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Abstract

This paper presents another definition of substitutes and complements. It follows a dual approach using the Luenberger's benefit function. The benefit function measures the amount of a reference bundle that an individual would be willing to give up to move from a given utility level to any bundle. Therefore the benefit function associates to any bundle of goods another bundle that lies on a given indifference curve. This enables one to derive an inverse demand function which is defined as the support price of this associated bundle. The classification of goods between complements and substitutes is then obtained by the comparative static properties of the support price. We present some examples which show that the proposed classification is different from the one obtained with another dual approach based on Deaton's distance function.

J.E.L Classification Numbers: D01, D11.

Key Words: Benefit Function, Complement, Substitute.

1 Introduction

As Newman (1994) put it:

> Economists have found it surprisingly hard to nail down the obvious intuition that in some rough unspecified way tea and coffee are substitutes, and bacon and eggs complements. Not that they can't do it. Quite the opposite, they have all too many ways of doing it, so much so that even if their less attractive inventions are discarded still an abundance is left, each with its own usefulness and charm.

There are two ways to determine complementarity and substitutability relationships based on the consumer problem. First, there is a primal approach which is the most popular one and relies on hicksian demand functions. Second, there is a dual approach which uses an "inverse" demand function (see Deaton (1979) and Newman (1994)). It relies upon Deaton's (1979) distance function (a version of the gauge function). By definition, the distance function associates to any bundle of goods another bundle that lies on a given indifference curve. The inverse demand function is defined as the support price of the associated bundle[1]. The classification of goods between complements and substitutes is then obtained by the comparative static properties of the support price (which measures how this price changes with quantities).

This paper proposes yet another definition of substitutes and complements. Our approach to classify goods will also rely on an inverse demand function, but instead of using the distance function, we will use the benefit function introduced by Luenberger (1992 a, b) (see also Allais (1943), Blackorby and Donaldson (1978), Chambers et al. (1995), Luenberger (1992) (a-b), (1995), (1996)).

The benefit function approach to demand analysis has been developped considerably in the recent years. It provides the basis for new econometric estimation of direct and inverse demand functions (especially because it allows for a richer set of parametrizations used in the dual approach). On this see Baggio and Chavas (forthcoming), Färe et al. (2008), and McLaren and Wong (2008).

The benefit function is based on a reference bundle g and allows a well-suited cardinal comparison of different bundles of goods. Let a bundle x and a reference utility level α be given. The benefit function $b(x, \alpha)$ measures

[1]We say that a price vector supports a commodity bundle x whenever every preferred bundle to x costs no less than x.

how many units of g an individual would be willing to give up to move from a utility level α to the bundle x.

The benefit function associates to any bundle of goods a bundle lying on a given indifference curve. That bundle of goods is obtained by a translation of the original bundle in the direction of the reference bundle g. Again, an inverse demand function may be defined as the support price of this associated bundle. The classification of goods between complements and substitutes is also obtained by the comparative static properties of the support price[2]. Since the benefit function has been introduced by Luenberger, it is natural to call our notion of substitutes and complements, Luenberger complementarity and substitutability. We show that the classification obtained is different from the one obtained with the distance function.

The plan of the paper is as follows. In the next section, we present what we call the Luenbergers's notion of substitute and complement goods. We study the link between the supergradients of the benefit function and the prices supporting the projection (i.e. $x - bg$). We also study a property of Luenberger's complements and substitutes assuming differentiability of the benefit function. We recall the Deaton's notion in section 3 and we compare it to Luenberger's. We also study the link between the distance function and the supporting prices of the projection. We also present an example showing that the Deaton and Luenberger do not always coincide. Section 4 offers some brief concluding remarks.

2 Luenberger's complements and subtitutes

Let us recall the formal definition of the benefit function. Let $X \subset \mathbb{R}^l$, be a consumption set and $U : X \to \mathbb{R}$ be a utility function. Let g be in $X \setminus \{0\}$. The benefit function $b : \mathbb{R}^l \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is defined as follows:

$$b(x, \alpha) \equiv \sup\{\lambda \mid U(x - \lambda g) \geq \alpha, x - \lambda g \in X\} \tag{1}$$

The benefit function may take the value $-\infty$ when there is no λ in $\mathbb{R}$ such that $x - \lambda g \in X$ and $U(x - \lambda g) \geq \alpha$.

In this paper we shall always assume that $g \geq 0$ and that the consumption set X is bounded from below (i.e. there exists $y \in \mathbb{R}^l$, such that $\forall x \in X, x \geq y$) .

[2]Färe et al. (2005) and (2007) follows a similar approach in the context of production theory. Here, we only consider consumer theory and we study the link bewteen the benefit function and the support prices (especially when the former is not differentiable). Moreover, we compare the Luenberger complements and substitutes with Deaton's.

This ensures that when the set $X^b(x,\alpha) \equiv \{\lambda : U(x-\lambda g) \geq \alpha, x-\lambda g \in X\}$ is not empty and closed then $b(x,\alpha) = \max\{\lambda : U(x-\lambda g) \geq \alpha, x-\lambda g \in X\}$ exists in $\mathbb{R}$.

Let $x \in X$. We say that $p \in \mathbb{R}^l_+$ supports x if it satisfies the following property: for all $x' \in X$, $U(x') \geq U(x) \Rightarrow p.x' \geq p.x$.

When $U(.)$ is quasi-concave and $b(x,\alpha)$ is finite, $x - b(x,\alpha)g$ has always a non-empty set of support prices (this is a consequence of the separating hyperplan Theorem).

From now on, we assume that X has a non-empty interior and that x is in the interior of X. We also assume that the support prices of $x - b(x,\alpha)g$ are such that $p.g. \neq 0$ and belong to a half-line. We therefore set $P(x,\alpha) = p/p.g$ and we assume that it is differentiable with respect to x (see figure 1).

We introduce next the notion of Luenberger's complement and substitute goods.

Definition 1. *Good j is said to be*

- *Luenberger-complement with good k at x if* $\frac{\partial P_j(x,\alpha)}{\partial x_k} > 0$,

- *Luenberger-substitute with good k at x if* $\frac{\partial P_j(x,\alpha)}{\partial x_k} < 0$,

- *Luenberger-independent with good k at x if* $\frac{\partial P_j(x,\alpha)}{\partial x_k} = 0$.

We shall remove the restriction at x whenever one of the sign conditions above is satisfied for all x at which $P(x,\alpha)$ is differentiable.

From the preceding definition and the assumptions made above, it follows that investigating the Luenberger' substitute or complementarity property of a given bundle of goods amounts to see how the support price of $x - b(x,\alpha)g$ changes with x.

To motivate this definition, consider the following argument. Assume that $\frac{\partial P_j(x,\alpha)}{\partial x_k} > 0$. This means that whenever a consumer gets more of good k, the price that he is willing to accept in order to have one more unit of good j increases. In this sense, the two goods can be seen as being complementary.

Our approach developped for the study of complements and substitutes in consumption theory, is parallel to that of Färe et al. (2005), (2007) in production theory. More exactly, these authors introduce a Morishima elasticity of transformation between desirable and (or between) undesirable outputs.

To do this they study the percent change in the shadow price ratio between two ouputs due to a percent change in the output ratio. The shadow price ratio is expressed in terms of the partial derivatives of the directional distance function (a version of the benefit function for analyzing production, introduced by Chambers et al. (1995)).

According to Katzner (1970) pages 146-147, there are five properties that a good definition of complementarity and substitutability between commodities might possess. 1) Intuition. 2) Symmetry. 3) Dimensionality (i.e. the relationship between commodities should not depend on the dimension of the commodity space). 4) Universality (any pair of commodities should be capable of being declared complements, substitutes, or independent under the proposed definition). 5) Observability ("It should be possible to determine the relationship in which any individual holds any pair of goods by observing his market activity"). As was observed by Katzner, few definitions satisfy simultaneously all five properties.

With respect to Katzner properties, the definition above of complements and substitues does not depend upon the dimensionality of the commodity space (more precisely, our notion does not depend on the choice of a particular commodity space). By construction, it satisfies the requirement of universality: it is always possible to say what is the relationship between two goods (at least locally)[3].

The remaining part of this section seeks to present some links between $P(x, \alpha)$ and the supergradient of $b(., \alpha)$ at x[4].

We let $\partial_x b(x, \alpha)$ denote the supergradient of the benefit function at x when $b(x, \alpha)$ is finite[5]: that is $p \in \partial_x b(x, \alpha) \Rightarrow b(x', \alpha) - b(x, \alpha) \leq p.(x' - x)$ for all $x' \in X$.

Theorem 1. *Assume that X is closed, $U(.)$ is continuous and $x - b(x, \alpha)g$ and x are both interior points of X. Let p be such that $p.g > 0$.*
Then $p' = p/p.g$ is in $\partial_x b(x, \alpha)$ if and only if p supports $x - b(x, \alpha)g$.

Proof. Let $p' = p/p.g$ be in $\partial_x b(x, \alpha)$. As $x - b(x, \alpha g)$ is an interior point and $U(.)$ is continuous, we have $U(x - b(x, \alpha)g)) \geq \alpha$. Now if p does not support

[3] The other properties will be studied at the end of the section.

[4] Färe et al. (2005) and (2006) assume directly that the directional distance function is differentiable. Here we address the issue of the non-smoothness of the benefit function.

[5] See Boyd and Vandengerghe (2008) for a definition of the subgradient for a non convex function.

$x - b(x,\alpha)g$, there exists a bundle z such that: $U(z) \geq U\big(x - b(x,\alpha)g\big) \geq \alpha$ and $pz < p(x - b(x,\alpha)g)$. So

$$(p/p.g).(z - x) < (p/p.g)(-b(x,\alpha)g) \tag{2}$$

But as $b(z,\alpha) \geq 0$ (by definition of the benefit function), this implies that:

$$(p/p.g).(z - x) < (b(z,\alpha) - b(x,\alpha) \tag{3}$$

which contradicts the fact that $p/p.g$ is in $\partial_x b(x,\alpha)$.
Now let p support $x - b(x,\alpha)g$. If $p' = p/p.g \notin \partial_x b(x,\alpha)$, there exists y in X, such that: $b(y,\alpha) - p'.y > b(x,\alpha) - p'.x$. This proves that: $p.\big(x - b(x,\alpha)g\big) > p.\big(y - b(y,\alpha)g\big)$ and $b(y,\alpha) > -\infty$. But then $U\big(y - b(y,\alpha)g\big) < U\big(x - b(x,\alpha)g\big)$. However, since X is closed and $U(.)$ is continuous, $U\big(y - b(y,\alpha)g\big) \geq \alpha$. Moreover since $x - b(x,\alpha)g$ is an interior point of X, $U(x - b(x,\alpha)g) = \alpha$. Thus,

$$\alpha \leq U(y - b(y,\alpha)g) < U(x - b(x,\alpha)g) = \alpha \tag{4}$$

which is a contradiction. □

To enlight the definition of Luenberger's complements and substitutes assume also for the remaining part of this section that $b(.,\alpha)$ is a twice continuously differentiable concave function at x[6]. From Theorem 1, we have $P(x,\alpha) \equiv \nabla_x b(x,\alpha)$. Thus, investigating the substitution or complement properties of goods boils down to study the sign of the second partial derivatives of $b(.,\alpha)$ at x. This is an approach followed in Baggio and Chavas (forthcoming).

Note that in this case the Luenberger's definition of complements and substitutes is perfectly symmetrical (that is, if a good j is Luenberger-complement with good k at x, the converse is also true). This follows at once from Schwarz's Theorem.

So, the Luenberger's definition satisfies the four first requirements proposed by Katzner (i.e. intuition, symmetry, dimensionality and universality). However it should be noted that the definition depends on the particular reference bundle g chosen.

It would be hard to assert that the Luenberger's definition of complement and subsitutes satisfies the fifth requirement, namely that of observability (the possibility to infer the relationship between commodities for an agent by only

[6]See Courtault et al. (2004 a,b) for a study of the differentiability of the benefit function.

observing his market activity). But this is also true for other definitions, and in particular for that of Deaton (see the next section).

In a two-dimension framework, we have:

Proposition 1. *Assume that* $X \subset \mathbb{R}^2$, $g \gg 0$, $U(.)$ *is quasi-concave and differentiable,* $b(., \alpha)$ *is twice continuously differentiable at* x *and that* x *and* $x - b(x, \alpha)g$ *are interior points of* X. *Then, both goods are Luenberger-complement (or indifferent) at* x.

Proof. Since $U(.)$ is quasi-concave, $b(., \alpha)$ is concave (see, e.g. Luenberger (1995)). As a result, the differentiability assumption ensures that $\nabla^2_{xx} b(x, \alpha)$ is negative semi-definite. Since $\nabla_x P(x, u) = \nabla^2_{xx} b(x, u)$ it follows that $\nabla_x P(x, u)$ is also negative semi-definite. Moreover, since $x - b(x, \alpha)g$ is an interior point of X, $U(x - b(x, \alpha)g) = \alpha$ and it follows that:

$$P(x, \alpha).g = \nabla_x b(x, \alpha).g = 1 \tag{5}$$

so that:

$$\nabla_x P(x, \alpha)^T.g = 0 \tag{6}$$

In a two-dimension framework, the above equation implies:

$$\frac{\partial P_1}{\partial x_1} g_1 + \frac{\partial P_2}{\partial x_1} g_2 = 0 \Rightarrow \frac{\partial P_2}{\partial x_1} = -\frac{g_1}{g_2} \frac{\partial P_1}{\partial x_1} \tag{7}$$

$$\frac{\partial P_1}{\partial x_2} g_1 + \frac{\partial P_2}{\partial x_2} g_2 = 0 \Rightarrow \frac{\partial P_1}{\partial x_2} = -\frac{g_2}{g_1} \frac{\partial P_2}{\partial x_2} \tag{8}$$

Since $\nabla_x P(x, u)$ is negative semi-definite, this implies that $\frac{\partial P_2}{\partial x_1}$ and $\frac{\partial P_1}{\partial x_2}$ are non-negative and therefore the two goods are Luenberger-complement (or indifferent). □

Interestingly, when X is in $\mathbb{R}^n$ (for all n) and under the same other assumptions of the above Proposition, if $g = (1, 0, ..., 0)$ any good is Luenberger independent at x with the reference good. Indeed if $g = (1, 0, ..., 0)$, $P_1(x, \alpha) = 1$ for all x and α and so forall k, $\frac{\partial P_1(x,\alpha)}{\partial x_k} = 0$.

3 Deaton *vs* Luenberger's Complementarity and Substitutability

Let us compare the proposed definition of complements and substitutes with the alternative one given by Deaton[7]. Deaton uses a version of the gauge function, i.e. the distance function, as the main tool (see Deaton (1979)).

The distance function $d : \mathbb{R}^l_+ \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$, is defined by $d(x, \alpha) = \sup\{\lambda > 0 \mid U(\frac{x}{\lambda}) \geq \alpha\}$. Notice that $d(., \alpha)$ may take the value $+\infty$: if $U(x_1, x_2) = x_1 x_2$, $\alpha = 0$, whenever $z \gg 0$, $d(z, \alpha) = +\infty$.

As for the notion of Luenberger complements and substitutes, we can study the substitution property by looking at the way the support price of $x/d(x, \alpha)$ changes with x.

Assume that the set of support prices of $x/d(x, \alpha)$ is a half-line. Define $p^d(x, \alpha) = p/p.(x/d(x, \alpha))$ where p is a support price of $x/d(x, \alpha)$. The vector $p^d(x, \alpha)$ can be interpreted as an inverse (hicksian or compensated) vector demand function.

The two notions of support prices $P(x, \alpha)$ and $p^d(x, \alpha)$ are depicted in figure 1 for the case $l = 2$.

Following Newman (1994), (page 547, paragraph Hicks-Deaton), if i and j are complement in the rough every day sense, one would expect that as one has more j one would be willing to pay more for a marginal unit of i, while if they are substitutes one would be willing to pay less. Hence, assuming that $p^d(x, \alpha)$ is differentiable at x, a good i is a substitute for j if $\frac{\partial p_i^d(x,\alpha)}{\partial x_j} < 0$, and a complement is $\frac{\partial p_i^d(x,\alpha)}{\partial x_j} > 0$ (the goods are independent if the partial derivative is zero). This remarks justifies the Deaton's definition of complements and substitutes.

In the remaining part of this section, we will first present some relations between the support prices of $x/d(x, \alpha)$ and the supergradient of $d(x, \alpha)$; second, we will briefly compare the two notions of complements and substitutes.

[7] Another well known definition of substitutes and complements uses the Hicksian demands. These are defined as being the quantities that minimize expenditures under the constraint that utility must be as great as a reference level α. Clearly, $e(p, \alpha) = p.H(p, \alpha)$, where $H(.)$ denotes the Hicksian demand functions. A good i is a (Hicks-Allen) substitute (resp. complement of) for good j if $\frac{\partial H_i(x,\alpha)}{\partial p_j} > 0$ (resp. $\frac{\partial H_i(x,\alpha)}{\partial p_j} < 0$) (there are independent if the partial derivative is zero). Newman (1994) notes that two goods may be Hicks-Allen substitute but not substitute in the Deaton'sense.

Theorem 2. *Assume that X is closed cone and $U(.)$ is continuous, x and $x/d(x,\alpha)$ are interior points of X, $p.x > 0$ and $U(\frac{x}{d(x,\alpha)}) = \alpha$. Then $\frac{p}{p.\frac{x}{d(x,\alpha)}} \in \partial_x d(x,\alpha)$ if and only if p supports $x/d(x,\alpha)$.*

Proof. If p does not support $x/d(x,\alpha)$., there exists $z \in X$ such that:

$$U(z) \geq U(\frac{x}{d(x,\alpha)}) \tag{9}$$

and $pz < p\frac{x}{d(x,\alpha)}$. Since $U(x/d(x,\alpha)) = \alpha$ we have $U(z) \geq \alpha$ so $d(z,\alpha) \geq 1$. Since X is a cone and $\frac{p}{p.\frac{x}{d(x,\alpha)}} \in \partial_x d(x,\alpha)$, one has for all positive θ:

$$d(\theta z,\alpha) - d(x,\alpha) \leq \frac{p}{p.\frac{x}{d(x,\alpha)}}.(\theta z - x) \tag{10}$$

Let $\theta = d(x,\alpha)$. We obtain:

$$d(z,\alpha) - 1 \leq \frac{p}{p.\frac{x}{d(x,\alpha)}}.(z - \frac{x}{d(x,\alpha)}) \tag{11}$$

Since $pz < p\frac{x}{d(x,\alpha)}$, one gets:

$$d(z,\alpha) - 1 < 0 \tag{12}$$

which is a contradiction.
Now If $\frac{p}{p.\frac{x}{d(x,\alpha)}} \notin \partial_x d(x,\alpha)$, there exists z in X such that:

$$d(z,\alpha) - d(x,\alpha) > \frac{p}{p.\frac{x}{d(x,\alpha)}}.(z - x) \tag{13}$$

By assumption and definition of $d(.,\alpha)$, we have:

$$U(\frac{z}{d(z,\alpha)}) \geq \alpha = U(\frac{x}{d(x,\alpha)}) \tag{14}$$

By the support property, if follows that:

$$\frac{p}{p.\frac{x}{d(x,\alpha)}}.\frac{z}{d(z,\alpha)} \geq \frac{p}{p.\frac{x}{d(x,\alpha)}}\frac{x}{d(x,\alpha)} = 1 \tag{15}$$

So:

$$\frac{p.z}{p.\frac{x}{d(x,\alpha)}} \geq d(z,\alpha) \tag{16}$$

$$\Rightarrow \frac{p.z}{p.\frac{x}{d(x,\alpha)}} - \frac{p.x}{p.\frac{x}{d(x,\alpha)}} \geq d(z,\alpha) - d(x,\alpha) \tag{17}$$

$$\Rightarrow \frac{p}{p.\frac{x}{d(x,\alpha)}}.(z-x) \geq d(z,\alpha) - d(x,\alpha) \tag{18}$$

which is a contradiction. □

When $d(.,\alpha)$ is concave and differentiable at x, the gradient of d, $\nabla_x d(x,\alpha)$ is a support price of $x/d(x,\alpha)$. Moreover, the set of support prices is a half-line. Moreover, when $d(.,\alpha)$ is twice continuously differentiable at x, we can inspect the complement and substitution properties of goods by studying the sign of the matrix $\nabla_{xx} d(x,\alpha)$.

While close to that of Deaton, our definition of Luenberger complements and substitutes is a little bit more intuitive since the benefit function has a natural economic interpretation which is not the case for the gauge function[8].

The next example illustrates the fact that Deaton and Luenberger's definitions of complements and substitutes do not necessarily coincide.

Example. We already know that when g has only one non-zero coordinate, all goods are Luenberger-independent with the good corresponding to the non-zero coordinate. For instance, assume that $U : \mathbb{R}_+^{*2} \to \mathbb{R}$, $U(x) = -1/(x_1) - 1/(x_2)$ and $g = (1,0)$. Then, $d(x,\alpha) = -\frac{\alpha}{\frac{1}{x_1}+\frac{1}{x_2}}$ and it is easy to show that the two goods are Deaton complement. So the two notions do not always coincide.

4 Conclusion

In this paper, we have presented a new definition of complements and substitutes based upon the benefit function. We have compared this notion with an alternative one, namely that obtained with the Deaton distance function. We have provided a simple example where our notion differs sharply from that of Deaton.

[8]For a more complete comparison between the benefit function and the gauge function, see Luenberger (1992b), page 480.

While we add yet another definition to a rather large set of existing ones (see Newman's quotation in the introduction of this paper), we think that our definition has a more intuitive flavor than that proposed by Deaton. Indeed, the benefit function on which it is based has a greater economic interpretation than the distance function.
A natural extension of the present paper would be to study a relation similar to the Slutsky decomposition of a price effect between a Hicks-substitution effect and income effect.

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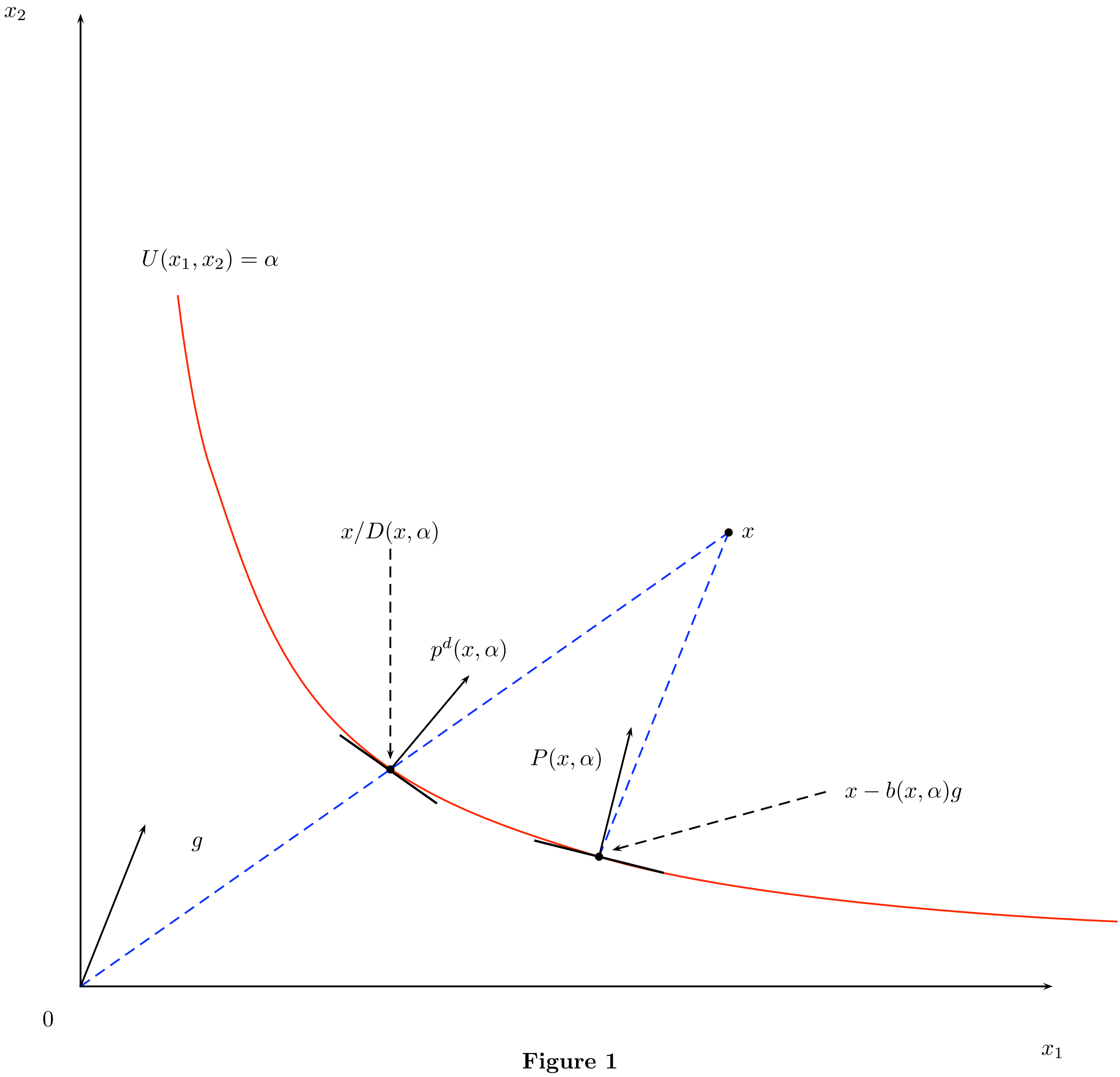

Figure 1