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Learners engaging with transformation geometry

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This article reports on a qualitative, interpretivist study that focused on the use of visualisation and analytic strategies by Grade 12 learners when working with problems based on transformation geometry. The research was conducted with 40 learners from a Grade 12 class at one high school in the north Durban area of Kwazulu-Natal. Participants completed a written task and a smaller sample of the participants engaged in investigative semi-structured interviews with the researchers. The framework for the study was based on transformations of semiotic representations as well as the visualiser/analyser model. The findings revealed that most learners performed treatments in the analytic mode when responding to the tasks, and showed limited movement across the two modes which are essential for a deepening of understanding. The study identified one learner, however, who was able to move flexibly between the modes and who displayed a deep understanding of the concepts. The article concludes by recommending that opportunities need to be created for learners to engage in transformation geometry activities which emphasise conversion.

Keywords: analysis; conversions; transformation geometry; transformations; treatments; visualisation

Introduction

Within the current (2011) South African curriculum, school mathematics incorporates transformation geometry, which was introduced in the Further Education and Training (FET) band in 2006. This strand allows learners to make connections across other geometries within the space, shape and measurement learning outcome, as well as with algebra and trigonometry. These connections are intended to create a more integrated, holistic knowledge of mathematics (DoE, 2003) and to allow for novel interpretations of the mathematics learnt in the other strands. It was this characteristic of the multiple connections between the strands that this study was focused on, to find out more about the strategies that Grade 12 learners used to solve problems in transformation geometry. By studying the learners' visual strategies, we also hope to develop an understanding of how learners' use of visualisation techniques could contribute to more effective problem solving. We also hope that the study will inform teachers and curriculum developers about possible pedagogic approaches that could be used successfully when teaching this strand of mathematics. Unfortunately the strand of transformation geometry has been removed from the FET band from 2012 onwards (DBE, 2011). However the strand remains in the curriculum for Grades R to 9, and will still provide rich learning opportunities for the learners in these grades.

This article reports on a qualitative, interpretivist study focusing on the use of visualisation and analytic strategies by Grade 12 learners when working with problems based on transformation geometry. It addresses the effectiveness to which learners use either strategy, or both.

Literature review

There is limited available research on learners' understanding and learning of transformation geometry. In one such study Edwards (1997) maintains that transformation geometry provides an ample opportunity for learners to develop their spatial visualisation skills and geometrical reasoning ability. She used transformation geometry tasks to document the learning path of the students in a computer based micro-world. Edwards (2003) identified a particular misconception about rotations. She found that instead of seeing rotation as mapping all the points of the plane around a centre point, the students in her study expected the shape to slide to the given centre point and then turn around it- showing that "they had a hard time seeing rotation as occurring 'at a distance' from the object" (Edwards, 2003:7). Another study (Sproule, 2005) was carried out with Grade 7 learners, and sought to identify the strategies that were best able to support learners in correctly completing reflection tasks. Sproule found that although measuring distances in the diagrams was the most common strategy used by the learners, those participants who folded along the axis of symmetry were the most successful. He speculated that the act of seeing the fold lines may have added to their success. In this study we envisioned investigating something similar, whether learners' use of visualisation strategies could improve their success in solving transformation geometry problems. Other strategies that were utilised (Sproule, 2005) in solving the problems on reflections were the use of grid lines, measuring, drawing in marks, turning the figure, mental folds and using the mirror. Whilst Sproule focused primarily on completing reflection tasks, we developed a transformation geometry task that encompassed reflection, rotation, translation, enlargement and investigations on the Cartesian plane. With the increased scope of our tasks we anticipated that the strategies employed by the learners to complete our tasks would be different, because we were of the opinion that the tools available would influence the learners' way of thinking. The transformation geometry strand provides an opportunity for assessing the use of skills and abilities in merging algebraic and geometric ideas. This strand of the mathematics curriculum also encourages a visual as well as an analytic approach, and provides a context for combining algebra and geometry. The combination of approaches, if any, would be more prevalent with Grade 12 learners, since by Grade 12, learners would have been exposed to both visual and analytical strategies.

The visual approach is one which advocates investigations and discovery of properties via concrete manipulations, models and diagrams. Within learning environments, opportunities that exploit the visual mode of thinking ought to be sought (McLoughlin, 1997). A diagram provided on the Cartesian plane may elicit visualisation strategies provided the student is able to recognise what is given. However these strategies can be supported by the analytic approach which is characterised by general formulae to describe the results of transformations on figures that are situated within the Cartesian plane. In a situation where written instructions about a general point are given, we were curious about whether the mode of presentation would elicit analytic or visual strategies.

The analytic approach seemed to be compelling to learners because of the ready availability of formulae. Nevertheless, using generalised formulae to work out problems based on transformation geometry is not as simple as learners perceive it to be. One of the important contributors to learners' success in working with transformations is their procedural fluency in algebraic procedures. Procedural fluency is a facet of mathematical proficiency and is described as "skill in carrying out procedures flexibly, accurately, efficiently and appropriately" (Kilpatrick, Swafford & Findell, 2001:107). Because analytic strategies in transformation geometry are dependent on the use of algebraic rules, learners' misapplication of rules can be largely attributed to poor algebraic skills. Many errors uncovered in the study [not reported in this paper] were related to learners' inefficient use of algebraic manipulations (Naidoo & Bansilal, 2010). Although learners may know the rules for working out the results of translations, rotations and reflections, their poor algebraic skills may let them down.

In order to successfully use analytic techniques, learners would need to firstly know the rules, understand the meaning of the algebra used in the rules, and be able to apply the rules using the given points. For these learners who struggle with the basics of algebra, they would never be able to move past the first step, and would perform disastrously in questions in transformation geometry, because of their poor background knowledge.

An alternative approach is the use of visual strategies. These are ways of improving communication by using scaffolds that can be seen. There are different types of scaffolds which include natural environment cues, body language, traditional tools and specifically designed tools. In empirical research with high school students in the US, Presmeg (2006) identified five types of imagery, which are concrete imagery, kinaesthetic imagery (of physical movement), dynamic imagery, memory images and pattern imagery. She found that concrete imagery was the most prevalent, with dynamic imagery (where the image itself is moved or transformed) being used effectively but very rarely. This study will show evidence of one learner exhibiting the use of dynamic imagery.

Dreyfus (1991:33) argued that "the status of visualisation in mathematics should be and can be upgraded from that of a helpful learning aid to that of a fully recognised tool for learning and proof". His words convey a sense that many mathematics educators accord visualisation a low status in their classrooms, which may explain why many learners are reluctant to use visualisation in mathematics. These words also suggest that teachers do not clearly distinguish between visualisation skills as such, and the strategy of visualisation as a reasoning process or tool.

Breen described two types of thinking as (1997:97) "The one was a tendency towards abstraction … The other was a tendency towards intuitive understanding which stresses processes of visualisation and imagery". Here, there is an inference that visualisation is relegated to the senses, as if not involved in reasoning.

In fact, even the well-known and widely applied "Van Hiele model of geometric thought" specifies a visual reasoning level as an initial level of geometric understanding. This is followed by a level of analysis at a higher level, then informal deduction, formal deduction and finally the highest level of rigour (Crowley, 1987). All levels are sequential and hierarchical and teaching activities are often designed according to the levels. The descriptions of the first three levels are given below (Crowley, 1987:3)

Level 1 (Visualisation): The object is seen as a whole, individual properties are therefore not distinguished.
Level 2 (Analysis): The object can be identified by the properties; each property is seen in isolation so properties of figures are not compared.
Level 3 (Informal Deduction): The objects are still determined by their properties; however the relationships between properties and figures evolve.

Whilst the van Hiele levels of thinking are focused on geometry, they are valuable when working with transformation geometry, which deals with transformation of geometric figures. In this study, learners were asked to carry out transformation on triangles, and thus needed to recognise and understand the properties of the triangles. The question arises as to whether the movement from visualisation to analysis is a progression to a higher level of thinking as set out in Van Hiele's model. To answer more effectively, it will be necessary to clarify the terms 'visualisation' and 'analysis'. It is pertinent to note that the use of these terms visualisation and analysis are similar to that of Zazkis, Dautermann and Dubinsky (1996) whose model is discussed in the next section. However the model proposed by Zazkis *et al*. (1996) added further spiral movements between the two interacting modes of thought of visualisation and analysis, whereas Van Hiele's model suggests that analysis is a higher level of understanding than the visual reasoning level.

Note that visual skills such as being able to "see" in your mind the image when a point is rotated 90°, say, is different from visualisation as a reasoning strategy (as used by Zazkis *et al*., 1996), which can be used on any level of thinking employing the visual skills. It is possible that visual skills can be employed in visualisation strategies up to the highest level of reasoning (Hoffer, 1981). This study will provide evidence of one learner who has been able to use his visual skills as a tool for reasoning about the effect of transformations on figures.

Theoretical framework

In analysing the students' performance we drew upon the Visualizer/Analyzer (VA) model as developed by Zazkis *et al*. (1996). We also used Duval's (2003) framework concerning transformations of semiotic representations. Duval (2006) points out that the part played by the semiotic systems of representation is not only to designate mathematical objects or to communicate but also to work on mathematical objects and with them. A semiotic system is characterised by a set of elementary signs, a set of rules for the production and transformation of signs and an underlying meaning structure deriving from the relationship between the signs within the system (Ernest, 2006). Duval (2003) asserted that two different types of transformations of semiotic representations may occur during any mathematical activity. The first type, called *treatments*, involves transformations from one semiotic representation to another within the same system. The second type, called *conversions*, involve changing the system but conserving the reference to the same objects. This resonates strongly with the principles of transformation geometry in that this strand presents opportunities for mathematical activities where one representation may be converted to another.

Mathematical activities which are *conversions* involve working with two semiotic representations, each of which preserves the objects under scrutiny. However, the content associated with the objects in each representation is different. An example of a conversion is the movement from the algebraic notation of an equation to its graphical representation. Duval (2003:6) asserts that the content (properties) of a representation of an object depends more on the register of the representation than on the object represented. Thus passing from one register to another is not only changing the means of *treatment* (hence transformation of representation); it is also making explicit other properties or other aspects of the same object. Hence working with each representation offers different perspectives, resulting in a strengthening of the understanding of the concept.

The VA model as developed by Zazkis *et al*. (1996) specified the two elements — visualisation and analysis as two interacting (and not hierarchical) modes of thought. In this paper we use a condensed version of the definitions of visualisation and analysis as specified by Zazkis *et al*. (1996:441-442). We view an act of *visualisation* as a mental construction of external objects or processes, or an external construction of mental objects or processes of the individual. An act of *analysis* or analytic thinking is any mental manipulation of objects or processes with or without the aid of symbols. In this study, the acts of visualiaation are mainly related to external constructions of figures described by Cartesian coordinates, while acts of analysis involve mainly mental manipulation with the aid of symbols (which could be algebraic or points of the Cartesian plane). Associated with the acts of visualisation and of analysis, are accompanying semiotic representations which allow these acts to be mediated between individuals.

The VA model (Zazkis *et al.*, 1996) can then be seen to describe a series of *conversion*-type activities between visual and analytic representations, each of which are mutually dependent in problem solving, rather than unrelated opposites. In the model of Zazkis *et al.* (1996) the thinking begins with an act of visualisation, V_1, which could entail the learner looking at some "picture" and constructing mental processes or objects based on this "picture". The next step is an act of analysis, A_1, which consists of some kind of coordination of the objects and processes constructed in step V_1. This analysis can lead to new constructions. In a subsequent act of visualisation, V_2, the learner returns to the same "picture" used in V_1, but as a result of the analysis in A_1, the picture has now changed. As the movement between the V and A is repeated, each act of analysis is based on the previous act of visualisation. This act of analysis is used to produce a new richer visualisation which is then subjected to a more sophisticated analysis. This thus creates a spiral effect as illustrated in Figure 1 (adapted from Zazkis *et al.*, 1996:447).

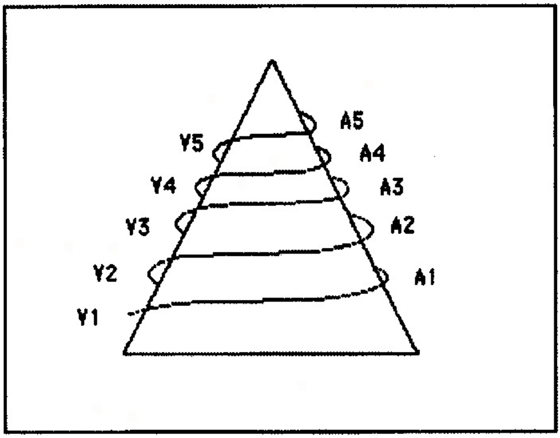

Figure 1 The spiral effect of visualisation and analysis

In this model the acts of analysis deepens the acts of visualisation and vice versa. It is also important to note that according to this model, as the horizontal motion in the model is repeated, the acts of visualisation and analysis become successively closer for the individual. At first the passage from one to the other may represent a major mental effort, but gradually the two kinds of thought become more interrelated and the movement between visualisation and analysis becomes less of a concern. On a similar note, Siegler (2003) asserts that a learner's level of thinking can be viewed as a staircase. Each stair on the upper level represents a new approach to thinking; as you move higher up the staircase, the thinking and approach to a problem become more advanced and sophisticated.

Research design

We believe that it is important to understand how learners combine visual and analytical approaches in solving tasks in transformation geometry. Therefore the purpose of this paper is to explore strategies Grade 12 learners employed when working with tasks in transformation geometry. The corresponding research question is: To what extent do learners move between the visual and analytic modes of thinking when working with transformation geometry? We hope then to set out some implications for the teaching of transformation geometry.

This qualitative study utilised a case study methodology and can be viewed as an "instrumental case study" because we wanted to gain more insight into the strategies the class used when solving transformation geometry tasks (Stake 2008:122). The participants were a class of 40 Grade 12 learners from a high school in the northern Durban area. This school is an urban school and draws learners from a lower to middle socio-economic community. The participants had completed the section on transformation geometry in Grade 11 and at the time of the study were being reacquainted with the section in Grade 12, as required by the syllabus. The transformation geometry tasks used in the study were based on Grade 11 and 12 exam type questions.

Data for the study were generated from the 40 learners' responses to a set of transformation geometry tasks and semi-structured investigative interviews with six of the learners. The interviews were designed to probe learners' reasoning about the strategies they used. The data was analysed by analysing the solutions to the six questions, noting the explanations and the strategies employed. The strategies employed by learners to answer the task were identified in terms of whether students used formulae or visual strategies or a combination of these. Thereafter six learners were selected, based on an initial analysis of their written responses to determine which warranted further investigation.

During the semi-structured interviews the learners within the sample were asked to explain their solutions to each question as they revisited their tasks. The researchers prompted them where necessary to clarify their thinking as well as to ascertain that the researchers understood the strategies they were using. Learners were also asked to confirm their explanations by demonstrating how they answered certain questions from the task. The interviews were audio-taped and then transcribed by the two researchers. The analysis of the interviews was then carried out in conjunction with the learners' written responses which formed the basis of the interview questions. We used the technique of open coding (Henning, 2004:131) in the analysis of the written responses; this refers to "naming and categorising phenomena through close examination of the data" and to fracturing them into "concepts and categories". We then grouped together concepts at a higher, more abstract level, using the data from the interviews where possible, as confirmation of our categorisation. Such an analysis took "place on two levels — the actual words used by the respondents and the conceptualization of these words by the researcher" (Henning, 2004:132).

The task comprised six questions, three of which are under scrutiny in this article. The first two questions were selected because the analysis of the learners' responses raised pertinent issues about the use of visual and analytic strategies when performing rigid transformations of figures. The third question was selected because it required reasoning and reflections about the properties of translations and rotations and could not easily be answered by the use of a formula only.

In terms of ethical considerations, approval for protocols was obtained from the university, and the parents of the learners were given an informed consent letter which asked for permission to use their assessment responses for this research study. Some limitations of the study were that not all the learners were interviewed. Scheduling interviews with all learners would nit have been possible because of the time constraints n Grade 12 learners, so it was decided to carry out only six interviews, thus limiting our generalisation about the strategies used by all the learners.

In terms of reliability and external validity, the test items were carefully selected after much deliberation between the researchers. The researchers ensured that the questions chosen were ones that the students would have encountered in learning. The language used was sufficiently basic so that most learners would be able to understand the words used and hence these participants' responses may represent a close approximation of how other learners with similar backgrounds would respond to the selected questions. Nonetheless, no broad generalisations are made.

Results and discussion

In this section, we discuss the learners' written and interview responses to Questions 4 and 6 (see Table 1) in an attempt to describe the extent to which the learners moved between acts of

Table 1 Questions from the transformation geometry task

<table>
<tr><th>Question 4</th><th>Question 6</th></tr>
<tr><td>$P(2;1)$, $Q(-2;3)$ and $R(-4;-1)$ are the vertices of a triangle. Find the images specified and sketch the figure formed.
(a) Determine the images P',Q',R' of P,Q,R after the translation $(x;y) \to (x+2;y-1)$.
(b) Find the images P'',Q'',R'' of P,Q,R after the reflection about the x-axis.
(c) Find the images P''',Q''',R''' of P,Q,R after the rotation about the origin through an angle of -90^0.</td><td>Investigate whether each of the figures in Question 4 above are isosceles, equilateral or right angled. Fill in the following table.
<table>
<tr><th>Triangle</th><th>Isosceles (Y/N)</th><th>Equilateral (Y/N)</th><th>Right-angled</th><th>Reason</th></tr>
<tr><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td></tr>
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visualisation and that of analysis. The interview excerpts are taken from interviews with six learners whom we have named Ken, Nishan, Lucy, May, Ann, and Jack.

The kinds of transformations required in Question 4 change the location or position but preserve the size and shape of the figures (rigid transformations). Question 6 was designed partly to find out whether learners understood the preservation of lengths and angles under the given transformations.

From the interview responses it was evident that some learners reduced the transformation exercise (Q4) to a *treatment* activity without any visualisation activities. The learners performed the transformations on a point by point basis using the algebraic rules. These algebraic rules are shown in Table 2.

Table 2 Transformation geometry rules

Transformation	Rule
Reflection	
About the x axis	(x, y) becomes $(x, -y)$
About the y axis	(x, y) becomes $(-x, y)$
About $y = x$	(x, y) becomes (y, x)
About $y = -x$	(x, y) becomes $(-y, -x)$
Rotation	
90º about the origin	(x, y) becomes $(-y, x)$
180º about the origin	(x, y) becomes $(-x, -y)$
270º about the origin	(x, y) becomes $(y, -x)$
–90º about the origin	(x, y) becomes $(y, -x)$
Translation	
$T_{a,b}\,(x, y)$	$(x + a, y + b)$

Some of the responses of the learners revealed their approaches:

Ken: "*... you have got to translate these coordinates to the rules they have given you and you have got to plot your points, this is what I did. I worked out the coordinates first and then plotted...*"

Nishan: *"... reflection about the x axis, okay for b I don't think I have the formula or the process on how to reflect about the x axis at that time so I did not answer that question but I think reflection about the x axis, umph ... I am not too sure. Okay basically in c I wrote it as the question says, rotate it minus 90 degrees, for me it was anticlockwise and I basically interchanged x and y and making x negative 2, but I don't think that is the correct answer; I used the rule, I found the point and then I drew it..."*

Andrea: *"... I worked out the points..."*

Lucy: *"... I did work out the points and I wrote it down at first and then I followed the points and I sketched it..."*

May: *"...whenever I have something like this, I always write down the rule. I have a way of remembering the rule..."*

The above comments revealed (as was also supported by their written responses) that those five learners carried out a *treatment* using the algebraic rules without doing any visualisation. For Q6, after working out the new points, most learners then plotted the points on the Cartesian system and joined them up, with some making use of different colours to distinguish between the four different triangles. Plotting the points and joining them up constituted their first act of visualisation even though it was in response to the instruction. The written responses of May and Ann revealed that they drew the triangles in different colours, while Nishan drew the triangles on separate axes, allowing him to see them as distinct figures. Two learners revealed that they used different colours merely to distinguish between the various triangles.

Lucy: *"... the colours bring it more in perspective ... if it's in the same colour you won't be able to tell what has been rotated, what has been moved..."*

Ken: *"... I had to use different colours because if I didn't I would not be able to differentiate... Well if you use the same colour pen, you wouldn't be able to differentiate between the original triangle and the one that you did rotate or translate...*"

The preceding comments revealed that the learners used the colours more as a tool to distinguish the different triangles and not as a tool for analysis of the process. The colours allowed them to guess possible properties of the transformed triangles. However, it was evident that the use of colours was not intended to help them understand the properties of the transformed triangle in relation to the properties of the original triangle.

When responding to Question 6, with respect to the properties of the new triangles, most of the learners then performed an act of analysis (A_1,) on each of the images of the triangles they had transformed. Based on the responses to the interview questions, learners indicated various ways of identifying the properties:

Ken: *"... in the first one now all the sides are equal here… I probably what I did was, I can see that not all the sides are equal and ... uh... it couldn't have been an equilateral, the reasoning though I don't know how I got that..."*

Nishan: *"... I worked it out on a rough page and to me it looked like an isosceles triangle and it was right angled as to you can see here PQ = QR, that means that there was a right angle which indicates to me that it was an isosceles, in b , eh, .., well b I didn't work out, I just randomly guessed and in c it also looked like an isosceles triangle coming from a and if it was isosceles it will also be right angled, to me none of them looked like an equilateral, I did take one thing into account whether it was right angled, if it was then it could not be equilateral so that was the process that I used..."*

Lucy: *"... Okay basically by looking at your sketch you would be able to determine whether what kind of triangle it is, whether either two sides are equal or not and it also tells you whether it is right angled because of the way you sketched it ... and the fact that the two lines bisecting are adjacent we learnt in school that when the gradients, when you multiply it and they give you negative 1 so that will tell you that the lines are adjacent..."*

Ann: *"... By looking at the triangle I felt that the sides were equal; you can see that it was isosceles. I looked at the triangle and I thought it was 90 degrees. I then measured the angle… I found question 6 was a bit challenging and I did not really know what to do there, and after I plotted the points I could not work out you know it was hard and as you can see I left some of it blank..."*

May: *"... isosceles? I said no, because it's not equal. For isosceles, two sides are equal, one side not…I'm not sure if I used the rules and found the lengths, but they seem equal. That why we took it as equal..."*

The above demonstrates that the learners executed a superficial analysis of the properties. Most learners considered all four triangles as having different properties that needed to be investigated. Most looked at the visualisation of each of the triangles and made inferences based on what they saw. They judged the equality of lines on the basis of what they saw. Others measured the sides or angles of the triangles they drew. There were three learners (not interviewed) who indicated in their written responses that the 2nd and 3rd triangles were right-angled, but not the first. There were also three learners (not interviewed) who indicated in their written responses that the 1st triangle was isosceles but not the other two. These responses exhibited that those learners did not comprehend that certain transformations do not alter the size and shape of a figure. Some learners laboriously worked out the lengths of each of the sides of each of the triangles. This demonstrated a limited understanding of the properties of transformations, because they did not understand the differences between rigid and non-rigid transformations. This could be due to the fact that while some learners learn abstract mathematics concepts from the time of their initial learning, some learners fail to grasp concepts and others grasp these abstract concepts but cannot connect them to procedures (Siegler, 2003).

We now present the case of one student, who we call Jack, who appeared comfortable with the VA movement between the two different representations. He revealed an integrated analytic-visual approach. When explaining how he worked out a 90º rotation of the point (2; 3), Jack said he didn't use the rules, but worked it out each time — in this case he said it would be ($y; –x$) and then offered to explain how he worked it out.

Jack: *"... (2; 3) plotted there [pointing to the actual location] will be rotated clockwise 90º, will go to the 4th quad, so you will see the spacing from the x-axis to be 2 units, it will be the same [pointed out a turning motion of the x-axis turning clockwise into the y-axis]. So it will be –2. The y-value will become –2, so that's –x on that point [pointing to the second coordinate on the answer that he had written down]. And the spacing from here [pointing to y-axis]. So the spacing here [pointing to the x-axis] will be 3 units, so that will become y..."*

Jack's use of visual and analytic thinking of the concept of a rotation is very balanced because he moved from the visual to the analytic and back again performing a series of *conversion*–types of movement. Given the point (2; 3) [V_1] he analysed the meaning of the two coordinates [A_1] by identifying them as being in the first quadrant. He then transformed each coordinate by applying a 90º rotation of the axes [V_2]. He proceeded to analyse the resulting coordinate in terms of its location [A_2] and identified the transformed point in terms of its old position and the sign of the new coordinates in terms of its position [V_3]. This horizontal movement between the two perspectives permitted the development of greater insight. The acts of visualisation assisted him in distinguishing the effect of the rotations on the point, while the act of analysis assisted him in being more precise about the location of the points. This corresponds with Duval's (2003) statement that the passing from one representation to another makes explicit other properties or other aspects of the same object.

Jacks' response to Q6 revealed his understanding of the effects of transformations on a figure (in this case a triangle):

Interviewer:	If I asked you, is this [original triangle] isosceles, what would you have done?
Jack:	*I would have checked the lengths of the sides.*
Interviewer:	Would you have done it for each of your triangles?
Jack:	*No, because one is a transformation of the triangle, so the same property is being translated.*
Interviewer:	And what about it being right angled?
Jack:	*You have to find the gradients, and see if it is a negative reciprocal.*
Interviewer:	And would you have done it for all?
Jack:	*No, because the one is a transformation of the other, so the same property is being translated.*
Interviewer:	What properties would be the same?
Jack:	*If it's not being enlarged, then the area would be the same, and the sides would be the same.*

The above conversation reveals that Jack's understanding was clear that when a figure was transformed without changing its size (by a reflection, rotation or translation), properties related to the dimensions of the triangles remain the same. He understood that the figure that was being transformed was a triangle, and did not look at the points in isolation. He thus did not repeat the investigations on each of the transformed triangles of Question 6. Other learners were not so certain and had to be probed before they realised that it was unnecessary to investigate each of the triangles.

Another question (see Table 3) revealed the differences in the way the five interviewees approached the question as compared to Jack:

Table 3 Questions from the transformation geometry task

1. Find the image of a point (*a;b*) after
   a) a rotation through 90º about the origin, and then
   b) a translation of the point 2 units to the left and 5 units up.
2. Would you get the same result that you did in b) above, if you did the translation first and thereafter the rotation?

We look at the learners' responses in the interview:

Ann: *"... it won't make a difference..."*
Ken: *"... It would not make a difference..."*
Nishan: *" ... a – 2 and b + 5 and you get to change y – x and the final answer will be b+5 and –a –2. Yes you would get the same answer..."*
Lucy: *"... It would not have made much of a difference because if you move the point as the example said 2 units to the left wherever the point is rotate it 90 degrees. So I don't think b would have really made a difference..."*
May: *"... No, I would expect it to be the same, because we rotate it about 90º. And then we use the rule, it would be the same..."*
Jack: *"... If you rotate and translate, you get to a point and if you translate and rotate you will get to a different point..."*

Here again, Jack demonstrates a deeper insight into the properties of transformation and realises that the operations of translation and rotation by 90° are not commutative. It is clear that the order of the two operations makes a difference: The result of a translation of the point (*a, b*) by 2 units to the left and 5 units up, followed by a rotation of 90° is (*b*+5; –*a*+2). The result of a 90° rotation of the point (*a; b*) followed by a translation by 2 units to the left and 5 units up is (*b*–2; –*a*+5); showing that the two operations are not commutative. It is interesting that only Jack immediately rebutted the equality of the two results.

The other students seem to have difficulty with 'seeing' the rotation. It is possible that like the students in Edwards (2003) study, they did not recognise that the rotation would occur some distance away from the original figure. For example Lucy's comment "wherever the point is, rotate it 90°' suggests that she sees it turning on itself. This could be as a result of experience; when we look at a rotating wheel or spinning top, or even when we rotate, we turn around a centre point located within our bodies or the object (Edwards, 2003). Jack's responses suggest that he had a deeper understanding of the concepts of transformations. Zazkis *et al*. (1996:444) commented that their "observations reveal that students who mix, harmonise, and synthesise the strategies usually have a more mature understanding of the problems." This is similar to the case of Jack who was able to move effortlessly between the acts of visualisations and analysis. Many other learners were stuck within one representation that utilised only the algebraic rules. Those learners had performed the transformations in the one representation — as a *treatment* (Siegler, 2003).

Our study has not been able to identify whether Jack's deeper understanding is a result of, or whether it contributes to, his facility in moving across different representations. However we believe that it is a bit of both. As his understanding improves, it is easier for him to move across different representations. As he moves across different representations, the different aspects emphasised by each contribute to a deeper understanding.

Our thinking is supported by Zakis *et al*. (1996:455) who suggests that "moving across [the visualisation and analysis processes] in order to move up, at a rate appropriate for (the learners' needs), may help them to make the connections necessary". Although moving across may not be easy or conventional, the authors advise that the effort "might facilitate a richer and more useful understanding of complex ideas". Duval (2003) concurs that moving across is not conventional because "*conversions* plays no intrinsic roles in mathematical processes of justification or proof"; they don't attract much attention "as if it were only a matter of an activity which is lateral and obvious and precedes the 'real' mathematical activity". However Duval (2003) emphasises the crucial role of *conversion*-type activities in developing understanding: "from a cognitive point of view it is, on the contrary, the activity of conversion which appears to be the fundamental representational transformation, the one which leads to the mechanisms underlying understanding".

Concluding remarks

In this article we looked at the ways in which a class of Grade 12 learners responded to tasks based on transformation geometry. In particular we considered the interview responses of six learners in trying to understand to what extent they utilised visual and analytic modes of representation when engaging in these tasks. Our data revealed that most learners performed *treatments*, mainly in the analytic mode, when responding to the tasks and there was little evidence of movement across the two modes. Such movement Zazkis *et al*. (1996) identify as a mechanism to contribute to deeper understanding. Duval (2003) emphasised that *conversion*-type activities which involve movement across different representation are essential for a deepening of understanding. In this study we identified one learner who displayed great facility in moving across different representations. We argued that his deep understanding of the concepts of transformation was both supported by, and also contributed to, his skill in moving between the visual and analytic representations.

The implications are that teachers should provide opportunities for learners to engage in activities which emphasise *conversion*, instead of just concentrating on *treatment*-type problems. Perhaps these types of activities should be part of the investigations of learners to originally discover the rule so that when they apply the rule as *treatments* later on, it will be with understanding. It would then also be possible for them to apply this type of reasoning again if they have forgotten the rule and can therefore not apply the *treatment*. As Zazkis *et al*. (1996:444) caution, even though both visual and analytical strategies may be available to learners, learners "often have difficulty making connections between them" and may (1996:455) "resist working with strategies they find uncomfortable or challenging." However, the effort is worthwhile because of the strong possibility of its facilitating a richer and more useful understanding.

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