Edge Cover Domination in Mangoldt Graph

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ABSTRACT

In their recent study of arithmetic graphs associated with certain arithmetic functions, the authors have introduced a new class of arithmetic graphs associated with Mangoldt function $\Lambda(n)$, $n \geq 1$, an integer and studied their basic properties and vertex cover. In this paper the edge cover, edge domination set, bondage number, non – bondage number and their parameters have been obtained for these graphs.

Key words: Vertex cover, Edge cover, Edge domination, Bondage number, Non-Bondage number.

[**AMS (MOS) Subject Classification:** 68R05]

1. INTRODUCTION

It was during 1850's, a study of dominating sets in graphs started purely as a problem in the game of chess. Chess enthusiasts in Europe considered the problem of determing the minimum number of queens that can be placed on a chess board so that all the squares are either attacked by a queen or occupied by a queen. A precise notion of a dominating set, that is present in the current literature can be said to be given by Berge (1962) and Ore (1962) Since then a number of graph theorists, Allan and Laskar (1978) Allan et al., (1984); Cockayne and Hedetniemi (1977); Haynes and Slater (1998); Kulli and Sigarkant (1992) etc., have studied various domination parameters of graphs. In a graph G the entire vertex set V trivially covers every edge of G in the sense that every edge of G is incident with some vertex of V. In practical problems one may be interested in a subset of V with minimum cordinality which covers every edge of G. In the same way a subset of the edge E of G with minimum cordinality which covers every vertex of G plays a crucial roll. These considerations lead to the concepts of vertex cover, vertex covering number, edge cover, edge covering number and other domination parameters of a graph, which constitute the domination theory of graphs. Domination theory of graphs has many applications in Engineering and Communication Networks. For these applications, the arithmetic graphs associated with certain number theoretic arithmetic functions like the Euler totient function $\Phi(n)$, the divison function $d(n)$, the quadratic residue function and the Mangoldt function $\Lambda(n)$, $n \geq 1$, an integer come handy. These arithmetic graphs have been studied in detail by Madhavi (2003).

Maheswari and Madhavi (2008) have studied the basic properties and vertex cover of the Mangoldt graph associated with Mangoldt function $\Lambda(n)$. This paper is devoted for the study of edge domination of a Mangoldt graph. In this study, we have followed Bondy and Murty (1979); and Harary (1969) for graph theory and Apostol (1989) for number theory terminology and notations not explained here.

2. THE MANGOLDT GRAPH AND ITS PROPERTIES

In this section we introduce a new class of arithmetic graph, namely, Mangoldt graph associated with the Mangoldt arithmetic function $\Lambda(n)$, $n \geq 1$ an integer (Apostol, 1989) and briefly outline its basic properties, whose proofs can be found in Maheswari and Madhavi (2008).

2.1. Definition

Let $n \geq 1$ be an integer. The Mangoldt function $\Lambda(n)$ is defined as follows:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The following is the table of the values of $\Lambda(n)$ for $n = 1, 2, \ldots, 10$.

n :	1	2	3	4	5	6	7	8	9	10
$\Lambda(n)$:	0	log2	log3	log2	log5	0	log7	log2	log3	0

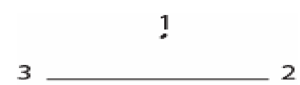
2.2. Definition

Let $n \geq 1$ be an integer. The Mangoldt graph M_n is defined as the graph whose vertices are the elements of the set $\{1, 2, \ldots, n\}$ and two distinct vertices x, y are adjacent (or (x, y) is an edge) if and only if $\Lambda(x \cdot y) = 0$ or $x \cdot y$ is not a power of a prime.

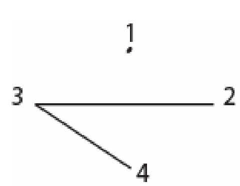
The Mangoldt graphs M_1, M_2, M_3, M_4, M_6, M_7, M_8, M_{10} and M_{11} are given below:

M_1

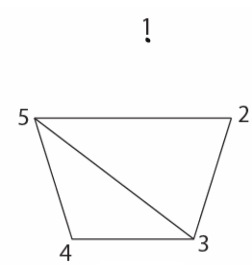
M_2



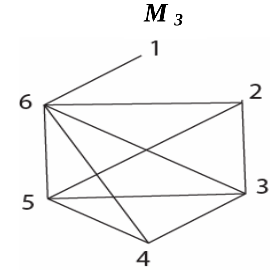
M_3



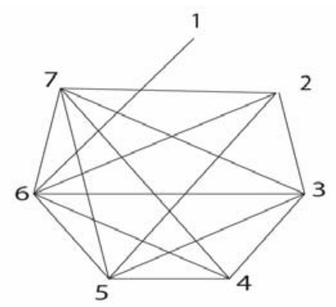
M_4



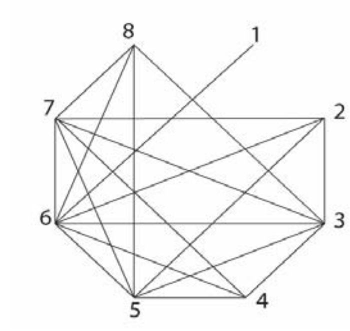
M_5



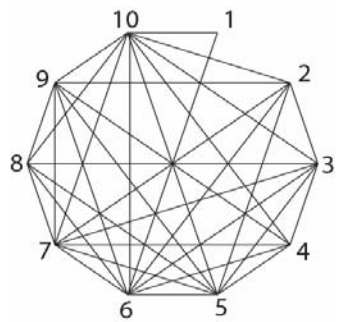
M_6



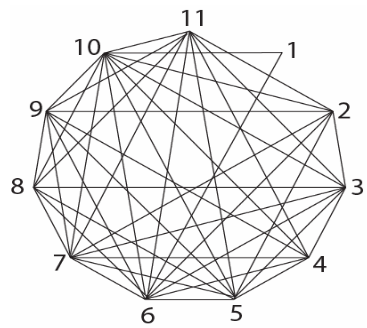
M $_7$



M $_8$



M_{10}



M_{11}

In M_{10} there is an edge between *2* and *7; 3* and *8; 4* and *10* since *2×7; 3×8* and *4×10* are not powers of a single prime and there is no edge between *1* and *2; 2* and *4; 2* and *8,* since *1×2; 2×4* and *2×8* are powers of the prime *2*. Similarly there is no edge between *1* and *3; 3* and *9* as their products are the powers of the prime *3*.

The following theorem gives a formula for the number of edges in M_n which is useful in finding the non – bondage number of M_n in section 4.

2.3. Theorem

Let $n \geq 2$ be an integer, $p_1, p_2, p_3, ... p_t$ prime numbers $\leq n$ and α_i the largest positive integer such that $p_i^{\alpha_i} \leq n$. The number of edges ε in the graph M_n is given by

$$\varepsilon = {}^nC_2 - [({}^{\alpha_1+1}C_2) + ({}^{\alpha_2+1}C_2) + ... + ({}^{\alpha_t+1}C_2)\,]$$

Proof: M_n has n vertices $1, 2, \ldots, n$. The total number of edges that can be formed with these vertices is ${}^nC_2$. If p_i is any prime $\leq n$ then there is no edge between any pair of vertices among 1, $p_i, p_i^2, ..., p_i^{\alpha_i}$ in M_n. There are $({}^{\alpha_i+1}C_2)$ such pairs and the total number of edges of M_n is ${}^nC_2 - [({}^{\alpha_1+1}C_2) + ({}^{\alpha_2+1}C_2) + ... + ({}^{\alpha_i+1}C_2)\,]$.

In Maheswari and Madhavi (2008) the authors have obtained the basic properties of the Mangoldt graph and studied its vertex domination parameters. For completeness they are stated below without proofs, which can be found in Maheswari and Madhavi (2008).

2.4. Theorem

For $n \leq 5$, M_5 is a disconnected graph.

2.5. Theorem

For $n \geq 6$, M_n is neither a bipartite graph nor a tree.

2.6. Theorem

For $n \geq 6$, the graph M_n is a connected graph.

2.7. Theorem

For $n \geq 10$, M_n is Hamiltonian.

2.8. Definition

A set S of vertices of a graph G is called a **vertex cover** of G if every edge of G is incident with some vertex in S. **A minimum vertex cover** is the one with minimum cardinality and the cardinality of a minimum vertex cover of a graph G is called the **covering number** of G and it is denoted by $\beta(G)$.

2.9. Theorem

Let $n > 2$ be an integer for every prime $p \leq n$, the set

$$\{1,2,\ldots n\} - \{1, p, p^2, \ldots p^\alpha \mid p^\alpha \leq n \text{ but } p^{\alpha+1} > n\}$$

is a vertex cover and the set

$$\{1,2,\ldots,n\} - \{1,2,2^2,\ldots,2^t \mid 2^t \leq n \text{ but } 2^{t+1} > n\}$$

is the minimum vertex cover of the Mangoldt graph M_n. Further the vertex covering number $\beta(M_n) = n - (t+1)$, where t is a positive integer such that $2^t \leq n$ but $2^{t+1} > n$.

In M_{10} the sets $\{1,2,3\ldots,10\} - \{1,2,2^2,2^3\} = \{3,5,6,7,9,10\}$; $\{1,2,3,\ldots,10\} - \{1,3,3^2\} = \{2,4,5,6,7,8,10\}$; $\{1,2,3,\ldots,10\} - \{1,5\} = \{2,3,4,6,7,8,9,10\}$ and $\{1,2,3,\ldots 10\} - \{1,7\} =$

$\{2,3,4,5,6,8,9,10\}$ are vertex covers . Among these the set $\{3,5,6,7,9,10\}$ is the minimum vertex cover of M_{10} and thus its covering number $\beta(M_{10}) = 6$. In the same way one can see that $\{1,2, . . . ,11\}–\{1,2,2^2,2^3\}=\{3,5,6,7,9,10,11\}$ is the minimum vertex cover of M_{11} and $\beta(M_{11}) = 7$.

2.10. Definition

A **dominating set** of graph G is a subset D of the vertex set V such that each vertex of $V–D$ is adjacent to at least one vertex of D. A **minimum dominating** set is one with minimum cardinality. The **domination number** $\gamma(G)$ is the cardinality of a minimum dominating set.

2.11. Theorem

If $n \geq 6$, then for each positive integer $r \leq n$ which is not a power of a single prime, the singleton set $\{r\}$ is a minimum dominating set of M_n and $\gamma (M_n) = 1$.

2.12. Example

In the graph M_{10} the sets $\{6\}$ and $\{10\}$ are the minimum dominating sets and $\gamma(M_{10}) =1$. Similarly $\{6\}$ and $\{10\}$ are minimum dominating sets of the graph M_{11} and $\gamma(M_{11}) =1$

3. EDGE DOMINATION IN MANGOLDT GRAPH

Just as a (minimum) vertex cover in a graph is the set of vertices in G which (minimum cardinality and) covers every edge of G, a (minimum) edge cover of a graph is the set of edges (with minimum cardinality and) covers all vertices of the graph.

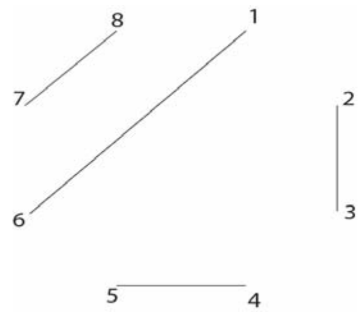
3.1. Definition

A set F of edges of a graph G is called an **edge cover** of G if every vertex of G is incident with some edge in F. A **minimum edge cover** is one with minimum cardinality. The number of edges in a minimum edge cover of G is called the **edge covering number** of G and it is denoted by $\beta^{|}(G)$.

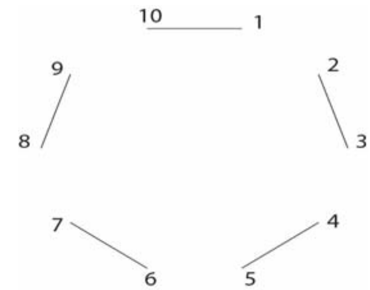
3.2. Example

The edge sets $\{(2,3),(4,5),(7,8),(6,1)\}$; $\{ (2,3), (4,5), (6,7), (8,9), (10,1)\}$ and $\{(2,3), (4,5), (6,7), (8,9), (10,11), (6,1)\}$ *or* $\{(2,3), (4,5), (6,7), (8,9), (10,11), (10,1)\}$ are respectively minimum edge

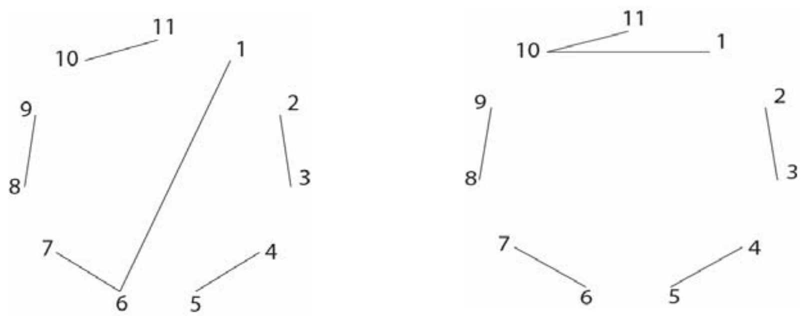
– covers of M_8, M_{10} and M_{11}. Observe that $\beta^{|}\ (M_8)=4$, $\beta^{|}(M_{10})=5$ and $\beta^{|}(M_{11})= 6$. Graphically they are represented below



Minimum edge – cover of M_8



Minimum edge – cover of M_{10}



Minimum edge – covers of M_{11}

3.3. Theorem

If $n \geq 6$ the minimum edge cover of the Mangoldt graph M_n is given by the set

$\{(2, 3), (4, 5),\ldots, (n–4, n–3), (n–1, n), (n–2, 1)\}$ if $n = 2^r$ and $r > 1$,

$\{(2, 3), (4 , 5),\ldots, (n–2, n–1), (n,1)\}$, if n is an even number and $n \neq 2^r$ and

$\{(2, 3), (4, 5),\ldots, (n–1, n), (r,1)\}$, if n is an odd number and $r \leq n$, a positive integer which is not a power of a single prime .

Proof : Suppose $n = 2^r$ where $r >1$ is an integer. Clearly there is an edge between u and $u + 1$ for $2\leq u \leq n–1$. Further $n–2 = 2^r –2 = 2 (2^{r-1} – 1)$ and $2^{r-1} –1$ is an odd number since $n >5$. So $n – 2$ is not a power of a single prime and there exists an edge between $n–2$ and 1 and $F=\{(2,3), (4,5), . . .,(n–4, n–3), (n–1, n), (n–2, 1)\}$forms an edge cover of M_n. Since the end vertices of the edges in F are distinct and n is even, $| F | = n/2$. So $n/2 \geq$ the cardinality of the minimum edge covering of M_n. Since n is even the number of distinct pairs of distinct vertices is

$n/2$ and thus the cardinality of the minimum edge cover of M_n is $\geq n/2$. These show that F is a minimum edge cover of M_n.

If n is an even number which is not a power of 2 then the set of edges $F = \{(2,3),(4,5), \ldots, (n-2, (n-1), (n,1)\}$ forms an edge covering of M_n. As above it can be seen that F is a minimum edge cover of M_n with cardinality $n/2$.

If n is an odd number then for any composite number $r \leq n$ which is not a power of a prime, the set of edges $F=\{(2,3), (4,5),\ldots, (n-1,n), (r,1)\}$ is an edge cover of M_n with cardinality $(½)(n-1)+1 = (n+1)/2$. On the other hand since n is odd, if n vertices of M_n are paired into $(n-1)/2$ distinct pairs of distinct vertices, one vertex is leftout. So the minimum number of edges one needs to cover all the vertices of M_n is $(½)(n-1)+1 = (n+1)/2 = |F|$ so that F is a minimum edge cover of M_n. ■

The following corollary is immediate.

3.4. Corollary

If $n \geq 6$, the edge covering number is $n/2$ if $n = 2^r$, $r > 1$ or n is even and $n \neq 2^r$ and $(n+1)/2$ if n is odd.

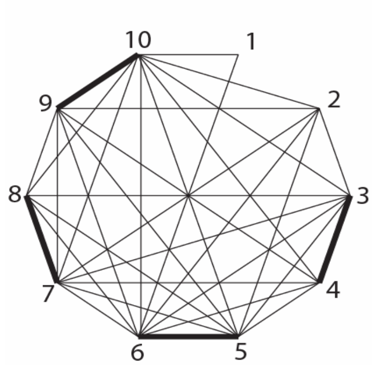
Another concept which plays a crucial role in the applications of graph theory is the **edge dominating set.** This is the set of edges of a graph G such that every edge of the graph not in this set is adjacent to atleast one edge of this set.

3.5. Definition

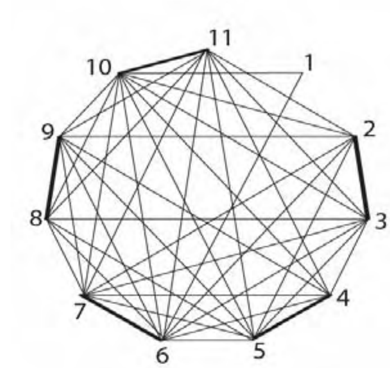
A subset F of the edge set E in a graph G is an **edge dominating set** of G if each edge of E not in F (that is in $E-F$) is adjacent to at least one edge in F. The minimum cardinality among all edge dominating sets of G is called an **edge domination number** of G and is denoted by $\gamma^{/}(G)$.

3.6. Example

The edge sets $\{(3,4), (5,6), (7,8), (9,10)\}$ and $\{(2,3), (4,5), (6,7), (8,9), (10,11)\}$ respectively represent the edge dominating sets of M_{10} and M_{11} and they are represented below by the thickened lines in the graphs of M_{10} and M_{11}. Edge Dominating sets of M_{10} and M_{11} are given below.



M_{10}



M_{11}

It is evident that $\gamma^{|}(M_{10}) = 4 = (10 - 2)/2$ and that $\gamma^{|}(M_{11}) = 5 = (11 - 1)/2$. Observe that *10* is even and *11* is odd. The following theorem establishes these facts for M_n for even and odd values of *n*.

3.7. Theorem

The edge dominating set of the Mangoldt graph M_n , $n > 5$, is the set of edges $\{(3,4), (5,6), \ldots, (n-1, n)\}$ if *n* is even and $\{(2, 3), (4,5), \ldots, (n-1,n)\}$ if *n* is odd .

Proof: *Case 1:* Let *n* be even. Consider the edge set $E_1=\{(3,4),(5,6),\ldots,(n-1,n)\}$.

Let $(r, s) \in E - E_1$. If $r = 1$ then *s* must be a product of powers of at least two distinct primes.

If *s* is even then $(s-1, s) \in E_1$ and it is adjacent to (r, s).

If *s* is odd then $(s, s+1) \in E_1$ and it is adjacent to (r, s).

If $r = 2$ then *s* must be either a prime $\neq 2$ or a product of distinct primes.

In the first case if $s > 2$ is a prime then the edge $(s, s+1)$ is in E_1 and it is adjacent to (r,s). In the second case if *s* is a product of distinct primes then (r,s) is adjacent to $(s, s+1)$ in E_1 when *s* is even or adjacent to $(s, s+1)$ in E_1 when *s* is odd. So let $r \geq 3$ and $s \neq r+1$. Suppose *r* is odd. Then $r = 2t+1$ for some positive integer *t* and the edge $(2t+1, 2t+2)$ in E_1 is adjacent with (r, s). Suppose *r* is even. Then $r = 2t$ for some integer $t \geq 1$ and the edge $(2t, 2t-1)$ in E_1 is adjacent with $(2t, s)$. In this case E_1 is an edge dominating set of M_n. Any edge in E_1 is not adjacent to any one of the remaining edges in E_1 since, if $(i,i+1), (j, j+1), i \neq j$ are any two edges in E_1 then $i \neq j, j+1$ and $j \neq i, i+1$. So, if we delete the edge $(i, i+1)$ from E_1 then the remaining edge set $E_1^{|} =$

$E_1 - \{i, i+1\}$ can not be an edge dominating set since the edge $\{i, i+1\} \in E - E_1^{|}$ and it is not adjacent to any of the edges in $E_1^{|}$. So, E_1 is the minimum edge dominating set.

Case 2: Let n be odd. Consider the edge set $E_2 = \{(2,3), (4,5),.., (n-1,n)\}$. Let $(r,s) \in E - E_2$. The case $r = 2$ can be dealt as in case 1. So let $r > 2$ and $s \neq r + 1$. If r is odd then $(r, s) = (2t+1, s)$ for some positive integer t and the edge $(2t+2, 2t+1)$ is in E_2 and this is adjacent with $(2t+1, s)$. If r is even then $(r, s) = (2t, s)$ for some positive integer t and the edge $(2t, 2t+1)$ is in E_2 and this is adjacent with the edge $(2t, s)$. Thus E_2 is an edge dominating set. As in case 1 one can see that E_2 is the minimum edge dominating set.

The following corollary is immediate.

3.8. Corollary

If $n \geq 6$, the edge domination number $\gamma^{|}(M_n)$ *is* $(n-2)/2$, if n is even and $(n-1)/2$, if n is odd.

3.9. Remarks

In Example 3.6 we have drawn the graphs of M_{10} and M_{11} and indentified their edge dominating sets and edge domination numbers. But by the Theorem 3.7, one gets the edge dominating set of M_{10} as $\{(3,4), (5,7), (7,8), (9,10)\}$, since 10 is even and that of M_{11} as $\{(2,3),(4,5),(6,7),(8,9),(10,11)\}$ since 11 is odd. Also by the Corollary 3.8, we get $\gamma^{|}(M_{10}) = 1(10-2)/2 = 4$ and $\gamma^{|}(M_{11}) = (11-1)/2 = 5$. For large values of n one can get these parameters with the help of these results without drawing the graphs.

4. THE BONDAGE NUMBER OF A MANGOLDT GRAPH

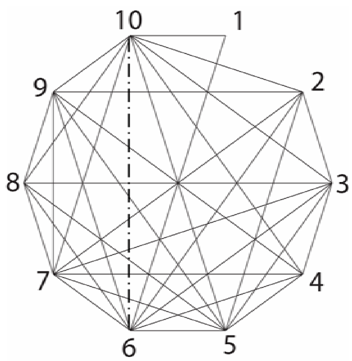
Deletion of a set F of edges from a graph G (retaining the end vertices) may result in increasing the vertex covering number of G. Among all these sets F, sets with minimum cardinality have importance and this leads to the concept of bondage number of G.

4.1. Definition

Let F be a set of edges of G such that $\gamma(G-F) > \gamma(G)$. Then the **bondage number *b (G)*** of G is defined to be the minimum number of edges in F.

4.2. Example

Consider the graph M_{10}. By the Theorem 2.8, $\gamma(M_{10}) = 1$. For the edge set $F = \{(6,10)\}$ the domination number $\gamma(M_{10} - F) = 2$, since the vertex set $\{(6,10\}$ is a minimum vertex cover of $M_{10} - F$. So the bondage number $b(M_{10}) = |F| = 1$. The broken line denotes the edge $(6,10)$ in F, that is deleted from M_{10}. Graph of $M_{10} - F$ is given below.



Graph of $M_{10} - F$

The number theoretic lemma given below gives a positive integer N associated with a given integer $n \geq 2$. The bondage number of M_n is given in terms of N.

4.3. Lemma

Let $n \geq 2$ be an integer. Let $p_1, p_2, ..., p_t$ be the primes $\leq n$ and let $\alpha_1, \alpha_2, \ldots \alpha_t$ be the largest positive integers such that $p_i^{\alpha_i} \leq n$ for $1 \leq i \leq t$. Then the number N of numbers $\leq n$ which are products of more than two distinct prime powers is given by

$$N = n - \{\ \alpha_1 + \alpha_2 + ... + \alpha_t + 1\ \}.$$

Proof : Let $n \geq 2$ be an integer and let $p_1, p_2, ..., p_t$ be primes $\leq n$ and for each i, let α_i be the largest positive integer such that $p_i^{\alpha_i} \leq n$ for $1 \leq i \leq t$. Among the numbers $1, 2, 3, \ldots\ldots, n$ the numbers which are powers of the prime i are $p_i, p_i^2, ...p_i^{\alpha_i}, 1 \leq i \leq t$ and the number of such numbers is α_i. So the total number of numbers (among $1, 2, 3, \ldots, n$), which are powers of a single prime $\leq n$ is $\alpha_1 + \alpha_2 + ... + ..\alpha_t$. Apart from these the number 1 is also not a power of two or more distinct prime powers. So the number of numbers less than or equal to n and which are products of two or more than two distinct prime powers is N.

4.4. Theorem

For $n \geq 6$ the bondage number $b(M_n)$ is $\lceil N/2 \rceil$ where $\lceil N/2 \rceil$ is the smallest integer $> N/2$ and N is the positive integer given in Lemma 4.3.

Proof: Let $\{m_1, m_2, m_3, \ldots, m_N\}$ be the set of all integers $\leq n$ which are products of two or more distinct prime powers and let

$$F = \begin{cases} \{(m_1, m_2), (m_3, m_4), \ldots, (m_{N-1}, m_N)\}, & \text{if } N \text{ is even,} \\ \{(m_1, m_2), (m_3, m_4), \ldots, (m_{N-2}, m_{N-1}), (m_N, m_i)\} \text{for some } i, 1 \leq i \leq N-1\}, & \text{if } N \text{ is odd.} \end{cases}$$

It is clear that $|F| = \lceil N/2 \rceil$. Consider the graph $M_n – F$. For any prime $p_j \leq n, 1 \leq j \leq t$ and positive integer α such that $p_j^{\alpha} \leq n$ and for any m_i, $1 \leq i \leq N$ consider the set $\{ m_i, p_j^{\alpha} \}$. The vertex set V of M_n can be written as

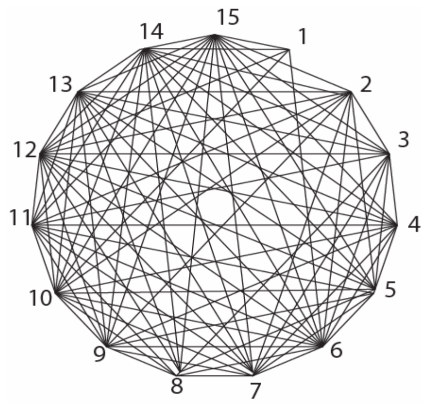
$$V = \{1\} \cup \{ m_1, m_2, \ldots, m_N \} \cup \{ p_1, p_1^2, \ldots, p_1^{\alpha_1}; p_2, p_2^2, \ldots, p_2^{\alpha_2}; \ldots; p_t, p_t^2, \ldots, p_t^{\alpha_t} \}.$$

Any vertex r of M_n is adjacent to p_j^{α} if $r = m_h$, $1<h\leq N$ and to m_i if $r = p_k^{\beta}$. So $\{m_i, p_i^{\alpha}\}$ is a dominating set of $M_n – F$. It is also a minimum dominating set. For, if we delete m_i from $\{m_i, p_i^{\alpha}\}$ then the singleton set $\{ p_j^{\alpha} \}$ is not a dominating set of $M_n–F$ since the vertex $p_j^{\gamma} \leq n$ where $\gamma \neq \alpha$ is not adjacent to p_j^{α} and if we delete p_j^{α} from the set S then the singleton set $\{ m_i \}$ is not a dominating set of $M_n– F$ since either m_{i-1} or m_{i+1} is not adjacent with m_i depending on the choice of i.

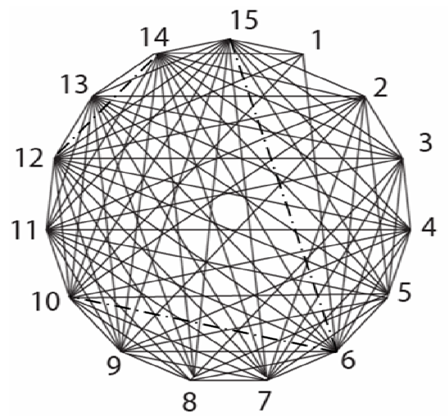
By the Theorem 2.11, $\gamma(M_n) = 1$ so that $\gamma(M_n–F) = 2 > 1 = \gamma(M_n)$. If $F_1 = F–\{(m_i, m_{i+1})\}$ then it is easy to see that either the singleton set $\{ m_i \}$ or $\{m_{i+1}\}$ is a dominating set of $(M_n–F_1)$ and thus $\gamma(M_n–F_1) = 1 = \gamma(M_n)$ or the domination number is not increased. Thus F is a minimum set of edges of M_n such that $\gamma(M_n–F) > \gamma(M_n)$ so that the bondage number $b(M_n) = |F| = \lceil N/2 \rceil$.

4.5. Example

Consider the graph M_{15} given below.



M_{15}



M_{15} - F

The primes ≤ 15 are $2,3,5,7$, 11 and 13. The largest powers α_i of these primes such that $p_i^{\alpha_i} \leq 15$ are respectively $3,2,1,1,1,1$ so that

$$N = 15 - (3+2+1+1+1+1+1) = 5 \text{ and } \lceil N/2 \rceil = \lceil 5/2 \rceil = 3.$$

Also $m_1 = 6$, $m_2=10$, $m_3 = 12$, $m_4 = 14$ and $m_5 = 15$ and $F = \{(m_1,m_2), (m_3,m_4), (m_5,m_1)\} = \{(6,10),(12,14),(15,6)\}$. These edges are marked with dotted lines in the graph of $M_{15} - F$, which is got by deleting these edges from M_{15}.

Cleary each of the singleton sets $\{6\}$ or $\{10\}$ or $\{12\}$ or $\{14\}$ or $\{15\}$ is a minimum vertex cover of M_{15} so that $\gamma(M_{15}) = 1$.

The singleton set $\{6\}$ is not a vertex cover of $M_{15} - F$, since 10 and 15 are not adjacent to 6 in this graph. In the same way each of the other singleton sets are not vertex covers of M_{15}–F. However each of the vertex sets $\{6,2\}$, $\{6,4\}$, $\{6,8\}$, $\{6,9\}$, $\{6,5\}$, $\{6,7\}$, $\{6,11\}$ and $\{6,13\}$ is a minimum vertex cover of $M_{15} - F$ and the cardinality of each of these sets is 2 so that $\gamma(M_{15} - F) = 2 > 1 = \gamma(M_{15})$. So the bondage number $b(M_{15}) = |F|=3$. Observe that $\lceil N/2 \rceil = 3$.

4.6. Remark

One may take $F = \{(m_1,m_2), (m_3,m_4), (m_5,m_2) = \{(6,10), (12,14), (15,10)$ or $F = \{(m_1,m_2), (m_3,m_4), (m_5,m_3) = \{(6,10), (12,14), (15,12)$ or $F = \{(m_1,m_2), (m_3,m_4), (m_5,m_4) = \{(6,10), (12,14), (15,14)\}$ and still get the same bondage number of $M_{15} - F$.

4.7. Remark

Theorem 4.4 gives the formula for bondage number of M_n, namely, $b(M_n) = \lceil N/2 \rceil$, where N is the positive integer associated with n given in Lemma 4.3. One need not draw the graph and identify the minimum set of edges F such $\gamma(M_n) < \gamma(M_n - F)$. This formula comes handy for large values of n for which sketching of the graph M_n is unvieldy. Once the value of n is given one can calculate N and $\lceil N/2 \rceil$ gives $b(M_n)$.

4.8. Definition

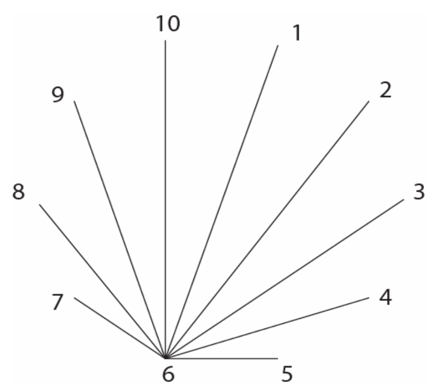
The **non - bondage number $b'(G)$** of a graph G is the maximum cardinality among all sets of edges $F \subseteq E$ such that $\gamma(G - F) = \gamma(G)$.

4.9. Example

For the Mangoldt graph M_{10} the set $F = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,7), (6,8), (6,9), (6,10)\}$ is a minimum set of edges such $\gamma(G) = 1 = \gamma(M_{10})$, where G is the graph (given below) whose vertex set is $\{1,2, . . . ,10\}$and the edge set F, since the singleton vertex set $\{6\}$ is the vertex cover of G. So $F^| = E - F$ where E is the edge set of M_{10}, is the set with minimum cardinality such that $\gamma(M_{10} - F^|) = \gamma(M_{10})$. Hence the non – bondage number $b^|(M_{10})$ of M_{10} is $|F^|| = | E - F| = |E| - |F|$. But by the Theorem 2.3

$$|E| = {}^{10}C_2 - \left({}^4C_2 + {}^3C_2 + {}^2C_2 + {}^2C_2\right) = 34 \text{ and } |F| = 9.$$

So $b^|(M_{10}) = 34 - 9 = 25$. The edge set $\{(10,1), (10,2), . . . ,(10,9)\}$ in place of F will also yield the same result.



G

In the following theorem we shall denote the graph G with vertex set V and edge set E by $G(V,E)$.

4.10. Theorem

For $n \geq 6$, the non - bondage number $b^{|}(M_n)$ of the Mangoldt graph M_n is $\varepsilon - (n-1)$ where ε is the number of edges of M_n.

Proof: Let V and E respectively denote the vertex set and edge set of M_n so that $M_n = G\ (V,E)$. Suppose E_1 is the maximum set of edges in G such that $\gamma\,(G\ (V,\ E-E_1)) = \gamma(G)$. Then $b^{|}\ (M_n) = b^{|}\ (G) = |\ E_1|$. Further $E_2 = E - E_1$ is the minimum set of edges in G such that

$$\gamma(G\ (V,\ E_2)) = \gamma\,(G).$$

For any composite number r which is a product of more than two distinct prime powers, the set $E_2 = \{(r,s)|1 \leq s \leq n,\ s \neq r\}$ of edges is such that $\gamma\,(G\ (V,\ E_2)) = \gamma(G)$. Since $|\ E_2\ |= n-1$ we have $|\ E_1| = |\ E\ | - |\ E_2\ | = \varepsilon - (n-1)$.

The following example gives the importance of the theorem 4.10 in determining the non – bondage number of M_n for any integer $n \geq 6$.

4.11. Example

Consider the graph M_{15} given in the Example 4.5. By the Theorem 2.3 the number of edges in M_{15} is given by $\boldsymbol{\varepsilon} = {}^{15}C_2 - \left({}^{4}C_2 + {}^{3}C_2 + {}^{2}C_2 + {}^{2}C_2 + {}^{2}C_2 + {}^{2}C_2\right) = 92$. So by the Theorem 4.10 the non – bondage number $b^{|}(M_{15})$ is given by

$$b^{|}\ (M_{15}) = \boldsymbol{\varepsilon} - (15 - 1) = 92 - 9 = 83.$$

5. CONCLUSION

In Maheswari and Madhavi (2008) it is shown that the Mangoldt graph M_n is Hamiltonian for $n \geq 10$. Maheswari and Madhavi, 2009 have obtained formulae for the number of triangles, the cycles of smallest length and the Hamilton cycles, the cycles of longest length in Cayley graphs associated with the set of Quadratic residues modulo a prime p. The formula for the number of triangles in a Mangoldt graph M_n. (Maheswari and Madhavi, 2008) is also obtained. So, it is interesting to find formula for Hamilton cycles in M_n. Further one may carry out studies on these arithmetic graphs about other domination parameters li ke independent domination number, total domination number and others.

6. ACKNOWLEDGEMENTS

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