphi(x,t)\, u(x,t)\, dt + g_0(t) = e^{1+t} \tag{25}$$

where $\varphi(x,t) = 1$ and $g_0(t) = e^t$

$$u(1,t) = \int_0^1 \psi(x,t)\, u(x,t)\, dt + g_1(t) = \frac{1}{2}e^{1+t} \tag{26}$$

where $\psi(x,t) = \dfrac{1}{2}$ and $g_1(t) = \dfrac{1}{2}e^t$

To solve this problem, equation (23) is equivalent to the following integral equation

$$u = e^x + \int_0^t \left(u_{xx} + u^2 - u_x^2\right) dt \tag{27}$$

Set $u_0 = e^x$ and $N(u) = \int_0^t \left(u_{xx} + u^2 - u_x^2\right) dt$.

Following the algorithm (9) – (11), the successive approximations are

$$u_1 = N(u_0) = e^x t \tag{28}$$

$$u_2 = N(u_0 + u_1) - N(u_0) = e^x t + \frac{1}{2}e^x t^2 - e^x \tag{29}$$

$$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1) = e^x t^2 + \frac{1}{6}e^x t^3 - e^x - e^x t \tag{30}$$

Thus, the approximate solution of Eqs. (23)–(26) after 3rd iteration is

$$\sum_{i=0}^{3} u_i = \frac{1}{6}e^x t^3 + \frac{3}{2}e^x t^2 + e^x t - e^x \tag{31}$$

which is very close to exact solution $u(x,t) = e^{x+t}$ obtained by Cheniguel & Reghioua (2013)

Example 3
Consider the following nonlinear reaction-diffusion equation:

$$u_t = \frac{1}{6}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right)$$

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$$0 < x, y, z < 1, \quad 0 < t < T \tag{32}$$

Subject to the initial condition:

$$u(x,y,z,0) = x^2y^2z^2, \quad 0 < x, y, z < 1, \; 0 < t < T \tag{33}$$

and the boundary conditions:

$$u(0,y,z,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)\,dx\,dy\,dz + g_1 = \frac{1}{27}e^t, \quad g_1 = 0 \tag{34}$$

$$u(1,y,z,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)\,dx\,dy\,dz + g_2 = \frac{1}{27}e^t + \frac{1}{2}t, \quad g_2 = \frac{1}{2}t \tag{35}$$

$$u(x,0,z,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)\,dx\,dy\,dz + g_3 = \frac{1}{27}\left(e^x + 1\right), \quad g_3 = \frac{1}{27} \tag{36}$$

$$u(x,1,z,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)\,dx\,dy\,dz + g_4 = \frac{1}{27}\left(e^t + 3\right), \quad g_4 = \frac{1}{9} \tag{38}$$

$$u(x,y,0,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)\,dx\,dy\,dz + g_5 = \frac{1}{27}e^t + \frac{1}{6}, \quad g_5 = \frac{1}{6} \tag{39}$$

$$u(x,y,1,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)\,dx\,dy\,dz + g_6 = \frac{1}{27}e^t + \frac{1}{5}t, \quad g_6 = \frac{1}{5}t \tag{40}$$

To solve this problem, equation (32) is equivalent to the following integral equation

$$u = x^2y^2z^2 + \int_0^t \frac{1}{6}\left(x^2u_{xx} + y^2u_{yy} + z^2u_{zz}\right)dt \tag{41}$$

Set $u_0 = x^2y^2z^2$ and

$N(u) = \int_0^t \frac{1}{6}\left(x^2u_{xx} + y^2u_{yy} + z^2u_{zz}\right)d$. Following the algorithm (9) – (11), the successive approximations are

$$u_1 = N(u_0) = x^2y^2z^2t \tag{42}$$

$$u_2 = N(u_0+u_1) - N(u_0) = \frac{1}{2}x^2y^2z^2t^2 + x^2y^2z^2t - x^2y^2z^2t \tag{43}$$

$$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1) = \frac{1}{6}x^2y^2z^2t^3 + x^2y^2z^2t^2 - x^2y^2z^2t \tag{44}$$

Thus, the approximate solution of Equations (32)–(40) after 3rd iteration is

$$\sum_{i=0}^{3} u_i = \frac{1}{6}x^2y^2z^2t^3 + \frac{3}{2}x^2y^2z^2t^2 + x^2y^2z^2t - x^2y^2z^2 \tag{43}$$

which is very close to exact solution $u(x,y,z,t) = x^2y^2z^2e^t$ obtained by Cheniguel & Reghioua (2013)

Example 4

Consider the IBVP

$$u_{tt} = \left(u^{-1}u_x\right)_x \quad 0 < x < 1, \quad 0 < t < T \tag{44}$$

with the initial condition:

$$u(x,0) = \frac{1}{(1+x)^2} \quad u_t(x,0) = 0, \quad 0 < x < 1, \; 0 < t < T \tag{45}$$

and the boundary conditions:

$$u(0,t) = \int_0^1 \varphi(x,t)\,u(x,t)\,dt + g_0(t) = 1 + \frac{1}{2}t \tag{46}$$

where $\varphi(x,t) = 1$ and $g_0(t) = \frac{1}{2}$

$$u(1,t) = \int_0^1 \psi(x,t)\,u(x,t)\,dt + g_1(t) = 1 + \frac{5}{40}t \tag{47}$$

where $\psi(x,t) = \frac{1}{4}$ and $g_1(t) = \frac{7}{8}$

To solve this problem, equation (44) is equivalent to the following integral equation

$$u = \frac{1}{(1+x)^2} + \int_0^t\int_0^t \left(u^{-1}u_x\right)_x dt\,dt \tag{48}$$

Set $u_0 = \frac{1}{(1+x)^2}$ and $N(u) = \int_0^t\int_0^t \left(u^{-1}u_x\right)_x dt\,dt$.

Following the algorithm (9) – (11), the successive approximations are

$$u_1 = N(u_0) = \frac{t^2}{(1+x)^2} \tag{49}$$

$$u_2 = N(u_0+u_1) - N(u_0) = \frac{t^2}{(1+x)^2}{}^{4} - \frac{1}{(1+x)^2} \tag{50}$$

$$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1) = -\frac{1}{(1+x)^2} \tag{51}$$

Thus, the approximate solution of Equations (44) - (47) after 3rd iteration is

$$\sum_{i=0}^{3} u_i = -\frac{1}{(1+x)^2} + \frac{2t^2}{(1+x)^2} \tag{52}$$

which is very close to exact solution $u(x,t) = \frac{1-t^2}{(1+x)^2}$

obtained by Cheniguel & Reghioua (2013)

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RESULTS

Table 1: Pointwise error obtained between NIM results and exact solutions in **Example 1**

$$h_x = \frac{1}{10}, \quad h_t = \frac{1}{250}$$

x_i	u_{ex}	u_{HPM} 3-iterates	u_{NIM}	$\lvert u_{ex} - u_{NIM} \rvert$
0.0	2.56E-10	-1.92E-3	4.09E-13	2.5E-10
0.1	0.000100	-9.2E-5	0.000100	0
0.2	0.001600	0.001584	0.001600	0
0.3	0.008100	0.007908	0.008100	0
0.4	0.025617	0.025408	0.025617	0
0.5	0.062500	0.062308	0.062528	2.8E-05
0.6	0.129600	0.129410	0.129641	4.1E-05
0.7	0.240100	0.239910	0.240158	5.8E-05
0.8	0.409600	0.409410	0.409677	7.7E-05
0.9	0.656100	0.655910	0.656201	1.0E-05
1.0	1.000000	0.999981	1.000128	1.2E-05

Table 2: Pointwise error obtained between NIM results and exact solutions in **Example 2**

$$h_x = \frac{1}{10}, \quad h_t = \frac{1}{250}$$

x_i	u_{ex}	u_{HPM} 3-iterates	u_{NIM}	$\lvert u_{ex} - u_{NIM} \rvert$
0.0	1.004008	1.004009	1.003983	2.5E-05
0.1	1.109600	1.109600	1.109573	2.7E-05
0.2	1.226268	1.226300	1.226268	0
0.3	1.355269	1.355301	1.355236	3.3E-05
0.4	1.497803	1.497807	1.497768	3.5E-05
0.5	1.655329	1.655302	1.655289	4.0E-05
0.6	1.828421	1.829405	1.829378	9.2E-05
0.7	2.021823	2.021801	2.021775	4.8E-05
0.8	2.234460	2.234500	2.234407	5.3E-05
0.9	2.469461	2.469503	2.469402	5.9E-05
1.0	2.729176	2.729205	2.729111	6.5E-05

Table 3: Pointwise error obtained between NIM results and exact solutions in **Example 3**

$$h_x = \frac{1}{10}, \quad h_t = \frac{1}{250}$$

x_i, y_i, z_i	u_{ex}	u_{HPM} 3-iterates	u_{NIM}	$\lvert u_{ex} - u_{NIM} \rvert$
0.0,0.0,0.0	0	0	0	0
0.1,0.1,0.1	0.000001	0.000001	0.000001	0
0.2,02,0.2	0.000064	0.000064	0.000064	0
0.3,0.3,0.3	0.000731	0.000731	0.000731	0
0.4,0.4,0.4	0.004112	0.004112	0.004112	0
0.5,0.5,0.5	0.015687	0.015687	0.015687	0
0.6,0.6,0.6	0.046842	0.046842	0.046842	0
0.7,0.7,0.7	0.118120	0.118120	0.118117	3.0E-06
0.8,0.8,0.8	0.263194	0.263194	0.263188	6.0E-06
0.9,0.9,0.9	0.533571	0.533571	0.533558	1.3E-05
1.0,1.0,1.0	1.004008	1.004008	1.003983	2.5E-05

Table 4: Pointwise error obtained between NIM results and exact solutions in **Example 4**

$$h_x = \frac{1}{10}, \quad h_t = \frac{1}{250}$$

x_i	u_{ex}	u_{HPM} 3-iterates	u_{NIM}	$\lvert u_{ex} - u_{NIM} \rvert$
0.0	1.000001	1.000000	0.999968	3.3E-05
0.1	0.826459	0.826450	0.826419	4.0E-05
0.2	0.694455	0.694442	0.694422	3.3E-05
0.3	0.591725	0.591723	0.591697	2.8E-05
0.4	0.510212	0.510200	0.510187	2.5E-05
0.5	0.444451	0.444444	0.444430	2.1E-05
0.6	0.390631	0.390631	0.390612	1.9E-05
0.7	0.346026	0.346022	0.346009	1.7E-05
0.8	0.308646	0.308641	0.308632	1.4E-05
0.9	0.277012	0.277013	0.276999	1.3E-05
1.0	0.250004	0.250010	0.249992	1.2E-05

Conclusion

In this paper, we applied the new iterative method by Versha & Sachin (2008) and modified new iterative method (MNIM) by Yaseen & Samraiz (2012) for solving linear and nonlinear initial boundary value problem (IBVP) with non local conditions. The method is applied directly without using linearization, discretization, perturbation or restrictive assumption in comparison with other existing methods. The exact and approximate solutions were obtained by using the initial conditions only. The results as shown in the tables and graphs illustrate the stability and convergence of the method even at third iteration for IBVP. Thus we conclude the method used in this paper can be considered as an efficient alternative for solving IBVP with non-local conditions.

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