

ON A NUMERICAL ALGORITHM FOR UNCERTAIN SYSTEM

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ABSTRACT

A numerical method for computing stable control signals for system with bounded input disturbance is developed. The algorithm is an elaboration of the gradient technique and variable metric method for computing control variables in linear and non-linear optimization problems. This method is developed for an integral quadratic problem subject to a dynamic system with input bounded uncertainty.

Key words: Stable Control, Gradient Technique, Variable Metric Method and Input bounded Uncertainty.

INTRODUCTION

This paper is concerned with the development of a numerical method for the computation of controls for system with bounded input disturbances. Existing numerical methods for computing control signals for optimisation problems do not consider the situation when uncertainties are involved in the system, this work is set to give consideration to such problem and for a start we shall develop this method for an integral quadratic problem subject to a dynamical system with input bounded uncertainty.

Formulation of Problem:

Consider

$$\text{Min} \int_0^T \{x(t)Qx(t) + u(t)Ru(t)\} dt \quad \dots (1)$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t) + Cv(t) \quad \dots (2)$$

and

$$x(0) = x^0, \text{ given, } t \in [0, T], \quad \dots (3)$$

$x(t) \in X \subset \mathbb{R}^N$, is the state vector $u(t) \in U \subset \mathbb{R}^M$ is the control vector and $v(t) \in V \subset \mathbb{R}^M$ is the uncertain vector.

$U \cap V = \phi$. T is the final time. $Q \in \mathbb{R}^{N \times N}$ is positive semi-definite (symmetric) $R \in \mathbb{R}^M$ is a positive definite matrix.

We state the following assumptions:

A_1 : We assume that A is a stable matrix and if otherwise there exist a matrix

$$K \in \mathbb{R}^{M \times N} \text{ such that } \bar{A} = A + B^T K.$$

Given that PD^N denotes the set of positive symmetric members of $\mathbb{R}^{N \times N}$

Then

$$P\bar{A} + \bar{A}^T P + Q_1 = 0, Q_1 \in PD^N \quad \dots (4)$$

A_2 : $v(t)$ is a Caratheodory function and given $\bar{m}, m \in \mathbb{R}$, $m \leq v(t) \leq \bar{m}$

A_3 : (A, B) , is controllable.

For our subsequent development of this paper, we now consider

$$\dot{x}(t) = Ax(t) \quad \dots (5)$$

Let $\Phi(t_0, t)$ be a transition matrix for (5) then we can write the solution for equations (2) and (3) as

$$x(t) = \Phi(t_0, t)x_0 + \int_0^t \Phi(t, \sigma) \{Bu(\sigma) + Cv(\sigma)\} d\sigma \quad \dots (6)$$

We calculate the transition matrix using the following formula

$$f(A) = \sum_{k=1}^n f(\lambda_k) \prod_{j=1}^n \left(\frac{A - \lambda_j I}{\lambda_i - \lambda_j} \right) \quad \dots (7)$$

Where

λ_k denotes the eigen-value of the matrix A .

Let $B[\mathbb{R}^N, \mathbb{R}^M]$ be the set of bounded linear map between

\mathbb{R}^N and \mathbb{R}^M . We now define a map

$$L: B[\mathbb{R}^N, \mathbb{R}^M] \rightarrow B[\mathbb{R}^N, \mathbb{R}^M]$$

Be such that

$$Lu = \int_0^t \Phi(t, \sigma) Bu(\sigma) d\sigma \quad \dots (8)$$

And similarly define

$$Kv = \int_0^t \Phi(t, \sigma) Cv(\sigma) d\sigma \quad \dots (9)$$

For arbitrary elements $w_1, w_2 \in \mathbb{R}^N$, define the inner product on \mathbb{R}^N as

$$\langle w_1, w_2 \rangle = \int_0^t w_1(t)^T w_2(t) dt \quad \dots (10)$$

$$\text{Let } \Phi(t_0, t)x_0 = z_1 \quad \dots (11)$$

Therefore

$$x(t) = z_1 + Lu + Kv \quad \dots (12)$$

We can now write (1) as

$$\begin{aligned} J(u, v) &= \langle x(t), Qx(t) \rangle + \langle u(t), Ru(t) \rangle \\ &= \langle z_1 + Lu + Kv, Q(z_1 + Lu + Kv) \rangle + \langle u, Ru \rangle \end{aligned} \quad \dots (13)$$

Therefore,

$$\begin{aligned} J(u, v) &= \langle u, (LQ^T L + R)u \rangle + \langle v, KQ^T Lu \rangle + 2 \langle u, LQ^T Kv \rangle \\ &\quad + 2 \langle z_1, Q^T Lu \rangle + 2 \langle v, K^T Qz_1 \rangle + \langle v, KQ^T Kv \rangle \end{aligned} \quad \dots (14)$$

We state the following remarks

Remark (1): $J(u, v)$ is convex with respect to $u \in U \subset \mathbb{R}^M$

Proof: Consider

$$\langle u, (LQ^T L + R)u \rangle = (\langle Lu, LQ^T u \rangle + \langle u, Ru \rangle) > 0$$

This implies convexity.

Remark (2): Under the assumption that $u(t)$ is a minimizer for $J(u, v)$ and $v(t)$ is a maximizer, then $J(u, v)$ is concave with respect to $v \in \mathbb{R}^M$

Proof: Let Γ_i be l -dimensional row vector of matrix $KQ^T K$. Denote the norm of v by $\|v\|$. Since v is assumed bounded there exist \tilde{m} such that $\|v\| \leq \tilde{m}$. Now consider

$$\langle v, KQ^T Kv \rangle = \sum_{i=1}^l (\Gamma_i v)^2 \leq \|v\|^2 \sum_{i=1}^l \|\Gamma_i\|^2 \leq m^2 \text{Tr}[KQ^T K]^2 = d^2 > 0 \quad \dots (15)$$

Assumption:

A_4 : By Minimax theorem (Abiola, 2009) we assume $v(t) = -m^0 u(t)$ where $\underline{m} < v(t) < \overline{m}$, and

$m^0 = \frac{1}{2}(\overline{m} + \underline{m})$. Using A_4 , we write $J(u, v)$ as

$$\begin{aligned} J(u) &= \langle u, (LQ^T L + R) + ((m^0)^2 - 2m^0)(KQ^T L)u \rangle \\ &\quad + 2 \langle z_1, Q^T Lu \rangle - 2m^0 \langle u, K^T Qz_1 \rangle \end{aligned} \quad \dots (16)$$

We state the following proposition from (14).

Proposition (1)

For every $m^0 > 0$, $\{(LQ^T L + R) + (m^0)^2 - 2m^0(KQ^T L)\} > 0$

The prove of the above proposition is an obvious fact.

Numerical Method

The problem under consideration is the minimization of $J(u)$ defined in (16) with respect to the control signal $u(t) \in \mathbb{R}^n$.

$J(u)$ is non-linear, continuous and twice differentiable. We wish to find numerically u^* such that:

$$J(u^*) < J(u) \quad \forall u : \|u - u^*\| < \delta$$

Suppose $\nabla J(u)$ denotes the gradient of $J(u)$, and we also set

$$\nabla J(u) = g(u) \quad \dots (17)$$

Let

$$\nabla^2 J(u) = \bar{A}(u) \quad \dots (18)$$

Where $\bar{A}(u)$ denotes the Hessian matrix at point u . we assume that $\bar{A}(u)$ is non-degenerate at point u^* . The first order condition for minimum is that

$$\nabla J(u) = 0 \quad \dots (19)$$

A second order condition for minimum is that

$$\nabla^2 J(u) = \bar{A}(u) \geq 0 \quad \dots (20)$$

We see from proposition (1) that $\bar{A}(x) > 0$

One way of minimizing an unconstrained optimization problem is by application of Newton's Method, which is described briefly below:

We consider the problem described as:

$$\text{Min } F(x), x \in \mathbb{R}^n \quad \dots (21)$$

Find x^* such that $\forall x \|x - x^*\| < \delta$ and $F(x^*) < F(x)$.

Suppose $\nabla F(x) = g(x)$ is the gradient of $F(x)$ at point x and $\nabla^2 F(x) = \bar{A}(x)$ is the Hessian matrix, then the function $F(x)$ at each iteration is approximated by a quadratic model ϕ and effectively minimized as:

$$\phi(x + p) = F(x) + g(x)^T P + \frac{1}{2} (P^T A(x) P) \quad \dots (22)$$

Using the first order condition for minimum, we have

$$\phi'(x + p) = \nabla F(x) + \bar{A}(x) P = 0 \quad \dots (23)$$

Solving equation (23) we have

$$\bar{A}(x) p = -\nabla F(x) \quad \dots (24)$$

On the basis of (24) we state the Newton's Algorithm.

Newton's Algorithm

Step 1 Calculate $F(x_k), g(x_k), \bar{A}(x_k)$

Step 2. Check if $\|g(x_k)\| < \varepsilon$ for a predetermined ε , if so stop, else

Step3. Set $\bar{A}(x_k)P_k = -g(x_k)$

Step4. Set $x_{k+1} = x_k + P_k$

Step5. Set $k = k + 1$, and go to Step1.

The Newton's algorithm described above can be shown to have second order convergence (Dixon, 1975). This is because it utilizes quadratic model which is minimized at each iteration. The global convergence of Newton's method is not guaranteed because the iteration can diverge from arbitrary starting points. Since Newton's equations have to be solved at each iteration, it breaks down when $\bar{A}(x_k)$ is singular. However, to circumvent the drawback posed by the possibility of singularity of $\bar{A}(x_k)$, a variable metric algorithm was proposed (Dixon, 1975)

The scheme involves the following:

$$x_{k+1} = x_k - \alpha_k P^k g(x_k) \quad \dots (25)$$

x_0, P^0 are guessed arbitrarily. α_k is determined by a suitable line search method. P^0 is a positive definite matrix and normally P^k is intended to be approximation of the inverse of the Hessian matrix $\nabla^2 F(x)$.

This matrix is updated using information obtained in each iterative step as follows:

$$P^{k+1} = (P^k, v^k, y^k, z^k) \quad \dots (26)$$

Where $v^k = x_{k+1} - x_k, y^k = g_{k+1} - g_k$, and z^k is a vector parameter.

Variants of the variable metric algorithm are derived in the way in which the appropriate inverse matrices are constructed. A lot of methods for determining P^k have been proposed. For example (Dixon, 1975) proposed an updating rule for P^k such that

$$P^{k+1} = P^k + D \quad \dots (27)$$

where D is required to be very small as much as possible in some norm. He suggested the following Frobenious norm, defined as:

$$\gamma = \|D\|^2 = Tr[\bar{W}D\bar{W}D^T] \quad \dots (28)$$

\bar{W} is a positive definite matrix. D is assumed symmetric such that

$$D^T - D = 0 \quad \dots (29)$$

The quasi-newton condition defined as:

$$\begin{cases} Dy - r = 0 \\ r = \bar{W} - P^k y \end{cases} \quad \dots (30)$$

is also satisfied.

Minimising (28) subject to (29) and (30) using Lagrange method an analytical solution for D was derived as:

$$D = \frac{\{\theta(w - Py + b\theta)^T + (W - Py + b\theta)\theta^T\}}{y^T \theta} \quad \dots (31)$$

where

$$\theta = W^{-1}y, \quad b = \frac{1/2(y^T W - y^T Py)}{y^T W^{-1}y}$$

We next consider a method for constructing an approximation inverse for the Hessian matrix which also takes into consideration the uncertainty nature of the problem under investigation. This is the crux of this paper.

Minimising Algorithm

Consider the problem defined in (1) – (3) and let

$$H(x, u, v, \lambda) = x(t)^T Qx(t) + u(t)^T Ru(t) + \lambda^T (Ax(t) + Bu(t) + Cv(t)) \quad \dots (32)$$

be the system Hamiltonian.

Compute the gradient of (32) with respect to $u(t)$ as follows:

$$H(x, u + h, v, \lambda) = x(t)^T Qx(t) + (u + h)(t)^T R(u + h)(t) + \lambda^T (Ax(t) + B(u + h)(t) + Cv(t)) \quad \dots (33)$$

Since R is symmetric then $R^T = R$

Therefore,

$$H(x, u + h, v, \lambda) - H(x, u, v, \lambda) = 2u(t)^T Rh + \lambda^T Bh \quad \dots (34)$$

Now,

$$\lim_{h \rightarrow 0} \frac{H(x, u + h, v, \lambda) - H(x, u, v, \lambda)}{\|h\|} = 2u(t)^T R + \lambda^T B \quad \dots (35)$$

Therefore

$$\frac{\partial H}{\partial u} = 2u(t)^T R + \lambda^T B \quad \dots (36)$$

$$\text{We now choose } \lambda(t) = 2Px(t) \quad \dots (37)$$

P is a positive definite matrix calculated from (4), we then have

$$\frac{\partial H}{\partial u} = 2(u(t)^T R + B^T Px(t)) \quad \dots (38)$$

This is expressible as

$$\nabla H(u) = 2 \left\{ u(t)R + BP^T \left[\Phi(t_0, t)x_0 + \int_0^t \Phi(t, \sigma)Bu(\sigma)d\sigma - \bar{m} \int_0^t \Phi(t, \sigma)Cu(\sigma)d\sigma \right] \right\} \quad \dots (39)$$

We can now propose the following algorithm:

Define $J(u)$ by equation (16), $\nabla H(u)$ by (39), then

Step 1: Choose u_0 arbitrarily and compute $J(u_0)$, $\nabla H(u_0)$ and set

$$\nabla H(u_k) = g_k, \text{ and set } g_k = -F_k, k = 0$$

Step 2: If $\|g_k\| \leq \varepsilon$, for a predetermined ε , stop, else set

Step 3: $u_{k+1} = u_k + \alpha_k W_k F_k(u_k)$, where

$$\alpha_k = \frac{\langle g_k, g_k \rangle}{\langle F_k, \bar{A} F_k \rangle}$$

$$\bar{A} = \left\{ (LQ^T L + R) + ((m^0)^2 - 2m^0)(KQ^T L) \right\}$$

$$W_k = \left| \frac{P^{-1}}{\left[\int_0^t \{ \nabla H(u_k) P^{-1} \nabla H(u_k) \} d\tau \right]^{\frac{1}{2}}} \right|$$

To avoid the possibility of W_k being negative definite, P is a positive definite matrix calculated from:

$$PA + A^T P + Q = 0$$

Updating of sequences is by gradient technique, which is defined as follows:

$$g_{k+1} = g_k + \alpha_k \bar{A} F_k$$

$$F_{k+1} = -g_{k+1} + \beta_k F_k$$

$$\beta_k = \frac{\langle g_{k+1}, g_{k+1} \rangle}{\langle g_k, g_k \rangle}$$

Step 4: Set $k = k + 1$ and go to step 1

The convergence of this algorithm is similar to conjugate gradient algorithm given by Ibiejugba & Abiola (1985)

In what follows we show the procedure to derive the operator W_k

Suppose at iteration k and $k + 1$

$$u_{k+1} - u_k = \Delta u_k \quad \dots (40)$$

Let $S(u)$ be such that

$$S(u) = \int_0^T \left\{ \left(\frac{\partial H}{\partial u} \right) \Delta u_k(t) \right\} dt \quad \dots (41)$$

where $\left(\frac{\partial H}{\partial u} \right) = \nabla H(u)$ which was defined in (39)

Let a distance function be defined between u_{k+1} and u_k as follows:

$$(\delta S)^2 = \int_0^T (\Delta u_k) P (\Delta u_k) d\tau \quad \dots (42)$$

P is a positive definite matrix calculated as before. We consider only when

$\delta S > 0$, and suppose there is an infinitesimal change in Δu_k which is defined as

$$\int_0^T \left\{ \frac{d(\Delta u_k)}{ds} P \frac{d(\Delta u_k)}{ds} \right\} d\tau = 1 \quad \dots (43)$$

Then the corresponding change in (41) is given as

$$\frac{dS(u)}{ds} = \int_0^T \left\{ \left(\frac{\partial H}{\partial u} \right)^T \frac{d(\Delta u_k)}{ds} \right\} d\tau \quad \dots (44)$$

We now minimise (44) by choosing $\frac{d(\Delta u_k)}{ds}$ as control. In doing, this we use the

Lagrange multiplier technique and let η be the multiplier such that:

$$S_\eta = \int_0^T \left\{ \left(\frac{\partial H}{\partial u} \right)^T \left(\frac{d(\Delta u_k)}{ds} \right) + \eta^T \left(\frac{d(\Delta u_k)}{ds} + P \frac{d(\Delta u_k)}{ds} \right) \right\} d\tau \quad \dots (45)$$

We define the Hamiltonian as follows:

$$H_\eta = \left[\left(\frac{\partial H}{\partial u} \right)^T \left(\frac{d(\Delta u_k)}{ds} \right) + \eta^T \left(\frac{d(\Delta u_k)}{ds} + P \frac{d(\Delta u_k)}{ds} \right) \right] \quad \dots (46)$$

$$\frac{\partial H_\eta}{\left(\frac{d(\Delta u_k)}{ds} \right)} = \frac{\partial H}{\partial u} + 2\eta^T P \frac{d(\Delta u_k)}{ds} \quad \dots (47)$$

Setting (47) equals to zero implies

$$\frac{\partial H_\eta}{\partial \left(\frac{d(\Delta u_k)}{ds} \right)} = 0 \quad \dots (48)$$

Therefore,

$$\frac{d(\Delta u_k)}{ds} = \frac{1}{2\eta} \frac{P^{-1} \partial H}{\partial u} \quad \dots (49)$$

Using (49) in (43) we have:

$$\int_0^T \left[\left(\frac{\partial H}{\partial u} \right) P^{-1} \left(\frac{\partial H}{\partial u} \right) \right]^{\frac{1}{2}} d\tau = 2\eta \quad \dots (50)$$

Therefore

$$\frac{d\Delta u_k}{ds} = \frac{-P^{-1} \left(\frac{\partial H}{\partial u} \right)}{\left[\int_0^T \left\{ \left(\frac{\partial H}{\partial u} \right) P^{-1} \left(\frac{\partial H}{\partial u} \right) \right\} d\tau \right]^{\frac{1}{2}}} \quad \dots (51)$$

Numerical Illustration

Following Corless & Leitmann (1981), consider the following system defined by:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_1(t) + u_2(t) + v_2(t) \end{cases} \quad \dots (52)$$

The objective function is defined as

$$\text{Minimize } J(u, v, x) = \frac{1}{2} \int (u_1^2(t) + u_2^2(t)) dt$$

From the above, it is easy to see that

$$R = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B = C = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We also note that A is unstable. We therefore select $K = (k_1 \quad k_2)$ such that $k_1 = 0, k_2 < 0$

$$\bar{A} = A + BK = \begin{pmatrix} 0 & 1 \\ -1 & k_2 \end{pmatrix}$$

$$\text{Let } Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then if

$P\bar{A} + \bar{A}^T P + Q = 0$, we can compute

$$P = \begin{pmatrix} \frac{1}{2}k_2 - k_2^{-1} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -k_2^{-1} \end{pmatrix}$$

Suppose $k_2 = -\frac{1}{2}$ then, $P = \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}$

which inverse is computed to be:

$$P = \begin{pmatrix} \frac{8}{15} & -\frac{2}{15} \\ -\frac{2}{15} & \frac{8}{15} \end{pmatrix}$$

Applying the proposed algorithm, we have Table 1:

TABLE 1. CONTROL AND OBJECTIVE FUNCTION

Iteration	Time	Control u	Objective function
1	1	-44.193350	2592.106
2		45.114150	5597.7884
1	2	-31.34126	1372.045
2		-47.49781	2853.739
1	3	-23.35895	851.0466
2		26.9681	1857.691
1	4	-19.87427	629.528
2		24.146440	1389.023
1	5	34.4169	606.081
1	6	23.739	587.460
1	7	21.9034	502.018
1	8	19.6593	422,2720
1	9	18.934	383.1967
1	10	17.881	343.480
1	11	13.191	296.097
1	12	2,3015	13,71743

Conclusion

We achieve the minimum of the objective function in just 12 iterations which can be considered as good enough. In our subsequent paper we shall consider the application of this algorithm water quality control system where uncertainties are involved. Furthermore we have demonstrated in this paper that it is possible to derive a numerical algorithm for some class of control problems where uncertainties are involved.

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