Modelling and Analysis of a Holling Type II Stage Structured Predator-Prey System in the Presence of Harvesting

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Abstract

The effect of harvesting and predation on a Holling type II stage-structured predator-prey system with assumption that harvesting and predation happen only to mature preys was investigated. Stability of interior point was analysed using Routh Hurwith Criterion. Numerical simulations were carried out. The results show that harvesting and predation have strong negative impacts on population dynamics of stage structured predator-prey system.

Keywords: Predator-prey System, Stage Structure, Holling type II, Harvesting

Introduction

The dynamic relationships between predators and their preys have long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to their universal existence and importance (Kar 2005). Predator and prey interact by means of predation. A predator may prey on immature or mature individuals. A predator-prey system which includes the stages of growth for species is called stage structured predator-prey system. Much attention has been paid to biological systems with stage structures (Xu et al. 2004). This is due to the fact that there are many species whose individual members have a life history taking them through two stages, immature and mature. Thus considering stage structure in population, corresponds with the natural phenomenon (Zhang and Zhang 2016). The current study investigated a stage structured predator-prey system with Holling type II functional response, in which prey population is assumed to have two stages, immature and mature. By functional response we mean how predator responds to the prey density. In Holling type II functional response, the predator spends more time in searching for the prey at low prey densities, while the predator spends more time handling the prey at high prey densities.

In ecology, species live in communities where two or more species either share some basic requirements or compete for resources, habitats or territories (Andre 2014). However there are many factors that affect the dynamics of predator-prey interactions such as diseases, harvesting, prey refuge, among others (Naji and Majeed 2016). Many biological species have been driven to extinction and many others are at the verge of extinction due to several external forces such as over-exploitation, over-predation, environmental pollution and mismanagement of the habitats (Dubey 2007, Sagamiko et al. 2015). In order to protect these species, appropriate measures such as restriction on harvesting and creating reserved zones should be adopted in efforts to decrease the interactions of these species with external forces (Dubey 2007). Kar (2006) pointed out that harvesting has a strong impact on the dynamical evolution of a population subjected to it.

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Several studies have been conducted on the stage structured predator-prey system such as those by Georgescu and Hsieh (2007), Kar and Chattopadhyay (2010), Tian and Xu (2011), Gakkhar and Gupta (2017), Naji and Majeed (2016), but the aspect of logistic growth to both species with harvesting efforts applied to prey species has not been considered. Thus this study investigated the predator-prey system subjected to Holling type II functional response with stage structure to prey species, in the presence of harvesting.

Materials and Methods
Formulation of the model

Consider one prey and one predator system. Let *x(t)* and *y(t)* represent the population density of prey species and predator species at time *t*, respectively. The prey species population is divided into two sub-populations, namely the immature and mature sub-populations. Now let $x_1(t)$ and $x_2(t)$ represent the population densities of immature and mature prey species at time *t*, respectively. The population density of prey species at time *t* is given by $x(t) = x_1(t) + x_2(t)$. For clarity x, x_1, x_2 and y will be used throughout this study instead of $x(t)$, $x_1(t)$, $x_2(t)$ and $y(t)$, respectively.

The study adopts Holling type II functional response to the consumption of prey by individual predator. The model is developed from the following assumptions:
i. All external factors do not affect the system.
ii. The immature preys depend on mature preys.
iii. The immature prey population density grows logistically with intrinsic growth rate $r_1 > 0$ and environmental carrying capacity $k_1 > 0$ and they become mature at the rate $\sigma > 0$ while experiencing natural death at the rate $\mu_1 > 0$.
iv. The matured preys are harvested at constant per capita rate $h > 0$, and face natural death at the per capita rate $\mu_2 > 0$. They are

also captured by predators at the rate $a > 0$ (a is the predation rate), and predator's handling time is $\frac{b}{a} > 0$.

v. The predator depends on matured prey species with conversion factor $c > 0$. Also predator has different sources of food which leads to the population density of predators grow logistically with the rate $r_3 > 0$ and environmental carrying capacity $k_3 > 0$ in the absence of preys, and face natural death at the per capita rate $\mu_3 > 0$.

Following the above assumptions, the model equations will be described as

$$\frac{dx_1}{dt} = r_1 x_2 (1 - \frac{x_2}{k_1}) - (\mu_1 + \sigma)x_1,$$

$$\frac{dx_2}{dt} = \sigma x_1 - (\mu_2 + h)x_2 - \frac{ayx_2}{1 + bx_2},$$

$$\frac{dy}{dt} = r_3 y \left(1 - \frac{y}{k_3}\right) - \mu_3 y + c\left(\frac{ayx_2}{1 + bx_2}\right),$$

$$x_1, x_2, y > 0.$$

$$(1)$$

The model system (1) is solved in the region
$$\Omega = \{(x_1, x_2, y) \in R_{+}^3, \quad (x_1(0) = x_{10}, x_2(0) = x_{20}, y(0) = y_0)\}.$$

Boundedness of the System

Lemma 1 *The solutions* $(x_1, x_2, y) \in R_3^+$ *of the system (1) subjected to the given initial conditions are ultimately bounded in* R_3^+.

Proof 1: let $(x_1, x_2, y) \in R_3^+$ be any solution of the system (1) with initial conditions $x_1(0) = x_{10} \geq 0, x_2(0) = x_{20} \geq 0,$ and $y(0) = y_0 \geq 0$, we define a time dependent function which describes the total population density of species, as

$$\rho(t) = x_1 + cx_2 + y. \qquad (2)$$

Calculating the derivative of $\rho(t)$ along the solutions of the system (1), yields

$$\frac{d\rho}{dt} = \frac{dx_1}{dt} + c\frac{dx_2}{dt} + \frac{dy}{dt}.$$

Substituting equations of the system (1) into equation (3), results into

$$\frac{d\rho}{dt} = r_1 x_2 \left(1 - \frac{x_2}{k_1}\right) + r_3 y \left(1 - \frac{y}{k_3}\right) - (\mu_1 + \sigma - c\sigma)x_1 - c(\mu_2 + h)x_2 - \mu_3 y.$$

Let $a_0 = \mu_1 + \sigma - c\sigma$ and $b_0 = \mu_2 + h$, we have

$$\frac{d\rho}{dt} = r_1 x_2 \left(1 - \frac{x_2}{k_1}\right) + r_3 y \left(1 - \frac{y}{k_3}\right) - a_0 x_1 - cb_0 x_2 - \mu_3 y.$$

For any arbitrary λ chosen arbitrary, we have

$$\frac{d\rho}{dt} + \lambda\rho = r_1 x_2 \left(1 - \frac{x_2}{k_1}\right) + r_3 y \left(1 - \frac{y}{k_3}\right) - (a_0 - \lambda)x_1 - c(b_0 - \lambda)x_2 - (\mu_3 - \lambda)y.$$

Choose $\lambda = \min(a_0, b_0, \mu_3)$, we have

$$\frac{d\rho}{dt} + \lambda\rho \le r_1 x_2 \left(1 - \frac{x_2}{k_1}\right) + r_3 y \left(1 - \frac{y}{k_3}\right)$$

Now $\max\left(r_1 x_2 \left(1 - \frac{x_2}{k_1}\right)\right) = \frac{k_1 r_1}{4}$ and

$\max\left(r_3 y \left(1 - \frac{y}{k_3}\right)\right) = \frac{k_3 r_3}{4}$ this implies that

$$\frac{d\rho}{dt} + \lambda\rho \le \frac{k_1 r_1}{4} + \frac{k_3 r_3}{4}.$$

Let $K = \frac{k_1 r_1}{4} + \frac{k_3 r_3}{4}$, thus we have

$$\frac{d\rho}{dt} + \lambda\rho \le K.$$

Using integrating factor $I = e^{\lambda t}$, we have

$$\rho(t) \le \frac{K}{\lambda} + ce^{-\lambda t},$$

at $t = 0$, $\rho(0) = \rho_0$ thus

$$\rho(t) \le \frac{K}{\lambda} + \left(\rho_0 - \frac{K}{\lambda}\right)e^{-\lambda t} \le \max\left\{\rho_0, \frac{K}{\lambda}\right\}$$

Therefore

$$0 \le \rho(t) \le \frac{K}{\lambda} + \left(\rho_0 - \frac{K}{\lambda}\right)e^{-\lambda t}.$$

As $t \xrightarrow{(3)} \infty$, it gives

$$0 \le \rho(t) \le \frac{K}{\lambda},$$

hence ρ is bounded in $\mathbb{R}_3^+$, implies x_1, x_2, and y are bounded. This completes the proof. $\square$

Equilibrium solutions of the system
This section establishes conditions for existence of equilibrium points of the system (1). By equating $x_1' = x_2' = y' = 0$, the system (1) becomes

$$r_1 \tilde{x}_2 \left(1 - \frac{\tilde{x}_2}{k_1}\right) - (\mu_1 + \sigma)\tilde{x}_1 = 0,$$

$$\sigma \tilde{x}_1 - (\mu_2 + h)\tilde{x}_2 - \frac{a\tilde{y}\tilde{x}_2}{1 + b\tilde{x}_2} = 0, \qquad (4)$$

$$r_3 \tilde{y} \left(1 - \frac{\tilde{y}}{k_3}\right) - \mu_3 \tilde{y} + c\left(\frac{a\tilde{y}\tilde{x}_2}{1 + b\tilde{x}_2}\right) = 0.$$

Solving for $\tilde{y}$ from the third equation of the system (4), we get

$$\tilde{y} = 0, \qquad (5)$$

or

$$\tilde{y} = \frac{k_3}{r_3}\left(r_3 - \mu_3 + \frac{ca\tilde{x}_2}{1 + b\tilde{x}_2}\right). \qquad (6)$$

In the absence of predators, we have $\tilde{y} = 0$. Thus, substituting $\tilde{y} = 0$ into second equation of the system (4), yields

$$\sigma \tilde{x}_1 - (\mu_2 + h)\tilde{x}_2 = 0,$$

implying that

$$\tilde{x}_2 = \left(\frac{\sigma}{\mu_2 + h}\right)\tilde{x}_1. \qquad (7)$$

Now, substituting equation (7) into the first equation of the system (4), yields

$$\tilde{x}_1 \left(\frac{\sigma r_1}{\mu_2 + h} \right) \left(1 - \frac{\sigma}{(\mu_2 + h)k_1} \tilde{x}_1 \right) - (\sigma + \mu_1)\tilde{x}_1 = 0,$$

implying that

$$\tilde{x}_1 = 0 \tag{8}$$

or

$$\tilde{x}_1 = \frac{k_1(\mu_2 + h)}{\sigma}(1 - Q_0), \tag{9}$$

where $Q_0 = \left(\frac{(\mu_1 + \sigma)(\mu_2 + h)}{\sigma r_1} \right).$

Thus, when $\tilde{x}_1 = 0$, implies $\tilde{x}_2 = 0$ (according to equation (7)). Hence the system (1) has trivial equilibria $E_0(0,0,0)$.

Similarly, when $\tilde{x}_1 \neq 0$, implies $\tilde{x}_2 = k_1(1 - Q_0)$. Hence the system (1) has predator free equilibria $E_1(\tilde{x}_1, \tilde{x}_2, 0)$. For positive E_1 we need $\tilde{x}_1 > 0$ and $\tilde{x}_2 > 0$, implies the equilibria E_1 exists in R_3^+ if

$$Q_0 < 1. \tag{10}$$

Therefore, in absence of predator, the threshold value Q_0 should be less than one, for the predator free equilibrium E_1 to exist; otherwise prey population goes to extinction. The Q_0 is the ratio between maximum death rate to the maximum growth rate of prey, it describes the average decrease of prey population. The results agree with the model assumptions that, prey population grows logistically in absence of predators. The following theorem summarizes the results.

Theorem 1 *The predator free equilibrium point E_1 of the system (1) exists for $\tilde{x}_1 > 0$ and $\tilde{x}_2 > 0$, if $Q_0 < 1$, otherwise it does not exist.*
In the presence of predator population, we have $\tilde{y} \neq 0$. Now, substituting equation (6) into the second equation of the system (4), yields

$$\sigma \tilde{x}_1 - (\mu_2 + h)\tilde{x}_2 - \left(\frac{ack_3 \tilde{x}_2}{r_3(1 + b\tilde{x}_2)} \right) \left(r_3 - \mu_3 + \frac{ac\tilde{x}_2}{1 + b\tilde{x}_2} \right) = 0,$$

Making $\tilde{x}_1$ the subject, we get

$$\tilde{x}_1 = \frac{1}{\sigma} \left((\mu_2 + h)\tilde{x}_2 + \left(\frac{ack_3 \tilde{x}_2}{r_3(1 + b\tilde{x}_2)} \right) \left(r_3 - \mu_3 + \frac{ac\tilde{x}_2}{1 + b\tilde{x}_2} \right) \right). \tag{11}$$

Now, substituting equation (11) into the first equation of the system (4), yields

$$r_1 \tilde{x}_2 \left(1 - \frac{\tilde{x}_2}{k_1} \right) - \frac{\mu_1 + \sigma}{\sigma} \left((\mu_2 + h)\tilde{x}_2 + \left(\frac{ack_3 \tilde{x}_2}{r_3(1 + b\tilde{x}_2)} \right) \left(r_3 - \mu_3 + \frac{ac\tilde{x}_2}{1 + b\tilde{x}_2} \right) \right) = 0,$$

implying that

$$\tilde{x}_2 = 0, \tag{12}$$

or

$$\tilde{x}_2^3 + a_2 \tilde{x}_2^2 + a_1 \tilde{x}_2 + a_0 = 0, \tag{13}$$

where

$$a_0 = \left(\frac{1}{b^2} \right) \left[D_1(r_3 - \mu_3) - k_1(1 - Q_0) \right],$$

$$a_1 = \left(\frac{1}{b^2} \right) \left[1 - 2k_1 b(1 - Q_0) - D_1 b(r_3 - \mu_3) - acD_1 \right],$$

$$a_2 = \frac{2}{b} - k_1(1 - Q_0),$$

$$D_1 = \frac{ak_1 k_3(\mu_1 + \sigma)}{\sigma r_1 r_3}.$$

For $\tilde{x}_2 = 0$, implies $\tilde{x}_1 = 0$ (according to equation (11)) and $\tilde{y} = k_3 \left(1 - \frac{\mu_3}{r_3} \right)$ (according to equation (6)). Thus, the system (1) has prey free equilibria $E_2(0, 0, \tilde{y})$.

For positive E_2 we need $\tilde{y} > 0$, implies that the prey free equilibria exists in R_3^+ if $r_3 > \mu_3$. Therefore in absence of prey population, the growth rate r_3 of predator population should be greater than its death rate μ_3 for the prey free equilibrium E_2 to exist, otherwise the population goes to extinction. This agrees with the model assumption that the predator has different sources of food which leads to its population to grow logistically in

absence of preys. The following theorem summarizes the results.

Theorem 2 *The prey free equilibrium point* E_2 *of the system (1) exists for* $\tilde{y} > 0$*, if* $r_3 > \mu_3$*, otherwise it does not exist.*

For $\tilde{x}_2 \neq 0$, we have the cubic equation (13). Using Sturm's theorem together with Descate's Rule of Signs to discuss the possible real zeros of the cubic equation (13).

The Sturm sequence corresponding to equation (13) is given by

$$p_0(\tilde{x}_2) = \tilde{x}_2^3 + a_2\tilde{x}_2^2 + a_1\tilde{x}_2 + a_0,$$

$$p_1(\tilde{x}_2) = 3\tilde{x}_2^2 + 2a_2\tilde{x}_2 + a_1,$$

$$p_2(\tilde{x}_2) = (\frac{2}{9}a_2^2 - \frac{2}{3}a_1)\tilde{x}_2 + (\frac{1}{9}a_1a_2 - a_0)\quad,$$

and

$$p_3(\tilde{x}_2) = \frac{1}{w_1^2}(2a_2w_1w_2 - 3w_2^2 - w_1^2a_2),$$

where $\qquad w_1 = \frac{2}{9}a_2^2 - \frac{2}{3}a_1 \qquad$ and

$$w_2 = \frac{1}{9}a_1a_2 - a_0.$$

For the roots to be biologically feasible to this study, the cubic equation (13) should have positive real zeros. So now $\tilde{x}_2 \in (0, \alpha)$ and $\alpha \in \mathrm{I}^+$. Evaluating Sturm's sequence at $\tilde{x}_2 = \alpha$ gives,

$$p_0(\alpha) = \alpha^3 + a_2\alpha^2 + a_1\alpha + a_0,$$

$$p_1(\alpha) = 3\alpha^2 + 2a_2\alpha + a_1,$$

$$p_2(\alpha) = (\frac{2}{9}a_2^2 - \frac{2}{3}a_1)\alpha + (\frac{1}{9}a_1a_2 - a_0)\quad,$$

and

$$p_3(\alpha) = \frac{1}{w_1^2}(2a_2w_1w_2 - 3w_2^2 - w_1^2a_2).$$

Descate's Rule of Signs gives six cases on Sturm's sequence evaluated at $\tilde{x}_2 = \alpha$, for cubic equation (13) to have real roots in $(-\infty, \alpha]$.

Case I

If $p_0(\alpha) > 0$, $p_1(\alpha) > 0$, $p_2(\alpha) > 0$ and $p_3(\alpha) > 0$, then the equation (13) will have no positive real zeros in R^+. Now $p_0(\alpha) > 0$ and $p_1(\alpha) > 0$ if $Q_0 > 1$, $r_3 < \mu_3$ and $acD_1 < 1$ where $Q_0 = \left(\frac{(\mu_1 + \sigma)(\mu_2 + h)}{\sigma r_1}\right)$.

And $p_2(\alpha) > 0$ if $Q_0 < 1$, $r_3 > \mu_3$ and $Q_1 > 2$, where $Q_1 = bk_1(1 - Q_0)$. Since $p_0(\alpha) > 0$, $p_1(\alpha) > 0$ and $p_2(\alpha) > 0$ cannot happen simultaneously, then this case is not biologically feasible to this study.

Case II

If $p_0(\alpha) < 0$, $p_2(\alpha) > 0$, $p_3(\alpha) > 0$ with $p_1(\alpha) > 0$ or $p_1(\alpha) < 0$, then the equation (13) will have exactly one positive real zero in R^+. This condition is biologically feasible to this study if $p_1(\alpha) < 0$. So $p_0(\alpha) < 0$ if $\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0 < 0$, $p_1(\alpha) < 0$ if $3\alpha^2 + 2a_2\alpha + a_1 < 0$, $p_2(\alpha) > 0$ if $Q_0 < 1$, $r_3 > \mu_3$ and $Q_1 > 2$, and $p_3(\alpha) > 0$ if $Q_2 > 0$, where $Q_2 = 2a_2w_1w_2 - 3w_2^2 - w_1^2a_2$. Thus, system (1) has one positive interior equilibrium E_3. Therefore, Co-existence equilibrium point E_3 exists in R_3^+ if the above conditions hold under case II. The following theorem summarizes the results.

Theorem 3 *An equilibrium solution* $E_3(\tilde{x}_1, \tilde{x}_2, \tilde{y})$ *of the system (1) exists for* $\tilde{x}_1 > 0$*,* $\tilde{x}_2 > 0$ *and* $\tilde{y} > 0$ *if* E_1 *and* E_2 *also exists with* $Q_1 > 2$ *and* $Q_2 > 0$ *otherwise it does not exist.*

Theorem 4 *Three equilibrium solutions* E_1, E_2 *and* E_3 *of the system (1) co-exist if* $Q_0 < 1$, $r_3 > \mu_3$, $Q_1 > 2$ *and* $Q_2 > 0$.

Case III

If $p_0(\alpha) > 0$, $p_1(\alpha) > 0$, $p_2(\alpha) > 0$ and $p_3(\alpha) < 0$, then equation (13) will have exactly one positive real zero in $\mathbb{R}^+$. Since $p_0(\alpha) > 0$, $p_1(\alpha) > 0$ and $p_2(\alpha) > 0$ cannot happen simultaneously, then this case is not biologically feasible to this study.

Case IV

If $p_0(\alpha) < 0$, $p_2(\alpha) > 0$, $p_3(\alpha) < 0$ with $p_1(\alpha) > 0$ or $p_1(\alpha) < 0$, then equation (13) will have two or no positive real zeros. This condition is biologically feasible to this study if $p_1(\alpha) < 0$. So $p_3(\alpha) < 0$ if $Q_2 < 0$.

Thus, the system of equations (1) may have two or zero positive interior equilibria $E_{3,4}$.

Therefore, co-existence equilibrium point $E_{3,4}$ exists in $\mathbb{R}_3^+$ if the above conditions hold under case IV. The following theorem summarizes the results.

Theorem 5: *Four equilibrium solutions* E_1, E_2, E_3 *and* E_4 *of the system (1) co-exist if* $Q_0 < 1$, $r_3 > d_3$, $Q_1 > 2$ *and* $Q_2 < 0$.

Lemma 2: *The system of equations (1) has two interior equilibrium points.*

Theorem 6: *If the system has two interior equilibrium points, then there exists backward bifurcation.*

Case V

If $p_0(\alpha) > 0$, $p_1(\alpha) < 0$, $p_2(\alpha) > 0$ and $p_3(\alpha) > 0$, then the equation (13) will have two or zero (none) positive real zeros in $\mathbb{R}^+$. Since $p_0(\alpha) > 0$ and $p_2(\alpha) > 0$ cannot happen simultaneously, then this case is not biologically feasible to this study.

Case VI

If $p_0(\alpha) > 0$, $p_1(\alpha) < 0$, $p_2(\alpha) > 0$ and $p_3(\alpha) < 0$, then equation (13) will have three or one positive real zeros in $\mathbb{R}^+$. Since $p_0(\alpha) > 0$ and $p_2(\alpha) > 0$ cannot happen simultaneously, then this case is not biologically feasible to this study.

Local stability of the equilibrium points

The stability of each equilibrium point is studied by computing Jacobian matrix and finding the eigenvalues evaluated at each equilibrium point. The real parts of eigenvalues of the Jacobian matrix must be negative for stability of equilibrium points. The corresponding Jacobian matrix for the system (1) is

$$J = \begin{pmatrix} -(\mu_1 + \sigma) & -\dfrac{2r_1\tilde{x}_2}{k_1} + r_1 & 0 \\ \sigma & -\left(\mu_2 + h + \dfrac{a\tilde{y}}{(1+b\tilde{x}_2)^2}\right) & -\dfrac{a\tilde{x}_2}{1+b\tilde{x}_2} \\ 0 & \dfrac{ca\tilde{y}}{(1+b\tilde{x}_2)^2} & (r_3 - \mu_3) - \dfrac{2r_3\tilde{y}}{k_3} + \dfrac{ca\tilde{x}_2}{1+b\tilde{x}_2} \end{pmatrix}.$$

Local stability of boundary equilibrium points

The Jacobian matrix of the system (1) evaluated at equilibrium point $E_0(0,0,0)$ is given by

$$J_0 = \begin{pmatrix} -(\mu_1 + \sigma) & r_1 & 0 \\ \sigma & -(\mu_2 + h) & 0 \\ 0 & 0 & (r_3 - \mu_3) \end{pmatrix}.$$

The corresponding characteristics polynomial is

$$(r_3 - \mu_3 - \lambda)\left[\lambda^2 + \rho_1\lambda + \rho_2\right] = 0,$$

where

$$\rho_1 = (\sigma + \mu_1 + \mu_2 + h),$$
$$\rho_2 = -\sigma r_1(1 - Q_0),$$

where $Q_0 = \left(\dfrac{(\mu_1 + \sigma)(\mu_2 + h)}{\sigma r_1}\right)$.

By using Routh-Hurwitz criteria, the necessary conditions for equilibrium point E_0 of the system (1) to be locally asymptotically stable

are $\rho_1 > 0$ and $\rho_2 > 0$, implies $r_3 < \mu_3$ and $Q_0 > 1$, otherwise the population free equilibrium point E_0 of the system (1) is unstable. The following theorem summarizes the results.

Theorem 7 *The equilibrium point E_0 of the system (1) is locally asymptotically stable if $Q_0 > 1$ and $r_3 < \mu_3$, otherwise is unstable.*

The Jacobian matrix evaluated at predator free equilibrium point

$$E_1\left(\frac{k_1(\mu_2 + h)}{\sigma}(1 - Q_0), k_1(1 - Q_0), 0\right) \quad \text{of the}$$

system (1) is given by

$$J_2 = \begin{pmatrix} -(\mu_1 + \sigma) & r_1(2Q_0 - 1) & 0 \\ \sigma & -(\mu_2 + h) & \dfrac{ak_1(1 - Q_0)}{bk_1(Q_0 - 1) - 1} \\ 0 & 0 & r_3 - \mu_3 + \dfrac{ack_1(1 - Q_0)}{bk_1(1 - Q_0) + 1} \end{pmatrix}.$$

The corresponding characteristic equation is

$$\left(r_3 - \mu_3 + \frac{ack_1(1 - Q_0)}{bk_1(1 - Q_0) + 1} - \lambda\right)(\lambda^2 + \rho_1\lambda - \rho_2) = 0,$$

By Routh-Hurwitz criteria, the necessary conditions for predator free equilibrium point E_1 of the system (1) to be locally asymptotically stable are $\rho_1 > 0$, $\rho_2 < 0$, $\mu_3 > r_3$ and $M < 1$, where

$$M = \left(\frac{ack_1(1 - Q_0)}{\left[bk_1(1 - Q_0) + 1\right](\mu_3 - r_3)}\right) \quad . \quad \text{This}$$

would imply $r_3 < \mu_3$, $Q_0 < 1$, and $M < 1$, otherwise the predator free equilibrium point E_1 of the system (1) is unstable. The following theorem summarizes the results.

Theorem 8 *The predator free equilibrium point E_1 of the system (1) is locally asymptotically stable if $r_3 < \mu_3$, $Q_0 < 1$, and $M < 1$, otherwise is unstable.*

The Jacobian matrix of the system (1) evaluated at prey free equilibrium point $E_2\left(0, 0, k_3\left(1 - \dfrac{\mu_3}{r_3}\right)\right)$ is given by

$$J_1 = \begin{pmatrix} -(\mu_1 + \sigma) & r_1 & 0 \\ \sigma & -\left((\mu_2 + h) + ak_3\left(1 - \dfrac{\mu_3}{r_3}\right)\right) & 0 \\ 0 & ack_3\left(1 - \dfrac{\mu_3}{r_3}\right) & -(r_3 - \mu_3) \end{pmatrix}.$$

The corresponding characteristic polynomial is

$$(-(r_3 - \mu_3) - \lambda)(\lambda^2 + \rho_3\lambda + \rho_4) = 0,$$

where

$$\rho_3 = (\mu_1 + \sigma + \mu_2 + h) + ak_3\left(1 - \frac{\mu_3}{r_3}\right),$$

$$\rho_4 = -\sigma r_1(1 - Q_0) + ak_3(\mu_1 + \sigma)\left(1 - \frac{\mu_3}{r_3}\right)$$

By using Routh-Hurwitz criteria, the necessary conditions for prey free equilibrium point E_2 of system (1) to be locally asymptotically stable are $r_3 > \mu_3$ and $\rho_{3,4} > 0$, implies $Q_0 > 1$ and $r_3 > \mu_3$. Otherwise the prey free equilibrium point E_2 of system (1) is unstable. The following theorem summarizes the results.

Theorem 9 *The prey free equilibrium point E_2 of the system (1) is locally asymptotically stable if $Q_0 > 1$ and $r_3 > \mu_3$, otherwise is unstable.*

Local stability of interior equilibrium point(s)

The Jacobian matrix of the system (1) evaluated at Co-existence equilibrium point $E_3(\tilde{x}_1, \tilde{x}_2, \tilde{y})$ is given by

$$
J_3 = \begin{pmatrix}
-(\mu_1 + \sigma) & -\dfrac{2r_1\tilde{x}_2}{k_1} + r_1 & 0 \\[3mm]
\sigma & -(\mu_2 + h + \dfrac{a\tilde{y}}{(1+b\tilde{x}_2)^2}) & -\dfrac{a\tilde{x}_2}{1+b\tilde{x}_2} \\[3mm]
0 & \dfrac{ca\tilde{y}}{(1+b\tilde{x}_2)^2} & (r_3 - \mu_3) - \dfrac{2r_3\tilde{y}}{k_3} + \dfrac{ca\tilde{x}_2}{1+b\tilde{x}_2}
\end{pmatrix}.
$$

Substituting $\tilde{y} = \dfrac{k_3}{r_3}\left(r_3 - \mu_3 + \dfrac{ca\tilde{x}_2}{1+b\tilde{x}_2}\right)$ from equation (6) into J_3, yields:

$$
J_3 = \begin{pmatrix}
-(\mu_1 + \sigma) & -\dfrac{2r_1\tilde{x}_2}{k_1} + r_1 & 0 \\[3mm]
\sigma & -\left(\mu_2 + h + \dfrac{ak_3}{r_3(1+b\tilde{x}_2)^2}\left(r_3 - \mu_3 + \dfrac{ca\tilde{x}_2}{1+b\tilde{x}_2}\right)\right) & -\dfrac{a\tilde{x}_2}{1+b\tilde{x}_2} \\[3mm]
0 & \dfrac{ack_3}{r_3(1+b\tilde{x}_2)^2}\left(r_3 - \mu_3 + \dfrac{ac\tilde{x}_2}{1+b\tilde{x}_2}\right) & (\mu_3 - r_3) - \dfrac{ca\tilde{x}_2}{1+b\tilde{x}_2}
\end{pmatrix}.
$$

The corresponding characteristics polynomial is given by

$$\lambda^3 + \gamma_1\lambda^2 + \gamma_2\lambda + \gamma_3 = 0,$$

where

$$\gamma_1 = (\mu_1 + \sigma + \mu_2 + h) + \left(1 + \dfrac{ak_3}{r_3(1+b\tilde{x}_2)^2}\right)\left(r_3 - \mu_3 + \dfrac{ac\tilde{x}_2}{1+b\tilde{x}_2}\right),$$

$$\gamma_2 = \left(\mu_1 + \sigma + \mu_2 + h + \dfrac{ack_3\tilde{x}_2}{r_3(1+b\tilde{x}_2)^3} + \dfrac{ak_3(\mu_1+\sigma)}{r_3(1+b\tilde{x}_2)^2}\right)\left(r_3 - \mu_3 + \dfrac{ac\tilde{x}_2}{1+b\tilde{x}_2}\right) + \dfrac{ak_3}{r_3(1+b\tilde{x}_2)^2}\left(r_3 - \mu_3 + \dfrac{ac\tilde{x}_2}{1+b\tilde{x}_2}\right)^2 +$$

$$\sigma r_1(1-Q_0), \text{ and}$$

$$\gamma_3 = \left(\dfrac{a^2ck_3(\mu_1+\sigma)\tilde{x}_2}{r_3(1+b\tilde{x}_2)^3} + \sigma r_1(1-Q_0)\right)\left(r_3 - \mu_3 + \dfrac{ac\tilde{x}_2}{1+b\tilde{x}_2}\right) + \dfrac{ak_3}{r_3(1+b\tilde{x}_2)^2}\left(r_3 - \mu_3 + \dfrac{ac\tilde{x}_2}{1+b\tilde{x}_2}\right)^2.$$

By using Routh Hurwitz criteria, the necessary conditions for equilibrium point E_3 of the system (1) to be locally asymptotically stable are $\gamma_1 > 0$, $\gamma_3 > 0$ and $\gamma_1\gamma_2 > \gamma_3$ otherwise the equilibrium point E_3 of the system (1) is unstable. The above conditions are satisfied if $Q_0 < 1$, $r_3 > \mu_3$ and $a < 1$. The following theorem summarizes the results.

Theorem 10: *The equilibrium point E_3 of the system (1) is locally asymptotically stable if $Q_0 < 1$, $r_3 > \mu_3$ and $a < 1$, otherwise is unstable.*

Results

The effect of harvesting and predation on stage structured predator-prey system is investigated by using the following parameter values: $r_1 = 1.82$, $r_3 = 0.008$, $\mu_1 = 0.001$, $\mu_2 = 0.05$, $\mu_3 = 0.009$, $k_1 = 300$, $k_3 = 50$, $\sigma = 1.5$, $h = 0.5$, $a = 0.674$, $b = 1$, $c = 0.125$. The results are presented in Figures 1-4.

Discussion

The effect of harvesting on the structured predator-prey system was simulated and analysed numerically and the discussion of results is as follows.

Figure 1 shows the phase portraits for co-existence of equilibrium points and that there are orbitally asymptotically stable periodic solutions in the presence of both harvesting and predation. Figure 2(a) shows the increase of immature prey population resulting from the absence of predation and harvesting to matured prey species. The increase of immature prey population follows the assumption that all immature prey species are being raised by their matured prey species. Similarly from the Figure 2(b) it is observed that there is an increase in matured prey population to the carrying capacity at different maturity rates. There is also a decrease of matured prey population when there is no immature prey species that become matured (that is, when $\sigma = 0$). Figure 2(c) shows the increase of predator population caused by other sources of food.

Figures 3(a) and 3(b) show the increase of prey population densities at low predation and harvesting. These observations follow the above results that, low harvesting and predation have less effect on prey population densities. Similarly, we observe the decrease of prey population density at high predation and harvesting. These observations follow the results that high harvesting and predation cause severe effect on prey population densities. Also, Figure 3(c) shows the increase of predator population due to consuming matured prey species. At high maturity rate, it is observed from Figures 4(a) and 4(b) the increase of prey populations to its carrying capacity.
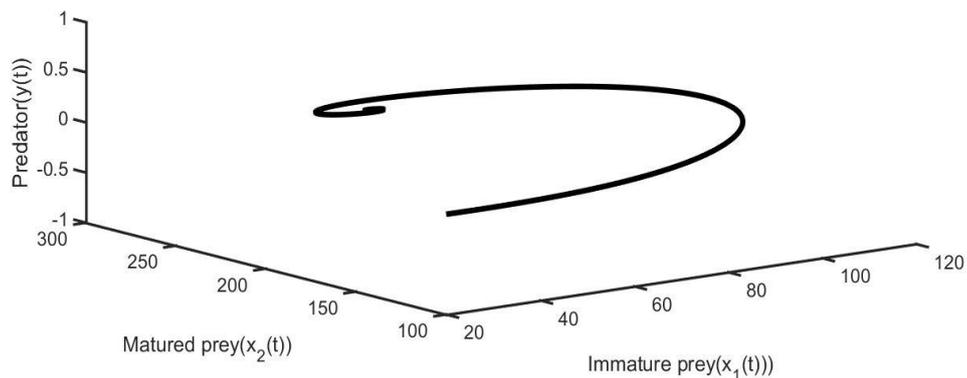

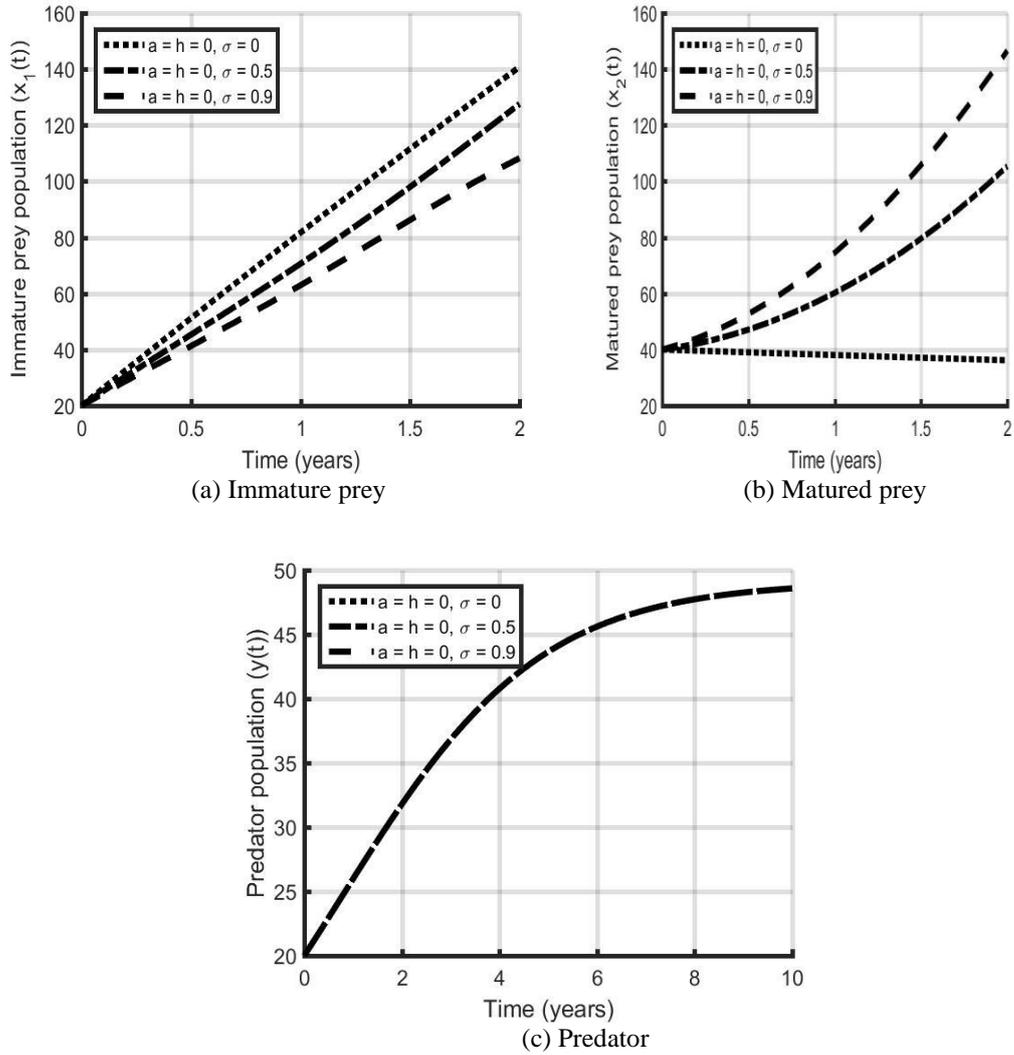**Figure 1:** Phase diagram for the dynamical behaviour of interior equilibrium point E_3.

(a) Immature prey

(b) Matured prey

(c) Predator

Figure 2(a)-(c): The effect of Stage structure on predator and prey population densities.
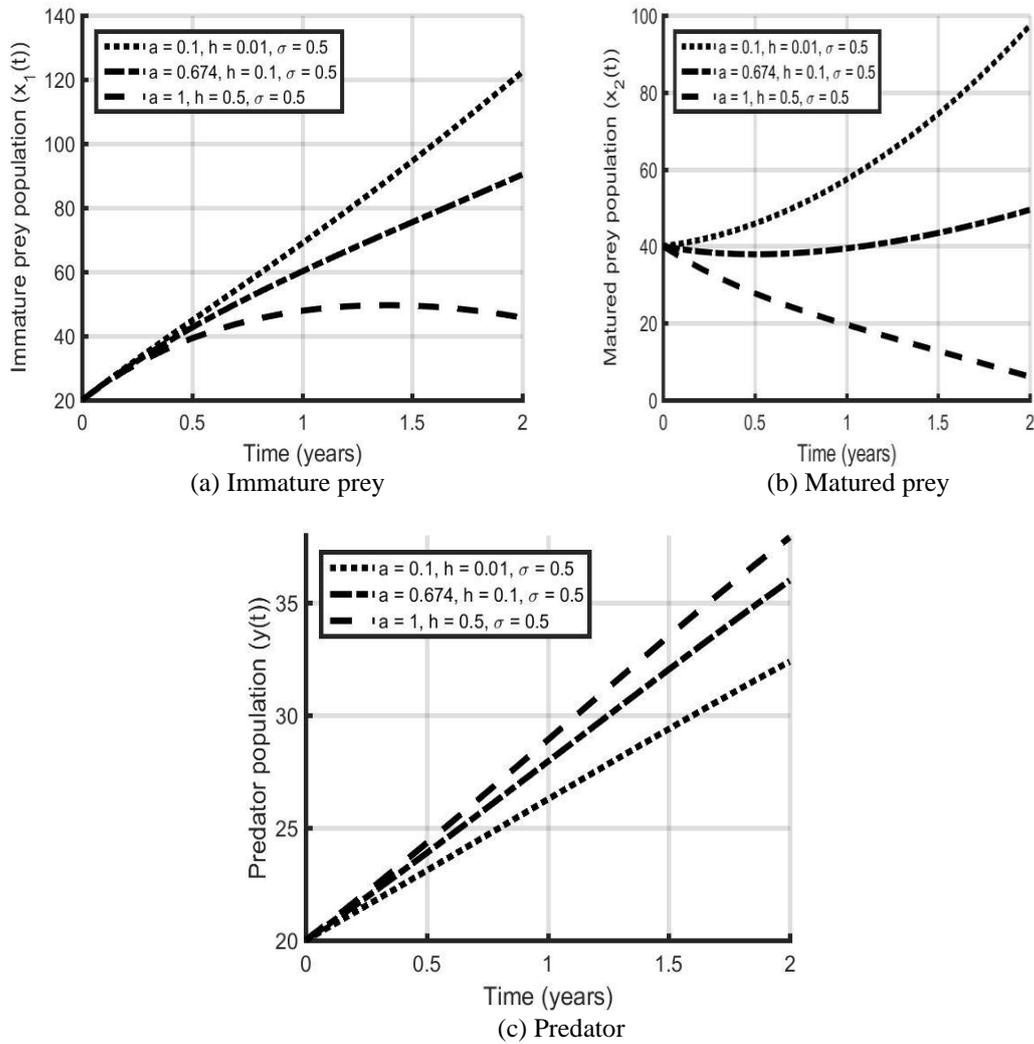
(a) Immature prey

(b) Matured prey

(c) Predator

Figure 3(a)-(c): The effect of harvesting and predation on predator and prey population densities at $\sigma = 0.5$.

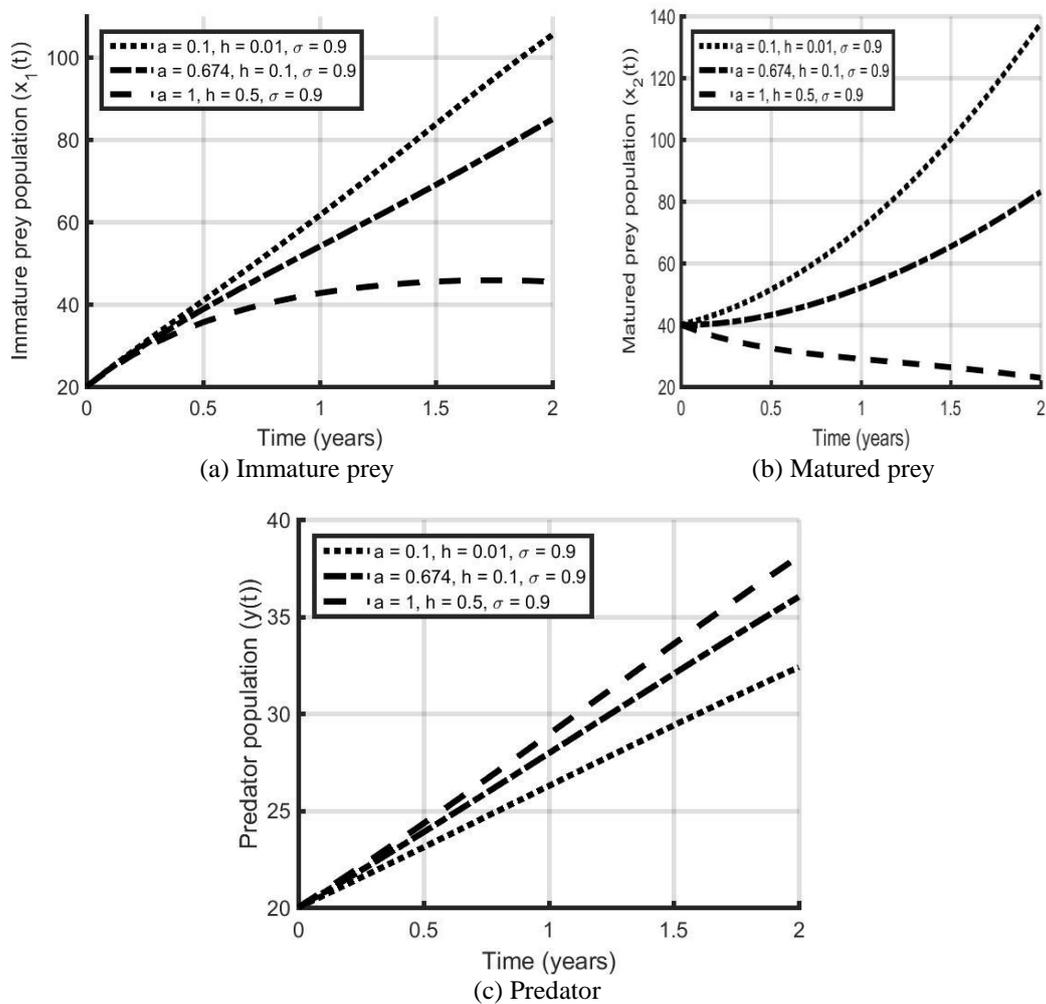(a) Immature prey

(b) Matured prey

(c) Predator

Figure 4(a)-(c): The effect of harvesting and predation on predator and prey population densities at $\sigma = 0.9$.

Conclusion

In summary this study concludes that the population of preys and predators cannot exceed their environmental carrying capacity. It has been shown analytically that, in the absence of prey population, the predator's growth rate should be greater than its death rate for it to persist otherwise the population will perish. Also in the absence of predator population, the average decrease of prey population should be less than one, for the prey population to persist, otherwise the population will perish. The two populations co-exist if the threshold value is less than one. Numerical results have shown that the rate at which immature preys become mature plays a crucial role in the existence of prey species. Harvesting and predation have negative impacts on population dynamics of stage structured prey-predator system. Thus, the study suggests that, future work should include life stages of predator species.

References
Andre MD 2014 *Modelling population dynamics*. The University of Amsterdam.

Dubey B 2007 A prey-predator model with a reserved area. *Nonlin. Anal.: Model. Contr.* 12: 479-494.

Gakkhar S and Gupta K 2017 Dynamics of a stage-structured predator-prey model. *Int. J. Appl. Phys. Math.* 7(1): 24-32.

Georgescu P and Hsieh YH 2007 Global dynamics of predator-prey model with stage structure for predator. *SIAM J. Appl. Math.* 67(5): 1379-1395.

Kar TK 2005 Stability analysis of a prey predator model incorporating a prey refuge. C*ommunicat. Nonlin. Sci. Numer. Simulat.* 10(6): 681-691.

Kar TK 2006 Modelling and Analysis of a harvested prey predator system incorporating a prey refuge. *J. Comput. Appl. Math.* 185: 19-33.

Kar TK and Chattopadhyay SK 2010 A dynamic reaction model of a prey-predator system with stage-structure for predator. *Modern Applied Science* 4(5): 183.

Naji RK and Majeed SJ 2016 The dynamical analysis of a prey-predator model with are fuge-stage structure prey population. *Int. J. Different. Equat.* 2016: 2010464.

Sagamiko TD, Shaban N, Nahonyo CL and Makinde OD 2015 Optimal control of a threatened wildebeest-lion prey-predator system in the Serengeti ecosystem. *Open J. Ecol.* 5: 110-119.

Tian X and Xu R 2011 Global dynamics of a predator-prey system with Holling type II functional response. *Nonlinear Analysis: Modelling and Control* 16(2): 242-253.

Xu R, Chaplain MAJ and Davidson FA 2004 Persistence and global stability of a ratio-dependent predator-prey model with stage structure. *Appl. Math. Comput.* 158: 729-744.

Zhang LM and Zhang CF 2016 Permanence, periodicity and extinction of a delayed biological system with stage structured preference for predator. *J. Appl. Math. Phys.* 4: 546-560.