SUBORBITAL GRAPHS OF THE SYMMETRIC GROUP S_n ACTING ON UNORDERED *r*-ELEMENT SUBSETS

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Abstract
In this paper we construct the suborbital graphs of the symmetric group S_n acting on unordered *r*-element subsets of $X = \{1, 2, 3, ..., n\}$, $X^{(r)}$ $(r, n \in \mathbb{N})$ and analyse their properties. It is shown that the suborbital graphs are undirected, connected if $r < \frac{1}{2}n$, and have girth three if $n \geq 3r$.

Key words: Symmetric group, *r*-element subsets, suborbital graphs

1.0 Introduction

In 1967, Sims [11] introduced the idea of suborbital graphs of a permutation group *G* acting on a set *X*; these are graphs (possibly directed) with vertex-set *X*, on which *G* induces automorphisms.

Many interesting graphs like the Petersen graph, the Coxeter graph and the Biggs-Smith graph can be realized as suborbital graphs of some group acting on a given set (Neumann 1977; Kamuti 1992; Bon and Cohen 1989). Suborbital graphs of subgroups of the modular group PSL(2, $\mathbb{Z}$) acting on the rational projective line $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ have been studied by several authors (Besenk *et al.,* 2010; Keskin and Demirtürk 2009; Guler *et al.,* 2008; Keskin 2004; Akbas and Baskan 1996). In this paper we construct and investigate properties of the symmetric group S_n acting on the set $X^{(r)}$ of *r*-element subsets from the set *X* = {1, 2, ..., *n*}.

2.0 Definitions and Preliminary Results

A (simple) graph is an ordered pair *H* = (*V*, *E*), where *V* is a finite, non-empty set of objects called vertices, and *E* is a (possibly empty) set of 2-subsets of *V* called edges. The set *V* is called the vertex set of *H*, and *E* is called the edge set of *H*. If *e* = {*u*, *v*}∈*E*(*H*), we say that vertices *u* and *v* are adjacent in *H*, and that *e* joins or connects *u* and *v*. The edge *e* is said to be incident with *u* (and *v*), and vice versa. The following important facts arise from carefully considering what the definition of a graph says.

- *E* is a set. Therefore two vertices are either adjacent, or not adjacent, period.
  There can be at most one edge joining any two vertices.
- The elements of *E* are subsets of *V* of size 2. Therefore no vertex can be adjacent to itself. Edges join pairs of distinct vertices.

There is no requirement that the edge set be non-empty. Therefore the minimum number of edges a graph can have is zero. If the graph has *n* vertices, then the maximum number of edges it can have equals the number of two element subsets of *V* , which is $\binom{n}{2}$. A graph with *n* vertices has $\binom{n}{2}$ edges if every pair of distinct vertices is an edge. Such a graph is called a complete graph on *n* vertices. We represent graphs by pictures in the plane by associating a point with each vertex and joining points corresponding to adjacent vertices by a (possibly curved) line segment. How the vertices and edges are drawn is unimportant, the same graph can have many pictures. What is important is what the vertices are (i.e., *V*), and which pairs of vertices are adjacent (i.e., *E*).

Two graphs are equal if they have the same vertex set and the same edge set. But there are other ways in which two graphs could be regarded the same. For example, one could regard two graphs as being the same if it is possible to rename

the vertices of one and obtain the other. If this happens we call the graphs isomorphic. (Formally, two graphs J and H are isomorphic if there is a 1-1 correspondence $f : V(J) \to V(H)$ such that
$\{x, y\} \in E(J) \Leftrightarrow \{f(x), f(y)\} \in E(\mathrm{H})$.)

The relation $\Re$ on the set of all graphs defined by $J \Re H$ if and only if J and H are isomorphic (i.e., the vertices of J can be renamed so as to obtain H) is an equivalence relation, and the equivalence classes are collections of graphs which are the same in this sense.

The degree of a vertex x of a simple graph H is the number of edges that contain x. We use deg(x) to denote the degree of the vertex x. If H is a graph with n vertices, then for any vertex x, $0 \le \deg(x) \le n - 1$. For any graph H, the sum of the degrees of the vertices equals twice the number of edges (i.e., $\sum_{x \in V} \deg(x) = 2|E|$). Notice that this says the sum of the vertex degrees is an even number. The minimum degree of H denoted by $\delta(H)$, is the smallest number of edges incident with a point of H while the maximum degree of H, denoted by $\Delta(H)$, is the largest such number. If $\delta(H) = \Delta(H) = r$, G is called regular of degree r.

A walk in a simple graph H is a sequence $v_0v_1...v_k$ of vertices such that consecutive vertices in the sequence are adjacent (i.e., $\{v_{i-1}, v_i\} \in E$ for $i = 1, 2, ..., k$). The integer k is called the length of the walk. Notice that k equals the number of vertices in the walk minus one. Thinking of the picture of a graph, it is the number of edges that would be traversed if you started at v_0 and travelled to v_1 along $\{v_0, v_1\}$, then to v_2 along $\{v_1, v_2\}$ and so on until v_n is reached. Observe that a walk is any sequence of consecutive adjacent vertices. It may or may not end where it starts, and may contain the same vertex many times. Also notice that the sequence consisting of a single vertex is a walk (of length zero).

A path in a simple graph H is a walk in H that contains no repeated vertices. Notice that every path is a walk, but the converse is false. Also, since every path is a walk, it has a length (as before).

A graph H is called connected if for all pairs of vertices u and v there is a walk that starts at u and ends at vertex v; otherwise H is disconnected. A walk in a graph H is called closed if its first and last vertex are the same. Since a closed walk is a walk, it has a length as above. Also, notice that a closed walk may or may not contain repeated vertices other than the first and last (which are the same). A closed walk of length at least three in which all vertices are distinct except the first and last is called a cycle. The length of the shortest cycle (if any) in H is called the girth of H. Every cycle is a closed walk, but not every closed walk is a cycle.
A tree is a connected graph that contains no cycles.
A leaf of a tree is a vertex of degree one.

Let X be a nonempty set and G be a group. We say that G acts on the left of X if for each $x \in X$ and $g \in G$ there corresponds a unique element $gx \in X$ such that, for all $x \in X$ and $g_1, g_2 \in G$

a) $(g_1g_2)x = g_1(g_2)x$
b) $1.x = x$, where 1 is the identity in G.

The action of G from the right can be defined in a similar way.

Let G act on X. Then X is partitioned into disjoint equivalence classes (with respect to an equivalence relation) called orbits or transitivity classes of the action. For each $x \in X$, the orbit containing x is denoted by $\mathrm{Orb}_{G(x)}$. Thus,
$\mathrm{Orb}_{G(x)} = \{gx | g \in G\}$
The action of a group G on the set X is said to be transitive if for each pair of points $x, y \in X$, there exists $g \in G$ such that $gx = y$; in other words, if the action has only one orbit.
Suppose that G acts transitively on X. For each subset Y of X and each $g \in G$, let $gY = \{gy | y \in Y\} \subseteq X$. A subset Y of X is said to be a block for the action if, for each $g \in G$, either $gY = Y$ or $gY \cap Y = \emptyset$. In particular, $\emptyset$, X and all 1-element subsets of X are obviously blocks. These are called the trivial blocks. If these are the only blocks, then we say that G acts primitively on X. Otherwise, G acts imprimitively.
Suppose G acts on X, then G acts on $X \times X$ by $g(x, y) = (gx, gy)$, $g \in G$, $x, y \in X$. If $O \subseteq X \times X$ is a G-orbit, then for a fixed $x \in X$, $\Delta = \{y \in X | (x, y) \in O\}$ is a G_x-orbit. Conversely, if $\Delta \subseteq X$ is a G_x-orbit, then $O = \{(gx, gy) | g \in G, y \in \Delta\}$ is a G-orbit on $X \times X$. We say Δ corresponds to O. The G-orbits on $X \times X$ are called suborbitals.
Let Δ be an orbit of G_x on X. Define $\Delta^* = \{gx | g \in G, x \in g\Delta\}$, then Δ^* is also an orbit of G_x and is called the G_x -orbit paired with Δ. If $\Delta^* = \Delta$, then Δ is called a self-paired orbit of G_x.
Let $O_i \subseteq X \times X$, ($i = 0, 1, 2, ..., r - 1$) be a suborbital. Then we form a graph Γ_i, by taking X as the points of Γ_i and including a directed line from x to y ($x, y \in X$) if and only if $(x, y) \in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i.
Now $O_i^* = \{(x, y) | (y, x) \in O_i\}$ is also a G-orbit. Let Γ_i^* be the suborbital graph corresponding to the suborbital O_i^* and let the suborbit Δ_i ($i = 0, 1, ..., r - 1$) correspond to the suborbital O_i. Then Γ_i is undirected if Δ_i is self-paired and Γ_i is directed if Δ_i is not self-paired.

Theorem 2.1 (Nyaga, 2011)
If $n \geq 2r$, the rank of S_n acting on $X^{(r)}$ is $r + 1$.

Theorem 2.2 (Nyaga, 2011)
The suborbits $\Delta_0, \Delta_1, \Delta_2, ..., \Delta_{r-1}, \Delta_r$ of S_n acting on $X^{(r)}$ are self paired.

Remark 2.3
The $r + 1$ suborbits of S_n given in Theorem 2.2 are defined by:
$\mathrm{Orb}_{G\{1,2,3,\ldots,r\}}\{1, 2, 3, ..., r\} = \Delta_0$, the trivial orbit.

$\mathrm{Orb}_{G\{1,2,3,\ldots,r\}}\{1, r+1, r+2, \ldots, 2r-1\} = \Delta_1$, the orbit containing exactly one of 1, 2, 3,..., r.
$\mathrm{Orb}_{G\{1,2,3,\ldots,r\}}\{1, 2, r+1, \ldots, 2r-2\} = \Delta_2$, the orbit containing exactly two of 1, 2, 3,..., r.
$\mathrm{Orb}_{G\{1,2,3,\ldots,r\}}\{1, 2, 3, r+1, \ldots, 2r-3\} = \Delta_3$, the orbit containing exactly three of 1, 2, 3,..., r.
$\mathrm{Orb}_{G\{1,2,3,\ldots,r\}}\{1, 2, \ldots, r-1, r+1\} = \Delta_{r-1}$, the orbit containing exactly $r-1$ of 1, 2, 3,..., r.
$\mathrm{Orb}_{G\{1,2,3,\ldots,r\}}\{r+1, r+2, \ldots, 2r\} = \Delta_r$, the orbit containing none of 1, 2, 3,..., r.

Theorem 2.4 (Nyaga, 2011)
If $2 \leq r < \frac{1}{2}n$, then the action of S_n on $X^{(r)}$ is primitive.

Theorem 2.5 (Sims, 1967)
Let G be transitive on X. Then G is primitive if and only if each suborbital graph Γ_i, $i = 1, 2, \ldots, r$ is connected.

3.0 Main Results

The construction of the suborbital graphs corresponding to the suborbits of S_n is as follows:
The suborbital O_1 corresponding to the suborbit Δ_1 is
$O_1 = \{[g\{1, 2, 3, \ldots, r\}, g\{1, r+1, r+2, \ldots, 2r-1\}]\}$, where $g \in S_n$. Therefore, in Γ_1, the suborbital graph corresponding to O_1, there is an edge from vertex A to B if and only if
$|A \cap B| = 1$.
The suborbital O_2 corresponding to the suborbit Δ_2 is
$O_2 = \{[g\{1, 2, 3, \ldots, r\}, g\{1, 2, r+1, \ldots, 2r-2\}]\}$, $g \in S_n$. Therefore, in Γ_2, the suborbital graph corresponding to O_2, there is an edge from vertex A to B if and only if $|A \cap B| = 2$.
The suborbital O_3 corresponding to the suborbit Δ_3 is
$O_3 = \{[g\{1, 2, 3, \ldots, r\}, g\{1, 2, 3, r+1, \ldots, 2r-3\}]\}$, $g \in S_n$. Therefore, in Γ_3, the suborbital graph corresponding to O_3, there is an edge from vertex A to B if and only if $|A \cap B| = 3$.
Continuing with this argument, the suborbital O_{r-1} corresponding to the suborbit Δ_{r-1} is
$O_{r-1} = \{[g\{1, 2, 3, \ldots, r\}, g\{1, 2, 3, \ldots, r-1, r+1\}]\}$, $g \in S_n$. Therefore, in Γ_{r-1}, the suborbital graph corresponding to O_{r-1}, there is an edge from vertex A to B if and only if
$|A \cap B| = r-1$.
The suborbital O_r corresponding to the suborbit Δ_r is
$O_r = \{[g\{1, 2, 3, \ldots, r\}, g\{r+1, r+2, \ldots, 2r\}]\}$, $g \in S_n$. Therefore, in Γ_r, the suborbital graph corresponding to O_r, there is an edge from vertex A to B if and only if $|A \cap B| = 0$

Theorem 3.1

a) $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_r$ are undirected.
b) If $n \geq 3r$, $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_r$ have girth 3.
c) $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_r$ are connected if $n > 2r$

Proof

a) Using Theorem 2.2, Δ_i, $i = 1, 2, 3, \ldots, r$ are self-paired, implying that $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_r$ are undirected.
b) Let $X = \{1, 2, 3, \ldots, n\}$ and suppose that $n \geq 3r$. Then there exists three unordered r-element subsets of X, say A, B, and C such that, $|A \cap B| = |A \cap C| = |B \cap C| = 1$;
$|A \cap B| = |A \cap C| = |B \cap C| = 2$; ... $|A \cap B| = |A \cap C| = |B \cap C| = r - 1$;
$|A \cap B| = |A \cap C| = |B \cap C| = 0$. Thus in each case A, B, and C are adjacent vertices in $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_r$ respectively. Therefore if $n \geq 3r$, $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_r$ have girth 3.
c) By Theorem 2.4, G acts primitively on $X^{(r)}$ if $2 \leq r < \frac{1}{2}n$. So, by Theorem 2.5, $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_r$ are connected if $n > 2r$.

Remark 3.2

Theorem 3.1 (b) means that if $n < 3r$, then some Γ_i, $i = 1, 2, \ldots, r$ may be of girth 3 but not all of them. This theorem gives a sufficient condition for all of Γ_i to have girth 3.

Example 3.3

Let $G = S_5$ acting on $X^{(2)}$. We shall base our discussions on Δ_1, and Δ_2. The suborbital O_1 corresponding to the suborbit Δ_1 is $O_1 = \{[g\{1, 2\}, g\{1, 3\}] \mid g \in G\}$. The suborbital graph Γ_1 corresponding to the suborbital O_1 has 2-element subsets A and B from X adjacent if and only if $|A \cap B| = 1$. Secondly, the suborbital O_2 corresponding to the suborbit Δ_2 is
$O_2 = \{[g\{1, 2\}, g\{3, 4\}] \mid g \in G\}$. The suborbital graph Γ_2 corresponding to O_2 has 2-element subsets A and B from X adjacent if and only if $|A \cap B| = 0$. We construct Γ_1 as in Figure 1.

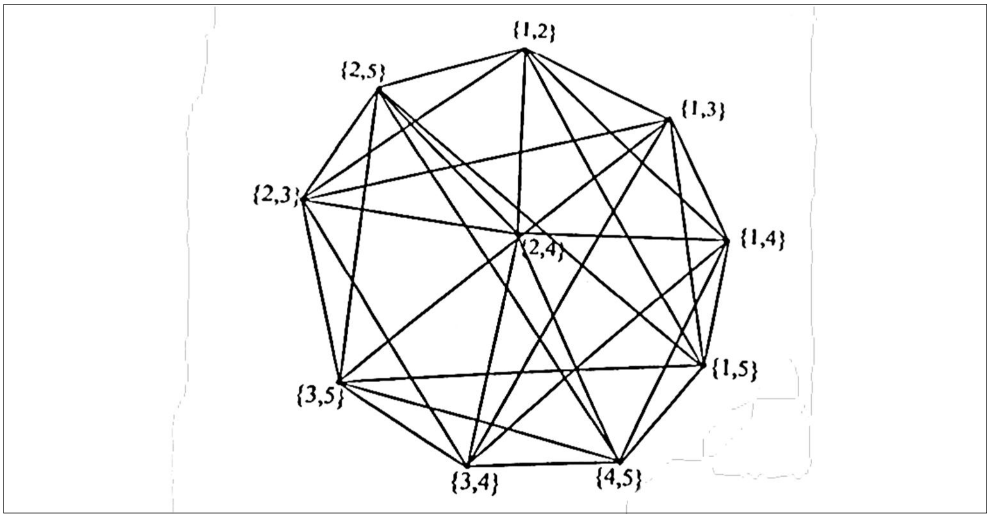

Figure 1: Γ_1, the suborbital graph of S_5 acting on $X^{(2)}$ corresponding to Δ_1

From Figure 1, Γ_1 is connected, regular of degree 6 and has girth 3.
We also construct Γ_2 as in Figure 2

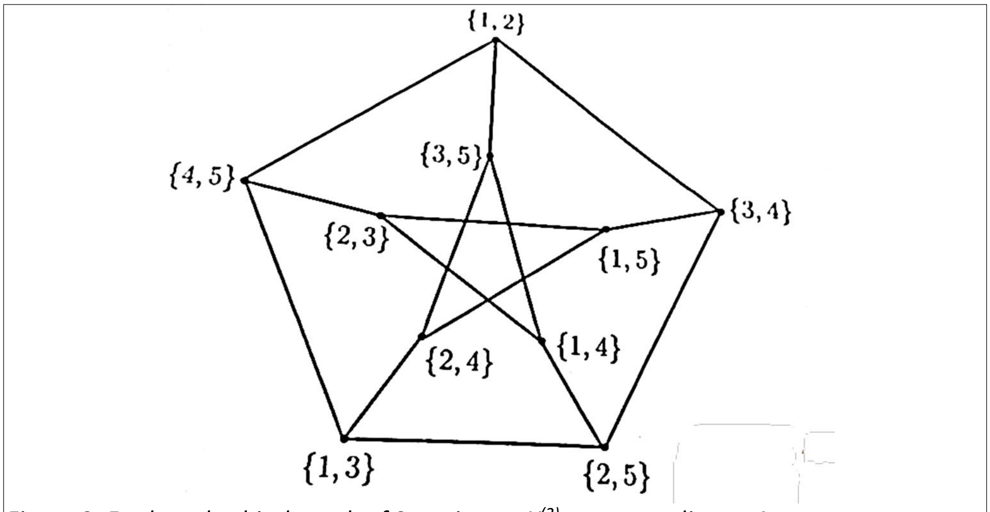

Figure 2: Γ_2, the suborbital graph of S_5 acting on $X^{(2)}$ corresponding to Δ_2

From Figure 2, Γ_2 is regular of degree 3. It is connected and has girth 5.

References

Akbas, M. and Baskan, T. (1996). Suborbital graphs for the normalizer of *$\Gamma_0(N)$. Tr. J. of Mathematics*, **20**, pp. 379-387.

Bensenk, M., G¨uler, B. ¨ O., Deˇger, A. H., Kader, S. (2010). Conditions to be a forest for normalizer. *Int. Journal of Math. Analysis,* **4,** 1635-1643.

Bon, J. V. and Cohen, A. M. (1979). Linear groups and distance-transitive graphs. *Europ. J. Combinatorics* **10**: 399-411.

Guler, B. O., Kader, S., Besenk, M. (2008). On suborbital graphs of congruence Subgroup $\Gamma_0(N)$. *International Journal of Computational and Mathematical Sciences* **2(3),** pp. 153-156.

Kamuti, I. N., (1992). *Combinatorial formulas, invariant and structures associated with primitive permutation representations of PSL (2, q) and PGL (2, q)* Ph.D. Thesis, University of Southampton.

Keskin, R. (2004). Suborbital graphs for the normalizer $\Gamma_0(m)$. *Europ. J. Combinatorics*, **27**, pp. 193-206.

Keskin, R. and Demirt¨urk, B. (2009). On the suborbital graphs for the normalizer of $\Gamma_0(N)$. *The Electronic Journal of Combinatorics*. **16** #R116.

Neumann, P. M. (1977). *Finite permutation groups. Edge-coloured graphs and matrices, Topics in group theory and computation*. Edited by P. J. Cameron, Academic Press, London, San Francisco.

Nyaga, L. N. (2012). *Ranks, subdegrees, and suborbital graphs of the symmetric group S_n acting on unordered r-element subsets*. Ph.D. Thesis, JKUAT.

Nyaga, L. N., Kamuti, I. N., Mwathi, C. W., Akanga, J. R. (2011). Ranks and subdegrees of the symmetric group S_n acting on unordered r-element subsets. *International Journal of Pure and Applied Mathematics* vol. 3(2), 2011. pp. 147-163.

Sims, C. C. (1967). Graphs and finite permutation groups. *Math. Zeitschrift* **95**, 76-86.